Chapter XII.

COMMUTATIVE SEMIGROUP COHOMOLOGY.

Commutative semigroup cohomology assigns abelian groups $H^n(S, \mathbb{G})$ to a commutative semigroup S and an abelian group valued functor $\mathbb{G}$ on S.

Other cohomology theories have been considered for commutative semigroups (see the introduction to Section 4). The theory we call commutative semigroup cohomology is of particular interest because $H^2(S, \mathbb{G})$ classifies commutative group coextensions of S by $\mathbb{G}$, that is, $H^2(S, \mathbb{G})$ coincides with the extension group $\mathrm{Ext}\,(S, \mathbb{G})$ in Chapter V; moreover, if S is complete group-free and $\mathbb{G}$ is Schützenberger, then, as we saw in Chapter V, $H^2(S, \mathbb{G})$ classifies complete commutative semigroups T with $T/\mathcal{H} \cong S$ and Schützenberger functor isomorphic to $\mathbb{G}$. These results make cohomology an important part of the structure theory of commutative semigroups.

Commutative semigroup cohomology is an instance of triple cohomology, which provides a definition in dimensions $n \geqq 3$ as well as valuable properties.

After a brief account of triple cohomology and two sections of preliminary results, this chapter defines commutative semigroup cohomology, and gives simpler definitions in low dimensions.

1. TRIPLE COHOMOLOGY.

This section gives, without proofs, the definition and main properties of triple cohomology. We follow Beck [1967] and Barr & Beck [1969] but have renumbered cohomology groups in the more traditional fashion. We assume a general knowledge of category theory and triples, from, say, MacLane [1971]; Grillet [1999] also has a short account of triples and the tripleability of varieties.

1. The minimal requirements for cohomology are: a category $\mathcal{C}$; a functor $\mathbb{V} : \mathcal{C} \longrightarrow \mathcal{C}$ (normally denoted by $\mathbb{G}$, but we use $\mathbb{G}$ for abelian group valued functors); a natural transformation $\epsilon : \mathbb{V} \longrightarrow 1_{\mathcal{C}}$; and a contravariant functor $\mathbb{A}$

from $\mathcal{C}$ to the category Ab of abelian groups and homomorphisms.

For every $0 \leqq i \leqq n$ the natural transformation $\epsilon : \mathbb{V} \longrightarrow 1_{\mathcal{C}}$ induces a natural transformation $\epsilon^{n,i} = \mathbb{V}^{n-i}\, \epsilon\, \mathbb{V}^i : \mathbb{V}^{n+1} \longrightarrow \mathbb{V}^n$. To each object C of $\mathcal{C}$ can then be assigned a complex of abelian groups

$$0 \longrightarrow \mathbb{A}\mathbb{V}C \longrightarrow \cdots \longrightarrow \mathbb{A}\mathbb{V}^n C \xrightarrow{\delta_n} \mathbb{A}\mathbb{V}^{n+1}C \longrightarrow \cdots$$

where $\delta_n = \sum_{0 \leqq i \leqq n} (-1)^i \mathbb{A}\epsilon_C^{n,i} : \mathbb{A}\mathbb{V}^n C \longrightarrow \mathbb{A}\mathbb{V}^{n+1}C$. That $\delta_{n+1} \circ \delta_n = 0$ for all $n \geqq 1$ follows by a standard argument from the equalities $\epsilon^{n,j} \circ \epsilon^{n+1,\,i} = \epsilon^{n,i} \circ \epsilon^{n+1,\,j+1}$, which hold for all $0 \leqq i \leqq j \leqq n$. The n-th cohomology group of C with coefficients in $\mathbb{A}$ is

$$H^n(C,\mathbb{A}) = \operatorname{Ker} \delta_n \,/\, \operatorname{Im} \delta_{n-1} \quad \text{if } n \geqq 2, \quad \text{and} \quad H^1(C,\mathbb{A}) = \operatorname{Ker} \delta_1.$$

(In Beck [1967], Barr & Beck [1969], these groups are H^{n-1} and H^0.)

In what follows $\mathbb{V}$ and ϵ arise from an adjunction $(\mathbb{F},\mathbb{U},\eta,\epsilon) : \mathcal{U} \longrightarrow \mathcal{C}$, where $\mathbb{F} : \mathcal{U} \longrightarrow \mathcal{C}$ is a left adjoint of $\mathbb{U} : \mathcal{C} \longrightarrow \mathcal{U}$, and $\eta : 1_{\mathcal{U}} \longrightarrow \mathbb{U}\mathbb{F}$, $\epsilon : \mathbb{F}\mathbb{U} \longrightarrow 1_{\mathcal{C}}$ are the corresponding natural transformations. Then $\mathbb{T} = \mathbb{U}\mathbb{F}$, η, and $\mu = \mathbb{U}\epsilon\mathbb{F}$ constitute a triple on $\mathcal{U}$; $\mathbb{V} = \mathbb{F}\mathbb{U}$, ϵ, and $\nu = \mathbb{F}\eta\mathbb{U}$ constitute a cotriple on $\mathcal{C}$.

If for example $\mathcal{C}$ is the category of commutative semigroups and homomorphisms, and $\mathcal{U}$ is the category of *Sets* of sets and mappings, then the free c.s. functor $\mathbb{F} : Sets \longrightarrow \mathcal{C}$ is a left adjoint of the forgetful or underlying set functor $\mathbb{U} : \mathcal{C} \longrightarrow Sets$; η_X embeds a set X into the free c.s. on X; ϵ is described by Lemma 4.1 below. Like all varieties, $\mathcal{C}$ is tripleable over *Sets*.

2. To obtain the Beck cohomology groups of an object S of $\mathcal{C}$, one applies the above to the category $\overline{\mathcal{C}} = \mathcal{C}{\downarrow}S$ of objects over S; abelian group objects of $\overline{\mathcal{C}}$ provide coefficient functors (Beck [1967]). The details are as follows.

Recall that an **object over** S in $\mathcal{C}$ is an ordered pair (C,π) of an object C of $\mathcal{C}$ and a morphism $\pi : C \longrightarrow S$. A morphism $\gamma : (C,\pi) \longrightarrow (D,\rho)$ of objects over S is a morphism $\gamma : C \longrightarrow D$ in $\mathcal{C}$ such that $\rho \circ \gamma = \pi$.

$$\begin{tikzcd} C \arrow[r, "\gamma"] \arrow[d, "\pi"'] & D \arrow[dl, "\rho"] \\ S & \end{tikzcd}$$

Every adjunction $(\mathbb{F},\mathbb{U},\eta,\epsilon) : \mathcal{U} \longrightarrow \mathcal{C}$ lifts to an adjunction $(\overline{\mathbb{F}},\overline{\mathbb{U}},\overline{\eta},\overline{\epsilon}) : \overline{\mathcal{U}} = \mathcal{U}{\downarrow}\mathbb{U}S \longrightarrow \mathcal{C}{\downarrow}S = \overline{\mathcal{C}}$; namely,

$$\overline{\mathbb{F}}(X,\pi) = (\mathbb{F}X,\overline{\pi}),\ \ \overline{\mathbb{U}}(C,\rho) = (\mathbb{U}C,\mathbb{U}\rho),\ \ \overline{\eta}_{(X,\pi)} = \eta_X,\ \ \overline{\epsilon}_{(C,\rho)} = \epsilon_C,$$

where $\overline{\pi} : \mathbb{F}X \longrightarrow S$ is the morphism such that $\mathbb{U}\overline{\pi} \circ \eta_X = \pi$. In particular, the cotriple $(\mathbb{V} = \mathbb{F}\mathbb{U},\ \epsilon,\ \nu = \mathbb{F}\eta\mathbb{U})$ induced by $(\mathbb{F},\mathbb{U},\eta,\epsilon)$ lifts to the cotriple $(\overline{\mathbb{V}} = \overline{\mathbb{F}}\overline{\mathbb{U}},\ \overline{\epsilon},\ \overline{\nu} = \overline{\mathbb{F}}\overline{\eta}\overline{\mathbb{U}})$ induced by $(\overline{\mathbb{F}},\overline{\mathbb{U}},\overline{\eta},\overline{\epsilon})$ on $\overline{\mathcal{C}} = \mathcal{C}{\downarrow}S$. By definition,

$$\overline{\mathbb{V}}(C,\pi) = \overline{\mathbb{F}}(\mathbb{U}C,\mathbb{U}\pi) = (\mathbb{V}C,\overline{\pi}),\ \text{where}\ \mathbb{U}\overline{\pi} \circ \eta_{\mathbb{U}C} = \mathbb{U}\pi;$$

$\overline{\epsilon}_{(C,\pi)} = \epsilon_C$; and $\overline{\nu}_{(C,\pi)} = \overline{\mathbb{F}}\,\overline{\eta}_{(\mathbb{U}C,\mathbb{U}\pi)} = \mathbb{F}\,\eta_{\mathbb{U}C} = \nu_C$. In particular, $\overline{\epsilon}^{n,i}_{(C,\pi)} = \epsilon^{n,i}_C$. If $\mathcal{C}$ is tripleable over $\mathcal{U}$ (if the adjunction $(\mathbb{F},\mathbb{U},\eta,\epsilon)$ is tripleable), then $\overline{\mathcal{C}}$ is tripleable over $\overline{\mathcal{U}}$ ($(\overline{\mathbb{F}},\overline{\mathbb{U}},\overline{\eta},\overline{\epsilon})$ is tripleable).

3. An **abelian group object** of a category $\mathcal{C}$ is an object G of $\mathcal{C}$ such that every $\mathrm{Hom}_{\mathcal{C}}(C,G)$ is a set, together with an abelian group operation $+$ on every set $\mathrm{Hom}_{\mathcal{C}}(C,G)$, such that $\mathbb{A} = \mathrm{Hom}_{\mathcal{C}}(-,G)$ is a (contravariant) abelian group valued functor on $\mathcal{C}$; equivalently, such that

$$(g+h)\circ\gamma \ = \ (g\circ\gamma)+(h\circ\gamma)$$

for all morphisms $g,h : D \longrightarrow G$ and $\gamma : C \longrightarrow D$ of $\mathcal{C}$.

A morphism $\varphi : G \longrightarrow G'$ of abelian group objects of $\mathcal{C}$ is a morphism of $\mathcal{C}$ such that the mapping $\mathrm{Hom}_{\mathcal{C}}(C,\varphi) : \mathrm{Hom}_{\mathcal{C}}(C,G) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C,G')$ is a homomorphism of abelian groups for every object C of $\mathcal{C}$; equivalently,

$$\varphi\circ(g+h) \ = \ (\varphi\circ g)+(\varphi\circ h)$$

for all C and $g,h : C \longrightarrow G$. Then $\mathrm{Hom}_{\mathcal{C}}(-,\varphi)$ is a natural transformation of abelian group valued functors.

When $\mathcal{C}$ has finite products, abelian group objects of $\mathcal{C}$ can also be defined by means of suitable morphisms $m : G\times G \longrightarrow G$, $e : 0 \longrightarrow G$, $i : G \longrightarrow G$, as in MacLane [1971, first edition] (see also Lemma 2.2 below).

If for example G is an abelian group in the usual sense, then every Hom_{Sets} (C,G) is an abelian group under pointwise addition, and this makes G an abelian group object of $Sets$ since $(g+h)\circ\gamma = (g\circ\gamma)+(h\circ\gamma)$ for all $g,h : C \longrightarrow G$ and $\gamma : D \longrightarrow C$. If conversely G is an abelian group object of $Sets$, then $\mathrm{Hom}_{Sets}(\{1\},G)$ is an abelian group, and the bijection $\mathrm{Hom}_{Sets}(\{1\},G) \longrightarrow G$ is readily used to make G an abelian group so that pointwise addition on every $\mathrm{Hom}_{Sets}(C,G)$ is the given operation.

In general, an **action** $\boldsymbol{.}$ of an abelian group object G of $\mathcal{C}$ on an object A of $\mathcal{C}$ assigns to every object C of $\mathcal{C}$ a group action

$$\mathrm{Hom}_{\mathcal{C}}(C,G)\times\mathrm{Hom}_{\mathcal{C}}(C,A) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C,A),\ \ (g,a) \longmapsto g\,.\,a$$

of the abelian group $\mathrm{Hom}_{\mathcal{C}}(C,G)$ on $\mathrm{Hom}_{\mathcal{C}}(C,A)$ which is natural in C; equivalently,

$$(g\,.\,a)\circ\gamma \;=\; (g\circ\gamma)\,.\,(a\circ\gamma)$$

for all $\gamma : C \longrightarrow D$, $g : D \longrightarrow G$, and $a : D \longrightarrow A$.

If for instance G is an abelian group which acts on a set X in the usual sense, then, for every set C, the abelian group $\mathrm{Hom}_{Sets}(C,G)$ acts pointwise on $\mathrm{Hom}_{Sets}(X,G)$, and this is an abelian group object action in $Sets$ since $(g\,.\,a)\circ\gamma = (g\circ\gamma)\,.\,(a\circ\gamma)$ for all $g : C \longrightarrow G$, $a : C \longrightarrow X$, and $\gamma : D \longrightarrow C$.

4. The ingredients for Beck cohomology are: an adjunction $(\mathbb{F},\mathbb{U},\eta,\epsilon) : \mathcal{U} \longrightarrow \mathcal{C}$; an object S of $\mathcal{C}$; and an abelian group object $\overline{G} = (G,\rho)$ of $\overline{\mathcal{C}} = \mathcal{C}{\downarrow}S$. For any object $\overline{T} = (T,\pi)$ of $\overline{\mathcal{C}}$, the **Beck cohomology** groups or **triple cohomology** groups of $\overline{T}$ with coefficients in $\overline{G}$ are its cohomology groups calculated from the cotriple $\overline{\mathbb{V}}$ above and coefficient functor $\mathbb{A} = \mathrm{Hom}_{\overline{\mathcal{C}}}(-,\overline{G})$. The triple cohomology groups of S are those of $(S, 1_S)$.

For a more detailed definition, let $\overline{T}_n = (T_n, \pi_n) = \overline{\mathbb{V}}^n\overline{T}$. Then $T_0 = T$, $\pi_0 = \pi$, and $(T_{n+1}, \pi_{n+1}) = \overline{\mathbb{V}}(T_n, \pi_n)$, so that $T_{n+1} = \mathbb{F}\mathbb{U}T_n$ and $\pi_{n+1} : T_{n+1} \longrightarrow S$ is the morphism such that $\mathbb{U}\pi_{n+1}\circ\eta_{\mathbb{U}T_n} = \mathbb{U}\pi_n$.

An n-**cochain** is an element of

$$C^n(\overline{T},\overline{G}) \;=\; \mathbb{A}\overline{\mathbb{V}}^n(\overline{T}) \;=\; \mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{T}_n,\overline{G}),$$

that is, a morphism $u : T_n \longrightarrow G$ of $\mathcal{C}$ such that $\rho\circ u = \pi_n$. Next, $\epsilon_{\overline{T}}^{n,i} = \epsilon_T^{n,i} : T_{n+1} \longrightarrow T_n$ and $\mathbb{A}\epsilon_T^{n,i} = \mathrm{Hom}_{\overline{\mathcal{C}}}(\epsilon_T^{n,i},\overline{G}) : C^n(\overline{T},\overline{G}) \longrightarrow C^{n+1}(\overline{T},\overline{G})$ sends $u : T_n \longrightarrow G$ to $u\circ\epsilon_T^{n,i}$. Hence

$$\delta_n \;=\; \textstyle\sum_{0\leqq i\leqq n}(-1)^i\mathbb{A}\epsilon_T^{n,i} : C^n(\overline{T},\overline{G}) \longrightarrow C^{n+1}(\overline{T},\overline{G})$$

sends $u : T_n \longrightarrow G$ to

$$\delta_n u \;=\; \textstyle\sum_{0\leqq i\leqq n}(-1)^i(u\circ\epsilon_T^{n,i}).$$

By definition, $H^n(\overline{T},\overline{G})$ is the n-th homology group of the cochain complex

$$0 \longrightarrow C^1(\overline{T},\overline{G}) \longrightarrow \dots \longrightarrow C^n(\overline{T},\overline{G}) \xrightarrow{\delta_n} C^{n+1}(\overline{T},\overline{G}) \longrightarrow \dots$$

An n-**cocycle** is an element of $Z^n(\overline{T},\overline{G}) = \mathrm{Ker}\,\delta_n \subseteq C^n(\overline{T},\overline{G})$. An n-**coboundary** is an element of $B^n(\overline{T},\overline{G}) = \mathrm{Im}\,\delta_{n-1} \subseteq Z^n(\overline{T},\overline{G})$ if $n \geqq 2$,

$B^1(\overline{T},\overline{G}) = 0$ if $n = 1$. The Beck cohomology groups of $\overline{T}$ are the groups

$$H^n(\overline{T},\overline{G}) \;=\; Z^n(\overline{T},\overline{G}) \,/\, B^n(\overline{T},\overline{G}).$$

The Beck cohomology groups of S are $H^n(S,\overline{G}) = H^n((S,1_S),\overline{G})$.

A morphism $\tau : \overline{T} \longrightarrow \overline{T}'$ in $\overline{\mathcal{C}}$ induces a homomorphism

$$\tau^n \;=\; \mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{V}}^n\tau,\overline{G}) : C^n(\overline{T}',\overline{G}) \longrightarrow C^n(\overline{T},\overline{G})$$

which sends $u \in C^n(\overline{T},\overline{G})$ to $u \circ \overline{\mathbb{V}}^n\tau$. Since composition with $\overline{\mathbb{V}}^n\tau$ preserves sums and $\epsilon_T^{n,i}$ is natural in T, we have $\tau^{n+1}(\delta_n u) = \delta_n(\tau^n u)$ for all $u \in C^n(\overline{T}',\overline{G})$; that is, $(\tau^n)_{n \geqq 1}$ is a chain transformation. Hence τ^n takes $Z^n(\overline{T}',\overline{G})$ into $Z^n(\overline{T},\overline{G})$, takes $B^n(\overline{T}',\overline{G})$ into $B^n(\overline{T},\overline{G})$, and induces a homomorphism $H^n(\tau,\overline{G}) : H^n(\overline{T}',\overline{G}) \longrightarrow H^n(\overline{T},\overline{G})$. Thus $H^n(-,\overline{G})$ is a contravariant abelian group valued functor on $\overline{\mathcal{C}}$.

Similarly, a morphism $\varphi : \overline{G} \longrightarrow \overline{G}'$ of abelian group objects induces a homomorphism $\varphi^n = \mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{T}_n,\varphi) : C^n(\overline{T},\overline{G}) \longrightarrow C^n(\overline{T},\overline{G}')$ which sends $u \in C^n(\overline{T},\overline{G})$ to $\varphi \circ u$. Again $\varphi^{n+1}(\delta_n u) = \delta_n(\varphi^n u)$ for all $u \in C^n(\overline{T},\overline{G})$, and φ induces a homomorphism $H^n(\overline{T},\varphi) : H^n(\overline{T},\overline{G}) \longrightarrow H^n(\overline{T},\overline{G}')$. Thus $H^n(\overline{T},-)$ is a functor. In fact $H^n(\overline{T},\varphi)$ is natural in $\overline{T}$, so $H^n(-,-)$ is a bifunctor.

5. The main properties of Beck cohomology are as follows.

Theorem 1.1. *When $\overline{T} = \overline{\mathbb{F}}\overline{X}$ for some object $\overline{X}$ of $\overline{\mathcal{U}}$, then $H^n(\overline{T},\overline{G}) = 0$ for all $n \geqq 2$; also $H^1(\overline{\mathbb{V}}\overline{C},\overline{G}) \cong C^1(\overline{C},\overline{G})$ for every object $\overline{C}$ of $\overline{\mathcal{C}}$.*

A sequence $\overline{G} \longrightarrow \overline{G}' \longrightarrow \overline{G}''$ of abelian group objects and morphisms of $\overline{\mathcal{C}}$ is **short $\mathbb{V}$-exact** in case

$$0 \longrightarrow \mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{V}}\overline{C},\overline{G}) \longrightarrow \mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{V}}\overline{C},\overline{G}') \longrightarrow \mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{V}}\overline{C},\overline{G}'') \longrightarrow 0$$

is a short exact sequence (in Ab) for every object $\overline{C}$ of $\overline{\mathcal{C}}$.

Theorem 1.2. *Every short $\mathbb{V}$-exact sequence $\mathcal{E} : \overline{G} \longrightarrow \overline{G}' \longrightarrow \overline{G}''$ of abelian group objects of $\mathcal{C}\downarrow S$ induces an exact sequence*

$$\cdots\; H^n(\overline{T},\overline{G}) \longrightarrow H^n(\overline{T},\overline{G}') \longrightarrow H^n(\overline{T},\overline{G}'') \longrightarrow H^{n+1}(\overline{T},\overline{G}) \;\cdots$$

which is natural in $\mathcal{E}$.

Theorems 1.1 and 1.2 constitute Theorem 2 of Beck [1967].

Up to natural isomorphisms, $H^n(\overline{T},-)$ is the only sequence for which Theorems 1.1 and 1.2 hold (Barr and Beck [1969], Theorem 3.3). Another useful characterization of $H^n(\overline{T},\overline{G})$ was given by Barr and Beck [1969] (Proposition 11.2). We give the contravariant version in Wells [1978]:

Theorem 1.3. *Let*

$$\mathbb{C}: \ 0 \longrightarrow \mathrm{Hom}_{\overline{\mathcal{C}}}(-,\overline{G}) \longrightarrow \mathbb{C}^1 \longrightarrow \cdots \longrightarrow \mathbb{C}^n \longrightarrow \mathbb{C}^{n+1} \longrightarrow \cdots$$

be a complex of abelian group valued contravariant functors on $\overline{\mathcal{C}}$ *and* $\mathbb{H}^n$ *be the n-th homology functor of* $\mathbb{C}$. *Assume that* $\mathbb{H}^n(\overline{\mathbb{V}}\overline{T}) = 0$ *for all* $\overline{T}$ *and* $n \geqq 2$, *that* $\mathbb{H}^1(\overline{\mathbb{V}}\overline{T})$ *is naturally isomorphic to* $C^1(\overline{T},\overline{G})$, *and that there is for each* $n \geqq 1$ *a natural transformation* $\tau^n : \mathbb{C}^n \circ \overline{\mathbb{V}} \longrightarrow \mathbb{C}^n$ *such that* $\tau^n \circ \mathbb{C}^n\overline{\epsilon} = 1_{\mathbb{C}^n}$. *Then* $\mathbb{H}^n$ *is naturally isomorphic to* $H^n(-,\overline{G})$.

6. The fourth property of Beck cohomology requires additional definitions. As above, let $(\mathbb{F},\mathbb{U},\eta,\epsilon) : \mathcal{U} \longrightarrow \mathcal{C}$ be an adjunction, S be an object of $\mathcal{C}$, and $\overline{G}$ be an abelian group object of $\mathcal{C}\downarrow S$. A **Beck extension** of $\overline{G}$ by S (called a $\overline{G}$-module in Beck [1967], Definition 6) is an object $\overline{E} = (E,\pi)$ of $\overline{\mathcal{C}} = \mathcal{C}\downarrow S$ together with an action $\centerdot$ of $\overline{G}$ on $\overline{E}$ such that

(BE1) $\mathbb{U}\pi \circ \sigma = 1_{\mathbb{U}S}$ for some $\sigma : \mathbb{U}S \longrightarrow \mathbb{U}E$;

(BE2) for every object $\overline{C}$ of $\overline{\mathcal{C}}$, the action of $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{E})$ preserves projection to S: $\pi \circ (g \centerdot e) = \pi \circ e$ whenever $g : \overline{C} \longrightarrow \overline{G}$ and $e : \overline{C} \longrightarrow \overline{E}$ in $\overline{\mathcal{C}}$;

(BE3) for every object $\overline{C}$ of $\overline{\mathcal{C}}$, $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ acts simply and transitively on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{E})$: for every $e, f : \overline{C} \longrightarrow \overline{E}$, there exists a unique $g : \overline{C} \longrightarrow \overline{G}$ such that $g \centerdot e = f$.

A morphism $\varphi : \overline{E} \longrightarrow \overline{F}$ of Beck extensions of $\overline{G}$ by S is a morphism in $\overline{\mathcal{C}}$ which preserves the action of $\overline{G}$: $\varphi \circ (g \centerdot e) = g \centerdot (\varphi \circ e)$ for all $g : \overline{C} \longrightarrow \overline{G}$ and $e : \overline{C} \longrightarrow \overline{E}$.

Beck ([1967], Theorem 6) proved that $H^2(S,\overline{G})$ **classifies** Beck extensions of $\overline{G}$ by S in the following sense:

Theorem 1.4. *When* $\mathcal{C}$ *is tripleable over* $\mathcal{U}$, *there is a one-to-one correspondence between elements of* $H^2(S,\overline{G})$ *and isomorphy classes of Beck extensions of* $\overline{G}$ *by* S, *which is natural in* $\overline{G}$.

2. ABELIAN GROUP OBJECTS.

For a more concrete definition of triple cohomology for commutative semigroups we investigate abelian group objects in the category of commutative semigroups over a given commutative semigroup. The main result is from Grillet [1991C], [1995C]; a similar result for monoids was proved by Wells [1978].

1. Let S be a commutative semigroup.

Proposition 2.1. *Let $\mathcal{C}$ be the category of commutative semigroups. An abelian group object of $\mathcal{C}{\downarrow}S$ is a split commutative group coextension of S. The category $\mathcal{O}$ of abelian group objects of $\mathcal{C}{\downarrow}S$ is equivalent to the category $\mathcal{A}$ of abelian group valued functors on $H(S)$.*

Proof. First we show:

Lemma 2.2. *Let G be an abelian group object of a category $\mathcal{C}$ with finite products. Let p_1, $p_2 : G \times G \longrightarrow G$ be the projections and*

$$m = p_1 + p_2 : G \times G \longrightarrow G.$$

Then $g + h = m \circ (g,h)$ whenever $g,h : C \longrightarrow G$.

Proof. $(g,h) : C \longrightarrow G \times G$ is the morphism such that $p_1 \circ (g,h) = g$ and $p_2 \circ (g,h) = h$. Then $g + h = (p_1 \circ (g,h)) + (p_2 \circ (g,h)) = (p_1 + p_2) \circ (g,h) = m \circ (g,h)$. □

To probe c.s. over S we use the additive semigroup $\mathbb{N}^+$ and the following construction. For every $a \in S$ let $\pi_a : \mathbb{N}^+ \longrightarrow S$ be the unique homomorphism such that $\pi_a(1) = a$, namely $\pi_a(n) = a^n$; then $\overline{\mathbb{N}}_a^+ = (\mathbb{N}^+, \pi_a)$ is a c.s. over S.

Lemma 2.3. *Let $\overline{T} = (T, \tau)$ be a commutative semigroup over S. For every $a \in S$ let $T_a = \{t \in T \mid \tau t = a\}$. Evaluation at 1 is a bijection $\varphi \longmapsto \varphi(1)$ of $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{N}}_a^+, \overline{T})$ onto T_a.*

Proof. If $\varphi : \overline{\mathbb{N}}_a^+ \longrightarrow \overline{T}$ is a morphism, then $\tau \circ \varphi = \pi_a$ and $\varphi(1) \in T_a$. Conversely there is for every $t \in T_a$ a unique semigroup homomorphism $\bar{t} : \mathbb{N}^+ \longrightarrow T$ such that $\bar{t}(1) = t$, namely, $t \longmapsto t^n$; then $\tau \circ \bar{t} = \pi_a$ and $\bar{t}$ is a morphism $\overline{\mathbb{N}}_a^+ \longrightarrow \overline{T}$ in $\overline{\mathcal{C}}$. This defines mutually inverse bijections. □

2. Now let $\mathcal{C}$ be the category of c.s. and $\overline{\mathcal{C}} = \mathcal{C}{\downarrow}S$. Let $\overline{G} = (G, \rho)$ be an abelian group object of $\overline{\mathcal{C}} = \mathcal{C}{\downarrow}S$; in particular, G is a c.s., which we write multiplicatively, and ρ is a multiplicative homomorphism.

We use Lemma 2.2 to construct a partial addition on G such that addition on every $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ is pointwise. In $\overline{\mathcal{C}}$, the direct product $\overline{P} = (P,\pi) = \overline{G} \times \overline{G}$ and its projections p_1, $p_2 : \overline{P} \longrightarrow \overline{G}$ are given by the pullback

$$\begin{array}{ccc} P & \xrightarrow{p_1} & G \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \rho} \\ G & \xrightarrow[\rho]{} & S \end{array}$$

where $P = \{(x,y) \in G \times G \mid \rho x = \rho y\}$, $p_1(x,y) = x$, and $p_2(x,y) = y$. Then $m = p_1 + p_2 : P \longrightarrow G$ is a partial addition on G; $x + y = m(x,y)$ is defined if and only if $\rho x = \rho y$, if and only if x and y belong to the same set $G_a = \rho^{-1}a$. Thus m provides an addition on every G_a.

Lemma 2.4. *When $\overline{G} = (G,\rho)$ is an abelian group object of $\overline{\mathcal{C}}$:*

$$(x+y)(z+w) \;=\; xz + yw \tag{1}$$

whenever $\rho x = \rho y$ and $\rho z = \rho w$; addition on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ is pointwise; every G_a is an abelian group under addition; for every $a \in S$, $\overline{g} \longmapsto \overline{g}(1)$ is an additive isomorphism of $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{N}}_a^+,\overline{G})$ onto G_a; and ρ is surjective.

Proof. Since m is a multiplicative homomorphism, $(x+y)(z+w) = m(x,y)\,m(z,w) = m(xz,yw) = xz + yw$ whenever $\rho x = \rho y$ and $\rho z = \rho w$.

Let $\overline{C} = (C,\pi)$ and $g,h : \overline{C} \longrightarrow \overline{G}$. By Lemma 2.2, $g + h = m \circ (g,h)$. Now $(g,h)(c) = (g(c),h(c)) \in P$ for every $c \in C$, and $(g+h)(c) = m(g(c),h(c)) = g(c) + h(c)$. Thus, addition on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ is pointwise. Then the bijection $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{N}}_a^+,\overline{G}) \longrightarrow G_a$, $\varphi \longmapsto \varphi(1)$ in Lemma 2.3 is an additive isomorphism. Hence G_a is an abelian group under addition. In particular, $G_a \neq \emptyset$ and ρ is surjective. $\square$

Lemma 2.5. *Let p_a be the identity element of G_a under addition. An abelian group valued functor $\mathbb{A}\overline{G} = (G,\gamma)$ on $H(S)$ is defined by*

$$\gamma_{a,t}\, g \;=\; g p_t \tag{2}$$

for every $a \in S$, $t \in S^1$, $g \in G_a$, where $p_t = 1 \in G^1$ if $t = 1 \in S^1$. Moreover, $\overline{G}$ is a split group coextension of S by $\mathbb{A}\overline{G}$.

Proof. If $t = 1 \in S^1$, then $\gamma_{a,t}$ is the identity on G_a; otherwise $\gamma_{a,t}\, g \in G_{at}$, since ρ is a homomorphism, and $\gamma_{a,t}$ is an additive homomorphism, since

$$gp_t + hp_t \;=\; (g+h)(p_t + p_t) \;=\; (g+h)p_t$$

for all $g,h \in G_a$, by (1). For all $t,u \in S$ we have

$$p_t p_u \;=\; (p_t + p_t)(p_u + p_u) \;=\; p_t p_u + p_t p_u$$

by (1), which in the abelian group G_{tu} implies $p_t p_u = p_{tu}$. This also holds if $t = 1$ or $u = 1$. Hence $\gamma_{at,u} \circ \gamma_{a,t} = \gamma_{a,tu}$ for all $t,u \in S^1$. Thus $\mathbb{A}\overline{G} = (G,\gamma)$ is an abelian group valued functor on $H(S)$.

The abelian group G_a acts simply and transitively on itself by left addition: $g \centerdot x = g + x$. By (1),

$$(g \centerdot x)\, y \;=\; (g+x)(p_b + y) = gp_b + xy = (\gamma_{a,b} g) \centerdot xy$$

for all $g,x \in G_a$ and $y \in G_b$. Thus $(G,\rho,\centerdot)$ is a commutative group coextension of S by $\mathbb{A}\overline{G}$, which splits since $p_a p_b = p_{ab}$ for all $a,b \in S$. □

Lemma 2.6. *Let* $\varphi : \overline{G} \longrightarrow \overline{H} = (H,\sigma)$ *be a morphism of abelian group objects. Then* $\varphi(G_a) \subseteq H_a$ *for every* $a \in S$ *and* $\mathbb{A}\varphi = (\varphi_{|G_a})_{a \in S}$ *is a natural transformation from* $\mathbb{A}\overline{G}$ *to* $\mathbb{A}\overline{H}$.

Proof. By definition, φ is a multiplicative homomorphism, $\sigma \circ \varphi = \rho$, and $\varphi_* = \mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\varphi)$ is a homomorphism for every $\overline{C}$. In particular $\varphi(G_a) \subseteq H_a$ for every $a \in S$. Let $\varphi_a = \varphi_{|G_a} : G_a \longrightarrow H_a$ be the restriction of φ to G_a. For every $a \in S$, Lemma 2.4 provides additive isomorphisms $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{N}}_a^+,\overline{G}) \cong G_a$, $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{N}}_a^+,\overline{H}) \cong H_a$. The square

$$\begin{array}{ccc}
\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{N}}_a^+,\overline{G}) & \xrightarrow{\cong} & G_a \\
{\scriptstyle\varphi_*}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_a} \\
\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{N}}_a^+,\overline{H}) & \xrightarrow[\cong]{} & H_a
\end{array}$$

commutes, since $(\varphi_*(\overline{g}))(1) = (\varphi \circ \overline{g})(1) = \varphi(\overline{g}(1)) = \varphi_a(\overline{g}(1))$ for all $\overline{g} : \overline{\mathbb{N}}_a^+ \longrightarrow \overline{G}$. Since φ_* is an additive homomorphism, it follows that φ_a is an additive homomorphism. In particular $\varphi_a(p_a) = q_a$, the identity element of the abelian group H_a. Since φ is a multiplicative homomorphism, we have $\varphi_{at}(gp_t) = (\varphi_a g)\, q_t$ for all $a,t \in S$ and $g \in G_a$; thus $\mathbb{A}\varphi = (\varphi_a)_{a \in S}$ is a natural transformation from $\mathbb{A}\overline{G}$ to $\mathbb{A}\overline{H}$. □

We now have a functor $\mathbb{A} : \mathcal{O} \longrightarrow \mathcal{A}$.

3. Conversely let (G,γ) be an abelian group valued functor on $H(S)$. Let

g^t denote $\gamma_{a,t}g$ when $g \in G_a$ and $t \in S^1$. As in Theorem V.4.1, there is a split commutative group coextension $(E, \rho, .)$ of S by (G, γ), in which E is the disjoint union $\bigcup_{a \in S} \left(G_a \times \{a\}\right)$ with multiplication

$$(g,a)(h,b) = (g^b + h^a,\, ab),$$

projection $\rho : (g,a) \longmapsto a$ to S, and action $g \centerdot (h,a) = (g+h,\, a)$ of G_a on $E_a = G_a \times \{a\}$. Then $\overline{G} = (E, \rho)$ is a c.s. over S.

Let $\overline{C} = (C, \pi)$ be an object of $\overline{\mathcal{C}}$ and $\overline{g}, \overline{h} : \overline{C} \longrightarrow \overline{G}$ be morphisms in $\overline{\mathcal{C}}$. Then $\rho \circ \overline{g} = \rho \circ \overline{h} = \pi$. For every $c \in C$ there exist unique $g_c, h_c \in G_a$ such that $\overline{g}(c) = (g_c, a)$ and $\overline{h}(c) = (h_c, a)$, where $a = \pi c$. Define

$$(\overline{g} + \overline{h})(c) = (g_c + h_c,\, a).$$

Lemma 2.7. *With the addition defined above, $\overline{G}$ is an abelian group object over S. Moreover, $\mathbb{A}\overline{G} \cong (G, \gamma)$.*

Proof. Let $\overline{g}, \overline{h} : \overline{C} \longrightarrow \overline{G}$ be morphisms in $\overline{\mathcal{C}}$. Since $\overline{g}$ and $\overline{h}$ are multiplicative homomorphisms, we have

$$(g_{cd},\, ab) = \overline{g}(cd) = \overline{g}(c)\,\overline{g}(d) = (g_c, a)(g_d, b) = (g_c^b + g_d^a,\, ab),$$

where $a = \pi c$ and $b = \pi d$, and $g_{cd} = g_c^b + g_d^a$; similarly $h_{cd} = h_c^b + h_d^a$. Hence

$$\begin{aligned}(\overline{g} + \overline{h})(c)\,(\overline{g} + \overline{h})(d) &= (g_c + h_c,\, a)(g_d + h_d,\, b) \\ &= (g_c^b + h_c^b + g_d^a + h_d^a,\, ab) \\ &= (g_{cd} + h_{cd},\, ab) = (\overline{g} + \overline{h})(cd).\end{aligned}$$

and $\overline{g} + \overline{h}$ is a multiplicative homomorphism. We now have an addition on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C}, \overline{A})$. It is immediate that $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C}, \overline{A})$ is an abelian group, in which the identity element is $c \longmapsto (0, \pi c)$ and the opposite of $\overline{g} : c \longmapsto (g_c, \pi c)$ is $c \longmapsto (-g_c, \pi c)$. Moreover $(\overline{g} + \overline{h}) \circ \delta = (\overline{g} \circ \delta) + (\overline{h} \circ \delta)$ for every morphism $\delta : \overline{D} \longrightarrow \overline{C}$. Thus $\overline{G}$ is an abelian group object of $\overline{\mathcal{C}}$.

We show that $\mathbb{A}\overline{G} \cong (G, \gamma)$. Since $\overline{G}$ is an abelian group object, it induces, as in the first part of the proof, an abelian group addition on every E_a. For every $a \in S$, Lemma 2.4 provides an additive isomorphism $\overline{g} \longmapsto \overline{g}(1)$ of $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{N}_a^+}, \overline{G})$ onto E_a. For every $(g,a), (h,a) \in E_a$ we now have homomorphisms $\overline{g}, \overline{h} : \overline{\mathbb{N}_a^+} \longrightarrow \overline{G}$ such that $\overline{g}(1) = (g, a)$ and $\overline{h}(1) = (h, a)$; since addition on

$\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{N}}_a^+, \overline{G})$ is pointwise,

$$(g,a)+(h,a) \;=\; \overline{g}(1)+\overline{h}(1) \;=\; (\overline{g}+\overline{h})(1) \;=\; (g+h,\, a).$$

Thus $g \longmapsto (g,a)$ is an isomorphism of G_a onto E_a.

In particular, the identity element of E_a is $p_a = (0,a)$. The homomorphism $\epsilon_{a,t}$ in the functor $\mathbb{A}\overline{G} = (E,\epsilon)$ is then given by (2): $\epsilon_{a,t}(g,a) = (g,a)(0,t) = (g^t,\, at)$ for all $t \in S$. Thus $\mathbb{A}\overline{G}$ is naturally isomorphic to (G,γ). □

Lemma 2.8. *Let $(\varphi_a)_{a\in S}$ be a natural transformation from (G,γ) to (H,δ). Then $\varphi : \overline{G} \longrightarrow \overline{H}$, defined by $\varphi(g,a) = (\varphi_a g,\, a)$, is a morphism of abelian group objects.*

Proof. Since $(\varphi_a)_{a\in S}$ is a natural transformation, we have $(\varphi_a g)^t = \varphi_{at}(g^t)$ whenever $g \in G_a$. Hence φ is a multiplicative homomorphism:

$$\begin{aligned}\varphi((g,a)(h,b)) &= \varphi(g^b + h^a,\, ab) = (\varphi_{ab} g^b + \varphi_{ab} h^a,\, ab)\\ &= ((\varphi_a g)^b + (\varphi_b h)^a,\, ab) = \varphi(g,a)\,\varphi(h,b)\end{aligned}$$

and a morphism in $\overline{\mathcal{C}}$. Moreover, for any $\overline{g}, \overline{h} : \overline{C} \longrightarrow \overline{G}$, we have, with the notation as above, $\varphi(\overline{g}(c)) = (\varphi_a g_c, a)$, $\varphi(\overline{h}(c)) = (\varphi_a h_c, a)$, and

$$(\varphi\circ\overline{g}+\varphi\circ\overline{h})(c) \;=\; (\varphi_a g_c + \varphi_a h_c,\, a) \;=\; \varphi(g_c + h_c,\, a) \;=\; \varphi\big((\overline{g}+\overline{h})(c)\big);$$

thus φ is a morphism of abelian group objects. □

We now have a functor $\mathbb{O} : \mathcal{A} \longrightarrow \mathcal{O}$; $\mathbb{O}(G,\gamma)$ is $\overline{G}$ in Lemma 2.7, and Lemma 2.8 constructs $\mathbb{O}(\varphi_a)_{a\in S} = \varphi$.

4. We saw that $\mathbb{A}\mathbb{O}G$ is isomorphic to G. If conversely $\overline{G} = (G,\pi)$ is any abelian group object of $\overline{\mathcal{C}}$, then G is a split group coextension of S by $\mathbb{A}\overline{G}$ and $\overline{G}$ is isomorphic to $\mathbb{O}\mathbb{A}\overline{G}$ as a semigroup over S and as an abelian group object. If not exhausted our reader will verify that these isomorphisms are natural, which completes the proof of Proposition 2.1. □

3. BECK EXTENSIONS.

Continuing Section 2 we now investigate Beck extensions in the category of commutative semigroups over a given commutative semigroup. The main result is from Grillet [1991C], [1995C]; a similar result for monoids was proved by Wells [1978].

1. Let S be a c.s. and $\mathcal{C}$ be the category of c.s. and homomorphisms.

Proposition 3.1. *Let $\overline{G} = (G,\rho)$ be an abelian group object of $\mathcal{C}{\downarrow}S$ and $\mathbb{G} = \mathbb{A}\overline{G} = (G,\gamma)$ be the corresponding abelian group valued functor on $H(S)$. A Beck extension of $\overline{G}$ by S is a commutative group coextension of S by $\mathbb{G}$. The category $\mathcal{B}$ of Beck extensions of $\overline{G}$ by S is isomorphic to the category $\mathcal{E}$ of commutative group coextensions of S by $\mathbb{G}$.*

A morphism in $\mathcal{E}$ (necessarily an isomorphism) is an equivalence of commutative group coextensions.

Proof. The proof is rather similar to that of Proposition 2.1. First we show:

Lemma 3.2. *Let G be an abelian group object of a category $\mathcal{C}$ with finite products and $\centerdot$ be an action of G on an object E of $\mathcal{C}$. Let $p_1 : G \times E \longrightarrow G$, $p_2 : G \times E \longrightarrow E$ be the projections and*

$$q = p_1 \centerdot p_2 : G \times E \longrightarrow E.$$

Then $g \centerdot e = n \circ (g,e)$ for all $g : C \longrightarrow G$ and $e : C \longrightarrow E$.

Proof. $(g,e) : C \longrightarrow G \times E$ is the morphism such that $p_1 \circ (g,e) = g$ and $p_2 \circ (g,e) = e$. Since $\centerdot$ is an action, $g \centerdot e = (p_1 \circ (g,e)) \centerdot (p_2 \circ (g,e)) = (p_1 \centerdot p_2) \circ (g,e) = n \circ (g,e)$. $\square$

2. Now let $\overline{G}$ be an abelian group object over a c.s. S and $\overline{E} = (E,\pi)$ be a Beck extension of $\overline{G}$ by S, so that there is an action of $\overline{G}$ on $\overline{E}$ and (BE1), (BE2), (BE3) hold. In particular, E is a c.s., which we write multiplicatively, and π is a multiplicative homomorphism. By (BE1), $\pi \circ \sigma = 1_{\mathbb{U}S}$ for some mapping $\sigma : S \longrightarrow E$; hence π is surjective, and (E,π) is a coextension of S.

We use Lemma 3.2 to construct a partial action of G on E such that $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ acts pointwise on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{E})$ for every $\overline{C} \in \overline{\mathcal{C}}$. In $\overline{\mathcal{C}}$, the direct product $\overline{P} = (P,r) = \overline{G} \times \overline{E}$ and its projections $p_1 : \overline{P} \longrightarrow \overline{G}$, $p_2 : \overline{P} \longrightarrow \overline{E}$ are given by the pullback

$$\begin{array}{ccc} P & \xrightarrow{p_1} & G \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \rho} \\ E & \xrightarrow[\pi]{} & S \end{array}$$

where $P = \{(g,x) \in G \times E \mid \rho g = \pi x\}$, $p_1(g,x) = g$, and $p_2(g,x) = x$. Then $q = p_1 \centerdot p_2 : P \longrightarrow E$ is a partial action of G on E, for which $g \centerdot x = q(g,x)$ is defined if and only if $\rho g = \pi x$, if and only if $g \in G_a = \rho^{-1}a$ and $x \in E_a = \pi^{-1}a$ for some $a \in S$; thus q provides a set action of G_a on E_a, for every $a \in S$.

Lemma 3.3. *When $\overline{E} = (E,\pi)$ is a Beck extension of S by $\overline{G} = (G,\rho)$:*

$$(g \,.\, x)(h \,.\, y) \;=\; gh \,.\, xy \tag{3}$$

whenever $\rho g = \pi x$ and $\rho h = \pi y$; $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ acts on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{E})$ pointwise; the action of G_a on E_a is a simply transitive group action; and $\mathbb{E}\overline{E} = (E,\pi,.)$ is a commutative group coextension of S by $\mathbb{G}$.

Proof. Since q is a multiplicative homomorphism, we have $(g \,.\, x)(h \,.\, y) = q(g,x)\, q(h,y) = q(gh,xy) = gh \,.\, xy$ whenever $\rho g = \pi x$ and $\rho h = \pi y$.

Let $\overline{C} = (C,\sigma)$ and $g : \overline{C} \longrightarrow \overline{G}$, $e : \overline{C} \longrightarrow \overline{E}$ be morphisms in $\overline{\mathcal{C}}$. Then $(g,e)(c) = (g(c),e(c)) \in P$ for every $c \in C$, and Lemma 3.2 yields $g \,.\, e = n \circ (g,e)$ and

$$(g \,.\, e)(c) \;=\; q(g(c),e(c)) \;=\; g(c) \,.\, h(c).$$

Thus $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ acts on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{E})$ pointwise.

For every $a \in S$, Lemmas 2.4 and 2.3 provide an additive isomorphism $\overline{g} \longmapsto \overline{g}(1)$ of $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{N}}_a^+,\overline{G})$ onto G_a and a bijection $\overline{e} \longmapsto \overline{e}(1)$ of $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{N}}_a^+,\overline{E})$ onto E_a. Moreover $(\overline{g} \,.\, \overline{e})(1) = \overline{g}(1) \,.\, \overline{e}(1)$, since $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ acts on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{E})$ pointwise. Therefore the action of G_a on E_a is a group action, since the action of $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{E})$ is a group action; and G_a acts simply and transitively on E_a, by (BE3).

Let p_a be the identity element of G_a. Then $p_a \,.\, x = x$ for all $x \in E_a$, and (3), (2) yield

$$(g \,.\, x)y \;=\; (g \,.\, x)(p_b \,.\, y) \;=\; gp_b \,.\, xy \;=\; \gamma_{a,b} g \,.\, xy$$

whenever $g \in G_a$, $x \in E_a$, and $y \in E_b$. Thus $\mathbb{E}\overline{E} = (E,\pi,.)$ is a commutative group coextension of S by $\mathbb{G}$. □

Lemma 3.4. *Every morphism $\varphi : \overline{E} \longrightarrow \overline{E}' = (E',\pi')$ of Beck extensions of S by $\overline{G}$ is an equivalence of commutative group coextensions of $\mathbb{E}\overline{E}$ to $\mathbb{E}\overline{E}'$.*

Proof. By definition, φ is a multiplicative homomorphism of E into E'; $\pi' \circ \varphi = \pi$; and $\varphi \circ (\overline{g} \,.\, \overline{e}) = \overline{g} \,.\, (\varphi \circ \overline{e})$ for all $\overline{g} : \overline{C} \longrightarrow \overline{G}$ and $\overline{e} : \overline{C} \longrightarrow \overline{E}$. In particular, $\varphi(E_a) \subseteq E'_a$ for every $a \in S$. For every $g \in G_a$ and $x \in E_a$, Lemma 2.3 yields morphisms $\overline{g} : \overline{\mathbb{N}}_a^+ \longrightarrow \overline{G}$ and $\overline{x} : \overline{\mathbb{N}}_a^+ \longrightarrow \overline{E}$ such that $\overline{g}(1) = g$ and $\overline{x}(1) = x$. Then $\varphi \circ (\overline{g} \,.\, \overline{x}) = \overline{g} \,.\, (\varphi \circ \overline{x})$ and evaluation at 1 yields $\varphi(g \,.\, x) = g \,.\, \varphi(x)$. Thus φ preserves the action of G_a. Then φ is a bijection of E_a onto E'_a, since G_a acts simply and transitively on E_a and on E'_a, and

φ is an equivalence of commutative group coextensions. $\square$

We now have a functor $\mathbb{E} : \mathcal{B} \longrightarrow \mathcal{E}$.

3. Conversely let $(E,\pi,.)$ be a commutative group coextension of S by G. Then $\overline{E} = (E,\pi)$ is a c.s. over S. By (1), (2),

$$gh \;=\; (g+p_a)(p_b+h) \;=\; gp_b+p_a h \;=\; \gamma_{a,b}g+\gamma_{b,a}h \;=\; g^b+h^a$$

for all $g \in G_a$ and $h \in G_b$; then

$$(g\,.\,x)(h\,.\,y) \;=\; g^b\,.\,(x\,(h\,.\,y)) \;=\; (g^b+h^a)\,.xy \;=\; gh\,.\,xy \qquad (4)$$

for all $g \in G_a$, $h \in G_b$, $x \in E_a$, and $y \in G_b$, since E is a group coextension.

Let $\overline{C} = (C,\tau)$ be an object of $\overline{\mathcal{C}}$ and $\overline{g} : \overline{C} \longrightarrow \overline{G}$, $\overline{e} : \overline{C} \longrightarrow \overline{E}$ be morphisms in $\overline{\mathcal{C}}$. Then $\rho \circ g = \pi \circ e = \tau$ and $\overline{g}(c)\,.\,\overline{e}(c)$ is defined in E for every $c \in C$. Define

$$(\overline{g}\,.\,\overline{e})(c) \;=\; \overline{g}(c)\,.\,\overline{e}(c).$$

Lemma 3.5. *With the action defined above,* $\mathbb{B}(E,\pi,.) = \overline{E}$ *is a Beck extension of* S *by* $\overline{G}$. *Moreover,* $\mathbb{E}\overline{E}$ *is the given coextension* $(E,\pi,.)$.

Proof. Since $\overline{g}$ and $\overline{e}$ are multiplicative homomorphisms, we have, by (4),

$$\big(\overline{g}(c)\,.\,\overline{e}(c)\big)\big(\overline{g}(d)\,.\,\overline{e}(d)\big) \;=\; \overline{g}(c)\overline{g}(d)\;.\;\overline{e}(c)\overline{e}(d) \;=\; \overline{g}(cd)\,.\,\overline{e}(cd)$$

where $a = \tau c$, $b = \tau d$; hence $\overline{g}\,.\,\overline{e}$ is a multiplicative homomorphism. Now $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ acts on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{E})$; this is a group action since addition on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ is pointwise. (BE1) holds since π is surjective. Also

$$\pi\big((\overline{g}\,.\,\overline{e})(c)\big) \;=\; \pi\big((\overline{g}(c)\,.\,\overline{e}(c)\big) \;=\; \pi\big(\overline{e}(c)\big)$$

for all $c \in C$ and (BE2) holds.

We show that $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ acts simply and transitively on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{E})$. Let $\overline{e},\overline{f} : \overline{C} \longrightarrow \overline{E}$. For every $c \in C$ there exists a unique $\overline{g}(c) \in G_a$ such that $\overline{f}(c) = \overline{g}(c)\,.\,\overline{e}(c)$, where $a = \tau c$. Since $\overline{e}$ and $\overline{f}$ are multiplicative homomorphisms, we have

$$\begin{aligned}\overline{g}(cd)\,.\,\overline{e}(cd) \;&=\; \overline{f}(cd) \;=\; \overline{f}(c)\overline{f}(d)\\ &=\; \big(\overline{g}(c)\,.\,\overline{e}(c)\big)\big(\overline{g}(d)\,.\,\overline{e}(d)\big) \;=\; \overline{g}(c)\overline{g}(d)\;.\;\overline{e}(cd)\end{aligned}$$

for all $c,d \in C$, by (4). Hence $\overline{g}$ is a homomorphism. Then $\overline{g}$ is the unique morphism $\overline{C} \longrightarrow \overline{G}$ such that $\overline{g}\,.\,\overline{e} = \overline{f}$. Thus $\overline{E}$ (with the action . of $\overline{G}$ on $\overline{E}$) is a Beck extension of S by $\overline{G}$.

Since $\overline{E}$ is a Beck extension, it induces, as in the first part of the proof, a simply transitive group action of G_a on E_a, which makes $\overline{E}$ a commutative group coextension $\mathbb{E}\overline{E}$ of S by G. Let $a \in S$. For every $g \in G_a$ and $e \in E_a$, Lemma 2.3 yields morphisms $\overline{g} : \overline{C}_a \longrightarrow \overline{G}$ and $\overline{e} : \overline{C}_a \longrightarrow \overline{G}$ such that $\overline{g}(1) = g$ and $\overline{e}(1) = e$. Then the action of G_a on E_a in $\mathbb{E}\overline{E}$ satisfies

$$g \centerdot e \;=\; \overline{g}(1) \centerdot \overline{e}(1) \;=\; (\overline{g} \centerdot \overline{e})(1);$$

so does the action of G_a on E_a in the given coextension $(E, \pi, .)$. Hence the two actions coincide and $\mathbb{E}\overline{E} = (E, \pi, .)$. $\square$

Lemma 3.6. *Every equivalence of commutative group coextensions from* $(E, \pi, .)$ *to* $(E', \pi', .)$ *is a morphism of Beck extensions from* $\mathbb{B}(E, \pi, .)$ *to* $\mathbb{B}(E', \pi', .)$.

Proof. Let φ be an equivalence of commutative group coextensions from $(E, \pi, .)$ to $(E', \pi', .)$. Then φ is a multiplicative homomorphism and preserves projection to S and action of G. For every $\overline{g} : \overline{C} \longrightarrow \overline{G}$ and $\overline{e} : \overline{C} \longrightarrow \overline{E}$ we then have $\varphi\big(\overline{g}(c) \centerdot \overline{e}(c)\big) = \overline{g}(c) \centerdot \varphi(\overline{e}(c))$ and $\varphi \circ (\overline{g} \centerdot \overline{e}) = \overline{g} \centerdot (\varphi \circ \overline{e})$. Thus φ is a morphism of Beck extensions. $\square$

4. We now have a functor $\mathbb{B} : \mathcal{E} \longrightarrow \mathcal{B}$. We saw (Lemma 3.5) that $\mathbb{E}\mathbb{B} = 1_{\mathcal{E}}$. If conversely $\overline{E}$ is a Beck extension of $\overline{G}$ by S, then the action of $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C}, \overline{G})$ on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C}, \overline{E})$ is the pointwise action induced by the action of G on $\mathbb{E}\overline{E}$ and coincides with the action of $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C}, \overline{G})$ on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C}, \mathbb{B}\mathbb{E}\overline{E})$. Hence $\mathbb{B}\mathbb{E}\overline{E} = \overline{E}$. Thus $\mathbb{B}$ and $\mathbb{E}$ are mutually inverse isomorphisms. $\square$

4. COMMUTATIVE SEMIGROUP COHOMOLOGY.

Triple cohomology in the category $\mathcal{C}$ of commutative semigroups provides a good definition of commutative semigroup cohomology. This section brings a more concrete definition, based on the results in Sections 2 and 3. The results are from Grillet [1991C], [1995C].

Other cohomology theories have been considered for commutative semigroups. Inasaridze extended the construction of n-extensions and Ext^n from abelian groups to commutative cancellative monoids [1964], [1965] and to commutative Clifford semigroups [1964], [1967]. Kruming [1982] characterized finite commutative semigroups whose Eilenberg-MacLane cohomology vanishes; Novikov [1990] showed that cancellative c.s. with this property are subsemigroups of $\mathbb{N}$.

See also Carbonne [1983]. For a survey of semigroup cohomology in general, see Grillet & Novikov [2002].

1. Let F_X be the free c.s. on a set X, which we write multiplicatively. For what follows it is best to regard the elements of F_X as **commutative words** in X, which are nonempty unordered sequences $[x_1, \dots, x_m]$ of elements of X; unordered means

$$[x_{\sigma 1}, \dots, x_{\sigma m}] \;=\; [x_1, \dots, x_m]$$

for every permutation σ. It is customary to write $[x_1, \dots, x_m]$ as a product $x_1 \cdots x_m$, but this would quickly become very confusing in what follows. Multiplication in F_X is by concatenation:

$$[x_1, \dots, x_m]\,[y_1, \dots, y_n] \;=\; [x_1, \dots, x_m, y_1, \dots, y_n].$$

The injection $\eta_X : X \longrightarrow F_X$ takes $x \in X$ to $[x] \in F_X$. Every mapping of a set X into a c.s. factors uniquely through η_X.

The free c.s. functor $\mathbb{F} : \mathit{Sets} \longrightarrow \mathcal{C}$ assigns to a set X the free c.s. $\mathbb{F}X = F_X$ on X, and assigns to a mapping $f : X \longrightarrow Y$ the unique homomorphism $\mathbb{F}f : \mathbb{F}X \longrightarrow \mathbb{F}Y$ such that $\mathbb{F}f \circ \eta_X = \eta_Y \circ f$; $\mathbb{F}f$ sends $[x_1, x_2, \dots, x_m] = [x_1]\,[x_2] \cdots [x_m] \in \mathbb{F}X$ to $[ft_1, ft_2, \cdots, ft_m] \in \mathbb{F}Y$. $\mathbb{F}$ is a left adjoint of the forgetful functor $\mathbb{U} : \mathcal{C} \longrightarrow \mathit{Sets}$.

Lemma 4.1. *In the adjunction* $(\mathbb{F}, \mathbb{U}, \eta, \epsilon) : \mathit{Sets} \longrightarrow \mathcal{C}$, $\epsilon_T : \mathbb{F}\mathbb{U}T \longrightarrow T$ *sends* $[t_1, t_2, \dots, t_m] \in \mathbb{F}\mathbb{U}T$ *to* $t_1 t_2 \cdots t_m \in T$, *for every c.s.* T.

Proof. Since $(\mathbb{F}, \mathbb{U}, \eta, \epsilon) : \mathit{Sets} \longrightarrow \mathcal{C}$ is an adjunction, we have $\mathbb{U}\epsilon_T \circ \eta_{\mathbb{U}T} = 1_{\mathbb{U}T}$ (see e.g. Theorem IV.1 in MacLane [1971]). Thus $\epsilon_T : \mathbb{F}\mathbb{U}T \longrightarrow T$ is the homomorphism such that $\epsilon_T\,[t] = \epsilon_T\,(\eta_{\mathbb{U}T}\, t) = t$ for all $t \in T$. Hence ϵ_T sends $[t_1, t_2, \dots, t_m] = [t_1]\,[t_2] \cdots [t_m] \in \mathbb{F}\mathbb{U}T$ to $t_1 t_2 \cdots t_m \in T$. $\square$

In the cotriple $(\mathbb{V}, \epsilon, \nu)$ induced by the adjunction $(\mathbb{F}, \mathbb{U}, \eta, \epsilon) : \mathit{Sets} \longrightarrow \mathcal{C}$, $\mathbb{V} = \mathbb{F}\mathbb{U}$ sends a c.s. T to the free c.s. $\mathbb{V}T = \mathbb{F}\mathbb{U}T$ on the set T. If $f : T \longrightarrow T'$ is a homomorphism, then $\mathbb{V}f : [x_1, \dots, x_m] \longmapsto [fx_1, \dots, fx_m]$. Lemma 4.1 describes $\epsilon_T : \mathbb{V}T \longrightarrow T$; ν will not be used.

For every c.s. S the cotriple $(\mathbb{V}, \epsilon, \nu)$ lifts to a cotriple $(\overline{\mathbb{V}}, \overline{\epsilon}, \overline{\nu})$ on $\overline{\mathcal{C}} = \mathcal{C}{\downarrow}S$; if $\overline{T} = (T, \tau)$ is a c.s. over S, then $\overline{\mathbb{V}}\overline{T} = (\mathbb{V}T, \overline{\tau})$, where $\mathbb{U}\overline{\tau} \circ \eta_{\mathbb{U}T} = \mathbb{U}\tau$; that is, $\overline{\tau}[t] = \tau t$ for every $t \in T$ and

$$\overline{\tau}[x_1, \dots, x_m] \;=\; \tau x_1 \cdots \tau x_m \;=\; \tau\,(x_1 \cdots x_m)$$

for all $m > 0$ and $x_1, \dots, x_m \in T$.

2. The next Lemma describes $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{V}}\overline{T}, \overline{G})$ when $\overline{G}$ is an abelian group

object of $\overline{\mathcal{C}}$. When $\mathbb{G} = (G,\gamma)$ is an abelian group valued functor on $H(S)$, and $\overline{T} = (T,\tau)$ is a c.s. over S, let

$$C(\overline{T},\mathbb{G}) \;=\; \prod\nolimits_{t\in T} G_{\tau t}\,;$$

$C(\overline{T},\mathbb{G})$ consists of all families $u = (u_t)_{t\in T}$ such that $u_t \in G_{\tau t}$ for all $t \in T$, under pointwise addition (that is, 1-cochains on T with values in $\mathbb{G}$).

Lemma 4.2. *Let $\overline{G} = (G,\rho)$ be an abelian group object of $\overline{\mathcal{C}}$ and $\mathbb{G}$ be the corresponding abelian group valued functor on $H(S)$. For every object $\overline{T}$ of $\overline{\mathcal{C}}$ there is a natural isomorphism* $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{V}T},\overline{G}) \cong C(\overline{T},\mathbb{G})$.

Proof. By Proposition 2.1 we may assume that $\overline{G}$ is the split commutative group coextension $\bigcup_{a\in S} G_a \times \{a\}$ of S by the corresponding abelian group valued functor $\mathbb{G}$. Then addition on $\mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{C},\overline{G})$ is as follows: if $f,g:\overline{C} \longrightarrow \overline{G}$ and $f(c) = (f_c,a)$, $g(c) = (g_c,a)$, then $(f+g)(c) = (f_c+g_c,\, a)$.

We have $\overline{\mathbb{V}T} = (\mathbb{V}T,\overline{\tau})$, where $\mathbb{V}T$ is the free c.s. on the set T and $\overline{\tau}[x_1,\ldots,x_m] = \tau(x_1\cdots x_m)$ for all $m > 0$ and $x_1,\ldots,x_m \in T$, in particular $\overline{\tau}[t] = \tau t$ for all $t \in T$. If $f : \overline{\mathbb{V}T} \longrightarrow \overline{G}$ is a morphism in $\overline{\mathcal{C}}$, then $\rho \circ f = \overline{\tau}$ and there is for every $t \in T$ a unique $u_t \in G_{\overline{\tau}[t]} = G_{\tau t}$ such that $f[t] = (u_t,\tau t)$.

Since $\overline{\mathbb{V}T}$ is free on T there is for every $u = (u_t)_{t\in T} \in C(\overline{T},\mathbb{G})$ a unique semigroup homomorphism $f : \overline{\mathbb{V}T} \longrightarrow \overline{G}$ such that $f[t] = (u_t,\tau t)$. Then

$$f\,[x_1,\ldots,x_m] \;=\; (u_{x_1},\tau x_1)\cdots(u_{x_m},\tau x_m) \;=\; \left(\textstyle\sum_{1\leqq i\leqq m} u_{x_j}^{\tau\widehat{x}_j}\,,\; \tau x\right),$$

where $\widehat{x}_j = x_1\cdots x_{j-1}\,x_{j+1}\cdots x_m$ and $x = x_1\cdots x_m$, for all $m > 0$ and $x_1,\ldots,x_m \in T$. (If $m = 1$, then $\widehat{x}_j$ is an empty product and $\widehat{x}_j = 1 \in T^1$.) This provides a bijection $\Theta : \mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{\mathbb{V}T},\overline{G}) \longrightarrow C(\overline{T},\mathbb{G})$. If $f[t] = (u_t,\tau t)$ and $g[t] = (v_t,\tau t)$, then $(f+g)[t] = (u_t+v_t,\,\tau t)$; hence Θ is an isomorphism. It is immediate that Θ is natural in $\overline{T}$ and $\mathbb{G}$. □

3. The triple cohomology of $\overline{T} = (T,\tau) \in \overline{\mathcal{C}}$, and the triple cohomology of S, which is that of $(S,1_S)$, can now be obtained as follows. Let $\overline{G} = (G,\rho)$ be an abelian group object of $\overline{\mathcal{C}}$.

Let $\overline{T}_n = (T_n,\pi_n) = \overline{\mathbb{V}}^n\overline{T}$. Then $T_0 = T$, $\pi_0 = \tau$ ($T_0 = S$, $\pi_0 = 1_S$ for the cohomology of S); $T_{n+1} = \mathbb{V}T_n$ is the semigroup of all commutative words $[x_1,\ldots,x_m]$ with $m > 0$ and $x_1,\ldots,x_m \in T_n$, and

$$\pi_{n+1}[x_1,\ldots,x_m] \;=\; \pi_n(x_1\cdots x_m) \in S. \qquad (\pi)$$

For all $0 \leqq i \leqq n$ the morphisms $\overline{\epsilon_{T}^{n,i}} = \epsilon_{T}^{n,i} : \overline{T}_{n+1} \longrightarrow \overline{T}_{n}$ in $\overline{\mathcal{C}}$, $\epsilon_{T}^{n,i} = \mathbb{V}^{n-i}\,\epsilon_{\mathbb{V}^{i}T}$ may be defined by induction:

$$\epsilon^{n,n}\,[x_1,\ldots,x_m] \;=\; \epsilon_{\mathbb{V}^{n}T}\,[x_1,\ldots,x_m] \;=\; x_1\cdots x_m \in T_n \qquad (\epsilon')$$

for all $[x_1,\ldots,x_m] \in T_{n+1}$, by Lemma 4.1; when $i < n$, $\epsilon_{T}^{n,i} = \mathbb{V}\,\epsilon_{T}^{n-1,\,i}$ and

$$\epsilon^{n,i}\,[x_1,\ldots,x_m] \;=\; [\epsilon^{n-1,\,i}x_1,\;\ldots,\;\epsilon^{n-1,\,i}x_m] \in T_n \qquad (\epsilon'')$$

for all $[x_1,\ldots,x_m] \in T_{n+1}$.

The equality

$$\pi_n \circ \epsilon^{n,i} \;=\; \pi_{n+1} \qquad (\pi\epsilon)$$

is proved by induction on n: for every $[x_1,\ldots,x_m] \in T_{n+1}$,

$$\pi_n\,\epsilon^{n,n}\,[x_1,\ldots,x_m] \;=\; \pi_n\,(x_1\cdots x_m) \;=\; \pi_{n+1}\,[x_1,\ldots,x_m]$$

and

$$\begin{aligned}\pi_n\,\epsilon^{n,i}\,[x_1,\ldots,x_m] \;&=\; \pi_n\,[\epsilon^{n-1,i}x_1,\;\ldots,\;\epsilon^{n-1,i}x_m] \\ &=\; \pi_{n-1}\,(\epsilon^{n-1,i}x_1\,\cdots\,\epsilon^{n-1,i}x_m) \;=\; \pi_{n-1}\,\epsilon^{n-1,i}\,(x_1\cdots x_m) \\ &=\; \pi_n\,(x_1\cdots x_m) \;=\; \pi_{n+1}\,[x_1,\ldots,x_m]\end{aligned}$$

for all $i < n$, since $\epsilon^{n,i}$ is a homomorphism.

An n-cochain is an element of $C^n(\overline{T},\overline{G}) = \mathrm{Hom}_{\overline{\mathcal{C}}}(\overline{T}_n, \overline{G})$. The coboundary homomorphism

$$\delta_n \;=\; \textstyle\sum_{0\leqq i\leqq n}(-1)^i\mathrm{Hom}_{\overline{\mathcal{C}}}(\epsilon_{T}^{n,i},\overline{G}) : C^n(\overline{T},\overline{G}) \longrightarrow C^{n+1}(\overline{T},\overline{G})$$

sends $v : \overline{T}_n \longrightarrow \overline{G}$ to

$$\delta_n v \;=\; \textstyle\sum_{0\leqq i\leqq n}(-1)^i(v\circ\epsilon_{T}^{n,i}).$$

An n-cocycle is an element of $Z^n(\overline{T},\overline{G}) = \mathrm{Ker}\,\delta_n \subseteq C^n(\overline{T},\overline{G})$; if $n \geqq 2$, an n-coboundary is an element of $B^n(\overline{T},\overline{G}) = \mathrm{Im}\,\delta_{n-1} \subseteq Z^n(\overline{T},\overline{G})$. The Beck cohomology groups of $\overline{T}$ are the groups

$$H^n(\overline{T},\overline{G}) \;=\; Z^n(\overline{T},\overline{G})\,/\,B^n(\overline{T},\overline{G})$$

where $n \geqq 2$, and $H^1(\overline{T},\overline{G}) = Z^1(\overline{T},\overline{G})$.

By Proposition 2.1, $\overline{G}$ is a split commutative group coextension of S by the corresponding abelian group valued functor $\mathbb{G}$. By Lemma 4.2, there is a natural isomorphism Θ of $C^n(\overline{T},\overline{G}) = \mathrm{Hom}_{\overline{\mathbb{C}}}(\overline{T}_n, \overline{G})$ onto

$$C^n(S,\mathbb{G}) = C(\overline{T}_{n-1},\mathbb{G}) = \prod_{t\in T_{n-1}} G_{\pi_{n-1}t};$$

when $v : \overline{T}_n \longrightarrow \overline{G}$, then $\Theta v = u = (u_t)_{t\in T_{n-1}} \in C^n(S,\mathbb{G})$ is given by

$$v[t] = (u_t, \pi_{n-1}t).$$

When $u = (u_t)_{t\in T_{n-1}} \in C^n(S,\mathbb{G})$, then $\Theta^{-1}u = v : \overline{T}_n \longrightarrow \overline{G}$ is given by

$$v[x_1,\dots,x_m] = \left(\textstyle\sum_{1\leqq j\leqq m} u_{x_j}^{\pi_{n-1}\widehat{x}_j}, \pi_{n-1}x\right),$$

where $\widehat{x}_j = x_1\cdots x_{j-1}\, x_{j+1}\cdots x_m$ and $x = x_1\cdots x_m$, for all $m > 0$ and $x_1,\dots,x_m \in T_{n-1}$; then $\delta'_n u = \Theta(\delta_n v) \in C^{n+1}(S,\mathbb{G})$ is given for all $t = [x_1,\dots,x_m] \in T_n$ by

$$\begin{aligned}
\left((\delta'_n u)_t, \pi_n t\right) &= (\delta_n v)[t] = \textstyle\sum_{0\leqq i\leqq n} (-1)^i \,(v\,\epsilon^{n,i}[t]) \\
&= \textstyle\sum_{0\leqq i\leqq n-1} (-1)^i\,(v\,[\epsilon^{n-1,i}t]) + (-1)^n\, vt \\
&= \textstyle\sum_{0\leqq i\leqq n-1} (-1)^i \left(u_{\epsilon^{n-1,i}t}, \pi_{n-1}\,\epsilon^{n-1,i}t\right) \\
&\quad + (-1)^n \left(\textstyle\sum_{1\leqq j\leqq m} u_{x_j}^{\pi_{n-1}\widehat{x}_j}, \pi_{n-1}x\right), \\
&= \textstyle\sum_{0\leqq i\leqq n-1} (-1)^i \left(u_{\epsilon^{n-1,i}t}, \pi_n t\right) \\
&\quad + (-1)^n \left(\textstyle\sum_{1\leqq j\leqq m} u_{x_j}^{\pi_{n-1}\widehat{x}_j}, \pi_{n-1}x\right),
\end{aligned}$$

since $\pi_{n-1}\circ\epsilon^{n-1,i} = \pi_n$, where $\widehat{x}_j = x_1\cdots x_{j-1}x_{j+1}\cdots x_m$ and $x = x_1\cdots x_m$, so that $\pi_{n-1}x = \pi_n t$. Thus

$$(\delta'_n u)_t = \textstyle\sum_{0\leqq i\leqq n-1} (-1)^i\, u_{\epsilon^{n-1,i}t} + (-1)^n \sum_{1\leqq j\leqq m} u_{x_j}^{\pi_{n-1}\widehat{x}_j}$$

for all $t = [x_1,\dots,x_m] \in T_n$ and we have proved:

Theorem 4.3. *Let S be a commutative semigroup. Up to natural isomorphisms, the Beck cohomology of S has coefficients in an abelian group valued functor $\mathbb{G}$ on $H(S)$, and $H^n(S,\mathbb{G})$ is the n-th homology group* $\mathrm{Ker}\,\delta_n / \mathrm{Im}\,\delta_{n-1}$ *of the complex*

$$0 \longrightarrow C^1(S,\mathbb{G}) \longrightarrow \cdots \xrightarrow{\delta_{n-1}} C^n(S,\mathbb{G}) \xrightarrow{\delta_n} C^{n+1}(S,\mathbb{G}) \longrightarrow \cdots$$

where $C^n(S,\mathbb{G}) = \prod_{t\in T_{n-1}} G_{\pi_{n-1}t}$ and

$$(\delta_n u)_t \;=\; \textstyle\sum_{0 \leqq i \leqq n-1} (-1)^i \, u_{\epsilon^{n-1,i}\,t} \;+\; (-1)^n \sum_{1 \leqq j \leqq m} u_{x_j}^{\pi_{n-1}\widehat{x}_j} \qquad (\delta)$$

for all $t = [x_1, \ldots, x_m] \in T_n$, *where* $\widehat{x}_j = x_1 \cdots x_{j-1}\, x_{j+1} \cdots x_m$.

If $m = 1$, then $\widehat{x}_j$ is an empty product and $\widehat{x}_j = 1 \in T^1_{n-1}$.

4. Theorem 4.3 describes $H^1(S,\mathbb{G})$ and $H^2(S,\mathbb{G})$ as follows.

A 1-cochain $u \in C^1(S,\mathbb{G})$ is a family $u = (u_a)_{a \in S}$ such that $u_a \in G_a$ for all $a \in S$ ($u \in \prod_{s \in S} G_a$). When $t = [a_1, \ldots, a_m] \in T_1$, so that $m > 0$ and $a_1, \ldots, a_m \in S$, and $a = a_1 \cdots a_m$, then $\pi_1 t = \pi_0(a_1 \cdots a_m) = a$ by (π) and $\epsilon^{0,0} t = a_1 \cdots a_m = a = \pi_1 t$ by (ϵ'). Thus (δ) reads

$$(\delta_1 u)_t \;=\; u_a \;-\; \textstyle\sum_{1 \leqq j \leqq m} u_{a_j}^{\widehat{a}_j}$$

for all $t = [a_1, \ldots, a_m] \in T_1$, where $a = a_1 \cdots a_m$ and $\widehat{a}_j = a_1 \cdots a_{j-1}\, a_{j+1} \cdots a_m$. Hence u is a 1-cocycle if and only if

$$u_{a_1 \cdots a_m} \;=\; \textstyle\sum_{1 \leqq j \leqq m} u_{a_j}^{\widehat{a}_j}$$

for all $a_1, \ldots, a_m \in S$; equivalently, if $u_{ab} = u_a^b + u_b^a$ for all $a, b \in S$.

A 2-cochain $u \in C^2(S,\mathbb{G}) = \prod_{x \in T_1} G_{\pi_1 x}$ is a family $u = (u_x)_{x \in T_1}$ such that $u_x \in G_{\pi_1 x}$ for all $x \in T_1$. When $A = [x_1, \ldots, x_m] \in T_2$, so that $m > 0$ and $x_1, \ldots, x_m \in T_1$, and $x = x_1 \cdots c_m$, then $\pi_2 t = \pi_1(x_1 \cdots x_m) = \pi_1 x$, $\epsilon^{1,1} t = x_1 \cdots x_m = x \in T_1$ by (ϵ'), and

$$\epsilon^{1,0} t \;=\; [\epsilon^{0,0} x_1, \;\ldots,\; \epsilon^{0,0} x_m] \;=\; [\pi_1 x_1, \;\ldots,\; \pi_1 x_m]$$

by (ϵ''). Thus (δ) reads

$$(\delta_2 u)_{[x_1, \ldots, x_m]} \;=\; u_{[\pi x_1, \ldots, \pi x_m]} \;-\; u_{x_1 \cdots x_m} \;+\; \textstyle\sum_{1 \leqq j \leqq m} u_{x_j}^{\pi \widehat{x}_j}$$

for all $x_1, \ldots, x_m \in T_1$, where $\pi = \pi_1$ and $\widehat{x}_j = x_1 \cdots x_{j-1}\, x_{j+1} \cdots x_m$. Hence u is a 2-cocycle if and only if

$$u_{x_1 \cdots x_m} \;=\; u_{[\pi x_1, \ldots, \pi x_m]} \;+\; \textstyle\sum_{1 \leqq j \leqq m} u_{x_j}^{\pi \widehat{x}_j}$$

for all $m > 0$ and $x_1, \ldots, x_m \in T_1$; u is a 2-coboundary if and only if $u = \delta v$ for some 1-cochain v:

$$u_{[a_1, \ldots, a_m]} \;=\; v_{a_1 \cdots a_m} \;-\; \textstyle\sum_{1 \leqq j \leqq m} v_{a_j}^{\pi \widehat{a}_j} \,.$$

In the above, m is unbounded; computing $H^2(S,\mathbb{G})$ by Theorem 4.3 is therefore an infinite task, even when S is finite. Fortunately, more effective techniques are just around the corner.

5. Commutative semigroup cohomology inherits a number of properties from triple cohomology in general. Theorems 1.1, 1.2, and 4.3 yield:

Theorem 4.4. *When S is a free commutative semigroup, then $H^n(S,\mathbb{G}) = 0$ for all $n \geqq 2$.*

Theorem 4.5. *Every short exact sequence $\mathcal{G} : \mathbb{G} \longrightarrow \mathbb{G}' \longrightarrow \mathbb{G}''$ of abelian group valued functors on $H(S)$ induces an exact sequence*

$$\cdots\ H^n(S,\mathbb{G}) \longrightarrow H^n(S,\mathbb{G}') \longrightarrow H^n(S,\mathbb{G}'') \longrightarrow H^{n+1}(S,\mathbb{G})\ \cdots$$

which is natural in $\mathcal{G}$.

Proposition 3.1 and Theorem 1.4 yield

Theorem 4.6. *There is a one-to-one correspondence between elements of $H^2(S,\mathbb{G})$ and equivalence classes of commutative group coextensions of S by $\mathbb{G}$, which is natural in $\mathbb{G}$.*

Theorem 4.6 provides a bijection between $H^2(S,\mathbb{G})$ and the abelian group $\operatorname{Ext}(S,\mathbb{G})$ from Chapter V. In the next section we shall prove that the two groups are in fact isomorphic. Hence $H^2(S,\mathbb{G})$ can be calculated from factor sets and split factor sets, which one may assume are normalized.

6. We conclude this section by constructing a projective complex of which $H^n(S,\mathbb{G})$ is the cohomology; this takes place in the abelian category $\mathcal{A}$ of abelian group valued functors on $H(S)$.

Let $\overline{T} = (T,\pi)$ be a c.s. over S. For every $a \in S$ let $K_a = K_a(\overline{T})$ be the free abelian group generated by the set

$$X_a \;=\; X_a(\overline{T}) \;=\; \{(t,z) \in T \times S^1 \mid (\pi t)z = a\}.$$

When $v \in S^1$, $(t,z) \in X_a$ implies $(t,zv) \in X_{av}$ and there is a unique homomorphism $\kappa_{a,v} : K_a \longrightarrow K_{av}$ such that

$$\kappa_{a,v}(t,z) \;=\; (t,zv)$$

whenever $(t,z) \in X_a$. We see that $\kappa_{a,1}$ is the identity on K_a and that $\kappa_{av,w} \circ \kappa_{a,v} = \kappa_{a,vw}$. Thus $\mathbb{K}\overline{T} = (K,\kappa)$ is an abelian group valued functor on $H(S)$.

If $f : \overline{T} \longrightarrow \overline{T}'$ is a morphism in $\overline{\mathcal{C}}$, then $(t,z) \in X_a(\overline{T})$ implies $(ft,z) \in X_a(\overline{T}')$ and f induces a homomorphism $(\mathbb{K}f)_a : K_a(\overline{T}) \longrightarrow K_a(\overline{T}')$. It is immediate that $\mathbb{K}f : \mathbb{K}\overline{T} \longrightarrow \mathbb{K}\overline{T}'$ is a natural transformation. This constructs a functor $\mathbb{K} : \overline{\mathcal{C}} \longrightarrow \mathcal{A}$.

Lemma 4.7. *For every object $\overline{T} = (T, \pi)$ of $\overline{\mathcal{C}}$ and abelian group valued functor $\mathbb{G}$ on $H(S)$ there is an isomorphism $\mathrm{Hom}_{\mathcal{A}}(\mathbb{K}\overline{T}, \mathbb{G}) \cong C(\overline{T}, \mathbb{G})$ which is natural in $\overline{T}$ and $\mathbb{G}$. Hence $\mathbb{K}\overline{T}$ is projective.*

Lemma 4.7 shows that the functor $C(\overline{T}, -)$ is representable. Hence the isomorphism $\mathrm{Hom}_{\mathcal{A}}(\mathbb{K}\overline{T}, \mathbb{G}) \cong C(\overline{T}, \mathbb{G})$ determines $\mathbb{K}$ uniquely up to natural isomorphism.

Proof. Let $\varphi : K \longrightarrow \mathbb{G}$ be a natural transformation, where $K = \mathbb{K}\overline{T}$. For every $t \in T$, $(t, 1) \in X_{\pi t}$ and

$$\overline{\varphi}_t \;=\; \varphi_{\pi t}(t, 1) \;\in\; G_{\pi t}\,.$$

This constructs $\overline{\varphi} \in C(\overline{T}, \mathbb{G}) = \prod_{t \in T} G_{\pi t}$. Since φ is natural we have

$$\varphi_a(t, z) \;=\; \varphi_a\, \kappa_{\pi t, z}(t, 1) \;=\; \gamma_{\pi t, z}\, \varphi_{\pi t}(t, 1) \;=\; \gamma_{\pi t, z}\, \overline{\varphi}_t$$

for all $(t, z) \in X_a$. Thus φ is uniquely determined by $\overline{\varphi}$.

Conversely let $u \in C(\overline{T}, \mathbb{G})$. Define $\varphi_a : K_a \longrightarrow G_a$ by:

$$\varphi_a(t, z) \;=\; \gamma_{\pi t, z}\, u_t$$

for all $(t, z) \in X_a$. Then

$$\gamma_{a,v}\, \varphi_a(t, z) = \gamma_{a,v}\, \gamma_{\pi t, z}\, u_t = \gamma_{\pi t, vz}\, u_t = \varphi_{av}(t, vz) = \varphi_{av}\, \kappa_{a,v}(t, z)$$

for all $(t, z) \in X_a$. Hence $\varphi : K \longrightarrow \mathbb{G}$ is a natural transformation. We see that $\overline{\varphi} = u$. Thus $\mathrm{Hom}_{\mathcal{A}}(K, \mathbb{G}) \cong C(\overline{T}, \mathbb{G})$. It is immediate that this isomorphism is natural in $\overline{T}$ and $\mathbb{G}$.

If $\sigma : \mathbb{G} \longrightarrow \mathbb{G}'$ is an epimorphism in $\mathcal{A}$, then every $\sigma_a : G_a \longrightarrow G'_a$ is surjective, $C(\overline{T}, \sigma) = \prod_{t \in T} \sigma_{\pi t} : \prod_{t \in T} G_{\pi t} \longrightarrow \prod_{t \in T} G'_{\pi t}$ is surjective, and $\mathrm{Hom}_{\mathcal{A}}(K, \sigma)$ is an epimorphism. Thus K is projective in $\mathcal{A}$. $\square$

Proposition 4.8. *$H^n(S, \mathbb{G})$ is the n-th cohomology group of the projective complex*

$$0 \longleftarrow \mathbb{C}_1(S) \longleftarrow \cdots \longleftarrow \mathbb{C}_n(S) \longleftarrow \mathbb{C}_{n+1}(S) \longleftarrow \;\cdots$$

where $\mathbb{C}_n(S) = \mathbb{K}\overline{T}_{n-1}$ and $\partial : \mathbb{C}_{n+1}(S) \longrightarrow \mathbb{C}_n(S)$ is given by

$$\partial_a(t, z) \;=\; \textstyle\sum_{0 \leqq i \leqq n-1} (-1)^i\, (\epsilon^{n-1,i}\, t,\, z) \;+\; (-1)^n \sum_{1 \leqq j \leqq m} (x_j,\, (\pi_{n-1}\widehat{x}_j)\, z)$$

for all $(t,z) \in X_a$, *where* $t = [x_1,\dots,x_m] \in T_n$ *and* $\widehat{x}_j = x_1 \cdots x_{j-1}\, x_{j+1} \cdots x_m$.

Proof. By Lemma 4.7 there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(\mathbb{C}_n(S),\mathbb{G}) = \mathrm{Hom}_{\mathcal{A}}(\mathbb{K}\overline{T}_{n+1},\mathbb{G}) \xrightarrow{\cong} C(\overline{T}_{n+1},\mathbb{G}) = C^n(S,\mathbb{G}).$$

We show that these isomorphisms take the coboundary homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\partial,\mathbb{G}) : \mathrm{Hom}_{\mathcal{A}}(\mathbb{C}_n,\mathbb{G}) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathbb{C}_{n+1},\mathbb{G}),$$

$\varphi \longmapsto \varphi \circ \partial$, to the coboundary homomorphism δ_n in Theorem 4.3.

Let $u \in C^n(S,\mathbb{G})$. To u corresponds the natural transformation φ from $\mathbb{K}\overline{T}_{n-1}$ to $\mathbb{G}$ defined by

$$\varphi_a(t,z) \;=\; \gamma_{\pi t,z}\, u_t \;=\; u_t^z$$

for all $(t,z) \in X_a(\overline{T}_{n-1})$. Let $v \in C^{n+1}(S,\mathbb{G})$ correspond to $\varphi \circ \partial$. For every $t = [x_1,\dots,x_m] \in T_n$,

$$\begin{aligned}
v_t &= \varphi_{\pi t}\big(\partial_{\pi t}(t,1)\big)\\
&= \varphi_{\pi t}\big(\textstyle\sum_{0 \leqq i \leqq n-1} (-1)^i\, (\epsilon^{n-1,i}\, t,\, 1)\\
&\qquad + (-1)^n \textstyle\sum_{1 \leqq j \leqq m} (x_j,\, (\pi_{n-1}\widehat{x}_j)\, 1)\big)\\
&= \textstyle\sum_{0 \leqq i \leqq n-1} (-1)^i\, u_{\epsilon^{n-1,i}\, t} \;+\; (-1)^n \textstyle\sum_{1 \leqq j \leqq m} u_{x_j}^{\pi_{n-1}\widehat{x}_j}\\
&= (\delta_n u)_t.
\end{aligned}$$

Thus $v = \delta_n u$. □

The complex in Proposition 4.8 is not very barlike, since the generators (t,v) of $\mathbb{C}_n(S)$ include sequences $t \in T_{n-1}$ of unbounded length when $n \geqq 2$. It is not known in general whether there is a commutative "bar" complex in which t is replaced by a sequence of length n. Results in the next section indicate how the first groups of such a complex might be constructed.

7. Simpler chains can be used when coefficient functors are constant, or nearly constant. We call an abelian group valued functor $\mathbb{G} = (G,\gamma)$ on S **constant** when there is an abelian group A such that $G_a = A$ and $\gamma_{a,t} = 1_A$ for all $a \in S$ and $t \in S^1$. Then $\mathbb{G}$ and A may be identified, and we denote $C^n(S,\mathbb{G})$ by $C^n(S,A)$, and similarly for B^n, Z^n, and H^n. Constant functors are thin and surjecting. Cohomology with constant coefficients is the commutative analogue of the Eilenberg-MacLane cohomology for monoids.

When $\overline{T} = (T, \pi)$ is a c.s. over S let $K(\overline{T})$ be the free abelian group generated by the set T. For every abelian group G we have $\mathrm{Hom}\,(K(\overline{T}), G) \cong \prod_{t \in T} G = C(\overline{T}, G)$. Hence $H^n(S, G)$ is, as in Proposition 4.8, the n-th cohomology group of the complex

$$C(S): \ 0 \longleftarrow C_1(S) \longleftarrow \cdots \longleftarrow C_n(S) \longleftarrow C_{n+1}(S) \longleftarrow \ \cdots$$

where $C_n(S) = K(\overline{T}_{n-1})$ and $\partial : C_{n+1}(S) \longrightarrow C_n(S)$ is given by

$$\partial t \ = \ \textstyle\sum_{0 \leqq i \leqq n-1} (-1)^i \, \epsilon^{n-1,i}\, t \ + \ (-1)^n \sum_{1 \leqq j \leqq m} x_j$$

for all $t = [x_1, \ldots, x_m] \in T_n$. Since $C(S)$ is a complex of free abelian groups, there is for every abelian group G a Universal Coefficient Theorem

$$H^n(S, G) \ \cong \ \mathrm{Ext}\,(H_{n-1}(S), G) \oplus \mathrm{Hom}\,(H_n(S), G)$$

where $H_n(S)$ is the n-th homology group of $C(S)$, with $H_1(S) = C^1(S)/\mathrm{Im}\,\partial^1$ and $H_0(S) = 0$ (MacLane [1963]). We leave the details to our tireless reader.

When S has a zero element, then $H^n(S, G) = 0$ for every abelian group G, at least when $n \leqq 3$. This can be remedied by using functors that are not quite constant. When S has a zero element, an abelian group valued functor $\mathbb{G}$ on S is **almost constant** when there exists an abelian group A such that $G_a = A$ for all $a \neq 0$, $G_0 = 0$, and $\gamma_{a,t} = 1_A$ whenever $at \neq 0$. Then $\mathbb{G}$ may be identified with the abelian group A, and we denote the cohomology groups $H^n(S, \mathbb{G})$ by $H^n_0(S, A)$, and similarly for cochains, cocycles, and coboundaries. Almost constant functors are thin and surjecting.

As noted in Grillet [1974], almost constant functors arise naturally in the construction of homogeneous elementary semigroups. The Universal Coefficient Theorem can be saved if in the above we replace $K(\overline{T})$ by the almost constant functor $\mathbb{K}^0(\overline{T})$ in which the abelian group is the free abelian group generated by $\{t \in T \mid \pi t \neq 0\}$. Again we leave the details to our reader.

5. SYMMETRIC COCHAINS.

Commutative cohomology cries out for an equivalent description in which n-cochains are functions of n variables. This has been found only for $n \leqq 4$.

1-cochains already are functions $u = (u_a)_{a \in S}$ of one variable $a \in S$.

1. In dimension 2, Theorem 4.6 provides a one-to-one correspondence be-

tween the elements of $H^2(S,\mathbb{G})$ and the elements of the abelian group $\operatorname{Ext}(S,\mathbb{G})$ of all equivalence classes of commutative group coextensions of S by $\mathbb{G}$, which is also the abelian group of all equivalence classes of commutative factor sets on S with values in $\mathbb{G}$.

We now construct a more direct connection between factor sets and 2-cocycles, which induces an isomorphism $H^2(S,\mathbb{G}) \cong \operatorname{Ext}(S,\mathbb{G})$.

Let T_1 be the free commutative semigroup on the set S and $\pi = \pi_1 : T_1 \longrightarrow S$, so that $\pi[a_1, a_2, \dots, a_m] = a_1 a_2 \cdots a_m$ for all $a_1, \dots, a_m \in S$. As we saw in Section 4, a 2-cochain $u \in C^2(S,\mathbb{G}) = \prod_{x \in T_1} G_{\pi x}$ is a family $u = (u_x)_{x \in T_1}$ such that $u_x \in G_{\pi x}$ for all $x \in T_1$; a 2-cocycle is a 2-cochain u such that

$$u_{x_1 \cdots x_m} = u_{[\pi x_1, \dots, \pi x_m]} + \textstyle\sum_{1 \leqq j \leqq m} u_{x_j}^{\pi \widehat{x_j}} \tag{Z}$$

for all $m > 0$ and $x_1, \dots, x_m \in T_1$, where $\widehat{x_j} = x_1 \cdots x_{j-1}\, x_{j+1} \cdots x_m$; a 2-coboundary is a 2-cochain u (necessarily a 2-cocycle) of the form $u = \delta v$,

$$(\delta v)_{[a_1, \dots, a_l]} = v_{a_1 \cdots a_l} - \textstyle\sum_{1 \leqq i \leqq l} v_{a_i}^{\pi \widehat{a_i}} \tag{B}$$

for some 1-cochain v, where $\widehat{a_i} = a_1 \cdots a_{i-1}\, a_{i+1} \cdots a_l$.

A **symmetric 2-cochain** on S with values in $\mathbb{G}$ is a family $s = (s_{a,b})_{a,b \in S}$ such that $s_{a,b} \in G_{ab}$ and

$$s_{b,a} = s_{a,b}$$

for all $a, b \in S$. Under pointwise addition, symmetric 2-cochains constitute an abelian group $SC^2(S,\mathbb{G}) = \prod_{a,b \in S} G_{ab}$. A **symmetric 2-cocycle** on S with values in $\mathbb{G}$ is a symmetric 2-cochain s such that

$$s_{a,b}^c + s_{ab,c} = s_{a,bc} + s_{b,c}^a \tag{A}$$

for all $a, b, c \in S$; equivalently, a commutative factor set on S with values in $\mathbb{G}$ as defined in Section V.4. A **symmetric 2-coboundary** on S with values in $\mathbb{G}$ is a symmetric 2-cochain (necessarily, a symmetric 2-cocycle) s of the form

$$s_{a,b} = v_a^b + v_b^a - v_{ab}$$

for some 1-cochain v; equivalently, a split factor set. Under pointwise addition, symmetric 2-coboundaries and 2-cocycles form abelian groups $SB^2(S,\mathbb{G}) \subseteq SZ^2(S,\mathbb{G}) \subseteq SC^2(S,\mathbb{G})$.

We saw in Section V.4 that

$$\operatorname{Ext}(S,\mathbb{G}) \cong SZ^2(S,\mathbb{G}) / SB^2(S,\mathbb{G}).$$

2. Restriction to $S \times S$ yields a **trimming homomorphism** Σ of $C^2(S,\mathbb{G})$ into $SC^2(S,\mathbb{G})$; when u is a 2-cochain, Σu is given by

$$(\Sigma u)_{a,b} = u_{[a,b]}$$

for all $a,b \in S$; Σu is symmetric since $[b,a] = [a,b]$ in T_1.

Lemma 5.1. *When u is a 2-cocycle, then $u_{[a]} = 0$ for all $a \in S$ and*

$$u_{[a_1,\dots,a_l]} = \textstyle\sum_{1 \leqq i \leqq l-1} u^{a''_{i+1}}_{[a'_i,\, a_{i+1}]} \tag{Z$'$}$$

for all $l > 0$ and $a_1,\dots,a_l \in S$, where $a'_i = a_1 \cdots a_i$ and $a''_{i+1} = a_{i+2} \cdots a_l$.

Proof. When $m = 1$ and $x_1 = [a]$, then $\pi x_1 = [a]$, (Z) reads: $u_{x_1} = u_{x_1} + u^1_{x_1}$, and $u_{x_1} = 0$.

Now let $i \geqq 1$ and $a_1,\dots,a_{i+1} \in S$. With $m = 2$, $x_1 = [a_1,\dots,a_i]$, and $x_2 = [a_{i+1}]$, then $\pi x_1 = a'_i$, $\pi x_2 = a_{i+1}$, $u_{x_2} = 0$, and (Z) reads

$$u_{[a_1,\dots,a_{i+1}]} = u_{[a'_i,\, a_{i+1}]} + u^{a_{i+1}}_{[a_1,\dots,a_i]}\,. \tag{Z$''$}$$

(Z$'$) is proved by induction on l, using (Z$''$). First, (Z$'$) is trivial when $l = 1$ (then the right hand side is the empty sum 0) and when $l = 2$ (then a''_2 is the empty product and $a''_2 = 1 \in S^1$). If $l = 3$, then (Z$''$) yields

$$u_{[a_1,a_2,a_3]} = u^{a_3}_{[a_1,a_2]} + u_{[a'_2,a_3]}$$

and (Z$'$) holds. If $l > 3$, then, with $b''_{i+1} = a_{i+2} \cdots a_{l-1}$, (Z$''$) and the induction hypothesis yield

$$\begin{aligned} u_{[a_1,\dots,a_l]} &= u^{a_l}_{[a_1,\dots,a_{l-1}]} + u_{[a'_{l-1},\, a_l]} \\ &= \left(\textstyle\sum_{1 \leqq i \leqq l-2} u^{b''_{i+1}}_{[a'_i,\, a_{i+1}]}\right)^{a_l} + u_{[a'_{l-1},\, a_l]} \\ &= \textstyle\sum_{1 \leqq i \leqq l-1} u^{a''_{i+1}}_{[a'_i,\, a_{i+1}]}. \ \square \end{aligned}$$

Lemma 5.1 shows that $\Sigma : Z^2(S,\mathbb{G}) \longrightarrow SC^2(S,\mathbb{G})$ is injective.

Lemma 5.2. $\operatorname{Im}\Sigma = SZ^2(S,\mathbb{G})$.

Proof. Let u be a 2-cocycle. With $m = 2$, $x_1 = [a]$, and $x_2 = [b,c]$, (Z) reads

$$u_{[a,b,c]} = u_{[a,\, bc]} + u^a_{[b,c]}$$

(since $u_{[a]} = 0$). With $m = 2$, $x_1 = [a,b]$, and $x_2 = [c]$, (Z) reads

$$u_{[a,b,c]} = u_{[ab,c]} + u^c_{[a,b]} \,.$$

Therefore $\Sigma u \in SZ^2(S,G)$.

Conversely let $s \in SZ^2(S,\mathbb{G})$. Since Σ is injective, there is at most one $u \in Z^2(S,\mathbb{G})$ such that $\Sigma u = s$, and it is given by (Z′). Accordingly, define $t = (t_{a_1,\dots,a_l})_{l>0,\, a_1,\dots,a_l \in S}$ by

$$t_{a_1,\dots,a_l} = \textstyle\sum_{1 \leqq i \leqq l-1} s^{a''_{i+1}}_{a'_i,\, a_{i+1}}$$

for all $l > 0$ and $a_1,\dots,a_l \in S$, where $a'_i = a_1 \cdots a_i$ and $a''_{i+1} = a_{i+2} \cdots a_l$. In particular, $t_a = 0$ (if $l = 1$, then the right hand side is the empty sum 0) and $t_{a,b} = s_{a,b}$, for all $a,b \in S$. Also

$$\begin{aligned} t_{a_1,\dots,a_l,b} &= \textstyle\sum_{1 \leqq i \leqq l-1} s^{a''_{i+1}b}_{a'_i,\, a_{i+1}} + s_{a'_l,b} \\ &= \big(\textstyle\sum_{1 \leqq i \leqq l-1} s^{a''_{i+1}}_{a'_i,\, a_{i+1}}\big)^b + s_{a'_l,b} \\ &= t^b_{a_1,\dots,a_l} + t_{a'_l,b} \end{aligned}$$

for all $l > 0$ and $a_1,\dots,a_l, b \in S$; thus t satisfies (Z″).

We show by induction on $l > 0$ that

$$t_{a_{\sigma 1},\dots,a_{\sigma l}} = t_{a_1,\dots,a_l} \tag{P}$$

for every $a_1,\dots,a_l \in S$ and permutation σ. This is trivial if $l = 1$ and follows from $t_{a,b} = s_{a,b}$ if $l = 2$. When $l > 2$, σ is a product of transpositions of the form $(j\ \ j+1)$ and it suffices to prove (P) when $\sigma = (j\ \ j+1)$. If $j < l-1$, then (P) follows from the induction hypothesis and (Z″). If $j = l-1$, then, with $a'_{l-2} = b$, $a_{l-1} = c$, $a_l = d$, we have $a'_{l-1} = bc$, $a''_{l-1} = d$, and

$$\begin{aligned} t_{a_1,\dots,a_l} &= \textstyle\sum_{1 \leqq i \leqq l-3} s^{a''_{i+1}}_{a'_i,\, a_{i+1}} + s^d_{b,c} + s_{bc,d}\,, \\ t_{a_{\sigma 1},\dots,a_{\sigma l}} &= \textstyle\sum_{1 \leqq i \leqq l-3} s^{a''_{i+1}}_{a'_i,\, a_{i+1}} + s^c_{b,d} + s_{bd,c}\,, \end{aligned}$$

and (P) holds, since $s \in SZ^2(S,\mathbb{G})$ yields

$$s^d_{b,c} + s_{bc,d} = s^d_{c,b} + s_{cb,d} = s_{c,bd} + s^c_{b,d} = s^c_{b,d} + s_{bd,c} \,.$$

By (P), a 2-cochain $u \in C^2(S,\mathbb{G})$ is well defined by

$$u_{[a_1,\dots,a_l]} \;=\; t_{a_1,\dots,a_l}$$

for all $l > 0$ and $a_1,\dots,a_l \in S$. We show that $u \in Z^2(S,\mathbb{G})$; that is,

$$u_{x_1\cdots x_m} \;=\; u_{[\pi x_1,\dots,\pi x_m]} \;+\; \textstyle\sum_{1\leqq j\leqq m} u_{x_j}^{\pi\widehat{x}_j} \tag{Z}$$

holds for all $m > 0$ and $x_1,\dots,x_m \in T_1$, with $\widehat{x}_j = x_1\cdots x_{j-1}\,x_{j+1}\cdots x_m$.

(Z) is trivial if $m = 1$; for $m > 1$ we proceed by induction on m. Assume that (Z) holds for m and let $y_1,\dots,y_m, z \in T_1$. Let $x = y_1\cdots y_m = [a_1,\dots,a_l]$, $z = [c_1,\dots,c_n]$, $\pi y_j = b_j$, $\pi x = a = a_1\cdots a_n = b_1\cdots b_m$, and $\pi z = c = c_1\cdots c_n$. By the induction hypothesis,

$$u_x \;=\; u_{[b_1,\dots,b_l]} \;+\; \textstyle\sum_{1\leqq j\leqq m} u_{y_j}^{\widehat{b}_j}$$

where $\widehat{b}_j = b_1\cdots b_{j-1}\,b_{j+1}\cdots b_m$. Hence

$$\begin{aligned}
u_{xz} \;&=\; u_{[a_1,\dots,a_l,\, c_1,\dots,c_n]} \\
&=\; \textstyle\sum_{1\leqq i\leqq l-1} s_{a'_i,\, a_{i+1}}^{a''_{i+1}c} \;+\; \textstyle\sum_{0\leqq k\leqq n-1} s_{ac'_k,\, c_{k+1}}^{c''_{k+1}} \\
&=\; u_x^c \;+\; s_{a,c_1}^{c''_1} \;+\; \textstyle\sum_{1\leqq k\leqq n-1} s_{ac'_k,\, c_{k+1}}^{c''_{k+1}} \\
&=\; u_{[b_1,\dots,b_l]}^c \;+\; \textstyle\sum_{1\leqq j\leqq m} u_{y_j}^{\widehat{b}_j c} \;+\; s_{a,c_1}^{c''_1} \\
&\qquad +\; \textstyle\sum_{1\leqq k\leqq n-1} \big(-\, s_{a,c'_k}^{c_{k+1}} \;+\; s_{a,\, c'_k c_{k+1}} \;+\; s_{c'_k\,,c_{k+1}}^{a}\big)^{c''_{k+1}}
\end{aligned}$$

by the induction hypothesis and (A). Hence

$$\begin{aligned}
u_{xz} \;&=\; u_{[b_1,\dots,b_l]}^c \;+\; \textstyle\sum_{1\leqq j\leqq m} u_{y_j}^{\widehat{b}_j c} \;+\; s_{a,c_1}^{c''_1} \;-\; \textstyle\sum_{1\leqq k\leqq n-1} s_{a,c'_k}^{c''_k} \\
&\qquad +\; \textstyle\sum_{2\leqq k\leqq n} s_{a,c'_k}^{c''_k} \;+\; \big(\textstyle\sum_{1\leqq k\leqq n-1} s_{c'_k\,,c_{k+1}}^{c''_{k+1}}\big)^{a} \\
&=\; u_{[b_1,\dots,b_l]}^c \;+\; \textstyle\sum_{1\leqq j\leqq m} u_{y_j}^{\widehat{b}_j c} \;+\; s_{a,c} \;+\; u_z^a \\
&=\; u_{[b_1,\dots,b_l,c]} \;+\; \textstyle\sum_{1\leqq j\leqq m} u_{y_j}^{\widehat{b}_j c} \;+\; u_z^a
\end{aligned}$$

by (Z”), and (Z) holds for $m+1$. Thus $u \in Z^2(S,\mathbb{G})$. Then $\Sigma u = s$. □

Lemma 5.3. $\Sigma\,\big(B^2(S,\mathbb{G})\big) = SB^2(S,\mathbb{G})$.

Proof. When $v \in C^1(S,\mathbb{G})$, then (B) yields $(\delta v)_{[a,b]} = v_{ab} - v_a^b - v_b^a$. □

3. Lemmas 5.1, 5.2, and 5.3 provide an isomorphism $Z^2(S,\mathbb{G}) \cong SZ^2(S,\mathbb{G})$ which is natural in G, sends $B^2(S,\mathbb{G})$ to $SB^2(S,\mathbb{G})$, and induces an isomorphism $Z^2(S,\mathbb{G}) \,/\, B^2(S,\mathbb{G}) \cong SZ^2(S,\mathbb{G}) \,/\, SB^2(S,\mathbb{G})$. This proves:

Theorem 5.4. *For every commutative semigroup S there is an isomorphism $H^2(S,\mathbb{G}) \cong \operatorname{Ext}(S,\mathbb{G})$ which is natural in $\mathbb{G}$.*

The isomorphism $H^2(S,\mathbb{G}) \cong SZ^2(S,\mathbb{G}) \,/\, SB^2(S,\mathbb{G})$ can be refined, using Proposition V.4.5, into an isomorphism

$$H^2(S,\mathbb{G}) \;\cong\; NSZ^2(S,\mathbb{G}) \,/\, NSB^2(S,\mathbb{G}),$$

where $NSZ^2(S,\mathbb{G})$ and $NSB^2(S,\mathbb{G})$ are the groups of symmetric 2-cocycles and 2-coboundaries s that are normalized ($s_{e,a} = 0$ whenever $e^2 = e \geqq_{\mathcal{H}} a$). Normalization can be confined to a single idempotent; when applied to the identity element, it yields:

Corollary 5.5. *When the commutative semigroup S has no identity element, then $H^2(S,\mathbb{G}) \cong H^2(S^1,\mathbb{G}')$, where $\mathbb{G}'$ extends $\mathbb{G}$ to S^1 so that $G'_1 = 0$.*

The study of $H^2(S,\mathbb{G})$ may therefore be limited to monoids. Theorems 5.6 and 5.7 below have similar corollaries for dimensions 3 and 4. It is probable that Corollary 5.5 extends to all dimensions.

When S is finite, the computation of $H^2(S,\mathbb{G})$ using Theorem 5.4 is a finite task, since a symmetric 2-cocycle consists of finitely many group elements subject to finitely many conditions, and a symmetric 2-coboundary is determined by finitely many group elements. Thus Theorem 5.4 is a marked improvement upon Theorem 4.3. But further improvement would not hurt.

For example, let S be the Volkov nilsemigroup

$$S \;=\; \langle\, a,b \;\big|\; a^3 = a^2b = ab^2 = b^4 = 0,\; a^2 = ab = b^3 \,\rangle.$$

(Example III.3.6); the elements of S are a, b, $c = b^2$, $d = a^2 = ab = b^3$, and 0. A normalized symmetric 2-cochain s on S consists of $s_{a,a}$, $s_{a,b}$, $s_{a,c}$, $s_{a,d}$, $s_{a,0}$, $s_{b,b}$, $s_{b,c}$, $s_{b,d}$, $s_{b,0}$, $s_{c,c}$, $s_{c,d}$, $s_{c,0}$, $s_{d,d}$, $s_{d,0}$, and $s_{0,0} = 0$ (since s is normalized).

Then s is a symmetric 2-cocycle if and only if

$$s^z_{x,y} \;+\; s_{xy,z} \;\;=\;\; s_{x,yz} \;+\; s^x_{y,z} \qquad\qquad A(x,y,z)$$

holds in G_{xyz} for all $x,y,z \in S$. We note that $A(x,y,z)$ is trivial if $x = z$ and remains unchanged if x and z are interchanged; moreover, $A(x,y,z)$ follows from

$$s_{y,x}^{z} + s_{xy,z} = s_{y,xz} + s_{x,z}^{y} \qquad A(y,x,z)$$

and

$$s_{y,z}^{x} + s_{yz,x} = s_{y,zx} + s_{z,x}^{y} \qquad A(y,z,x).$$

Hence it suffices to state $A(x,y,z)$ when $x \leqq y$ and $x < z$ in the lexicographic order $a < b < c < d < 0$ on S. If $x = a$, there are 5 choices for y and 4 choices for z; if $x = b$, there are 4 choices for y and 3 choices for z; if $x = c$, there are 3 choices for y and 2 choices for z; if $x = d$, there are 2 choices for y and 1 choice for z; this yields $20 + 12 + 6 + 2 = 40$ conditions.

We leave the actual conditions and their solutions to northern readers, who are blessed with long winter evenings. $H^2(S,\mathbb{G})$ will be computed in the next chapter when $\mathbb{G}$ is thin, using more efficient methods.

4. A **symmetric** 3-**cochain** on S with values in $\mathbb{G}$ is a family $t = (t_{a,b,c})_{a,b,c\in S}$ such that $t_{a,b,c} \in G_{abc}$ and

$$t_{c,b,a} = -t_{a,b,c} \quad \text{and} \quad t_{a,b,c} + t_{b,c,a} + t_{c,a,b} = 0$$

for all $a,b,c \in S$. These conditions are satisfied by the coboundary

$$(\delta s)_{a,b,c} = s_{a,b}^{c} - s_{a,bc} + s_{ab,c} - s_{b,c}^{a}$$

of every symmetric 2-cochain $s \in SC^2(S,\mathbb{G})$. Under pointwise addition, symmetric 3-cochains constitute an abelian group $SC^3(S,\mathbb{G}) \subseteq \prod_{a,b,c\in S} G_{abc}$.

A **symmetric** 3-**cocycle** on S with values in $\mathbb{G}$ is a symmetric 3-cochain t such that

$$t_{a,b,c}^{d} - t_{a,b,cd} + t_{a,bc,d} - t_{ab,c,d} + t_{b,c,d}^{a} = 0$$

for all $a,b,c,d \in S$. This condition is satisfied by the coboundary $t = \delta s$ of every symmetric 2-cochain s; such a coboundary is a **symmetric** 3-**coboundary** on S with values in G. Under pointwise addition, symmetric 3-coboundaries and 3-cocycles form abelian groups $SB^3(S,\mathbb{G}) \subseteq SZ^3(S,\mathbb{G}) \subseteq SC^3(S,\mathbb{G})$. The following result is due to Grillet [1991C], [1997C]:

Theorem 5.6. *For every commutative semigroup S there is an isomorphism $H^3(S,\mathbb{G}) \cong SZ^3(S,\mathbb{G}) \,/\, SB^3(S,\mathbb{G})$ which is natural in $\mathbb{G}$.*

A **symmetric** 4-**cochain** on S with values in $\mathbb{G}$ is a family

$u = (t_{a,b,c,d})_{a,b,c,d \in S}$ such that $u_{a,b,c,d} \in G_{abcd}$ and

$$u_{a,b,b,a} = 0, \quad u_{d,c,b,a} = -u_{a,b,c,d}\,,$$
$$u_{a,b,c,d} - u_{b,c,d,a} + u_{c,d,a,b} - u_{d,a,b,c} = 0\,,$$
$$u_{a,b,c,d} - u_{b,a,c,d} + u_{b,c,a,d} - u_{b,c,d,a} = 0$$

for all $a,b,c,d \in S$. These conditions are satisfied by the coboundary

$$(\delta t)_{a,b,c,d} \;=\; t^{d}_{a,b,c} \;-\; t_{a,b,cd} \;+\; t_{a,bc,d} \;-\; t_{ab,c,d} \;+\; t^{a}_{b,c,d}$$

of every symmetric 3-cochain $s \in SC^3(S,\mathbb{G})$ (this is shown in the next section). Under pointwise addition, symmetric 4-cochains constitute an abelian group $SC^4(S,\mathbb{G}) \subseteq \prod_{a,b,c,d\in S} G_{abcd}$.

A **symmetric** 4-**cocycle** on S with values in $\mathbb{G}$ is a symmetric 4-cochain u such that

$$u^{e}_{a,b,c,d} \;-\; u_{a,b,c,de} \;+\; u_{a,b,cd,e} \;-\; u_{a,bc,d,e} \;+\; u_{ab,c,d,e} \;-\; u^{a}_{b,c,d,e} \;=\; 0$$

for all $a,b,c,d,e \in S$. A **symmetric** 4-**coboundary** on S with values in $\mathbb{G}$ is the coboundary $u = \delta t$ of a symmetric 3-cochain t. Under pointwise addition, symmetric 4-coboundaries and 4-cocycles form abelian groups $SB^4(S,\mathbb{G}) \subseteq SZ^4(S,\mathbb{G}) \subseteq SC^4(S,\mathbb{G})$. The following result is due to Grillet [2001H]:

Theorem 5.7. *For every commutative semigroup S there is an isomorphism $H^4(S,\mathbb{G}) \cong SZ^4(S,\mathbb{G}) \,/\, SB^4(S,\mathbb{G})$ which is natural in $\mathbb{G}$.*

The author's proofs of Theorem 5.6 and 5.7 are computational like the proof of Theorem 5.4, but much longer, and very likely to tax the reader's patience (even during long winter evenings). Better proofs would rely on Theorem 1.3, or on the underlying spectral sequence argument, to show that symmetric cochains define the same cohomology. This requires a general definition of symmetric cochains and a proof that $SZ^n(S,\mathbb{G}) = SB^n(S,\mathbb{G})$ when S is free. The author has proofs of this last fact when $n \leqq 3$, which are, unfortunately, entirely comparable in length and spirit to the proofs of Theorems 5.4 and 5.6.

Theorems 5.6 and 5.7 strongly suggest that symmetric cochains, cocycles, and coboundaries can be defined in every dimension n so that $H^n(S,\mathbb{G}) \cong SZ^n(S,\mathbb{G}) \,/\, SB^n(S,\mathbb{G})$. How to do this is still an open problem. In the above, the coboundary homomorphisms for symmetric cochains are essentially the same as in Leech cohomology and can be defined in all dimensions. The symmetry conditions in dimension $n+1$ can be defined by induction as all the symmetry conditions inherited by coboundaries of symmetric n-cochains. The difficulty lies in proving that $H^n(S,\mathbb{G}) \cong SZ^n(S,\mathbb{G}) \,/\, SB^n(S,\mathbb{G})$ (or that $SZ^n(S,\mathbb{G}) = SB^n(S,\mathbb{G})$ when S is free). There the matter rests, for now.