Quillen cohomology and Hochschild cohomology

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1 Introduction

In their initial work ([4], [5], [1]), Michel André and Daniel Quillen described a cohomology theory applicable in very general algebraic categories. The term "Quillen," or "André-Quillen" cohomology has a uniform meaning. In contrast, the notion of Hochschild cohomology, while making appearances in diverse algebraic settings, has not been given such a uniform treatment. In this paper I offer a proposal for a definition of Hochschild cohomology in Quillen's context.

There are close relations between the two forms of cohomology. Quillen cohomology is intrinsically simplicial—typically one forms resolutions in a nonadditive category—and is correspondingly hard to compute. Hochschild cohomology, in contrast, will be defined as the derived functors of an additive functor—a form of "global sections"—on an abelian category, and should be easier to compute. There is a map from Hochschild to Quillen cohomology, and a spectral sequence having it as an edge homomorphism. The spectral sequence gives obstructions to the map being an isomorphism. The vanishing of the obstruction groups is a "smoothness" condition. We offer an account of observations of Quillen which amount to the assertion that in the category of groups every object is smooth, and in the category of associative R algebras an object A is smooth provided that $\mathrm{Tor}_q^R(A, A) = 0$ for $q > 0$. This observation accounts for the fact that Quillen cohomology offers nothing new in those contexts. The proposed definition of Hochschild cohomology actually gives Shukla cohomology in the case of associative R algebras, and (up to the usual shift in dimension, about which we will have more to say) it gives ordinary group cohomology when applied to groups. André and Quillen proved that under appropriate finiteness conditions this general condition of smoothness coincides in commutative algebras with the usual one.

I also consider a couple of more novel examples: categories with fixed object set (following work of Baues and Dwyer), and "racks" (recovering work of Andru.. and Grana).

2 Definitions

Let $\mathbf{C}$ be an "algebraic category," i.e. a cocomplete category with a set of small projective generators. Quillen observes that for any object $C \in \mathbf{C}$, the category $\mathrm{Ab}(\mathbf{C}/C)$ is abelian, and that there is a left adjoint to the forgetful functor,

$$\mathrm{Ab}_C : \mathbf{C}/C \to \mathrm{Ab}(\mathbf{C}/C).$$

He also exhibits a model category structure on the category of simplicial objects $s\mathbf{C}/C$.

Let $X_\bullet \to C$ be a cofibrant replacement for C in $\mathbf{C}$ (or, equivalently, of $1 : C \to C$ in $\mathbf{C}/C$). The Quillen homology object or "cotangent complex" is

$$L_C = \mathrm{Ab}_C X_\bullet \in \mathrm{Ho}\,(s\mathrm{Ab}(\mathbf{C}/C)).$$

Its homotopy is the Quillen homology

$$HQ_s(C) = \pi_s(L_C).$$

If $M \in \mathrm{Ab}(\mathbf{C}/C)$, the Quillen cohomology of C with coefficients in M is

$$HQ^*(C; M) = \pi^*(\mathrm{Hom}_{\mathbf{C}/C}(X_\bullet, M)).$$

This cohomology can be rewritten as

$$HQ^*(C; M) = \pi^*(\mathrm{Hom}_{\mathrm{Ab}(\mathbf{C}/C)}(\mathrm{Ab}_C X_\bullet, M)).$$

This expresses the Quillen cohomology as the derived functors of a composite:

$$B \mapsto \mathrm{Ab}_C B \mapsto \mathrm{Hom}_{\mathrm{Ab}(\mathbf{C}/C)}(\mathrm{Ab}_C B, M)$$

$$\mathbf{C}/C \to \mathrm{Ab}(\mathbf{C}/C) \to \mathrm{Ab}(\mathbf{Sets})$$

The first functor is a left adjoint, and we receive a "universal coefficients" spectral sequence

$$E_2^{st} = \mathrm{Ext}^s_{\mathrm{Ab}(\mathbf{C}/C)}(HQ_t(C), M) \Longrightarrow HQ^{s+t}(C; M).$$

Regarded as a functor on $\mathrm{Ab}(\mathbf{C}/C)$,

$$\mathrm{Hom}_{\mathbf{C}/C}(C, M) = \mathrm{Hom}_{\mathrm{Ab}(\mathbf{C}/C)}(\mathrm{Ab}_C C, M)$$

could be called the group of "global sections." I would like to call its derived functors the *Hochschild cohomology* of C with coefficients in M:

$$HH^*(C; M) = \mathrm{Ext}^*_{Ab(\mathbf{C}/C)}(\mathrm{Ab}_C C, M)$$

The edge homomorphism in the universal coefficients spectral sequence is a natural map

$$HH^*(C; M) \to HQ^*(C; M) \qquad (1)$$

and the spectral sequence gives obstructions to this map being an isomorphism. The initial terms give

$$HH^0(C; M) \xrightarrow{\cong} HQ^0(C; M)$$

and the exact sequence

$$0 \to HH^1(C; M) \to HQ^1(C; M) \to \mathrm{Hom}_{\mathrm{Ab}(\mathbf{C}/C)}(HQ_1(C), M) \to$$

$$\to HH^2(C; M) \to HQ^2(C; M)$$

More generally, if the Quillen homology object L_C is *discrete*, then the map (1) is an isomorphism. In this case we will call the object C *smooth*. To be provocative we will also call C *étale* if $L_C = 0$.

The next couple of sections will evoke fairly standard arguments for the first two clauses in the following theorem.

Theorem (Quillen, André). (1) If $\mathbf{C}$ is the category of groups, every object is smooth, but only the trivial group is étale. (2) If $\mathbf{C}$ is the category of associative algebras over a commutative ring R, every object A such that

$$\mathrm{Tor}_q^R(A, A) = 0 \quad \text{for} \quad q > 0$$

is smooth, and if in addition $\mu : A \otimes_R A \to A$ is an isomorphism then A is étale. (3) If $\mathbf{C}$ is the category of commutative algebras over a Noetherian commutative ring R, then a finitely generated R algebra C is smooth (resp. étale) if and only if it is smooth (resp. étale) in the usual sense of commutative algebra.

We end this section with a comment about why one focuses on $\mathrm{Ab}_A A$ rather than the more general $\mathrm{Ab}_A B$ for an object $p : B \to A$ over A. The composition functor $\mathbf{C}/B \to \mathbf{C}/A$ has a right adjoint $p^* : \mathbf{C}/A \to \mathbf{C}/B$, sending $C \to A$ to $B \times_A C \to B$. This right adjoint lifts to a functor $p^* : \mathrm{Ab}(\mathbf{C}/A) \to \mathrm{Ab}(\mathbf{C}/B)$, which is right adjoint to a functor $p_* : \mathrm{Ab}(\mathbf{C}/B) \to \mathrm{Ab}(\mathbf{C}/A)$. Since the diagram

$$\begin{array}{ccc} \mathbf{C}/B & \xleftarrow{p^*} & \mathbf{C}/A \\ \uparrow & & \uparrow \\ \mathrm{Ab}(\mathbf{C}/B) & \xleftarrow{p^*} & \mathrm{Ab}(\mathbf{C}/A) \end{array}$$

commutes, the diagram of left adjoints commutes as well, showing that there is an isomorphism natural in $C \to B$

$$p_* \mathrm{Ab}_B C = \mathrm{Ab}_A C.$$

In particular, $\mathrm{Ab}_A B = p_* \mathrm{Ab}_B B$.

We will identify the functor p_* in examples below.

further comments

Lemma. If $\mathbf{A}$ is an additive category, every object of $\mathbf{A}$ has a unique unital product, namely the abelian group structure with unit given by the unique map from the terminal object and product given by the fold map $A \coprod A \to A$ composed with the inverse of the isomorphism $A \coprod A \to A \times A$.

Lemma. If $\mathbf{A}$ is abelian and $A \in \mathbf{A}$, then the functor sending an object $p : B \to A$ of $\mathbf{A}/A$ to $\ker p \in \mathbf{A}$ establishes an equivalence from the category of sectioned objects over A to the category $\mathbf{A}$.

3 Groups

Let $\mathbf{C} = \mathbf{Gp}$. We'll recall what a group object over $G \in \mathbf{Gp}$ turns out to be. It's a group H and a homomorphism $p : H \to G$; a "unit" homomorphism $e : G \to H$ such that $pe = 1_G$; and more. Already we can say that we have a split extension. Let $K = \ker p$. The extension is determined by the action of G on K determined by e:—$G \times K \to H$ by $(g,k) \mapsto k \cdot eg$ is a group isomorphism, if we put the group structure

$$(g,k)(g',k') = (gg', k \cdot {}^{g}k')$$

on the product, where ${}^{g}k' = (eg)k'(eg)^{-1}$. Write $G\tilde{\times}K$ for this group.

Now for the group structure: this is a group homomorphism

$$\mu : G \times K \times K = (G \times K) \times_G (G \times K) \to G \times K.$$

Being a homomorphism says that

$$\mu(g,k,k')\mu(g_1,k_1,k_1') = \mu(gg_1, k \cdot {}^{g}k_1, k' \cdot {}^{g}k_1').$$

μ is supposed to define on $G \times K$ the structure of a group over G; so it should be unital:

$$\mu(g,1,k) = (g,k) = \mu(g,k,1).$$

Thus

$$\mu(g,k,k_1') = \mu(1,k,1) \cdot \mu(g,1,k_1') = (1,k) \cdot (g,k_1') = (g,kk_1').$$

This says that the map $K \to p^{-1}(g)$ sending k to $k \cdot e(g) = e(g) \cdot k$ is a group isomorphism. We see that the group structure map μ is determined by the ordinary group product on K, and is independent of the action of G. The group structure is abelian provided K is abelian.

Thus: an abelian group object in the category of groups over G is precisely a split extension of G with abelian kernel. That is, for any group G, the category of abelian groups over G is simply the category of $\mathbb{Z}G$ modules.

Under this identification, the abelianization $\mathrm{Ab}_G H$ of an object $H \to G$ in $\mathbf{Gp}/G$ is the $\mathbb{Z}G$ module which is characterized by the existence of a natural isomorphism

$$\mathrm{Map}_G(H, G\tilde{\times}M) \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathrm{Ab}_G H, M)$$

A map $H \to G\tilde{\times}M$ over G is of the form $h \mapsto (ph, \varphi h)$ where $\varphi(hh') = \varphi h + p(h)\varphi h'$; that is, it's the same thing as a *crossed homomorphism* $H \to M$, $\varphi \in Z^1(H;M)$. This means that φ is a $\mathbb{Z}H$ module homomorphism $\varphi : \mathbb{Z}H[H] \to M$ with the property that $\varphi([hh'] - [h] - p(h)[h']) = 0$. Thus $\mathrm{Ab}_G H = \mathbb{Z}H[H]/([hh'] - [h] - p(h)[h'])$, which implies that

$$\mathrm{Ab}_G H = \mathbb{Z}G \otimes_{\mathbb{Z}H} I_H.$$

where

$$I_H = \ker(\epsilon : \mathbb{Z}H \to \mathbb{Z}).$$

The abelianization comes equipped with a map from H over G, which translates to the universal crossed homomorphism to a G module, $d : H \to \mathbb{Z}G \otimes_{\mathbb{Z}H} I_H$, sending $h \mapsto 1 \otimes (h-1)$.

The functor $p^* : \mathrm{Ab}(\mathbf{Gp}/G) \to \mathrm{Ab}(\mathbf{Gp}/H)$ simply regards a $\mathbb{Z}G$ module as a $\mathbb{Z}H$ module via the map $p : \mathbb{Z}H \to \mathbb{Z}G$. Its left adjoint is given by $p_* M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M$, and the isomorphism $\mathrm{Ab}_G H = p_* \mathrm{Ab}_H H$ is in view.

Lemma 3.1 If X is a free group then $I\mathbb{Z}X$ is a free module over $\mathbb{Z}X$. If X is the free group on the set S, then $I\mathbb{Z}X$ is the free $\mathbb{Z}X$ module on the set $\{s - 1 : s \in S\}$.

Proof. We are claiming that the $\mathbb{Z}X$ module I_X freely generated by the restriction to S of $d : X \to I_X$. For any $\mathbb{Z}X$ module M, the map d induces $Z^1(X; M) = \mathrm{Hom}_{\mathbb{Z}X}(I_X, M) \to \mathrm{Map}(S, M)$, which we must see is bijective. Certainly, any crossed homomorphism is determined by its restriction to S. On the other hand, knowing $\varphi|S$ determines φ on the set S^{-1} of inverses of elements of S: $\varphi(1) = 0$ implies that

$$\varphi(s^{-1}) = -s^{-1}\varphi(s).$$

Then the only possible choice of value of φ on the product $x_1 \cdots x_n$, where $x_i \in S \coprod S^{-1}$, is

$$\varphi(x_1 \cdots x_n) = \varphi(x_1) + x_1\varphi(x_2) + \cdots + x_1 \cdots x_{n-1}\varphi(x_n).$$

This is well-defined, since it is compatible with the cancellations $ss^{-1} = 1$ and $s^{-1}s = 1$, and it does define a crossed homorphism, as you can check by splitting $x_1 \cdots x_n$ into a product and using the formula. qed

Proposition 3.2 If $X_\bullet \to H$ is a cofibrant replacement and $H \to G$ is a monomorphism, then

$$\mathbb{Z}G \otimes_{\mathbb{Z}X_\bullet} I\mathbb{Z}X_\bullet \to \mathbb{Z}G \otimes_{\mathbb{Z}H} I\mathbb{Z}H$$

is a weak equivalence. In particular,

$$L_G \xrightarrow{\sim} \mathrm{Ab}_G G$$

and G is smooth as a group.

Proof. In

$$\begin{array}{ccc} \mathbb{Z}G \otimes^L_{\mathbb{Z}X_\bullet} I\mathbb{Z}X_\bullet & \xrightarrow{\sim} & \mathbb{Z}G \otimes_{\mathbb{Z}X_\bullet} I\mathbb{Z}X_\bullet \\ \downarrow \sim & & \downarrow \\ \mathbb{Z}G \otimes^L_{\mathbb{Z}H} I\mathbb{Z}H & \longrightarrow & \mathbb{Z}G \otimes_{\mathbb{Z}H} I\mathbb{Z}H. \end{array}$$

the top arrow is a weak equivalence by the lemma and the Corollary in Quillen's II.6.10. From the fact that $X_\bullet \to H$ is a weak equivalence it follows that $\mathbb{Z}X_\bullet \to \mathbb{Z}H$ is a weak equivalence. (For a startling proof of this standard fact, see [3], p. 161.) From this and the five lemma it follows that $I\mathbb{Z}X_\bullet \to I\mathbb{Z}H$ is a weak equivalence. Therefore the left vertical is a weak equivalence. Finally, the assumption that $H \to G$ is a monomorphism implies that $\mathbb{Z}G$ is free over $\mathbb{Z}H$, which implies that the bottom arrow is an equivalence. Thus the right arrow is too. qed

4 Associative algebras

Let $\mathbf{C} = R{-}\mathbf{Alg}$. We'll recall what a group object in $R{-}\mathbf{Alg}/A$ is. It consists of an R algebra map $p : B \to A$; a unit R algebra map $e : A \to B$; and more. Already we have a split extension. Write K for the kernel of p; it's a two sided ideal in B, and the splitting gives an A bimodule structure over R on K by $a \cdot k \cdot a' = e(a)ke(a')$. The map $A \oplus K \to B$ sending (a, k) to $ea + k$ is an A bimodule isomorphism, and in these terms the multiplication in B is given by $(a, k)(b, l) = (ab, al + kb + kl)$.

Conversely, given an A bimodule M over R, define an R algebra structure on $A \oplus M$ by requiring $(a, m)(a', m') = (aa', am' + ma')$ and $1 = (1, 0)$, and the projection $A \oplus M \to M$ is an algebra map with a kernel of square zero. Moreover, defining μ by $\mu((a, k), (a, l)) = (a, k + l)$ gives us an abelian group structure on this object of $R{-}\mathbf{Alg}/A$.

There is also a unital multiplication, $\mu : B \times_A B \to B$. In terms of the splitting, the unital condition asserts $\mu((a, 0), (a, k)) = (a, k) = \mu((a, k), (a, 0))$. The fact that μ is additive implies then that $\mu((a, k), (a, l)) = (a, k + l))$; it is entirely determined by the additive structure of K. The fact that it is an algebra map implies that $K^2 = 0$, and we are in the situation described in the previous paragraph.

This establishes an equivalence of categories between $R{-}\mathbf{Alg}/A$ and the category of A bimodules over R.

Let $B \to A$ be an R algebra over A, and M an A bimodule over R. A map of R algebras over A from B to $A \oplus M$ is the same as an R derivation

$\phi : B \to M$. The abelianization of B as an R algebra over A is thus the A bimodule over R that represents the functor sending the A bimodule M to $\mathrm{Der}_R(B, M)$. According to the pushforward formula,

$$\mathrm{Ab}_A B = A \otimes_B \mathrm{Ab}_B B \otimes_B A$$

so it will be enough to determine $\mathrm{Ab}_B B$. An R-linear map $B \to M$ extends uniquely to a B-bimodule homomorphism $B \otimes_R B \otimes_R B \to M$, and the map $B \to M$ is a derivation precisely when the extended map sends all elements of the form $a \otimes bb' \otimes c - ab \otimes b' \otimes c - a \otimes b \otimes b'c$ to zero. $\mathrm{Ab}_B B$ is thus the cokernel of the map $B \otimes_R B \otimes_R B \otimes_R B \to B \otimes_R B \otimes_R B$ described by this formula.

This map is a differential in the Hochschild complex

$$0 \leftarrow B \leftarrow B \otimes_R B \leftarrow B \otimes_R B \otimes_R B \leftarrow \cdots$$

This complex is exact (split exact as a sequence of one sided B-modules), so our cokernel is canonically isomorphic to

$$I_B = \ker(\mu : B \otimes_R B \to B)$$

The universal derivation $d : B \to I_B R$ is given by $d : b \mapsto b \otimes 1 - 1 \otimes b$.

Lemma 4.1 If X is a free R algebra then I_X is free as an X bimodule over R. If X is the free R algebra generated by the set S, then I_X is the free X bimodule over R generated by $\{s \otimes 1 - 1 \otimes s : s \in S\}$.

Proof. The composite $\mathrm{Hom}_{X^e}(I_X, M) \cong \mathrm{Der}_R(X, M) \to \mathrm{Map}(S, M)$ sends f to $s \mapsto f(s \otimes 1 - 1 \otimes s)$. We wish to show that this composite is bijective, for any X bimodule M. Certainly, any R derivation $\delta : X \to M$ is determined by its restriction to S. Conversely, given a map $\delta : S \to M$, the only possible choice of value of an R derivation extending δ on the product $x_1 \cdots x_n$, where $x_i \in S$, is

$$\delta(x_1 \cdots x_n) = \delta(x_1)x_2 \cdots x_n + x_1\delta(x_2)x_3 \cdots x_n + \cdots + x_1 \cdots x_{n-1}\delta(x_n).$$

This formula does define an R derivation, as you can check by splitting $x_1 \cdots x_n$ into a product and using the formula. qed

An A bimodule M over R can be regarded as an $A^e = A \otimes_R A^{\mathrm{op}}$ module, and if $A \to B$ is an R algebra map then the B bimodule $B \otimes_A M \otimes_A B$ over R corresponds to the B^e module $B^e \otimes_{A^e} M$. B^e if flat as a right A^e module if and only if B is flat as both left and right A module.

Let $X_\bullet \to B$ be a cofibrant replacement for B as an R algebra.

Lemma 4.2 If $\mathrm{Tor}_q^R(B,B) = 0$ for $q > 0$, then $X_\bullet^e \to B^e$ and $I_{X_\bullet} \to I_B$ are weak equivalences.

Proof. Write B^{he} for the "homotopy extension" of B, $B^{he} = B \otimes_R^L B$. In

$$\begin{array}{ccc} X_\bullet^{he} & \xrightarrow{\sim} & X_\bullet^e \\ \downarrow\sim & & \downarrow \\ B^{he} & \longrightarrow & B^e \end{array}$$

the top arrow is an equivalence, and the left arrow is an equivalence since $X_\bullet \to B$ is. The hypothesis guarantees that the bottom arrow is an equivalence, so the right arrow is.

In this situation, then, the middle vertical in

$$\begin{array}{ccccc} I_{X_\bullet} & \longrightarrow & X_\bullet \otimes X_\bullet & \longrightarrow & X_\bullet \\ \downarrow\sim & & \downarrow & & \downarrow \\ I_B & \longrightarrow & B \otimes B & \longrightarrow & B \end{array}$$

is an equivalence, and so the left vertical is as well. qed

Proposition 4.3 Let $X_\bullet \to B$ be a cofibrant replacement and assume that $B \to A$ makes A flat as both right and left B module and that $\mathrm{Tor}_q^R(B,B) = 0$ for $q > 0$. Then

$$A \otimes_{X_\bullet} I_{X_\bullet} \otimes_{X_\bullet} A \to A \otimes_B I_B \otimes_B A$$

is a weak equivalence. In particular, if $\mathrm{Tor}_q^R(A,A) = 0$ for $q > 0$ then

$$L_A \xrightarrow{\sim} \mathrm{Ab}_A A$$

and A is smooth as an associative R algebra. Under these conditions, A is étale as an associative R algebra exactly when the multiplication map $A \otimes_R A \to A$ is bijective.

Proof. In

$$\begin{array}{ccc} A^e \otimes_{X_\bullet^e}^L I_{X_\bullet} & \xrightarrow{\sim} & A^e \otimes_{X_\bullet^e} I_{X_\bullet} \\ \downarrow\sim & & \downarrow \\ A^e \otimes_{B^e}^L I_B & \longrightarrow & A^e \otimes_{B^e} I_B \end{array}$$

the top arrow is an equivalence since I_X is free as an $X_\bullet^e$ module. Under the Tor condition, the left arrow is an equivalence by the Lemma and a spectral sequence. Under the assumption that A^e is flat over B^e, the bottom arrow is an equivalence, and then the right arrow is too. qed

5 Monoids

Fix a monoid X and consider an abelian group object A in the category $\mathbf{Mon}/X$ of monoids over X. We have a projection map $p : A \to X$ and a unit section $0 : X \to A$. It must be a monoid map so $0_1 = 1 \in A_1$. For each $x \in X$ let $A_x = p^{-1}(x)$. The abelian structure imposes an abelian group structure on each A_x. The fact that the abelian group structure map, $+$, is a map of monoids (whose structure map we denote by juxtaposition) translates to

$$ab + a'b' = (a + a')(b + b'), \qquad a, a' \in A_x, \quad b, b' \in A_y. \tag{2}$$

With $b = 0_y$ and $a' = 0_x$ this gives $a0_y + 0_x b' = ab'$. For any $x, y \in X$ write

$$\alpha_x = 0_x \cdot : A_y \to A_{xy}, \qquad \beta_y = \cdot 0_y : A_x \to A_{xy}.$$

Then

$$ab = \alpha_x b + \beta_y a.$$

Also

$$\alpha_1 b = b, \qquad \beta_1 a = a. \tag{3}$$

With $a = a' = 0_x$, (2) implies that α_x is a group homomorphism; similarly, β_y is a group homomorphism. Also, A_1 contains the unit 1 for the monoid A, and $\alpha_x(1) = 0_x \cdot 1 = 0_x$ and $\beta_x(1) = 1 \cdot 0_x = 0_x$, so we have

$$\alpha_x(1) = 0, \qquad \beta_y(1) = 0. \tag{4}$$

The associativity formula $(ab)c = a(bc)$ translates to

$$\alpha_{xy} c + \beta_z \alpha_x b + \beta_z \beta_y a = \alpha_x \alpha_y c + \alpha_x \beta_z b + \beta_{yz} a.$$

Taking two of the elements to be zero leads to the following three equations:

$$\alpha_{xy} = \alpha_x \alpha_y, \qquad \alpha_x \beta_z = \beta_z \alpha_x, \qquad \beta_{yz} = \beta_z \beta_y \tag{5}$$

and together these imply the long relation.

So $\mathrm{Ab}(\mathbf{Mon}/X)$ can be identified with the category whose objects are X-indexed abelian groups A together with homomorphisms $\alpha_x : A_y \to A_{xy}, \beta_y : A_x \to A_{xy}$, satisfying (3), (4), and (5). Thus $\mathrm{Ab}(\mathbf{Mon}/X)$ is the category of functors to abelian groups from the category $\mathrm{Fac}(X)$ having X as object set, and

$$\mathrm{Fac}(x, y) = \{(u, v) \in X \times X : y = uxv\}.$$

In this notation $\alpha_u = (u,1)$, $\beta_v = (1,v)$. The category of functors from this is the same as the category of modules over a ringoid R_X obtained by taking the free abelian group of all morphism sets in $\mathrm{Fac}(X)$. This has been considered by Dwyer and by Baues in the greater generality of categories with fixed object set.

In these terms, a section of the projection map is a family $c_x \in A_x$ of elements with the property that

$$c_{xy} = \alpha_x c_y + \beta_y c_x, \qquad c_1 = 0_1.$$

The abelianization of X as an object over itself, $\mathrm{Ab}_X X$, can be described as an R_X module in terms of generators and relations, and in fact there is a canonical contractible simplicial R_X module

$$R_X(1,-) \Leftarrow \oplus_{x\in X} R_X(x,-) \Leftarrow \oplus_{x,y\in X} R_X(xy,-) \Leftarrow \cdots \tag{6}$$

such that $\mathrm{Ab}_X X$ is the quotient of $\oplus_{x\in X} R_X(x,-)$ by the image of the differential. We describe the simplicial operators by giving the images of the "fundamental classes," that is, the identity classes. Denote by $\iota_{x_1,\dots,x_n}$ the identity class in the summand indexed by $(x_1,\dots,x_n)$. Then

$$d_j \iota_{x_1,\dots,x_n} = \begin{cases} \alpha_{x_1}\iota_{x_2,\dots,x_n} & \text{for } j=0, \\ \iota_{x_1,\dots,x_j x_{j+1},\dots,x_n} & \text{for } 0<j<n, \\ \beta_{x_n}\iota_{x_1,\dots,x_{n-1}} & \text{for } j=n, \end{cases}$$

$$s_j \iota_{x_1,\dots,x_{n-1}} = \iota_{x_1,\dots,x_j,1,x_{j+1},\dots,x_{n-1}}.$$

There is a contracting homotopy s_{-1} as well. It is not a map of modules, and so to define it I must give its value on a basis of $R_X(x_1\cdots x_{n-1}, y)$ for each $y \in X$:

$$s_{-1}(u,v)\iota_{x_1,\dots,x_{n-1}} = (1,v)\iota_{u,x_1,\dots,x_{n-1}}.$$

(There is also

$$s_n(u,v)\iota_{x_1,\dots,x_{n-1}} = (u,1)\iota_{x_1,\dots,x_{n-1},v},$$

giving a "right" contracting homotopy.)

Thus

$$R^\bullet\Gamma(X;A) = \mathrm{Ext}^\bullet_{R_X}(\mathrm{Ab}_X X, A)$$

may be computed as the cohomology of a complex

$$0 \to \prod_{x\in X} A_x \to \prod_{x,y\in X} A_{xy} \to \cdots .$$

This simplicial object (6) is augmented to the R_X module $\mathbb{Z}_X$ given by the constant functor on $\mathrm{Fac}(X)$ with value $\mathbb{Z}$, by means of $\epsilon\iota_1 = 1$, and the contracting homotopy extends to $s_{-1} : \mathbb{Z}_X(y) \to R_X(1, y)$ by $s_{-1}1 = \beta_y$. Thus the abelianization fits into a short exact sequence

$$0 \to \mathrm{Ab}_X X \to R_X(1, -) \to \mathbb{Z}_X \to 0.$$

This story extends pretty directly to the case of the categories with fixed object set.

6 Lie algebras

Suppose **C** is the category of Lie algebras over a ring K: vector spaces with a bilinear product satisfying

$$[x, x] = 0\,, \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

This forces $[x, y] = -[y, x]$.

The category of abelian objects over L is equivalent to the category of L-modules, or what is the same, the category of UL-modules. A derivation is a derivation in the usual sense, so the UL-module of Lie differentials is

$$UL \otimes L/\{ia \otimes b - ib \otimes a - 1 \otimes [a, b]\}$$

where $i : L \to UL$ is the canonical map. This is

$$\mathrm{coker}\,(d : UL \otimes \Lambda^2 L \to UL \otimes L)$$

where d is the UL-linear extension of the map $a \wedge b \mapsto ia \otimes b - ib \otimes a - 1 \otimes [a, b]$. This is a differential in the Chevalley-Eilenberg complex, which (assuming L is projective as K-module) is exact. Consequently the Beck module of Lie differentials is given by

$$\mathrm{Ab}_L L = I_L = \ker\,(\epsilon : UL \to K)$$

As in associative algebras, a derivation on a free Lie algebra is freely determined by its values on the generating set; so I_L is a free UL module if L is a free Lie algebra.

But now assume that self-brackets vanish. Then L embeds into its universal enveloping algebra. The Poincaré-Birkoff-Witt theorem tells us that the

Lie filtration on UL has $\mathrm{Sym}(L)$ as associated graded, and so depends only on the underlying vector space of L. Therefore U preserves weak equivalences because its associated graded object does by Dold's theorem.

Let $X_\bullet \to L$ be a cofibrant replacement. Then $UX_\bullet \to UL$ is an equivalence. Therefore $I_{X_\bullet} \to I_L$ is an equivalence.

The cotangent complex is given by

$$\mathbf{L}_L = UL \otimes_{UX_\bullet} I_{X_\bullet}$$

We have a diagram

$$\begin{array}{ccc} UL \otimes^L_{UX_\bullet} I_{X_\bullet} & \to & UL \otimes_{UX_\bullet} I_{X_\bullet} \\ \downarrow & & \downarrow \\ UL \otimes^L_{UL} I_L & \to & I_L \end{array}$$

The top map is an isomorphism because $I_{X_\bullet}$ is free over $X_\bullet$; the left one by one of Quillen's spectral sequences; and the bottom one, since UL is free over UL.

This shows that the cotangent object is discrete.

Hence for any L-module M, the universal coefficient spectral sequence collapses to the isomorphism

$$\mathrm{Ext}^*_{UL}(I_L, M) = HQ^*(L; M)$$

The short exact sequence $0 \to I_L \to UL \to K \to 0$ then shows that the Quillen cohomology of L with coeffients in M coincides up to a shift in dimension with the Chevalley-Eilenberg cohomology.

One may also study L_*Q on the category of Lie algebras, where

$$QL = L/[L, L]$$

is what is traditionally called abelianization. This is not correct terminology in general, but notice that the map $L \to IUL$ carries $[x, y]$ to $xy - yx \in (IUL)^2$ and hence induces a map

$$QL \to IUL/(IUL)^2 = K \otimes_{UL} IUL = U(0) \otimes_{UL} IUL = \mathrm{Ab}_0(L)$$

If L is free on the vector space V then $QL = V$, $UL = \mathrm{Tens}(V)$, and $IUL/(IUL)^2 = V$, so in that case this map is an isomorphism.

So if $X_\bullet$ is a cofibrant replacement for L, then

$$L_nQ(L) = \pi_n(QX_\bullet) = \pi_n(K \otimes_{UX_\bullet} IUX_\bullet) = HQ_n(L; K)$$

where K is regarded as a UL-module through the augmenation.

7 Extended Lie algebras

In characteristic 2, you may want to replace the axiom $[x, x] = 0$ with the weaker assumption (equivalent in characteristic not 2) that $[x, y] = [y, x]$. Call these "extended Lie algebras," $\mathbf{Lie^e}$. The map $e : x \mapsto [x, x]$ is linear. The set of self-brackets forms an ideal R, one with the property that $[R, L] = 0$. Then the universal map to the Lie algebra underlying an associative algebra kills this ideal. Write L^r for L/R. The category $\mathrm{Ab}(\mathbf{Lie^e}/L)$ is equivalent to the category of left $U(L^r)$-modules, and the module of Kähler differentials is $IU(L^r)$.

The free functor $\mathbf{Mod}_k \to \mathbf{Lie^e}$ can be seen as $Lie(V) \oplus eLie(V)$.

The module of indecomposables QL is of course equal to QL^r, and as before $L_n Q^{\mathbf{Lie^e}}(L) = HQ_n(L; K)$.

Let E be the exterior algebra on one generator e. It acts on an extended Lie algebra, with $ex = [x, x]$. The forgetful functor $\mathbf{Lie^e} \to \mathbf{Mod}_E$ has a left adjoint F. We should be able to compute $L_* Q^{\mathbf{Lie^e}}(FV)$.

8 Hochschild resolutions

So we have the "Hochschild resolutions"

$$I_A \leftarrow A \otimes A \otimes A \leftarrow A \otimes A^{\otimes 2} \otimes A \leftarrow \cdots$$

$$I_G \leftarrow \mathbb{Z}[G]\langle G \rangle \leftarrow \mathbb{Z}[G]\langle G^2 \rangle \leftarrow \cdots$$

$$I_L \leftarrow UL \otimes L \leftarrow UL \otimes \Lambda^2(L) \leftarrow \cdots$$

9 σ-algebras

Fix a prime p. A σ-algebra is a commutative algebra R together with a self map σ such that $r \mapsto r^p - p\sigma(r)$ is a ring endomorphism. A module over (R, σ) consists in an R module M together with an additive self map τ such that

$$\tau(rx) = r^p \tau(x) - p\sigma(r)(x + \tau(x)) \,.$$

A global section of (M, τ) is a derivation $d : R \to M$ such that

$$d(\sigma r) = r^{p-1} dr - \tau(dr) \,.$$

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