Chapter XIV.

SEMIGROUPS WITH ZERO COHOMOLOGY.

Like other cohomology theories, commutative semigroup cohomology gives rise to the following problem:

(1) For which commutative semigroups S does $H^n(S,\mathbb{G}) = 0$ for all $n \geqq 2$ and all $\mathbb{G}$?

By Theorem XII.4.4, free commutative semigroups have this property.

The special role of H^2 suggests two additional problems:

(2) For which commutative semigroups S does $H^2(S,\mathbb{G}) = 0$ for all $\mathbb{G}$?

(3) For which finite (more generally, complete) group-free semigroups S does $H^2(S,\mathbb{G}) = 0$ whenever $\mathbb{G}$ is thin and surjecting (or thin, finite, and surjecting)?

Since $H^2(S,\mathbb{G}) \cong \operatorname{Ext}(S,\mathbb{G})$, $H^2(S,\mathbb{G})$ vanishes for all $\mathbb{G}$ if and only if every commutative group coextension of S splits. Free commutative semigroups and free commutative monoids have this property; so do semilattices, by Proposition V.4.4, and free abelian groups, by Proposition V.4.6. Problem (2) asks if there are any other semigroups with this property.

When S is complete group-free, then $H^2(S,\mathbb{G})$ vanishes whenever $\mathbb{G}$ is thin and surjecting if and only if every exact $\mathcal{H}$-coextension of S splits; equivalently, if every complete semigroup T with $T/\mathcal{H} \cong S$ splits as a coextension of S. By Proposition V.4.4, semilattices have this property. Problem (3) asks what other complete group-free semigroups have this property.

From the point of view of the structure and construction of commutative semigroups, problem (2) is at this time more interesting than problem (1), and problem (3) is most interesting of all.

Problem (1) is still unsolved, but problems (2) and (3) have been solved in some major particular cases. In this chapter we solve problem (2) when S is finite group-free and, after some preliminary results, we solve problem (3) when S is a finite nilmonoid. The results are due to the author [1997Z], [2001Z]. Problem

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(3) was also solved by the author [2000T] for semigroups with two generators, in which case the solutions are nilmonoids or semilattices.

1. GROUP-FREE MONOIDS.

In this section, S is a finite commutative group-free monoid. We show that $H^2(S,\mathbb{G}) = 0$ for all $\mathbb{G}$ if and only if S is a semilattice. This was proved by the author [1997Z].

1. If S is a semilattice, then $H^2(S,\mathbb{G}) = 0$ for all $\mathbb{G}$ by Proposition V.4.4. We now let S be a finite commutative group-free semigroup, but not a semilattice, and cook up a functor $\mathbb{G}$ such that $H^2(S,\mathbb{G}) \neq 0$. By Corollary XII.5.5 we may assume that S is a monoid.

Call an abelian group valued functor $\mathbb{G}$ on $H(S)$ **selective** if there exists an element c of S such that $G_c \neq 0$ and $G_s = 0$ for all $s \neq c$. Then $\gamma_{s,t} = 0$ unless $s = st = c$, in which case $\gamma_{s,t}$ is an endomorphism of G_c; and $\mathbb{G}$ is thin if and only if $\gamma_{s,t}$ is the identity on G_c when $s = st = c$.

Lemma 1.1. *If S is partially free, but not a semilattice, then $H^2(S,\mathbb{G}) \neq 0$ for some thin finite surjecting and selective functor $\mathbb{G}$.*

Proof. By Theorem XIII.5.6 there is for every thin functor $\mathbb{G} = (G,\gamma)$ an isomorphism

$$H^2(S,\mathbb{G}) \cong \bigoplus\nolimits_{c\in \operatorname{Irr}(S)} \left(G_{e(c)} / \operatorname{Im} \gamma^c_{e(c)}\right),$$

where $\operatorname{Irr}(S)$ is the set of all irreducible elements of S and $e(c)$ is the idempotent in the archimedean component of c. If S is not a semilattice, then $\operatorname{Irr}(S)$, which generates S, contains an element c which is not idempotent. Then $e = e(c) \neq c$. Let $\mathbb{G}$ be the thin selective functor in which $G_e \neq 0$ is any nontrivial finite abelian group and $G_s = 0$ for all $s \neq e$. $\mathbb{G}$ is also finite and surjecting. Moreover $H^2(S,\mathbb{G}) \neq 0$, since its direct summand $G_{e(c)} / \operatorname{Im} \gamma^c_{e(c)} \cong G_e \neq 0$. □

2. We now let the finite monoid S be group-free but not partially free. Green's preorder $\leqq_{\mathcal{H}}$ is a partial order relation on S, which we denote as before by just $\leqq$. Let $\pi : F = F_X \longrightarrow S$ be the standard presentation of S and $\mathcal{C} = \ker \pi$. Then π induces a bijection of X onto $\operatorname{Irr}(S)$, and F is finitely generated. The direction set $\mathcal{D}$, extent cells E_A, and trace congruences $\mathcal{C}_A$ of $\mathcal{C}$ are as in Chapter X.

Since S is not partially free, one of the trace congruences $\mathcal{C}_B$ is not a Rees congruence and has a nontrivial class other than the ideal $B\backslash H_B$; then there is a $\mathcal{C}$-class $C \subseteq E_B$ whose projection $p'_B C \subseteq H_B$ is not trivial. Let $c \in S$ be maximal (under $\leqq$) such that the $\mathcal{C}$-class $C = \pi^{-1}c$ has a nontrivial projection $p'_B C \subseteq H_B$ (where $B \in \mathcal{D}$ is determined by $C \subseteq E_B$).

Lemma 1.2. *C does not contain elements a, b such that $p'_B a < p'_B b$; hence $0 \notin p'_B C$.*

Proof. If $a,b \in C$ and $p'_B a < p'_B b$, then $p'_B a \; \mathcal{C}_B \; p'_B b = p'_B a + t$ for some $t \in B'$, $t > 0$; hence

$$p'_B a \;\; \mathcal{C}_B \;\; p'_B a + t \;\; \mathcal{C}_B \;\; p'_B a + 2t \;\; \mathcal{C}_B \;\; \cdots \;\; \mathcal{C}_B \;\; p'_B a + kt$$

for all $k > 0$ and $p'_B C \subseteq H_B$ contains $p'_B a + kt$ for all $k > 0$. This contradicts the finiteness of H_B (Lemma X.4.1).

Assume $0 \in p'_B C$. Since $p'_B C$ is nontrivial, C contains elements a, b such that $p'_B a = 0 \neq p'_B b$; then $p'_B a < p'_B b$, which we just saw is impossible. □

Lemma 1.3. *The element c is not idempotent and is not irreducible; hence $C \cap X = \varnothing$.*

Proof. If c is idempotent, then c, $2c \in C$ with either $p'_B c < p'_B(2c)$ or $p'_B c = 0$, which is impossible by Lemma 1.2.

Since $\pi : F \longrightarrow S$ is the standard presentation, c is irreducible if and only if $c = \pi y$ for some $y \in X$, if and only if $C \cap X \neq \varnothing$. Assume that C contains some $y \in X$. Then $y \notin B$, otherwise $0 = p'_B y \in p'_B C$. Since $p'_B C$ is nontrivial there is some $a = \sum_{x \in X} a_x x \in C$ with $p'_B a \neq y$. By Lemma 1.2 we cannot have $y = p'_B y < p'_B a$; therefore $y \nleq p'_B a$, $y \nleq a$, $a_y = 0$, and $a = \sum_{x \in X,\, x \neq y} a_x x$. Also $a \neq 0$ by Lemma 1.2. Since π is injective on X this makes $c = \pi a$ a product of irreducible elements $\pi x \neq \pi y$. If $|a| = \sum_{x \in X} a_x > 1$, then $c = \pi y$ is not irreducible; otherwise $|a| = 1$, $a = x \neq y$, and π is not injective on X; this is the required contradiction. □

3. We now call upon the overpath method. Let $\preccurlyeq$ be any total order on X in which $X\backslash B$ precedes $X \cap B$ ($x \prec y$ for all $x \in X\backslash B$ and $y \in X \cap B$). Order $G = G_X$ lexicographically: let $\sum_{x \in X} a_x x \sqsubset \sum_{x \in X} b_x x$ if and only if there exists $t \in X$ such that $a_x = b_x$ for all $x \prec t$ but $a_t < b_t$. (Then $x \sqsubset y$ in G if and only if $x \succ y$ in X.) Then $\sqsubseteq$ is a compatible total order on G, and induces a compatible well order on F. Since $X\backslash B$ precedes $X \cap B$, $p'_B a \sqsubset p'_B b$ implies $a \sqsubset b$.

Let $\epsilon(c)$ be the least idempotent $e \geqq_{\mathcal{H}} c$ of S.

Lemma 1.4. *If $m \in M$ and $m \leqq a \in C$, then either $p'_B m = p'_B q(m)$, or $m \in C$ and $p'_B m \sqsupset p'_B q(m)$. If $m \in M$ and $m \leqq a \in F$, where $c < \pi a \leqq \epsilon(c)$ in S, then $p'_B m = p'_B q(m)$.*

Proof. In either case $\epsilon(c) = e(\pi a)$ and $a \in E_B$ (Proposition X.3.4). Assume $p'_B m \neq p'_B q(m)$. Then $p'_B m \sqsupset p'_B q(m)$ (otherwise $m \sqsubset q(m)$). Since $m \leqq a \in E_B$, we have $m \in E_D$ for some $D \in \mathfrak{D}$, $D \subseteq B$ by (E2). Then $X\backslash B \subseteq X\backslash D$; since $X\backslash B$ precedes $X \cap B$, $p'_B m \sqsupset p'_B q(m)$ implies $p'_D m \sqsupset p'_D q(m)$. Thus the $\mathcal{C}$-class $C_m \subseteq E_D$ has a nontrivial projection $p'_D C$. Since $\pi m \geqq c$ the choice of c implies $\pi m = c$, and $m \in C$. Then $c \not< \pi a$, since $\pi a \leqq \pi m$. □

In what follows

$$\begin{aligned} M_B &= \{m \in M \mid p'_B m = p'_B q(m)\} \quad \text{and} \\ M_C &= \{m \in M \mid m \in C \text{ and } p'_B m \sqsupset p'_B q(m)\}. \end{aligned}$$

By Lemma 1.4, when $m \in M$ and $m \leqq a$, then $m \in M_B \cup M_C$ if $a \in C$, and $m \in M_B$ if $c < \pi a \leqq \epsilon(c)$.

Lemma 1.5. *Let p be an overpath from $a \in C$ to b. If $p'_B a = p'_B b$, then p consists solely of elements of M_B. If $p'_B a \neq p'_B b$, then p consists of elements of M_B and one element m of M_C such that $p'_B v(m) = p'_B(a - b)$.*

Proof. Let $p : m^1, \dots, m^k$ be an overpath from $a \in C$ to b and

$$a = p^0 \xrightarrow{m^1} p^1 \xrightarrow{m^2} \dots \xrightarrow{m^k} p^k = b$$

be the corresponding path, so that $p^{i-1} \geqq m^i$ and $p^i - p^{i-1} = v(m^i)$ for all $i > 0$. Since $a \in C$, then $m^i \leqq p^{i-1} \in C$ and $m^i \in M_B \cup M_C$ for all i, by Lemma 1.4. Also

$$a - b = \textstyle\sum_i v(m^i);$$

since $p'_B v(m^i) = 0$ when $m^i \in M_B$,

$$p'_B a - p'_B b = \textstyle\sum_{m^i \in M_C} p'_B v(m^i),$$

with $p'_B v(m^i) \sqsupset 0$ in G since $p'_B m^i \sqsupset p'_B q(m^i)$ for all $m^i \in M_C$.

If $p'_B a = p'_B b$, then $0 = \sum_{m^i \in M_C} p'_B v(m^i)$ is a sum of positive elements of G (ordered by $\sqsubseteq$), which is not possible unless the sum is empty; hence p consists solely of elements of M_B.

If $p'_B a \neq p'_B b$, then $\sum_{m^i \in M_C} p'_B v(m^i) \neq 0$ and there is some $m^i \in M_C$. Let j be the least i such that $m^i \in M_C$. For all $i < j$ we have $m^i \in M_B$ for all $i < j$ and $p'_B p^i - p'_B p^{i-1} = p'_B v(m^i) = 0$. Therefore $p'_B a = p'_B p^0 = p'_B p^{j-1} \leqq p'_B m^j$. Since $m^j \in C$ this implies $p'_B a = p'_B m^j$, by Lemma 1.2. Also $q(m^j) = q(a) \sqsubseteq b$, so that $p'_B q(m^j) \sqsubseteq p'_B b$ (otherwise $q(m^j) \sqsupset b$) and $p'_B q(m^j) = p'_B b$ by Lemma 1.2. (In fact, $p'_B q(m^j) = p'_B q(a)$.) Hence $p'_B a - p'_B b = p'_B v(m^j)$ and

$$\sum_{m^i \in M_C,\, m^i \neq m^j} p'_B v(m^i) \;=\; 0.$$

As before, this sum must be empty. Hence m^j is the only element of M_C which appears in p. □

Corollary 1.6. $M_C \neq \varnothing$.

Proof. $p'_B a \neq p'_B q(a)$ for some $a \in C$, since $p'_B C$ is nontrivial; there exists an overpath from a to $q(a)$, which by Lemma 1.5 includes some $m \in M_C$. □

4. Now let $\mathbb{G}$ be the thin selective functor in which G_c is any finite abelian group and $G_s = 0$ for all $s \neq c$. Then $\mathbb{G}$ is finite, but not surjecting since $\operatorname{Im} \gamma_c^{\epsilon(c)} = 0$. By Lemma 1.3, $G_s = 0$ when s is idempotent and when s is irreducible. Hence $\prod_{x \in X} G_{\pi x} = 0$ and $MB^1(S, \mathbb{G}) = 0$. We use Lemma 1.5 to construct nontrivial minimal cocycles.

Lemma 1.7. *Given $g_x \in G_c$ for every $x \in X \backslash B$ let*

$$u_m \;=\; \begin{cases} 0 \in G_{\pi m} & \text{if } m \notin M_C, \\ \sum_{x \in X \backslash B} v(m)_x g_x \in G_c & \text{if } m \in M_C. \end{cases}$$

Then u is a minimal cocycle.

Proof. Let $a \in F$ and p be an overpath from a to b. We show that

$$u_{a;p;b} \;=\; \sum_{m \in M,\, \pi m \geqq \pi a} p_m \, \gamma_{\pi a}^{\pi m} u_m \;\in\; G_{\pi a}$$

is independent of path (where p_m is the number of appearances of m in p).

If $a \notin C$, then $G_{\pi a} = 0$ and $u_{a;p;b} = 0$. Now let $a \in C$. Since $u_m = 0$ when $m \notin M_C$ and $\pi m = \pi a$ when $m \in M_C$ we have

$$u_{a;p;b} \;=\; \sum_{m \in M_C} p_m \, u_m \,.$$

We now invoke Lemma 1.5. If $p'_B a = p'_B b$, then p consists solely of elements of M_B and $u_{a;p;b} = 0$. If $p'_B a \neq p'_B b$, then p consists of elements of M_B and one element n of M_C such that $p'_B v(n) = p'_B (a - b)$. Then $v(n)_x = a_x - b_x$

for all $x \in X\backslash B$ and

$$u_{a;p;b} = u_n = \textstyle\sum_{x\in X\backslash B} (a_x - b_x)\, g_x .$$

In either case $u_{a;p;b}$ depends only on a and b. □

5. Since $M_C \neq \emptyset$ (Corollary 1.6) it is possible to choose the finite abelian group G_c and $g_x \in G_c$ so that $u_m \neq 0$ for some $m \in M_C$. (For instance take any $n \in M_C$; then $v(n)_x \neq 0$ for some $x \in X\backslash B$; let G_c be cyclic of order p, where p does not divide $v(n)_x$, and let $g_x \neq 0$, $g_y = 0$ for all $y \neq x$; then $u_n = v(n)_x\, g_x \neq 0$.) Then $MZ^1(S,\mathbb{G}) \neq 0$; since we saw that $MB^1(S,\mathbb{G}) = 0$ it follows that $H^2(S,\mathbb{G}) \neq 0$, and we have proved that $H^2(S,\mathbb{G}) \neq 0$ for some $\mathbb{G}$ if S is not partially free. Since $\mathbb{G}$ is thin finite and selective we have in fact proved:

Theorem 1.8. *For a finite group-free commutative semigroup S the following conditions are equivalent:*

(1) $H^2(S,\mathbb{G}) = 0$ *for all* $\mathbb{G}$*;*

(2) $H^2(S,\mathbb{G}) = 0$ *whenever* $\mathbb{G}$ *is thin, finite, and selective;*

(3) S *is a semilattice.*

2. THE ZERO GROUP.

In this section we assume that S has a zero element; for instance, that S is finite group-free. We study how $H^2(S,\mathbb{G})$ depends on the zero group G_0. This yields necessary conditions that $H^2(S,\mathbb{G})$ vanish when $\mathbb{G}$ is Schützenberger.

1. When S has a zero element, an abelian group valued functor $\mathbb{G}$ on $H(S)$ is **almost null** if $G_a = 0$ for all $a \neq 0$ and **reduced** if $G_0 = 0$.

When $\mathbb{G}$ is thin and almost null, Proposition V.4.7 provides isomorphisms $H^2(S,\mathbb{G}) \cong \mathrm{PHom}(S\backslash 0, G_0) \cong \mathrm{Hom}(G(S\backslash 0), G_0)$; the partial homomorphism φ which corresponds to the cohomology class cls s of $s \in SZ^2(S,\mathbb{G})$ sends $a \in S\backslash 0$ to $\varphi(a) = s_{a,0}$.

Proposition 2.1. *Let S have a zero element. For every abelian group valued functor $\mathbb{G}$ on $H(S)$ there is a short exact sequence*

$$0 \longrightarrow \mathbb{G}' \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}'' \longrightarrow 0$$

which is natural in $\mathbb{G}$, *in which* $\mathbb{G}'$ *is almost null and* $\mathbb{G}''$ *is reduced. If* $\mathbb{G}$ *is thin, then* $\mathbb{G}'$ *and* $\mathbb{G}''$ *are thin. If* $\mathbb{G}$ *is finite, then* $\mathbb{G}'$ *and* $\mathbb{G}''$ *are finite. If* S *is complete group-free and* $\mathbb{G}$ *is thin and surjecting, then* $\mathbb{G}'$ *and* $\mathbb{G}''$ *are thin and surjecting.*

Proof. $\mathbb{G}' = (G',\gamma')$ and $\mathbb{G}'' = (G'',\gamma'')$ are defined as follows. Let $G'_a = 0$ for all $a \neq 0$ and $G'_0 = G_0$; let $\gamma'_{a,t} = 0$ if $a \neq 0$ and $\gamma'_{0,t} = \gamma_{0,t}$. Then $\mathbb{G}'$ is almost null, and $\mathbb{G}'$ is thin if $\mathbb{G}$ is thin (so that $\gamma_{0,t}$ is the identity on G_0 for all t), and is finite if $\mathbb{G}$ is finite. If S is complete group-free and $\mathbb{G}$ is thin and surjecting, then $\mathbb{G}'$ is thin and surjecting.

Let $G''_a = G_a$ for all $a \neq 0$ and $G''_0 = 0$; let $\gamma''_{a,t} = 0$ if $at = 0$, $\gamma''_{a,t} = \gamma_{a,t}$ if $at \neq 0$. Then $\mathbb{G}''$ is reduced, and is thin (finite, surjecting) if $\mathbb{G}$ is thin (finite, surjecting).

The exact sequence $\mathbb{G}' \xrightarrow{\alpha} \mathbb{G} \xrightarrow{\beta} \mathbb{G}''$ is defined as follows: if $a \neq 0$, then $\alpha_a = 0$ and $\beta_a = 1_{G_a}$; $\alpha_0 = 1_{G_0}$ and $\beta_0 = 0$. The following diagrams commute:

$$\begin{tikzcd}
G'_a = 0 \arrow[r] \arrow[d, "\gamma'_{a,t}"'] & G_a \arrow[r, equal] \arrow[d, "\gamma_{a,t}"'] & G''_a \arrow[d, "\gamma''_{a,t}"] \\
G'_{at} = 0 \arrow[r] & G_{at} \arrow[r, equal] & G''_{at}
\end{tikzcd}$$

whenever $a, at \neq 0$;

$$\begin{tikzcd}
G'_a = 0 \arrow[r] \arrow[d, "\gamma'_{a,t}"'] & G_a \arrow[r, equal] \arrow[d, "\gamma_{a,t}"'] & G''_a \arrow[d, "\gamma''_{a,t}"] \\
G'_{at} \arrow[r, equal] & G_0 \arrow[r] & 0 = G''_0
\end{tikzcd}$$

whenever $a \neq 0$ and $at = 0$; and

$$\begin{tikzcd}
G'_0 \arrow[r, equal] \arrow[d, "\gamma'_{0,t}"'] & G_0 \arrow[r] \arrow[d, "\gamma_{0,t}"'] & 0 = G''_0 \arrow[d, "\gamma''_{0,t}"] \\
G'_0 \arrow[r, equal] & G_0 \arrow[r] & 0 = G''_0 .
\end{tikzcd}$$

Thus α and β are natural transformations. Naturality in $\mathbb{G}$ is similar. □

2. By Theorem XII.4.5 there is an exact sequence

$$H^2(S,\mathbb{G}') \longrightarrow H^2(S,\mathbb{G}) \longrightarrow H^2(S,\mathbb{G}'')$$

which is natural in $\mathbb{G}$, in which the homomorphisms are induced by α and β.

Proposition 2.2. *If* S *has a zero element, then* $H^2(S,\mathbb{G}) = 0$ *whenever* $\mathbb{G}$ *is thin and surjecting if and only if*

(N) $H^2(S,\mathbb{G}) = 0$ *whenever* $\mathbb{G}$ *is thin and almost null, and*

(R) $H^2(S,\mathbb{G}) = 0$ *whenever* $\mathbb{G}$ *is thin, surjecting, and reduced;*

also, $H^2(S,\mathbb{G}) = 0$ *whenever* $\mathbb{G}$ *is thin finite and surjecting if and only if*

(Nf) $H^2(S,\mathbb{G}) = 0$ *whenever* $\mathbb{G}$ *is thin finite and almost null, and*

(Rf) $H^2(S,\mathbb{G}) = 0$ *whenever* $\mathbb{G}$ *is thin finite surjecting and reduced.*

Proof. In the exact sequence

$$0 \longrightarrow \mathbb{G}' \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}'' \longrightarrow 0$$

in Proposition 2.1, if $\mathbb{G}$ is thin (finite, surjecting), then $\mathbb{G}'$ and $\mathbb{G}''$ are thin (finite, surjecting). If therefore $H^2(S,\mathbb{G}) = 0$ whenever $\mathbb{G}$ is thin and surjecting, then (N) and (R) hold; if $H^2(S,\mathbb{G}) = 0$ whenever $\mathbb{G}$ is thin finite and surjecting, then (Nf) and (Rf) hold.

If conversely (N) and (R) hold, and $\mathbb{G}$ is thin and surjecting, then $H^2(S,\mathbb{G}') = 0$, $H^2(S,\mathbb{G}'') = 0$, and the exact sequence

$$H^2(S,\mathbb{G}') \longrightarrow H^2(S,\mathbb{G}) \longrightarrow H^2(S,\mathbb{G}'')$$

shows that $H^2(S,\mathbb{G}) = 0$. Conditions (Nf) and (Rf) similarly imply $H^2(S,\mathbb{G}) = 0$ whenever $\mathbb{G}$ is thin finite and surjecting. □

Conditions (N) and (Nf) are easily settled when S is finite. Let $\pi : F = F_X \longrightarrow S$ be a presentation of S, $\mathcal{C}$ be the congruence induced by π, and $Z = \pi^{-1}0$ be the zero class. In the following result, K is the subgroup of G generated by all differences $a - b$ with $a \mathrel{\mathcal{C}} b$ and $a,b \notin Z$; relative to any compatible well order on F, K is also generated by all defining vectors $v(m)$ with $m \notin Z$.

Proposition 2.3. *Let* S *be a finite commutative semigroup with a zero element. The following conditions on* S *are equivalent:*

(N) $H^2(S,\mathbb{G}) = 0$ *whenever* $\mathbb{G}$ *is thin and almost null;*

(Nf) $H^2(S,\mathbb{G}) = 0$ *whenever* $\mathbb{G}$ *is thin finite and almost null;*

(K) *there is a presentation of* S *in which* $K = G$*;*

(K+) $K = G$ *in every presentation of* S.

Proof. By Proposition V.4.7, $H^2(S,\mathbb{G}) \cong \operatorname{Hom}(G(S\backslash 0), G_0)$ whenever $\mathbb{G}$

is thin and almost null. Now $G(S\backslash 0) \cong G/K$ by Proposition XIII.4.2, in any presentation of S. Hence (K) implies (N), which in turn implies (Nf). If on the other hand S is finite and $G/K \neq 0$, then $G/K \cong G(S\backslash 0)$ is finitely generated, there exists a finite abelian group A such that $\mathrm{Hom}(G/K,\ A) \neq 0$, and (Nf) does not hold; therefore (Nf) implies (K+). □

3. NILMONOIDS.

In this section, we characterize finite commutative nilmonoids S such that $H^2(S, \mathbb{G}) = 0$ whenever $\mathbb{G}$ is thin finite and surjecting; equivalently, every elementary semigroup T such that $T/\mathcal{H} \cong S$ splits as a coextension of S. This result is due to the author [2001Z].

1. To explore nilmonoids we use certain simple coefficient functors. When J is an ideal of a commutative semigroup S, an abelian group valued functor $\mathbb{G} = (G, \gamma)$ on $H(S)$ is **semiconstant over** J when there is an abelian group A such that

$$G_s = A \text{ for all } s \in S\backslash J,\ G_s = 0 \text{ for all } s \in J,$$

$\gamma_{s,t} = 1_A$ if $st \notin J$, and $\gamma_{s,t} = 0$ if $st \in J$. If $J = \varnothing$, then $\mathbb{G}$ is constant. If S has a zero element, then almost constant functors are semiconstant over 0; conversely one can view a semiconstant functor $\mathbb{G}$ on S over an ideal $J \neq \varnothing$ as an almost constant functor on S/J. In general, a semiconstant functor is thin, surjecting, and (if S has a zero element and $J \neq \varnothing$) reduced.

Let $\pi : F \longrightarrow S$ be any presentation of S, $\mathcal{C} = \ker \pi$ be the congruence induced by π, and $\sqsubseteq$ be a compatible well order on F. Let J be an ideal of S. Let M_J and X_J be the sets

$$M_J = \{m \in M \mid \pi m \notin J\}, \quad X_J = \{x \in X \mid \pi x \notin J\}.$$

Lemma 3.1. *When $\mathbb{G}$ is semiconstant over J, then:*

(1) *a minimal cochain u is determined by its values $(u_m)_{m \in M_J}$ on M_J;*

(2) *u is a minimal cocycle if and only if*

$$\textstyle\sum_{m \in M_J} r_m\, u_m = \sum_{m \in M_J} s_m\, u_m$$

for every positive relation $\sum_{m \in M_J} r_m\, v(m) = \sum_{m \in M_J} s_m\, v(m)$ which is realized in a $\mathcal{C}$-class $C \subsetneqq \pi^{-1} J$;

(3) *u is a minimal coboundary if and only if there exists $g = (g_x)_{x \in X_J}$ such that $g_x \in A$ for all $x \in X_J$ and*

$$u_m = \textstyle\sum_{x \in X_J} v(m)_x \, g_x$$

for all $m \in M_J$;

(4) *when u is a minimal coboundary, then*

$$\textstyle\sum_{m \in M_J} r_m \, u_m = 0$$

for every vector relation $\sum_{m \in M_J} r_m \, v(m) = 0$ which holds in G.

Proof. (1) is clear since $u_m = 0$ whenever $\pi m \notin J$.

(2) follows from Proposition XIII.4.6: u is a minimal cocycle if and only if

$$Z(a,r,s) : \textstyle\sum_{m \in M_a} r_m \, \gamma_{\pi a}^{\pi m} u_m = \sum_{m \in M_a} s_m \, \gamma_{\pi a}^{\pi m} u_m$$

whenever $a \in S$ and the positive relation $\sum_{m \in M} r_m \, v(m) = \sum_{m \in M} s_m \, v(m)$ is realized in C_a, where $M_a = \{ m \in M \mid \pi m \geqq_{\mathcal{H}} \pi a \}$. If $\pi a \in J$, then $Z(a,r,s)$ is trivial. If $\pi a \notin J$, then $M_a \subseteq M_J$ and $Z(a,r,s)$ is equivalent to

$$\textstyle\sum_{m \in M_J} r_m \, u_m = \sum_{m \in M_J} s_m \, u_m$$

since $r_m > 0$ implies $\pi m \geqq_{\mathcal{H}} \pi a$ and similarly for $s_m > 0$.

(3) follows from the definition of δg: when $g = (g_x)_{x \in X}$, then $u = \delta g$ is given by

$$u_m = \textstyle\sum_{x \in X,\, x \leqq m} m_x \, \gamma_{\pi m}^{\pi x} g_x - \sum_{x \in X,\, x \leqq q(m)} q(m)_x \, \gamma_{\pi m}^{\pi x} g_x$$

for all $m \in M$. Again $u_m = 0$ if $m \notin M_J$. If $m \in M_J$, then $x \leqq m$ implies $x \in X_J$, and so does $x \leqq q(m)$, and

$$\begin{aligned} u_m &= \textstyle\sum_{x \in X_J,\, x \leqq m} m_x \, g_x - \sum_{x \in X_J,\, x \leqq q(m)} q(m)_x \, g_x \\ &= \textstyle\sum_{x \in X_J} (m_x - q(m)_x) \, g_x = \sum_{x \in X_J} v(m)_x \, g_x , \end{aligned}$$

since $m_x > 0$ implies $x \leqq m$ and similarly for $q(m)_x > 0$.

(4) follows from (3). Assume that $\sum_{m \in M_J} r_m \, v(m) = 0$ holds in G. Then $\sum_{m \in M_J} r_m \, v(m)_x = 0$ for every $x \in X$. If u is a minimal coboundary, then

$$\textstyle\sum_{m \in M_J} r_m \, u_m = \sum_{m \in M_J} \sum_{x \in X_J} r_m \, v(m)_x \, g_x = 0. \ \square$$

2. As a consequence of Lemma 3.1, we prove:

Lemma 3.2. *Let J be a nonempty ideal of S which contains every element $s \in S$ such that a nontrivial positive relation is realized in $C_s = \pi^{-1} s$. If* (Rf) *holds, then the defining vectors* $v(m)$ *with* $m \in M_J$ *are distinct and linearly independent.*

Proof. Let A be any finite abelian group and $\mathbb{G}$ be the corresponding semi-constant functor over J, which is thin finite surjecting and reduced.

Every minimal cochain $u = (u_m)_{m \in M}$ is a minimal cocycle: if p and q are overpaths from a to b, then either p and q consist of the same elements of M, in which case $u_{a;p;b} = u_{a;q;b}$, or $\sum_{m \in M} p_m\, v(m) = \sum_{m \in M} q_m\, v(m)$ is a nontrivial relation which is realized in $C_a = C_b$, in which case $\pi a = \pi b \in J$, $G_{\pi a} = 0$, and $u_{a;p;b} = u_{a;q;b} = 0$. Hence $MZ^1(S, \mathbb{G}) = MC^1(S, \mathbb{G}) \cong \prod_{m \in M_J} A$.

If there is a nontrivial vector relation $r : \sum_{m \in M_J} r_m\, v(m) = 0$ between the vectors $v(m)$ with $m \in M_J$, then

$$\textstyle\sum_{m \in M_J} r_m\, u_m \;=\; 0$$

for every minimal coboundary u, by Lemma 3.1. If A is a cyclic group of suitable prime order p, then p does not divide every nonzero r_m and there is a minimal cochain u such that $\sum_{m \in M_J} r_m u_m \neq 0$; for instance, let $u_m \neq 0$, where p does not divide r_m, and $u_n = 0$ for all $n \neq m$). Then u is a minimal cocycle but not a minimal coboundary, $H^2(S, \mathbb{G}) \cong MZ^1(S, \mathbb{G}) \,/\, MB^1(S, \mathbb{G}) \neq 0$, and (Rf) does not hold. □

3. We now let S be a finite nilmonoid and assume that F is finitely generated. A thin abelian group valued functor $\mathbb{G}$ on S is surjecting if and only every homomorphism γ_s^1 is surjective.

A vector relation $\sum_{m \in M} r_m\, v(m) = 0$ is **reachable** in a $\mathcal{C}$-class C when it follows from relations that are realized in C (when it is a linear combination with integer coefficients of relations that are realized in C). By Proposition XIII.4.5, every vector relation is reachable in some $\mathcal{C}$-class and is reachable in the zero class $Z = \pi^{-1}0$.

Let J be a nonempty ideal of S. Let K_J be the subgroup of G generated by all defining vectors $v(m)$ with $m \in M_J$. G is a finitely generated free abelian group and so is $K_J \subseteq G$. A **defining basis** of K_J (relative to J) is a subset B of M_J such that

(1) the defining vectors $v(m)$ with $m \in B$ are distinct and constitute a basis of K_J, so that for every $m \in M_J \backslash B$ there is a unique vector relation $v(m) = \sum_{n \in B} r_n\, v(n)$ (with integer coefficients); and

(2) for every $m \in M_J \backslash B$ the relation $v(m) = \sum_{n \in B} r_n \, v(n)$ is reachable in C_m.

If $m \notin B$ and the relation $v(m) = \sum_{n \in B} r_n \, v(n)$ is reachable in a $\mathcal{C}$-class $C_s = \pi^{-1} s$, then some vector relation containing m is realized in C_s and $\pi m \geqq_{\mathcal{H}} s$; thus (2) states that the relation $v(m) = \sum_{n \in B} r_n \, v(n)$ is reachable in the highest possible $\mathcal{C}$-class.

Lemma 3.2 shows that in some cases M_J itself is a defining basis of K_J. Our main lemma is:

Lemma 3.3. *Let S be a finite nilmonoid and J be a nonempty ideal of S. If* (Rf) *holds, then K_J has a defining basis.*

Proof. We assume (Rf) and proceed by induction on $|S \backslash J|$. Let J_0 be the set of all $s \in S$ such that a nontrivial positive relation is realized in $C_s = \pi^{-1} s$. Then J_0 is an ideal of S, since a relation which is realized in C_s is realized in every C_{st}. By Lemma 3.2, K_J has a defining basis whenever J contains J_0. This kickstarts the induction.

For the general case we expand $S \backslash J$ from the bottom, which matches what writing this book is doing to the author. Let J be a nonempty ideal of S. Assume that K_J has a defining basis B (relative to J) and that $J \neq 0$. Let s be a maximal element of J (under $\leqq_{\mathcal{H}}$), so that $J' = J \backslash \{s\}$ is an ideal of S and s is a minimal element of $S \backslash J' = (S \backslash J) \cup \{s\}$. We construct a defining basis of $K_{J'}$.

We have $M_{J'} = M_J \cup M_s$, where

$$M_s = M \cap C_s = \{ m \in M \mid \pi m = s \}.$$

Since S is a nilmonoid, the $\mathcal{C}$-class C_s, which is not the zero class, cannot contain comparable elements $a < b$. Therefore an overpath $p : m^1, \ldots, m^k$ from $a \in C_s$ to $b \in C_s$ contains at most one element of M_s which must be its last element m^k: if $a = p^0, \ldots, p^k = b$ is the corresponding path and $m^j \in M_s$, then $p^{j-1} \leq m^j$ implies $p^{j-1} = m^j$ in C_s, and then $p^j = q(m^j)$, so that the path $p^0, \ldots, p^j$ has reached the least element of C_s and continues no further.

Therefore the positive relations which are realized in C_s contain at most two elements of M_s and are of three types:

(a) relations $\sum_{n \in M_J} r_n \, v(n) = \sum_{m \in M_J} s_m \, v(m)$ containing no element of M_s;

(b) relations $v(m_1) + \sum_{n\in M_J} r_n\, v(n) = \sum_{m\in M_J} s_m\, v(m)$ containing one element m_1 of M_s, with coefficient 1;

(c) relations $v(m_1) + \sum_{n\in M_J} r_n\, v(n) = v(m_2) + \sum_{m\in M_J} s_m\, v(m)$ containing two elements $m_1 \neq m_2$ of M_s, with coefficients 1.

From $B \cup M_s$ we extract a defining basis $B \cup D$ of $K_{J'}$. Starting from $D = M_s$ we trim D, one element at a time, as follows. If $m_1 \in D$ appears in a relation $v(m_1) = \sum_{n\in M_J} r_n\, v(n)$ of type (b), then remove m_1 from D and replace $v(m_1)$ by $\sum_{n\in M_J} r_n\, v(n)$ in every other relation (of type (b) or (c)) in which $v(m_1)$ appears; this yields relations of type (a) or (b) which are reachable in C_s. If $m_1 \in D$ appears in a relation $v(m_1) = v(m_2) + \sum_{n\in M_J} r_n\, v(n)$ of type (c) (with $m_1 \neq m_2$), then remove m_1 from D and replace $v(m_1)$ by $v(m_2) + \sum_{n\in M_J} r_n\, v(n)$ in every other relation (of type (b) or (c)) in which $v(m_1)$ appears; this yields relations of type (b) or (c) which are reachable in C_s. This process terminates since M_s is finite. Then all relations of type (b) or (c) have been used and D has the following properties:

(A) no relation $v(m_1) + \sum_{n\in M_J} r_n\, v(n) = \sum_{m\in M_J} s_m\, v(m)$ with $m_1 \in D$, or $v(m_1) + \sum_{n\in M_J} r_n\, v(n) = v(m_2) + \sum_{n\in M_J} s_n\, v(n)$ with $m_1, m_2 \in D$ and $m_1 \neq m_2$, can be realized in C_s;

(B) for every $m \in M_s \backslash D$ there is a relation $v(m) = \sum_{n\in M_J \cup D} r_n\, v(n)$ which is reachable in C_s; in particular,

(C) the defining vectors $v(m)$ with $m \in M_J \cup D$ generate $K_{J'}$.

These properties imply:

(A*) If $\pi a \notin J'$, then no nontrivial relation

$$R:\ \sum_{m\in B\cup D} r_m\, v(m) \;=\; \sum_{m\in B\cup D} s_m\, v(m)$$

can be realized in C_a. Indeed assume that R can be realized in C_a. If $r_m \neq s_m$ for some $m \in D$, then $\pi m \geqq_{\mathcal{H}} \pi a$, $\pi a = s$ since s is minimal in $S\backslash J'$, the given relation R is realized in C_s, and R is of the form $v(m_1) + \sum_{n\in M_J} r_n\, v(n) = \sum_{n\in M_J} s_n\, v(n)$ with $m_1 \in D$, or of the form $v(m_1) + \sum_{n\in M_J} r_n\, v(n) = v(m_2) + \sum_{n\in M_J} s_n\, v(n)$ with $m_1, m_2 \in D$ and $m_1 \neq m_2$, which contradicts (A). Therefore $r_m = s_m$ for all $m \in D$; then $r_m = s_m$ for all $n \in B \cup D$, since the vectors $v(n)$ with $n \in B$ are distinct and linearly independent, and R is trivial.

(B*) for every $m \in M_{J'} \backslash (B \cup D)$ there is a relation $v(m) = \sum_{n\in B\cup D}$

$r_n\, v(n)$ which is reachable in C_m. This follows from (B) if $m \in M_s$; if $m \in M_J$ there is a relation $v(m) = \sum_{n\in B} r_n\, v(n)$ which is reachable in C_m since B is a defining basis of K_J.

(C*) the defining vectors $v(m)$ with $m \in B \cup D$ generate $K_{J'}$.

We show that the defining vectors $v(m)$ with $m \in B \cup D$ are distinct and linearly independent; then (B*) and (C*) show that $B \cup D$ is a defining basis of $K_{J'}$ (relative to J').

As in the proof of Lemma 3.2, let A be any finite abelian group and $\mathbb{G}$ be the corresponding semiconstant functor over J', which is thin finite surjecting and reduced.

By Lemma 3.1, a minimal cochain $u = (u_m)_{m\in M}$ is determined by its values $(u_m)_{m\in M_{J'}}$ on $M_{J'}$, and $u = (u_m)_{m\in M}$ is a minimal cocycle if and only if $\sum_{m\in M_{J'}} r_m\, u_m = \sum_{m\in M_{J'}} s_m\, u_m$ whenever the positive relation $\sum_{m\in M_{J'}} r_m\, v(m) = \sum_{m\in M_{J'}} s_m\, v(m)$ is realized in a $\mathcal{C}$-class $C \not\subseteq \pi^{-1}J'$. Hence u is a minimal cocycle if and only if $\sum_{m\in M_{J'}} r_m\, u_m = 0$ whenever the vector relation $\sum_{m\in M_{J'}} r_m\, v(m) = 0$ is reachable in a $\mathcal{C}$-class $C \not\subseteq \pi^{-1}J'$.

For every $m \in M_{J'}$ there is by (B*) a relation $v(m) = \sum_{n\in B\cup D} r_n\, v(n)$ which is reachable in C_m. Hence every minimal cocycle u satisfies $u_m = \sum_{n\in B\cup D} r_n\, u_n$. Therefore a minimal cocycle u is uniquely determined by its values $(u_n)_{n\in B\cup D}$ on $B \cup D$, which can be chosen arbitrarily since no nontrivial vector relation $\sum_{n\in B\cup D} r_n\, v(n) = 0$ can be realized in any $\mathcal{C}$-class $C \not\subseteq \pi^{-1}J'$, by (A*). Thus $MZ^1(S,\mathbb{G}) \cong \prod_{n\in B\cup D} A$ has $|A|^{|B\cup D|}$ elements.

If there is a nontrivial vector relation $r : \sum_{n\in B\cup D} r_n\, v(n) = 0$ between the vectors $v(n)$ with $n \in B \cup D$, then

$$\sum_{n\in B\cup D} r_n\, u_n \;=\; 0$$

holds for every minimal coboundary u, by Lemma 2.3. If A is a cyclic group of suitable prime order p, then p does not divide every r_n and there is a minimal cocycle u such that $\sum_{n\in B\cup D} r_n\, u_n \neq 0$; for instance, let $u_m \neq 0$, where p does not divide r_m, and $u_n = 0$ for all $n \neq m$, $n \in B \cup D$. Then u is not a minimal coboundary and $H^2(S,\mathbb{G}) = MZ^1(S,\mathbb{G}) \,/\, MB^1(S,\mathbb{G}) \neq 0$. If therefore (Rf) holds, then there can be no nontrivial relation $\sum_{n\in B\cup D} r_n\, v(n) = 0$. Hence $B \cup D$ is a defining basis of $K_{J'}$. □

4. We can now prove:

Theorem 3.4. *For a finite commutative nilmonoid S the following conditions are equivalent:*

(1) $H^2(S,\mathbb{G}) = 0$ *whenever* $\mathbb{G}$ *is thin and surjecting;*

(2) $H^2(S,\mathbb{G}) = 0$ *whenever* $\mathbb{G}$ *is thin finite and surjecting;*

(3) *in some presentation* $\pi : F \longrightarrow S$ *(with F finitely generated), G has a defining basis (relative to* 0*);*

(4) *in every presentation* $\pi : F \longrightarrow S$ *(with F finitely generated), G has a defining basis (relative to* 0*).*

If for example S is the Volkov nilmonoid (Example XII.4.9), we saw in Section XIII.4 that G has a defining basis; we also saw that $H^2(S,\mathbb{G}) = 0$ whenever $\mathbb{G}$ is thin finite and surjecting.

Proof. If (2) holds, then $G = K = K_0$ by Lemma 3.1 and $G = K_0$ has a defining basis by Lemma 3.2, applied to the ideal $J = \{0\}$. Thus (2) implies (4). It remains to show that (3) implies (1).

Assume that G has a defining basis B relative to 0, in some presentation $\pi : F \longrightarrow S$ where F is finitely generated. Then $K = G$ and $H^2(S,\mathbb{G}) = 0$ whenever $\mathbb{G}$ is thin and almost null, by Proposition 2.2. Now let $\mathbb{G}$ be thin, surjecting, and reduced ($G_0 = 0$).

Since B is a defining basis of G there is for every $m \in M_0 \backslash B$ a relation

$$v(m) = \textstyle\sum_{n\in B} r_n\, v(n) = \sum_{n\in B,\, r_n\neq 0} r_n\, v(n)$$

which is reachable in C_m; in particular $\pi n \geqq_{\mathcal{H}} \pi m$ when $r_n \neq 0$. Then

$$u_m = \textstyle\sum_{n\in B,\, r_n\neq 0} r_n\, \gamma_{\pi m}^{\pi n} u_n$$

for every minimal cocycle u. Also $u_m = 0$ whenever $\pi m = 0$ since $\mathbb{G}$ is reduced. Thus a minimal cocycle is determined by its values $(u_n)_{n\in B}$ on B. (The latter can be chosen arbitrarily, as readily shown, so that $MZ^1(S,\mathbb{G}) \cong \prod_{n\in B} G_{\pi n}$.)

Since the vectors $v(n)$ with $n \in B$ constitute a basis of G, their coordinate matrix $\big(v(n)_x\big)_{n\in B,\, x\in X}$ has an inverse, which is an integer matrix $\big(t_{n,x}\big)_{n\in B,\, x\in X}$ such that

$$\textstyle\sum_{x\in X} v(m)_x\, t_{n,x} = 1 \text{ if } m = n, \enspace 0 \text{ if } m \neq n$$

for all $m,n \in B$. Let u be any minimal cocycle. Since $\mathbb{G}$ is surjecting there is for every $n \in B$ some $h_n \in G_1$ such that $u_n = \gamma_{\pi n}^1 h_m$. For every $x \in X$ let

$$g_x \;=\; \textstyle\sum_{n\in B} t_{n,x}\, \gamma^{1}_{\pi x} h_n \;\in\; G_{\pi x}\,.$$

Since $v(m)_x \neq 0$ implies $x \leqq m$ or $x \leqq q(m)$, and $\pi x \geqq_{\mathcal{H}} \pi m$, we have

$$\begin{aligned}(\delta g)_m \;&=\; \textstyle\sum_{x\in X,\, x\leqq m} m_x\, \gamma^{\pi x}_{\pi m} g_x \;-\; \sum_{x\in X,\, x\leqq q(m)} q(m)_x\, \gamma^{\pi x}_{\pi m} g_x \\ &=\; \textstyle\sum_{x\in X,\, \pi x\geqq \pi m} v(m)_x\, \gamma^{\pi x}_{\pi m} g_x \\ &=\; \textstyle\sum_{x\in X,\, \pi x\geqq \pi m} \sum_{n\in B}\; v(m)_x\, t_{n,x}\, \gamma^{1}_{\pi m} h_n \\ &=\; \textstyle\sum_{x\in X} \sum_{n\in B}\; v(m)_x\, t_{n,x}\, \gamma^{1}_{\pi m} h_n \;=\; \gamma^{1}_{\pi m} h_m \;=\; u_m\end{aligned}$$

for every $m \in B$. Since u and δg are minimal cocycles, this implies $u = \delta g$. Thus $H^2(S,\mathbb{G}) = 0$ whenever $\mathbb{G}$ is thin, surjecting, and reduced. By Proposition 2.1, $H^2(S,\mathbb{G}) = 0$ whenever $\mathbb{G}$ is thin and surjecting. □

Theorem 3.4 does not extend immediately to every finite group-free semigroup S. If indeed S has two generators and $H^2(S,\mathbb{G}) = 0$ whenever $\mathbb{G}$ is thin finite and surjecting, then S is a either a semilattice or a nilmonoid (Grillet [2000T]). Theorem XIII.5.6 also shows that a partially free semigroup S does not in general satisfy $H^2(S,\mathbb{G}) = 0$ whenever $\mathbb{G}$ is thin finite and surjecting, even though G always has a defining basis, by Lemmas XIII.5.1 and XIII.5.3.

5. We conclude this section with some examples. All examples have two generators c and d; in their standard presentation we let $X = \{x, y\}$, with $\pi x = c$, $\pi y = d$, and order F lexicographically, with $ix + jy \sqsubset kx + ly$ if and only if either $i < k$, or $i = k$ and $j < l$.

Example 3.5. Let S be the nilmonoid with the presentation

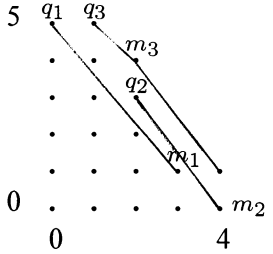
$$S \cong \langle\, c, d \;\big|\; c^3 d = d^5,\; c^4 = c^2 d^3,\; c^5 = c^3 d^2 = c^2 d^5 = d^6 = 0 \,\rangle.$$

The nontrivial $\mathcal{C}$-classes (other than the zero class) are $\{5y,\, 3x + y\}$, $\{2x + 3y,\, 4x\}$, and $\{x + 5y,\, 2x + 4y,\, 4x + y\}$. Hence M_0 consists of $m_1 = 3x + y$, $m_2 = 4x$, $m_3 = 2x + 4y$. We have $q_1 = q(m_1) = 5y$, $q_2 = q(m_2) = 2x + 3y$, $q_3 = q(m_3) = x + 5y$, and $v(m_1) = 3x - 4y$, $v(m_2) = 2x - 3y$, $v(m_3) = x - y$. Thus $v(m_1)$ and $v(m_2)$ constitute a basis of G (since $\begin{vmatrix} 3 & -4 \\ 2 & -3 \end{vmatrix} = -1$).

There are two paths from $4x + y$ to $x + 5y$:

$$4x + y \xrightarrow{m_1} x + 5y \quad\text{and}\quad 4x + y \xrightarrow{m_2} 2x + 4y \xrightarrow{m_3} x + 5y;$$

the corresponding overpaths are m_1 and m_2, m_3. Thus the relation $v(m_3) = v(m_1) - v(m_2)$ is realized in C_{m_3}, and $\{m_1, m_2\}$ is a defining basis of G. Therefore $H^2(S,\mathbb{G}) = 0$ whenever $\mathbb{G}$ is thin finite and surjecting. □

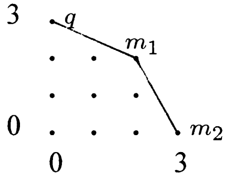


Example 3.5

Example 3.6. Let S be the nilmonoid with the presentation

$$S \cong \langle\, c,d \mid c^3 = c^2d^2 = d^3,\ c^4 = c^3d = cd^3 = d^4 = 0 \,\rangle.$$

There is one nontrivial $\mathcal{C}$-class (other than the zero class): $\{3y,\ 2x+2y,\ 3x\}$. Hence M_0 consists of $m_1 = 2x+2y$ and $m_2 = 3x$, with $q = q(m_1) = q(m_2) = 3y$ and $v(m_1) = 2x - y$ and $v(m_2) = 3x - 3y$.



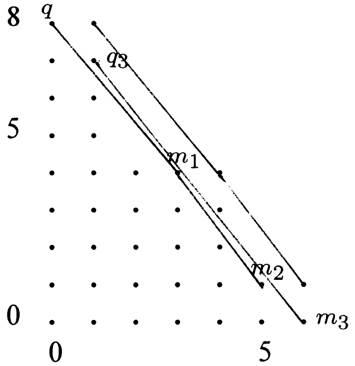
Example 3.6

The defining vectors $v(m_1)$ and $v(m_2)$ are linearly independent but do not constitute a basis of G (since $\begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} = -3$). Hence $K \neq G$ and (Nf) does not hold: there exists an almost null functor $\mathbb{G}$ such that $H^2(S,\mathbb{G}) \neq 0$. Since $G/K \cong \mathbb{Z}_3$ the almost null functor with $G_0 = \mathbb{Z}_3$ has $H^2(S,\mathbb{G}) \cong \mathrm{Hom}(\mathbb{Z}_3, \mathbb{Z}_3) \cong \mathbb{Z}_3$. However, $\{m_1, m_2\}$ is a defining basis of K. □

Example 3.7. Let S be the nilmonoid with the presentation

$$S \cong \langle\, c,d \mid c^6 = cd^7,\ c^5d = c^3d^4 = d^8,\ c^7 = c^5d^2 = c^2d^5 = d^9 = 0 \,\rangle.$$

The nontrivial $\mathcal{C}$-classes (other than the zero class) are $\{8y,\ 3x+4y,\ 5x+y\}$, $\{x+7y,\ 6x\}$, and $\{x+8y,\ 4x+4y,\ 6x+y\}$. Hence M_0 consists of $m_1 = 3x+4y$, $m_2 = 5x+y$, and $m_3 = 6x$. We have $q = q(m_1) = q(m_2) = 8y$, $q_3 = q(m_3) = x+7y$, and $v(m_1) = 3x-4y$, $v(m_2) = 5x-7y$, $v(m_3) = 5x-7y$; $v(m_1)$ and $v(m_2)$ constitute a basis of G, since $\begin{vmatrix} 3 & -4 \\ 5 & -7 \end{vmatrix} = -1$.



Example 3.7

The only $\mathcal{C}$-class with two overpaths (other than the zero class) is $C = \{x + 8y,\ 4x + 4y,\ 6x + y\}$, which does not contain m_2 or m_3. The relation $v(m_3) = v(m_2)$ is realized in C but it is not reachable in C_{m_3}. Therefore $\{m_1, m_2\}$ is not a defining basis of G. Similarly $\{m_1, m_3\}$ is not a defining basis of G. Therefore G does not have a defining basis and (Rf) does not hold; $H^2(S, \mathbb{G}) \neq 0$ for some functor $\mathbb{G}$ which is thin finite surjecting and reduced.

The proof of Lemma 3.2 provides such a functor. Let $t = c^5 d = c^3 d^4 = d^8$, so that $J_0 = \{0, t\}$ and πm_1, πm_2, $\pi m_3 \notin J_0$. Since $v(m_1)$, $v(m_2)$, $v(m_3)$ are not linearly independent, there is a finite abelian group A such that $H^2(S, \mathbb{G}) \neq 0$ when G is semiconstant with $G_0 = G_t = 0$ and $G_s = A$ for all other $s \in S$. The proof of Lemma 3.2 shows that every minimal cochain is a minimal cocycle, whereas a minimal coboundary u must satisfy $u_{m_2} = u_{m_3}$; accordingly $H^2(S, \mathbb{G}) \cong A$ and any finite abelian group $A \neq 0$ serves. □

Example 3.8. If we delete the relation $c^6 = cd^7$ from the presentation of S in Example 3.7, we obtain a nilmonoid

$$S \cong \langle\, c, d \mid c^5 d = c^3 d^4 = d^8,\ c^7 = c^5 d^2 = c^2 d^5 = d^9 = 0 \,\rangle$$

for which M_0 consists only of $m_1 = 3x + 4y$ and $m_2 = 5x + y$, and is a defining basis of G. Then $H^2(S, \mathbb{G}) = 0$ whenever $\mathbb{G}$ is thin finite and surjecting.