Semigroup Forum
DOI 10.1007/s00233-014-9683-z

RESEARCH ARTICLE

Functor homology and homology of commutative monoids

Revaz Kurdiani · Teimuraz Pirashvili

Received: 29 July 2014 / Accepted: 12 December 2014
© Springer Science+Business Media New York 2015

Abstract We show that methods of functor homology can be applied to monoids.

Keywords Cohomology of commuatative monoids · Harrison homology · Functor homology

The aim of this work is to clarify the relationship between homology theory of commutative monoids constructed á la Quillen [16,17] and technology of Γ-modules as it was developed in [12–15]. It should be pointed out that the cohomology theory of commutative monoids was first constructed by P.-A. Grillet in the series of papers [2–7] (see also the recent work [1]). So our results shed light on Grillet theory. For instance, we relate the commutative monoid (co)homology with André–Quillen (co)homology of corresponding monoid algebra. For another application we mention the Hodge decomposition for commutative monoid (co)homology which is an immediate consequence of our main result.

In Sect. 1 we recall the basics of Γ-modules and their relation with commutative algebra (co)homology. In Sect. 2 we construct an analogue of Kähler differentials for commutative monoids. In Sect. 3 we construct the homology theory for commutative monoids and we prove our main result, which states that commutative monoid homology is a particular case of the functor homology developed in [13].

Communicated by László Márki.

R. Kurdiani
Ivane Javakhishvili Tbilisi State University, Tbilisi, Georgia

T. Pirashvili (✉)
Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, UK
e-mail: tp59@le.ac.uk

1 Γ-modules and commutative algebra (co)homology

1.1 Notations

Let K be a field which is fixed in the whole paper. In what follows all vector spaces are defined over K. Moreover, $\otimes_K$ and Hom_K are denoted by $\otimes$ and Hom respectively. Let **Vect** be the category of vector spaces. For any set S, we let $K[S]$ be the free vector space spanned by S. For a vector space V, its dual vector space is denoted by $V^\sharp$.

1.2 Generalities on functor categories

Let $\mathbf{C}$ be a small category. A *left* $\mathbf{C}$-*module* is a covariant functor $\mathbf{C} \to \mathbf{Vect}$, while a *right* $\mathbf{C}$-*module* is a contravariant functor $\mathbf{C}^{op} \to \mathbf{Vect}$. The category of all left $\mathbf{C}$-modules is denoted by $\mathbf{C}$-*mod*, while the category of all right modules is denoted by *mod*-$\mathbf{C}$. It is well-known that the categories $\mathbf{C}$-*mod* and *mod*-$\mathbf{C}$ are abelian categories with sufficiently many projective and injective objects. For any object c of the category $\mathbf{C}$ one defines the left $\mathbf{C}$-module $\mathbf{C}^c$ and the right $\mathbf{C}$-module $\mathbf{C}_c$ by

$$\mathbf{C}^c(x) = K[\mathsf{Hom}_{\mathbf{C}}(c, x)], \text{ and } \mathbf{C}_c(x) = K[\mathsf{Hom}_{\mathbf{C}}(x, c)].$$

It follows from the Yoneda lemma that the collection $\mathbf{C}^c$ (resp. $\mathbf{C}_c^\sharp$), where $c \in Ob(\mathbf{C})$ form a system of projective (resp. injective) (co)generators of the category of left $\mathbf{C}$-modules.

Our main example is the case when $\mathbf{C} = \Gamma$. Where Γ is the category of finite pointed sets and pointed maps. For any integer $n \geq 0$, we let $[n]$ be the set $\{0, 1, \dots, n\}$ with basepoint 0. We can and we will assume that objects of Γ are sets $[n]$, $n \geq 0$. The category of all left Γ-modules is denoted by Γ-*mod*, while the category of all right modules is denoted by *mod*-Γ. Projective generators of the category of left Γ-modules are objects Γ^n, $n \geq 0$, where

$$\Gamma^n(X) = K[X^n].$$

1.3 Hochschild and Harrison (co)homology of Γ-modules

The definition of these objects are based on the following pointed maps (see [9] and [10]). For any $0 \leq i \leq n+1$, one defines a map

$$\epsilon^i : [n+1] \to [n], \; 0 \leq i \leq n+1,$$

by

$$\epsilon^i(j) = \begin{cases} j & j \leq i, \\ j-1 & j > i \leq n, \\ 0 & j = i = n+1. \end{cases}$$

For a left Γ-module F the Hochschild homology $HH_*(F)$ is defined as the homology of the chain complex

$$F([0]) \leftarrow F([1]) \leftarrow F([2]) \leftarrow \cdots \leftarrow F([n]) \leftarrow \cdots$$

where the boundary map $\partial : F([n+1]) \to F([n])$ is given by $\sum_{i=0}^{n+1}(-1)^i F(\epsilon^i)$.

Quite similarly for a right Γ-module T one defines the Hochschild cohomology $HH^*(T)$ as the cohomology of the following cochain complex

$$T([0]) \to T([1]) \to \cdots \to T([n]) \to T([n+1]) \to \cdots$$

where the coboundary map $\delta : T([n]) \to T([n+1])$ is given by $\delta = \sum_{i=0}^{n+1}(-1)^i T(\epsilon^i)$.

We have the following obvious fact.

Lemma 1.1 *Let F be a left Γ-module, then $HH_0(F) = F([0])$ and*

$$HH_1(F) = \mathsf{Coker}(\partial : F([2]) \to F([1])).$$

Similraly, if T is a right Γ-module, then $HH^0(T) = T([0])$ and

$$HH^1(T) = \ker(\delta : T([1]) \to T([2])).$$

Let Σ_n be the symmetric group on n letters, it acts as a group of automorphisms on $[n]$. For integers $p_1, \cdots, p_k \geq 1$, with $k \geq 2$ and $n = p_1 + \cdots + p_k$, we set

$$sh_{p_1,\cdots,p_k} = \sum \mathsf{sgn}(\sigma)\sigma \in \mathbb{Z}[\Sigma_n]$$

where $\sigma \in \Sigma_n$ is running over all $(p_1, \cdots, p_k)$-shuffles. Each $sh_{p_1,\cdots,p_k}$ induces a map $T([n]) \to T([n])$, called the shuffle map. Let us denote by $\tilde{T}_n$ the intersection of the kernels of all shuffle maps. These groups form a subcomplex of the Hochschild cochain complex, called Harrison cochain complex [9]. The groups $Harr^n(T), n \geq 0$ are defined as the cohomology of the Harrison cochain complex.

By duality we have also Harrison homology of left Γ-module.

The following is a theorem due to Loday [9]. For alternative approach see [12].

Theorem 1.2 *If K is a field of characteristic zero, then for any left Γ-module F and right Γ-module T there exist the so called Hodge decompositions:*

$$HH_n(F) \cong \bigoplus_{i=1}^{n} HH_n^{(i)}F), \quad n > 0,$$

$$HH^n(T) \cong \bigoplus_{i=1}^{n} HH_{(i)}^n(T), \quad n > 0,$$

for suitable defined $HH_n^{(i)}(F)$ and $HH_{(i)}^n(T)$. Moreover, for $i = 1$ one has

$$Harr_n(F) = HH_n^{(1)}(F), \quad Harr^n(T) = HH_{(1)}^n(T), \quad n > 0.$$

1.4 André–Quillen (co)homology of Γ-modules

We recall some material from [13].

A partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ is a sequence of natural numbers $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$. The sum of the partition λ is given by $s(\lambda) := \lambda_1 + \cdots + \lambda_k$, while the group $\Sigma(\lambda)$ is a product of the corresponding symmetric groups

$$\Sigma(\lambda) := \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k}.$$

which is identified with the Young subgroup of $\Sigma_{s(\lambda)}$. Let us observe that $\Sigma_n = Aut_\Gamma([n])$ and therefore Σ_n acts on $F([n])$ and $T([n])$ for any left Γ-module F and right Γ-module T.

Let

$$0 \to F_1 \to F \to F_2 \to 0$$

be an exact sequence of left Γ-modules. It is called a *$\mathcal{Y}$-exact sequence* if for any partition λ with $s(\lambda) = n$ the induced map

$$F([n])^{\Sigma(\lambda)} \to F_2([n])^{\Sigma(\lambda)}$$

is surjective. The class of $\mathcal{Y}$-exact sequences is proper in the sense of MacLane [11].

A left Γ-module F is *$\mathcal{Y}$-projective*, if the functor $\mathsf{Hom}_\Gamma(F, -)$ takes $\mathcal{Y}$-exact sequences to exact sequences. For example $S^n\Gamma^1$ is a $\mathcal{Y}$-projective [13]. Here S^n denotes the n-th symmetric power. According to [13] for any left Γ-module F there is a $\mathcal{Y}$-exact sequence

$$0 \to F_1 \to F_0 \to F \to 0$$

with $\mathcal{Y}$-projective F_0. Hence one can take relative left derived functors of the functor $HH_1 : \Gamma - mod \to K\text{-}mod$. The values of these derived functors on a left Γ-module F is denoted by $\pi^{\mathcal{Y}}{}_*(F)$. So by the definition the functors $\pi^{\mathcal{Y}}{}_*$ are uniquely defined (up to isomorphism) by the following properties

Lemma 1.3 *There exist a unique family of functors* $\pi^{\mathcal{Y}}{}_n : \Gamma - mod \to K - mod$, $n \geq 0$*, such that*

(i) $\pi^{\mathcal{Y}}{}_0(F) = HH_1(F)$.

(ii) *For any $\mathcal{Y}$-exact sequence*

$$0 \to F_1 \to F \to F_2 \to 0$$

there is a long exact sequence

$$\cdots \to \pi^{\mathcal{Y}}{}_{n+1}(F_2) \to \pi^{\mathcal{Y}}{}_n(F_1) \to \cdots \to \pi^{\mathcal{Y}}{}_0(F_1)$$
$$\to \pi^{\mathcal{Y}}{}_0(F) \to \pi^{\mathcal{Y}}{}_0(F_2) \to 0.$$

(iii) *The functor $\pi^{\mathcal{Y}}{}_n$ vanishes on $\mathcal{Y}$-projective objects, $n \geq 1$.*

By a dual argument for any right Γ-module T one obtains K-modules $\pi_{\mathcal{Y}}{}^*(T)$. In more detail, let

$$0 \to T_1 \to T \to T_2 \to 0$$

be an exact sequence of right Γ-modules. It is called a *$\mathcal{Y}$-exact sequence* if for any partition λ with $s(\lambda) = n$ the induced map

$$T_1([n])_{\Sigma(\lambda)} \to T_2([n])_{\Sigma(\lambda)}$$

is injective. The class of $\mathcal{Y}$-exact sequences of right Γ-modules is also proper. A right Γ-module T is *$\mathcal{Y}$-injective* if the functor $\mathsf{Hom}_\Gamma(-, T)$ takes $\mathcal{Y}$-exact sequences to exact sequences. By an argument dual to one given in [13] for any right Γ-module T there is a $\mathcal{Y}$-exact sequence

$$0 \to T \to T_0 \to T_1 \to 0$$

with $\mathcal{Y}$-injective T_0. Hence one can take relative left derived functors of the functor $HH^1 : mod-\Gamma \to K\text{-}mod$. The values of these derived functors on a right Γ-module T are denoted by $\pi_{\mathcal{Y}}{}^*(T)$.

Lemma 1.4 *If K is a field of characteristic zero, then for any left Γ module F and right Γ-module T one has isomorphisms*

$$\pi^{\mathcal{Y}}{}_*(F) \cong Harr_{*-1}(F), \quad \pi_{\mathcal{Y}}{}^*(T) = Harr^{*-1}(T).$$

Proof In characteristic zero, any short exact sequence of Γ-modules is also a $\mathcal{Y}$-exact sequences. Thus, $\pi^{\mathcal{Y}}{}_*(F) = Tor_*^\Gamma(t, F)$ and $\pi_{\mathcal{Y}}{}^*(T) = Ext_\Gamma^*(t, T)$, where t is the same right Γ-module as in [12,13,15]. Now, the result follows from [12, Proposition 2.2] and [12, Corollary 2.9]. □

1.5 Γ-modules and commutative algebras

The classical Hochschild cohomology (as well as the Harrison or Andre–Quillen (co)homology) of commutative algebras is a particular case of the cohomology of Γ-modules [9,12,13]. We recall the corresponding results. Let R be a commutative K-algebra and A be an R-module. We have a left and right Γ-modules $\mathcal{L}_*(R, A)$ and $\mathcal{L}^*(R, A)$ defined on objects by

$$\mathcal{L}^*(R, A)([n]) := \mathsf{Hom}(R^{\otimes n}, A), \quad \mathcal{L}_*(R, A)([n]) := R^{\otimes n} \otimes A.$$

For a pointed map $f : [n] \to [m]$, the action of $\mathcal{L}^*(R, A)$ on f is given by

$$f^*(\psi)(a_1 \otimes \cdots \otimes a_n) = b_0 \psi(b_1 \otimes \cdots \otimes b_m)$$

while for the functor $\mathcal{L}_*(R, A)$ one has:

$$f_*(a_0 \otimes \cdots \otimes a_n) = b_0 \otimes \cdots \otimes b_m,$$

where $b_j = \prod_{f(i)=j} a_i,\ \ j = 0, \ldots, n.$

Then one has [9]:

$$HH_*(\mathcal{L}_*(R, A)) = HH_*(R, A), \quad HH^*(\mathcal{L}^*(R, A)) = HH^*(R, A)$$
$$Harr_n(\mathcal{L}_*(R, A)) = Harr_m(R, A), \quad Harr^n(\mathcal{L}^*(R, A)) = Harr^m(R, A),$$

where $HH_*(R, A)$ and $Harr_*(R, A)$ (resp. $HH^*(R, A)$, $Harr^*(R, A)$) are the Hochschild and Harrison (co)homology of R with coefficients in A.

By [13] a similar result is also true for André–Quillen (co)homology of commutative rings. In order to state this result, let us first recall the definition of André–Quillen (co)homology [17].

Let **SCA** be category of simplicial commutative K-algebras and let **SS** be the category of simplicial sets and let U : **SCA** $\to$ **SS** be a forgetful functor. According to [16] there is a unique closed model category structure on the category **SCA** such that a morphism $f : X_* \to Y_*$ of **SCA** is weak equivalence (resp. fibration) if $U(f)$ is a weak equivalence (resp. fibration) of simplicial sets. A simplicial commutative K-algebra X_* is called *free* if each X_n is a free commutative K-algebra with a base S_n, such that degeneracy operators $s_i : X_n \to X_{n+1}$ maps S_n to S_{n+1}, $0 \le i \le n$. Thanks to [16] any free simplicial commutative K-algebra is cofibrant and any cofibrant object is a retract of a free simplicial commutative K-algebra.

We let $C^*(V^*)$ be the cochain complex associated to a cosimplicial K-module V^*. Let R be a commutative K-algebra and let A be an R-module. Then the André–Quillen cohomology of R with coefficients in A is defined by (see [17]):

$$\mathsf{D}^*(R, A) := H^*(C^*(\mathsf{Der}(P_*, A))),$$

where $P_* \to R$ is a cofibrant replacement of the K-algebra R considered as a constant simplicial K-algebra and Der denotes the K-module of all K-derivations.

The André–Quillen homology of R with coefficients in A is defined by

$$\mathsf{D}_*(R, A) := H_*(C_*(A \otimes_{P_*} \Omega^1_{P_*})),$$

where Ω^1_R is the Kähler differentials of a commutative K-algebra R.

The main result of [13] claims that there are natural isomorphisms

$$\pi^{\mathcal{Y}}{}_*(\mathcal{L}_*(R, A)) = D_*(R, A), \quad \pi_{\mathcal{Y}}{}^*(\mathcal{L}^*(R, A)) = D^*(R, A).$$

2 The category $\mathcal{H}(C)$ associated to a commutative monoid C

2.1 **Definition**

Let C be a commutative monoid. Define the category $\mathcal{H}(C)$ as follows. Objects of $\mathcal{H}(C)$ are elements of C. A morphism from an element $a \in C$ to an element b is a pair

(c, a) of elements of C such that $b = ca$. To simplify notations we write $a \stackrel{c}{\to} ac$ for a morphism $(a, c) : a \to b = ac$. If $a \stackrel{c}{\to} ac$ and $ac \stackrel{d}{\to} acd$ are morphisms in $\mathcal{H}(C)$, then the composite of these morphisms in $\mathcal{H}(C)$ is $a \stackrel{cd}{\longrightarrow} acd$.

It is clear that $1 \in C$ is an initial object of $\mathcal{H}(C)$.

According to Sect. 1.2 a *left* $\mathcal{H}(C)$*-module* is a covariant functor $\mathcal{H}(C) \to \mathbf{Vect}$, similarly a *right* $\mathcal{H}(C)$*-module* is a contravariant functor $\mathcal{H}(C)^{op} \to \mathbf{Vect}$. We let $\mathcal{H}(C)$-*mod* be the category of left $\mathcal{H}(C)$-modules, while *mod*-$\mathcal{H}(C)$ denotes the category of right $\mathcal{H}(C)$-modules. If M is a left $\mathcal{H}(C)$-module, then the value of M on the element $a \in C$ (considered as object of $\mathcal{H}(C)$) is denoted by $M(a)$. Moreover if $a, b, c \in C$ and $b = ca$, then we have an induced map $c_* : M(a) \to M(b)$, with obvious properties $1_* = \mathsf{Id}$ and $(c_1c_2)_* = c_{1*}c_{2*}$.

Quite similarly, if N is a right $\mathcal{H}(C)$-module, then the value of N on the element $a \in C$ is denoted by $N(a)$. Moreover if $a, b, c \in C$ and $b = ca$, then we have an induced map $c^* : N(b) \to N(a)$, with obvious properties $1^* = \mathsf{Id}$ and $(c_1c_2)^* = c_2^*c_1^*$.

The categories $\mathcal{H}(C)$-*mod* and *mod*-$\mathcal{H}(C)$ are abelian categories with enough projective and injective objects. For any element a of C we let C^a and C_a be respectively the left and right $\mathcal{H}(C)$-modules defined by

$$C^a(x) = K[(x : a)] = \bigoplus_{c \in (x:a)} K, \quad \text{and} \quad C_a(x) = K[(a : x)] = \bigoplus_{c \in (a:x)} K.$$

Here for elements $a, b \in C$ we let $(b : a)$ be the set of all elements $c \in C$ such that $b = ac$. By Sect. 1.2 left $\mathcal{H}(C)$-modules C^a, $a \in C$ form a family of projective generators of the category $\mathcal{H}(C)$-*mod*. Similarly $C_a, a \in C$ form a family of projective generators of the category *mod*-$\mathcal{H}(C)$ and the modules $C_a^\sharp$, $a \in C$ form a family of injective cogenerators of $\mathcal{H}(C)$-*mod*.

Let N be a right $\mathcal{H}(C)$-module and M be a left $\mathcal{H}(C)$-module. We let $N \otimes_{\mathcal{H}(C)} M$ be the vector space generated by elements of the form $x \otimes y$, where $x \in N(a)$, $y \in M(a)$, $a \in C$. These elements are subject to the following relations

$$\begin{aligned}
(x_1 + x_2) \otimes y &= x_1 \otimes y + x_2 \otimes y,\\
x \otimes (y_1 + y_2) &= x \otimes y_1 + x \otimes y_2,\\
kx \otimes y &= x \otimes ky,\\
c^*(z) \otimes y &= z \otimes c_*(y).
\end{aligned}$$

Here $k \in K$, $c \in C$, $x, x_1, x_2 \in N(a)$, $y, y_1, y_2 \in M(a)$, $z \in N_{ca}$. Then one has

$$N \otimes_{\mathcal{H}(C)} C^a \cong N(a), \quad C_a \otimes_{\mathcal{H}(C)} M \cong M(a).$$

If $f : C \to C'$ is a homomorphism of monoids, then f induces a functor $\mathcal{H}(f) : \mathcal{H}(C) \to \mathcal{H}(C')$ in an obvious way. Thus for any left $\mathcal{H}(C')$-module M one has a left $\mathcal{H}(C)$-module $f^*(M)$, which is given by

$$f^*(M)(i) = M(f(i)).$$

In this way one obtains a functor f^* from the category of (left or right) modules over $\mathcal{H}(C')$ to the category of modules over $\mathcal{H}(C)$.

2.2 $K[C]$-modules and $\mathcal{H}(C)$-modules

We let $K[C]$ be the monoid algebra of the monoid C. Any $K[C]$-module A gives rise to the left $\mathcal{H}(C)$-module $j^*(A)$ which is defined by

$$j^*(A)(a) = A$$

and for $b = ca$, the induced morphisms

$$A = j^*(A)(a) \xrightarrow{c_*} j^*(A)(b) = A$$

is simply the multiplication by c.

If M is a left $\mathcal{H}(C)$-module, we let $j_*(M)$ be the following $K[C]$ module. As a K-module one has

$$j_*(M) = \bigoplus_{x \in C} M(x),$$

The action of C is defined as follows: for $x \in C$, $a \in M(x)$ and $c \in C$ one has

$$ci_x(a) = i_{cx}(c_*(a)).$$

Here i_x is the canonical inclusion $M(x) \to j_*(M)$, $x \in C$.

Lemma 2.1 *The functor j_* is a left adjoint functor to j^*.*

Proof For a left $\mathcal{H}(C)$-module M and a left $K[C]$-module A, an element

$$\xi \in \mathsf{Hom}_{\mathcal{H}(C)}(M, j^*(A))$$

is given by the family of K-module homomorphisms $\xi_a : M(a) \to A$, $a \in A$ such that for any $c \in C$ the following

$$\begin{array}{ccc} M(a) & \xrightarrow{\xi_a} & A \\ {\scriptstyle c_*}\downarrow & & \downarrow{\scriptstyle c} \\ M(ac) & \xrightarrow[\xi_{ac}]{} & A \end{array}$$

is a commutative diagram. The homomorphisms ξ_a, $a \in C$, defines a homomorphism of K-modules

$$\hat{\xi} : j_*(M) = \bigoplus_{a \in C} M(a) \to A$$

which clearly is a $K[M]$-homomorphism. So, $\xi \mapsto \hat{\xi}$ gives rise to a homomorphism

$$\mathsf{Hom}_{\mathcal{H}(C)}(M, j^*(A)) \to \mathsf{Hom}_{K[C]}(j_*(M), A)$$

which is obviously an isomorphism. □

2.3 Derivations, differentials and (co)homology in the theory of commutative algebras

Let C be a commutative monoid and let M be a left $\mathcal{H}(C)$-module. A *derivation* $\delta : C \to M$ of C with values in M is a function which assigns to each element $a \in C$ an element $\delta(a) \in M(a)$, such that

$$\delta(ab) = a_*(\delta(b)) + b_*(\delta(a)).$$

The abelian group of all derivations of C with values in M is denoted by $\mathsf{Der}(C, M)$.

We claim that there exist a universal derivation. In fact we construct a left $\mathcal{H}(C)$-module Ω_C, called *differentials* of a monoid C. It is a left $\mathcal{H}(C)$-module generated by symbols $da \in \Omega_C(a)$ one for every element $a \in C$, subject to relations

$$d(ab) = a_*(d(b)) + b_*(d(a))$$

for every a and $b \in C$. It follows from the construction that $a \mapsto da$ is a derivation, which is clearly universal one, in the sense that for any derivation $\delta : C \to, M$ there is a unique homomorphism of $\mathcal{H}(C)$-modules $\delta^* : \Omega_C \to M$ such that $\delta(a) = \delta^*(da)$. Thus for any left $\mathcal{H}(C)$-module M one has a canonical isomorphism

$$\mathsf{Der}(C, M) \cong Hom_{\mathcal{H}(C)}(\Omega_C, M).$$

Lemma 2.2 *One has an isomorphism of $K[C]$-modules*

$$j_*(\Omega_C) = \Omega^1_{K[C]}$$

Here $j_* : \mathcal{H}(C) - mod \to K[C]$-mod is the functor constructed in Sect. 2.2 and $\Omega^1_{K[C]}$ is the Kähler differentials of the K-algebra $K[C]$.

Proof Let A be a $K[C]$-module. Then we have

$$\mathsf{Der}(C, j^*(A)) = Hom_{\mathcal{H}(C)}(\Omega_C, j^*(A)) = Hom_{K[C]}(j_*(\Omega), A).$$

On the other hand

$$\mathsf{Der}(C, j^*(A)) = \mathsf{Der}(K[C], A) = Hom_{K[C]}(\Omega^1_{K[A]}, A)$$

and the result follows from the Yoneda lemma. □

2.4 The case $C = \mathbb{N}$

If C is the free abelian monoid with a generator t, then a left $\mathcal{H}(C)$-module is nothing but a diagram of vector spaces

$$M = (M_0 \xrightarrow{t} M_1 \xrightarrow{t} M_2 \xrightarrow{t} M_3 \xrightarrow{t} \cdots)$$

In particular the projective object C^n corresponds to the diagram

$$0 \to 0 \to \cdots \to 0 \to K \xrightarrow{1} K \xrightarrow{1} \cdots$$

where the first nontrivial term appears at the place n.

Quite similarly a right $\mathcal{H}(C)$-module is nothing but a diagram of vector spaces

$$N = (\cdots \xrightarrow{t} N_3 \xrightarrow{t} N_2 \xrightarrow{t} N_1 \xrightarrow{t} N_0).$$

In particular the projective object C_n corresponds to the diagram

$$\cdots \to 0 \to 0 \to K \xrightarrow{1} K \xrightarrow{1} \cdots \xrightarrow{1} K$$

where the first nontrivial term appears at the place n.

One easily observes that for any left $\mathcal{H}(C)$-module M one has an isomorphism

$$\mathsf{Der}(C, M) \cong M_1$$

which is given by $\delta \mapsto \delta(t)$. This follows from the fact that $\delta(t^n) = nt^{n-1}\delta(t)$. Thus

$$\Omega_C = C^1 = (0 \to K \xrightarrow{1} K \xrightarrow{1} K \xrightarrow{1} \cdots).$$

2.5 Product of two monoids

Let C be a product of two monoids: $C = C_1 \times C_2$. Then $\mathcal{H}(C) = \mathcal{H}(C_1) \times \mathcal{H}(C_2)$. Assume M_1 and M_2 are (say left) $\mathcal{H}(C_1)$ and $\mathcal{H}(C_2)$-modules respectively. Then one can form a $\mathcal{H}(C)$-module $M_1 \boxtimes M_2$ as follows:

$$M_1 \boxtimes M_2(x_1, x_2) = M_1(x_1) \otimes M_2(x_2).$$

Lemma 2.3 *For any elements $c_1 \in C$ and $c_2 \in C_2$, one has*

$$C^{(c_1,c_2)} = C^{c_1} \boxtimes C^{c_2}$$

and

$$C_{(c_1,c_2)} = C_{c_1} \boxtimes C_{c_2}.$$

Proof By definition one has

$$\begin{aligned}C^{c_1}\boxtimes C^{c_2}(x_1,x_2)&=C^{c_1}(x_1)\otimes C^{c_2}(x_2)\left(\bigoplus_{a_1\in C_1;a_1c_1=x_1}K\right)\otimes\left(\bigoplus_{a_2\in C_2;a_2c_2=x_2}K\right)\\&=\bigoplus_{(a_1,a_2)(c_1,c_2)=(x_1,x_2)}K\\&=C^{(c_1,c_2)}(x_1,x_2).\end{aligned}$$

Similarly for the second isomorphism. □

We have homomorphisms

$$\iota_1:C_1\to C,\ \iota(c_1)=(c_1,1),\quad \iota_2:C_2\to C,\ \iota(c_2)=(1,c_2).$$

For any left $\mathcal{H}(C)$-module M we set

$$M^{(1)}=\iota_1^*(M),\quad M^{(2)}=\iota_2^*(M).$$

Lemma 2.4 *For any left $\mathcal{H}(C)$-module M one has*

$$\mathsf{Der}(C,M)\cong\mathsf{Der}(C_1,M^{(1)})\oplus\mathsf{Der}(C_1,M^{(2)}).$$

Proof This easily follows from the fact $(c_1,c_2)=(c_1,1)(1,c_2)$. □

We also have projections $\pi_1:C\to C_1$ and $\pi_2:C\to C_2$, given respectively by $\pi_i(c_1,c_2)=c_i$, $i=1.2$.

Lemma 2.5 *For any left $\mathcal{H}(C_i)$-module X_i, $i=1,2$ and any left $\mathcal{H}(C)$-module M, one has isomorphisms*

$$\mathsf{Hom}_{\mathcal{H}(C)}(\pi_1^*X_1,M)\cong\mathsf{Hom}_{\mathcal{H}(C_1)}(X_1,M^{(1)})$$

and

$$\mathsf{Hom}_{\mathcal{H}(C)}(\pi_2^*X_2,M)\cong\mathsf{Hom}_{\mathcal{H}(C_2)}(X_2,M^{(2)}).$$

Proof Let $\eta\in\mathsf{Hom}_{\mathcal{H}(C)}(\pi_1^*X_1,M)$. Thus η is a collection of homomorphisms of K-modules

$$\eta_{(a_1,a_2)}:X_{a_1}\to M_{(a_1,a_2)}$$

such that for any elements $c_1 \in C_1$, $c_2 \in C_2$ the following diagram commutes

$$\begin{array}{ccc} X_{a_1} & \xrightarrow{\eta_{(a_1,a_2)}} & M_{(a_1,a_2)} \\ {\scriptstyle c_{1*}}\downarrow & & \downarrow{\scriptstyle (c_1,c_2)_*} \\ X_{a_1c_1} & \xrightarrow[\eta_{(a_1c_1,a_2c_2)}]{} & M_{(a_1c_1,a_2c_2)} \end{array}$$

it follows that $\eta_{(a_1,a_)} = (1, a_2)_* \circ \eta_{(a_1,1)}$. It is clear that the family of homomorphisms $\eta_{a_1,1}$, $a_1 \in C_1$ defines the morphism $\hat{\eta} \in \mathsf{Hom}_{\mathcal{H}(C_1)}(X_1, M^{(1)})$ and the previous equality shows that $\eta \mapsto \hat{\eta}$ is really a bijection. □

Corollary 2.6 *If* $C = C_1 \times C_2$*, then*

$$\Omega_C = \pi_1^*\Omega_{C_1} \oplus \pi_2^*\Omega_{C_1},$$

where $\pi_i : C \to C_i$, $i = 1, 2$ *is the canonical projection.*

Proof For any $\mathcal{H}(C)$-module M one has

$$\begin{aligned} \mathsf{Hom}_{\mathcal{H}(C)}(\Omega_C, M) &= \mathsf{Der}(C, M) \\ &= \mathsf{Der}(C_1, M^{(1)}) \oplus \mathsf{Der}(C_2, M^{(2)}) \\ &= \mathsf{Hom}_{\mathcal{H}(C_1)}(\Omega_{C_1}, M^{(1)}) \oplus \mathsf{Hom}_{\mathcal{H}(C_2)}(\Omega_{C_2}, M^{(2)}) \\ &= \mathsf{Hom}_{\mathcal{H}(C)}(\pi_1^*\Omega_{C_1}, M) \oplus \mathsf{Hom}_{\mathcal{H}(C)}(\pi_2^*\Omega_{C_2}, M) \\ &= \mathsf{Hom}_{\mathcal{H}(C)}(\pi_1^*\Omega_{C_1} \oplus \pi_2^*\Omega_{C_2}, M) \end{aligned}$$

and the result follows from the Yoneda lemma. □

3 Commutative monoid (co)homology and Γ-modules

3.1 Γ-modules related to monoids

Let C be a commutative monoid and let N be a right $\mathcal{H}(C)$-module. Define left Γ-module $\mathsf{G}_*(C, N)$ as follows. On objects it is given by

$$\mathsf{G}_*(C, N)([n]) = \bigoplus_{(a_1,\cdots,a_n)\in C^n} N(a_1 \dots a_n).$$

In order, to extend the definition on morphism, we let

$$\iota_{(a_1,\dots,a_n)} : N(a_1 \dots a_n) \to \mathsf{G}_*(C, N)([n])$$

be the canonical inclusion. Let $f : [n] \to [m]$ be a pointed map. Then the homomorphism

$$f_* : \mathsf{G}_*(C, N)([n]) \to \mathsf{G}_*(C, N)([m])$$

is defined by

$$f_* \iota_{(a_1,\dots,a_n)}(x) = \iota_{(b_1,\dots,b_m)}((b_0)_*(x)),$$

where $x \in N(a_1 \cdots a_n)$ and

$$b_j = \prod_{f(i)=j} a_i, \quad j = 0, \dots, n.$$

Here we used the convention that $b_0 = 1$ provided $f^{-1}(\{0\}) = \{0\}$.

Quite similarly, let M be a left $\mathcal{H}(C)$-module. Define a right Γ-module $\mathsf{G}^*(C, M)$ as follows. On objects it is given by

$$\mathsf{G}^*(C, M)([n]) = \prod_{(a_1,\cdots,a_n)\in C^n} M(a_1 \dots a_n).$$

Thus $\eta \in \mathsf{G}(C, M)([n])$ is a function which assigns to any n-tuple of elements $(a_1, \dots, a_n)$ of C an element $\eta(a_1, \dots, a_n) \in M_{a_1 \dots a_n}$. Let $f : [n] \to [m]$ be a pointed map and $\xi \in \mathsf{G}(C, M)([m])$. Then the function $f^*(\xi) \in \mathsf{G}(C, M)([n])$ is given by

$$f^*(\xi)(a_1, \dots, a_n) = b_{0*}(\xi(b_1, \dots, b_m)).$$

Lemma 3.1 *Let $C = \mathbb{N}$ be a free commutative monoid with a generator t, and let C_n be the standard projective right $\mathcal{H}(C)$-module, $n \geq 0$, see Sect. 2.4. Then one has an isomorphism of left Γ-modules*

$$\mathsf{G}_*(C, C_n) = \bigoplus_{k=0}^{n} S^k \circ \Gamma^1$$

In particular, $\mathsf{G}_*(C, C_n)$ is $\mathcal{Y}$-projective.

Proof Since $\Gamma^1([m])$ is a free K-module spanned on $x_1, \cdots, x_m$, it follows that $S^k \circ \Gamma^1([m])$ is a free K-module spanned by all monomials of degree k on the variables $x_1, \cdots, x_m$. On the other hand we have

$$\mathsf{G}_*(C, C_n)([m]) = \bigoplus_{k=0}^{n} \bigoplus_{n_1+\dots+n_m=k} K.$$

To see the expected isomorphism, it is enough to assign to a basis element of $\bigoplus_{n_1+\dots+n_m=k} K$ the monomial $x_1^{n_1} \dots x_m^{n_m}$. □

Lemma 3.2 *Let $C = C_1 \times C_2$ be product of two monoids and N_i be right $\mathcal{H}(C_i)$ modules, $i = 1, 2$. Then one has*

$$\mathsf{G}_*(C, N_1 \boxtimes N_2) = \mathsf{G}_*(C_1, N_1) \otimes \mathsf{G}_*(C_2, N_2).$$

The proof is straightforward.

Corollary 3.3 *Let C be a finitely generated free commutative monoid and let N be a projective object in the category of right $\mathcal{H}(C)$-modules. Then $\mathsf{G}_*(C, N)$ is a $\mathcal{Y}$-projective left Γ-module.*

Proof Since, any projective object is a retract of a direct sum of standard projective modules C_c, it is enough to restrict ourself with the case when $N = C_c$. Assume $C = \mathbb{N}^k$. We will work by induction on k. If $k = 1$, then the result was already established, see Lemma 3.1. Rest follows from Lemma 2.4 and Lemma 3.2 and the fact that tensor product of two $\mathcal{Y}$-projective objects is $\mathcal{Y}$-projective see [13]. □

3.2 Homology and cohomology of commutative monoids

Let **CM** be the category of all commutative monoids and let **SCM** be the category of all simplicial commutative monoids. There is a forgetful functor $U' : \mathbf{SCM} \to \mathbf{SS}$. By [16] there is a unique closed model category structure on the category **SCM** such that a morphism $f : X_* \to Y_*$ of **SCM** is a weak equivalence (resp. fibration) if $U'(f)$ is a weak equivalence (resp. fibration) of simplicial sets. A simplicial commutative monoid X_* is called *free* if each X_n is a free commutative monoid with a base Y_n, such that degeneracy operators $s_i : X_n \to X_{n+1}$ maps Y_n to Y_{n+1}, $0 \le i \le n$. According to [16] any free simplicial commutative monoid is cofibrant and any cofibrant object is a retract of a free simplicial commutative monoid.

If $C' \to C$ is a morphism of commutative monoids then it gives rise to a functor $\mathcal{H}(C') \to \mathcal{H}(C)$, which allows us to consider any left or right $\mathcal{H}(C)$-module as a module over $\mathcal{H}(C')$. In particular if $P_* \to C$ is an augmented simplicial monoid and M is a left $\mathcal{H}(C)$-module, one can considered M as a left $\mathcal{H}(P_k)$-module, for all $k \ge 0$. The same holds for right $\mathcal{H}(C)$-modules.

Let M be a left $\mathcal{H}(C)$-module. Then the Grillet cohomology of C with coefficients in M is defined by

$$\mathsf{D}^*(C, M) := H^*(C^*(\mathsf{Der}(P_*, M))),$$

where $P_* \to C$ is a cofibrant replacement of the monoid C considered as a constant simplicial monoid.

Let N be a right $\mathcal{H}(C)$-module. Then the Grillet homology of C with coefficients in N is defined by

$$\mathsf{D}_*(C, N) := H_*(C_*(\Omega_{P_*} \otimes_{\mathcal{H}(P_*)} N))),$$

where $P_* \to C$ is a cofibrant replacement of the monoid C considered as a constant simplicial monoid.

The definition of the cohomology essentially goes back to Grillet (see [2–5]), but the definition of the Grillet homology is new.

By comparing the definition we obtain the following basic fact, which is missing in (see [2–5]).

Lemma 3.4 *Let C be a commutative monoid and A be a $K[C]$-module. Then one has the isomorphisms:*

$$\mathsf{D}^*(C, j^*(A)) \cong \mathsf{D}^*(K[C], A),$$
$$\mathsf{D}_*(C, j^*(A)) \cong \mathsf{D}_*(K[C], A).$$

Proof The isomorphism in the dimension zero is the obvios one, compare with Lemma 2.2. Rest follows from the fact that if $P_* \to C$ is a cofibrant replacement of C in the category **SCM**, then $K[P_*] \to K[C]$ is a cofibration replacement of $K[C]$ in the category **SCA**. □

3.3 The main Theorem

Now we are in the situation to state our main theorem, which relates Grillet (co)homology of the monoid M with the Andre–Quillen (co)homology of the Γ-modules $\mathsf{G}_*(C, N)$ and $\mathsf{G}^*(C, M)$.

Theorem 3.5 *Let C be a commutative monoid, M be a left and N be a right $\mathcal{H}(C)$-modules. Then one has the following isomorphisms*

$$\mathsf{D}^*(C, M) = \pi_{\mathcal{Y}}{}^*(\mathsf{G}^*(C, M)),$$
$$\mathsf{D}_*(C, N) = \pi^{\mathcal{Y}}{}_*(\mathsf{G}_*(C, N)).$$

The proof is based on several steps. The idea is to reduce the theorem to the case when M is a free commutative monoid with one generator. In this case, the theorem is proved using direct computation. We need some lemmas.

Lemma 3.6 *Let C be a commutative monoid, N be a right $\mathcal{H}(C)$-module. Then one has a natural isomorphism*

$$HH_1(\mathsf{G}_*(C, N)) \cong N \otimes_{\mathcal{H}(C)} \Omega_C.$$

Proof Thanks to Lemma 1.1 one has $HH_1(\mathsf{G}_*(C, N)$ is isomorphic to the cokernel of the map

$$\partial : \bigoplus_{a,b\in C} N(ab) \to \bigoplus_{a\in C} N(a)$$

As usual, we let $i_a : N(a) \to \bigoplus_{a\in C} N(a)$ be the canonical inclusion. For an element $x \in N(a)$, the class of $i_a(x)$ in $HH_1(\mathsf{G}_*(C, N))$ is denoted by $cl(a; x)$. Then

$$cl(a; x) \mapsto x \otimes da$$

defines the isomorphism $HH_1(\mathsf{G}_*(C,N)) \cong N \otimes_{\mathcal{H}(C)} \Omega_C$. □

Lemma 3.7 *Let C be a commutative monoid and let*

$$0 \to N_1 \to N \to N_2 \to 0$$

be a short exact sequence of right $\mathcal{H}(C)$-modules, then

$$0 \to \mathsf{G}_*(C,N_1) \to \mathsf{G}_*(C,N) \to \mathsf{G}_*(C,N_2) \to 0$$

is a $\mathcal{Y}$-exact sequence of left Γ-modules.

Proof For a partition λ of n and a set P we denote by P^λ the set of orbits of the cartesian product P^n under the action of the group $\Sigma(\lambda) \subset \Sigma_n$. In particular we have a set C^λ. For any element $\mu \in C^\lambda$ we put $N_\mu := N(a_1 \cdots a_n)$, where $(a_1, \cdots, a_n) \in \mu$. Since

$$\mathsf{G}_*(C,N)([n])^{\Sigma(\lambda)} = \bigoplus_{\mu \in C^\lambda} N_\mu$$

the result follows. □

By the same argument we have also the following.

Lemma 3.8 *Let $f : D \to C$ be a surjective homomorphism of commutative monoids, then for any right $\mathcal{H}(C)$-module N the induced morphism of left Γ-modules*

$$\mathsf{G}_*(D,N) \to \mathsf{G}_*(C,N)$$

is a $\mathcal{Y}$-epimorphism.

Proof In the notation of the proof of Lemma 3.7 the map $D^\lambda \to C^\lambda$ is surjective and the result follows. □

Lemma 3.9 *Let $\epsilon : X_* \to C$ be a simplicial resolution in the category of commutative monoids and N be a right $\mathcal{H}(C)$-module. Then the associated chain complexes of the simplicial left Γ-module $\mathsf{G}_*(X_*,N) \to \mathsf{G}_*(C,N)$ is a $\mathcal{Y}-$resolution.*

Proof Since $X_*^\lambda \to C^\lambda$ is a weak equivalence the result follows. □

Lemma 3.10 *Let C be a free commutative monoid, N be a projective right $\mathcal{H}(C)$-module and M be an injective left $\mathcal{H}(C)$-module. Then for any $i > 0$ one has*

$$\pi_i^{\mathcal{Y}}(\mathsf{G}_*(C,N)) = 0, \quad \pi^i_{\mathcal{Y}}(\mathsf{G}^*(C,M)) = 0.$$

Proof It suffices to consider the cases when $N = C_c$ and $M = C_c^\sharp$, for an element $c \in C$. This is because the family $(C_c)_{c\in C}$ (resp. $(C_c^\sharp)_{c\in C}$) is a family of projective (resp. injective) (co)generators. Since $\pi^i_{\mathcal{Y}}(\mathsf{G}_*(C,N^\sharp)) = (\pi_i^{\mathcal{Y}}(\mathsf{G}_*(C,N)))^\sharp$, it suffices to consider only $\pi_i^{\mathcal{Y}}(\mathsf{G}_*(C,N))$. Since homology commutes with direct limits, one can assume that C is finitely generated. Then by Lemma 3.2 we can reduce to the case when C has one generator. In this case the result follows from Lemma 3.1. □

3.4 Proof of Theorem 3.5

We give proof only for homology. A dual argument works for cohomology. Thanks to Lemma 3.6, the theorem is true in dimension $i = 0$. Next, consider the case when C is a free monoid. In this case $\mathsf{D}_i(C, -) = 0$, if $i > 0$. On the other hand, $\pi_i^{\mathcal{Y}}(\mathsf{G}_*(C, F)) = 0$ provided F is projective, thanks to Lemma 3.10. By Lemma 3.7, the functor $\pi_*^{\mathcal{Y}}(\mathsf{G}_*(C, -))$ assigns the long exact sequence to a short exact sequence of right $\mathcal{H}(C)$-modules. Therefore, we can consider such an exact sequence associated to a short exact sequence of right $\mathcal{H}(C)$-modules

$$0 \to N_1 \to F \to N \to 0$$

with projective F. Since the result is true if $i = 0$, one obtains, by induction on i, that $\pi_i^{\mathcal{Y}}(\mathsf{G}_*(C, -)) = 0$ provided $i > 0$ and C is a free commutative monoid. Thus the theorem is true in this case.

Now consider the general case. Let $P_* \to C$ be a free simplicial resolution in the category of commutative monoids. Then we have

$$N \otimes_{\mathcal{H}(P_*)} \Omega \cong \pi^{\mathcal{Y}}{}_0(\mathsf{G}_*(C, N)).$$

Thanks to Lemma 3.9, $C_*(\mathsf{G}_*(P_*, N)) \to \mathsf{G}_*(C, N)$ is a $\mathcal{Y}$-resolution consisting of $\pi_*^{\mathcal{Y}}$-acyclic objects and the result follows.

3.5 Applications

Let C be a commutative monoid, M be a left $\mathcal{H}(C)$-module and N be a right $\mathcal{H}(C)$-module. For the Γ-modules $\mathsf{G}_*(C, N)$ and $\mathsf{G}^*(C, M)$ one can apply the reach theory of functor homology developed in [9, 12, 13]. For example, if one applies the Hochschild cohomology theory to $\mathsf{G}^*(C, M)$, one recovers Leech cohomology [8] $H^*(C, M)$. On the other hand, D^*-theory is nothing but Grillet cohomology [2–7]. Hence, by Theorem 1.2 we have the following result.

Corollary 3.11 *Let C be a commutative monoid and let M be a left $\mathcal{H}(C)$-module. If K is a field of characteristic zero, then the Grillet cohomology is a direct summand of Leech cohomology. In more concrete terms, the Leech cohomology $H^*(C, M)$ has a decompostion:*

$$H^n(C, M) \cong \bigoplus_{i=1}^{n} H^n_{(i)}(C, M), \quad n > 0,$$

such that $H^n_{(1)}(C, M) \cong D^{n-1}(C, M)$.

Of course we can also apply the Hochschild homology theory to $\mathsf{G}_*(C, N)$. The corresponding theory should be considered as a dual of Leech cohomology and hence we will call them *Leech homology* and denote it by $H_*(C, N)$. It should be made clear that $H_*(C, -)$ and $H^*(C, -)$ have different domain categories.

If one applies Harrison theories to Γ-modules $\mathsf{G}_*(C, N)$ and $\mathsf{G}^*(C, M)$, one obtains the groups $Harr_*(C, N)$ and $Harr^*(C, M)$. For example, $Harr^*(C, M)$ is defined using the following cochain complex $C^*_{Harr}(C, M)$, whose n-dimensional cochains are functions f which assign to any n-tuple $(c_1, \dots, c_n) \in C^n$ an element $f(c_1, \dots, c_n) \in M(c_1 \dots c_n)$ such that f is zero on any shuffles. That is, for any integers $p_1, \dots, p_k$ such that $k \geq 2$ and $n = p_1 + \dots + p_k$ one has

$$\sum_\sigma \mathsf{sgn}(\sigma) f(c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(n)}) = 0$$

where $\sigma \in \Sigma_n$ is running over all $(p_1, \dots, p_k)$-shuffles. The coboundary map is given as in Leech theory:

$$\begin{aligned}(df)(c_0, \dots, c_n) = {} & c_{0*} f(c_1, \dots, c_n) \\ & + \sum_i (-1)^i f(c_0, \dots, c_i c_{i+1}, \dots, c_n) \\ & + (-1)^n c_{n*} f(c_0, \dots, c_{n-1}).\end{aligned}$$

Now Lemma 1.4 tells us that if K is a field of characteristic zero, then we have

$$D_*(C, N) = Harr_{*+1}(C, N), \quad D^*(C, M) = Harr^{*+1}(C, M)$$

In particular, this solves the cocycle problem for Grillet cohomology [6, line-1, p.3425] in the case of characteristic zero.

Remark In this paper we have restricted ourself to the case when K is a field. This is because in our main references [12,13,15] there is such a restriction. However, Theorem 3.5 is valid for an arbitrary commutative ring K. With the same argument: of course now the role of the linear dual $V^\sharp = \mathsf{Hom}(V, K)$ is played by $\mathsf{Hom}_K(V, J)$, where J is an injective cogenerator of the category of K-modules.

Acknowledgments The second author is very grateful to Antonio Cegarra for interesting communication on the subject. The authors would like to thank the referee for his careful reading and many valuable suggestions. Research was partially supported by the GNSF Grant ST08/3-387. The second author acknowledgement the support given by the University of Leicester in granting academic study leave.

References

1. Calvo-Cervera, M., Cegarra, A.M., Heredia, B.A.: On the third cohomology group of commutative monoids. arXiv.1406.6835
2. Grillet, P.A.: Commutative semigroup cohomology. Semigroup Forum **43**, 247–252 (1991)
3. Grillet, P.A.: The commutative cohomology of nilsemigroups. J. Pure Appl. Algebra **82**, 233–251 (1992)
4. Grillet, P.A.: Commutative semigroup cohomology. Commun. Algebra **23**, 3573–3587 (1995)
5. Grillet, P.A.: The commutative cohomology of finite semigroup. J. Pure Appl. Algebra **102**, 25–47 (1995)

6. Grillet, P.A.: Cocycles in commutative semigroup cohomology. Commun. Algebra **25**, 3425–3462 (1997)
7. Grillet, P.A.: The commutative cohomology of two-generator semigroups. Semigroup Forum **60**, 4–15 (2000)
8. Leech, J.: $\mathcal{H}$-coextensions of monoids. Memoirs A.M.S. **157**, 1–66 (1975)
9. Loday, J.-L.: Opérations sur l'homologie cyclique des algébres commutatives. Invent. Math. **96**, 205–230 (1989)
10. Loday, J.-L.: Cyclic Homology, Grund Math. Wiss, vol. 301, 2nd edn. Springer, Heidelberg (1998)
11. MacLane, S.: Homology. Classics in Mathematics. Springer, Berlin (1995). x+422 pp.
12. Pirashvili, T.: Hodge decomposition of higher order Hochschild homology. Ann. Sci. École Norm. Sup. (4) **33**(2), 151–179 (2000)
13. Pirashvili, T.: André-Quillen homology via functor homology. Proc. AMS **131**, 1687–1694 (2002)
14. Pirashvili, T.: Dold-Kan type theorem for Γ-groups. Math. Ann. **318**(2), 277–298 (2000)
15. Pirashvili, T., Richter, B.: Robinson-Whitehouse complex and stable homotopy. Topology **29**, 525–530 (2000)
16. Quillen, D.G.: Homotopical algebra. Lecture Notes in Mathematics No. 43. Springer, Berlin (1967). iv+156 pp.
17. Quillen, D.G.: On the (co)homology of commutative rings. AMS Proc. Sym. Pure Math. **17**, 65–87 (1970)