Multigroups over a commutative semigroup

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Abstract

This article introduces abelian multigroups, multirings, and multimodules over a commutative semigroup, with applications to cohomology. AMS classification: 20M50, 20M14.

Contents

1 Introduction

This article originates in an unknown referee's remark that the cohomology of commutative semigroups refines André-Quillen cohomology and should therefore be studied by the same methods.

Indeed this is the approach used by Kurdiani and Pirashvili in [7], using modules as coefficients. Unfortunately, the cohomology of a commutative semigroup S requires its coefficients to include one abelian group for each element of S [6], whereas a module can only assign the same abelian group to all elements of S. Applying André-Quillen methods ([1, 10, 11]; see also [12]) to the commutative semigroup S requires structures, such as the abelian group valued functors in [6], whose contents are spread among elements of S.

In Sections 2 through 12 we develop the rather boring basic algebra, adjunctions, and operations, of six such structures: multisets, abelian multigroups, and multimodules, which parallel sets, abelian groups, and modules; commutative multimonoids, multirings, and multialgebras, which parallel commutative monoids, rings, and algebras. It is clear that there are multimonoids, multirings, and multialgebras that are not necessarily commutative, but André-Quillen methods require commutativity.

We found that André-Quillen cohomology also requires a comonad (V, ϵ, ν) in which the elements of $V(X)$ can be written uniquely in more or less polynomial form. For technical reasons this rules out most of the comonads in Sections 2 to 12, including commutative multi-R-algebras, and Sections 13 through 16 develop the theory for commutative multirings only (without any input from Sections 9 to 12).

2 Multisets

In what follows, S is a given commutative semigroup or commutative monoid, and S^1 is the commutative monoid defined as follows: if S has an identity element, then $S^1 = S$; otherwise, $S^1 = S \cup \{1\}$.

Recall that a *set* X *over* S consists of a set X and a *projection* mapping of X into S, denoted regardless of X by $x \longmapsto x^\vee$. Projection to S decomposes X into a disjoint union $X = \bigcup_{a \in S} X_a$, where $X_a = \{ x \in X \mid x^\vee = a \}$.

Every subset Y of X inherits a projection to S which makes it a set over S, with a decomposition $Y = \bigcup_{a \in S} Y_a$, where $Y_a = Y \cap X_a$.

A *morphism* $f \colon X \longrightarrow Y$ of sets over S is a mapping $f \colon X \longrightarrow Y$ such that $(fx)^\vee = x^\vee$ for all $x \in X$; equivalently, $f(X_a) \subseteq Y_a$ for all $a \in S$.

1. Definition. A *multiset over* S is a set X over S together with an action $(x, t) \longmapsto x^t$ of S^1 on X such that

$$
\begin{aligned}
&(x^t)^\vee = x^\vee\, t\,,\\
&x^1 = x \quad \text{for all} \quad x\,, \quad \text{and}\\
&(x^t)^u = x^{tu} \quad \text{for all} \quad x, t, u\,.
\end{aligned}
$$

For example, if S has just one element, then a multiset over S is just a set. In general, S begets a multiset S on itself, in which $S_a = \{a\}$ and $a^t = at$.

Multisets over S have an equivalent definition as functors on the Leech category $\mathcal{H}(S)$, which is the small category (originally denoted by $\mathbb{R}(S)$ in [8]) whose objects are the elements of S, with one morphism $(a, t) \colon a \longrightarrow at$ for each $a \in S$ and $t \in S^1$. Composition is by $(at, u) \circ (a, t) = (a, tu)$; the identity morphism on $a \in S$ is $(a, 1)$.

Hence, a set-valued functor (X, χ) on $\mathcal{H}(S)$ assigns a set X_a to each $a \in S$ and a mapping $\chi_{a,t} \colon X_a \longrightarrow X_{at}$ to each $a \in S$ and $t \in S^1$, so that $\chi_{a,1}$ is the identity on X_a and $\chi_{at,u} \circ \chi_{a,t} = \chi_{a,tu}$ for all $a \in S$ and $t, u \in S^1$. We regard the sets X_a as pairwise disjoint, so that $X = \bigcup \big(X_a \mid a \in S \big)$, together with the projection $x^\vee = a$ whenever $x \in X_a$, is a set over S. An action of S^1 on X is then defined by

$$x^t \;=\; \chi_{a,t}\, x$$

and makes X a multiset over S. Conversely, a multiset X over S can be regarded as a set-valued functor on $\mathcal{H}(S)$, that assigns X_a to $a \in S$ and $\chi_{a,t} \colon x \longmapsto x^t$, $X_a \longrightarrow X_{at}$, to $a \in S$Z and $t \in S^1$.

A morphism $f \colon X \longrightarrow Y$ of multisets is a morphism of sets over S ($(fx)^\vee = x^\vee$ for all x) that preserves the action of S: $f(x^t) = (fx)^t$ for all x and t. Equivalently, the restrictions $f_a \colon X_a \longrightarrow Y_a$ of f constitute a natural transformation $f = (f_a)_{a \in S}$ from X to Y.

2. Submultisets. A *submultiset* Y of a multiset X is a subset Y of X that inherits the action of S: $x \in Y$ implies $x^t \in Y$. The action of S on X then induces an action of S on Y, which makes Y a multiset over S. Equivalently, a submultiset of X is a subfunctor.

For example, if $f\colon X \longrightarrow Y$ is a morphism of multisets, then $\operatorname{Im} f = f(X)$ is a submultiset of Y.

Given a multiset X and a subset Y of X, the submultiset $\overline{Y}$ of X *generated by* Y is the least submultiset of X that contains Y:

$$\overline{Y} \;=\; \{\, y^t \mid y \in Y,\, t \in S^1 \,\}\,.$$

For every $a \in S$,

$$\overline{Y}_a \;= \bigcup (\, Y_c^t \mid c \in S,\; t \in S^1,\; ct = a\,)\,.$$

3. Congruences. A *congruence* on a multiset X over S is an equivalence relation $\mathcal{C}$ on the set X such that

$$\begin{aligned} &x \;\mathcal{C}\; y \;\text{ implies }\; x^\vee = y^\vee \;\text{ and} \\ &x \;\mathcal{C}\; y \;\text{ implies }\; x^t = y^t \;\text{ for all }\; t \in S^1\,. \end{aligned}$$

For example, if $f\colon X \longrightarrow Y$ is a morphism of multisets over S, then the equivalence relation $\ker f$ on X induced by f is a congruence on X.

Conversely, if $\mathcal{C}$ is a congruence on a multiset X over S, then the quotient set $X/\mathcal{C}$ is a multiset over S, on which the projection to S and action of S^1 are well defined by $C(x)^\vee = x^\vee$ and $C(x)^t = C(x^t)$, where $C(x)$ denotes the $\mathcal{C}$-class of x, so that the projection $X \longrightarrow X/\mathcal{C}$ is a morphism of multisets over S.

If $f\colon X \longrightarrow Y$ is a morphism of multisets over S, then $X/\ker f \cong \operatorname{Im} f$.

4. Free multisets. There is in general no adjunction of sets to sets over S. Indeed the category of sets over S has products; the product of X and Y is given by the pullback (of sets) on the projections to S. Hence the forgetful functor from sets over S to sets does not preserve products (unless S has just one element), and has no left adjoint.

Similarly, in the category of multisets over S, the product of X and Y is given by the pullback (of sets) on p and q, which yields a multiset over S

$$P = \{\, (x,y) \in X \times Y \mid x^\vee = y^\vee \,\}$$

with projection $(x,y) \longmapsto x^\vee = y^\vee$ to S and action $(x,y)^t = (x^t,\, y^t)$ of S, so that the maps $(x,y) \longmapsto x$ and $(x,y) \longmapsto y$ are morphisms of multisets over S. Hence there is no adjunction of sets to multisets over S.

If S is a commutative monoid, however, there is an adjunction of sets over S to multisets over S. The *free multiset* X^+ *over* S *on* a set X over S is

$$X^+ \;=\; X \times S\,,$$

with projection $(x,a)^\vee = x^\vee a$ to S and action $(x,a)^t = (x, at)$ of S. Thus, for each $a \in S$,

$$(X^+)_a \;=\; \{\, (x,t) \mid x \in X,\; t \in S,\; x^\vee t = a \,\}\,.$$

For example, $\mathbb{Z}^+$ is a multiset over S whose elements (n,a) are the *integers over* S.

There is a canonical mapping $\eta\colon X \longrightarrow X^{+}$, $x \longmapsto (x,1)$, which is a morphism of sets over S. Moreover η is injective. Hence we may identify $x \in X$ and $\eta x = (x,a) \in X^{+}$. Then $(x,t) = (x,1)^t = x^t$ (in X^{+}, not in X).

Every morphism f of sets over S of X into a multiset Y over S *lifts* uniquely to a morphism of multisets over S from X^{+} to Y, meaning that there is a unique morphism $g\colon X^{+} \longrightarrow Y$ of multisets over S such that $g \circ \eta = f$. Indeed g is unique: $g\,(x,1) = fx$ implies $g\,(x,a) = g\,\big((x,1)^a\big) = \big(g\,(x,1)\big)^a = (fx)^a$. Conversely, define $g\colon X^{+} \longrightarrow Y$ by

$$g\,(x,a) \;=\; (fx)^a\,.$$

Then g is a morphism of multisets over S: (x,a) and $(fx)^a$ both project to $x^{\vee}\,a$, and $g\,\big((x,a)^t\big) = g\,(x,at) = (fx)^{at} = \big(g\,(x,a)\big)^t$.

In the adjunction of sets over S to multisets over S, the counit ϵ is found as follows. Let X be a multiset over S; let X^{+} be the free multiset on the underlying set over S of X. Then ϵ assigns to X the morphism $\epsilon_X\colon X^{+} \longrightarrow X$ of sets over S that lifts the identity 1_X on X. By the above, $\epsilon_X\,(x,a) = x^a$, as calculated in the multiset X.

3 Monoids over S

In this section we assume that S is a commutative monoid.

1. Monoids over S. Recall that a commutative *monoid M over S* consists of a commutative monoid M and a projection $x \longmapsto x^{\vee}$ which is a monoid homomorphism ($1^{\vee} = 1$ and $(xy)^{\vee} = x^{\vee}\,y^{\vee}$ for all x, y). In particular, a monoid over S is a set over S.

A *morphism* $f\colon M \longrightarrow N$ of commutative monoids over S is a morphism of sets over S ($(fx)^{\vee} = x^{\vee}$) which is also a monoid homomorphism.

For every subset X of a commutative monoid M over S there is a submonoid $\overline{X}$ of M (such that $1 \in \overline{X}$ and $xy \in \overline{X}$ for all $x, y \in \overline{X}$) *generated by* X, which is the least submonoid of M that contains X: namely

$$\overline{X} \;=\; \{\, x_1 x_2 \cdots x_n \mid n \geq 0,\;\; x_1,\, x_2, \ldots,\, x_n \in X \,\}\,,$$

it being understood that $x_1 x_2 \cdots x_n = 1 \in M$ if $n = 0$.

2. Free commutative monoids. The free commutative monoid $\mathbb{F} = \mathbb{F}(X)$ on a set X consists of monomials in X. To avoid confusion between products in $\mathbb{F}$ and products in other monoids we replace every element x of X by an indeterminate X_x (so that $\overline{X} = \{\, X_x \mid x \in X \,\}$ is a set with a bijection $x \longmapsto X_x$ of X to $\overline{X}$).

Accordingly, a *monomial* on a set X is a formal product $\mathrm{X}^m = \prod_{x \in X} X_x^{m(x)}$ of powers of elements of $\overline{X}$ (more rigorously, a mapping m of X into $\mathbb{N} = \{0, 1, \ldots, n, \ldots\}$, written as a product of powers, in which any term X_x^0 is regarded as an identity element and is ignored), with exponents $m(x)$ that are almost all zero, meaning that the set $\{\, x \in X \mid m(x) > 0 \,\}$ is finite.

The monomials on X, multiplied by $\mathrm{X}^m\,\mathrm{X}^n = \mathrm{X}^{m+n}$, constitute the free commutative monoid $\mathbb{F} = \mathbb{F}(X)$ on the set X. The identity element of $\mathbb{F}$ has $m(x) = 0$ for all $x \in X$.

The canonical map $\eta\colon X \longrightarrow \mathbb{F}$ sends $x \in X$ to X_x, viewed as the monomial $\mathrm{X}^m \in \mathbb{F}$ in which $m(x) = 1$ and $m(y) = 0$ for all $y \neq x$. Every monomial is now an actual product in $\mathbb{F}$:

$$\mathrm{x}^m \;=\; \prod\nolimits_{x \in X} X_x^{m(x)} \;=\; \prod\nolimits_{x \in X} (\eta x)^{m(x)} \,.$$

Since η is injective we may identify $x \in X$ and $\eta x \in \mathbb{F}$.

The universal property of $\mathbb{F}(X)$ and η states that every mapping f of X into a commutative monoid M extends to a unique monoid homomorphism φ of $\mathbb{F}$ into M: there is a unique monoid homomorphism $\varphi\colon \mathbb{F} \longrightarrow M$ such that $\varphi \circ \eta = f$, namely $\varphi\left(\prod_{x \in X} X_x^{m(x)}\right) = \prod_{x \in X} (fx)^{m(x)}$.

In the adjunction of sets to commutative monoids, the counit ϵ assigns to a commutative monoid M the monoid homomorphism $\epsilon_M : \mathbb{F}(M) \longrightarrow M$ that extends the identity on M ($\epsilon_M \circ \eta = 1_M$): $\epsilon_M \left(\prod_{x \in M} X_x^{m(x)} = \prod_{x \in M} x^{m(x)}\right.$, as calculated in M.

3. Free monoids over S on sets over S. The adjunction of sets to commutative monoids lifts to an adjunction of sets over S to commutative monoids over S. Given a set X over S, the *free commutative monoid over S on X* is the free commutative monoid $\mathbb{F} = \mathbb{F}(X)$ on the set X together with the unique monoid homomorphism that extends the projection of X to S: $(\mathrm{X}^m)^\vee = \prod_{x \in X} (x^\vee)^{m(x)}$.

The unit of this adjunction, $\eta\colon X \longrightarrow \mathbb{F}$ is inherited from $\mathbb{F}$ and sends $y \in X$ to the monomial X_y: $X_y = \mathrm{X}^m$, where $m(y) = 1$ and $m(x) = 0$ if $x \neq y$. Since $X_y^\vee = y^\vee$, η is a morphism of sets over S. Again η is injective and we may identify $x \in X$ and $\eta x \in \mathbb{F}$.

The universal property of the commutative monoid $\mathbb{F}(X)$ over S is inherited from that of the commutative monoid $\mathbb{F}$: if f is a morphism of sets over S of X into a commutative monoid M over S, then the unique monoid homomorphism of $\mathbb{F}$ to M that extends f is a morphism of commutative monoids over S.

The counit ϵ assigns to a commutative monoid M over S the morphism $\epsilon_M : \mathbb{F}(M) \longrightarrow M$ of commutative monoids over S that extends the identity on M: the evaluation map $\prod_{x \in M} X_x^{m(x)} \longmapsto \prod_{x \in M} x^{m(x)} \in M$.

4 Multimonoids

In this section we assume that S is a commutative monoid.

1. Definition. A *commutative multimonoid over S* is a commutative monoid M over S together with an action $(x, t) \longmapsto x^t$ of S on M such that M is also a multiset over S ($(x^t)^\vee = x^\vee\, t$, $x^1 = x$, and $(x^t)^u = x^{tu}$, for all x, t, u), $x \longmapsto x^\vee$ is a monoid homomorphism ($1^\vee = 1$ and $(xy)^\vee = x^\vee\, y^\vee$ for all x, y), and

$$(xy)^t \;=\; x^t\, y \;=\; x\, y^t \quad \text{for all } x, y \in M \text{ and } t \in S.$$

For example, if S has just one element, then a commutative multimonoid over S is just a commutative monoid. In general, S begets a *trivial* multimonoid S on itself, in which $S_a = \{a\}$, $a^t = at$, and ab is the given product in S.

The action of S on a commutative monoid M over S is determined by its action on 1 and the multiplication:

$$x^t \;=\; (x1)^t \;=\; x\,1^t\,.$$

In particular, $1^t\,1^u = 1\,(1^u)^t = 1^{tu}$ and $e\colon t \longmapsto 1^t$ is a monoid homomorphism. Moreover, e splits the projection to S, since $(1^t)^\vee = 1^\vee\,t = t$, and

$$x^t\,y^u \;=\; x\,1^t\,y\,1^u \;=\; xy\,1^{tu} \;=\; (xy)^{tu}\,.$$

Conversely, if M is a commutative monoid over S and $e\colon S \longrightarrow M$ splits its projection to S, then S acts on M by: $x^t = e(t)\,x$, and this action makes M a commutative multimonoid over S. Thus a commutative multimonoid over S can be regarded as a commutative monoid over S with a split projection to S.

A morphism $\varphi\colon M \longrightarrow N$ of multimonoids over S is a morphism φ of multisets ($(\varphi x)^\vee = x^\vee$ for all x and $\varphi\,(x^t) = (\varphi\,x)^t$ for all x, t) which is also a morphism of commutative monoids over S ($\varphi\,1 = 1$ and $\varphi\,(xy) = (\varphi\,x)(\varphi\,y)$ for all x, y).

2. Submultimonoids. A *submultimonoid* of a commutative multimonoid M is a submultiset N of M ($x \in N$ implies $x^t \in N$ for all t) that is also a submonoid of M ($1 \in N$, and $x, y \in N$ implies $xy \in N$). The multiplication on M then induces a multiplication on N, which makes N a commutative multimonoid over S.

For example, if $f\colon M \longrightarrow N$ is a morphism of commutative multimonoids, then $\operatorname{Im} f$ is a submultimonoid of N.

Every submultimonoid of M contains $\operatorname{Im} e$, where $e(t) = 1^t$ splits the projection to S. Conversely, a submonoid N of M that contains $\operatorname{Im} e$ is a submultimonoid of M, since $x \in N$ implies $x^t = e(t)\,x \in N$.

Given a multimonoid M and a subset X of M, the submultimonoid $\overline{X}$ of M *generated by* X is the least submultimonoid of M that contains X. Equivalently, $\overline{X}$ is the submonoid of M generated by $X \cup \operatorname{Im} e$. Since $\operatorname{Im} e$ is a submonoid,

$$\begin{aligned}\overline{X} \;&=\; \{\, x_1 x_2 \cdots x_n \mid n \geq 0,\;\; x_1,\, x_2, \dots,\, x_n \in X \cup \operatorname{Im} e \,\} \\ &=\; \{\, (x_1 x_2 \cdots x_n)\,1^t \mid n \geq 0,\;\; x_1,\, x_2, \dots,\, x_n \in X,\;\; t \in S \,\}\end{aligned}$$

(with $x_1 x_2 \cdots x_n = 1 \in M$ if $n = 0$).

3. Congruences. A *congruence* on a commutative multimonoid M over S is an equivalence relation $\mathcal{C}$ on the set X such that

$$\begin{aligned}&x \;\mathcal{C}\; y \;\text{ implies }\; x^\vee = y^\vee\,, \\ &x \;\mathcal{C}\; y \;\text{ implies }\; x^t = y^t \;\text{ for all }\; t \in S^1\,, \;\text{ and} \\ &x \;\mathcal{C}\; y \;\text{ implies }\; xz \;\mathcal{C}\; yz \;\text{ for all }\; z \in M\,.\end{aligned}$$

For example, if $\varphi\colon M \longrightarrow N$ is a morphism of multisets over S, then the equivalence relation $\ker\varphi$ on M induced by f is a congruence on M.

Conversely, if $\mathcal{C}$ is a congruence on a commutative multimonoid M over S, then the quotient set $M/\mathcal{C}$ is a commutative multimonoid over S, on which the projection to S, action of S^1, and multiplication are well defined by $C(x)^\vee = x^\vee$, $C(x)^t = C(x^t)$, and $C(x)\,C(y) = C(xy)$, where $C(x)$ denotes the $\mathcal{C}$-class of x, so that the projection $M \longrightarrow M/\mathcal{C}$ is a morphism of commutative multimonoids over S.

If $\varphi\colon M \longrightarrow N$ is a morphism of commutative multimonoids over S, then $M/\ker f \cong \operatorname{Im} f$.

4. Free multimonoids on monoids over S**.** There is an adjunction of commutative monoids over S to commutative multimonoids over S. Indeed, if M is a commutative monoid over S, then the free multiset over S on the set M from Section 2,

$$M^+ \;=\; M \times S$$

(with projection $(x,a)^\vee = x^\vee a$ to S and action $(x,a)^t = (x,at)$ of S) inherits from M and S a multiplication, namely the componentwise multiplication

$$(x,a)(y,b) \;=\; (xy,ab)\,,$$

which makes M^+ a commutative multimonoid over S, since

$$\big((x,a)(y,b)\big)^t \;=\; (xy,abt) \;=\; (x,a)\,(y,b)^t \;=\; (x,a)^t\,(y,b)\,.$$

For example, the integers over S constitute a commutative multimonoid $\mathbb{Z}^+$ over S.

The canonical map $\eta\colon x \longmapsto (x,1)$ is now a morphism of commutative multimonoids over S of M into M^+. Since η is injective we may identify $x \in M$ and $\eta x \in M^+$. Then $(x,a) = (x,1)^a = x^a$ in M^+ (but not in M).

Since M^+ is the free multiset over S on M, every morphism f of multisets over S from M to a commutative multimonoid N over S lifts uniquely to a morphism $g\colon (x,a) \longmapsto (fx)^a$ of multisets over S from M^+ to N. If f is a morphism of commutative monoids over S, then

$$g\,(xy,ab) \;=\; (f(xy))^{ab} \;=\; ((fx)(fy))^{ab} \;=\; (fx)^a\,(fy)^b$$

and g is a morphism of commutative multimonoids over S. Hence M^+ is the *free commutative multimonoid over* S *on* M.

In the adjunction of commutative monoids over S to commutative multimonoids over S, the counit assigns to a commutative multimonoid M over S the morphism $\epsilon_M\colon M^+ \longrightarrow M$ that lifts the identity on M: $\epsilon_M(x,a) = x^a$, as calculated in M.

5. Free multimonoids on sets over S**.** Composing the adjunction of sets over S to commutative monoids over S and the adjunction of commutative monoids over S to commutative multimonoids over S yields an adjunction of sets over S to commutative multimonoids over S.

Accordingly, the *free commutative multimonoid over* S *on* a set X over S is the free commutative multimonoid $\mathbb{F}^{+} = \mathbb{F}(X)^{+}$ over S on the free commutative monoid $\mathbb{F} = \mathbb{F}(X)$ over S on X. We saw in Section 3 that $\mathbb{F}$ consists of all monomials $\mathrm{X}^m = \prod_{x\in X} x^{m(x)}$ on X, multiplied by $\mathrm{X}^m\,\mathrm{X}^n = \mathrm{X}^{m+n}$, and projection $(\mathrm{X}^m)^\vee = \prod_{x\in X} (x^\vee)^{m(x)}$.

Hence $\mathbb{F}^{+}$ consists of all ordered pairs $(\mathrm{X}^m,\, a)$, where $a \in S$ and $\mathrm{X}^m \in \mathbb{F}$, with projection $(\mathrm{X}^m,\, a)^\vee = (\mathrm{X}^m)^\vee\, a$ to S, action $(\mathrm{X}^m,\, a)^t = (\mathrm{X}^m,\, at)$ of S, and multiplication $(\mathrm{X}^m,\, a)(\mathrm{X}^n,\, b) = (\mathrm{X}^{m+n},\, ab)$.

The canonical map $\eta\colon X \longrightarrow \mathbb{F}^{+}$ sends $x \in X$ to $(X_x,\, 1) \in \mathbb{F}^{+}$, where $X_x \in \mathbb{F}$ stands for the monomial $\prod_{y\in X} X_y^{m(y)}$ in which $m(x) = 1$ and $m(y) = 0$ for all $y \neq x$. Every element $(\mathrm{X}^m,\, a)$ of $\mathbb{F}^{+}$ is now a product

$$(\mathrm{X}^m,\, a) \;=\; \big(\textstyle\prod_{x\in X} (\eta\, x)^{m(x)}\big)(1, a)\,.$$

We see that η is a morphism of sets over S. Moreover, η is injective, and we may identify $x \in X$ and $\eta x \in \mathbb{F}^{+}$; then

$$(\mathrm{X}^m,\, a) \;=\; \big(\textstyle\prod_{x\in X} (\eta\, x)^{m(x)}\big)(1, a) \;=\; \big(\textstyle\prod_{x\in X} x^{m(x)}\big)^a$$

in $\mathbb{F}^{+}$ (not in $\mathbb{F}$).

Every morphism f of sets over S from X to a commutative multimonoid M over S extends uniquely to a morphism g of commutative monoids over S from $\mathbb{F}$ to M, which sends $\mathrm{X}^m = \prod_{x\in X} X_x^{m(x)}$ to $\prod_{x\in X} (fx)^{m(x)}$ and in turn lifts uniquely to a morphism φ of commutative multimonoids over S from $\mathbb{F}^{+}$ to M, namely

$$\varphi\,(\mathrm{X}^m,\, a) \;=\; (g\,\mathrm{X}^m)^a \;=\; \big(\textstyle\prod_{x\in X} (fx)^{m(x)}\big)^a\,.$$

In the adjunction of sets over S to commutative multimonoids over S, the counit ϵ assigns to a commutative multimonoid M over S the morphism $\epsilon_M : \mathbb{F}(M)^{+} \longrightarrow M$ of commutative monoids over S that lifts the identity on M:

$$\epsilon_M \left(\textstyle\prod_{x\in M} X_x^{m(x)},\, a\right) \;=\; \big(\textstyle\prod_{x\in M} x^{m(x)}\big)^a\,,$$

as calculated in M.

6. Free multimonoids on multisets over S**.** There is also an adjunction of multisets over S to commutative multimonoids over S.

Given a multiset X over S, the *free commutative multimonoids over* S *on* X is constructed as follows. Let $\overline{X} = \{\, X_x \mid x \in X \,\}$ be an isomorphic copy of X, with $X_x^\vee = x^\vee$ and $X_x^t = X_{x^t}$. The *half free* commutative monoid on X is the commutative monoid

$$\mathbb{H}(X) \;=\; \langle\, \overline{X} \mid X_x^t\, X_y = X_x\, X_y^t \;\text{ for all }\; x, y \in X,\; t \in S \,\rangle$$

generated by X subject to all defining relations $X_x^t\, X_y = X_x\, X_y^t$, where $x, y \in X$ and $t \in S$. Equivalently, $\mathbb{H}(X) = \mathbb{F}(X)/\approx$, where $\approx$ is the smallest congruence on $\mathbb{F}(X)$ such that $X_x^t\, X_y \approx X_x\, X_y^t$ for all $x, y \in X$ and $t \in S$.

Every element of $\mathbb{H}$ is the equivalence class $[X_1 X_2 \cdots X_n]$ of a commutative product $X_1 X_2 \cdots X_n$ of elements of $\overline{X}$. The action of S on $\mathbb{H}$ is well defined by

$$\begin{aligned}[X_1 X_2 \cdots X_n]^t &= [X_1^t\, X_2\, \cdots\, X_n] \\ &= [X_1\, X_2^t\, \cdots\, X_n] \;=\; \cdots \;=\; [X_1\, X_2\, \cdots\, X_n^t]\,.\end{aligned}$$

Projection to S is well defined by

$$[X_1 X_2 \cdots X_n]^\vee \;=\; X_1^\vee\, X_2^\vee\, \cdots\, X_n^\vee\,.$$

With this action of S and projection to S, $\mathbb{H}$ is a commutative multimonoids over S.

In $\mathbb{F}(X)$, the equivalence class $[X_x]$ of X_x consists of only X_x. The canonical mapping $\eta\colon X \longrightarrow \mathbb{H}(X)$ sends $x \in X$ to $[X_x]$.

Every morphism f of multisets over S from X to a commutative monoid M over S extends uniquely to a monoid homomorphism φ of $\mathbb{F}(X)$ into M. Since $x^t\, y = x\, y^t$ for all $x, y \in M$ we have $\varphi\,(X_x^t\, X_y) = \varphi\,(X_x\, X_y^t)$ for all $x, y \in X$ and $t \in S$. Therefore φ induces a monoid homomorphism ψ of $\mathbb{H}(X)$ into M such that

$$\psi\,[X_1 \cdots X_n] \;=\; \varphi\,(X_1 \cdots X_n)$$

for all $X_1, \ldots, X_n \in \overline{X}$. Then ψ is a morphism of commutative multimonoids over S, $\psi \circ \eta = f$, and ψ is unique with these properties.

Composing the adjunction $X \longmapsto X^+$ of sets over S to multisets over S and the adjunction $X^+ \longmapsto \mathbb{H}(X^+)$ of multisets over S to commutative multimonoids over S yields the adjunction $X \longmapsto \mathbb{F}(X)^+$ of sets over S to commutative multimonoids over S. Therefore there is a natural isomorphism

$$\mathbb{H}(X^+) \cong \mathbb{F}(X)^+\,.$$

5 Multigroups

In this section S is any commutative semigroup.

1. Definition. An *abelian multigroup over* S is a multiset G over S (with a projection to S and an action of S^1 on G such that $(x^t)^\vee = (x^\vee)\,t$, $x^1 = x$, and $(x^t)^u = x^{tu}$, for all x, t, u) together with an addition on each $G_a = \{\, x \in G \mid x^\vee = a \,\}$, such that

every G_a is an abelian group under addition, and
$(x+y)^t \;=\; x^t + y^t$ whenever $x^\vee = y^\vee$.

For example, if S has just one element, then a abelian multigroup over S is just an abelian group. Conversely, any abelian group G can be turned into a *constant* abelian multigroup over any commutative semigroup S, in which $G_a = G$ and $x^t = x$ for all a, t, x. A commutative semigroup S also begets a *trivial* abelian multigroup over itself, in which every $G_a = \{0_a\}$ is a trivial group and $0_a^t = 0_{at}$.

If G is an abelian group, then

$$G^{+} \;=\; G \times S\,,$$

with projection $(x,a)^{\vee} = a$ to S, action $(x,a)^t = (x, at)$ of S^1, and addition $(x,a) + (y,a) = (x+y,\, a)$ on each $(G^{+})_a$, is an abelian multigroup over S. (The projection $(x,a)^{\vee} = a$ differs from the projection $(x,a)^{\vee} = (x^{\vee})\, a$ in the similar constructions in Sections 2 and 4, since G lacks a projection to S.) For example,

$$\mathbb{Z}^{+} \;=\; \mathbb{Z} \times S\,,$$

is an abelian multigroup over S; the elements of $\mathbb{Z}^{+}$ are the *integers over* S.

Abelian multigroups over S have an equivalent definition as functors on the Leech category $\mathcal{H}(S)$. An abelian group valued functor (G,γ) on $\mathcal{H}(S)$ (called an $\mathcal{H}(S)$-*module* in [4]) assigns an abelian group G_a to each $a \in S$ and a homomorphism $\gamma_{a,t} \colon G_a \longrightarrow G_{at}$ to each $a \in S$ and $t \in S^1$, so that $\gamma_{a,1}$ is the identity on G_a and $\gamma_{at,u} \circ \gamma_{a,t} = \gamma_{a,tu}$ for all $a \in S$ and $t, u \in S^1$. We regard the groups G_a as pairwise disjoint, so that $G = \bigcup \big(G_a \mid a \in S \big)$, together with the projection $x \longmapsto a$ whenever $x \in G_a$, is a set over S. As in Section 2, an action of S^1 on G is then defined by

$$x^t \;=\; \gamma_{a,t}\, x$$

and makes G an abelian multigroup over S. Conversely, an abelian multigroup G over S can be regarded as an abelian group valued functor on $\mathcal{H}(S)$, that assigns G_a to $a \in S$ and $\gamma_{a,t} \colon x \longmapsto x^t$ to $a \in S$ and $t \in S^1$.

In general, the additions on the groups G_a constitute a partial addition on G, under which $x + y$ is defined if and only if $x^{\vee} = y^{\vee}$, in which case $(x+y)^{\vee} = x^{\vee} = y^{\vee}$. Generally, a sum $x_1 + \cdots + x_n$ of elements of G (actually, the sequence $x_1, \ldots, x_n$) is *homogeneous* if $x_1, \ldots, x_n$ all project to the same element of S; then the sum $x_1 + \cdots + x_n$ is defined in G and projects to that element.

Denote the identity element of G_a by 0_a (by 0, if a can be retrieved from context). Since $x \longmapsto x^t$ is a homomorphism of G_a into G_{at} we have

$$0_a^t \;=\; 0_{at} \quad (\text{or } 0^t = 0),$$

for all a, t.

A *morphism* φ of abelian multigroups over S from G to H is a morphism of multisets over S (a mapping of G into H such that $(\varphi\, x)^{\vee} = x^{\vee}$ and $\varphi\,(x^t) = (\varphi\, x)^t$, for all x, t) such that $\varphi\,(x+y) = \varphi x + \varphi y$ whenever $x + y$ is defined. Equivalently, the restrictions $\varphi_a \colon G_a \longrightarrow H_a$ constitute a natural transformation. Hence the category of abelian multigroups over S is isomorphic to the category of abelian group valued functors on $\mathcal{H}(S)$, and is an abelian category.

If G and H are abelian multigroups over S, then multigroup morphisms from G to H can be added pointwise. If φ and ψ are morphisms from G to H, then the pointwise sum $\varphi + \psi$ is defined, since $(\varphi x)^{\vee} = x^{\vee} = (\psi x)^{\vee}$ for all $x \in G$. This makes the set $\mathrm{Hom}\,(G,H)$ of all such morphisms an abelian group. Now

Hom (G, H) is not a multigroup; its elements, however, are basically families of homomorphisms indexed by S, and in this sense are already spread over S.

2. Submultigroups. A *submultigroup* K of an abelian multigroup G over S is a submultiset of G (a subset K of G such that $x \in K$ implies $x^t \in K$) such that $K_a = K \cap G_a$ is a subgroup of G_a for every $a \in S$; equivalently, a subfunctor of G. Then K, together with the projection to S, action of S^1, and additions that it inherits from G, is an abelian multigroup over S, and the inclusion mapping $K \longrightarrow G$ is a morphism of abelian multigroups over S.

For example, if $\varphi\colon G \longrightarrow H$ is a morphism of abelian multigroups, then $\operatorname{Im}\varphi = \varphi(G)$ is a submultigroup of H, and $\operatorname{Ker}\varphi = \{x \in G \mid \varphi x = 0\}$ is a submultigroup of G; these are the image and kernel of φ in the category of abelian multigroups over S.

In general, if K is a submultigroup of $G = (G, \gamma)$, then the inclusion morphism $K \longrightarrow G$ has a cokernel $G \longrightarrow G/K$, which can be described as follows. As a set, G/K is the disjoint union $G/K = \bigcup \big(G_a/K_a \mid a \in S \big)$. The action of S^1 on G induces an action of S^1 on G/K, under which $(x + K_a)^t = x^t + K_{at}$ whenever $x \in G_a$. This makes G/K an abelian multigroup over S, the *quotient multigroup* of G by K, that comes with a morphism $G \longrightarrow G/K$ of abelian multigroups over S, and a short exact sequence $0 \longrightarrow K \longrightarrow G \longrightarrow G/K \longrightarrow 0$ of abelian multigroups over S.

Equivalently, G/K is the quotient of G by the congruence $\mathcal{C}$ on G (in the obvious sense) defined by

$$x \;\mathcal{C}\; y \;\text{ if and only if }\; x^\vee = y^\vee \;\text{ and }\; x - y \in K\,.$$

If $\varphi\colon G \longrightarrow H$ is a morphism of multigroups, then $G/\operatorname{Ker}\varphi \cong \operatorname{Im}\varphi$.

Given a multigroup G and a subset X of G, there is a submultigroup $\overline{X}$ of G *generated by* X, which is the least submultigroup $\overline{X}$ of G that contains X. For each $a \in S$ let $X_a = X \cap G_a$ and let Y be the submultiset of G generated by X,

$$Y \;=\; \{\, x^t \mid x \in X,\, t \in S^1 \,\}\,.$$

For each $a \in S$, $\overline{X}_a$ is the subgroup of G_a generated by Y_a; its elements are (necessarily homogeneous) linear combinations $\sum \big(n_y y \;\big|\; y \in Y_a \big)$ of elements of Y_a with integer coefficients that are almost all zero. Indeed, $x \in \overline{X}$ implies $x^t \in \overline{X}$, since $y \in Y$ implies $y^t \in Y$.

3. Free abelian multigroups over multisets. Let X be a set and let C be a set with en element $0 \in C$. A formal linear combination of elements of X with coefficients in C that are almost all zero is a finite sum

$$c \;= \sum \big(c_x x \;\big|\; x \in C \big)\,;$$

equivalently, a mapping $c\colon X \longrightarrow C$ such that $\{\, c_x \mid c_x \neq 0 \,\}$ is finite, written as a sum in which every term $0x$ is ignored. For example, the free abelian group on X is the set of all linear combinations of elements of X with coefficients in

$\mathbb{Z}$, under pointwise addition

$$\sum\big(\, m_x x \bigm| x \in X \,\big) \; + \sum\big(\, n_x x \bigm| x \in X \,\big) \; = \sum\big(\, (m_x + n_x)\, x \bigm| x \in X \,\big)\,.$$

If X is a set over S, then a linear combination $c = \sum\big(\, c_x x \bigm| x \in C \,\big)$ is *homogeneous* if all $x \in X$ such that $c_x \neq 0$ have the same projection to S, which is then the projection of c. We denote by

$$C \dashv X$$

the set over S of all such homogeneous linear combinations of elements of X with coefficients in C.

The *free abelian multigroup over S on* a multiset X over S is $\mathbb{Z} \dashv X$. Addition is pointwise: if $g = \sum\big(\, g_x x \bigm| x \in X,\ x^\vee = g^\vee \big)$, $h = \sum\big(\, h_x x \bigm| x \in X,\ x^\vee = g^\vee \big)$, and $g^\vee = h^\vee$, then

$$g + h \; = \sum\big(\, (g_x + h_x)\, x \bigm| x \in X,\ x^\vee = g^\vee = h^\vee \,\big)\,.$$

If $g = \sum\big(\, g_x x \bigm| x \in X,\ x^\vee = g^\vee \,\big)$, then g^t is well defined for any $t \in S^1$ by

$$g^t \; = \sum\big(\, g_x x^t \bigm| x \in X,\ x^\vee = g^\vee \,\big)\,;$$

this sum can be put in standard form $\sum\big(\, h_y y \bigm| y \in X,\ y^\vee = (g^t)^\vee \,\big)$, where $h_y = \sum\big(\, g_x \mid x \in X,\ x^\vee = g^\vee,\ x^t = y \big)$.

With this partial addition and action of S, the set $\mathbb{Z} \dashv X$ over S becomes an abelian multigroup over S.

For each $a \in S$, $g \in (\mathbb{Z} \dashv X)_a$ if and only if $g = \sum\big(\, g_x x \bigm| x \in X,\ x^\vee = a \big)$. Thus $(\mathbb{Z} \dashv X)_a = \mathbb{Z} \dashv X_a$ is the free abelian group on X_a and is isomorphic to a subgroup of the free abelian group on the set X; but $(\mathbb{Z} \dashv X)_a$ cannot be defined as a subgroup of the latter since the pairwise disjoint groups $(\mathbb{Z} \dashv X)_a$ cannot have a common zero element.

The canonical map $\eta\colon X \longrightarrow \mathbb{Z} \dashv X$ is inherited from the adjunction of sets to abelian groups, and sends $x \in X$ to $\sum\big(\, n_y y \bigm| y \in X \,\big)$, where $n_x = 1$ and $n_y = 0$ if $y \neq x$. Since η is injective, we may identify $x \in X$ and $\eta x \in \mathbb{Z} \dashv X$. Then $g = \sum\big(\, g_x x \bigm| x \in X,\ x^\vee = g^\vee \big)$ is a linear combination within $\mathbb{Z} \dashv X$, $g = \sum\big(\, g_x\, (\eta\, x) \bigm| x \in X,\ x^\vee = g^\vee \big)$. Moreover, η is a morphism of multisets over S.

Every morphism f of multisets over S from X to an abelian multigroup G over S extends uniquely to a morphism φ of abelian multigroups over S from $\mathbb{Z} \dashv X$ to G such that $\varphi \circ \eta = f$, which sends

$$\begin{aligned} g \; &= \sum\big(\, g_x x \bigm| x \in X,\ x^\vee = g^\vee \big) \\ &= \sum\big(\, g_x\, (\eta\, x) \bigm| x \in X,\ x^\vee = g^\vee \big) \in \mathbb{Z} \dashv X \end{aligned}$$

to

$$\varphi g \; = \sum\big(\, g_x\, (f x) \bigm| x \in X,\ x^\vee = g^\vee \big) \in G\,.$$

In the adjunction of multisets over S to abelian multigroups over S, the counit ϵ assigns to an abelian multigroup G over S the morphism $\epsilon_G \colon \mathbb{Z} \dashv G \longrightarrow G$ that extends the identity on G. By the above,

$$\epsilon_G \left(\sum \left(n_x x \mid x \in G_a \right) \right) = \sum \left(n_x x \mid x \in G_a \right),$$

where the second sum is calculated in G_a.

4. Free abelian multigroups on sets over S. If S is a commutative monoid, then composing the adjunction of sets over S to multisets over S in Section 2 with the adjunction of multisets over S to abelian multigroups over S yields an adjunction of sets over S to abelian multigroups over S. Accordingly, the *free abelian multigroup over S on* a set $X = (X, p)$ over S is the free abelian multigroup $\mathbb{Z} \dashv X^+$ over S on X^+, where $X^+ = X \times S$ is the free multiset over S on X in Section 2, with projection $(x,a)^\vee = (x^\vee)\, a$ to S and action $(x,a)^t = (x, at)$ of S.

Thus F is the set of all homogeneous linear combinations

$$p = \sum \left(p_{x,t}\, (x,t) \mid x \in X,\ t \in S,\ (x^\vee)\, t = p^\vee \right)$$

of elements of $\mathbb{Z}^+$, with integer coefficients that are almost all zero. The action of S is

$$p^t = \left(\sum \left(p_y\, y \mid y \in X^+,\ y^\vee = p^\vee \right) \right)^t = \sum \left(p_y\, y^t \mid y \in X^+,\ y^\vee = p^\vee \right);$$

and addition in $\mathbb{Z} \dashv X^+$ is pointwise

$$\begin{aligned} p + q &= \sum \left(p_y\, y \mid y \in X^+,\ y^\vee = p^\vee \right) + \sum \left(q_y\, y \mid y \in X^+,\ y^\vee = q^\vee \right) \\ &= \sum \left((p_y + q_y)\, y \mid y \in X^+,\ y^\vee = p^\vee \right), \end{aligned}$$

defined if $p^\vee = q^\vee$.

The canonical map $\eta \colon X \longrightarrow F$ sends $x \in X$ to $(x,1) = \sum \left(n_y\, y \mid y \in X^+ \right)$, where $n_y = 1$ if $y = (x,1)$ and $n_y = 0$ otherwise. Then $(x,t) = (\eta x)^t$ and the typical element p of $\mathbb{Z} \dashv X^+$ is a homogeneous linear combination within $\mathbb{Z} \dashv X^+$:

$$p = \sum \left(p_{x,t}\, (\eta x)^t \mid x \in X,\ t \in S,\ (x^\vee)\, t = p^\vee \right).$$

If $f \colon X \longrightarrow G$ is a morphism of sets over S from X to an abelian multigroup G over S, then f extends uniquely to a morphism $g \colon X^+ \longrightarrow G$ of sets over S, $(x,a) \longmapsto (fx)^a$, which in turn extends uniquely to a morphism $\varphi \colon F \longrightarrow G$ of abelian multigroups over S such that $\varphi \circ \eta = f$, which sends

$$p = \sum \left(p_{x,t}\, (\eta x)^t \mid x \in X,\ t \in S,\ (x^\vee)\, t = p^\vee \right).$$

to

$$\varphi\, p = \sum \left(p_{x,t}\, (fx)^t \mid x \in X,\ t \in S,\ (x^\vee)\, t = p^\vee \right).$$

The counit assigns to an abelian multigroup G over S the morphism ϵ_G from $\mathbb{Z} \dashv G^+$ to G that extends the identity on G:

$$\begin{aligned} \epsilon_G\, p \;&=\; \epsilon_G \Big(\sum \big(p_{x,t}\, (x,t) \;\big|\; x \in G,\ t \in S,\ (x^\vee)\, t = p^\vee \big) \Big) \\ &=\; \sum \big(p_{x,t}\, x^t \;\big|\; x \in G,\ t \in S,\ (x^\vee)\, t = p^\vee \big)\,, \end{aligned}$$

where the second sum is calculated in G.

5. Presentations. Abelian multigroups over S can be presented as generated by the elements of a set over S (if S is a commutative monoid) or the elements of a multiset over S, subject to defining relations. A *defining relation* between the elements of a set or multiset X over S is a pair (A, B) of elements of the free abelian multigroup F on X, normally written as an equality $A = B$, which is *homogeneous*, meaning that A and B project to the same element $A^\vee = B^\vee$ of S.

A defining relation (A, B) between the elements of X *holds* in an abelian multigroup G over S *via* a morphism of sets over S or multisets over S from X to G if and only if $\varphi A = \varphi B$, where $\varphi\colon F \longrightarrow G$ is the morphism of abelian multigroups over S that extends f.

Let X be either a set over S or a multiset over S. The abelian multigroup over S *generated by* X *subject to* a set $\mathcal{R}$ of defining relations is the quotient

$$\langle\, X \mid \mathcal{R} \,\rangle \;\cong\; F/K\,,$$

where F is the free abelian multigroup over S on X and K is the submultigroup of F generated by all $A - B$ with $(A, B) \in \mathcal{R}$ (which are defined since all $(A, B) \in \mathcal{R}$ are homogeneous).

The abelian multigroup $\langle\, X \mid \mathcal{R} \,\rangle$ comes with a map η from X, which is the composition

$$\eta\colon X \xrightarrow{\eta^F} F \longrightarrow F/K\,.$$

If S is a commutative monoid and X is a set over S, then the typical element p of $F = \mathbb{Z} \dashv X^+$ is a homogeneous linear combination

$$p \;= \sum \big(p_{x,t}\, (\eta^F x)^t \;\big|\; x \in X,\ t \in S,\ (x^\vee)\, t = p^\vee \big)\,.$$

Hence the typical element q of $\langle\, X \mid \mathcal{R} \,\rangle$ is a homogeneous linear combination

$$q \;= \sum \big(q_{x,t}\, (\eta x)^t \;\big|\; x \in X,\ t \in S,\ (x^\vee)\, t = q^\vee \big)\,.$$

If X is a multiset over S, then the typical element p of $F = \mathbb{Z} \dashv X$ is a homogeneous linear combination $p = \sum \big(p_x\, \eta^F x \;\big|\; x \in X,\ x^\vee = p^\vee \big)$, and the typical element q of $\langle\, X \mid \mathcal{R} \,\rangle$ is a homogeneous linear combination

$$q \;= \sum \big(q_x\, \eta x \mid x \in X,\ x^\vee = p^\vee \big)\,.$$

In either case, every $(A, B) \in \mathcal{R}$ holds in $\langle\, X \mid \mathcal{R} \,\rangle$ via η, by definition.

The universal property of $\langle X \mid \mathcal{R} \rangle$ and η is: if f is a morphism of sets over S, or multisets over S, from X to a abelian multigroup G over S and every $(A, B) \in \mathcal{R}$ holds in G via f, then there is a unique morphism φ of abelian multigroups over S from $\langle X \mid \mathcal{R} \rangle$ to G such that $\varphi \circ \eta = f$.

6 Multirings.

In this section, S is a commutative monoid.

1. Definition. A *commutative multiring over* S is an abelian multigroup R over S (with a projection to S, an action of S on R, and an abelian group addition on each R_a, such that $(x^t)^\vee = x^\vee\, t$, $x^1 = x$, $(x^t)^u = x^{tu}$, and $(x+y)^t = x^t + y^t$ for all x, y, t, u) with a commutative monoid multiplication on R such that R is a commutative monoid over S ($(xy)^\vee = x^\vee\, y^\vee$ and $(xy)^t = x^t\, y = x\, y^t$ for all x, y, t), and such that

$$(x + y)\, z \;=\; xz + yz \quad \text{whenever} \quad x + y \quad \text{is defined.}$$

For example, if S has just one element, then a commutative multiring over S is just a commutative ring with an identity element. In general, S begets a commutative multiring R over itself, in which $R_a = \{0_a\}$ is the trivial group, $0_a^t = 0_{at}$, and $0_a\, 0_b = 0_{ab}$.

If R is any commutative ring with an identity element, then

$$R^+ \;=\; R \times S\,,$$

with projection $(r, a)^\vee = a$ to S, action $(r, a)^t = (r, at)$ of S, partial addition $(r, a) + (s, a) = (r + s,\, a)$ (as in Section 5) and multiplication $(r, a)(s, b) = (rs, ab)$, is a commutative multiring over S. In particular, $\mathbb{Z}^+$ is a commutative multiring over S.

In general, if R is a multiring, then we denote the (additive) identity element of the abelian group R_a by 0_a, or just 0. If $x \in R_b$, then $0_a\, x = (0_a + 0_a)\, x = (0_a\, x) + (0_a\, x)$, which in the abelian group R_{ab} implies

$$0_a\, x \;=\; 0_{ab}\,, \quad \text{for all} \quad x \in R_b\,.$$

In particular, $0_a\, 0_b = 0_{ab}$. Since the projection $R \longrightarrow S$ is surjective, the identity element 1 of R projects to the identity element of S: $1^\vee = 1$. In particular, R_1 is a commutative ring with an identity element.

Since R is also a commutative monoid over S, the identity element provides another monoid homomorphism that splits the projection, namely $t \longmapsto 1^t$. As in Section 2, this map and the multiplication on R completely determine the action of S, as $x^t = x\, 1^t$ for all x, t.

A *morphism* $Q \longrightarrow R$ of commutative multirings over S is a mapping $\varphi\colon Q \longrightarrow R$ that preserves projection to S, action of S^1, sums, products, and identity elements; equivalently, is both a morphism of commutative multimonoids over S and a morphism of abelian multigroups over S.

2. Ideals. An *ideal* of a commutative multiring R over S is a submultigroup I of R ($I_a = I \cap R_a$ is a subgroup of R_a for every $a \in S$, and $x \in I$ implies $x^t \in I$), such that $x \in I$ implies $xy \in I$ for all $y \in R$. A *proper* ideal is an ideal I such that $1 \notin I$; equivalently, $I \neq R$.

For example, if $\varphi\colon Q \longrightarrow R$ is a morphism of commutative multirings, then $\operatorname{Ker}\varphi = \{\, x \in Q \mid \varphi x = 0 \,\}$ is an ideal of Q, which is proper if $\varphi(1) \neq 0$ in R.

In general, the quotient multigroup R/I inherits a multiplication from R, which is well defined by: $(x + I_a)(y + I_b) = xy + I_{ab}$ whenever $x \in R_a$ and $y \in R_b$, since $x - x' \in I_a$, $y - y') \in I_b$ implies

$$xy - x'y' \;=\; x\,(y - y') + (x - x')\,y' \;\in I_{ab}\,.$$

With this multiplication R/I becomes a commutative multiring over S, and the projection $R \longrightarrow R/I$ becomes a morphism of commutative multirings over S.

Equivalently, R/I is the quotient of R by the congruence $\mathcal{C}$ on R (in the obvious sense) defined by

$$x \;\mathcal{C}\; y \;\text{ if and only if }\; x^{\vee} = y^{\vee} \;\text{ and }\; x - y \in I\,.$$

Given a commutative multiring R and a subset X of R, the ideal (X) of R *generated by* X is the least ideal of R that contains X. Let Y be the submultiset of G generated by X:

$$Y \;=\; \{\, x^t \mid x \in X,\; t \in S \,\}\,.$$

Let Z be the monoid ideal of R generated by Y:

$$Z \;=\; \{\, ry \mid r \in R,\;\; y \in Y \,\}\,.$$

Then $z \in Z$ implies $z^t \in Z$ and $-z = (-1)\,z \in I$. Hence $(X) = \overline{Z}$ is the submultigroup of R generated by Z: for each $a \in S$, $\overline{Z}_a$ is the set of all homogeneous sums $z_1 + \cdots + z_n$ ($= 0$ if $n = 0$) of elements $z_1, \ldots, z_n$ of Z_a.

Indeed any ideal of R that contains X must also contain Y, Z, and $\overline{Z}$. Conversely, $X \subseteq Z \subseteq \overline{Z}$. If $x = z_1 + \cdots + z_n \in \overline{Z}_a$, where $z_1, \ldots, z_n \in Z_a$, then $x^t = z_1^t + \cdots + z_n^t \in \overline{Z}_{at}$. If $r \in R_a$ and $x = z_1 + \cdots + z_n \in \overline{Z}_b$, then $rz_i \in Z_{ab}$ for all i and $rx = rz_1 + \cdots + rz_n \in \overline{Z}_{ab}$. Thus $\overline{Z}$ is an ideal of R.

3. Submultirings. A *submultiring* of a commutative multiring R over S is a subset Q of R which is both a submultimonoid of R ($x^t \in Q$, $1 \in Q$, and $xy \in Q$, for all $x, y \in Q$ and all t) and a submultigroup of R ($Q_a = Q \cap R_a$ is a subgroup of R_a for every $a \in S$, and $x \in Q$ implies $x^t \in Q$); in particular, Q contains $\operatorname{Im} e$, where $e(t) = 1^t$ splits the projection to S. Then Q, together with the projection to S, action of S, additions, and multiplication that it inherits from R, is an commutative multiring over S, and the inclusion mapping $Q \longrightarrow G$ is a morphism of commutative multirings over S.

For example, if $\varphi\colon T \longrightarrow R$ is a morphism of commutative multirings, then $\operatorname{Im}\varphi = \varphi\,(T)$ is a submultiring of R.

If X is a (multiplicative) submultimonoid of R, then the (additive) submultigroup $\overline{X}$ of R generated by X is a submultiring of R. Indeed $\overline{X}$ is the set of all homogeneous sums $\pm x_1 \pm \cdots \pm x_n$ of elements $x_1, \dots, x_n$ of X. Hence $\overline{X}$ is a submultimonoid of R: if $x = \pm x_1 \pm \cdots \pm x_m \in \overline{X}$ and $y = \pm y_1 \pm \cdots \pm y_n \in \overline{X}$, where $x_1, \dots, x_m, y_1, \dots, y_n \in X$, then $xy = \sum_{i,j} \pm x_i\, y_j \in \overline{X}$.

Given a commutative multiring R and a subset X of R, the submultiring $\overline{X}$ of R *generated by* X is the least submultiring of R that contains X. By the above, $\overline{X}$ is the submultigroup of R generated by the submultimonoid Y of R generated by X: the set of all homogeneous sums $\pm y_1 \pm \cdots \pm y_n$ of elements $y_1, \dots, y_n$ of

$$Y \;=\; \{\, (x_1 x_2 \cdots x_n)\, 1^t \mid n \geq 0,\;\; x_1, x_2, \dots, x_n \in X,\;\; t \in S \,\}.$$

4. Free multirings on multimonoids. There is an adjunction of commutative multimonoids over S to commutative multirings over S. If M is a commutative multimonoid over S, then the free abelian multigroup $\mathbb{Z} \dashv M$ over S on the multiset M serves as the *free commutative multiring over S on M*.

Indeed we saw in Section 5 that, for each $a \in S$, an element p of $\mathbb{Z} \dashv M$ is uniquely a homogeneous linear combination $p = \sum \big(p_x\, x \bigm| x \in M,\; x^\vee = p^\vee \big)$; that $t \in S$ acts on p by $p^t = \sum \big(p_x\, x^t \bigm| x \in M,\; x^\vee = p^\vee \big)$; and $p + q = \sum \big((p_x + q_x) x \bigm| x \in M,\; x^\vee = p^\vee \big)$ if $p^\vee = q^\vee$.

The multiplication on M extends to a multiplication on $\mathbb{Z} \dashv M$, under which the product of $p = \sum \big(p_x\, x \bigm| x \in M,\; x^\vee = p^\vee \big)$ and $q = \sum \big(q_x\, x \bigm| x \in M,\; x^\vee = q^\vee \big)$ is

$$pq \;= \sum \big(p_x\, q_y\, xy \bigm| x, y \in M,\; x^\vee = p^\vee,\; y^\vee = q^\vee \big),$$

where the homogeneous sum $\sum \big(p_x\, q_y\, xy \bigm| x, y \in M,\; x^\vee = p^\vee,\; y^\vee = q^\vee \big)$ can be put in standard form

$$\begin{aligned} pq \;&= \sum \big(r_z\, z \bigm| z \in M,\; z^\vee = (pq)^\vee \big), \quad \text{where} \\ r_z \;&= \sum \big(p_x\, q_y \bigm| x, y \in M,\; x^\vee = p^\vee,\; y^\vee = q^\vee,\; xy = z \big). \end{aligned}$$

With this multiplication, the abelian multigroup $\mathbb{Z} \dashv M$ over S becomes a commutative multiring over S; its identity element is that of M.

The canonical map $\eta\colon M \longrightarrow \mathbb{Z} \dashv M$ in Section 5 that sends $x \in M$ to $x \in \mathbb{Z} \dashv M$ (short for the linear combination $x = \sum \big(n_y\, y \bigm| y \in M \big)$ where $n_x = 1$ and $n_y = 0$ if $y \neq x$) is now a morphism of commutative multimonoids over S. Moreover, every $p \in \mathbb{Z} \dashv M$ now is a linear combination within $\mathbb{Z} \dashv M$:

$$\sum \big(p_x\, x \bigm| x \in M,\; x^\vee = p^\vee \big) \;= \sum \big(p_x\, (\eta x) \bigm| x \in M,\; x^\vee = p^\vee \big).$$

Since η is injective we may identify $x \in M$ and $\eta x \in \mathbb{Z} \dashv M$. Then $\sum \big(p_x\, x \bigm| x \in M,\; x^\vee = p^\vee \big)$ is a linear combination within $\mathbb{Z} \dashv M$.

Every morphism $\varphi\colon M \longrightarrow R$ of commutative multimonoids over S from M to a commutative multiring R over S extends uniquely to a morphism $\psi\colon \mathbb{Z} \dashv M \longrightarrow R$ of abelian multigroups over S,

$$\psi \big(\sum \big(p_x\, x \bigm| x \in M,\; x^\vee = p^\vee \big) \;= \sum \big(p_x\, (\varphi\, x) \bigm| x \in M,\; x^\vee = p^\vee \big),$$

which preserves products and is therefore a morphism of commutative multirings over S. Thus $\mathbb{Z} \dashv M$ is the free commutative multiring over S on M.

In the adjunction of commutative multimonoids over S to commutative multirings over S, the counit ϵ assigns to an commutative multiring R the morphism $\epsilon_R \colon \mathbb{Z} \dashv R \longrightarrow R$ of commutative multirings over S that extends the identity on R ($\epsilon_R \circ \eta = 1_R$). By the above, ϵ_R is the evaluation map

$$\epsilon_R \left(\sum \left(p_r\, r \mid r \in R,\ r^\vee = p^\vee \right) \right) = \sum \left(p_r\, r \mid r \in R,\ r^\vee = p^\vee \right),$$

where the first sum is an element of $\mathbb{Z} \dashv R$ and the second sum is calculated in R.

5. Free multirings on sets over S. The *free commutative multiring over S on* a set X over S is the free commutative multiring $\mathbb{Z}[X] = \mathbb{Z} \dashv \mathbb{F}(X)^+$ over S on the free commutative multimonoid $\mathbb{F}(X)^+$ over S on X.

As in Section 4 we regard $\mathbb{F}(X)$ as the commutative multimonoid of monomials $\mathrm{X}^m = \prod_{x\in X} X_x^{m(x)}$ on a set $\overline{X} = \{\, X_x \mid x \in X \,\}$ over S with an isomorphism $x \longmapsto X_x$ from X. The typical element of $\mathbb{Z} \dashv \mathbb{F}(X)^+$ is a homogeneous linear combination

$$p = \sum \left(p_{m,t}\, (\mathrm{X}^m,\, t) \mid \mathrm{X}^m \in \mathbb{F}(X),\ t \in S,\ (\mathrm{X}^m)^\vee\, t = p^\vee \right),$$

where $(\mathrm{X}^m)^\vee = \prod_{x\in X} (x^\vee)^{m(x)}$.

The canonical map $\eta^F \colon X \longrightarrow \mathbb{F}(X)$ sends $x \in X$ to the monomial $\mathrm{X}^m = \prod_{y\in X} X_y^{m(y)}$ in which $m(x) = 1$ and $m(y) = 0$ if $y \neq x$, which we identify with X_x. Hence the canonical map $\eta \colon X \longrightarrow \mathbb{Z}[X]$ sends $x \in X$ to the linear combination

$$\eta x = \sum \left(n_{m,t}\, (\mathrm{X}^m,\, 1) \mid \mathrm{X}^m \in \mathbb{F}(X) \right)$$

in which $n_{m,t} = 1$ if $\mathrm{X}^m = X_x$ and $t = 1 \in S$, otherwise $n_{m,t} = 0$. Since η is injective we may identify $X_x \in \overline{X}$ and $\eta x \in \mathbb{Z}[X]$. Every $\mathrm{X}^m \in \mathbb{F}(X)$ is now a product $\mathrm{X}^m = \prod_{x\in X} X_x^{m(x)}$ of elements of $\mathbb{F}(X)$. Hence every element of $\mathbb{Z}[X]$ is uniquely a homogeneous linear combination

$$p = \sum \left(p_{m,t}\, (\textstyle\prod_{x\in X} X_x^{m(x)},\, t) \mid \prod_{x\in X} X_x^{m(x)} \in \mathbb{F}(X),\ t \in S,\ (\mathrm{X}^m)^\vee\, t = p^\vee \right),$$

of elements of $\mathbb{Z}[X]$.

Every morphism f of sets over S from X to a commutative multiring R over S then extends uniquely to a morphism of commutative multimonoids over S from $\mathbb{F}(X)^+$ to R that sends $(\prod_{x\in X} X_x^{m(x)},\, t)$ to $\left(\prod_{x\in X} (fx)^{m(x)}\right)^t$, thence extends uniquely to a morphism φ of commutative multirings over S from $\mathbb{Z}[X]$ to R that sends

$$p = \sum \left(p_{m,t}\, (\textstyle\prod_{x\in X} X_x^{m(x)},\, t) \mid \mathrm{X}^m \in \mathbb{F}(X),\ t \in S,\ (\mathrm{X}^m)^\vee\, t = p^\vee \right),$$

to

$$\varphi p = \sum \left(p_{m,t}\, \left(\textstyle\prod_{x\in X} (fx)^{m(x)}\right)^t \,\middle|\, \mathrm{X}^m \in \mathbb{F}(X),\ t \in S,\ (\mathrm{X}^m)^\vee\, t = p^\vee \right),$$

as calculated in R.

The counit ϵ in this adjunction sends a commutative multiring R over S to the morphism ϵ_R of commutative multirings over S from $\mathbb{Z}[R]$ to R that extends the identity on R. By the above, ϵ_R sends

$$p \;= \sum \big(p_{m,t} \, (\textstyle\prod_{r\in R} X_r^{m(r)}, \; t) \;\big|\; \mathrm{X}^m \in \mathbb{F}(R),\; t \in S,\; (\mathrm{X}^m)^\vee \, t = p^\vee \,\big) \,,$$

to

$$\epsilon_R p \;= \sum \Big(p_{m,t} \, \big(\textstyle\prod_{r\in R} r^{m(r)}\big)^t \;\Big|\; \mathrm{X}^m \in \mathbb{F}(R),\; t \in S,\; (\mathrm{X}^m)^\vee \, t = p^\vee \,\Big) \,,$$

as calculated in R.

6. Free multirings on multisets. The *free commutative multiring over S on* a multiset X over S is the free commutative multiring $\mathbb{Z} \dashv \mathbb{H}(X)^+$ over S on the free commutative multimonoid $\mathbb{H}(X)^+$ over S on X.

As in Section 4, $\mathbb{H}(X) = \mathbb{F}(X)/\approx$, where $\approx$ is the smallest congruence on $\mathbb{F}(X)$ such that $X_x^t \, X_y \approx X_x \, X_y^t$ for all $x, y \in X$ and $t \in S$. The typical element Y of $\mathbb{H}(X)$ is the equivalence class $Y = [X_1 X_2 \cdots X_n]$ of a commutative product $X_1 X_2 \cdots X_n$ of elements of $\overline{X} = \{ X_x \mid x \in X \}$; equivalently, $Y = [\mathrm{X}^m]$ for some $\mathrm{X}^m \in \mathbb{F}(X)$.

Accordingly, the typical element of $\mathbb{Z} \dashv \mathbb{H}(X)^+$ is a homogeneous linear combination

$$p \;= \sum \big(p_{Y,t} \, (Y, t) \;\big|\; Y \in \mathbb{H}(X),\; t \in S,\; Y^\vee \, t = p^\vee \,\big) \,.$$

If X only has one element, or if $x^t = x$ for all x, t, then $\mathbb{H}(X) = \mathbb{F}(X)$; otherwise $\approx$ is not the equality on $\mathbb{F}(X)$, hence an element Y of $\mathbb{H}(X)$ cannot be written in the form $Y = [\mathrm{X}^m]$ for some unique monomial X^m.

7 Multimodules

In this section, S is a commutative monoid and R is a given commutative multiring over S with an identity element.

1. Multimodules. A *multi-R-module over S* is an abelian multigroup M over S on which R acts so that

$$\begin{aligned}
&(rx)^\vee = r^\vee \, x^\vee \quad \text{for all} \quad r, x \,, \\
&1x = x \quad \text{for all} \quad x \,, \\
&(rx)^t = r^t \, x = r \, x^t \quad \text{for all} \quad r, x, t \,, \\
&(r+s) \, x = rx + sx \quad \text{whenever} \quad r+s \quad \text{is defined} \,, \\
&r \, (x+y) = rx + ry \quad \text{whenever} \quad x+y \quad \text{is defined} \,, \quad \text{and} \\
&r \, (sx) = (rs) \, x \quad \text{for all} \quad r, s, x \,.
\end{aligned}$$

For example, if S has just one element, then R is just a commutative ring and a multi-R-module over S is just an R-module. In general, the trivial abelian multigroup over S (in which $M_a = \{0_a\}$ for all $a \in S$) is a *trivial* multi-R-module over S for any multiring R over S. Every commutative multiring R on S is also a multi-R-module over S, in which R acts on itself by multiplication in R.

Every abelian multigroup G over S is a multi-$\mathbb{Z}^+$-module over S, where $\mathbb{Z}^+ = \mathbb{Z} \times S$ is the multiring of integers over S (with projection $(n,t) \longrightarrow t$ to S and $(n,t)^u = (n, tu)$, $(n,t) + (m,t) = (n+m,\, t)$, $(n,t)(m,u) = (nt, mu)$, as in Section 6). Indeed $\mathbb{Z}$ already acts on G since every abelian group G_a is a $\mathbb{Z}$-module; then $\mathbb{Z}^+$ acts on G by

$$(n,t)\, x \;=\; n\, x^t \;=\; (nx)^t$$

for all x, n, t.

If R is a commutative ring with an identity element and M is an R-module, then M^+ is a multi-R^+-module, in which R^+ acts on M^+ by $(r,a)(x,b) = (rx, ab)$.

In general, if M is a multi-R-module, where R is a commutative multiring over S, then we denote the identity element of the abelian group M_a by 0_a or just 0 (like the identity element of R_a). If $x \in M_b$, then $0_a\, x = (0_a + 0_a)\, x = (0_a\, x) + (0_a\, x)$, which in the abelian group M_{ab} implies

$$0_a\, x \;=\; 0_{ab}\,, \quad \text{whenever} \;\; x \in M_b\,.$$

Similarly,

$$r\, 0_b \;=\; 0_{ab}\,, \quad \text{whenever} \;\; r \in R_a\,.$$

A morphism $\varphi\colon M \longrightarrow N$ of multi-R-modules is a morphism of multigroups ($(\varphi x)^\vee = x^\vee$, $\varphi\,(x^t) = (\varphi x)^t$, and $\varphi\,(x+y) = \varphi x + \varphi y$ whenever $x+y$ is defined) that preserves the action of R:

$$\varphi\,(rx) \;=\; r\,(\varphi x) \;\; \text{for all} \;\; r \in R \;\; \text{and} \;\; x \in M\,.$$

Under pointwise addition the morphisms from M to N constitute an abelian group $\mathrm{Hom}_R\,(M,N)$. If $R = \mathbb{Z}^+$, then a morphism of multi-R-modules over S is just a morphism of abelian multigroups over S, and $\mathrm{Hom}_R\,(M,N) = \mathrm{Hom}\,(M,N)$.

2. Submultimodules. A *submultimodule* N of a multi-R-module M is a submultigroup of M that admits the action of R: if $r \in R$ and $x \in N$, then $rx \in N$. Then N inherits the action of R and is a multi-R-module.

For example, if $\varphi\colon M \longrightarrow N$ is a morphism of multi-R-modules, then $\mathrm{Im}\,\varphi$ is a submultimodule of N and $\mathrm{Ker}\,\varphi$ is a submultimodule of M.

In general, if N is a submultimodule of M, then the action $M_b \longrightarrow M_{ab}$ of $r \in R_a$ induces an action $M_b/N_b \longrightarrow M_{ab}/N_{ab}$, which makes the quotient multigroup M/N a multi-R-module and the projection $M \longrightarrow M/N$ a morphism of multi-R-modules. This results in a short exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ of multi-R-modules.

If $\varphi\colon M \longrightarrow N$ is a morphism of multi-R-modules, then $M/\mathrm{Ker}\,\varphi \cong \mathrm{Im}\,\varphi$.

Given a multi-R-module M and a subset X of M, there is a least submultimodule $\overline{X}$ of M that contains X, the submultimodule of M *generated by* X, which is constructed as follows.

Let $Y = \{\, x^t \mid x \in X,\ t \in S \,\}$ be the submultiset of M generated by X, as in Section 2. Then $y \in Y$ implies $y^t \in Y$ for all $t \in S$.

Let $Z = RY = \{\, ry \mid r \in R,\ y \in Y \,\}$, so that $z \in Z$ implies $z^t \in Z$ for all $t \in S$ and $rz \in Z$ for all $r \in R$, in particular $-z = (-1)\, z \in Z$.

Then $\overline{X}$ is the submultigroup of M generated by Z: for each $a \in S$, the elements of $\overline{X}_a$ are all (possibly empty, but necessarily homogeneous) sums $z_1 + \cdots + z_n$ of elements $z_1, \ldots, z_n$ of Z_a.

Indeed any submultimodule of M that contains X must also contain Y, Z, and $\overline{X}$. Conversely, $\overline{X}$ is a submultimodule of M. Indeed, $\overline{X}$ is a submultigroup of M, since $z \in Z_a$ implies $-z \in Z_a$. If x is a sum $x = z_1 + \cdots + z_n$ of elements $z_1, \ldots, z_n$ of Z_a, then $x^t = z_1^t + \cdots + z_n^t$ is a sum of elements of Z_{at}, and $rx = rz_1 + \cdots + rz_n$ is a sum of elements of $Z_{r^\vee a}$.

8 Modules

In this section R is a commutative multiring over a commutative monoid S.

1. Definition. Commutative multirings also act on ordinary abelian groups. In detail, an *R-module* M is an abelian group on which R acts so that

$$\begin{aligned}
&1x = x \ \text{ for all } \ x \in M\,, \\
&r\,(x+y) = rx + ry \ \text{ for all } \ r \in R \ \text{ and } \ x, y \in M\,, \\
&(r+s)\,x = rx + sx \ \text{ whenever } \ r+s \ \text{ is defined, and} \\
&r\,(sx) = (rs)\,x \ \text{ for all } \ r, s \in R \ \text{ and } \ x \in M\,.
\end{aligned}$$

An action of S on every R-module M is then defined by

$$x^t \;=\; 1^t\, x \ \text{ for all } \ x \in M \ \text{ and } \ t \in S$$

and satisfies

$$\begin{aligned}
&(x+y)^t = x^t + y^t\,, \\
&x^1 = x\,, \\
&(x^t)^u = (1^u\, 1^t)\, x = 1^{ut}\, x = x^{tu}\,, \ \text{ and} \\
&(rx)^t = r^t\, x = r\,(x^t)\,,
\end{aligned}$$

for all r, x, y, t, u, where the last equalities hold since

$$\begin{aligned}
(rx)^t \;&=\; (1^t r)\, x \;=\; (1\, r^t)\, x \\
&=\; (r\, 1^t)\, x \;=\; r\,(x^t)\,.
\end{aligned}$$

This makes M an S-module (and a $\mathbb{Z}[S]$-module, where $\mathbb{Z}[S]$ is the ordinary semigroup ring), but not a multimodule.

A *homomorphism* $\varphi\colon M \longrightarrow N$ of R-modules is homomorphism of abelian groups ($\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x, y \in M$) that preserves the action of R:

$$\varphi(rx) \;=\; r\,\varphi(x)$$

for all $r \in R$ and $x \in M$. The homomorphisms of M into N then constitute an R-module $\mathrm{Hom}_R(M, N)$, in which addition and action of R are pointwise.

2. Hom sets. If G and H are abelian multigroups over S, then we saw in Section 5 that the morphisms from G to H constitute an abelian group $\mathrm{Hom}(G, H)$ under pointwise addition.

More generally, the set or multiset morphisms of a set or multiset X over S into an abelian multigroup G over S constitute an abelian group $\mathrm{Map}(X, G)$ under pointwise addition. The universal property of the free multigroup F on X yields a natural isomorphism $\mathrm{Map}(X, G) \cong \mathrm{Hom}(F, G)$, part of the adjunction in Section 5.

R-modules are even more similar to ordinary modules than multi-R-modules over S, which makes them of little interest as far as this paper is concerned, except that Hom sets of multi-R-modules over S are R-modules.

Indeed let M and N be multi-R-modules over S. If φ and ψ are morphisms from M to N, then the pointwise sum $\varphi + \psi\colon x \longmapsto \varphi x + \psi x$ is defined, since $(\varphi x)^\vee = x^\vee = (\psi x)^\vee$ for all $x \in M$. Moreover, $(\varphi + \psi)(x + y) = (\varphi + \psi)(x) + (\varphi + \psi)(y)$ whenever $x + y$ is defined; $(\varphi + \psi)(x^t) = \big((\varphi + \psi)(x)\big)^t$ for all x, t; and $(\varphi + \psi)(rx) = r\,(\varphi + \psi)(x)$ for all r, x. Thus $\varphi + \psi$ is a morphism from M to N. This addition makes $\mathrm{Hom}_R(M, N)$ an abelian group.

Furthermore, R acts pointwise on $\mathrm{Hom}_R(M, N)$: if $r \in R$ and $\varphi\colon M \longrightarrow N$ is a morphism, then $r\varphi$ is defined by

$$(r\,\varphi)(x) \;=\; r\,\varphi(x)$$

for all $x \in M$. Then

$$\begin{aligned}
&(r\,\varphi)(x+y) = (r\,\varphi)(x) + (r\,\varphi)(y) \;\text{ whenever }\; x+y \;\text{ is defined,}\\
&(r\,\varphi)(x^t) = r\,\varphi(x^t) = r\,(\varphi x)^t = \big((r\,\varphi)(x)\big)^t \;\text{ for all }\; x, t\,, \;\text{ and}\\
&(r\,\varphi)(sx) = r\,\varphi(sx) = rs\,\varphi(x) = s\,\big((r\,\varphi)(x)\big) \;\text{ for all }\; s, x\,,
\end{aligned}$$

so that $r\varphi$ is a morphism from M to N. With this action of R, $\mathrm{Hom}_R(M, N)$ becomes an R-module.

There is also an action of S on $\mathrm{Hom}_R(M, N)$, defined by $\varphi^t = 1^t\,\varphi$; then

$$\varphi^t(x) \;=\; (1^t\,\varphi)(x) \;=\; 1^t\,(\varphi(x)) \;=\; (\varphi(x))^t$$

for all x, t.

More generally, the morphisms or multiset morphisms of a set or multiset X over S into a multi-R-module M over S constitute an R-module $\mathrm{Map}(X, M)$ under pointwise addition and action of R, which also has an action of S. The universal property of the free multi-R-module F on M yields a natural isomorphism $\mathrm{Map}(X, M) \cong \mathrm{Hom}_R(F, M)$ of R-modules, part of the adjunction in Section 12.

9 Multialgebras.

In this section S is a commutative monoid and R is a commutative multiring over S.

1. Definition. A *commutative multi-R-algebra over S* is a multi-R-module A over S: a multigroup A over S on which R acts so that

$$
\begin{aligned}
&(rx)^\vee = r^\vee x^\vee \ \text{ for all } \ r \in R \ \text{ and } \ x \in A\,,\\
&1x = x \ \text{ for all } \ x \in A\,,\\
&(rx)^t = r^t x = r\,x^t \ \text{ for all } \ r \in R \ \text{ and } \ x \in A\,,\\
&(r+s)\,x = rx + sx \ \text{ whenever } r+s \text{ is defined in } R \text{ and } x \in A\,,\\
&r\,(x+y) = rx + ry \ \text{ whenever } r \in R \text{ and } x+y \text{ is defined in } A, \text{ and}\\
&r\,(sx) = (rs)\,x \ \text{ for all } \ r, s \in R \ \text{ and } \ x \in A\,,
\end{aligned}
$$

together with a commutative and associative monoid multiplication on A such that

$$
\begin{aligned}
&(xy)^\vee = x^\vee y^\vee \ \text{ whenever } \ x \in A_a \ \text{ and } \ y \in A_b\,,\\
&(xy)^t \ = \ x^t\,y \ = \ x\,y^t \ \text{ for all } \ x, y \in A \ \text{ and } \ t \in S\,,\\
&(x+y)\,z \ = \ xz + yz \ \text{ whenever } x+y \text{ is defined in } A, \text{ and}\\
&r\,(xy) \ = \ (rx)\,y \ = \ x\,(ry) \ \text{ for all } \ r \in R \ \text{ and } \ x, y \in A\,.
\end{aligned}
$$

In particular, a commutative multi-R-algebra over S is both a multi-R-module over S and a commutative multiring over S.

If S has only one element, then R is an ordinary ring and a commutative multi-R-algebra over S is an ordinary commutative R-algebra. Any commutative multiring R over S is a commutative multi-R-algebra over S, in which R acts on itself by multiplication in R. Any commutative multiring R over S is also a commutative multi-$\mathbb{Z}^+$-algebra, in which the action of $\mathbb{Z}^+$ on R is the action that makes the abelian multigroup R a multi-$\mathbb{Z}^+$-module, $(n,t)\,r = n\,r^t$.

In general, the multiplication on a commutative multi-R-algebra A over S and the actions $e\colon t \longmapsto 1^t$ and $r \longmapsto r1$ of S and R on the identity element 1 of A determine their actions on all of A: for all x, t, r,

$$x^t = (1x)^t = 1^t\,x \ \text{ and } \ rx = r\,(1x) = (r1)\,x\,.$$

If A is a commutative multi-R-algebra over S, then every multi-A-module M over S is also a multi-R-module, in which $r \in R$ acts on M by $rx = (r1)\,x$ for all $x \in M$, where $1 \in A$ (so that $r1 \in A$).

A *morphism* $\varphi\colon A \longrightarrow B$ of commutative multi-R-algebras over S is a mapping that preserves the actions of S and R, existing sums, and products; equivalently, a morphism of multi-R-modules over S which is also a morphism of commutative multirings over S.

2. Ideals. An *ideal* of a commutative multi-R-algebra A over S is a submultimodule I of A (closed under existing additions and the actions of S and R)

which is also an ideal of the commutative multiring A ($x \in I$ implies $xy \in I$ for all $y \in A$). A *proper* ideal is an ideal I such that $1 \notin I$; equivalently, $I \neq A$.

For example, if $\varphi\colon A \longrightarrow B$ is a morphism of commutative multi-R-algebras, then $\operatorname{Ker}\varphi = \{\, x \in A \mid \varphi x = 0 \,\}$ is an ideal of A, which is proper if $\varphi(1) \neq 0$.

In general, the quotient multigroup A/I is both a quotient multimodule and a quotient multiring and is therefore a commutative multi-R-algebra over S; the map $A \longrightarrow A/I$ is a morphism of commutative multi-R-algebras over S. If $\varphi\colon A \longrightarrow B$ is a morphism of commutative multi-R-algebras over S, then $\operatorname{Im}\varphi \cong A/\operatorname{Ker}\varphi$.

Given a commutative multi-R-algebra A and a subset X of A, the ideal (X) of A *generated by* X is the least ideal of A that contains X. Let

$$Y \;=\; AX \;=\; \{\, zx \mid z \in A, \;\; x \in X \,\}\,.$$

If $y = zx \in Y$, then $-y = (-z)x \in Y$, $y^t = z^t\, y \in Y$, and $ry = (rz)\, x \in Y$.

Then (X) is the set of all possibly empty homogeneous sum $y_1 + \cdots + y_n$ of elements $y_1, \ldots, y_n$ of Y (that all project to the same element of S). Indeed if $x = y_1 + \cdots + y_n$ is a homogeneous sum of elements of Y, then so are $x^t = y_1^t + \cdots + y_n^t$, $rx = ry_1 + \cdots + ry_n$, and $zx = zy_1 + \cdots + zy_n$, for all $t \in S$, $r \in R$, and $z \in A$.

3. Submultialgebras. A *submultialgebra* of a commutative multi-R-algebra A over S is a subset of A which is closed under the actions of S and R, existing sums, and products; equivalently, a subset of A which is both a submultimodule and a submultiring of A.

For example, if $\varphi\colon A \longrightarrow B$ is a morphism of commutative multi-R-algebras over S, then $\operatorname{Im}\varphi$ is a submultialgebra of B.

If X is a subset of A, the submultialgebra $\overline{X}$ of A *generated by* X is the least submultialgebra of A that contains X. It can be described as follows.

Let Y be the submonoid of A generated by X:

$$Y \;=\; \{\, x_1 x_2 \cdots x_n \mid x_1,\, x_2, \ldots,\, x_n \in X \,\}\,,$$

with $x_1 x_2 \cdots x_n = 1$ if $n = 0$. Let

$$V \;=\; RY \;=\; \{\, ry \mid r \in R, \;\; y \in Y \,\}\,.$$

If $v = ry \in V$, then $v^t = r^t\, y \in V$ and $sv = (sr)\, y \in V$, in particular $-v \in V$. If $v = ry$ and $w = sz \in V$, where $y, z \in Y$, then

$$vw \;=\; (ry)(sz) \;=\; r\,(y\,(sz)) \;=\; r\,((sz)y) \;=\; r\,(s\,(zy)) \;=\; (rs)(yz) \;\in V\,.$$

Then $\overline{X}$ is the submultigroup of A generated by V: the set of all possibly empty homogeneous sums of elements of V. Indeed if $x = v_1 + \cdots + v_n \in \overline{X}$ is a homogeneous sum of elements of V, then so are $x^t = v_1^t + \cdots + v_n^t$ and $rx = rv_1 + \cdots + rv_n$; if $x = v_1 + \cdots + v_n \in \overline{X}$ and $y = w_1 + \cdots + w_m \in \overline{X}$ are homogeneous sums of elements of V, then so is $xy = \sum_{i,j} v_i\, w_j$.

10 The box product

This section defines a basic construction for multistructures with no additivity.

1. Definition. The cartesian product $X \times Y$ of two multisets over S is a set over S, with projection $(x,y)^\vee = x^\vee\, y^\vee$, but it lacks an adequate action of S. That honor is reserved for the box product $X \,\square\, Y$, which is basically a tensor product of S-sets.

Let X and Y be multisets over S. Let $\approx$ be the smallest equivalence relation on $X \times Y$ such that $(x^t,\, y) \approx (x,\, y^t)$ for all $x \in X$, $y \in Y$, and $t \in S^1$. The *box product* of X and Y is the set

$$X \,\square\, Y \;=\; (X \times Y)/\approx .$$

The *box map* $\beta\colon X \times Y \longrightarrow X \,\square\, Y$ is the surjection that sends $(x,y) \in X \times Y$ to its equivalence class $x \,\square\, y$.

The universal property of $X \,\square\, Y$ and β is as follows. A mapping f of $X \times Y$ into a set Z is *balanced* if and only if

$$f(x^t, y) \;=\; f(x, y^t)$$

for all $x \in X$, $y \in Y$, and $t \in S^1$; equivalently, the equivalence relation induced by f on $X \times Y$ contains $\approx$. The box map β is balanced, by definition of $\approx$. Conversely, every balanced mapping $f\colon X \times Y \longrightarrow Z$ factors uniquely through β (there exists a unique mapping $g\colon X \,\square\, Y \longrightarrow Z$ such that $g \circ \beta = f$), since the equivalence relation induced by f contains $\approx$.

We use this universal property to make $X \,\square\, Y$ a multiset over S. The projection $X \times Y \longrightarrow S$ of $X \times Y$ is balanced, since

$$(x^t)^\vee\, y^\vee \;=\; x^\vee\, t\, y^\vee \;=\; x^\vee\, (y^t)^\vee$$

for all x, y, t, and induces a projection of $X \,\square\, Y$ to S, which is the unique mapping such that

$$(x \,\square\, y)^\vee \;=\; x^\vee\, y^\vee$$

for all x, y. Next, for any $t \in S^1$, the mapping $(x,y) \longmapsto x^t \,\square\, y$ is balanced, since

$$(x^u)^t \,\square\, y \;=\; (x^t)^u \,\square\, y \;=\; x^t \,\square\, y^u$$

for all x, y, u. Hence there is a unique mapping $x \,\square\, y \longmapsto (x \,\square\, y)^t$ such that

$$(x \,\square\, y)^t \;=\; x^t \,\square\, y \;\;(=\; x \,\square\, y^t)$$

for all x, y. Uniqueness implies $(x \,\square\, y)^1 = x \,\square\, y$ and $\big((x \,\square\, y)^t\big)^u = (x \,\square\, y)^{tu}$ for all x, y. With this projection to S and action of S^1, $X \,\square\, Y$ is now an multiset over S.

If $f\colon X \longrightarrow X'$ and $g\colon Y \longrightarrow Y'$ are morphisms of multisets over S, then $(x,y) \longmapsto fx \,\square\, gy$ is balanced, and there is a unique mapping $f \,\square\, g\colon X \,\square\, Y \longrightarrow X' \,\square\, Y'$ such that

$$(f \,\square\, g)(x \,\square\, y) \;=\; fx \,\square\, gy \;\text{ for all }\; x \in X \;\text{ and }\; y \in Y .$$

Moreover, $f \,\Box\, g$ is a morphism of multisets over S, since

$$\big((f \,\Box\, g)(x \,\Box\, y)\big)^\vee \;=\; (fx)^\vee\,(gy)^\vee \;=\; x^\vee\, y^\vee \;=\; (x \,\Box\, y)^\vee$$

and

$$\big((f \,\Box\, g)(x \,\Box\, y)\big)^t \;=\; (fx)^t \,\Box\, gy \;=\; f(x^t) \,\Box\, gy \;=\; (f \,\Box\, g)\big((x \,\Box\, y)^t\big)\,,$$

for all x, y, t. This makes the box product a bifunctor to multisets over S.

Finally, if S is a commutative monoid, then there is for each multiset X over S a natural isomorphism $S \,\Box\, X \cong X$, where S is the trivial multiset over S, with projection $a^\vee = a$ and action $a^t = at$ of S. Indeed $(a, x) \longmapsto x^a$ is a balanced mapping of $S \times X$ into X, and induces a mapping of $S \,\Box\, X$ into X that sends $a \,\Box\, x$ to x^a and is in fact a morphism of multisets over S. The inverse isomorphism sends $x \in X$ to $1 \,\Box\, x \in S \,\Box\, X$.

2. Associativity. More generally, the box product $X_1 \,\Box\, X_2 \,\Box \cdots \Box\, X_n$ of a sequence $X_1,\, X_2, \dots,\, X_n$ of multisets over S is the quotient set

$$X_1 \,\Box\, X_2 \,\Box \cdots \Box\, X_n \;=\; \big(X_1 \times X_2 \times \cdots \times X_n\big) / \approx\,,$$

where $\approx$ is the smallest equivalence relation on the cartesian product $X_1 \times \cdots \times X_n$ such that

$$(x_1^t,\, x_2,\, \dots,\, x_n) \;\approx\; (x_1,\, x_2^t,\, \dots,\, x_n) \;\approx\; \cdots \;\approx\; (x_1,\, x_2,\, \dots,\, x_n^t)$$

for all $x_i \in X_i$ and $t \in S^1$.

The *box map*

$$\beta\colon X_1 \times X_2 \times \cdots \times X_n \longrightarrow X_1 \,\Box\, X_2 \,\Box \cdots \Box\, X_n$$

is the surjection that sends $(x_1,\, \dots,\, x_n) \in X_1 \times \cdots \times X_n$ to its equivalence class $x_1 \,\Box \cdots \Box\, x_n$.

The universal property of $X_1 \,\Box\, X_2 \,\Box \cdots \Box\, X_n$ and β is as follows. A mapping f of $X_1 \,\Box\, X_2 \,\Box \cdots \Box\, X_n$ into a set Y is *balanced* if and only if

$$f(x_1^t,\, x_2,\, \dots,\, x_n) \;=\; f(x_1,\, x_2^t,\, \dots,\, x_n) \;=\; \cdots \;=\; f(x_1,\, x_2,\, \dots,\, x_n^t)$$

for all $x_i \in X_i$ and $t \in S^1$; equivalently, the equivalence relation induced by f on $X_1 \times \cdots \times X_n$ contains $\approx$. The box map β is balanced, by definition of $\approx$. Conversely, every balanced mapping $f\colon X_1 \times \cdots \times X_n \longrightarrow Y$ factors uniquely through β (there exists a unique mapping $g\colon X_1 \,\Box \cdots \Box\, X_n \longrightarrow Y$ such that $g \circ \beta = f$), since the equivalence relation induced by f contains $\approx$.

We use this universal property to make $X_1 \,\Box\, X_2 \,\Box \cdots \Box\, X_n$ a multiset over S. The projection $X_1 \times X_2 \times \cdots \times X_n \longrightarrow S$ that sends $(x_1, x_2, \dots, x_n)$ to $x_1^\vee\, x_2^\vee \cdots x_n^\vee$ is balanced, since $(x^t)^\vee = x^\vee t$ for every $x \in X_i$. Therefore it induces a projection of $X_1 \,\Box\, X_2 \,\Box \cdots \Box\, X_n$ to S, which is the unique mapping such that

$$(x_1 \,\Box\, x_2 \,\Box \cdots \Box\, x_n)^\vee \;=\; x_1^\vee\, x_2^\vee \,\cdots\, x_n^\vee$$

for all $x_i \in X_i$. Next, for any $t \in S^1$, the mapping $(x_1, x_2, \dots, x_n) \longmapsto x_1^t \,\square\, x_2 \,\square \cdots \square\, x_n$ is balanced. Hence there is a unique mapping

$$x_1 \,\square\, x_2 \,\square \cdots \square\, x_n \longmapsto (x_1 \,\square\, x_2 \,\square \cdots \square\, x_n)^t$$

such that

$$(x_1 \,\square\, x_2 \,\square \cdots \square\, x_n)^t \;=\; x_1^t \,\square\, x_2 \,\square \cdots \square\, x_n \;\;(=\; x_1 \,\square\, x_2^t \,\square \cdots \square\, x_n \;=\; \cdots)$$

for all x, y. Uniqueness implies $(x_1 \,\square \cdots \square\, x_n)^1 = x_1 \,\square \cdots \square\, x_n$ and

$$\big((x_1 \,\square \cdots \square\, x_n)^t\big)^u \;=\; (x_1 \,\square \cdots \square\, x_n)^{tu}$$

for all x_i, t, u. With this projection to S and action of S^1, $X_1 \,\square \cdots \square\, X_n$ is now an multiset over S.

Now let X, Y, and Z be multisets over S. For each $z \in Z$, $(x, y) \longmapsto x \,\square\, y \,\square\, z$ is a balanced map of $X \times Y$ into $X \,\square\, Y \,\square\, Z$ and induces a mapping of $X \,\square\, Y$ into $X \,\square\, Y \,\square\, Z$ that sends $x \,\square\, y$ to $x \,\square\, y \,\square\, z$; then $(x \,\square\, y,\, z) \longmapsto x \,\square\, y \,\square\, z$ is a balanced map of $(X \,\square\, Y) \times Z$ into $X \,\square\, Y \,\square\, Z$ and induces a mapping of $(X \,\square\, Y) \,\square\, Z$ into $X \,\square\, Y \,\square\, Z$ that sends $(x \,\square\, y) \,\square\, z$ to $x \,\square\, y \,\square\, z$, and is in fact a morphism of multisets over S. Conversely, $(x, y, z) \longmapsto (x \,\square\, y) \,\square\, z$ is a balanced map of $X \times Y \times Z$ into $(X \,\square\, Y) \,\square\, Z$ and induces the inverse isomorphism.

Thus there are natural isomorphisms

$$(X \,\square\, Y) \,\square\, Z \;\cong\; X \,\square\, Y \,\square\, Z \;\cong\; X \,\square (Y \,\square\, Z)\,.$$

There are also natural isomorphisms $Y \,\square\, X \cong X \,\square\, Y$. In fact, multisets over S constitute a symmetric monoidal category (as defined in [9]); the identity object is the trivial multiset S over S.

3. Multimonoids. If S is a commutative monoid and X, Y are commutative multimonoids over S, then $X \,\square\, Y$ inherits a multiplication from X and Y. The multiplication on X is a balanced mapping of $X \times X$ into X, since $x^t\, y = (xy)^t = x\, y^t$ for all x, y, t. Hence the multiplications on X and Y induce a map

$$(X \,\square\, Y) \,\square (X \,\square\, Y) \overset{\cong}{\longrightarrow} (X \,\square\, X) \,\square (Y \,\square\, Y) \longrightarrow X \,\square\, Y$$

and a multiplication on $X \,\square\, Y$ such that

$$(x \,\square\, y)(x' \,\square\, y') \;=\; xx' \,\square\, yy'$$

for all x, x', y, y', which makes $X \,\square\, Y$ a commutative multimonoid over S.

The commutative multimonoid $X \,\square\, Y$ comes with canonical morphisms of commutative multimonoids over S

$$\lambda\colon X \longrightarrow X \,\square\, Y,\; x \longmapsto \overline{x} = x \,\square\, 1 \;\text{ and }\; \rho\colon Y \longrightarrow X \,\square\, Y,\; y \longmapsto \overline{y} = 1 \,\square\, y\,.$$

Then

$$x \,\square\, y \;=\; (x \,\square\, 1)(1 \,\square\, y) \;=\; \overline{x}\,\overline{y}\,,$$

so that $X \,\square\, Y = \overline{X}\,\overline{Y}$, where $\overline{X} = \operatorname{Im} \lambda$ and $\overline{Y} = \operatorname{Im} \rho$.

Unfortunately there is no evidence that λ and ρ are injective.

11 The mixed product

The mixed product is similar to the box product but allows an addition in its first variable.

1. Definition. Let G be an abelian multigroup over S and X be a multiset over S. The *mixed product* $G \boxslash X$ of G and X is the abelian multigroup over S generated by $G \times X$, subject to all homogeneous defining relations

$$\begin{aligned} (g^t,\, x) &= (g,\, x^t) \quad \text{for all } \ g, x, t, \quad \text{and} \\ (g+h,\, x) &= (g,x) + (h,x) \quad \text{whenever } \ g+h \ \text{ is defined}, \end{aligned}$$

in the statement of which each $(g,x) \in G \times X$ has been identified with its image $\eta\,(g,x)$ in the free abelian multigroup $\mathbb{Z} \dashv (G \times X)$ over S on the multiset $G \times X$.

Equivalently, $G \boxslash X$ is the abelian multigroup over S generated by the multiset $G \,\Box\, X$ over S, subject to all homogeneous defining relations

$$(g+h) \,\Box\, x \;=\; (g \,\Box\, x) + (h \,\Box\, x) \quad \text{whenever } \ g+h \ \text{ is defined},$$

in the statement of which each $g \,\Box\, x \in G \,\Box\, X$ has been identified with its image $\eta\,(g \,\Box\, x)$ in the free abelian multigroup $F = \mathbb{Z} \dashv (G \,\Box\, X)$ over S on the multiset $G \,\Box\, X$. Thus

$$G \boxslash X \;=\; F/K\,,$$

where K is the submultigroup of F generated by all

$$k(g,h,x) \;=\; (g+h) \,\Box\, x \;-\; (g \,\Box\, x) \;-\; (h \,\Box\, x)$$

such that $g, h \in G$, $x \in X$, and $g+h$ is defined. The maps $G \,\Box\, X \xrightarrow{\eta} F \longrightarrow G \boxslash X$ are morphisms of multisets over S.

The abelian multigroup $G \boxslash X$ comes with a canonical *mixed map*

$$\mu\colon G \times X \xrightarrow{\beta} G \,\Box\, X \xrightarrow{\eta} F \longrightarrow G \boxslash X\,, \;\; (g,x) \longmapsto g \boxslash x\,.$$

The universal property of $G \boxslash X$ and μ is as follows. Recall that $G \times X$ is a set over S with projection $(g,x)^\vee = g^\vee x^\vee$ to S. A morphism f of sets over S from $G \times X$ to an abelian multigroup K over S is *balanced and left additive* if and only if

$$\begin{aligned} f(g^t, x) &= f(g, x^t) \quad \text{for all } \ g, x, t, \quad \text{and} \\ f(g+h,\, x) &= f(g,x) + f(h,x) \quad \text{whenever } \ g+h \ \text{ is defined}; \end{aligned}$$

equivalently, the defining relations of $G \boxslash X$ hold in K via f. The mixed map μ is balanced and left additive by definition of $G \boxslash X$:

$$\begin{aligned} g^t \boxslash x &= g \boxslash x^t \quad \text{for all } \ g, x, t, \quad \text{and} \\ (g+h) \boxslash x &= (g \boxslash x) + (h \boxslash x) \quad \text{whenever } \ g+h \ \text{ is defined}. \end{aligned}$$

In fact,

$$g^t \boxslash x \;=\; g \boxslash x^t \;=\; (g \boxslash x)^t \quad \text{for all} \quad g, x, t\,,$$

since $g^t \square x = g \square x^t = (g \square x)^t$ in $G \square X$ and $G \square X \longrightarrow G \boxslash X$ is a morphism of multisets over S. Conversely, every balanced and left additive morphism f of sets over S from $G \times X$ to an abelian multigroup H over S factors uniquely through μ (there exists a unique morphism $\varphi\colon G \boxslash X \longrightarrow H$ of abelian multigroups over S such that $\varphi \circ \mu = f$).

Since F is the free abelian multigroup over S on the multiset $G \square X$, every $p \in F$ is uniquely a homogeneous linear combination with integer coefficients

$$p \;= \sum \big(p_y\, y \mid y = g \square x,\; g^\vee x^\vee = a\big)$$

for some $a = p^\vee \in S$. Hence every element of $G \boxslash X$ is a homogeneous linear combination with integer coefficients

$$p \;= \sum \big(p_{g,x}\, (g \boxslash x) \mid g \in G,\; x \in X,\; g^\vee x^\vee = a\big)$$

for some $a = p^\vee \in S$. Since $\boxslash$ is left additive we also have $n\,(g \boxslash x) = ng \boxslash x$ for all $n \in \mathbb{Z}$, with $(ng)^\vee = g^\vee$. Therefore every element z of $G \boxslash X$ is a homogeneous sum

$$z \;=\; (g_1 \boxslash x_1) + \cdots + (g_n \boxslash x_n)\,,$$

where $n \geq 0$ and $g_i^\vee x_i^\vee = a = z^\vee$ for all i. Then

$$z^t \;=\; (g_1^t \boxslash x_1) + \cdots + (g_n^t \boxslash x_n) \;=\; (g_1 \boxslash x_1^t) + \cdots + (g_n \boxslash x_n^t)$$

for all $t \in S^1$.

If $\varphi\colon G \longrightarrow H$ is a morphism of abelian multigroups over S and $\psi\colon X \longrightarrow Y$ is a morphism of multisets over S, then $(g, x) \longmapsto \varphi g \boxslash \psi x$ is balanced and left additive and there is a unique morphism $\varphi \boxslash \psi\colon G \boxslash X \longrightarrow H \boxslash Y$ of abelian multigroups over S such that

$$(\varphi \boxslash \psi)(g \boxslash x) \;=\; \varphi g \boxslash \psi x \quad \text{for all} \quad g \in G \quad \text{and} \quad x \in X\,.$$

This makes the mixed product a bifunctor.

2. Associativity. For any abelian multigroup G and multisets X, Y over S, there is a natural isomorphism

$$(G \boxslash X) \boxslash Y \;\cong\; G \boxslash (X \square Y)\,.$$

Indeed for each $g \in G$, $(x, y) \longmapsto (g \boxslash x) \square y$ is a balanced mapping of $X \times Y$ into $(G \boxslash X) \square Y$ and induces a morphism $\varphi_g\colon X \square Y \longrightarrow (G \boxslash X) \square Y$ of abelian multigroups over S such that

$$\varphi_g\,(x \square y) \;=\; (g \boxslash x) \square y$$

for all x, y. In turn, $(g, z) \longmapsto \varphi_g\, z$ is a balanced and left additive mapping of $G \times (X \,\Box\, Y)$ into $(G \boxslash X) \,\Box\, Y$ and induces a morphism $\theta \colon G \boxslash (X \,\Box\, Y) \longrightarrow (G \boxslash X) \boxslash Y$ of abelian multigroups over S such that

$$\theta\bigl(g \boxslash (x \,\Box\, y)\bigr) \;=\; (g \boxslash x) \,\Box\, y$$

for all g, x, y.

Conversely, for each $y \in Y$, $(g, x) \longmapsto g \boxslash (x \,\Box\, y)$ is a balanced and left additive mapping of $G \times X$ into $G \boxslash (X \,\Box\, Y)$ and induces a morphism $\psi_y \colon G \boxslash X \longrightarrow G \boxslash (X \,\Box\, Y)$ of abelian multigroups over S such that

$$\psi_y\,(g \boxslash x) \;=\; g \boxslash (x \,\Box\, y)$$

for all g, x. In turn, $(z, y) \longmapsto \psi_y\, z$ is a balanced and left additive mapping of $(G \boxslash X) \times Y$ into $G \boxslash (X \,\Box\, Y)$ and induces a morphism $\zeta \colon (G \boxslash X) \boxslash Y \longrightarrow G \boxslash (X \,\Box\, Y)$ of abelian multigroups over S such that

$$\zeta\bigl((g \boxslash x) \boxslash y\bigr) \;=\; g \boxslash (x \,\Box\, y)$$

for all g, x, y. Then θ and ζ are mutually inverse isomorphisms.

3. Multimonoid rings. If S is a commutative monoid, then there is an adjunction of commutative multimonoids over S to commutative multi-R-algebras over S, for any R over S. The *free multi-R-algebra over S on* a commutative multimonoid M over S is constructed as follows.

Since both R and M are commutative multimonoids over S, their box product $R \,\Box\, M$ as multisets over S is also a commutative multimonoid over S, in which $(r \,\Box\, x)(s \,\Box\, y) = rs \,\Box\, xy$. Then the free abelian multigroup $F = \mathbb{Z} \dashv (R \,\Box\, M)$ over S on the multiset $R \,\Box\, M$ is also the free commutative multiring over S on the commutative multimonoid $R \,\Box\, M$, as in Section 6. The identity element of $R \,\Box\, M$ and F is $1 \,\Box\, 1$.

As an abelian multigroup over S, $R \boxslash M = F/K$, where K is the submultigroup of F generated by all

$$k(r, s, x) \;=\; (r + s) \,\Box\, x \;-\; r \,\Box\, x \;-\; s \,\Box\, x$$

such that $r, s \in R$, $x \in M$, and $r + s$ is defined (so that $k(r, s, x)$ is defined). If $t \,\Box\, y \in R \,\Box\, M$, then

$$k(r, s, x)(t \,\Box\, y) \;=\; k(rt, st, xy)\,.$$

Therefore K is an ideal of F; $R \boxslash M = F/K$ is a commutative multiring over S; and the projection $F \longrightarrow R \boxslash M$ is a morphism of commutative multiring over S, in particular

$$(r \boxslash x)(s \boxslash y) \;=\; rs \boxslash xy$$

for all r, s, x, y. The identity element of $R \boxslash M$ is $1 \boxslash 1$.

Every $r \in R$ now acts on $R \boxslash M$ by: $rp = (r \boxslash 1)\, p$ for all $p \in R \boxslash M$, in particular

$$r\,(s \boxslash x) \;=\; (r \boxslash 1)(s \boxslash x) \;=\; rs \boxslash x\,.$$

With this action, $R \boxslash M$ becomes a commutative multi-R-algebra over S. In fact, $R \boxslash M$ is the free commutative multi-R-algebra over S on the commutative multimonoid M over S.

Indeed $R \boxslash M$ comes with a canonical map $\eta\colon M \longrightarrow R \boxslash M$, $x \longmapsto 1 \boxslash x$, which is a morphism of commutative multimonoids over S.

The universal property of $R \boxslash M$ and η is proved as follows. Let φ be a morphism of commutative multimonoids over S from M to a commutative multi-R-algebra A over S. If $\psi\colon R \boxslash M \longrightarrow A$ is a morphism of commutative multi-R-algebras over S and $\psi \circ \eta = \varphi$, then $\psi\,(r \boxslash x) = r\,\varphi x$ for all r and x. Conversely,

$$(r, x) \longmapsto r\,(\varphi x)$$

is a balanced and left additive morphism of multisets over S from $R \times M$ to A and induces a unique morphism ψ of abelian multigroups over S from $R \boxslash M$ to A such that

$$\psi\,(r \boxslash x) \;=\; r\,(\varphi x)$$

for all r and x. Then

$$\begin{aligned}\psi\,\big((r \boxslash x)(s \boxslash y)\big) \;&=\; rs\,\varphi\,(xy) \;=\; \big(\psi\,(r \boxslash x)\big)\big(\psi\,(s \boxslash y)\big) \quad\text{and}\\ \psi\,\big(r\,(s \boxslash y)\big) \;&=\; rs\,\varphi y \;=\; r\,\psi\,(s \boxslash y)\,,\end{aligned}$$

for all r, s, x, y; hence ψ is a morphism of commutative multi-R-algebras over S.

In this adjunction of commutative multimonoids to commutative multi-R-algebras, the counit ϵ assigns to a commutative multi-R-algebra A over S the morphism $\epsilon_A\colon R \boxslash A \longrightarrow A$ induced by the identity on A. By the above,

$$\epsilon_A\,(r \boxslash a) \;=\; ra$$

as calculated in A, for all $r \in R$ and $a \in A$.

Since a commutative multi-$\mathbb{Z}^+$-algebra over S is the same as a commutative multiring over S, the above yields an adjunction of commutative multimonoids over S to commutative multirings over S, in which the *free commutative multiring over* S *on* a commutative multimonoid M over S is $\mathbb{Z}^+ \boxslash M$. From Section 6 it follows that $\mathbb{Z}^+ \boxslash M \cong \mathbb{Z} \dashv M$.

A natural isomorphism $\mathbb{Z}^+ \boxslash M \cong \mathbb{Z} \dashv M$ is found as follows. The mapping $((n, a),\, x) \longmapsto n x^a$ of $\mathbb{Z}^+ \times M$ into $\mathbb{Z} \dashv M$ is balanced and left additive and induces a morphism of commutative multirings over S from $\mathbb{Z}^+ \boxslash M$ to $\mathbb{Z} \dashv M$ such that

$$(n, a) \boxslash x \longmapsto n x^a$$

for all n, a, x. Since every $p \in \mathbb{Z} \dashv M$ is uniquely a homogeneous linear combination $p = \sum \big(\, p_x\, x \;\big|\; x \in M_a \,\big)$, where $a = p^{\vee}$, the map that sends

$$p \;= \sum \big(\, p_x\, x \;\big|\; x \in M_a \,\big) \;= \sum \big(\, p_x\, x \mid x \in M_a,\ p_x \neq 0\big) \;\in \mathbb{Z} \dashv M$$

to $\sum\big((p_x,1)\boxslash x \mid x\in M_a,\ p_x\neq 0\big)\in \mathbb{Z}^+\boxslash M$, where $a=p^\vee$, is well defined, and provides the inverse isomorphism.

4. Free multialgebras over sets over S**.** If S is a commutative monoid and R is a commutative multiring over S, then composing the adjunction of sets over S to commutative multimonoids over S with the adjunction of commutative multimonoids over S to commutative multi-R-algebras over S yields an adjunction of sets over S to commutative multi-R-algebras over S.

If R is a commutative multiring over S and X is a set over S, then the free commutative multimonoid over S on X is $\mathbb{F}(X)^+ = \mathbb{F}\times S$, where $\mathbb{F}=\mathbb{F}(X)$ is the free commutative monoid on the set X. When every element x of X is replaced by an indeterminate X_x, the elements of $\mathbb{F}(X)$ are all monomials $\mathrm{X}^m=\prod_{x\in X} X_x^{m(x)}$). In $\mathbb{F}(X)^+=\mathbb{F}\times S$,

$$(\mathrm{X}^m,\, a)^\vee \;=\; \big(\textstyle\prod_{x\in X}(x^\vee)^{m(x)}\big)\, a \;\text{ and }\; (\mathrm{X}^m,\, a)^t = (\mathrm{X}^m,\, at)\,.$$

Accordingly, the *free commutative multi-*R*-algebra over* S *on* the set X over S is the free commutative multi-R-algebra $R\boxslash\mathbb{F}(X)^+$ over S on the free commutative multimonoid $\mathbb{F}(X)^+$ over S.

It comes with a canonical map

$$\eta\colon X\longrightarrow R\boxslash\mathbb{F}(X)^+,\; x\longmapsto 1\boxslash(x,1)\,,$$

which is a morphism of sets over S. There is no general evidence that η is injective. However, there is a simpler canonical map that is injective, which sends $x\in X$ to $(x,1)\in\mathbb{F}(X)^+$, where x is short for the monomial X^m in which $m(x)=1$ and $m(y)=0$ for all $y\neq x$. Hence we can identify x and $(x,1)$. Then $\eta x = 1\boxslash x$, $(\mathrm{X}^m,\,1)=\prod_{x\in X} X_x^{m(x)}$ in $\mathbb{F}^+$, and

$$1\boxslash(\mathrm{X}^m,\,1) \;=\; 1\boxslash\textstyle\prod_{x\in X} X_x^{m(x)} \;=\; \prod_{x\in X}(\eta x)^{m(x)} \;=\; (\eta\mathrm{x})^m$$

in $R\boxslash\mathbb{F}(X)^+$, for every $\mathrm{X}^m\in\mathbb{F}(X)$, where $(\eta\mathrm{x})^m=\prod_{x\in X}(\eta x)^{m(x)}$.

The typical element of $R\boxslash\mathbb{F}(X)^+$ is a homogeneous sum

$$P \;=\; r_1\boxslash(\mathrm{X}^{m_1},\,a_1) \;+\; \cdots \;+\; r_n\boxslash(\mathrm{X}^{m_n},\,a_n)$$

where $n\geq 0$, $r_1,\,\dots,\,r_n\in R$, $\mathrm{X}^{m_1},\,\dots,\,\mathrm{X}^{m_n}\in\mathbb{F}(X)$, $a_1,\,\dots,\,a_n\in S$, and $r_i^\vee\,(\mathrm{X}^{m_i})^\vee\, a_i^\vee = P^\vee$ for all i. This simplifies to a polynomial-like form. Indeed

$$\begin{aligned} r\boxslash(\mathrm{X}^m,\,a) \;&=\; r^a\boxslash(\mathrm{X}^m,\,1) \;=\; \big(r^a\boxslash(1,1)\big)\big(1\boxslash(\mathrm{X}^m,\,1)\big)\\ &=\; r^a\,\big(1\boxslash(1,1)\big)\big(1\boxslash(\mathrm{X}^m,\,1)\big) \;=\; r^a\,(\eta\mathrm{x})^m\,, \end{aligned}$$

since $1\boxslash(1,1)$ is the identity element of $R\boxslash\mathbb{F}(X)^+$; hence the typical element P of $R\boxslash\mathbb{F}(X)^+$ is now a homogeneous polynomial with coefficients in R and variables in $\operatorname{Im}\eta$:

$$P \;=\; r_1\textstyle\prod_{x\in X}(\eta x)^{m_1(x)} \;+\; \cdots \;+\; r_n\prod_{x\in X}(\eta x)^{m_n(x)}\,,$$

where $n \geq 0$, $r_1, \ldots, r_n \in R$, $\mathrm{X}^{m_1}, \ldots, \mathrm{X}^{m_n} \in \mathbb{F}(X)$, and $r_i^\vee (\mathrm{x}^{m_i})^\vee = P^\vee$ for all i. However, there is no evidence that P can be written uniquely in this form, or that η is injective.

The universal property of $R \boxslash \mathbb{F}(X)^+$ and η is: every morphism f of sets over S from X to a commutative multi-R-algebra A over S induces a unique morphism $\varphi\colon R \boxslash \mathbb{F}(X)^+ \longrightarrow A$ of commutative multi-R-algebras over S such that $\varphi \circ \eta = f$. If

$$P \;=\; r_1 \textstyle\prod_{x\in X} (\eta x)^{m_1(x)} \;+\; \cdots \;+\; r_n \prod_{x\in X} (\eta x)^{m_n(x)}\,,$$

then

$$\varphi P \;=\; r_1 \textstyle\prod_{x\in X} (fx)^{m_1(x)} \;+\; \cdots \;+\; r_n \prod_{x\in X} (fx)^{m_n(x)}\,,$$

as calculated in A.

In the adjunction of sets over S to commutative multi-R-algebras over S, the counit ϵ assigns to a commutative multi-R-algebra A over S the morphism $\epsilon_A\colon R \boxslash \mathbb{F}(A)^+ \longrightarrow A$ of commutative multi-R-algebras over S induced by the identity on A: if

$$P \;=\; r_1 \textstyle\prod_{a\in A} (\eta a)^{m_1(a)} \;+\; \cdots \;+\; r_n \prod_{a\in A} (\eta a)^{m_n(a)} \;\in R[A]\,,$$

then

$$\epsilon_A P \;=\; r_1 \textstyle\prod_{a\in A} a^{m_1(a)} \;+\; \cdots \;+\; r_n \prod_{a\in A} a^{m_n(a)}\,,$$

as calculated in A.

12 The tensor product

The tensor product is similar to the mixed product but allows an addition in both variables.

1. Definition. Let G and H be abelian multigroups over S. The *tensor product* $G \otimes H$ of G and H is the abelian multigroup over S generated by the set $G \times H$ over S, subject to all homogeneous defining relations

$$\begin{aligned} (x^t,\, y) \;&=\; (x,\, y^t) \quad \text{for all} \quad x, y, t, \\ (x+y,\, z) \;&=\; (x, z) + (y, z) \quad \text{whenever} \quad x+y \quad \text{is defined}\,, \quad \text{and} \\ (x,\, y+z) \;&=\; (x, y) + (x, z) \quad \text{whenever} \quad y+z \quad \text{is defined}\,, \end{aligned}$$

in the statement of which each $(x, y) \in G \times H$ has been identified with its image $\eta\,(x, y)$ in the free abelian multigroup $\mathbb{Z} \dashv (G \times H)$ over S on the set $G \times H$ over S.

Equivalently, $G \otimes H$ is the abelian multigroup over S generated by the multiset $G \,\square\, H$ over S, subject to all homogeneous defining relations

$$\begin{aligned} (x+y) \,\square\, z \;&=\; (x \,\square\, z) + (y \,\square\, z) \quad \text{whenever} \quad x+y \quad \text{is defined}\,, \quad \text{and} \\ x \,\square (y+z) \;&=\; (x \,\square\, y) + (x \,\square\, z) \quad \text{whenever} \quad y+z \quad \text{is defined}\,, \end{aligned}$$

in the statement of which each $x \,\Box\, y \in G \,\Box\, H$ has been identified with its image $\eta\,(x \,\Box\, y)$ in the free abelian multigroup $F = \mathbb{Z} \dashv (G \,\Box\, H)$ over S on the multiset $G \,\Box\, H$. Thus

$$G \otimes X \;=\; F/K\,,$$

where K is the submultigroup of F generated by all

$$k(x,y,z) \;=\; (x+y) \,\Box\, z \;-\; (x \,\Box\, z) \;-\; (y \,\Box\, z)$$

such that $x, y \in G$, $z \in H$, and $x+y$ is defined, and all

$$l(x,y,z) \;=\; x \,\Box (y+z) \;-\; (x \,\Box\, y) \;-\; (x \,\Box\, z)$$

such that $x \in G$, $y, z \in H$, and $y+z$ is defined. The maps $G \,\Box\, H \xrightarrow{\eta} F \longrightarrow G \otimes H$ are morphisms of multisets over S.

The abelian multigroup $G \otimes H$ comes with a canonical *tensor map*

$$\tau\colon G \times H \xrightarrow{\beta} G \,\Box\, H \xrightarrow{\eta} F \longrightarrow G \otimes H\,,\;\; (x,y) \longmapsto x \otimes y\,.$$

The universal property of $G \otimes H$ and τ is as follows. Recall that $G \times H$ is a set over S with projection $(x,y)^\vee = x^\vee\, y^\vee$ to S. A morphism f of sets over S from $G \times H$ to an abelian multigroup K over S is *balanced and biadditive* if and only if

$$\begin{aligned} f(x^t, y) \;&=\; f(x, y^t) \;\;\text{for all}\;\; x, y, t, \\ f(x+y,\, z) \;&=\; f(x,z) + f(y,z) \;\;\text{whenever}\;\; x+y \;\;\text{is defined}\;\;\text{and} \\ f(x,\, y+z) \;&=\; f(x,y) + f(x,z) \;\;\text{whenever}\;\; y+z \;\;\text{is defined}\,; \end{aligned}$$

equivalently, the defining relations of $G \otimes H$ hold in K via f. The tensor map τ is balanced and biadditive by definition of $G \otimes H$:

$$\begin{aligned} x^t \otimes y \;&=\; x \otimes y^t \;\;\text{for all}\;\; x, y, t, \\ (x+y) \otimes z \;&=\; (x \otimes z) + (y \otimes z) \;\;\text{whenever}\;\; x+y \;\;\text{is defined}\;\;\text{and} \\ x \otimes (y+z) \;&=\; (x \otimes z) + (y \otimes z) \;\;\text{whenever}\;\; y+z \;\;\text{is defined}\,. \end{aligned}$$

In fact,

$$x^t \otimes y \;=\; x \otimes y^t \;=\; (x \otimes y)^t \;\;\text{for all}\;\; x, y, t\,,$$

since $x^t \,\Box\, y = x \,\Box\, y^t = (x \,\Box\, y)^t$ in $G \,\Box\, H$ and $G \,\Box\, H \longrightarrow G \otimes H$ is a morphism of multisets over S. Conversely, every balanced and biadditive morphism f of sets over S from $G \times H$ to an abelian multigroup K over S factors uniquely through τ (there exists a unique morphism $\varphi\colon G \otimes H \longrightarrow K$ of abelian multigroups over S such that $\varphi \circ \tau = f$).

For example, if R is a commutative multiring over S, then the multiplication on R is a balanced and biadditive mapping of $R \times R$ into R and induces (in fact, can be replaced by) a morphism $R \otimes R \longrightarrow R$ of abelian multigroups over S. Similarly, if M is a multi-R-module over S, then the action of R on M induces (and can be replaced by) a morphism $R \otimes M \longrightarrow M$ of abelian multigroups over S.

For every abelian multigroup G over S there is a natural isomorphism

$$\mathbb{Z}^{+} \otimes G \;\cong\; G\,,$$

where $\mathbb{Z}^{+} = \mathbb{Z} \times S$ with projection $(n,a)^{\vee} = a$, addition $(m,a) + (n,a) = (m + n\,a)$, and action $(n,a)^{t} = (n,at)$ of S. Indeed $((n,a),\,x) \longmapsto nx^{a}$ is a balanced and biadditive mapping of $\mathbb{Z}^{+} \times G$ into G, and induces a morphism of abelian multigroups over S from $\mathbb{Z}^{+} \otimes G$ to G that sends $(n,a) \otimes x$ to nx^{a} (namely, the action of $\mathbb{Z}^{+}$ on G as a multi-$\mathbb{Z}^{+}$-module). The inverse isomorphism sends $x \in X$ to $(0,1) \otimes x \in \mathbb{Z}^{+} \otimes G$.

Since $F = \mathbb{Z} \dashv (G \,\square\, H)$ is the free abelian multigroup over S on the multiset $G \,\square\, H$, every $p \in F$ is uniquely a homogeneous linear combination with integer coefficients

$$p \;= \sum \bigl(p_y\, y \bigm| y = x \,\square\, y,\ x^{\vee}\, y^{\vee} = p^{\vee}\bigr)\,.$$

Hence every element of $G \otimes H$ is a homogeneous linear combination with integer coefficients

$$p \;= \sum \bigl(p_{x,y}\,(x \otimes y) \bigm| x \in G,\ y \in H,\ x^{\vee}\, y^{\vee} = p^{\vee}\bigr)\,.$$

Since $\otimes$ is left additive we also have $n\,(x \otimes y) = nx \otimes y$ for all $n \in \mathbb{Z}$, with $(nx)^{\vee} = x^{\vee}$. Therefore every element z of $G \otimes H$ is a homogeneous sum

$$z \;=\; (x_1 \otimes y_1) + \cdots + (x_n \otimes y_n)\,,$$

where $n \geq 0$ and $x_i^{\vee}\, y_i^{\vee} = a = z^{\vee}$ for all i. Then

$$z^{t} \;=\; (x_1^{t} \otimes y_1) + \cdots + (x_n^{t} \otimes y_n) \;=\; (x_1 \otimes y_1^{t}) + \cdots + (x_n \otimes y_n^{t})$$

for all $t \in S^{1}$.

If $\varphi\colon G \longrightarrow H$ and $\psi\colon G' \longrightarrow H'$ are morphisms of multisets over S, then $(x,y) \longmapsto \varphi x \otimes \psi y$ is balanced and biadditive and there is a unique morphism $\varphi \otimes \psi\colon G \boxslash H \longrightarrow G' \otimes H'$ of abelian multigroups over S such that

$$(\varphi \otimes \psi)(x \otimes y) \;=\; \varphi x \otimes \psi y \quad \text{for all} \quad x \in G \quad \text{and} \quad y \in H\,.$$

This makes the mixed product a bifunctor.

2. Associativity. For any abelian multigroups G, H, K and multiset X, there are natural isomorphisms

$$(G \otimes H) \boxslash X \;\cong\; G \otimes (H \boxslash X) \quad \text{and} \quad (G \otimes H) \otimes K \;\cong\; G \otimes (H \otimes K)\,.$$

This is proved like the previous associativity properties. There are also natural isomorphisms $H \otimes G \cong G \otimes H$. Thus abelian multigroups over S constitute a symmetric monoidal category (as defined in [9]); the identity object is $\mathbb{Z}^{+}$.

3. Multirings. The tensor product of commutative multirings over S is a commutative multiring over S. Indeed, any two commutative multirings R and

T have a tensor product $R \otimes T$ as abelian multigroups over S, which is also an abelian multigroup over S. Let

$$\mu \colon R \otimes R \longrightarrow R \quad \text{and} \quad \nu \colon T \otimes T \longrightarrow T$$

be the morphisms of abelian multigroups over S induced by the multiplications on R and T, $\mu\,(r \otimes s) = rs$ and $\nu\,(t \otimes u) = tu$. Then $\mu \otimes \nu$ induces a morphism of abelian multigroups over S

$$(R \otimes T) \otimes (R \otimes T) \overset{\cong}{\longrightarrow} (R \otimes R) \otimes (T \otimes T) \overset{\mu \otimes \nu}{\longrightarrow} R \otimes T$$

and a multiplication on $R \otimes T$, under which

$$(r \otimes t)(s \otimes u) \;=\; rs \otimes tu$$

for all $r, s \in R$ and $t, u \in T$. If $v \in S$, then

$$\big((r \otimes t)(s \otimes u)\big)^v \;=\; (r \otimes t)^v\,(s \otimes u) \;=\; (r \otimes t)\,(s \otimes u)^v\,,$$

which implies $(pq)^v = p^v\,q = p\,q^v$ for all $p, q \in R \otimes T$. Hence $R \otimes T$ is a commutative multiring over S.

4. Multimodules. Now let S be a commutative monoid and let R be a commutative multiring over S.

If M is a multi-R-module over S and G is an abelian multigroup over S, then $M \otimes G$ is a multi-R-module over S; the action α of R on M induces an action of R on $M \otimes G$:

$$R \otimes (M \otimes G) \;\overset{\cong}{\longrightarrow}\; (R \otimes M) \otimes G \;\overset{\alpha \otimes 1}{\longrightarrow}\; M \otimes G$$

such that $r\,(x \otimes y) = rx \otimes y$ for all r, x, y.

In particular, $R \otimes G$ is a multi-R-module over S. In fact it is the *free multi-R-module over S on* the abelian multigroup G. The canonical map $\eta \colon G \longrightarrow R \otimes G$ sends $x \in G$ to $1 \otimes x$. If f is a morphism of abelian multigroups over S from G to a multi-R-module M over S, then $(r, x) \longmapsto r\,(fx)$ is a balanced and bi-additive mapping of $R \times G$ into M and induces a morphism $\varphi \colon R \otimes G \longrightarrow M$ of abelian multigroups over S such that $\varphi\,(r \otimes x) = r\,(fx)$ for all r, x. Then $\varphi\,\big(r\,(s \otimes x)\big) = rs\,f(x) = r\,\varphi\,(r \otimes x)$ for all r, s, x, so that φ is a morphism of multi-R-modules over S, and $\varphi \circ \eta = f$. Conversely, if $\psi \colon R \otimes G \longrightarrow M$ is a morphism of multi-R-modules over S, and $\psi \circ \eta = f$, then $\psi\,(r \otimes x) = \psi\,\big(r\,(1 \otimes x)\big) = r\,(fx)$ for all r, x, and $\psi = \varphi$.

For two multi-R-modules M and N over S, however, one needs a tensor product of M and N such that $rx \otimes y = x \otimes ry$ for all r, x, y. This is the *tensor product $M \otimes_R N$ over R*: the abelian multigroup over S generated by $M \,\square\, N$ subject to all defining relations

$$\begin{aligned}
(x + y) \,\square\, z \;&=\; (x \,\square\, z) + (y \,\square\, z) \quad \text{whenever} \quad x + y \quad \text{is defined}\,, \\
x \,\square (y + z) \;&=\; (x \,\square\, y) + (x \,\square\, z) \quad \text{whenever} \quad y + z \quad \text{is defined}\,, \quad \text{and} \\
rx \,\square\, y \;&=\; x \,\square\, ry\,,
\end{aligned}$$

which comes with a canonical *tensor map*

$$\tau\colon M \times N \xrightarrow{\beta} M \,\Box\, N \xrightarrow{\eta} M \otimes_R N\,,\ (x,y) \longmapsto x \otimes y\,.$$

Every element z of $M \otimes_R N$ is a homogeneous sum

$$z \;=\; (x_1 \otimes y_1) + \cdots + (x_n \otimes y_n)\,,$$

where $n \geq 0$, $x_1, \ldots, x_n \in M$, $y_1, \ldots, y_n \in N$, and $x_i^\vee\, y_i^\vee = z^\vee$ for all i.

If $R = \mathbb{Z}^+$, then multi-R-modules over S are just abelian multigroups over S and $\otimes_R$ is the tensor product $\otimes$ of abelian multigroups over S.

A mapping f of $M \times N$ into an abelian multigroup G over S is *fully balanced and biadditive* if it is balanced and biadditive and

$$f(rx, y) \;=\; f(x, ry)$$

for all r, x, y. The tensor map τ to $M \otimes_R N$ is fully balanced and biadditive. Conversely, if $f\colon M \times N \longrightarrow G$ is fully balanced and biadditive, then f induces a unique morphism $\varphi\colon M \otimes_R N \longrightarrow G$ of abelian multigroups over S such that $\varphi \circ \tau = f$ (such that $f(x,y) = \varphi\,(x \otimes y)$ for all x, y).

In particular, for every $r \in R$, the mapping $(x,y) \longmapsto rx \otimes y$ of $M \times N$ into $M \otimes_R N$ is fully balanced and biadditive and induces an endomorphism of abelian multigroups over S that sends $x \otimes y$ to $rx \otimes y$. Hence R acts on $M \otimes_R N$ so that

$$r\,(x \otimes y) \;=\; rx \otimes y \;\;(=\; x \otimes ry)$$

for all r, x, y, whereby $M \otimes_R N$ becomes a multi-R-module over S.

There are natural isomorphisms $R \otimes_R M \cong M$, $M \otimes_R N \cong N \otimes_R M$, and $(M \otimes_R N) \otimes_R Q \cong M \otimes_R (N \otimes_R Q)$. Thus, multi-$R$-modules over S constitute a symmetric monoidal category.

5. Free multimodules. By the above, the *free multi-R-module F over S on* a set X over S is the free multi-R-module over S on the free abelian multigroup $\mathbb{Z} \dashv X^+$ over S on X, that is $F = R \otimes (\mathbb{Z} \dashv X^+)$. The canonical mapping $\eta\colon X \longrightarrow F$ sends $x \in X$ to $1 \otimes (x, 1)$.

The typical element z of F is a finite sum

$$z \;=\; (r_1 \otimes p_1) + \cdots + (r_n \otimes p_n)\,,$$

where $n \geq 0$, $r_1, \ldots, r_n \in R$, $r_i^\vee\, p_i^\vee = z^\vee$ for all i, and $p_1, \ldots, p_n \in \mathbb{Z} \dashv X^+$, so that each p_i is a homogeneous linear combination

$$\begin{aligned} p \;&= \sum \left(p_y\, y \;\middle|\; y \in X^+,\ y^\vee = p^\vee \right) \\ &= \sum \left(p_{x,c}\,(x,c) \;\middle|\; (x,c) \in X^+,\ (x,c)^\vee = p^\vee \right), \end{aligned}$$

where $a = y^\vee$, of elements of $(X^+)_a$ with integer coefficients. Now

$$r \otimes p \;= \sum \left(p_{x,c}\,(r \otimes (x,c)) \;\middle|\; (x,c) \in X^+,\ (x,c)^\vee = p^\vee \right).$$

Since

$$n\,(r \otimes (x,c)) \;=\; n\,(r^c \otimes (x,1)) \;=\; nr^c \otimes (x,1)$$

for all $n \in \mathbb{Z}$, it follows that every element of F is a homogeneous sum

$$z \;=\; (r_1 \otimes x_1) + \cdots + (r_n \otimes x_n) \;=\; r_1\,(\eta\,x_1) + \cdots + r_n\,(\eta\,x_n)\,,$$

where $n \geq 0$, $r_1, \ldots, r_n \in R$, $x_1, \ldots, x_n \in X$, $r_i^{\vee}\, x_i^{\vee} = z^{\vee}$ for all i, each $x \in X$ has been identified with $(x,1) \in X^{+}$, and $\eta x = 1 \otimes X$. Unfortunately there is no evidence that z can be written in this form uniquely.

Multi-R-modules can now have presentations as generated by the elements of a set over S subject to defining relations. A *defining relation* between the elements of a set X over S is a pair (A,B) of elements of the free multi-R-module F on X, normally written as an equality $A = B$, which is *homogeneous*, meaning that A and B project to the same element $A^{\vee} = B^{\vee}$ of S.

A defining relation (A,B) between the elements of Y *holds* in a multi-R-module M *via* a morphism f of Y into M if and only if $\varphi\,A = \varphi\,B$, where $\varphi\colon F \longrightarrow M$ is the morphism that extends f.

The multi-R-module *generated by* a set X over S *subject to* a set $\mathcal{R}$ of defining relations is the quotient

$$\langle\, X \mid \mathcal{R} \,\rangle \;\cong\; F/K\,,$$

where F is the free multi-R-module on X and K is the submultimodule of F generated by all $A-B$ with $(A,B) \in \mathcal{R}$ (where $A-B$ is defined since $(A,B) \in \mathcal{R}$ is homogeneous).

The multi-R-module $\langle\, X \mid \mathcal{R} \,\rangle$ comes with a canonical map η from X, which is the composition

$$\eta\colon X \xrightarrow{\;\eta\;} F \longrightarrow F/K\,.$$

By definition, every $(A,B) \in \mathcal{R}$ holds in $\langle\, X \mid \mathcal{R} \,\rangle$ via η. The universal property of $\langle\, X \mid \mathcal{R} \,\rangle$ and η is: if f is a morphism of sets over S from X to a multi-R-module M and every $(A,B) \in \mathcal{R}$ holds in M via f, then there is a unique multi-R-module morphism φ from $\langle\, X \mid \mathcal{R} \,\rangle$ to M such that $\varphi \circ \eta = f$.

13 Derivations

In this section, S is a commutative monoid and R is a commutative multiring over S (with an identity element).

1. Derivations. If A is a commutative multi-R-algebra over S and M is a multi-A-module over S, then the action of A on M induces an action of R on M, namely

$$r\,x = (r1)\,x\,, \quad \text{for all } r \in R \text{ and } x \in M,$$

which makes M a multi-R-module over S. Similarly, A-modules are, in particular, R-modules.

A *derivation* of a commutative multi-R-algebra A over S into a multi-A-module M over S is a mapping $D\colon A \longrightarrow M$ such that

$$\begin{aligned} &D(x)^{\vee} = x^{\vee} \text{ for all } x\,, \\ &D(x^t) = D(x)^t \text{ for all } x, t\,, \\ &D(rx) = r\,D(x) \text{ for all } r, x\,, \\ &D(x+y) = D(x) + D(y) \text{ whenever } x+y \text{ is defined, and} \\ &D(xy) \;=\; x\,D(y) + y\,D(x) \text{ for all } x, y \in A. \end{aligned}$$

In particular, D is a morphism of multi-R-modules over S. Also, $D(1) = D(11) = D(1) + D(1)$, so that $D(1) = 0$. Hence $D(r1) = r\,D(1) = 0$ and $D(1^t) = D(1)^t = 0$, for all $r \in R$ and $t \in S$.

The set $\mathrm{Der}\,(A, M)$ of all derivations of A into M is an A-module under pointwise addition and action of A ($(yD)(x) = y\,D(x)$ for all $x, y \in A$). In particular, $\mathrm{Der}\,(A, M)$ is an R-module.

If $\varphi\colon M \longrightarrow N$ is a morphism of multi-A-modules over S and $D\colon A \longrightarrow M$ is a derivation, then $\varphi \circ D\colon A \longrightarrow N$ is a derivation. This yields an A-module homomorphism (in particular, an R-module homomorphism)

$$\mathrm{Der}\,(A, \varphi)\colon \mathrm{Der}\,(A, M) \longrightarrow \mathrm{Der}\,(A, N)\,, \quad D \longmapsto \varphi \circ D\,.$$

If $\psi\colon A \longrightarrow B$ is a morphism of commutative multi-R-algebras over S and $D\colon B \longrightarrow M$ is a derivation, then $D \circ \psi\colon A \longrightarrow N$ is a derivation. This yields an A-module homomorphism

$$\mathrm{Der}\,(\psi, M)\colon \mathrm{Der}\,(B, M) \longrightarrow \mathrm{Der}\,(A, M)\,, \quad D \longmapsto D \circ \psi\,.$$

With these maps, $\mathrm{Der}\,(-,-)$ is now a bifunctor to A-modules.

2. The universal derivation. Let Y be a set over S with an isomorphism $d\colon A \longrightarrow Y$ of sets over S, and let Ω be the multi-A-module over S generated by Y subject to

$$\begin{aligned} d\,(r+s) \;&=\; dr + ds \text{ whenever } r+s \text{ is defined in } A\text{, and} \\ d\,(rs) \;&=\; r\,ds + s\,dr \text{ for all } r, s \in A\,. \end{aligned}$$

These defining relations are homogeneous. The elements of Ω are the *Kähler differentials* of A.

The multi-A-module Ω comes with a derivation $d\colon A \longrightarrow \Omega$. The universal property of Ω yields a universal property of d: if $D\colon A \longrightarrow M$ is any derivation, then the elements $D(x)$ of M satisfy the defining relations of Ω and there is a unique multi-A-module morphism $\varphi\colon \Omega \longrightarrow M$ that sends dx to $D(x)$ for every $x \in A$; equivalently, $D = \varphi \circ d$. Hence $D \longmapsto \varphi$ is a natural isomorphism

$$\mathrm{Der}\,(A, M) \cong \mathrm{Hom}_A\,(\Omega, M)\,.$$

In particular, $\mathrm{Der}\,(A, -)$ is a left exact functor.

3. Redefinition. Next, we want for each set X over S a commutative multi-R-algebra A over S such that, for every multi-A-module M over S, there is a natural isomorphism of A-modules

$$\mathrm{Der}\,(A, M) \;\cong\; \mathrm{Map}\,(X, M)$$

which is natural in X and M. This is where things go wrong with commutative multi-R-algebras. The free commutative multi-R-algebra over S on X in Section 11 has a canonical map η which shows no inclination to be injective, nor do its elements show any inclination to be put in polynomial form uniquely.

The free commutative multiring over S on X in Section 6 suffers none of these defects. From here on therefore we deal with commutative multirings over S rather than commutative multi-R-algebras over S. This amounts to setting $R = \mathbb{Z}^+$ throughout.

Accordingly, let R be a commutative multiring over S. A *derivation* of R into a multi-R-module M over S is a mapping $D\colon R \longrightarrow M$ such that

$$\begin{aligned}
&D(x)^\vee = x^\vee \quad \text{for all} \quad x\,,\\
&D(x^t) = D(x)^t \quad \text{for all} \quad x, t\,,\\
&D(x+y) = D(x) + D(y) \quad \text{whenever} \quad x+y \quad \text{is defined,} \quad \text{and}\\
&D(xy) \;=\; x\,D(y) + y\,D(x) \quad \text{for all} \quad x, y \in A.
\end{aligned}$$

In particular, D is a morphism of abelian multigroups over S. As before, $D(1) = D(11) = D(1) + D(1)$, so that $D(1) = 0$, $D(r1) = r\,D(1) = 0$, and $D(1^t) = D(1)^t = 0$, for all $r \in R$ and $t \in S$.

The set $\mathrm{Der}\,(R, M)$ of all derivations of R into M is an R-module under pointwise addition and action of R ($(rD)(x) = r\,D(x)$ for all $x, y \in R$).

If $\varphi\colon M \longrightarrow N$ is a morphism of multi-R-modules over S and $D\colon A \longrightarrow M$ is a derivation, then $\varphi \circ D\colon R \longrightarrow N$ is a derivation. This yields an R-module homomorphism

$$\mathrm{Der}\,(R, \varphi)\colon \mathrm{Der}\,(R, M) \longrightarrow \mathrm{Der}\,(R, N)\,, \quad D \longmapsto \varphi \circ D\,.$$

If $\psi\colon Q \longrightarrow R$ is a morphism of commutative multirings over S and $D\colon R \longrightarrow M$ is a derivation, then M is a multi-Q-module, on which Q acts by $qx = (\psi q)\,x$, and $D \circ \psi\colon Q \longrightarrow M$ is a derivation. This yields an R-module homomorphism

$$\psi^* = \mathrm{Der}\,(\psi, M)\colon \mathrm{Der}\,(R, M) \longrightarrow \mathrm{Der}\,(Q, M)\,, \quad D \longmapsto D \circ \psi\,.$$

With these maps, $\mathrm{Der}\,(-,-)$ becomes a bifunctor to R-modules. Moreover, $\mathrm{Der}\,(R,-)$ is left exact.

4. Polynomials. In Section 6 we saw that the free commutative multiring over S on a set X over S is $\mathbb{Z}[X] = \mathbb{Z} \dashv \mathbb{F}(X)^+$, where $\mathbb{F}(X)$ is the free commutative monoid on the set X and $\mathbb{F}(X)^+ = \mathbb{F} \times S$. An element y of $\mathbb{F}(X)^+$ is an ordered pair

$$y \;=\; (\mathrm{X}^m,\, t) \;=\; \big(\prod_{x \in X} (X_x^{m(x)},\, 1)\big)^t\,,$$

in which $t \in S$ and $\mathrm{X}^m = \prod_{x \in X} X_x^{m(x)} \in \mathbb{F}(X)$ is a monomial on an isomorphic copy $\overline{X} = \{ X_x \mid x \in X \}$ of X. Every element p of $\mathbb{Z}[X]$ is a unique homogeneous linear combination of elements of $\mathbb{F}(X)^+$ with integer coefficients

$$p \;=\; \sum \big(p_y \, y \bigm| y \in \mathbb{F}(X)^+,\; y^\vee = p^\vee \big) \,.$$

For every set X over S and multi-$\mathbb{Z}[X]$-module M over S there is a natural isomorphism

$$\Theta \colon \mathrm{Der}\,(\mathbb{Z}[X],\, M) \;\cong\; \mathrm{Map}\,(X, M)$$

that sends a derivation$D$ of $\mathbb{Z}[X]$ into M to

$$d \;=\; \Theta(D) \;=\; D \circ \eta \colon X \xrightarrow{\;\eta\;} \mathbb{Z}[X] \xrightarrow{\;D\;} M \,,$$

so that

$$dx \;=\; D(X_x, 1) \,.$$

Then d is a morphism of sets over S, since D is a derivation. Moreover, Θ preserves pointwise sums and every action of $r \in R$ on M, and is a homomorphism of R-modules.

We show that Θ is injective. For every $D \in \mathrm{Der}\,(\mathbb{Z}[X],\, M)$, induction yields

$$\begin{aligned} &D\,(X_1 X_2 \cdots X_n,\, 1) \;=\; D\big((Y_1,\, 1)(Y_2,\, 1) \cdots (Y_n,\, 1)\big) \\ &\quad = \textstyle\sum_i \big((Y_1,\, 1) \cdots (Y_{i-1},\, 1)\; (Y_{i+1},\, 1) \cdots (Y_n,\, 1)\; D\,(Y_i,\, 1)\big) \end{aligned}$$

for all $n \geq 0$ and $Y_1, Y_2, \ldots, Y_n \in \overline{X}$ (with $Y_1 Y_2 \cdots Y_n = 1$ if $n = 0$ so that $D\,(Y_1 Y_2 \cdots Y_n,\, 1) = 0$ since $(1,1)$ is the identity element of $\mathbb{Z}[X]$). If $d = \Theta\,(D) = 0$, then $D\,(Y_1 Y_2 \cdots Y_n,\, 1) = 0$ for all $n \geq 0$ and $Y_1, Y_2, \ldots, Y_n \in \overline{X}$,

$$D\,(\mathrm{X}^m,\, t) \;=\; D\big((\mathrm{X}^m,\, 1)^t\big) \;=\; \big(D\,(\mathrm{X}^m,\, 1)\big)^t \;=\; 0$$

for all $y = (\mathrm{X}^m,\, t) \in \mathbb{F}(X)^+$, and $D(p) = 0$ for all $p \in \mathbb{Z}[X]$.

Conversely, every morphism $d \colon x \longmapsto dx$ of sets over S of X into M extends, more or less willingly, to a derivation $\widehat{d}$ of $\mathbb{Z}[X]$ into M, so that Θ is surjective.

For every $n > 0$ and $x \in X$, define

$$\widehat{d}\,(Y^n) \;=\; n\, Y^{n-1}\, dx \,,$$

where $Y = (X_x,\, 1)$, so that $(X_x^n,\, 1) = Y^n$, and $\widehat{d}\,(Y^n) = 0_1$ if $n = 0$. In particular, $\widehat{d}\,(Y) = dx$.

If $n > 0$, then

$$\widehat{d}\,(Y^n)^\vee \;=\; (Y^{n-1})^\vee\, (dx)^\vee \;=\; (Y^\vee)^n \;=\; (Y^n)^\vee \,,$$

since $(dx)^\vee = x^\vee = Y^\vee$. The equality $(\widehat{d}\,(Y^n))^\vee = (Y^n)^\vee$ also holds if $n = 0$, since $(1,1)^\vee = 1 = 0_1^\vee$.

In addition,

$$\begin{aligned}\widehat{d}\,(Y^m\,Y^n) \;&=\; (m+n)\,Y^{m+n-1}\,dx \\ &=\; n\,Y^m\,Y^{n-1}\,dx \;+\; m\,Y^n\,Y^{m-1}\,dx \\ &=\; Y^m\,\widehat{d}\,(Y^n) \;+\; Y^n\,\widehat{d}\,(Y^m)\,,\end{aligned}$$

for all $m, n > 0$. The equality

$$\widehat{d}\,(Y^m\,Y^n) \;=\; Y^m\,\widehat{d}\,(Y^n) \;+\; Y^n\,\widehat{d}\,(Y^m)$$

also holds if $m = 0$ or $n = 0$, since $(1, 1)$ is the identity element of $\mathbb{Z}[X]$.

For each $p = (\mathrm{X}^m,\, 1) = \prod_{x\in X}(X_x^{m(x)},\, 1)$ define

$$\widehat{d}\,(p) \;= \sum \big((\textstyle\prod_{x\in X,\ x\neq y} Y_x)\,\widehat{d}\,(Y_y) \;\big|\; y \in X,\ m(y) > 0\big)\,,$$

where $Y_x = (X_x^{m(x)},\, 1)$, so that $(\mathrm{X}^m,\, 1) = \prod_{x\in X} Y_x$; this sum is homogeneous since $(\widehat{d}\,Y_x)^\vee = Y_x^\vee = x^\vee$. Then

$$\widehat{d}\,(p)^\vee \;=\; \big(\textstyle\prod_{x\in X,\ x\neq y}(x^\vee)^{m(x)}\big)\,(y^\vee)^{m(y)} \;=\; \textstyle\prod_{x\in X}(x^\vee)^{m(x)} \;=\; p^\vee\,.$$

If $p = (\mathrm{X}^m,\, 1) = \prod_{x\in X} Y_x$ and $q = (\mathrm{X}^n,\, 1) = \prod_{x\in X} Z_x$, where $Z_x = (X_x^{n(x)},\, 1)$, then

$$\widehat{d}\,(p) \;= \sum \big(\textstyle\prod_{x\in X,\ x\neq y} Y_x)\,\widehat{d}\,(Y_y) \;\big|\; y \in X,\ m(y)+n(y) > 0\big)\,,$$

since $m(y) = 0$ implies $\prod_{x\in X,\ x\neq y} Y_x)\,\widehat{d}\,(Y_y) = 0$; similarly,

$$\widehat{d}\,(q) \;= \sum \big(\textstyle\prod_{x\in X,\ x\neq y} Z_x)\,\widehat{d}\,(Z_y) \;\big|\; y \in X,\ m(y)+n(y) > 0\big)\,.$$

Hence

$$\begin{aligned}\widehat{d}\,(pq) \;&=\; \widehat{d}\,(\mathrm{X}^{m+n},\, 1) \\ &=\; \sum \big((\textstyle\prod_{x\in X,\ x\neq y} Y_x\,Z_x)\,\widehat{d}\,(Y_y\,Z_y) \;\big|\; y \in X,\ m(y)+n(y) > 0\big) \\ &=\; \sum \big((\textstyle\prod_{x\in X,\ x\neq y} Y_x\,Z_x)\,Y_y\,\widehat{d}\,(Z_y) \;\big|\; y \in X,\ m(y)+n(y) > 0\big) \\ &\quad+\; \sum \big((\textstyle\prod_{x\in X,\ x\neq y} Y_x\,Z_x)\,Z_y\,\widehat{d}\,(Y_y) \;\big|\; y \in X,\ m(y)+n(y) > 0\big) \\ &=\; \big(\textstyle\prod_{x\in X} Y_x\big) \sum \big((\textstyle\prod_{x\in X,\ x\neq y} Z_x)\,\widehat{d}\,(Z_y) \;\big|\; y \in X,\ m(y)+n(y) > 0\big) \\ &\quad+\; \big(\textstyle\prod_{x\in X} Z_x\big) \sum \big((\textstyle\prod_{x\in X,\ x\neq y} Y_x)\,\widehat{d}\,(Y_y) \;\big|\; y \in X,\ m(y)+n(y) > 0\big) \\ &=\; p\,\widehat{d}\,(q) \;+\; q\,\widehat{d}\,(p)\,.\end{aligned}$$

For each $r = (\mathrm{X}^m,\, a) \in \mathbb{F}(X)^+$, $r = p^a$ where $p = (\mathrm{X}^m,\, 1)$, define

$$\widehat{d}\,(r) \;=\; \widehat{d}\,(p)^a\,.$$

Then
$$\widehat{d}(r)^{\vee} \;=\; (\widehat{d}(p))^{\vee}\, a \;=\; p^{\vee}\, a \;=\; r^{\vee}\,.$$
If $t \in S$, then $r^t = (\mathrm{x}^m,\, at) = p^{at}$ and
$$\widehat{d}(r^t) \;=\; \widehat{d}(p^{at}) \;=\; \widehat{d}(p)^{at} \;=\; (\widehat{d}(p)^a)^t \;=\; \widehat{d}(r)^t\,.$$
If also $s \in \mathbb{F}(X)^{+}$, $s = q^b$ where $q = (\mathrm{x}^n,\, 1)$, then $rs = (\mathrm{x}^{m+n},\, ab) = (pq)^{ab}$ and
$$\begin{aligned}\widehat{d}(rs) \;&=\; \widehat{d}(pq)^{ab} \;=\; \big(p\,\widehat{d}(q) \;+\; q\,\widehat{d}(p)\big)^{ab}\\ &=\; p^a\,\widehat{d}(q)^b \;+\; q^b\,\widehat{d}(p)^a \;=\; r\,\widehat{d}(s) \;+\; s\,\widehat{d}(r)\,.\end{aligned}$$

Finally, if
$$p \;=\sum \big(\, p_r\, r \;\big|\; r \in \mathbb{F}(X)^{+},\; r^{\vee} = p^{\vee}\big)\,,$$
then define
$$\widehat{d}(p) \;=\sum \big(\, p_r\, \widehat{d}(r) \;\big|\; r \in \mathbb{F}(X)^{+},\; r^{\vee} = p^{\vee}\big)\,.$$
Then $\widehat{d}(p)^{\vee} = p^{\vee}$. If $u \in S$, then
$$p^u \;=\sum \big(\, p_r\, r^u \;\big|\; r \in \mathbb{F}(X)^{+},\; r^{\vee} = p^{\vee}\big)$$
and
$$\begin{aligned}\widehat{d}(p^u) \;&=\sum \big(\, p_r\, \widehat{d}(r^u) \;\big|\; r \in \mathbb{F}(X)^{+},\; r^{\vee} = p^{\vee}\big)\\ &=\sum \big(\, p_r\, \widehat{d}(r)^u \;\big|\; r \in \mathbb{F}(X)^{+},\; r^{\vee} = p^{\vee}\big)\\ &=\; \widehat{d}(p)^u\,.\end{aligned}$$
If $p + q$ is defined (if $p^{\vee} = q^{\vee}$), then
$$\widehat{d}(p+q) \;=\; \widehat{d}(p) \;+\; \widehat{d}(q)\,.$$
Finally, if also
$$q \;=\sum \big(\, q_s\, s \;\big|\; s \in \mathbb{F}(X)^{+},\; s^{\vee} = q^{\vee}\big)\,,$$
then
$$pq \;=\sum \big(p_r\, q_s\, rs \;\big|\; r \in \mathbb{F}(X)^{+},\; s \in \mathbb{F}(X)^{+},\; r^{\vee} = p^{\vee},\; s^{\vee} = q^{\vee}\big)$$

and

$$
\begin{aligned}
\widehat{d}\,(pq) \; &= \sum \big(p_r\, q_s\, \widehat{d}\,(rs) \;\big|\; r \in \mathbb{F}(X)^{+},\; s \in \mathbb{F}(X)^{+},\; r^{\vee} = p^{\vee},\; s^{\vee} = q^{\vee}\big) \\
&= \sum \big(p_r\, q_s\, r\, \widehat{d}\,(s) + s\, \widehat{d}\,(r) \;\big| \\
&\qquad\qquad r \in \mathbb{F}(X)^{+},\; s \in \mathbb{F}(X)^{+},\; r^{\vee} = p^{\vee},\; s^{\vee} = q^{\vee}\big) \\
&= \; \big[\sum \big(\, p_r\, r \;\big|\; r \in \mathbb{F}(X)^{+},\; r^{\vee} = p^{\vee}\big)\big] \\
&\qquad\quad \big[\sum \big(\, q_s\, \widehat{d}\,(s) \;\big|\; s \in \mathbb{F}(X)^{+},\; s^{\vee} = q^{\vee}\big)\big] \\
&\quad\; + \; \big[\sum \big(\, q_s\, s \;\big|\; s \in \mathbb{F}(X)^{+},\; s^{\vee} = q^{\vee}\big)\big] \\
&\qquad\quad \big[\sum \big(\, p_r\, \widehat{d}\,(r) \;\big|\; r \in \mathbb{F}(X)^{+},\; r^{\vee} = p^{\vee}\big)\big] \\
&= \; p\, \widehat{d}\,(q) \; + \; q\, \widehat{d}\,(r)\,.
\end{aligned}
$$

Thus $\widehat{d}$ is a derivation.

14 Cohomology of multirings

In this section, S is a commutative monoid and R is a commutative multiring over S.

1. The comonad. The adjunction of sets over S to commutative multirings over S begets a comonad (V, ϵ, ν) in which V sends a commutative multiring R over S to the free commutative multiring $\mathbb{Z}[R]$ over S on the set R over S.

As before we replace every element r of R with an indeterminate X_r, so that $\overline{X} = \{\, X_r \mid r \in R\,\}$ is isomorphic to R as a set over S. Then the typical element of $\mathbb{F}(R)$ is a monomial

$$
\mathrm{X}^m \;=\; \textstyle\prod_{r \in R} X_r^{m(r)}
$$

and every element of $\mathbb{Z}[R]$ is uniquely a homogeneous linear combination with integer coefficients

$$
\begin{aligned}
p \; &= \sum \big(\, p_z\, z \;\big|\; z \in \mathbb{F}(R)^{+},\; z^{\vee} = p^{\vee}\big) \\
&= \sum \big(\, p_{m,t}\, (\mathrm{X}^m,\, t) \;\big|\; \mathrm{X}^m \in \mathbb{F}(R),\; t \in S,\; (\mathrm{X}^m)^{\vee}\, t = p^{\vee}\big)\,.
\end{aligned}
$$

Every morphism $\varphi\colon R \longrightarrow T$ of sets over S extends uniquely to a morphism $\psi = \mathbb{Z}[\varphi]$ of commutative multirings over S from $\mathbb{Z}[R]$ to $\mathbb{Z}[T]$ that sends

$$
p \; = \sum \big(\, p_{m,t}\, (\mathrm{X}^m,\, t) \;\big|\; \mathrm{X}^m \in \mathbb{F}(R),\; t \in S,\; (\mathrm{X}^m)^{\vee}\, t = p^{\vee}\big)
$$

to

$$
\mathbb{Z}[\varphi]\,(p) \; = \sum \big(\, p_{m,t}\, (\mathbb{Z}[\varphi]\,(\mathrm{X}^m),\, t) \;\big|\; \mathrm{X}^m \in \mathbb{F}(R),\; t \in S,\; (\mathrm{X}^m)^{\vee}\, t = p^{\vee}\big)\,,
$$

where $\mathbb{Z}[T]$ is written with one indeterminate Y_s for each $s \in T$ and

$$
\mathbb{Z}[\varphi]\,(\mathrm{X}^m) \;=\; \mathbb{Z}[\varphi]\,\big(\textstyle\prod_{r \in R} X_r^{m(r)}\big) \;=\; \textstyle\prod_{r \in R} Y_{\varphi(r)}^{m(r)}\,,
$$

as calculated in $\mathbb{Z}[T]$.

If now $\varphi\colon R \longrightarrow T$ is a morphism of commutative multirings over S, and we write VT with one indeterminate Y_s for each $s \in T$, then $V\varphi = \mathbb{Z}[\varphi]$ is the morphism of commutative multirings over S from VR to VT such that $(V\varphi)(X_r) = Y_{\varphi r}$ for all $r \in R$.

In particular, $\epsilon_R\colon VR \longrightarrow R$ is the evaluation morphism of commutative multirings over S that sends X_r to r and sends

$$p \;=\sum \Big(p_{m,t}\, \big(\textstyle\prod_{r\in R} X_r^{m(r)},\, t\big) \;\Big|\; \textstyle\prod_{r\in R} X_r^{m(r)} \in \mathbb{F}(R),\ t \in S,\ \big(\prod_{r\in R} X_r^{m(r)}\big)^{\vee} t = p^{\vee} \Big)$$

to

$$\epsilon\,(p) \;=\sum \Big(p_{m,t}\, \big(\textstyle\prod_{r\in R} r^{m(r)}\big)^{t} \;\Big|\; \textstyle\prod_{r\in R} X_r^{m(r)} \in \mathbb{F}(R),\ t \in S,\ \big(\prod_{r\in R} X_r^{m(r)}\big)^{\vee} t = p^{\vee} \Big)\,,$$

as calculated in R.

The comultiplication ν assigns to R the morphism $\nu_R = V\eta\colon VR \longrightarrow VVR$ induced by the adjunction unit $\eta\colon R \longrightarrow VR$, viewed as a morphism of sets over S.

We note that every multi-R-module M over S is also a multi-VR-module. The action of R on M extends to an action of VR on M in which

$$px \;=\; \epsilon(p)\, x$$

for every $p \in VR$ and $x \in M$. This makes M a multi-VR-module since ϵ is a morphism of commutative multirings over S.

2. The resolution. For every commutative multiring R over S there is now an augmented simplicial commutative multi-R-algebra R^* with objects $R^0 = R$ and $R^n = V^n R$ (where $n \geq 0$ and $V^n = V \circ V \circ \cdots \circ V$), face maps

$$\epsilon_{n,i} \;=\; V^i\, \epsilon\, V^{n-i}\, R\colon R^{n+1} \longrightarrow R^n \quad (i = 0, 1, \ldots, n),$$

augmentation $\epsilon\,(R) = \epsilon_{0,0}\colon VR \longrightarrow R$, and degeneracy maps

$$\nu_{n,i} \;=\; V^i\, \nu\, V^{n-i-1} R\colon R^{n-1} \longrightarrow R^n \quad (i = 0, 1, \ldots, n-1),$$

that satisfy the simplicial identities.

Let M be a multi-R-module over S, hence also a multi-VR-module over S and a multi-R^n-module over S for every $n \geq 0$. Applying to R^* the contravariant functor $\mathrm{Der}\,(-, M)$ yields an augmented cosimplicial R-module $\mathrm{Der}\,(R^*, M)$ with objects $\mathrm{Der}\,(R^0, M) = \mathrm{Der}\,(R, M)$ and $\mathrm{Der}\,(R^n, M) = \mathrm{Der}\,(V^n R, M)$ if $n > 0$, face maps

$$d_{n,i} \;=\; \epsilon^*_{n,i} \;=\; \mathrm{Der}\,(\epsilon_{n,i}, M)\colon \mathrm{Der}\,(R^n, M) \longrightarrow \mathrm{Der}\,(R^{n+1}, M)$$

for $i = 0, 1, \ldots, n$, augmentation

$$d_{0,0} \;=\; \epsilon^* \;=\; \mathrm{Der}\,(\epsilon, M)\colon \mathrm{Der}\,(R, M) \longrightarrow \mathrm{Der}\,(VR, M)\,,$$

and degeneracy maps $\mathrm{Der}\,(\nu_{n,i},\, M)$ $(i = 0, 1, \ldots, n-1)$, that satisfy the cosimplicial identities.

A coboundary homomorphism

$$\delta_n\colon \mathrm{Der}\,(R^n,\, M) \longrightarrow \mathrm{Der}\,(R^{n+1},\, M)$$

is then defined by

$$\delta_n \;=\; d_{n,0} - d_{n,1} + d_{n,2} - \cdots + (-1)^n\, d_{n,n}\,.$$

In particular, $\delta_0 = d_{0,0}$. The simplicial identities imply $\delta_{n+1} \circ \delta_n = 0$ for all $n \geq 0$. This yields an augmented cochain complex of R-modules

$$\begin{aligned} 0 \longrightarrow & \mathrm{Der}\,(R, M) \xrightarrow{\epsilon^*} \mathrm{Der}\,(VR,\, M) \xrightarrow{\delta_1} \ldots \\ & \longrightarrow \mathrm{Der}\,(V^n R,\, M) \xrightarrow{\delta_n} \mathrm{Der}\,(V^{n+1} R,\, M) \ldots \end{aligned}$$

The *André-Quillen cohomology* of the commutative multiring R over S with coefficients in the multi-R-module M over S assigns to R and M the R-modules $H^0(R, M) = \mathrm{Ker}\,\delta_0 \,/\, \mathrm{Im}\, e$ and $H^n(R, M) = \mathrm{Ker}\,\delta_n \,/\, \mathrm{Im}\,\delta_{n-1}$, where $n > 0$.

The natural isomorphisms $\mathrm{Der}\,(R[X], M) \cong \mathrm{Map}\,(X, M)$ yield an isomorphic complex of R-modules

$$\begin{aligned} 0 \longrightarrow & \mathrm{Der}\,(R, M) \xrightarrow{e} \mathrm{Map}\,(R, M) \xrightarrow{d_1} \ldots \\ & \longrightarrow \mathrm{Map}\,(V^{n-1} R,\, M) \xrightarrow{d_n} \mathrm{Map}\,(V^n R,\, M) \ldots \end{aligned}$$

whose homology modules are natural isomorphic to the André-Quillen cohomology modules of R with coefficients in the multi-R-module M. In particular, $H^1(R, M) \cong \mathrm{Ker}\, d_1 \,/\, \mathrm{Im}\, e$. Without the augmentation, $H^1(R, M)$ would be simply $\mathrm{Ker}\, d_1$.

3. Three maps. The augmentation $e\colon \mathrm{Der}\,(R, M) \longrightarrow \mathrm{Map}\,(R, M)$ is the composite

$$e\colon \mathrm{Der}\,(R, M) \xrightarrow{\epsilon^*} \mathrm{Der}\,(VR, M) \xrightarrow{\cong} \mathrm{Map}\,(R, M)$$

of $\epsilon^* = \mathrm{Der}\,(\epsilon, M)$, which sends $D \in \mathrm{Der}\,(R, M)$ to $D \circ \epsilon$, and the isomorphism $\mathrm{Der}\,(VR, M) \longrightarrow \mathrm{Map}\,(R, M)$, which sends $D \circ \epsilon$ to the mapping $r \longmapsto (D \circ \epsilon)(X_r) = D(r)$. Thus $e\,(D)$ is D viewed as simply a mapping of R into M.

Next, $d_{1,0}\colon \mathrm{Map}\,(R, M) \longrightarrow \mathrm{Map}\,(VR, M)$ is the composite

$$d_{1,0}\colon \mathrm{Map}\,(R, M) \xrightarrow{\cong} \mathrm{Der}\,(VR, M) \xrightarrow{\epsilon_{1,0}^*} \mathrm{Der}\,(V^2 R, M) \xrightarrow{\cong} \mathrm{Map}\,(VR, M)$$

and sends a map f from R to M to the derivation $\widehat{f}$ of VR such that $\widehat{f}(X_a) = fa$ for all $a \in S$, thence to the derivation $\epsilon_{1,0}^*\,(\widehat{f}) = \widehat{f} \circ \epsilon_{1,0} = \widehat{f} \circ \epsilon_{VR}$, thence to the corresponding map from VR to M.

Let $V^2R = \mathbb{Z}[Y]$ with one indeterminate Y_p for every $p \in VR$. Then $\widehat{f} \circ \epsilon_{VR}$ sends $Y_p \in V^2R$ to $\widehat{f}(\epsilon\, Y_p) = \widehat{f}(p)$. Hence $d_{1,0}\, f$ is the map from VR to M that sends $p \in VR$ to $\widehat{f}(p)$; in other words,

$$d_{1,0}\, f \;=\; \widehat{f}\,.$$

Similarly, $d_{1,1}\colon \mathrm{Map}\,(R, M) \longrightarrow \mathrm{Map}\,(VR, M)$ is the composite

$$d_{1,1}\colon \mathrm{Map}\,(R, M) \xrightarrow{\cong} \mathrm{Der}\,(VR, M) \xrightarrow{\epsilon_{1,1}^*} \mathrm{Der}\,(V^2R, M) \xrightarrow{\cong} \mathrm{Map}\,(VR, M)$$

and sends a map f from R to M to the derivation $\widehat{f}$ such that $\widehat{f}(X_a) = fa$ for all $a \in S$, thence to the derivation $\epsilon_{1,1}^*\,(\widehat{f}) = \widehat{f} \circ \epsilon_{1,1} = \widehat{f} \circ V\epsilon_R$, thence to the corresponding map from VR to M.

With $V^2R = \mathbb{Z}[Y]$ as above, $V\epsilon_R\colon V^2R \longrightarrow VR$ is the morphism of commutative multirings over S that sends Y_p to $X_{\epsilon p}$ for every $p \in VR$. Hence $\widehat{f} \circ V\epsilon_R$ sends Y_p to $\widehat{f}(X_{\epsilon p})$ for every $p \in VR$. Therefore $d_{1,1}\, f$ is the map from VR to M that sends $p \in VR$ to $f\,(\epsilon p)$; equivalently,

$$d_{1,1}\, f \;=\; f \circ \epsilon_R\,.$$

15 Cohomology of $\mathbb{Z}^+$

In this section, S is a commutative monoid and $R = \mathbb{Z}^+$, so that every abelian multigroup G over S is a multi-R-module over S. The cohomology groups of $\mathbb{Z}^+$ are the homology groups of the complex

$$\begin{aligned} 0 \longrightarrow &\mathrm{Der}\,(\mathbb{Z}^+, G) \xrightarrow{\;e\;} \mathrm{Map}\,(\mathbb{Z}^+, G) \xrightarrow{d_1} \mathrm{Map}\,(V\mathbb{Z}^+, G) \cdots \\ &\longrightarrow \mathrm{Map}\,(V^{n-1}\mathbb{Z}^+,\, G) \xrightarrow{d_n} \mathrm{Map}\,(V^n\mathbb{Z}^+,\, G) \cdots \end{aligned}$$

First, $\mathrm{Der}\,(\mathbb{Z}^+, G) = 0$. Indeed, if $D \in \mathrm{Der}\,(\mathbb{Z}^+, G)$, then $D(1,1) = 0$, since $(1,1)$ is the identity element of $\mathbb{Z}^+$, and

$$D(n,a) \;=\; D\big(n\,(1,1)^a\big) \;=\; n\,D(1,1)^a \;=\; 0$$

for every $(n,a) \in \mathbb{Z}^+$. This does not help crime scene investigations.

A 1-cochain $u \in \mathrm{Map}\,(\mathbb{Z}^+, G)$ assigns $u(n,a) \in G_a$ to each $(n,a) \in \mathbb{Z}^+$. From Section 14 we know that $d_{1,0}u = \widehat{u}$ is the derivation of $V\mathbb{Z}^+ = \mathbb{Z}[\mathbb{Z}^+]$ that extends u, constructed in Section 13, whereas $d_{1,1}u = u \circ \epsilon$.

With every $(n,a) \in \mathbb{Z}^+$ replaced by an indeterminate $X_{n,a}$, the typical element p of $\mathbb{Z}[\mathbb{Z}^+]$ is a homogeneous linear combination

$$p \;= \sum \big(\, p_{m,t}\,(\mathrm{X}^m,\, t) \;\big|\; \mathrm{X}^m \in \mathbb{F}(\mathbb{Z}^+),\ t \in S,\ (\mathrm{X}^m)^\vee\, t = p^\vee \big)\,,$$

where $\mathrm{X}^m = \prod_{(n,a)\in\mathbb{Z}^+} X_{n,a}^{m(n,a)}$, and $(\mathrm{X}^m)^\vee = \prod_{(n,a)\in\mathbb{Z}^+} a^{m(n,a)}$. Then $\widehat{u}$ is the derivation of $\mathbb{Z}[\mathbb{Z}^+]$ such that

$$\widehat{u}\,(X_{n,a},\, 1) \;=\; u(n,a)$$

for all $(n,a) \in \mathbb{Z}^+$. On the other hand, ϵ is the morphism of commutative multirings over S such that

$$\epsilon\,(X_{n,a},\,1) \;=\; (n,a)$$

for all $(n,a) \in \mathbb{Z}^+$. Every multi-$\mathbb{Z}^+$-module G is a multi-$\mathbb{Z}[\mathbb{Z}^+]$-module, in which

$$p\,x \;=\; (\epsilon p)\,x$$

for every $p \in \mathbb{Z}[\mathbb{Z}^+]$ and $x \in G$; in particular,

$$(X_{n,a},\,1)\,x \;=\; (n,a)\,x \;=\; n\,x^a$$

for every $(n,a) \in \mathbb{Z}^+$.

We show that $Z^1_Q(\mathbb{Z}^+,G) = 0$. (So $H^1_Q(\mathbb{Z}^+,G) = 0$; maybe $H_Q\,(\mathbb{Z}^+,G) = 0$?) By the above, u is a 1-cocycle if and only if $(d_{1,0} - d_{1,1})\,u = d_1 u = 0$, if and only if $u \circ \epsilon = \widehat{u}$.

If u is a 1-cocycle, then $u \circ \epsilon$ is a derivation. Hence

$$\begin{aligned} u\,(1,1) \;&=\; u\big((1,1)(1,1)\big) \;=\; u\,(\epsilon\,(X_{1,1},\,1)\;\epsilon\,(X_{1,1},\,1) \\ &=\; u\,\big(\,\epsilon\,((X_{1,1},\,1)\,(X_{1,1}),\,1)\big) \;=\; \widehat{u}\,((X_{1,1},\,1)\,(X_{1,1},\,1)) \\ &=\; (X_{1,1},\,1)\,(\widehat{u}\,(X_{1,1},\,1)) \;+\; (X_{1,1},\,1)\,(\widehat{u}\,(X_{1,1},\,1)) \\ &=\; (X_{1,1},\,1)\,u\,(1,1) \;+\; (X_{1,1},\,1)\,u\,(1,1) \;=\; u\,(1,1) \;+\; u\,(1,1)\,, \end{aligned}$$

and $u\,(1,1) = 0$. Hence

$$\begin{aligned} u\,(1,a) \;&=\; u\,((1,1)^a) \;=\; u\,\big((\epsilon\,(X_{1,1},\,1))^a\big) \;=\; u\,\big(\,\epsilon\,((X_{1,1},\,1)^a)\big) \\ &=\; \widehat{u}\,((X_{1,1},\,1)^a) \;=\; (\widehat{u}\,(X_{1,1},\,1))^a \\ &=\; u\,(1,1)^a \;=\; 0 \end{aligned}$$

for all $a \in S$. Finally,

$$\begin{aligned} u\,(m+n,a) \;&=\; u\,(\epsilon\,(X_{m,a},\,1) + \epsilon\,(X_{n,a},\,1)) \\ &=\; u\,\big(\,\epsilon\,((X_{m,a},\,1) + (X_{n,a},\,1))\big) \\ &=\; \widehat{u}\,((X_{m,a},\,1) + (X_{n,a},\,1)) \\ &=\; \widehat{u}\,(X_{m,a},\,1) \;+\; \widehat{u}\,(X_{n,a},\,1) \;=\; u\,(m,a) + u\,(n,a) \end{aligned}$$

for all $m,n \in \mathbb{Z}$ and $a \in S$. Hence

$$u\,(n,a) \;=\; n\,u\,(1,a) \;=\; 0$$

for all n,a.

16 Cohomology of $\mathbb{Z}[S]$

In this section, S is a commutative monoid and $R = \mathbb{Z}[S]$, where S is the set over itself with projection $a^\vee = a$. With every $a \in S$ replaced by an indeterminate X_a, the typical element of $\mathbb{Z}[S]$ is a homogeneous linear combination

$$p \;= \sum \big(\,p_{m,t}\,(\mathrm{X}^m,\,t)\;\big|\;\mathrm{X}^m \in \mathbb{F}(S),\;t \in S,\;(\mathrm{X}^m)^\vee\,t = p^\vee\big)\,,$$

where $\mathrm{X}^m = \prod_{a\in S} X_a^{m(a)}$ and $(\mathrm{X}^m)^\vee = \prod_{a\in S} a^{m(a)}$.

1. Coefficients. Every abelian multigroup over S can serve as coefficients in the cohomology of $\mathbb{Z}[S]$. Indeed every abelian multigroup G over S is canonically a multi-$\mathbb{Z}[S]$-module over S, in which

$$p \cdot x \;=\; x^{(p^\vee)}$$

for every $x \in G$ and $p = (\mathrm{X}^m,\, t) \in \mathbb{F}(S)^+$. In particular, $(X_a,\, 1) \cdot x = x^a$ and $(1,1) \cdot x = x$. Then

$$\begin{aligned}
&(p \cdot x)^\vee \;=\; (x^{(p^\vee)})^\vee \;=\; p^\vee\, x^\vee\,,\\
&(p \cdot x)^t \;=\; (x^{(p^\vee)})^t \;=\; x^{p^\vee t} \;=\; p \cdot x^t\,,\\
&p^t \cdot x \;=\; x^{(p^t)^\vee} \;=\; x^{p^\vee t} \;=\; p \cdot x^t\,,\\
&p \cdot (x+y) \;=\; (x+y)^{(p^\vee)} \;=\; x^{(p^\vee)} + y^{(p^\vee)} \;=\; (p \cdot x) + (p \cdot y)\\
&\qquad \text{whenever}\;\; x+y \;\;\text{is defined}\,,\;\; \text{and}\\
&p \cdot (q \cdot x) \;=\; (x^{(q^\vee)})^{(p^\vee)} \;=\; x^{(pq)^\vee} \;=\; pq \cdot x\,,
\end{aligned}$$

for all $p, q \in \mathbb{F}(S)^+$ and $x, y \in G$.

If now $p \in \mathbb{Z}[S]$ is a homogeneous linear combination

$$p \;= \sum \big(p_y\, y \bigm| y \in \mathbb{F}(S)^+,\; y^\vee = p^\vee\big),$$

then $p \cdot x$ is well defined by

$$p \cdot x \;= \sum \big(p_y\, (y \cdot x) \bigm| y \in \mathbb{F}(S)^+,\; y^\vee = p^\vee\big),$$

since this is a homogeneous sum. Then $(p+q) \cdot x = (p \cdot x) + (q \cdot x)$ whenever $p^\vee = q^\vee$, and the five properties above also hold for all $p, q \in \mathbb{Z}[S]$ and $x, y \in S$.

2. Cochains. Since $\mathrm{Der}\,(\mathbb{Z}[S], G) \cong \mathrm{Map}\,(S, G)$, the André-Quillen cohomology groups of S are the homology groups of the complex

$$\begin{aligned}
0 \longrightarrow &\mathrm{Map}\,(S,G) \xrightarrow{\;e\;} \mathrm{Map}\,(\mathbb{Z}[S],G) \xrightarrow{\;d_1\;} \mathrm{Map}\,(V\mathbb{Z}[S],G) \longrightarrow \;\dots\\
&\longrightarrow \mathrm{Map}\,(V^{n-1}\mathbb{Z}[S],\, G) \xrightarrow{\;d_n\;} \mathrm{Map}\,(V^n\mathbb{Z}[S],\, G) \longrightarrow \;\dots
\end{aligned}$$

Thus a 0-cochain $u \in \mathrm{Map}\,(S,G)$ assigns $u(a) \in G_a$ to each $a \in S$ (thus, $\mathrm{Map}\,(S,G) = C^1_C(S,G)$); a 1-cochain $u \in \mathrm{Map}\,(\mathbb{Z}[S],G)$ assigns $u(p) \in G_{p^\vee}$ to each $p \in \mathbb{Z}[S]$; and so forth.

From Section 14 we know that

$$e\colon \mathrm{Map}\,(S,G) \xrightarrow{\;\cong\;} \mathrm{Der}\,(\mathbb{Z}[S],G) \xrightarrow{\;e\;} \mathrm{Map}\,(\mathbb{Z}[S],G)$$

sends a 0-cochain $u \in \mathrm{Map}\,(S,G)$ to the derivation $\widehat{u}$ of $\mathbb{Z}[S]$ that extends u, then to $\widehat{u}$ viewed as a morphism of sets over S from $\mathbb{Z}[S]$ to G. Hence a 1-coboundary is a derivation: $B^2_Q(S,G) = \mathrm{Der}\,(\mathbb{Z}[S],G)$.

Also, for each 1-cochain $u \in \mathrm{Map}\,(\mathbb{Z}[S],\, G)$, $d_{1,0}\, u = \widehat{u}$, whereas $d_{1,1}\, u = u \circ \epsilon_{\mathbb{Z}[S]}$. Hence

$$\delta u \;=\; \widehat{u} \;-\; (u \circ \epsilon)\,.$$

We write $V\mathbb{Z}[S] = \mathbb{Z}[\mathbb{Z}[S]]$ with one indeterminate Y_p for each $p \in \mathbb{Z}[S]$, so that the typical element of $V\mathbb{Z}[S]$ is

$$q \;=\; \sum \big(\, q_{m,t}\, (\mathrm{Y}^m,\, t) \;\big|\; \mathrm{Y}^m \in \mathbb{F}(\mathbb{Z}[S]),\; t \in S,\; (\mathrm{Y}^m)^\vee\, t = q^\vee \big)\,,$$

where $\mathrm{Y}^m = \prod_{p \in \mathbb{Z}[S]} Y_p^{m(p)}$ and $(\mathrm{Y}^m)^\vee = \prod_{p \in \mathbb{Z}[S]} (p^\vee)^{m(p)}$. Then

$$\epsilon q \;=\; \sum \big(\, q_{m,t}\, (\textstyle\prod_{p \in \mathbb{Z}[S]} p^{m(p)})^t \;\big|\; \mathrm{Y}^m \in \mathbb{F}(\mathbb{Z}[S]),\; t \in S,\; (\mathrm{Y}^m)^\vee\, t = q^\vee \big)\,,$$

in particular

$$\epsilon\, (Y_p,\, 1) \;=\; p\,.$$

The multi-$\mathbb{Z}[S]$-module G is also a multi-$V\mathbb{Z}[S]$-module, on which

$$q \cdot x \;=\; (\epsilon q) \cdot x \;=\; x^{(q^\vee)}$$

for all $q \in V\mathbb{Z}[S]$ and $x \in G$.

3. 1-cocycles. We prove that $Z^1_Q(S, G) \cong C^1_C(S, G)$.

First, let $u \in qZ^2(S, G)$ be a 1-cocycle, so that

$$u \circ \epsilon \;=\; \widehat{u}\,.$$

Recall that

$$\widehat{u}\, (Y_p,\, 1) \;=\; u\,(p)$$

for all $p \in \mathbb{Z}[S]$.

Since ϵ and $\widehat{u}$ preserve existing sums and the action of S, then so does u. In detail, for all $p, q \in \mathbb{Z}[S]$ and $t \in S$ we have

$$\begin{aligned} u\,(p^t) \;&=\; u\,\big((\epsilon\,(Y_p,\, 1)^t\big) \;=\; u\,\epsilon\,\big((Y_p,\, 1)^t\big) \\ &=\; \widehat{u}\,\big((Y_p,\, 1)^t\big) \;=\; \big(\widehat{u}\,(Y_p,\, 1)\big)^t \\ &= (u\,(p))^t \end{aligned}$$

and, if $p^\vee = q^\vee$,

$$\begin{aligned} u\,(p+q) \;&=\; u\,\big(\,\epsilon\,(Y_p,\, 1) + \epsilon\,(Y_q,\, 1)\big) \;=\; u\,\epsilon\,\big(\,(Y_p,\, 1) + (Y_q,\, 1)\big) \\ &=\; \widehat{u}\,\big((Y_p,\, 1) + (Y_q,\, 1)\big) \;=\; \widehat{u}\,(Y_p,\, 1) \;+\; \widehat{u}\,(Y_q,\, 1) \\ &=\; u\,(p) \;+\; u\,(q)\,. \end{aligned}$$

Hence, if

$$p \;=\; \sum \big(\, p_{m,t}\, (\mathrm{X}^m,\, t) \;\big|\; \mathrm{X}^m \in \mathbb{F}(S),\; t \in S,\; (\mathrm{X}^m)^\vee\, t = p^\vee \big)\,,$$

then

$$u\,(p) \;=\sum \big(\, p_{m,t}\, u\,(\mathrm{X}^m,\, t) \;\big|\; \mathrm{X}^m \in \mathbb{F}(S),\ t \in S,\ (\mathrm{X}^m)^\vee\, t = p^\vee \big)\,.$$

For all $p, q \in \mathbb{Z}[S]$ we also have

$$\begin{aligned} u\,(pq) \;&=\; u\,\big(\,\epsilon\,(Y_p,\,1)\;\epsilon\,(Y_q,\,1)\big) \;=\; u\,\epsilon\,\big(\,\epsilon\,(Y_p,\,1)\,(Y_q,\,1)\big) \\ &=\; \widehat{u}\,\big((Y_p,\,1)\,(Y_q,\,1)\big) \;=\; (Y_p,\,1)\,\widehat{u}\,(Y_q,\,1) \;+\; (Y_q,\,1)\,\widehat{u}\,(Y_p,\,1) \\ &=\; (Y_q,\,1)\,u\,(p) \;+\; (Y_p,\,1)\,u\,(q) \;=\; u\,(p)^{(q^\vee)} \;+\; u\,(q)^{(p^\vee)}\,. \end{aligned}$$

Since $(1,1)^\vee = 1 \in S$ this implies $u\,(1,1) = u\,\big((1,1)(1,1)\big) = u\,(1,1) + u\,(1,1)$ and

$$u\,(1,1) \;=\; 0\,.$$

Hence

$$u\,(1,t) \;=\; u\,((1,1)^t) \;=\; (u\,(1,1))^t \;=\; 0$$

for all $t \in S$. Moreover, for every $p_1,\, p_2, \ldots,\, p_n \in \mathbb{Z}[S]$, induction on $n > 0$ yields

$$\begin{aligned} u\,(p_1p_2\cdots p_n) \;&=\textstyle\sum_{i=1,\ldots,n} u\,(p_i)^{c_i}\,, \quad \text{where} \\ c_i \;&=\; (p_1\cdots p_{i-1}\,p_{i+1}\cdots p_n)^\vee\,. \end{aligned}$$

In particular,

$$u\,(X_a^n,\,1) \;=\; n\,u\,(X_a,\,1)^{(a^{n-1})}\,,$$

and

$$u\,(\mathrm{X}^m,\,1) \;=\; u\,\big(\textstyle\prod_{a\in S,\ m(a)>0} X_a^{m(a)},\,1\big) \;=\textstyle\sum_{a\in S,\ m(a)>0} m(a)\,u\,(X_a,\,1)^{c_a}\,,$$

where

$$c_a \;=\; \big(a^{m(a)-1}\textstyle\prod_{b\in S,\ b\neq a}(X_b^{m(b)},\,1)\big)^\vee \;=\; a^{m(a)-1}\textstyle\prod_{b\in S,\ b\neq a} b^{m(b)}\,.$$

Then $u\,(\mathrm{X}^m,\,t) \;=\; \big(u\,(\mathrm{X}^m)\big)^t$.

Thus u is completely determined by the 1-cochain $\overline{u}\colon a \longmapsto u\,(X_a,\,1)$. This yields an injective homomorphism $\Theta\colon u \longmapsto \overline{u}$ of $Z_Q^2(S,G)$ into $C_C^1(S,G)$.

We show that Θ is surjective. Let $v \in C_C^1(S,G)$. For each $\mathrm{X}^m \in \mathbb{F}(S)$ let

$$\begin{aligned} u\,(\mathrm{X}^m,\,1) \;&=\textstyle\sum_{a\in S,\ m(a)>0} m(a)\,v\,(a)^{c_a}\,, \quad \text{where} \\ c_a \;&=\; a^{m(a)-1}\textstyle\prod_{b\in S,\ b\neq a} b^{m(b)}\,. \end{aligned}$$

Then

$$\begin{aligned} u\,(\mathrm{X}^n,\,1) \;&=\textstyle\sum_{a\in S,\ n(a)>0} n(a)\,v\,(a)^{d_a}\,, \quad \text{where} \\ d_a \;&=\; a^{n(a)-1}\textstyle\prod_{b\in S,\ b\neq a} b^{n(b)}\,, \quad \text{and} \\ u\,(\mathrm{X}^{m+n},\,1) \;&=\textstyle\sum_{a\in S,\ m(a)+n(a)>0} (m(a)+n(a))\,v\,(a)^{e_a}\,, \quad \text{where} \\ e_a \;&=\; a^{m(a)+n(a)-1}\textstyle\prod_{b\in S,\ b\neq a} b^{m(b)+n(b)} \\ &=\; c(a)\textstyle\prod_{b\in S} b^{n(b)} \;=\; d(a)\textstyle\prod_{b\in S} b^{m(b)} \\ &=\; c(a)\,(\mathrm{X}^n)^\vee \;=\; d(a)\,(\mathrm{X}^m)^\vee\,. \end{aligned}$$

and

$$\begin{aligned} u\big((\mathrm{X}^m,\,1)(\mathrm{X}^n,\,1)\big) \;&=\; u\,(\mathrm{X}^{m+n},\,1) \\ =\; &\big(\textstyle\sum_{a\in S,\ m(a)>0} m(a)\,v\,(a)^{c_a}\big)^{((\mathrm{X}^n)^\vee)} \\ &+\;\big(\textstyle\sum_{a\in S,\ n(a)>0} n(a)\,v\,(a)^{d_a}\big)^{((\mathrm{X}^m)^\vee)} \\ =&\; u\,(\mathrm{X}^m,\,1)^{((\mathrm{X}^n,\,1)^\vee)} \;+\; u\,(\mathrm{X}^n,\,1)^{((\mathrm{X}^m,\,1)^\vee)}\,. \end{aligned}$$

Thus, if $p = (\mathrm{X}^m,\,1)$ and $q = (\mathrm{X}^n,\,1)$, then

$$u\,(pq) \;=\; (u\,(p))^{(q^\vee)} \;+\; (u\,(q))^{(p^\vee)}\,.$$

Next, define

$$u\,(\mathrm{X}^m,\,t) \;=\; \big(u\,(\mathrm{X}^m,\,1)\big)^t \;=\; \textstyle\sum_{a\in S,\ m(a)>0} m(a)\,v\,(a)^{c_a\,t}$$

for all $\mathrm{X}^m \in \mathbb{F}(S)$ and $t \in S$, where $c_a = a^{m(a)-1}\prod_{b\in S,\ b\neq a} b^{m(b)}$ as above.

If $y = (\mathrm{X}^m,\,s) \in \mathbb{F}(S)^+$ and $t \in S$, then

$$\begin{aligned} u\,(y^t) \;&=\; u\,(\mathrm{X}^m,\,st) \;=\; u\,(\mathrm{X}^m,\,1)^{st} \\ &=\; \big(u\,(\mathrm{X}^m,\,1)^s\big)^t \;=\; \big(u\,(y)\big)^t\,. \end{aligned}$$

If $y = (\mathrm{X}^m,\,s)$ and $z = (\mathrm{X}^n,\,t) \in \mathbb{F}(S)^+$, then

$$\begin{aligned} u\,(yz) \;&=\; u\,\big(u\,(\mathrm{X}^{m+n},\,1)\big)^{st} \\ &=\; u\,(\mathrm{X}^m,\,1)^{(s\,(\mathrm{X}^n,\,1)^\vee\,t)} \;+\; u\,(\mathrm{X}^n,\,1)^{(t\,(\mathrm{X}^m,\,1)^\vee\,s)} \\ &=\; u\,(\mathrm{X}^m,\,s)^{((\mathrm{X}^n,\,t)^\vee)} \;+\; u\,(\mathrm{X}^n,\,t)^{(\mathrm{X}^m,\,s)^\vee)} \\ &=\; u\,(y)^{(z^\vee)} \;+\; (u\,(z))^{(t^\vee)}\,. \end{aligned}$$

Finally, for each $p = \sum\big(p_y\,y \bigm| y \in \mathbb{F}(S)^+,\ y^\vee = p^\vee\big) \in \mathbb{Z}[S]$, define

$$u\,(p) \;= \sum\big(p_y\,u\,(y) \bigm| y \in \mathbb{F}(S)^+,\ y^\vee = p^\vee\big)\,.$$

Then $(u\,(p))^\vee = p^\vee$, so that $u \in \mathrm{Map}\,(\mathbb{Z}[S], G)$. Also,

$$u\,(p+q) \;=\; u\,(p) \;+\; u\,(q)$$

for all $p, q \in \mathbb{Z}[S]$ such that $p^\vee = q^\vee$;

$$\begin{aligned} u\,(p^t) \;&= \sum\big(p_y\,u\,(y^t) \bigm| y \in \mathbb{F}(S)^+,\ y^\vee = p^\vee\big) \\ &= \sum\big(p_y\,(u\,(y))^t \bigm| y \in \mathbb{F}(S)^+,\ y^\vee = p^\vee\big) \;=\; (u\,(p))^t \end{aligned}$$

for all $p \in \mathbb{Z}[S]$ and $t \in S$; and

$$\begin{aligned} u\,(pq) \;&= \sum\big(p_y\,q\,(z)\,u\,(yz) \bigm| y, z \in \mathbb{F}(S)^+,\ y^\vee = p^\vee,\ z^\vee = q^\vee\big) \\ &= \sum\big(p_y\,q\,(z)\,u\,(y)^{(z^\vee)} \;+\; (u\,(z))^{(y^\vee)} \bigm| \\ &\qquad y, z \in \mathbb{F}(S)^+,\ y^\vee = p^\vee,\ z^\vee = q^\vee\big) \\ &=\; \big[\sum\big(p_y\,u\,(y) \bigm| y \in \mathbb{F}(S)^+,\ y^\vee = p^\vee\big)\big]^{(q^\vee)} \\ &\quad+\; \big[\sum\big(q_z\,u\,(z) \bigm| z \in \mathbb{F}(S)^+,\ z^\vee = q^\vee\big)\big]^{(p^\vee)} \\ &= u\,(p)^{(q^\vee)} \;+\; (u\,(q))^{(p^\vee)} \end{aligned}$$

for all $p, q \in \mathbb{Z}[S]$.

These properties imply that $u \circ \epsilon$ is a derivation of $V\mathbb{Z}[S]$. Indeed

$$(u\,\epsilon\,(p))^{\vee} \;=\; (\epsilon\,(p))^{\vee} \;=\; p^{\vee}$$

for every $p \in V\mathbb{Z}[S]$;

$$u\,\epsilon\,(p^t)) \;=\; u\,(((\epsilon\,p)^t) \;=\; (u\,\epsilon\,p)^t$$

for all $p \in V\mathbb{Z}[S]$ and $t \in S$;

$$u\,\epsilon\,(p+q) \;=\; u\,(\epsilon\,p + \epsilon q) \;=\; u\,\epsilon\,p \;+\; u\,\epsilon\,q$$

for all $p, q \in V\mathbb{Z}[S]$ such that $p^{\vee} = q^{\vee}$; and

$$\begin{aligned} u\,\epsilon\,(pq) \;&=\; u\,\big((\epsilon\,p)(\epsilon q)\big) \\ &=\; u\,(\epsilon\,p)^{((\epsilon\,q)^{\vee})} \;+\; (u\,(\epsilon\,q))^{((\epsilon\,p)^{\vee})} \;=\; u\,(\epsilon\,p)^{(q^{\vee})} \;+\; (u\,(\epsilon\,q))^{(p^{\vee})} \end{aligned}$$

for all $p, q \in V\mathbb{Z}[S]$.

Since $u\,\epsilon\,(Y_p,\,1) = u\,(p) = \widehat{u}\,(Y_p,\,1)$ for all $p \in \mathbb{Z}[S]$, it follows that $u \circ \epsilon = \widehat{u}$; and u is a 1-cocycle. Moreover, $\overline{u} = v$ by definition. Thus $\Theta\colon Z^1_Q(S,G) \longrightarrow C^1_C(S,G)$ is surjective; and $Z^1_Q(S,G) \cong C^1_C(S,G)$.

Notation:

$x^\vee$: the projection of x to S

$\mathbb{F}(X)$: free commutative monoid on X

$\mathbb{H}(X)$: half free commutative monoid on X

X^+: $X \times S$

$\mathrm{Map}\,(X, Y)$: the set of all morphisms of sets over S from X to Y

$A \dashv B$: set of all homogeneous linear combinations of elements of B with coefficients in A

$A \,\square\, B$: box product of A and B

$A \boxslash B$: mixed product of A and B

$A \otimes B$: tensor product of A and B

C_Q, Z_Q, B_Q, H_Q: André-Quillen cohomology

C_C, Z_C, B_C, H_C: symmetric (commutative semigroup) cohomology

Adjunctions of commutative objects over S:

sets to multisets: X^+

sets to multimonoids: $\mathbb{F}(X)^+$

sets to multigroups: $\mathbb{Z} \dashv X^+$

sets to multirings: $\mathbb{Z}[X] = \mathbb{Z} \dashv \mathbb{F}(X)^+$

sets to multi-R-modules: $R \otimes (\mathbb{Z} \dashv X^+)$

sets to multi-R-algebras: $R \boxslash \mathbb{F}(X)^+$

multisets to multimonoids: $\mathbb{H}(X)$

multisets to multigroups: $\mathbb{Z} \dashv X$

multisets to multirings: $\mathbb{Z} \dashv \mathbb{H}(X)$

multisets to multi-R-modules: $R \otimes (\mathbb{Z} \dashv X)$

multisets to multi-R-algebras: $R \boxslash \mathbb{H}(X)^+$

multimonoids to multirings: $\mathbb{Z} \dashv M$

multimonoids to multi-R-algebras: $R \boxslash M$

multigroups to multi-R-modules: $R \otimes G$

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