SPIN STRUCTURES AND QUADRATIC FORMS ON SURFACES

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1. *Introduction*

In [2] Atiyah investigates a certain invariant of spin structures dating back originally to Riemann. This invariant is defined as the mod 2 dimension of a certain space of holomorphic sections associated to the spin structure, and does not depend on the complex structures involved. The original motivation of this paper was to give a simple topological definition of the invariant. It was found in the process that there is a natural lifting of mod 2 homology classes on the surface to mod 2 classes in its unit tangent bundle. This produced in its wake a natural correspondence between spin structures on the surface and quadratic forms (of symplectic type) on H_1. The Atiyah invariant of a spin structure can then be defined as the Arf invariant of the corresponding quadratic form.

2. *Preliminaries*

Let M be a smooth closed orientable surface of genus g. We use the following notations and abbreviations.

SCC means "simple closed curve". If α, β are SCCs, then $\alpha \simeq \beta$ means that the two curves are homologous mod 2.

H_1, H^1 mean $H_1(M, Z_2)$, $H^1(M, Z_2)$ respectively. The intersection form on H_1 (written as a dot product) is *symplectic*, that is, $x \cdot x = 0$ for all x. A basis a_i, b_i $(i = 1, \ldots, g)$ of H_1 is *symplectic* if $a_i \cdot a_j = b_i \cdot b_j = 0$, $a_i \cdot b_j = 0$ if $i \neq j$, and $a_i \cdot b_i = 1$.

The dual pairing of H_1 and H^1 is denoted $\langle \alpha, x \rangle$ for $\alpha \in H^1, x \in H_1$. The symplectic automorphism group of H_1 is denoted by Sp; it is a finite group isomorphic to $Sp(g, Z_2)$. The dual action of $h \in Sp$ on H^1 will be written on the *right*, so that $\alpha(gh) = (\alpha g)h$ for $\alpha \in H^1$ and $g, h \in Sp$. We have by definition $\langle \alpha h, x \rangle = \langle \alpha, hx \rangle$. If f is a homeomorphism of M, then the induced map on H_1 is denoted by f^*.

We write UM for the unit tangent bundle of M and $\tilde{H}_1$, $\tilde{H}^1$ for $H_1(UM, Z_2)$ and $H^1(UM, Z_2)$ respectively. An element of $\tilde{H}_1$ can be represented as a smooth closed curve in UM or equivalently by a *framed* closed curve in M, by which we shall mean a smooth closed curve in M and a smooth vector field along it (the rest of a true frame follows from this data and an orientation of M). If $i : S^1 \to UM$ and

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$p: UM \to M$ are the fiber inclusion and projection, then we have exact sequences

$$0 \longrightarrow Z_2 \xrightarrow{i^*} \tilde{H}_1 \xrightarrow{p^*} H_1 \longrightarrow 0$$

$$0 \longrightarrow H^1 \xrightarrow{p^*} \tilde{H}^1 \xrightarrow{i^*} Z_2 \longrightarrow 0.$$

The generator of Z_2 in the first sequence will be denoted by z; it is the "fiber class", and may be represented, for example, by a small circle in M with framing given by its tangent vector field. The generator of Z_2 in the second sequence we denote simply by $1 \in Z_2$. This sequence allows us to identify H^1 with the subspace of $\tilde{H}^1$ consisting of all δ such that $i^*(\delta) = 0$.

A *spin structure* on M is a class $\xi \in \tilde{H}^1$ such that $i^*(\xi) = 1$, or alternatively, such that $\langle \xi, z \rangle = 1$ (see [2; p. 55] or [3]). Intuitively we think of ξ as a function assigning a number mod 2 to each framed curve of M, subject to the usual homological conditions and also that the boundary of a disc in M, tangentially framed, receives a one. Note that $\tilde{H}^1$ is the disjoint union of $H^1 \subset \tilde{H}^1$ and the set Φ of spin structures.

3. *Lifting cycles on M to UM*

Let α be a smooth oriented SCC on M. There is an obvious lifting of α to UM given by framing it with its unit tangent vector field; we denote this lifting by $\vec{\alpha}$. If α is not oriented, then we have two such liftings possible, but note that they are homotopic: we simply rotate all the vectors of one framing uniformly through 180° to get the other framing. Thus the mod 2 homology class of the lifting depends only on α and not on its orientation.

THEOREM 1A. *Let $\{\alpha_i\}$ $(i = 1, \ldots m)$, $\{\beta_j\}$ $(j = 1, \ldots n)$ be two collections of (smooth) SCCs in M, the curves of each collection being mutually disjoint. If*

$$\sum_{i=1}^{m} \alpha_i \simeq \sum_{j=1}^{n} \beta_j$$

in M, then

$$\sum_{i=1}^{m} \vec{\alpha}_i + mz \simeq \sum_{j=1}^{n} \vec{\beta}_j + nz$$

in UM.

Proof. Since $\sum \alpha_i$ and $\sum \beta_j$ are homologous, we may pass from $\{\alpha_i\}$ to $\{\beta_j\}$ by a sequence of the following three operations:

(a) smooth isotopies of the collection;

(b) adding or removing the boundary of a smooth 2-disc which is disjoint from the collection;

(c) a *band change*, defined as follows: let u, v be disjoint arcs of the collection (they may lie in the same component) and suppose that w is a smooth arc, beginning at an interior point of u and perpendicular to it there, and ending likewise at v, being otherwise disjoint from all the curves of the collection. See Figure 1a. Then the band change along w is the result of replacing u, v by the smooth arcs x, y of Figure 1b.

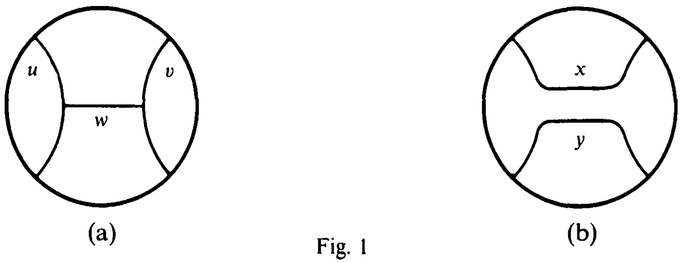


Fig. 1

To prove the theorem, it suffices then to show only that it holds when $\{\beta_j\}$ is related to $\{\alpha_i\}$ by a single move of the above types.

(a) A smooth isotopy changes neither the number of components m nor the homology classes of $\vec{\alpha}_i$.

(b) Adjoining or removing the boundary of a disc ∂D changes m by 1 mod 2, and to $\sum \vec{\alpha}_i$ adds $\overrightarrow{\partial D} = z$ mod 2, so the total change is again zero.

(c) If the band change is as illustrated in Figure 1, we need to distinguish two cases. First we define the notion of *parallel* and *antiparallel* orientations of u, v with respect to the arc w. This is defined pictorially in Figure 2.

Fig. 2

Case 1. u and v are in the same component and acquire *parallel* orientations from an orientation of this component. Then Figure 3 shows a sequence of homologies in UM converting u, v to the band changed result x, y of Figure 3f.

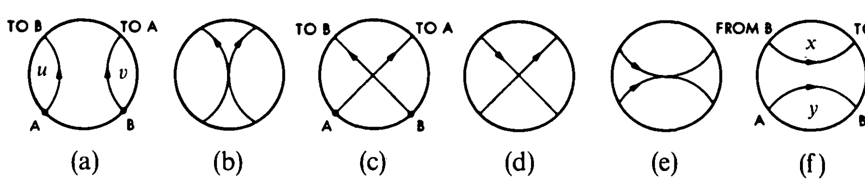
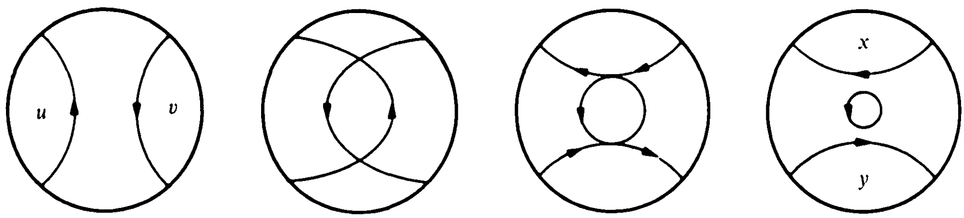


Fig. 3

The change from (c) to (d) is got by reversing the orientation of the component through B, which does not alter the lifting homologically. Thus no change has occurred in the sum of the tangential liftings. But note that x, y lie in the same component, as did u, v. Hence no change occurs in the number of components, and the theorem holds in this case.

Case 2. Either u and v lie in distinct components or are in the same component but acquire antiparallel orientations from it. In the former, we choose antiparallel orientations for u and v, so in either case we have the situation depicted in Figure 4a.



(a) (b) (c) (d)

Fig. 4

The complete figure shows a sequence of homologies in UM whose net result is to band change u, v to x, y and add a copy of z. In other words,

$$\sum_{i=1}^{m} \vec{\alpha}_i$$

has changed by the addition of z. But note that u, v are in the same component if and only if x, y are *not* so, and hence mz has also changed by z mod 2. There is, again, no total change, and the proof is complete.

COROLLARY. *Let α, β be smooth* SCCs *in M and suppose that $\alpha \simeq \beta$ in M; then $\vec{\alpha} \simeq \vec{\beta}$ in UM.*

Let now a be any homology class in H_1. We represent it by a collection of disjoint smooth SCCs $\{\alpha_i\}$ ($i = 1, \ldots, m$) and define $\tilde{a} \in \tilde{H}_1$ to be the homology class of

$$\sum_{i=1}^{m} \vec{\alpha}_i + mz .$$

The theorem shows that this depends on a alone; we call it the *canonical lifting of a*. Note in particular that $\tilde{0} = 0$: for if we represent $0 \in H_1$ by ∂D, then $\tilde{0} = \overrightarrow{\partial D} + z = 2z = 0$.

THEOREM 1B. *If $a, b \in H_1$ then $\widetilde{(a+b)} = \tilde{a} + \tilde{b} + (a \cdot b)z$.*

Remark. This formula makes precise the manner in which the canonical lifting fails to be a homomorphism.

Proof. Suppose first that $a \cdot b = 0$. Then we can represent a, b by *disjoint* smooth SCCs α, β. Choose a disc intersecting α, β in arcs u, v as in Figure 4 and orient α, β as shown there. Then in Figure 4d, x and y lie in the same component, and its homology class is $a+b$. Hence we have the equality

$$\overrightarrow{(a+b)}+z = \vec{a}+\vec{b} = \vec{a}+z+\vec{b}+z$$

that is,

$$\widetilde{(a+b)} = \tilde{a}+\tilde{b} = \tilde{a}+\tilde{b}+(a \cdot b)z .$$

Suppose on the other hand that $a \cdot b = 1$. in this case we may assume α, β to be SCCs which intersect (transversely) in just one point, as in Figure 5a, and then Figure 5 shows a sequence of homologies in UM connecting $\vec{a}+\vec{b}$ to $\vec{\gamma}$.

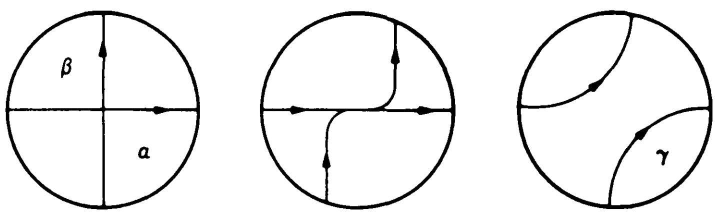


Fig. 5

Now the homology class of γ is $a+b$, so we get

$$\overrightarrow{(a+b)} = \vec{a}+\vec{b}$$

that is,

$$\widetilde{(a+b)} = \tilde{a}+\tilde{b}+z = \tilde{a}+\tilde{b}+(a \cdot b)z ,$$

which is what we required to prove.

THEOREM 1C. *If f is a diffeomorphism of M inducing f^* on H_1 and $\tilde{H}_1$, then* $\widetilde{(f^*a)} = f^*(\tilde{a})$.

Proof. If α is a smooth SCC representing a, then $f(\alpha)$ represents $f^*(a)$. But $\overrightarrow{f(\alpha)}$ is clearly the same as $f(\vec{\alpha})$, so $\overrightarrow{f^*(a)} = f^*(\vec{a})$. We get the required result from this by adding z to both sides and noting that $f^*(z) = z$ for any f.

LEMMA 1. *Suppose that f is a diffeomorphism of M which acts trivially on H_1. Then it also acts trivially on $\tilde{H}_1$, $\tilde{H}^1$ and Φ.*

Proof. We need only show that $f^* = 1$ on $\tilde{H}_1$. Let $\{e_i\}$ be a basis of H_1. Then $\tilde{e}_i$ and z generate $\tilde{H}_1$ (in fact, they actually form a basis). To see this, let $\tilde{x} \in \tilde{H}_1$ and let its projection to H_1 be x. Then $x = \sum m_i e_i$ and hence $\tilde{x}-\sum m_i \tilde{e}_i$ projects to 0 in H_1, which implies it is mz for some $m \in Z_2$.

Now by the hypothesis and Theorem 1C, we have $f^*(\tilde{e}_i) = \widetilde{f^*(e_i)} = \tilde{e}_i$. Also $f^*(z) = z$; this implies then that $f^* = 1$.

THEOREM 2. *The group Sp acts in a natural way on $\tilde{H}_1$ by the formula $h(\tilde{x}) = \widetilde{h(x)}$, $h(z) = z$ for $h \in Sp$. This action lifts that of Sp on H_1.*

Proof. We may lift h to a diffeomorphism f of M such that $f^* = h$. By the preceding lemma, the action of f^* on $\tilde{H}_1$ is independent of the lifting f and hence we may define the action of h to be that of f^*. But then

$$h(\tilde{x}) = f^*(\tilde{x}) = \widetilde{f^*(x)} = \widetilde{h(x)}.$$

This completes the proof.

We likewise get an induced action of Sp on $\tilde{H}^1$ and Φ, written on the right as usual. The Sp-action on Φ is an affine one, and it emphasizes the distinction between H^1 and its affine space Φ: for whereas $0 \in H^1$ is fixed by all $h \in Sp$, it is easy to see that, for $g \geqslant 2$, no spin structure of M is fixed by every h.

4. *Spin structures and quadratic forms on H_1*

By a *quadratic form* on the symplectic space H_1, we shall mean a function $\omega : H_1 \to Z_2$ such that

$$\omega(a+b) = \omega(a)+\omega(b)+a \cdot b.$$

Thus, a quadratic form in our restricted sense is just one whose associated bilinear form is the existing symplectic form on H_1. Note in particular that $\omega(0) = 0$.

LEMMA 2. *Let ω_1, ω_2 be two quadratic forms on H_1; then $\omega_1 - \omega_2$ is a homomorphism $H_1 \to Z_2$. Conversely, if ω is quadratic and $\theta : H_1 \to Z_2$ is a homomorphism then $\omega+\theta$ is quadratic.*

Proof. Indeed,

$$\begin{aligned}(\omega_1-\omega_2)(a+b) &= \omega_1(a+b)-\omega_2(a+b)\\ &= \omega_1(a)+\omega_1(b)+a\cdot b-\omega_2(a)-\omega_2(b)-a\cdot b\\ &= (\omega_1-\omega_2)(a)+(\omega_1-\omega_2)(b),\end{aligned}$$

showing $\omega_1 - \omega_2$ to be a homomorphism. The converse is proved similarly.

This lemma prompts the following definitions: if L is an abelian group, by an *affine space* over L we mean a transitive, free L-space, that is, a set K equipped with an "addition" $+ : L \times K \to K$ satisfying:

(a) $\lambda_1 + (\lambda_2 + k) = (\lambda_1 + \lambda_2) + k$ for $\lambda_i \in L$, $k \in K$;

(b) given any $k_1, k_2 \in K$, there is a *unique* $\lambda \in L$ such that $k_2 = \lambda + k_1$.

Thus K is practically like L, but with the distinction that it lacks a natural base point. The standard example for K is a non-trivial coset of L in some larger group $A \supset L$. For example, the set of spin structures Φ, being a coset of H^1 in $\tilde{H}^1$, is an affine space over H^1: we may "add" cohomology classes of M to spin structures.

Since the homomorphisms from H_1 to Z_2 are just the elements of H^1, Lemma 2 tells us that the set Ω of quadratic forms is also an affine space over H^1.

Two affine spaces K_1, K_2 over L are *affinely equivalent* if there is an $f : K_1 \to K_2$ such that $f(\lambda+k_1) = \lambda+f(k_1)$ for all $k_1 \in K_1$, $\lambda \in L$. This implies immediately that f is one to one and onto, and it is easy to see that any two affine spaces over L are equivalent. A related notion is an *(affine) automorphism* of K: this is a map $f : K \to K$ such that $f(\lambda+k) = f_0(\lambda)+f(k)$, where f_0 is some (necessarily unique) automorphism of L. Again, f will be bijective, and the set of such f forms a group. If f is a diffeomorphism of M then it induces a diffeomorphism of UM and hence automorphisms f^* of $\tilde{H}_1$ and $\tilde{H}^1$ (the action on the latter being written on the right). Since f^* takes $H^1 \subset \tilde{H}^1$ into itself, we have $\Phi f^* = \Phi$, and f^* is clearly an affine automorphism of Φ.

If now $h \in Sp$ and $\omega \in \Omega$, we define ωh by $\omega h(x) = \omega(hx)$. It is easy to see that ωh is quadratic on H_1 and that Sp acts, in this way, *affinely* on Ω. Since Φ is also an affine space over H^1 on which Sp acts affinely, it is affinely equivalent to Ω over H^1, and we are led to ask if they are equivalent in such a way as to preserve the Sp-action. The answer to this question is yes, as we see from the following considerations.

Let $\xi \in \Phi$. We define a function ω_ξ on H_1 by the formula $\omega_\xi(a) = \langle \xi, \tilde{a} \rangle$ for $a \in H_1$. This is defined, since $\tilde{a} \in \tilde{H}_1$ and $\xi \in \tilde{H}^1$. We have then:

$$\begin{aligned} \omega_\xi(a+b) &= \langle \xi, \widetilde{a+b} \rangle \\ &= \langle \xi, \tilde{a}+\tilde{b}+(a \cdot b)z \rangle \\ &= \langle \xi, \tilde{a} \rangle + \langle \xi, \tilde{b} \rangle + a \cdot b \quad (\text{since } \langle \xi, z \rangle = 1) \\ &= \omega_\xi(a)+\omega_\xi(b)+a \cdot b\,. \end{aligned}$$

Thus ω_ξ is indeed quadratic, and the assignment $\xi \to \omega_\xi$ gives us a function $Q : \Phi \to \Omega$.

THEOREM 3A. *The function Q is an affine equivalence, and hence a bijection of Φ to Ω.*

Proof. If $\delta \in H^1$ and $\xi \in \Phi$, then $Q(\xi+\delta) = \omega_{\xi+\delta}$ and

$$\omega_{\xi+\delta}(a) = \langle \xi+\delta, \tilde{a} \rangle = \langle \xi, \tilde{a} \rangle + \langle \delta, \tilde{a} \rangle.$$

But $\langle \delta, \tilde{a} \rangle = \langle \delta, a \rangle$, so we get

$$\omega_{\xi+\delta}(a) = \omega_\xi(a) + \langle \delta, a \rangle$$

for all $a \in H$. We may thus write $Q(\xi+\delta) = \omega_{\xi+\delta} = \omega_\xi+\delta = Q(\xi)+\delta$, which completes the proof.

THEOREM 3B. *The function Q commutes with the action of Sp on Φ, Ω, that is, $Q(\xi h) = (Q\xi)h$ for $\xi \in \Phi$, $h \in Sp$.*

Proof. $\omega_{\xi h}(a) = \langle \xi h, \tilde{a} \rangle = \langle \xi, h\tilde{a} \rangle = \langle \xi, \widetilde{ha} \rangle = \omega_\xi(ha) = (\omega_\xi h)(a)$.

As pointed out at the end of the previous section, no $\xi \in \Phi$ is fixed by all $h \in Sp$. More precisely, we have:

COROLLARY. *Let $\xi \in \Phi$ and $\omega = \omega_\xi$. The subgroup of Sp fixing ξ is precisely the orthogonal group 0_ω of ω.*

Proof. $g \in Sp$ fixes ξ if and only if it fixes ω.

5. *Atiyah's invariant on spin structures*

If L is a vector space over a field F and K is affine over L, we define $f: K \to F$ to be *linear* if $g(\lambda + k) = g_0(\lambda) + g(k)$ for some (necessarily unique) linear functional $g_0 : L \to F$. Polynomial functions on K are defined in the obvious way as sums of products of linear ones. If $x \in H_1$ then $\tilde{x} \in \tilde{H}_1$ gives a linear function on $\tilde{H}^1$ defined by $\tilde{\alpha} \to \langle \tilde{\alpha}, \tilde{x} \rangle$ for $\tilde{\alpha} \in \tilde{H}^1$. Restricted to Φ this gives us a linear function which we denote by $\bar{x}$. Let now a_i, b_i $(i = 1, \ldots, g)$ be a symplectic basis of H_1 and define the quadratic function α on Φ by

$$\alpha = \sum_{i=1}^{g} \bar{a}_i \bar{b}_i .$$

LEMMA 3. *$\alpha(\xi)$ is the Arf invariant of ω_ξ.*

Proof. We have

$$\alpha(\xi) = \sum_i \bar{a}_i(\xi) \bar{b}_i(\xi) = \sum_i \langle \xi, \tilde{a}_i \rangle \langle \xi, \tilde{b}_i \rangle = \sum_i \omega_\xi(a_i) \omega_\xi(b_i) .$$

COROLLARY 1. *The function α is independent of the choice of symplectic basis and is also invariant under the action of Sp on Φ.*

Proof. This follows from the same statements for the Arf invariant on Ω (proved by Arf in [**1**]) and the fact that the correspondence Q commutes with the Sp action.

COROLLARY 2. *Two spin structures are in the same Sp-orbit if and only if $\alpha(\xi_1) = \alpha(\xi_2)$.*

Proof. This is proved in the same way as the first corollary.

In [**2**], Atiyah constructs a Z_2 invariant of spin structures on a surface. This invariant dates back to Riemann, who discovered it via theta functions. Atiyah defines it thus: choose a complex structure on M; the spin structure ξ determines a holomorphic line bundle over M, which has then a finite dimensional vector space of holomorphic sections. This dimension, *mod* 2, is independent of the complex structure chosen and thus depends only on ξ. Note that although the invariant is defined by analytic data, it is a diffeomorphism invariant and hence it should be possible to compute it directly from topological data. We have then:

THEOREM 4. *The Atiyah invariant of ξ is just $\alpha(\xi)$.*

Proof. By its definition it is clear that Atiyah's invariant is invariant under the action of diffeomorphisms of M, that is, under the action of Sp on Φ, and hence it is constant on each of the (two) orbits of Sp in Φ. Now by Atiyah's Theorem 3 (see [**2**; pp. 49, 58]), his invariant is zero $2^{g-1}(2^g+1)$ times and one $2^{g-1}(2^g-1)$ times, that is, 0 on the larger orbit and 1 on the smaller. But the Arf invariant satisfies this same statement on Ω (*Proof.* $\omega \in \Omega$ is uniquely determined by its values α_i, β_i on a symplectic basis a_i, b_i, and its Arf invariant is then

$$\sum \alpha_i \beta_i .$$

The number of zeros of this expression is well known to be $2^{g-1}(2^g+1)$). Hence α satisfies the same statement on Φ, which implies that it is equal to Atiyah's invariant.

References

1. C. Arf, "Untersuchungen über quadratische Formen in Körpern der Charakteristik 2", *J. Reine Angew. Math.*, 183 (1941), 148–167.
2. M. Atiyah, "Riemann surfaces and spin structures", *Ann. Scient. Ecole Norm. Sup.*, V. 4 (1971). 47–62.
3. J. Milnor, "Remarks concerning spin manifolds", in *Differential and combinatorial topology* (Princeton, 1965).

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