Chapter XIII.

# THE OVERPATH METHOD.

We saw in Chapter XII that the cohomology group $H^2(S,\mathbb{G})$ can be computed by means of functions of two variables in $S$, namely, symmetric 2-cocycles. The overpath method computes $H^2(S,\mathbb{G})$ by means of functions of one variable, one for every defining relation of $S$ in any suitable presentation of $S$. This makes the computation of $H^2(S,\mathbb{G})$ a finite task whenever $S$ is finitely generated.

Applications compute $H^2(S,\mathbb{G})$ when $S$ is cyclic or, more generally, has only one defining relation, and when $S$ is partially free. This last application depends rather heavily on the construction of group-free congruences in Chapter X. We also show that strand bases give rise to minimal cocycles. The main results are from Grillet [1992], [1995F], [1996C], [1995P], [2000Z].

## 1. OVERPATHS.

The overpath method depends on certain properties of free commutative semigroups and congruences on these semigroups.

1. In what follows $F = F_X$ is the free commutative monoid on a set $X$. We return to the additive notation for $F$ and write the elements of $F$ as finite linear combinations $a = \sum_{x\in X} a_x x$ of elements of $X$, with the usual order:

$$a \leqq b \text{ if and only if } a_x \leqq b_x \text{ for all } x \in X;$$

the length of $a = \sum_{x\in X} a_x x$ is $|a| = \sum_{x\in X} a_x$.

**Proposition 1.1.** *On every free commutative monoid $F$ there exists an order relation $\sqsubseteq$ such that:*

(1) $(F,\sqsubseteq)$ *is well-ordered;*

(2) *if $a \leqq b$ in $F$, then $a \sqsubseteq b$;*

(3) *if $a \sqsubseteq b$ in $F$, then $a + c \sqsubseteq b + c$ for all $c \in F$.*



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**Proof.** Well-order $X$, then define $\sqsubseteq$ as follows: $a \sqsubset b$ if and only if either $|a| < |b|$, or $|a| = |b|$, $a \neq b$ and the least $x \in X$ such that $a_x \neq b_x$ satisfies $a_x > b_x$. (This is the **degree lexicographic order** on $F$.)

An element of $F$ of length $l$ is the sum $x_1 + \ldots + x_l$ of $l$ elements $x_1 \sqsubseteq x_2 \sqsubseteq \ldots \sqsubseteq x_l$ of $X$. Each set $F_l = \{a \in F \mid |a| = l\}$ is a subset of the lexicographic product $X^l = X \times \ldots \times X$ and is well-ordered by $\sqsubseteq$. Then $F$ is the ordinal sum of $F_0, F_1, \ldots, F_l, \ldots$ and is well-ordered by $\sqsubseteq$.

If $a \leqq b$ in $F$, then either $a = b$ or $|a| < |b|$; in either case $a \sqsubseteq b$.

Finally let $a \sqsubset b$ and $c \in F$; then either $|a| < |b|$, or $|a| = |b|$ and the least $x$ such that $a_x \neq b_x$ satisfies $a_x > b_x$. In the first case, $|a+c| < |b+c|$. In the second case, the least $x$ such that $a_x \neq b_x$ is also the least $x$ such that $a_x + c_x \neq b_x + c_x$, and satisfies $a_x + c_x > b_x + c_x$. In either case $a+c \sqsubseteq b+c$. □

We call an order relation $\sqsubseteq$ on $F$ a **compatible well order** when it has properties (1), (2), and (3) in Proposition 1.1. Explicit compatible well orders can be constructed in various ways, besides the degree lexicographic order, particularly if $X = \{x_1, x_2, \ldots, x_n\}$ is finite (see e.g. Adams & Loustaunau [1994]); if for instance $p_1, \ldots, p_n$ are the first $n$ prime numbers, then

$$a_1 x_1 + \cdots + a_n x_n \sqsubseteq b_1 x_1 + \cdots + b_n x_n \iff {p_1}^{a_1} \cdots {p_n}^{a_n} \leqq {p_1}^{b_1} \cdots {p_n}^{b_n}$$

is a compatible well order on $F$, the **prime order** on $F$ of Rosales [1995].

A **lexicographic order** $\sqsubseteq$ on $F$ is defined from a well order $\preccurlyeq$ on $X$ by: $a \sqsubset b$ if and only if $a \neq b$ and the least $x \in X$ such that $a_x \neq b_x$ has $a_x < b_x$. Then $a < b$ implies $a \sqsubset b$. Also, $x \prec y$ in $X$ implies $x \sqsupset y$ in $F$. (The usual definition requires $a_x > b_x$, but then $a < b$ does not imply $a \sqsupset b$.)

**Proposition 1.2.** *If $F$ is finitely generated, then every lexicographic order on $F$ is a compatible well order.*

**Proof.** If $a \sqsubset b$ and $c \in F$, then, in the above, the least $x$ such that $a_x \neq b_x$ is also the least $x$ such that $a_x + c_x \neq b_x + c_x$, so that $a_x + c_x > b_x + c_x$ and $a+c \sqsubset b+c$. If $X$ is finite, then $(F, \sqsubseteq)$ is a finite lexicographic product of copies of $\mathbb{N}^+$ and is well ordered. □

When $F$ is finitely generated, every total order $\sqsubseteq$ on $F$ with property (2) is a well order: if indeed there is an infinite sequence $a_1 \sqsupset a_2 \sqsupset \cdots \sqsupset a_n \sqsupset a_{n+1} \sqsupset \cdots$, then $A_n = \{t \in F \mid t \sqsupseteq a_n\}$ is an ideal of $F$ for every $n$, by (2), and $A_1 \subsetneqq A_2 \subsetneqq \cdots \subsetneqq A_n \subsetneqq A_{n+1} \subsetneqq \cdots$, a flagrant contradiction of Corollary VI.1.3; therefore the totally ordered set $(F, \sqsubseteq)$ satisfies the descending chain condition and is well ordered. Compatible well orders are then also known as a **linear admissible orders**.

If on the other hand $X$ is infinite, then the first generators $x_1 \prec x_2 \prec \cdots \prec x_n \prec \cdots$ of $X$ yield a nonempty subset $x_1 \sqsupset x_2 \sqsupset \cdots \sqsupset x_n \sqsupset \cdots$ of $F$ with no least element and $F$ is not well ordered by its lexicographic orders, even though they satisfy (2) and (3).

2. In what follows, $\sqsubseteq$ is any compatible well order on $F$.

Let $\mathcal{C}$ be a congruence on $F$. Under $\sqsubseteq$ the $\mathcal{C}$-class $C_a$ of $a \in F$ has a least element $q(a)$ (the **function minimum** of Rosales [1995]). By definition,

$$a \;\mathcal{C}\; q(a); \; c \;\mathcal{C}\; a \text{ implies } c \sqsupseteq q(a); \text{ and } a \;\mathcal{C}\; b \iff q(a) = q(b).$$

Then $F$ is the disjoint union $F = P \cup Q$, where $Q = Q(\mathcal{C}) = \{q(a) \mid a \in F\} = \{q \in F \mid a \;\mathcal{C}\; q \implies a \sqsupseteq q\}$ is the set of all least elements of all $\mathcal{C}$-classes, and

$$P = P(\mathcal{C}) = F \backslash Q = \{a \in F \mid a \sqsupset q(a)\}.$$

If $\mathcal{C}$ is the equality, then $P = \varnothing$.

**Lemma 1.3.** *$P$ is an ideal of $F$.*

**Proof.** When $a \in P$ and $c \in F$, then $a \sqsupset q(a)$, $a + c \;\mathcal{C}\; q(a) + c$, $a + c \sqsupset q(a) + c \sqsupseteq q(a + c)$, and $a + c \in P$. □

In what follows, $M = M(\mathcal{C})$ is the set of all minimal elements of $P$, under the usual order $\leqq$; since $(F, \leqq)$ satisfies the descending chain condition, $P$ is generated as an ideal of $F$ by $M$.

**Proposition 1.4.** *The congruence $\mathcal{C}$ is generated by all pairs $(m, q(m))$ with $m \in M(\mathcal{C})$.*

**Proof.** Let $\mathcal{M}$ be the congruence on $F$ generated by all pairs $(m, q(m))$ with $m \in M$. Then $\mathcal{M} \subseteq \mathcal{C}$, since $m \;\mathcal{C}\; q(m)$ for all $m \in M$. We show by artinian induction that $a \;\mathcal{M}\; q(a)$ for every $a \in F$ (this also follows from Proposition 1.5 below); then $a \;\mathcal{C}\; b$ implies $a \;\mathcal{M}\; q(a) = q(b) \;\mathcal{M}\; b$, so that $\mathcal{C} = \mathcal{M}$.

We have $a \;\mathcal{M}\; q(a)$ for all $a \in Q$ (then $a = q(a)$) and for all $a \in M$ (by definition of $\mathcal{M}$). Let $a \in P$. Then $a \geqq m$ for some $m \in M$ and $a = m + t$ for some $t \in F$. Let $b = q(m) + t$. Then $a \;\mathcal{M}\; b$, since $m \;\mathcal{M}\; q(m)$, $a \;\mathcal{C}\; b$ since $m \;\mathcal{C}\; q(m)$, and $a \sqsupset b$, since $m \sqsupset q(m)$. Then $b \;\mathcal{M}\; q(b)$ by the induction hypothesis, and $a \;\mathcal{M}\; b \;\mathcal{M}\; q(b) = q(a)$. □

Proposition 1.4 implies Rédei's Theorem. If indeed $F$ is finitely generated, then $M$ is finite by Dickson's Theorem (Corollary VI.1.3), and Proposition 1.4 shows that $\mathcal{C}$ is finitely generated (Grillet [1993R]).

Conversely Rosales [1995] devised an algorithm which constructs $q$ from any finite set of generators of $\mathcal{C}$; this provides an explicit algorithm for the solution of the word problem in any finite commutative presentation.

3. Given the congruence $\mathcal{C}$, we now regard the free c.m. $F$ as a directed graph with labeled edges, in which the vertices are the elements of $F$ and an edge $a \xrightarrow{m} b$ from $a$ to $b$, labeled by $m$, is an ordered pair $(a,m)$ such that $m \in M(\mathcal{C})$, $m \leqq a$, and $a - b = m - q(m)$. Then $a = m + t$ and $b = q(m) + t$, where $t = a - m = b - q(m) \in F$; hence $q(m) \leqq b$, $a \;\mathcal{C}\; b$ (since $m \;\mathcal{C}\; q(m)$), and $a \sqsupset b$ (since $m \sqsupset q(m)$).

A descending **path** from $a \in F$ to $b \in F$ is a sequence $a = p^0, \dots, p^k = b$ of elements of $F$ and edges

$$a = p^0 \xrightarrow{m^1} p^1 \xrightarrow{m^2} \dots \xrightarrow{m^k} p^k = b,$$

where $k \geqq 0$. (We index sequences of elements of $F$ by superscripts, to keep subscripts for coordinates.) Equivalently, a path from $a$ to $b$ consists of a sequence $a = p^0, \dots, p^k = b$ of elements of $F$ and a sequence $m^1, \dots, m^k$ of elements of $M(\mathcal{C})$ such that $m^i \leqq p^{i-1}$ and $p^{i-1} - p^i = m^i - q(m^i)$ for all $1 \leqq i \leqq k$. Then $q(m^i) \leqq p^i$, $p^0, \dots, p^k \in C_a$ and $p^0 \sqsupset \dots \sqsupset p^k$; in particular, $a \;\mathcal{C}\; b$ and $a \sqsupseteq b$ (with $a = b$ if $k = 0$). Also $a - b = \sum_{1 \leqq i \leqq k} (p^{i-1} - p^i) = \sum_{1 \leqq i \leqq k} \big(m^i - q(m^i)\big)$.

An **overpath** from $a \in F$ to $b \in F$ is the sequence $p : m^1, \dots, m^k \in M(\mathcal{C})$ of labels in a path

$$a = p^0 \xrightarrow{m^1} p^1 \xrightarrow{m^2} \dots \xrightarrow{m^k} p^k = b,$$

from $a$ to $b$. A path from $a$ to $b$ is determined by $p^0 = a$ and its overpath, since in the above the relation $p^{i-1} - p^i = m^i - q(m^i)$ determines $p^i$ from $p^{i-1}$ and $m^i$. In particular,

$$a - b \;=\; \textstyle\sum_{1 \leqq i \leqq k} \big(m^i - q(m^i)\big).$$

The empty sequence is an overpath from any $a \in F$ to itself. If

$$a = p^0 \xrightarrow{m^1} p^1 \xrightarrow{m^2} \dots \xrightarrow{m^k} p^k = b$$

is a path from $a$ to $b$, and

$$b = q^0 \xrightarrow{n^1} q^1 \xrightarrow{n^2} \dots \xrightarrow{n^k} q^k = c$$

is a path from $b$ to $c$, then

$$a = p^0 \xrightarrow{m^1} \dots \xrightarrow{m^k} p^k = b = q^0 \xrightarrow{n^1} \dots \xrightarrow{n^k} q^k = c$$

is a path from $a$ to $c$. Hence if $p : m^1,\dots,m^k$ is an overpath from $a$ to $b$, and $q : n^1,\dots,n^l$ is an overpath from $b$ to $c$, then $p + q : m^1,\dots,m^k, n^1,\dots,n^l$ is an overpath from $a$ to $c$.

Let $c \in F$. If $(a,m)$ is an edge from $a$ to $b$, then $(a + c, m)$ is an edge from $a + c$ to $b + c$. Hence if

$$a = p^0 \xrightarrow{m^1} p^1 \xrightarrow{m^2} \dots \xrightarrow{m^k} p^k = b$$

is a path from $a$ to $b$, then there is a path with the same labels from $a + c$ to $b + c$. Thus if $m^1,\dots,m^k$ is an overpath from $a$ to $b$, then $m^1,\dots,m^k$ is an overpath from $a + c$ to $b + c$.

The following result is a well-ordered version of Proposition I.2.9, and shows how $\mathcal{C}$ is generated by all pairs $(m, q(m))$ with $m \in M$.

**Proposition 1.5.** *For every $a \in F$, there exist a path from $a$ to $q(a)$ and an overpath from $a$ to $q(a)$.*

**Proof.** This is proved by artinian induction on $a$. If $a = q(a) \in Q$, then there is an empty path from $a$ to $q(a)$. Now let $a \in P$. Then $a \geqq m$ for some $m \in M$. Let $b = q(m) + t$, where $a = m + t$. Then $(a,m)$ is an edge from $a$ to $b$. Hence $a \,\mathcal{C}\, b$, $a \sqsupset b$, and the induction hypothesis yields a path from $b$ to $q(b)$. Adding $a \xrightarrow{m} b$ yields a path from $a$ to $q(a) = q(b)$. □

4. The process of well ordering $F$ to select "minimal" generators of $\mathcal{C}$ (as in Proposition 1.4) is reminiscent of Gröbner bases. Indeed let $K$ be a field and $K[X]$ be the polynomial ring with the set $X$ of commuting indeterminates. Ordering $F$ also orders the monomials $X^a = \prod_{x \in X} x^{a_x} \in K[X]$ (where $a = \sum_{x \in X} a_x x \in F$).

**Proposition 1.6.** *Let $\mathcal{C}$ be a congruence on $F$ and $I(\mathcal{C})$ be the ideal of $K[X]$ generated by all $X^a - X^b$ with $a \,\mathcal{C}\, b$. The set*

$$G(M) = \{X^m - X^{q(m)} \mid m \in M\}$$

*is a Gröbner basis of $I(\mathcal{C})$.*

**Proof.** First we note that $I = I(\mathcal{C})$ is generated by all $X^a - X^{q(a)}$, since $a \,\mathcal{C}\, b$ implies $q(a) = q(b)$ and $X^a - X^b = \left(X^a - X^{q(a)}\right) - \left(X^b - X^{q(b)}\right)$.

We show that the ideal $L(I)$ generated by the leading terms of polynomials in $I$ coincides with the ideal $L(G)$ generated by the leading terms of polynomials in $G(M)$; this is one of the criteria for Gröbner bases (see e.g. Adams & Loustaunau [1994], Theorem 1.6.2).

When $a \in P$, then $a \sqsupset q(a)$ and the leading term of $X^a - X^{q(a)}$ is $X^a$. Since $P$ is an ideal of $F$ by Lemma 1.3, $L(I)$ is generated by all $X^a$ with $a \in P$. Now $a \in P$ implies, as above, $a \geqq m$ for some $m \in M$, $a = m + t$ for some $t \in F$, and $X^a = X^m X^t \in L(G)$. Therefore $L(I) \subseteq L(G)$; conversely $L(G) \subseteq L(I)$ since $G(M) \subseteq I$. Thus $L(G) = L(I)$. $\square$

We give a direct proof that $I(\mathcal{C})$ is generated by $G(M)$. Let $J$ be the ideal of $K[X]$ generated by $G(M)$. We show by induction on $a$ that $X^a - X^{q(a)} \in J$ for all $a \in F$. When $a \in Q$, then $a = q(a)$ and $X^a - X^{q(a)} \in J$. Let $a \in P$. As in the proof of Proposition 1.4, $a \geqq m$ for some $m \in M$ and $a = m + t$ for some $t \in F$. Let $b = q(m) + t$. Then $a \sqsupset b$ since $m \sqsupset q(m)$, $X^b - X^{q(b)} \in J$ by the induction hypothesis, $X^a - X^b = X^t \left(X^m - X^{q(m)}\right) \in J$, and $X^a - X^{q(a)} = \left(X^a - X^b\right) - \left(X^b - X^{q(b)}\right) \in J$. Thus $X^a - X^{q(a)} \in J$ for all $a \in F$; therefore $I = J$.

## 2. MAIN RESULT.

The main result in this chapter is the computation of $H^2(S, \mathbb{G})$ by the overpath method. As a first application we find $H^2(S, \mathbb{G})$ when $S$ has a presentation with only one defining relation; for instance, when $S$ is cyclic. We also relate $H^2(S, \mathbb{G})$ to the strand bases in Chapter XI.

1. When $S$ is a commutative semigroup which does not have an identity element, we saw that $H^2(S, \mathbb{G}) \cong H^2(S^1, \mathbb{G}')$, where $\mathbb{G}'$ extends $\mathbb{G}$ to $H(S^1)$ so that $G'_1 = 0$ (Corollary XII.4.5). Hence we may as well start with a monoid $S$.

In what follows $S$ is a commutative monoid and $\mathbb{G} = (G, \gamma)$ is an abelian group valued functor on $H(S)$; $\pi : F \longrightarrow S$ is a surjective homomorphism, where $F$ is the free c.m. on some set $X$, and $\mathcal{C} = \ker \pi$; $\sqsubseteq$ is any compatible well order on $F$; $M$ and $q$ are as in Section 1. By Proposition 1.4, $\mathcal{C}$ is generated by all $(m, q(m))$ with $m \in M(\mathcal{C})$; this provides a presentation of $S$ as the c.m. generated by $X$ subject to all relations $m = q(m)$ with $m \in M$.

A **minimal cochain** on $S$ with values in $\mathbb{G}$ (short for minimal 1-cochain) is a family $u = (u_m)_{m \in M}$ such that $u_m \in G_{\pi m}$ for all $m \in M$.

Let $u$ be a minimal cochain. Let $a \in F$, $p : m^1, \ldots, m^k$ be an overpath from $a$ to $b$, and

$$a = p^0 \xrightarrow{m^1} p^1 \xrightarrow{m^2} \ldots \xrightarrow{m^k} p^k = b$$

be the corresponding path from $a$ to $b$, in which $p^0, \ldots, p^k \in C_a$, so that $\pi p^i = \pi a$ for all $i$. Define

$$u_{a;p} = \sum_{1 \leqq i \leqq k} u_{m^i}^{\pi t^i} \in G_{\pi a}$$

where $t^i = p^i - q(m^i) = p^{i-1} - m^i$. A **minimal cocycle** on $S$ with values in $\mathbb{G}$ is a minimal cochain $u$ such that $u_{a;p} = u_{a;q}$ whenever $p$ and $q$ are overpaths from $a$ to $q(a)$ (so that $u_{a;p}$ does not depend on $p$).

Let $g = (g_x)_{x \in X} \in \prod_{x \in X} G_{\pi x}$ be a family such that $g_x \in G_{\pi x}$ for every generator $x \in X$ of $F$. A minimal cochain $\delta g$ is defined by

$$(\delta g)_m = \sum_{x \in X,\, x \leqq m} m_x\, g_x^{\pi(m-x)} - \sum_{x \in X,\, x \leqq q(m)} q(m)_x\, g_x^{\pi(q(m)-x)}$$

for every $m = \sum_{x \in X} m_x x \in M$. A minimal cochain constructed in this fashion is a **minimal coboundary**. Under pointwise addition, minimal coboundaries, minimal cocycles, and minimal cochains constitute abelian groups

$$MB^1(S, \mathbb{G}) \subseteq MZ^1(S, \mathbb{G}) \subseteq MC^1(S, \mathbb{G}) = \prod_{m \in M} G_{\pi m}\,.$$

The main result in this chapter is:

**Theorem 2.1.** *For every commutative monoid $S$ there is an isomorphism*

$$H^2(S, \mathbb{G}) \cong MZ^1(S, \mathbb{G}) \,/\, MB^1(S, \mathbb{G})$$

*which is natural in $\mathbb{G}$.*

2. The proof of Theorem 2.1 occupies the next section. First we consider an example: when $S$ has a commutative presentation (as a semigroup or as a monoid)

$$S \cong \langle\, a_1, \ldots, a_n, \ldots \mid a_1^{r_1} a_2^{r_2} \cdots a_n^{r_n} = a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n} \,\rangle$$

with a single defining relation, in which we assume, not unreasonably, that $r_i + s_i > 0$ for all $i \leqq n$ and that $r_i \neq s_i$ for some $i$. Other examples are given in Grillet [2000T] and in Sections 4 and 5.

We can set up the surjective homomorphism $\pi : F = F_X \longrightarrow S$ so that $X$

contains distinct elements $x_1, \ldots, x_n$ such that $\pi x_1 = a_1, \ldots, \pi x_n = a_n$. Then $\mathcal{C} = \ker \pi$ is the congruence on $F$ generated by the single pair $(r, s)$, where

$$r = \textstyle\sum_{1 \leqq i \leqq n} r_i x_i \quad \text{and} \quad s = \sum_{1 \leqq i \leqq n} s_i x_i .$$

The congruence $\mathcal{C}$ is readily described:

**Lemma 2.2.** *$a \,\mathcal{C}\, b$ if and only if there exists a sequence $p^0, \ldots, p^k$ of elements of $F$ such that $k \geqq 0$, $a = p^0$, $p^k = b$, and either*

$$p^{i-1} - r = p^i - s \geqq 0 \quad \textit{for all } i \geqq 1 \tag{A}$$

*or*

$$p^{i-1} - s = p^i - r \geqq 0 \quad \textit{for all } i \geqq 1 \tag{B}$$

**Proof.** By Proposition I.2.9, $a \,\mathcal{C}\, b$ if and only if there exists a sequence $p^0, \ldots, p^k$ of elements of $F$ such that $k \geqq 0$, $a = p^0$, $p^k = b$, and, for every $i \geqq 1$, either

$$p^{i-1} - r = p^i - s \geqq 0 \tag{a}$$

or

$$p^{i-1} - s = p^i - r \geqq 0. \tag{b}$$

If (a) holds for $i < k$ and (b) holds for $i+1$, then $p^{i-1} - r = p^i - s = p^{i+1} - r$ and $p^i$, $p^{i+1}$ may be deleted from the sequence. Similarly if (b) holds for $i < k$ and (a) holds for $i+1$, then $p^{i-1} - s = p^i - r = p^{i+1} - s$ and again $p^i$, $p^{i+1}$ may be deleted from the sequence. After all such deletions, either (a) holds for all $i$, or (b) holds for all $i$. $\square$

Now let $\sqsubseteq$ be any compatible well order on $F$. We may assume that $r \sqsupset s$. When $a \,\mathcal{C}\, b$ and (A) holds, then in Lemma 2.2 $p^{i-1} - r = p^i - s \geqq 0$ implies $p^{i-1} = r + t$ and $p^i = s + t$, where $t = p^{i-1} - r = p^i - s \geqq 0$, so that $p^{i-1} \sqsupset p^i$ for all $i \geqq 1$ and $a \sqsupset b$; if (B) holds, then similarly $a \sqsubset b$. If therefore $a \in P$, so that $a \,\mathcal{C}\, q(a)$ and $a \sqsupset q(a)$, then (A) holds, $a - r = p^0 - r \geqq 0$, and $a \geqq r$. On the other hand, $r \,\mathcal{C}\, s$ and $r \sqsupset s$, so that $r \in P$. This proves:

**Lemma 2.3.** *$M(\mathcal{C})$ has just one element, namely $r$; and $q(r) = s$.*

Let

$$d = a_1^{r_1} a_2^{r_2} \cdots a_n^{r_n} = a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n} = \pi r = \pi s \in S.$$

By Lemma 2.3, a minimal cochain consists of $u \in G_d$, and $MC^1(S, \mathbb{G}) = G_d$. Moreover there is only one overpath from any $c \in F$ to $q(c)$, which is a sequence of $r$'s. Hence every minimal cochain is a minimal cocycle and $MZ^1(S, \mathbb{G}) = G_d$.

A minimal cochain $u \in G_d$ is a minimal coboundary if and only if there exists a family $g = (g_x)_{x \in X}$ such that $g_x \in G_{\pi x}$ for every $x \in X$ and

$$\begin{aligned} u &= \textstyle\sum_{x \in X,\, x \leqq r} r_x\, g_x^{\pi(r-x)} - \sum_{x \in X,\, x \leqq q(r)} q(r)_x\, g_x^{\pi(q(r)-x)} \\ &= \textstyle\sum_{i \leqq n,\, r_i > 0} r_i\, g_{x_i}^{\pi(r-x_i)} - \sum_{i \leqq n,\, s_i > 0} s_i\, g_{x_i}^{\pi(s-x_i)}. \end{aligned}$$

Let $\gamma_i : G_{a_i} \longrightarrow G_d$ be defined by:

$$\gamma_i = \begin{cases} r_i\, \gamma_{a_i, d_i'} - s_i\, \gamma_{a_i, d_i''} & \text{if } r_i, s_i > 0, \\ r_i\, \gamma_{a_i, d_i'} & \text{if } r_i > 0,\ s_i = 0, \\ -s_i\, \gamma_{a_i, d_i''} & \text{if } r_i = 0,\ s_i > 0, \end{cases}$$

where

$$\begin{aligned} d_i' &= a_1^{r_1} \cdots a_{i-1}^{r_{i-1}}\, a_i^{r_i - 1}\, a_{i+1}^{r_{i+1}} \cdots a_n^{r_n} \quad \text{(when } r_i > 0\text{), and} \\ d_i'' &= a_1^{s_1} \cdots a_{i-1}^{s_{i-1}}\, a_i^{s_i - 1}\, a_{i+1}^{s_{i+1}} \cdots a_n^{s_n} \quad \text{(when } s_i > 0\text{).} \end{aligned}$$

Then $u$ is a minimal coboundary if and only if there exist $g_i = g_{x_i} \in G_{a_i}$ such that $u = \sum_{1 \leqq i \leqq n} \gamma_i g_i$. Hence $MB^1(S, \mathbb{G}) = \sum_{1 \leqq i \leqq n} \operatorname{Im} \gamma_i$ and:

**Proposition 2.4.** *When $S$ has a commutative semigroup or monoid presentation with a single nontrivial defining relation, then, with the notation as above,*

$$H^2(S, \mathbb{G}) \cong G_d / \left(\textstyle\sum_{1 \leqq i \leqq n} \operatorname{Im} \gamma_i\right).$$

**Corollary 2.5.** *When $S = \langle\, a \mid a^r = a^{r+p} \,\rangle$ is cyclic with index $r$ and period $p$, then $H^2(S, \mathbb{G}) \cong G_{a^r} / \, p \operatorname{Im} \gamma_{a, a^{r-1}}$.*

Proposition 2.4 becomes simpler when $\mathbb{G}$ is thin. Then $d \leqq_{\mathcal{H}} a_i$ for all $i$ (since $r_i + s_i > 0$) and $\gamma_{a_i, d_i'} = \gamma_d^{a_i}$ when $r_i > 0$, $\gamma_{a_i, d_i''} = \gamma_d^{a_i}$ when $s_i > 0$, and $\gamma_i = (r_i - s_i)\, \gamma_d^{a_i}$ for all $i$. Hence

**Corollary 2.6.** *When $S$ has a commutative semigroup or monoid presentation with a single nontrivial defining relation and $\mathbb{G}$ is thin, then, with the notation as above,*

$$H^2(S, \mathbb{G}) \cong G_d / \left(\textstyle\sum_{1 \leqq i \leqq n} (r_i - s_i) \operatorname{Im} \gamma_d^{a_i}\right).$$

If for instance $S = \langle\, a \mid a^r = a^{r+p} \,\rangle$ is cyclic with index $r$ and period $p$ and $\mathbb{G} = A$ is constant ($G_s = A$ and $\gamma_{s,t} = 1_A$ for all $s \in S$ and $t \in S^1$), then

$H^2(S,\mathbb{G}) \cong A/pA$; thus $H^2(S,\mathbb{G}) \cong \mathrm{Ext}(H,A)$, where $H$ is the subgroup $\{a^k \mid k \geqq r\} \cong \mathbb{Z}_p$ of $S$.

3. Finally we show that strand bases in Chapter XI give rise to minimal cocycles. This result is from Grillet [1996C], [2001C].

In what follows, $\mathcal{C}$ is a subcomplete congruence on a free commutative monoid $F$ and $\mathcal{C}^*$ is its group-free hull; $\pi : F \longrightarrow S$ and $\pi^* : F \longrightarrow S^*$ are surjective homomorphisms which induce $\mathcal{C}$ and $\mathcal{C}^*$ respectively. If $\mathcal{C}$ is complete, then $S^* \cong S/\mathcal{H}$ and one expects the cohomology of $S^*$ to show up somewhere in the construction of $S$ and $\mathcal{C}$. Minimal cocycles provide this connection.

The direction set, extent cells, strand groups, strand bases, and notation are as in Chapter XI. Also $\sqsubseteq$ is a compatible well order on $F$; the mapping $q$ and sets $M$ and $Q$ are those of $\mathcal{C}^*$, not of $\mathcal{C}$.

**Lemma 2.7.** *Let $s$ be a strand base of $\mathcal{C}$. For every $m \in M(\mathcal{C}^*)$ let*

$$\overline{s}_m \;=\; s_m - m - s_{q(m)} + q(m) \;\in\; G_m .$$

*If $m^1, m^2, \dots, m^k$ is an overpath from $a$ to $b$, then*

$$s_a - s_b - a + b - \overline{s}_{m^1} - \cdots - \overline{s}_{m^k} \;\in\; R_a = R_b .$$

**Proof.** Let

$$a = p^0 \xrightarrow{m^1} p^1 \xrightarrow{m^2} \dots \xrightarrow{m^k} p^k = b$$

be a path from $a$ to $b$, so that $p^{i-1} = m^i + t^i$, $p^i = q(m^i) + t^i$ for some $t^i \in F$ and $p^{i-1} - p^i = m^i - q(m^i)$, for every $1 \leqq i \leqq k$. Then $R_a = R_{p^0} = R_{p^1} = \dots = R_{p^k} = R_b$ by (R2), since $a = p^0$, $p^1, \dots, p^k = b$ are all in the same $\mathcal{C}^*$-class, and

$$\begin{aligned} s_{p^{i-1}} - s_{p^i} - p^{i-1} + p^i - \overline{s}_{m^i} \;&=\; s_{p^{i-1}} - s_{p^i} - m^i + q(m^i) - \overline{s}_{m^i} \\ &=\; s_{p^{i-1}} - s_{p^i} - s_{m^i} + s_{q(m^i)} \;\in\; R_{p^i} = R_a \end{aligned}$$

by (S+) in Lemma XI.6.1. Adding these equalities yields $s_a - s_b - a + b - \overline{s}_{m^1} - \cdots - \overline{s}_{m^k} \in R_a$. □

With $b = q(a)$, Lemma 2.7 implies that a strand base of $\mathcal{C}$ is completely determined modulo strand groups by its values on $M \cup Q$.

2. The strand group functor $\mathbb{K} = (K, \psi)$ of $\mathcal{C}$ is the thin abelian group valued functor on $F/\mathcal{C}^*$ defined as follows (Section XI.4). To every $\mathcal{C}^*$-class

$C^*$, $K$ assigns the group $K_a = G_a/R_a$, which does not depend on the choice of $a \in C^*$. When $C^* \geqq_{\mathcal{H}} D^*$ in $F/\mathcal{C}^*$, then $a \leqq b$ for some $a \in C^*$, $b \in D^*$, $G_a \subseteq G_b$, $R_a \subseteq R_b$ by (R3), and $\psi^a_b : K_a \longrightarrow K_b$ sends $g + R_a$ to $g + R_b$ and does not depend on the choice of $a \in C^*$ and $b \in D^*$ (as long as $a \leqq b$). Since $S^* \cong F/\mathcal{C}^*$ we may regard $\mathbb{K}$ as a thin abelian group valued functor on $S^*$; then $\mathbb{K}$ is isomorphic to the extended Schützenberger functor of $S$, which is the usual Schützenberger functor if $S$ is complete (Proposition XI.4.8).

**Proposition 2.8.** *Let $\mathcal{C}$ be a subcomplete congruence on $F$ and $s$ be a strand base of $\mathcal{C}$. For every $m \in M(\mathcal{C}^*)$ let*

$$s^*_m \;=\; \bar{s}_m + R_m \;=\; s_m - m - s_{q(m)} + q(m) + R_m \;\in\; G_m/R_m.$$

*Then $s^*$ is a minimal 1-cocycle on $F/\mathcal{C}^*$ with values in the strand group functor $\mathbb{K}$ of $\mathcal{C}$. Moreover, two strand bases $s$ and $t$ define the same congruence if and only if $s^* = t^*$.*

**Proof.** $s^*$ is a minimal cochain. When $p : m^1, \dots, m^k$ is an overpath from $a$ to $b$ and

$$a = p^0 \xrightarrow{m^1} p^1 \xrightarrow{m^2} \dots \xrightarrow{m^k} p^k = b$$

is the corresponding path, then

$$s^*_{a;p;b} \;=\; \textstyle\sum_i \psi^{m_i}_{p^i} s^*_{m_i} \;=\; \sum_i \bar{s}_{m^i} + R_a \;=\; s_a - s_b - a + b + R_a$$

by Lemma 2.7. Hence $s^*_{a;p;b}$ is independent of path and $s^*$ is a minimal cocycle.

By Proposition XI.5.2, two strand bases $s$ and $t$ define the same congruence if and only if $a\ \mathcal{C}^*\ b$ implies $s_a - s_b - t_a + t_b \in R_a (= R_b)$. Since $m\ \mathcal{C}^*\ q(m)$ this implies

$$\big(s_m - m - s_{q(m)} + q(m)\big) - \big(t_m - m - t_{q(m)} + q(m)\big) \;\in\; R_m$$

for all $m \in M$ and $s^* = t^*$. Conversely assume that $s^* = t^*$. Then $s^*_{a;p;b} = t^*_{a;p;b}$ and

$$s_a - s_b - a + b + R_a \;=\; t_a - t_b - a + b + R_a$$

whenever $p$ is an overpath from $a$ to $b$. Hence $s_a - s_{q(a)} - t_a + t_{q(a)} \in R_a$ for all $a$. If $a\ \mathcal{C}^*\ b$, then $q(a) = q(b)$, $s_b - s_{q(a)} - t_b + t_{q(a)} \in R_b = R_a$, and $s_a - s_b - t_a + t_b \in R_a$. Thus $s$ and $t$ are equivalent. □

Proposition 2.8 embeds the set of equivalence classes of strand bases (and the set of all subcomplete congruence with the given strand groups) into the abelian group $MZ^1(F/\mathcal{C}^*, \mathbb{K})$.

## 3. PROOF OF MAIN RESULT.

1. In what follows, $S$ is a commutative monoid, $\mathbb{G} = (G,\gamma)$ is an abelian group valued functor on $H(S)$, $F = F_X$ is the free c.s. on a set $X$, and $\pi : F \longrightarrow S$ is a surjective homomorphism; we prove Theorem 2.1.

We begin the proof by lifting 1- and 2-cochains from $S$ to $F$.

The homomorphism $\pi : F \longrightarrow S$ extends to a functor $\pi : H(F) \longrightarrow H(S)$. Hence $\mathbb{G} = (G,\gamma)$ lifts to an abelian group valued functor $\mathbb{G}\pi = (\mathbb{G}\pi, \gamma\pi) = \mathbb{G} \circ \pi$ on $H(F)$; $\mathbb{G}\pi$ assigns $G_{\pi a}$ to $a \in F$ and $\gamma_{\pi a, \pi t}$ to $(a,t) : a \longrightarrow at$. Thus $g^t = g^{\pi t}$, where $g \in G_{\pi a}$, $g^t$ is provided by $\mathbb{G}\pi$, and $g^{\pi t}$ is provided by $\mathbb{G}$. Note that $\mathbb{G}\pi$ is thin, since $F$ is cancellative.

Every 1-cochain $u = (u_a)_{a\in S} \in C^1(S,\mathbb{G})$ lifts to a 1-cochain $\pi^* u = u \circ \pi \in C^1(F,\mathbb{G}\pi)$ defined by

$$(\pi^* u)_a \;=\; u_{\pi a} \;\in\; G_{\pi a}$$

for all $a \in F$. If $u$ is a 1-cocycle, so that $u_{ab} = u_a^b + u_b^a$ for all $a,b \in S$, then

$$(\pi^* u)_{ab} \;=\; u_{(\pi a)(\pi b)} \;=\; u_{\pi a}^{\pi b} + u_{\pi b}^{\pi a} \;=\; (\pi^* u)_a^b + (\pi^* u)_b^a$$

and $\pi^* u$ is a 1-cocycle; thus $\pi^* Z^1(S,\mathbb{G}) \subseteq Z^1(F,\mathbb{G}\pi)$.

Similarly every symmetric 2-cochain $u = (u_a)_{a\in S} \in SC^2(S,\mathbb{G})$ lifts to a symmetric 2-cochain $\pi^* u = u \circ \pi \in C^2(F,\mathbb{G}\pi)$ defined by

$$(\pi^* u)_{a,b} \;=\; u_{\pi a, \pi b} \;\in\; G_{\pi(ab)}$$

for all $a,b \in F$. If $u$ is a symmetric 2-cocycle, so that $u_{a,b}^c + u_{ab,c} = u_{a,bc} + u_{b,c}^a$ for all $a,b,c \in S$, then

$$\begin{aligned}(\pi^* u)_{a,b}^c \;+\; (\pi^* u)_{ab,c} \;&=\; u_{\pi a, \pi b}^{\pi c} \;+\; u_{(\pi a)(\pi b),\, \pi c}\\ &=\; u_{\pi a,\, (\pi b)(\pi c)} \;+\; u_{\pi b, \pi c}^{\pi a} \;=\; (\pi^* u)_{a,bc} + (\pi^* u)_{b,c}^a\end{aligned}$$

for all $a,b,c \in F$, and $\pi^* u$ is a symmetric 2-cocycle. If $u = \delta v$ is a symmetric 2-coboundary, so that $u_{a,b} = v_a^b - v_{ab} + v_b^a$ for all $a,b \in S$, then

$$(\pi^* u)_{a,b} \;=\; v_{\pi a}^{\pi b} \;-\; v_{(\pi a)(\pi b)} \;+\; v_{\pi b}^{\pi a} \;=\; (\delta \pi^* v)_{a,b}$$

for all $a,b \in S$, and $\pi^* u$ is a symmetric 2-coboundary. Thus

$$\pi^* SZ^2(S,\mathbb{G}) \subseteq SZ^2(F,\mathbb{G}\pi) \quad\text{and}\quad \pi^* SB^2(S,\mathbb{G}) \subseteq SB^2(F,\mathbb{G}\pi).$$

Since $H^2(F, \mathbb{G}\pi) = 0$ (Theorem XII.3.4), symmetric 2-cocycles on $S$ lift to symmetric 2-coboundaries on $F$ and can therefore be constructed by projecting the coboundaries of 1-cochains on $F$. This marks the birthplace of Theorem 2.1.

2. A 1-cochain $u \in C^1(F, \mathbb{G}\pi)$ is **consistent** (relative to $\pi$) when

$$\pi a = \pi b \quad \text{implies} \quad u_{a+c} - u_{b+c} = u_a^c - u_b^c \quad \text{for all } c \in F.$$

Under pointwise addition consistent 1-cochains form a subgroup $K^1(F, \mathbb{G}\pi)$ of $C^1(F, \mathbb{G}\pi)$. We shall see that consistent 1-cochains on $F$ are precisely those whose coboundaries project to symmetric 2-cocycles on $S$. First we show:

**Lemma 3.1.** *$K^1(F, \mathbb{G}\pi)$ contains $Z^1(F, \mathbb{G}\pi)$ and $\pi^* C^1(S, \mathbb{G})$.*

**Proof.** If $u \in Z^1(F, \mathbb{G}\pi)$, then $\pi a = \pi b$ implies $\gamma_{\pi a, \pi c} = \gamma_{\pi b, \pi c}$ and $u_a^c - u_{a+c} = u_c^a = u_c^b = u_b^c - u_{b+c}$.

If $u = \pi^* v$, then $\pi a = \pi b$ implies $u_a = v_{\pi a} = v_{\pi b} = u_b$, $\gamma_{\pi a, \pi c} = \gamma_{\pi b, \pi c}$, $\pi(a+c) = \pi(b+c)$, $u_{ac} = u_{bc}$, and $u_a^c - u_{a+c} = u_b^c - u_{b+c}$. □

When $u \in K^1(F, \mathbb{G}\pi)$, then $\pi a = \pi b$, $\pi c = \pi d$ imply $\gamma_{\pi c, \pi a} = \gamma_{\pi c, \pi b}$, $\gamma_{\pi b, \pi c} = \gamma_{\pi b, \pi d}$, and

$$u_a^c - u_{a+c} + u_c^a \;=\; u_b^c - u_{b+c} + u_c^b \;=\; u_b^d - u_{b+d} + u_d^b \,.$$

Hence a homomorphism $\Delta : K^1(F, \mathbb{G}\pi) \longrightarrow SC^2(S, \mathbb{G})$ is well defined by

$$(\Delta u)_{\pi a, \pi c} \;=\; u_a^c - u_{a+c} + u_c^a \;=\; (\delta u)_{a,c}$$

for all $a, c \in F$. It is immediate that $\Delta$ is natural in $\mathbb{G}$.

**Lemma 3.2.** $\operatorname{Im} \Delta = SZ^2(S, \mathbb{G})$.

**Proof.** $\Delta u$ is a symmetric 2-cocycle, since $\delta u$ is a cocycle and

$$\begin{aligned}(\Delta u)_{\pi a, \pi b}^{\pi c} + (\Delta u)_{(\pi a)(\pi b),\, \pi c} \;&=\; (\delta u)_{a,b}^c + (\delta u)_{a+b,\, c} \\ =\; (\delta u)_{a,\, b+c} + (\delta u)_{b,c}^a \;&=\; (\Delta u)_{\pi a,\, (\pi b)(\pi c)} + (\Delta u)_{\pi b, \pi c}^{\pi a}.\end{aligned}$$

Conversely let $s \in SZ^2(S, \mathbb{G})$. Then $\pi^* s \in SZ^2(F, \mathbb{G}\pi)$. Since $H^2(F, \mathbb{G}\pi) = 0$ we have $\pi^* s = \delta u$ for some $u \in C^1(F, \mathbb{G}\pi)$, so that

$$s_{\pi a, \pi c} \;=\; u_a^c - u_{a+c} + u_c^a$$

for all $a, c \in F$. If $\pi a = \pi b$, then

$$u_a^c - u_{a+c} + u_c^a \;=\; s_{\pi a, \pi c} \;=\; s_{\pi b, \pi c} \;=\; u_b^c - u_{b+c} + u_c^b,$$

$\gamma_{\pi c,\pi b} = \gamma_{\pi c,\pi a}$, $u_c^a = u_c^b$, and $u_a^c - u_{a+c} = u_b^c - u_{b+c}$. Thus $u \in K^1(F,\mathbb{G}\pi)$. We see that $\Delta u = s$. □

3. Lemma 3.2 shows that $H^2(S,\mathbb{G})$ is determined by consistent cochains.

**Lemma 3.3.** *When* $u \in K^1(F,\mathbb{G}\pi)$, *then* $\Delta u \in SB^2(S,\mathbb{G})$ *if and only if* $u \in Z^1(F,\mathbb{G}\pi) + \pi^* C^1(S,\mathbb{G})$.

**Proof.** If $u = v + \pi^* w$, where $v \in Z^1(F,\mathbb{G}\pi)$ and $w \in C^1(S,\mathbb{G})$, then $u \in K^1(F,\mathbb{G}\pi)$ by Lemma 3.1 and

$$(\Delta u)_{\pi a,\pi b} \;=\; v_a^b - v_{a+b} + v_b^a \;+\; w_{\pi a}^{\pi b} - w_{(\pi a)(\pi b)} + w_{\pi b}^{\pi a} \;=\; (\delta w)_{\pi a,\pi b}$$

for all $a,b \in F$, since $v \in Z^1(F,\mathbb{G}\pi)$, so that $\Delta u = \delta w \in SB^2(S,\mathbb{G})$.

Conversely assume $\Delta u = \delta w \in SB^2(S,\mathbb{G})$, where $w \in C^1(S,\mathbb{G})$. Then

$$u_a^b - u_{a+b} + u_b^a \;=\; w_{\pi a}^{\pi b} - w_{(\pi a)(\pi b)} + w_{\pi b}^{\pi a} \;=\; (\pi^* w)_a^b - (\pi^* w)_{a+b} + (\pi^* w)_b^a$$

for all $a,b \in F$, and $v = u - \pi^* w \in Z^1(F,\mathbb{G}\pi)$. □

**Corollary 3.4.** *There is an isomorphism*

$$H^2(S,\mathbb{G}) \;\cong\; K^1(F,\mathbb{G}\pi) \,/\, \big(Z^1(F,\mathbb{G}\pi) + \pi^* C^1(S,\mathbb{G})\big)$$

*which is natural in* $\mathbb{G}$.

**Proof.** The isomorphism follows from Lemmas 3.2 and 3.3, since $H^2(S,\mathbb{G}) \cong SZ^2(S,\mathbb{G}) \,/\, SB^2(S,\mathbb{G})$; it is natural in $\mathbb{G}$ since $\Delta$ is natural in $\mathbb{G}$. □

4. Now let $\sqsubseteq$ be any compatible well order on $F$ and $\mathcal{C} = \ker \pi$. We use $P$, $Q$, $M$, $q$ from Section 1 to trim consistent cochains.

A **partial 1-cochain** on $F$ with values in $\mathbb{G}$ is a 1-cochain $u = (u_a)_{a\in F} \in C^1(F,\mathbb{G}\pi)$ such that $u_c = 0$ for all $c \in Q$. (Thus $u$ is, in effect, a cochain on $P$ only.) Under pointwise addition, partial 1-cochains constitute a subgroup $P^1(F,\mathbb{G}\pi) \cong \prod_{a\in P} G_{\pi a}$ of $C^1(F,\mathbb{G}\pi)$. Consistent partial 1-cochains constitute an abelian group $KP^1(F,\mathbb{G}\pi) = K^1(F,\mathbb{G}\pi) \cap P^1(F,\mathbb{G}\pi)$.

When $u \in C^1(F,\mathbb{G}\pi)$, define $\Pi u$ by

$$(\Pi u)_a \;=\; u_a \;-\; u_{q(a)} \;\in\; G_{\pi a}$$

for all $a \in F$. If $a \in Q$, then $a = q(a)$ and $(\Pi u)_a = 0$. Thus $\Pi$ is a homomorphism of $C^1(F,\mathbb{G}\pi)$ into $P^1(F,\mathbb{G}\pi)$. In fact $\operatorname{Im}\Pi = P^1(F,\mathbb{G}\pi)$, since every partial cochain $v$ satisfies $v_{q(a)} = 0$ and $\Pi v = v$. We see that $\Pi$ is natural in $\mathbb{G}$.

**Lemma 3.5.** *When* $u \in C^1(F, \mathbb{G}\pi)$, *then* $u \in K^1(F, \mathbb{G}\pi)$ *if and only if* $\Pi u \in KP^1(F, \mathbb{G}\pi)$.

**Proof.** Let $u \in C^1(F, \mathbb{G}\pi)$ and $v = \Pi u$. Let $a, b, c \in F$ satisfy $\pi a = \pi b$. Let $r = q(a) = q(b)$ and $s = q(a+c) = q(b+c)$. Then

$$\begin{aligned} v_{a+c} \; - \; v_a^c \; &= \; (u_{a+c} - u_a^c) \; - \; (u_s - u_r^c), \quad \text{and} \\ v_{b+c} \; - \; v_b^c \; &= \; (u_{b+c} - u_b^c) \; - \; (u_s - u_r^c). \end{aligned}$$

If $u$ is consistent, then $u_{a+c} - u_a^c = u_{b+c} - u_b^c$ and $v_{a+c} - v_a^c = v_{b+c} - v_b^c$; hence $v$ is consistent. If conversely $v$ is consistent, then $v_{a+c} - v_a^c = v_{b+c} - v_b^c$ and $u_{a+c} - u_a^c = u_{b+c} - u_b^c$; hence $u$ is consistent. $\square$

5. Given a family $g = (g_x)_{x \in X} \in \prod_{x \in X} G_{\pi x}$ (with $g_x \in G_{\pi x}$ for all $x \in X$), define $Dg \in C^1(F, \mathbb{G}\pi)$ by

$$(Dg)_a \; = \; \textstyle\sum_{x \in X,\, x \leqq a} \; a_x \, g_x^{a-x} \; - \; \sum_{x \in X,\, x \leqq q(a)} \; q(a)_x \, g_x^{q(a)-x}$$

for all $a = \sum_{x \in X} a_x x \in F$. We see that $(Dg)_a \in G_{\pi a} = G_{\pi q(a)}$, and that $D$ is a homomorphism of $\prod_{x \in X} G_{\pi x}$ into $C^1(F, \mathbb{G}\pi)$ and is natural in $\mathbb{G}$.

**Lemma 3.6.** $\operatorname{Im} D \subseteq KP^1(F, \mathbb{G}\pi)$. *When* $u \in C^1(F, \mathbb{G}\pi)$, *then* $u \in Z^1(F, \mathbb{G}\pi) + \pi^* C^1(S, \mathbb{G})$ *if and only if* $\Pi u \in \operatorname{Im} D$.

**Proof.** If $a \in Q$, then $a = q(a)$ and $(Dg)_a = 0$; thus $Dg \in P^1(F, \mathbb{G}\pi)$. That $Dg \in K^1(F, \mathbb{G}\pi)$ can be proved directly but follows from Lemma 3.1 and the rest of the statement, as $u \in \operatorname{Im} D$ implies $\Pi u = u$ and $u \in Z^1(F, \mathbb{G}\pi) + \pi^* C^1(S, \mathbb{G}) \subseteq K^1(F, \mathbb{G}\pi)$.

Let $u = z + \pi^* w \in Z^1(F, \mathbb{G}\pi) + \pi^* C^1(S, \mathbb{G})$, where $z \in Z^1(F, \mathbb{G}\pi)$ and $w \in C^1(S, \mathbb{G})$, so that $u_a = z_a + w_{\pi a}$ for all $a \in F$. Since $z$ is a 1-cocycle we have $z_{a+b} = z_a^b + z_b^a$ for all $a, b \in F$; hence

$$z_a \; = \; \textstyle\sum_{x \in X,\, x \leqq a} \; a_x \, z_x^{a-x},$$

for every $a = \sum_{x \in X} a_x x \in F$. Since $\pi q(a) = \pi a$,

$$\begin{aligned} (\Pi u)_a \; &= \; z_a \; + \; w_{\pi a} \; - \; z_{q(a)} \; - \; w_{\pi q(a)} \\ &= \; \textstyle\sum_{x \in X,\, x \leqq a} \; a_x \, z_x^{a-x} \; - \; \sum_{x \in X,\, x \leqq q(a)} \; q(a)_x \, z_x^{q(a)-x} \; = \; (Dg)_a, \end{aligned}$$

where $g = (z_x)_{x \in X}$. Thus $\Pi u \in \operatorname{Im} D$.

Conversely assume $\Pi u \in \operatorname{Im} D$, so that there exists $g = (g_x)_{x \in X} \in \prod_{x \in X} G_{\pi x}$ such that

$$u_a - u_{q(a)} = \textstyle\sum_{x \in X,\, x \leqq a} a_x\, g_x^{a-x} - \sum_{x \in X,\, x \leqq q(a)} q(a)_x\, g_x^{q(a)-x}$$

for all $a \in F$. For every $a \in F$ let

$$z_a = \textstyle\sum_{x \in X,\, x \leqq a} a_x\, g_x^{a-x} \in G_{\pi a}.$$

If $a, b \in F$, and $x \leqq a + b$ (equivalently, $(a+b)_x > 0$), then $x \leqq a$ or $x \leqq b$ (or both); hence

$$\begin{aligned} z_{a+b} &= \textstyle\sum_{x \in X,\, x \leqq a,\, x \not\leqq b} a_x\, g_x^{a+b-x} \\ &\quad + \textstyle\sum_{x \in X,\, x \not\leqq a,\, x \leqq b} b_x\, g_x^{a+b-x} \\ &\quad + \textstyle\sum_{x \in X,\, x \leqq a,\, x \leqq b} (a_x + b_x)\, g_x^{a+b-x} = z_a^b + z_b^a\,; \end{aligned}$$

thus $z \in Z^1(F, \mathbb{G}\pi)$. Also $u_a - u_{q(a)} = z_a - z_{q(a)}$ for all $a$. Let $v_a = u_a - z_a$. Then $v \in C^1(F, \mathbb{G}\pi)$ and $v_a = v_{q(a)}$ for all $a$. Hence $\pi a = \pi b$ implies $v_a = v_b$ and $v = \pi^* w$, where $w \in C^1(S, \mathbb{G})$ is well defined by $w_{\pi a} = v_a$. Thus $u = z + \pi^* w \in Z^1(F, \mathbb{G}\pi) + \pi^* C^1(S, \mathbb{G})$. □

**Corollary 3.7.** *There is an isomorphism*

$$H^2(S, \mathbb{G}) \cong KP^1(F, \mathbb{G}\pi) / \operatorname{Im} D$$

*which is natural in* $\mathbb{G}$.

**Proof.** We saw that $\Pi : C^1(F, \mathbb{G}\pi) \longrightarrow P^1(F, \mathbb{G}\pi)$ is a surjective homomorphism. Now $K^1(F, \mathbb{G}\pi) = \Pi^{-1}\, KP^1(F, \mathbb{G}\pi)$ by Lemma 3.5 and $Z^1(F, \mathbb{G}\pi) + \pi^* C^1(S, \mathbb{G}) = \Pi^{-1} \operatorname{Im} D$ by Lemma 3.6; therefore

$$K^1(F, \mathbb{G}\pi) \,/\, \big(Z^1(F, \mathbb{G}\pi) + \pi^* C^1(S, \mathbb{G})\big) \cong KP^1(F, \mathbb{G}\pi) / \operatorname{Im} D.$$

This isomorphism is natural in $\mathbb{G}$ since $\Pi$ and $D$ are natural in $\mathbb{G}$. The natural isomorphism $H^2(S, \mathbb{G}) \cong KP^1(F, \mathbb{G}\pi) / \operatorname{Im} D$ then follows from Corollary 3.4. □

6. Recall that a minimal cochain on $S$ with values in $\mathbb{G}$ is a family $u = (u_m)_{m \in M}$ such that $u_m \in G_{\pi m}$ for all $m \in M$. Under pointwise addition, minimal cochains constitute a subgroup $M^1(F, \mathbb{G}\pi) \cong \prod_{m \in M} G_{\pi m}$ of $C^1(F, \mathbb{G}\pi)$.

Every partial 1-cochain $u = (u_a)_{a \in F}$ has a restriction $Ru = (u_m)_{m \in M}$ to $M$, which is a minimal cochain. This defines a restriction homomorphism

$R : P^1(F, \mathbb{G}\pi) \longrightarrow M^1(F, \mathbb{G}\pi)$ which is natural in $\mathbb{G}$.

**Lemma 3.8.** *$R$ is injective on $KP^1(F, \mathbb{G}\pi)$.*

**Proof.** Let $u = (u_a)_{a \in F}$ be a consistent partial cochain such that $Ru = 0$ (such that $u_m = 0$ for all $m \in M$). We use artinian induction on $a$ to prove that $u_a = 0$ for all $a \in P$ (so that $u = 0$). Already $u_a = 0$ for all $a \in Q$ and for all $a \in M$. Let $a \in P \backslash M$. Then $a > m$ for some $m \in M$, $a = m + c$ for some $c \in F$, and $b = q(m) + c$ satisfies $\pi b = \pi a$ and $b \sqsubset a$. Since $u$ is consistent we have $u_{m+c} - u_{q(m)+c} = u_m^c - u_{q(m)}^c$, with $u_m = 0$, $u_{q(m)} = 0$, and $u_{q(m)+c} = u_b = 0$ by the induction hypothesis; hence $u_a = u_{m+c} = 0$. □

7. Lemma 3.8 shows that a consistent partial cochain is determined by its restriction to $M$. Therefore Corollary 3.7 can be restated in terms of minimal cochains; this will yield the main result. First we reconstruct consistent partial cochains from their restrictions.

Let $u$ be a minimal cochain. When $a \sqsupseteq b$ in $F$ and $p : m^1, \dots, m^k$ is an overpath from $a$ to $b$, let

$$u_{a;p;b} = \textstyle\sum_{1 \leqq i \leqq k} u_{m^i}^{t^i} \in G_{\pi a},$$

where $t^i$ is obtained from the corresponding path

$$a = p^0 \xrightarrow{m^1} p^1 \xrightarrow{m^2} \dots \xrightarrow{m^k} p^k = b$$

by $t^i = p^i - q(m^i) = p^{i-1} - m^i$. Recall that $p^0, \dots, p^k \in C_a$, so that $\pi p^i = \pi a$ and $u_{m^i}^{t^i} \in G_{\pi a}$ for all $i$.

We denote $u_{a;p;q(a)}$ by $u_{a;p}$. If $a \in Q$, then $p$ is empty and $u_{a;p} = 0$. If $a = m \in M$, then $p = \{m\}$ and $u_{m;p} = u_m$.

When $p$ is an overpath from $a$ to $b$, then $p$ is an overpath from $a + c$ to $b + c$ for any $c \in F$; the corresponding path is

$$a + c = p^0 + c \xrightarrow{m^1} p^1 + c \xrightarrow{m^2} \dots \xrightarrow{m^k} p^k + c = b + c,$$

with $p^i + c - q(m^i) = p^{i-1} + c - m^i = t^i + c$, and

$$u_{a+c;\, p;\, b+c} = u_{a;p;b}^c .$$

If $a = b$, then $p$ is empty and $u_{a;p;b} = 0$. If $p : m^1, \dots, m^k$ is an overpath from $a$ to $b$, and $q : n^1, \dots, n^l$ is an overpath from $b$ to $c$, then $p + q$ :

$m^1,\dots,m^k,\ n^1,\dots,n^l$ is an overpath from $a$ to $c$, and

$$u_{a;\,p+q;\,c} \;=\; u_{a;p;b} \;+\; u_{b;q;c}\,.$$

In particular, $u_{a;p;b} = u_{a;p+q} - u_{b;q}$ when $q$ is an overpath from $b$ to $q(b) = q(a)$.

8. **Independence of path** for $u_{a;p;b}$ means that $u_{a;p;b} = u_{a;q;b}$ whenever $p$ and $q$ are overpaths from $a$ to $b$; and similarly for $u_{a;p}$. These properties are equivalent and characterize the restrictions of consistent partial cochains:

**Lemma 3.9.** *When $u \in M^1(F, \mathbb{G}\pi)$, then $u \in R(KP^1(F, \mathbb{G}\pi))$ if and only if $u_{a;p}$ is independent of path; and then $u_{a;p;b}$ is independent of path.*

**Proof.** First let $v \in KP^1(F, \mathbb{G}\pi)$ and $u = Rv$. Let $p : m^1,\dots,m^k$ be an overpath from $a$ to $b$; let

$$a = p^0 \xrightarrow{m^1} p^1 \xrightarrow{m^2} \cdots \xrightarrow{m^k} p^k = b$$

be the corresponding path and $t^i = p^i - q(m^i) = p^{i-1} - m^i$. Since $v \in KP^1(F, \mathbb{G}\pi)$ and $\pi m^i = \pi q(m^i)$ we have

$$v^{t^i}_{m^i} \;-\; v_{p^{i-1}} \;=\; v^{t^i}_{q(m^i)} \;-\; v_{p^i}$$

and $u^{t^i}_{m^i} = v_{p^{i-1}} - v_{p^i}$. Therefore

$$u_{a;p;b} \;=\; \textstyle\sum_{1 \leqq i \leqq k} u^{t^i}_{m^i} \;=\; v_{p^0} - v_{p^k} \;=\; v_a - v_b\,.$$

Hence $u_{a;p;b}$ is independent of path. In particular $u_{a;p}$ is independent of path.

Conversely let $u \in M^1(F, \mathbb{G}\pi)$. Assume that $u_{a;p}$ is independent of path. Then $v \in C^1(F, \mathbb{G}\pi)$ is well defined by

$$v_a \;=\; u_{a;p}$$

whenever $p$ is an overpath from $a$ to $q(a)$. If $a \in Q$, then $p$ is empty and $v_a = u_{a;p} = 0$; thus $v \in P^1(F, \mathbb{G}\pi)$. If $a = m \in M$, then $p = \{m\}$ and $v_m = u_{m;p} = u_m$; thus $u = Rv$. It remains to show that $v$ is consistent: $v_{a+c} - v_{b+c} = v^c_a - v^c_b$ whenever $\pi a = \pi b$ and $c \in F$.

First let $b = q(a)$. Let $p$ be an overpath from $a$ to $b$ and $q$ be an overpath from $b+c$ to $q(b+c) = q(a+c)$. Then $p$ is an overpath from $a+c$ to $b+c$ and $p+q$ is an overpath from $a+c$ to $q(a+c)$. Hence

$$\begin{aligned} v_{a+c} &= u_{a+c;\,p+q} \\ &= u_{a+c;\,p;\,b+c} + u_{b+c;\,q} = u^c_{a;p} + u_{b+c;\,q} = v^c_a + v_{b+c} \end{aligned}$$

and $v^c_a - v_{a+c} = -v_{b+c} = -v_{q(a)+c}$. If now we assume only $\pi a = \pi b$, then $q(a) = q(b)$ and

$$v^c_a - v_{a+c} = -v_{q(a)+c} = -v_{q(b)+c} = v^c_b - v_{b+c}\,.$$

Thus $v$ is consistent. □

9. Finally, recall that a minimal cocycle is a minimal cochain $u$ such that $u_{a;p}$ is independent of path. Under pointwise addition minimal cocycles constitute a subgroup $MZ^1(F, \mathbb{G}\pi)$ of $MC^1(F, \mathbb{G}\pi)$.

A minimal coboundary is a minimal cochain $u$ for which there exists $g = (g_x)_{x \in X} \in \prod_{x \in X} G_{\pi x}$ such that $u = RDg$; equivalently,

$$u_m = \sum_{x \in X,\, x \leqq m} m_x\, g_x^{m-x} - \sum_{x \in X,\, x \leqq q(m)} q(m)_x\, g_x^{q(m)-x}$$

for all $m \in M$. Under pointwise addition minimal coboundaries constitute a subgroup $MB^1(F, \mathbb{G}\pi)$ of $MC^1(F, \mathbb{G}\pi)$.

Lemma 3.9 shows that $R$ induces an isomorphism of $KP^1(F, \mathbb{G}\pi)$ onto $MZ^1(F, \mathbb{G}\pi)$. Since $\operatorname{Im} D \subseteq KP^1(F, \mathbb{G}\pi)$ it follows that $MB^1(F, \mathbb{G}\pi) = \operatorname{Im} RD \subseteq MZ^1(F, \mathbb{G}\pi)$. Then Corollary 3.7 yields

$$H^2(S, \mathbb{G}) \cong KP^1(F, \mathbb{G}\pi)/\operatorname{Im} D \cong MZ^1(F, \mathbb{G}\pi) \,/\, MB^1(F, \mathbb{G}\pi)$$

which is natural in $\mathbb{G}$ since $R$ is natural in $\mathbb{G}$. This proves Theorem 2.1. □

## 4. DEFINING VECTORS.

In this section we show that minimal cocycles are determined by relations between certain integer vectors, and that the computation of $H^2(S, \mathbb{G})$ is a finite task when $S$ is finitely generated and $\mathbb{G}$ is thin.

1. As before, $S$ is a commutative monoid, $F = F_X$ is the free commutative semigroup on a set $X$, and $\pi : F \longrightarrow S$ is a surjective homomorphism; $G = G_X$ is the free abelian group on $X$, whose elements are finite linear combinations $a = \sum_{x \in X} a_x\, x$ with integer coefficients and can be regarded as integer vectors.

By Proposition 1.4, $\mathcal{C}$ is generated by all pairs $(m,\, q(m))$ with $m \in M$, which may be regarded as defining relations of $S$. The **defining vectors** of $\mathcal{C}$

(or of $S$) are the integer vectors

$$v(m) = m - q(m) \in G$$

with $m \in M$.

**Proposition 4.1.** *The subgroup of $G$ generated by the defining vectors is the Rédei group $R$ of $\mathcal{C}$; the universal group of $S$ is isomorphic to $G/R$.*

**Proof.** Recall that the Rédei group of $\mathcal{C}$ is

$$R = \{a - b \in G \mid a \mathcal{C} b\}.$$

Since $\mathcal{C}$ is generated by all pairs $(m, q(m))$ with $m \in M$, it follows from Proposition I.2.9 that $a - b$ is a sum of differences $m - q(m)$ and $q(m) - m$ when $a \mathcal{C} b$, so that $a - b$ belongs to the subgroup $K$ of $G$ generated by the defining vectors. Hence $R \subseteq K$. Conversely every defining vector $v(m) = m - q(m)$ is in $R$, since $m \mathcal{C} q(m)$; hence $K \subseteq R$.

Since $\mathcal{C}$ is generated by all pairs $(m, q(m))$ with $m \in M$, $S \cup \{0\}$ is generated, as a commutative monoid with zero, by the set $X$ subject to all relations $m = q(m)$ $(m \in M)$, with $m, q(m) \neq 0$ in $S \cup \{0\}$. By Proposition III.3.4, $G(S)$ is the abelian group generated by $X$ subject to all relations $m = q(m)$; that is, $G(S) \cong G/K$. □

**Proposition 4.2.** *Let $S$ have a zero element and $Z = \pi^{-1}0 \subseteq F$ be the zero class. Let $K$ be the subgroup of $G$ generated by all defining vectors $v(m)$ with $m \notin Z$. Then $G/K$ is the universal abelian group $G(S\backslash 0)$ of the partial semigroup $S\backslash 0$.*

**Proof.** Since $Z$ is a $\mathcal{C}$-class, $m \in Z$ implies $q(m) \in Z$. Since $\mathcal{C}$ is generated by all pairs $(m, q(m))$ with $m \in M$, $S$ is generated, as a commutative monoid with zero, by the set $X$ subject to all relations $m = q(m)$ $(m \in M\backslash Z)$ and $m = 0$ $(m \in M \cap Z)$. By Proposition III.3.4, $G(S\backslash 0)$ is the abelian group generated by $X$ subject to all relations $m = q(m)$ $(m \in M\backslash Z)$; that is, $G(S\backslash 0) \cong G/K$. □

2. We now consider relations between defining vectors. We distinguish **vector relations**

$$R(r): \qquad \textstyle\sum_{m\in M} r_m\, v(m) = 0,$$

in which every $r_m$ is an integer and $r_m = 0$ for almost all $m$, and **positive relations**

$$R(r,s): \qquad \textstyle\sum_{m\in M} r_m\, v(m) = \sum_{m\in M} s_m\, v(m),$$

in which every $r_m$ and every $s_m$ is a nonnegative integer and $r_m = s_m = 0$ for almost all $m$. The two types are essentially equivalent.

**Lemma 4.3.** *Every positive relation is a trivial consequence of a finite sum of minimal positive relations; if $X$ is finite, then there are only finitely many minimal positive relations.*

**Proof.** When $R = R(r,s) : \sum_{m\in M} r_m\, v(m) = \sum_{m\in M} s_m\, v(m)$ is a positive relation, let the *weight* of $R$ be $\sum_{m\in M} (r_m + s_m)$; call $R$ *nontrivial* when $r_m \neq s_m$ for some $m$, and *essential* when it is nontrivial but there is no $m$ such that $r_m > 0$ and $s_m > 0$ (so that no cancellation is possible in $R(r,s)$). Every nontrivial positive relation can be simplified by cancellation in $G$ into an essential positive relation; hence every nontrivial positive relation $R(r,s)$ is a *trivial consequence* of an essential positive relation $R(e,f)$ ($r_m - e_m = s_m - f_m \geqq 0$ for all $m$).

Positive relations are ordered coefficientwise: $R(p,q) \leqq R(r,s)$ if and only if $p_m \leqq r_m$ and $q_m \leqq s_m$ for all $m$. A *minimal* positive relation is a minimal nontrivial positive relation. Minimal positive relations are essential.

When $R(p,q) < R(r,s)$, then $\sum_{m\in M} p_m\, v(m) = \sum_{m\in M} q_m\, v(m)$ and $\sum_{m\in M} r_m\, v(m) = \sum_{m\in M} s_m\, v(m)$ imply

$$\textstyle\sum_{m\in M} (r_m - p_m)\, v(m) \;=\; \sum_{m\in M} (s_m - q_m)\, v(m),$$

so that $R(r-p,\ s-q)$ is a positive relation; then $R(r,s)$ is the sum of $R(p,q)$ and $R(r-p,\ s-q)$. If $R(r,s)$ is essential, then so are $R(p,q)$ and $R(r-p,\ s-q)$. Thus an essential positive relation which is not minimal is a sum of essential positive relations of lesser weight. Therefore every essential positive relation is a finite sum of minimal positive relations, and every positive relation is a consequence of a finite sum of minimal positive relations.

Under pointwise addition, pairs $(r,s)$ of families $r = (r_m)_{m\in M}$, $s = (s_m)_{m\in M}$ of nonnegative integers constitute a finitely generated free commutative monoid $F'$. Minimal positive relations constitute an antichain of $F'$. If $X$ is finite, then, by Dickson's Theorem, $M$ is finite, all antichains of $F'$ are finite, and there are only finitely many minimal positive relations. □

3. Relations between defining vectors arise when there is more than one path from an element of $F$ to another. When $p : m^1, \dots, m^k$ is an overpath from $a$ to $b$, we saw that

$$a - b \;=\; \textstyle\sum_{1\leqq i\leqq k} \left(m^i - q(m^i)\right) \;=\; \sum_{1\leqq i\leqq k} v(m^i)$$

We write this equality as

$$a - b \;=\; \textstyle\sum_{m\in M} p_m\, v(m),$$

where $p_m$ is the number of appearances of $m$ in the sequence $p : m^1,\dots,m^k$.

When $p$ and $q$ are two overpaths from $a$ to $b$, then

$$R(p,q) : \textstyle\sum_{m\in M} p_m\, v(m) = a - b = \sum_{m\in M} q_m\, v(m)$$

is a positive relation and there is a vector relation

$$R(p-q) : \textstyle\sum_{m\in M} (p_m - q_m)\, v(m) \;=\; 0.$$

A relation between defining vectors is **realized at** $a \in F$ when it arises in this fashion from a pair of overpaths from $a$ to some $b$. Thus a vector relation $R(r)$ : $\sum_{m\in M} r_m\, v(m) = 0$ is realized at $a$ when there exist $b \in F$ and overpaths $p$ and $q$ from $a$ to $b$ such that $r_m = p_m - q_m$ for all $m$ (so that $R(r) = R(p-q)$); a positive relation $R(r,s) : \sum_{m\in M} r_m\, v(m) = \sum_{m\in M} s_m\, v(m)$ is realized at $a \in F$ when there exist $b \in F$ and overpaths $p$ and $q$ from $a$ to $b$ such that $r_m - p_m = s_m - q_m \geqq 0$ for all $m$ (so that $R(r,s)$ is a trivial consequence of $R(p,q)$).

**Lemma 4.4.** *When $R(r,s)$ is realized at $a$, then $r_m = s_m$ whenever $\pi m \ngeqq_{\mathcal{H}} \pi a$.*

**Proof.** Let $b \in F$ and $p$, $q$ be overpaths from $a$ to $b$ such that $r_m - p_m = s_m - q_m \geqq 0$ for all $m \in M$. If $p_m > 0$ or $q_m > 0$ (if $m$ appears in $p$ or in $q$), then $m \geqq c$ for some $c \in C_a$ and $\pi m \geqq_{\mathcal{H}} \pi a$. Therefore $\pi m \ngeqq_{\mathcal{H}} \pi a$ implies $p_m = q_m = 0$ and $r_m = s_m$. □

A relation of either kind is **realized in** a $\mathcal{C}$-class $C$ when it is realized at some $a \in C$ (then $b \in C$ in the above). In the case of a vector relation it may be assumed that $b = q(a)$, since an overpath from $b$ to $q(a) = q(b)$ can be added to $p$ and $q$ if necessary, without changing $p_m - q_m$.

The trivial relation $0 = 0$ is realized at every $a \in F$. A relation which is realized at $a$ is realized at $a + c$ for every $c \in F$, since overpaths from $a$ to $b$ are also overpaths from $a + c$ to $b + c$. More surprisingly:

**Proposition 4.5.** *Every relation between defining vectors is realized in some $\mathcal{C}$-class (in the zero class, if $S$ has a zero element).*

**Proof.** First we prove the following: for every family $r = (r_m)_{m\in M}$ of nonnegative integers (with $r_m = 0$ for almost all $m \in M$) there exists an overpath $p$ such that $p_m = r_m$ for all $m$. This is shown by induction on $|r| = \sum_{m\in M} r_m$. If $|r| = 0$, then the empty ovepath from any $a$ to $a$ serves. If $|r| > 0$, then $r_n > 0$ for some $n \in M$, and the induction hypothesis yields an overpath $p : m^1,\dots,m^k$ from some $a \in F$ to some $b \in F$ such that $p_m = r_m$

for all $m \neq n$ and $p_n = r_n - 1$. In $F$ there is an element $c$ such that $c \geqq b$ and $c \geqq n$ (for instance, $b \vee n$). Then $p : m^1, \ldots, m^k$ is an overpath from $a + (c - b)$ to $b + (c - b) = c$, $\{n\}$ is an overpath from $n + (c - n) = c$ to $q(n) + (c - n)$, and $q = p + \{n\} : m^1, \ldots, m^k, n$ is an overpath from $a + (c - b)$ to $q(n) + (c - n)$. We see that $q_m = r_m$ for all $m$.

If now $R(r, s)$ is a positive relation, then the above provides an overpath $p$ from some $a$ to some $b$ such that $p_m = r_m$ for all $m$ and an overpath $q$ from some $c$ to some $d$ such that $q_m = s_m$ for all $m$. Then

$$\begin{aligned} a - b &= \textstyle\sum_{m \in M} p_m \, v(m) = \sum_{m \in M} r_m \, v(m) \\ &= \textstyle\sum_{m \in M} s_m \, v(m) = \sum_{m \in M} q_m \, v(m) = c - d. \end{aligned}$$

In $F$ there is an element $e$ such that $e \geqq a$ and $e \geqq c$; if $S$ has a zero element, then the zero class $Z$ is an ideal of $F$ and we can arrange that $e \in Z$. Let $f = (e - a) + b = (e - c) + d$. Then $p$ and $q$ are overpaths from $e$ to $f$, and $p_m = r_m$, $q_m = s_m$ for all $m$. In particular $R(r, s)$ is realized in $C_e$. Thus every positive relation is realized in some $\mathcal{C}$-class (in the zero class, if $S$ has a zero element). Then so is every vector relation. □

4. We now show that minimal cocycles are determined by relations between the defining vectors, when the coefficient functor is thin.

Let $\mathbb{G} = (G, \gamma)$ be an abelian group valued functor on $H(S)$. When $u = (u_m)_{m \in M}$ is a minimal cochain and $p : m^1, \ldots, m^k$ is as overpath from $a$ to $b$, then

$$u_{a;p;b} = \textstyle\sum_{1 \leqq i \leqq k} u_{m^i}^{t^i}$$

where $t^i = p^i - q(m^i) = p^{i-1} - m^i$ is provided by the corresponding path

$$a = p^0 \xrightarrow{m^1} p^1 \xrightarrow{m^2} \cdots \xrightarrow{m^k} p^k = b.$$

If $\mathbb{G}$ is thin, then $\gamma_{\pi m^i, \pi t^i}$ depends only on $m^i$ and $\pi(m^i + t^i) = \pi p^{i-1} = \pi a$ and is denoted by $\gamma_{\pi a}^{\pi m^i}$; hence

$$\begin{aligned} u_{a;p;b} &= \textstyle\sum_{1 \leqq i \leqq k} u_{m^i}^{t^i} = \sum_{m \in M, \, \pi m \geqq_{\mathcal{H}} \pi a} p_m \, \gamma_{\pi a}^{\pi m} u_m \\ &= \textstyle\sum_{m \in M_a} p_m \, \gamma_{\pi a}^{\pi m} u_m , \end{aligned}$$

where

$$M_a = \{ m \in M \mid \pi m \geqq_{\mathcal{H}} \pi a \}.$$

Recall that $\pi m \geqq_{\mathcal{H}} \pi a$ when $m$ appears in $p$ (when $p_m > 0$). Thus $u$ is a minimal cocycle if and only if

$$Z(p,q,a): \qquad \textstyle\sum_{m\in M_a} p_m \, \gamma_{\pi a}^{\pi m} u_m \; = \; \sum_{m\in M_a} q_m \, \gamma_{\pi a}^{\pi m} u_m$$

whenever $p$ and $q$ are overpaths from $a \in F$ to $q(a)$.

**Proposition 4.6.** *When $\mathbb{G}$ is thin, a minimal cochain $u$ is a minimal cocycle if and only if*

$$Z(r,s,a): \qquad \textstyle\sum_{m\in M_a} r_m \, \gamma_{\pi a}^{\pi m} u_m \; = \; \sum_{m\in M_a} s_m \, \gamma_{\pi a}^{\pi m} u_m$$

*holds whenever $a \in F$ and the positive relation $R(r,s)$ is realized at $a$.*

**Proof.** If $R(r,s)$ is realized at $a$, then $R(r,s)$ is a trivial consequence of $R(p,q)$ for some overpaths $p$ and $q$ from $a$ to some $b \in C_a$; then $Z(r,s,a)$ is a trivial consequence of $Z(p,q,a)$, since $p_m = q_m = 0$ and $r_m = s_m$ when $m \notin M_a$ by Lemma 4.5. By cancellation in $G_{\pi a}$, $Z(r,s,a)$ holds if and only if $Z(p,q,a)$ holds. Similarly, when an overpath from $b$ to $q(b) = q(a)$ is added to $p$ and $q$ (to obtain overpaths from $a$ to $q(a)$), then $Z(p,q,a)$ is replaced by an equivalent condition. Hence $u$ is a minimal cocycle if and only if every $Z(r,s,a)$ holds. □

On the other hand, minimal coboundaries satisfy relations between defining vectors regardless of whether they are realized. Call a positive relation $R(r,s)$ **verifiable at** $a \in F$ when $r_m = s_m$ whenever $m \notin M_a$ (whenever $\pi m \ngeqq_{\mathcal{H}} \pi a$). By Lemma 4.4, every positive relation which is realized at $a$ is verifiable at $a$.

**Proposition 4.7.** *When $\mathbb{G}$ is thin and $u$ is a minimal coboundary, then*

$$Z(r,s,a): \qquad \textstyle\sum_{m\in M_a} r_m \, \gamma_{\pi a}^{\pi m} u_m \; = \; \sum_{m\in M_a} s_m \, \gamma_{\pi a}^{\pi m} u_m$$

*holds whenever $a \in F$ and the positive relation $R(r,s)$ is verifiable at $a$.*

**Proof.** Let $u$ be a minimal coboundary, so that

$$u_m \; = \; \textstyle\sum_{x\in X,\, x\leqq m} m_x \, g_x^{\pi(m-x)} \; - \; \sum_{x\in X,\, x\leqq q(m)} q(m)_x \, g_x^{\pi(q(m)-x)}$$

for all $m = \sum_{x\in X} m_x x \in M$, where $g_x \in G_{\pi x}$ for all $x \in X$. Since $\mathbb{G}$ is thin and $\pi m = \pi q(m)$,

$$u_m \; = \; \textstyle\sum_{x\in X,\, x\leqq m} m_x \, \gamma_{\pi m}^{\pi x} g_x \; - \; \sum_{x\in X,\, x\leqq q(m)} q(m)_x \, \gamma_{\pi m}^{\pi x} g_x$$

for all $m \in M$.

Let $R(r,s) : \sum_{m\in M} r_m \, v(m) = \sum_{m\in M} s_m \, v(m)$ be verifiable at $a$. Then $\sum_{m\notin M_a} r_m \, v(m) = \sum_{m\notin M_a} s_m \, v(m)$, since $r_m = s_m$ when $m \notin M_a$;

hence $\sum_{m \in M_a} r_m \, v(m) = \sum_{m \in M_a} s_m \, v(m)$ and

$$\textstyle\sum_{m \in M_a} r_m \, v(m)_x \;=\; \sum_{m \in M_a} s_m \, v(m)_x$$

for every $x \in X$. Let $X_a = \{x \in X \mid \pi x \geqq_{\mathcal{H}} \pi a\}$. Then $x \leqq m \in M_a$ implies $\pi x \geqq_{\mathcal{H}} \pi m \geqq_{\mathcal{H}} \pi a$ and $x \in X_a$; $x \leqq q(m) \in M_a$ implies $x \in X_a$; and

$$\begin{aligned}
&\textstyle\sum_{m \in M_a} \; r_m \; \gamma_{\pi a}^{\pi m} u_m \\
&\quad= \textstyle\sum_{m \in M_a} \sum_{x \in X,\, x \leqq m} \; r_m \; m_x \; \gamma_{\pi a}^{\pi x} g_x \\
&\qquad - \textstyle\sum_{m \in M_a} \sum_{x \in X,\, x \leqq q(m)} \; r_m \; q(m)_x \; \gamma_{\pi a}^{\pi x} g_x \\
&\quad= \textstyle\sum_{m \in M_a} \sum_{x \in X_a} \; r_m \; m_x \; \gamma_{\pi a}^{\pi x} g_x \\
&\qquad - \textstyle\sum_{m \in M_a} \sum_{x \in X_a} \; r_m \; q(m)_x \; \gamma_{\pi a}^{\pi x} g_x \,,
\end{aligned}$$

since $m_x = 0$ if $x \nleqq m$ and $q(m)_x = 0$ if $x \nleqq q(m)$,

$$\begin{aligned}
&\quad= \textstyle\sum_{m \in M_a} \sum_{x \in X_a} \; r_m \; v(m)_x \; \gamma_{\pi a}^{\pi x} g_x \\
&\quad= \textstyle\sum_{m \in M_a} \sum_{x \in X_a} \; s_m \; v(m)_x \; \gamma_{\pi a}^{\pi x} g_x \\
&\quad= \textstyle\sum_{m \in M_a} \sum_{x \in X,\, x \leqq m} \; s_m \; m_x \; \gamma_{\pi a}^{\pi x} g_x \\
&\qquad - \textstyle\sum_{m \in M_a} \sum_{x \in X,\, x \leqq q(m)} \; s_m \; q(m)_x \; \gamma_{\pi a}^{\pi x} g_x \\
&\quad= \textstyle\sum_{m \in M_a} \; r_m \; \gamma_{\pi a}^{\pi m} u_m \,,
\end{aligned}$$

since $m_x = 0$ if $x \nleqq m$ and $q(m)_x = 0$ if $x \nleqq q(m)$. Thus $u$ satisfies $Z(r,s,a)$. □

5. Computing $H^2(S, \mathbb{G})$ with Theorem 2.1 still looks like an infinite task even when $S$ is finite, since independence of path must be established at every $a \in F$. When $F$ is finitely generated and $\mathbb{G}$ is thin, we show that minimal cocycles are characterized by finitely many conditions $u_{a;p} = u_{a;q}$; hence the computation of $H^2(S, \mathbb{G})$ a finite task. It seems likely that this result holds even if $\mathbb{G}$ is not thin.

**Proposition 4.8.** *When $\mathbb{G}$ is thin and $X$ is finite, a minimal cochain $u$ is a minimal cocycle if and only if if satisfies finitely many conditions $Z(r,s,a)$, in which $a \in F$ and the positive relation $(r,s)$ is realized at $a$.*

**Proof.** Let $F''$ be the set of all ordered pairs $(r,a)$ where $r = (r_m)_{m \in M}$ is a family of nonnegative integers and $a \in F$. Under pointwise addition, $F''$ is a free commutative monoid $F'' \cong F_M \times F$.

Realizability yields a binary relation $\mathcal{R}$ on $F''$:

$$(r,a)\,\mathcal{R}\,(s,b) \iff a = b \text{ and } R(r,s) \text{ is realized at } a.$$

We see that $\mathcal{R}$ is reflexive and symmetric. If moreover $(r,a)\,\mathcal{R}\,(s,b)$, so that $R(r,s)$ is realized at $a$, then, for any $t$, $R(r+t,\ s+t)$ is a trivial consequence of $R(r,s)$ and is realized at $a$, $R(r+t,\ s+t)$ is realized at $a+c$ for any $c \in F$, and $(r+t,\ a+c)\ \mathcal{R}\ (s+t,\ b+c)$; thus $\mathcal{R}$ admits addition.

By Proposition I.2.9, the congruence $\overline{\mathcal{R}}$ on $F''$ generated by $\mathcal{R}$ is given by: $(r,a)\ \overline{\mathcal{R}}\ (s,b)$ if and only if there exist $k \geqq 0$ and $(r^0,a^0),\dots,(r^k,a^k) \in F''$ such that $(r,a) = (r^0,a^0)$, $(r^{i-1},a^{i-1})\ \mathcal{R}\ (r^i,a^i)$ for all $1 \leqq i \leqq k$, and $(r^k,a^k) = (s,b)$. Then $a = a^0 = \dots = a^k = b$ and the equalities

$$\textstyle\sum_{m\in M} r^0_m\, v(m) \;=\; \sum_{m\in M} r^1_m\, v(m) \;=\; \cdots \;=\; \sum_{m\in M} r^k_m\, v(m)$$

show that $R(r,s)$ is a consequence of $R(r^0,r^1)$, $R(r^1,r^2)$, ..., $R(r^{k-1},r^k)$.

By Proposition 4.6, a minimal cochain $u$ is a minimal cocycle if and only if it satisfies $Z(r,s,a)$ whenever $(r,a)\,\mathcal{R}\,(s,a)$. If $(r,a)\ \overline{\mathcal{R}}\ (s,a)$, then in the above $Z(r^0,r^1,a)$, $Z(r^1,r^2,a)$, ..., $Z(r^{k-1},r^k,a)$ hold in $G_{\pi a}$; by the equalities

$$\begin{aligned}\textstyle\sum_{m\in M_a} r^0_m\, \gamma^{\pi m}_{\pi a} u_m \;&=\; \textstyle\sum_{m\in M_a} r^1_m\, \gamma^{\pi m}_{\pi a} u_m\\ &=\; \cdots \;=\; \textstyle\sum_{m\in M_a} r^k_m\, \gamma^{\pi m}_{\pi a} u_m\,,\end{aligned}$$

$Z(r,s,a)$ is a consequence of $Z(r^0,r^1,a)$, $Z(r^1,r^2,a)$, ..., $Z(r^{k-1},r^k,a)$ and holds in $G_{\pi a}$ if $u$ is a minimal cocycle. Hence a minimal cochain $u$ is a minimal cocycle if and only if it satisfies $Z(r,s,a)$ whenever $(r,a)\ \overline{\mathcal{R}}\ (s,a)$.

Since $M$ is finite it follows from Rédei's Theorem that $\overline{\mathcal{R}}$ is finitely generated. Therefore a minimal cochain $u$ is a minimal cocycle if and only if it satisfies finitely many conditions $Z(r,s,a)$, with $a \in F$ and $(r,a)\ \overline{\mathcal{R}}\ (s,a)$, each of which is a consequence of finitely many conditions $Z(r,s,a)$, with $a \in F$ and $(r,s)$ realized at $a$. $\square$

A more explicit choice of conditions to verify is given in Grillet [1995F] but no longer seems particularly helpful.

6. We conclude this section with an example. More general examples are given in Grillet [2000T].

**Example 4.9.** Let $S$ be the commutative nilmonoid

$$S \;=\; \langle\, c,d \;\big|\; c^3 = c^2d = cd^2 = d^4 = 0,\ c^2 = cd = d^3\,\rangle;$$

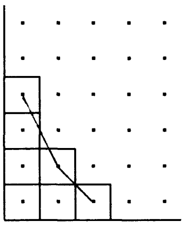
$S$ is the Volkov semigroup (Example III.3.6) with an identity adjoined.

Let $X = \{x,y\}$ and $\pi x = c$, $\pi y = d$. Then $\mathcal{C} = \ker \pi$ has four one element classes, one three element class $C = \{2x,\, x+y,\, 3y\} = \pi^{-1}(cd)$, and one infinite class $J = \pi^{-1}0$ which is the ideal of $F$ generated by $\{3x,\, 2x+y,\, x+2y,\, 4y\}$.

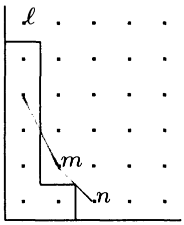
The lexicographic order $\sqsubseteq$ on $F$

$$ix + jy \sqsubset kx + ly \iff i < k, \quad \text{or} \quad i = k,\ j < l$$

is a compatible well order on $F$. Under $\sqsubseteq$ the least element of $C$ is $3y$; the least element of $J$ is $4y$.



Example 4.9




$Q$ and $M$

Thus $Q$ is the coideal generated by $4y$ and $x$; $M$ and the defining vectors are given by the table

| $m$ | $q(m)$ | $v(m)$ |
|---|---|---|
| $l = 5y$ | $4y$ | $y$ |
| $m = x + y$ | $3y$ | $x - 2y$ |
| $n = 2x$ | $3y$ | $2x - 3y$ |

The defining vectors $v(m)$ with $m \notin J$ are $v(m)$ and $v(n)$. They constitute a basis of $G$ since $\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} = 1$. Hence the universal abelian group of $S\backslash 0$ is trivial, by Proposition 4.2.

We see that $v(n) = 2v(m) + v(l)$. The only $\mathcal{C}$-class in which nontrivial positive relations are realized is $J$; by Proposition 4.5, every positive relation is realized in $J$. (More sophisticated examples are given in the next chapter.)

Let $\mathbb{G}$ be a thin abelian group valued functor on $H(S)$. A minimal cochain $u$ consists of $u_l \in G_0$, $u_m \in G_{cd}$, and $u_n \in G_{cd}$.

In $J$, $2x + y \xrightarrow{n} 4y$ and $2x + y \xrightarrow{m} x + 3y \xrightarrow{m} 5y \xrightarrow{l} 4y$ are paths from $a = 2x + y$ to $q(a) = 4y$. Hence $p : n$ and $q : m,m,l$ are overpaths from $a$ to $q(a)$ (and the relation $v(n) = 2v(m) + v(l)$ is realized at $2x + y$).

Therefore minimal cocycles satisfy

$$u_n^0 = u_{a;p} = u_{a;q} = 2u_m^0 + u_l ;$$

this is according to Proposition 4.6.

Conversely let $u$ be a minimal cochain such that $u_n^0 = 2u_m^0 + u_l$. Let $a \in F$ and let $p$ and $q$ be overpaths from $a$ to $q(a)$. We may assume that $a \in J$ (otherwise $p = q$). We have

$$\begin{aligned} a - q(a) &= p_l\, y + p_m\,(x - 2y) + p_n\,(2x - 3y) \\ &= q_l\, y + q_m\,(x - 2y) + q_n\,(2x - 3y); \end{aligned}$$

hence $p_m + 2p_n = q_m + 2q_n$ and $p_l - 2p_m - 3p_n = q_l - 2q_m - 3q_n$. Adding twice the first equality to the second yields $p_l + p_n = q_l + q_n$. Hence

$$\begin{aligned} u_{a;p} &= p_l\, u_l \;+\; p_m\, u_m^0 \;+\; p_n\, u_n^0 \\ &= (p_l + p_n)\, u_l \;+\; (p_m + 2p_n)\, u_m^0 \\ &= (q_l + q_n)\, u_l \;+\; (q_m + 2q_n)\, u_m^0 \\ &= q_l\, u_l \;+\; q_m\, u_m^0 \;+\; q_n\, u_n^0 \;=\; u_{a;q}\,. \end{aligned}$$

Thus minimal cocycles are characterized by the single condition $u_n^0 = 2u_m^0 + u_l$; this is according to Proposition 4.8. Hence there is an isomorphism $u \longmapsto (u_m, u_n)$ of $MZ^1(F, \mathbb{G}\pi)$ onto $G_{cd} \oplus G_{cd}$. A peek at minimal coboundaries suggests that $u_m$ and $u_n$ are uniquely determined by $g = u_n - 2u_m$ and $h = 2u_n - 3u_m$ (namely, $u_m = h - 2g$ and $u_n = 2h - 3g$), and provides a more useful isomorphism $\theta : u \longmapsto (u_n - 2u_m,\ 2u_n - 3u_m)$ of $MZ^1(F, \mathbb{G}\pi)$ onto $G_{cd} \oplus G_{cd}$.

Next, $u$ is a minimal coboundary if and only if there exist $g \in G_c$ and $h \in G_d$ such that $u_l = 5g^0 - 4g^0 = g^0$, $u_m = (g^d + h^c) - 3g^d = h^c - 2g^d$, and $u_n = 2h^c - 3g^d$. Then $\theta(u) = (g^d, h^c)$ and $\theta$ sends $MB^1(F, \mathbb{G}\pi)$ onto $\operatorname{Im} \gamma_{c,d} \oplus \operatorname{Im} \gamma_{d,c} \subseteq G_{cd} \oplus G_{cd}$. Hence

$$H^2(S, \mathbb{G}) \;\cong\; (G_{cd} / \operatorname{Im} \gamma_{c,d}) \oplus (G_{cd} / \operatorname{Im} \gamma_{d,c}).$$

If $\mathbb{G}$ is surjecting (as well as thin), then $\gamma_{1,c}$, $\gamma_{1,d}$, and $\gamma_{1,cd}$ are surjective; hence $\gamma_{c,d}$, and $\gamma_{d,c}$ are surjective and $H^2(S, \mathbb{G}) = 0$.

# 5. PARTIALLY FREE SEMIGROUPS.

Partially free semigroups were defined in Section X.6. At this time they constitute the only large class of finite commutative semigroups with a formula for $H^2(S,\mathbb{G})$: namely,

$$H^2(S,\mathbb{G}) \;\cong\; \bigoplus_{c\in \mathrm{Irr}\,(S)} G_{e(c)}/\mathrm{Im}\,\gamma^c_{e(c)}\,,$$

where $\mathbb{G}$ is thin, Irr $(S)$ is the set of all irreducible elements of $S$, and $e(c)$ is the idempotent in the archimedean component of $c$. This result is from Grillet [1995P].

Other formulas yield $H^2(S,\mathbb{G})$ when $S$ has one defining relation (Proposition 2.4) or is cyclic (Corollary 2.5). But no such formula seems to exist for semigroups with two generators (Grillet [2000T]).

1. Let $S$ be finite and partially free. By Corollary XII.5.5 we may assume that $S$ is a monoid. Since $S$ is group-free, $\leqq_{\mathcal{H}}$ is a partial order relation on $S$, which we denote by $\leqq$.

By Proposition X.2.2, $S$ is generated by Irr $(S)$ and has a standard presentation $\pi : F = F_X \longrightarrow S$, where $X$ is finite, $\pi$ is injective on $X$, and $\pi(X) = \mathrm{Irr}\,(S)$. The direction set $\mathcal{D}$, extent cells $E_A$, coideals $H_A$, and trace congruences $\mathcal{C}_A$ of the congruence $\mathcal{C}$ induced by $\pi$ are as in Chapter X. The idempotents of $S$ are all $e_A = \pi(I_A)$ with $A \in \mathcal{D}$, where $I_A = E_A \cap A$. Since $S$ is partially free, all trace congruences are Rees congruences and

$$a \;\mathcal{C}\; b \text{ if and only if } a,b \in E_A \text{ and } p'_A a = p'_A b \text{ for some } A \in \mathcal{D}.$$

Put any total order $\preccurlyeq$ on $X$ and order $F$ lexicographically: $\sum_{x\in X} a_x\, x \sqsubset \sum_{x\in X} b_x\, x$ if and only if there exists $t \in X$ such that $a_x = b_x$ for all $x \prec t$ and $a_t < b_t$. (Then $x \sqsubset y$ in $F$ if and only if $x \succ y$ in $X$.) Since $X$ is finite, $\sqsubseteq$ is a compatible well order on $F$.

We show that the defining vectors contain a basis of $G$.

**Lemma 5.1.** *Let $x \in X$, $c = \pi x$, and $D$ be the smallest element of $\mathcal{D}$ that contains $x$. There exists $m(x) \in M$ such that: $m(x) \in E_D \cap D$; $\pi m(x) = e(c)$; if $x \in A \in \mathcal{D}$, then $\pi m(x) \geqq \pi a$ for all $a \in E_A$; and*

$$v(m(x)) \;=\; x \;+\; \textstyle\sum_{y\in X} t_{x,y}\, y$$

*with $t_{x,y} = 0$ unless $y \prec x$ and $y \in D$.*

**Proof.** First take any $m \in I_D$, so that $\pi m = e_D$.

If $A \in \mathcal{D}$, $A \subset D$, then $x \notin A$ by the choice of $D$. By (E2), $kx \in E_D$ for some $k > 0$. Then $c^k = \pi(kx) = e_D$ and $\pi m = e(c)$.

If $x \in A \in \mathcal{D}$, then $D \subseteq A$, $\epsilon(\pi a) = e_A$ for all $a \in E_A$ by Proposition X.3.4, and $\pi m = e_D \geqq e_A \geqq \pi a$ by Proposition X.3.3.

We now choose $m(x) = m \in I_D$ as follows. Let $w$ be the least element of the $\mathcal{C}$-class $I_D$ under $\sqsubseteq$. Since $x \in D = D(w)$ we have $w + x \in I_D$. Let $m(x) = m$ be least under $\sqsubseteq$ such that $m \in I_D$ and $m_x = w_x + 1$. In particular, $w \sqsubset m \sqsubseteq w + x$.

We show that $m \in M$. First $m \notin Q = \{a \in F \mid q(a) = a\}$, since $w \mathcal{C} m$ and $w \sqsubset m$. To prove that $m$ is minimal with this property (under $\leqq$) it suffices to show that $m - y \in Q$ for every $y \in X$, $y \leqq m$. Note that $y \leqq m$ implies $y \in D$.

Assume $m - y \in E_D$. If $y \neq x$, then $(m-y)_x = m_x = w_x + 1$; since $m - y \in I_D$ and $m - y \sqsubset m$ this contradicts the choice of $m$. Therefore $y = x$. Hence $m - x \in I_D$, $m - x \sqsupseteq w$, $m \sqsupseteq w + x$, $m = w + x$, and $m - y = m - x = w \in Q$.

Now let $m - y \notin E_D$. Then $y \neq x$. Also $m - y \in E_B$ for some $B \in \mathcal{D}$, $B \subset D$. Then $D \nsubseteq B$ and $x \notin B$. Let $q = q(m-y)$. Then $q \sqsubseteq m - y$ and $q \mathcal{C} m - y$; hence $q \in E_B$, and $q \in D$ since $D$ is a union of $\mathcal{C}$-classes. Since $S$ is partially free we also have $p'_B q = p'_B(m-y)$. In particular $q_x = (m-y)_x = m_x$. Hence $q + y$ has the following properties: $q + y \in D$, $q + y \in E_D$ (since $q + y \ \mathcal{C}\ m$), $q + y \sqsubseteq m$, and $(q+y)_x = q_x = m_x = w_x + 1$. By the choice of $m$, $q + y = m$, and $m - y = q \in Q$.

This proves $m \in M$. We have $w = q(m)$. Hence $v(m)_x = m_x - w_x = 1$. Also $v(m)_y = m_y - w_y = 0$ if $y \notin D$, since $m, w \in D$. Since $w \sqsubset m$ there exists $t \in X$ such that $w_y = m_y$ for all $y \prec t$ and $w_t < m_t$; in particular $t \preccurlyeq x$. If $t \prec x$, then $w_y = m_y$ for all $y \prec t$ and $w_t < m_t$ implies $w + x \sqsubset m$, whereas $m \sqsubseteq w + x$; therefore $t = x$, and $v(m)_y = m_y - w_y = 0$ for all $y \prec x$. $\square$

Lemma 5.1 implies that $\big(v(m(x))\big)_{x \in X}$ is a basis of $G$. In fact:

**Corollary 5.2.** *For every* $A \in \mathcal{D}$, $\big(v(m(x))\big)_{x \in X \cap A}$ *is a basis of* $G_A$.

**Proof.** $G_A = G(A) \subseteq G$ is the free abelian group on $X \cap A$. When $m \in M \cap E_A$, then $q(m) \in E_A$ and $m \mathcal{C} q(m)$ implies $p'_A m = p'_A q(m)$ since $S$ is partially free; hence $v(m) \in G_A$. In particular $v(m(x)) \in G_A$ for all

$x \in X \cap A$. When $x \in X \cap A$, then $D \subseteq A$ in Lemma 5.1 and

$$v(m(x)) \;=\; x \;+\; \textstyle\sum_{y \in X \cap A,\, y \prec x} t_{x,y}\, y.$$

Hence the defining vectors $v(m(x))$ with $x \in X \cap A$ constitute a basis of $G_A$. □

2. By Corollary 5.2 there is for every $m \in M \cap E_A$ an equality

$$v(m) \;=\; \textstyle\sum_{x \in A} k_{m,x}\, v(m(x)) \;\in\; G_A$$

with integer coefficients $k_{m,x}$.

**Lemma 5.3.** *For every $m \in M \cap E_A$ the vector relation*

$$v(m) \;=\; \textstyle\sum_{x \in A} k_{m,x}\, v(m(x)) \tag{$*$}$$

*is realized in the $\mathcal{C}$-class $C_m$ of $m$.*

**Proof.** If $m = m(x)$ for some $x \in A$, then $(*)$ is trivial, and is realized in $C_m$. Hence we may assume that $m \neq m(x)$ for all $x \in A$.

We have $m + A \subseteq C_m$, since $m \in E_A$. Let $p : m^1, \ldots, m^k$ consist of $m$ and of $-k_{m,x}$ copies of $m(x)$ for every $x \in X \cap A$ with $k_{m,x} < 0$, arranged in any order. Let $q : n^1, \ldots, n^l$ similarly consist of $k_{m,x}$ copies of $m(x)$ for every $x \in X \cap A$ with $k_{m,x} > 0$, arranged in any order. Then $\sum_i v(m^i) = \sum_j v(n^j)$, by $(*)$. Also $p_m - q_m = 1$ and $q_{m(x)} - p_{m(x)} = k_{m,x}$ for every $x \in X \cap A$. If we can show that $p$ and $q$ are overpaths from some $a \in C_m$ to some $b \in C_m$, then they will realize $(*)$.

We have $m(x) \in A$ for all $x \in X \cap A$, and $m^i \in A$ for all $i > 1$. Also $v(m^i) \in G_A$ by Corollary 5.2. If $a \in m + A$ has sufficiently large coordinates, we can arrange by induction on $i$ that $p^{i-1} \geqq m^i$ and $p^i - m = p^{i-1} - m + v(m^i) \in A$ for all $i$. Then $p$ is an overpath from $a$ to $b = a + \sum_i v(m^i) \in m + A$. Similarly, if $a \in m + A$ has sufficiently large coordinates, then $q$ is an overpath from $a$ to $a + \sum_j v(n^j)$, and $\sum_j v(n^j) = b$ since $\sum_i v(m^i) = \sum_j v(n^j)$). □

3. Now let $\mathbb{G}$ be a thin abelian group valued functor on $H(S)$.

**Proposition 5.4.** *When $S$ is partially free and $\mathbb{G}$ is thin, a minimal cochain $u = (u_m)_{m \in M}$ is a minimal cocycle if and only if*

$$u_m \;=\; \textstyle\sum_{x \in X \cap A} k_{m,x}\, \gamma_{\pi m}^{\pi m(x)} u_{m(x)} \tag{$**$}$$

*whenever $A \in \mathcal{D}$ and $m \in M \cap E_A$.*

**Proof.** By Proposition 4.6, a minimal cocycle $u = (u_m)_{m \in M}$ must satisfy

$(**)$ for all $A \in \mathcal{D}$ and $m \in M \cap E_A$, since $(*)$ is realized at $C_m$ by Lemma 5.3. (Note that $\pi m(x) \geqq \pi m$ for all $x \in X \cap A$, by Lemma 5.1, since $m \in E_A$.)

Conversely assume that $(**)$ holds for all $A \in \mathcal{D}$ and $m \in M \cap E_A$. Let $p$ and $q$ be overpaths from $a$ to $b$. Then $\sum_{m \in M} p_m\, v(m) = b - a = \sum_{m \in M} q_m\, v(m)$, with $m \in M_a = \{m \in M \mid \pi m \geqq \pi a\}$ whenever $p_m > 0$ or $q_m > 0$. Also $a \,\mathcal{C}\, b$ and $a, b \in E_A$ for some $A \in \mathcal{D}$.

If $m$ appears in either $p$ or $q$, then $m \leqq c$ for some $c \in C_a \subseteq E_A$ and $m \in E_B$ for some $B \subseteq A$ by (E2). Then

$$v(m) = \sum_{x \in X \cap B} k_{m,x}\, v(m(x)) \quad \text{and} \quad u_m = \sum_{x \in X \cap B} k_{m,x}\, \gamma_{\pi m}^{\pi m(x)}\, u_{m(x)} \,.$$

(By Lemma 5.1, $\pi m(x) \geqq \pi m$ when $x \in X \cap B$ and $m \in E_B$). Let $k_{m,x} = 0$ when $m \in E_B$ and $x \in X \backslash B$. Then $v(m) = \sum_{x \in X \cap A} k_{m,x}\, v(m(x))$ and

$$\begin{aligned}
\sum_{m \in M} p_m\, v(m) &= \sum_{m \in M, x \in X \cap A} p_m\, k_{m,x}\, v(m(x)), \\
\sum_{m \in M} q_m\, v(m) &= \sum_{m \in M, x \in X \cap A} q_m\, k_{m,x}\, v(m(x)).
\end{aligned}$$

Since the defining vectors $v(m(x))$ with $x \in X \cap A$ constitute a basis of $G_A$ (Corollary 5.2), this implies

$$\sum_{m \in M} p_m\, k_{m,x} = \sum_{m \in M} q_m\, k_{m,x}$$

for all $x \in X \cap A$ and

$$\sum_{m \in M_a} p_m\, k_{m,x} = \sum_{m \in M_a} q_m\, k_{m,x}$$

for all $x \in X \cap A$, since $m \in M_a$ whenever $p_m > 0$ or $q_m > 0$. Since $\pi m(x) \geqq \pi a$ when $x \in A$ by Lemma 5.1, we now have

$$\begin{aligned}
u_{a;p;b} &= \sum_{m \in M_a} p_m\, \gamma_{\pi a}^{\pi m} u_m \\
&= \sum_{B \subseteq A} \sum_{m \in E_B \cap M_a} p_m\, \gamma_{\pi a}^{\pi m} u_m \\
&= \sum_{B \subseteq A} \sum_{m \in E_B \cap M_a} \sum_{x \in X \cap B} p_m\, k_{m,x}\, \gamma_{\pi a}^{\pi m(x)}\, u_{m(x)} \\
&= \sum_{B \subseteq A} \sum_{m \in E_B \cap M_a} \sum_{x \in X \cap A} p_m\, k_{m,x}\, \gamma_{\pi a}^{\pi m(x)}\, u_{m(x)} \\
&= \sum_{m \in M_a} \sum_{x \in X \cap A} p_m\, k_{m,x}\, \gamma_{\pi a}^{\pi m(x)}\, u_{m(x)} \\
&= \sum_{m \in M_a} \sum_{x \in X \cap A} q_m\, k_{m,x}\, \gamma_{\pi a}^{\pi m(x)}\, u_{m(x)} \\
&= u_{a;q;b} \,.
\end{aligned}$$

Thus $u_{a;p;b}$ is independent of path. □

4. Proposition 5.4 shows that a minimal cocycle is determined by its values on all $m(x)$. Hence

$$MZ^1(S,\mathbb{G}) \;\cong\; \bigoplus\nolimits_{x\in X} G_{\pi m(x)}\,;$$

the isomorphism $\Psi : MZ^1(S,\mathbb{G}) \longrightarrow \bigoplus_{x\in X} G_{\pi m(x)}$ sends $u = (u_m)_{m\in M} \in MZ^1(S,\mathbb{G})$ to $(u_{m(x)})_{x\in X}$. We now compute $MB^1(S,\mathbb{G})$.

**Lemma 5.5.** $\Psi\big(MB^1(S,\mathbb{G})\big) = \Theta\big(\bigoplus_{x\in X} \operatorname{Im}\gamma^{\pi x}_{\pi m(x)}\big)$, *where* $\Theta$ *is the automorphism of* $\bigoplus_{x\in X} G_{\pi m(x)}$ *defined for all* $v = (v_x)_{x\in X} \in \bigoplus_{x\in X} G_{\pi m(x)}$ *by*

$$(\Theta v)_x \;=\; v_x \;+\; \sum\nolimits_{y\in X,\, t_{x,y}\neq 0} t_{x,y}\, \gamma^{\pi m(y)}_{\pi m(x)}\, v_y\,.$$

**Proof.** Let $D$ be the smallest element of $\mathcal{D}$ containing $x$. By Lemma 5.1, $\pi x \geqq \pi m(x)$ and $v(m(x)) = x + \sum_{y\in X} t_{x,y}\, y$, where $t_{x,y} \neq 0$ implies $y \prec x$, $y \in D$, and (by Lemma 5.1 applied to $y$) $\pi m(y) \geqq \pi m(x)$, since $m(x) \in E_D$. Hence $\Theta$ is well defined. Since $t_{x,y} \neq 0$ implies $y \prec x$, the matrix of $\Theta$ is triangular with 1's on the diagonal and $\Theta$ is an isomorphism.

Let $u = \delta g \in MB^1(S,\mathbb{G})$, where $g = (g_x)_{x\in X} \in \bigoplus_{x\in X} G_{\pi x}$. Then

$$u_m \;=\; \sum\nolimits_{x\in X,\, x\leqq m} m_x\, \gamma^{\pi x}_{\pi m} g_x \;-\; \sum\nolimits_{x\in X,\, x\leqq q(m)} q(m)_x\, \gamma^{\pi x}_{\pi q(m)}\,.$$

Let $x \in X$, $D$ be the smallest element of $\mathcal{D}$ containing $x$, $y \in X$, and $m = m(x)$. By Lemma 5.1, $m \in I_D = E_D \cap D$, $y \leqq m$ implies $y \in D$, $q(m) \in I_D$, and $y \leqq q(m)$ implies $y \in D$. Since $\pi y \geqq \pi m(y) \geqq \pi m(x)$ for all $y \in D$ and $m - q(m) = v(m) = x + \sum_{y\in X} t_{x,y}\, y$, we have

$$\begin{aligned}
u_{m(x)} &= \sum\nolimits_{y\in X,\, y\leqq m} m_y\, \gamma^{\pi y}_{\pi m} g_y \;-\; \sum\nolimits_{y\in X,\, y\leqq q(m)} q(m)_y\, \gamma^{\pi y}_{\pi q(m)} \\
&= \sum\nolimits_{y\in X\cap D} m_y\, \gamma^{\pi y}_{\pi m} g_y \;-\; \sum\nolimits_{y\in X\cap D} q(m)_y\, \gamma^{\pi y}_{\pi q(m)} \\
&= \sum\nolimits_{y\in X\cap D} v(m)_y\, \gamma^{\pi y}_{\pi m} g_y \;=\; \gamma^{\pi x}_{\pi m} g_x \;+\; \sum\nolimits_{y\in D} t_{x,y}\, \gamma^{\pi y}_{\pi m} g_y \\
&= \gamma^{\pi x}_{\pi m} g_x \;+\; \sum\nolimits_{y\in X,\, t_{x,y}\neq 0} t_{x,y}\, \gamma^{\pi y}_{\pi m} g_y \;=\; (\Theta v)_x\,,
\end{aligned}$$

where $v = \big(\gamma^{\pi x}_{\pi m(x)}\, g_x\big)_{x\in X}$. Thus $\Psi u = \Theta v$. Hence $\Psi\big(MB^1(S,\mathbb{G})\big) = \Theta\big(\bigoplus_{x\in X} \operatorname{Im}\gamma^{\pi x}_{\pi m(x)}\big)$. $\square$

We can now prove

**Theorem 5.6.** *When* $S$ *is a finite partialy free commutative semigroup and*

$\mathbb{G}$ *is thin, there is an isomorphism*

$$H^2(S,\mathbb{G}) \cong \bigoplus\nolimits_{c\in \mathrm{Irr}\,(S)} G_{e(c)}/\mathrm{Im}\,\gamma^c_{e(c)}\,,$$

*which is natural in* $\mathbb{G}$.

**Proof.** We may assume that $S$ is a monoid. Since $\Psi$ and $\Theta$ are isomorphisms,

$$\begin{aligned} H^2(S,\mathbb{G}) &\cong MZ/MB \cong \Theta^{-1}\Psi MZ\,/\,\Theta^{-1}\Psi MB \\ &= \big(\bigoplus\nolimits_{x\in X} G_{\pi m(x)}\big)/\big(\bigoplus\nolimits_{x\in X} \mathrm{Im}\,\gamma^{\pi x}_{\pi m(x)}\big) \\ &= \bigoplus\nolimits_{x\in X} \big(G_{\pi m(x)}\,/\,\mathrm{Im}\,\gamma^{\pi x}_{\pi m(x)}\big) \\ &= \bigoplus\nolimits_{c\in \mathrm{Irr}\,(S)} G_{e(c)}/\mathrm{Im}\,\gamma^c_{e(c)}\,, \end{aligned}$$

since $\pi : F \longrightarrow S$ induces a bijection of $X$ onto $\mathrm{Irr}\,(S)$ and $x \in X$ implies $c = \pi x \in \mathrm{Irr}\,(S)$ and $\pi m(x) = e(c)$, by Lemma 5.1. The isomorphism is natural in $\mathbb{G}$, since $\Psi$ and $\Theta$ are natural in $\mathbb{G}$. □