Addition to Multigroups over a commutative semigroup

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Errata to subsection 6.6.

6. Free multirings on multisets. The *free commutative multiring over S on* a multiset X over S is the free commutative multiring $\mathbb{Z}[X] = \mathbb{Z} \dashv \mathbb{H}(X)$ over S on the free commutative multimonoid $\mathbb{H}(X)$ over S on X.

As in Section 4, $\mathbb{H}(X)$ is the half-free commutative monoid

$$\mathbb{H}(X) \ = \ \mathbb{F}(X)/\approx$$

on X, where $\approx$ is the smallest congruence on $\mathbb{F}(X)$ such that $X_x^t\, X_y \approx X_x\, X_y^t$ for all $x, y \in X$ and $t \in S$. The typical element Y of $\mathbb{H}(X)$ is the equivalence class $Y = [X_1 X_2 \cdots X_n]$ of a commutative product $X_1 X_2 \cdots X_n$ of elements of $\overline{X} = \{\, X_x \mid x \in X \,\}$; equivalently, $Y = [\mathrm{X}^m]$ for some $\mathrm{X}^m \in \mathbb{F}(X)$.

Accordingly, the typical element of $\mathbb{Z} \dashv \mathbb{H}(X)$ is a homogeneous linear combination

$$p \ = \sum \big(\, p_Y\, Y \;\big|\; Y \in \mathbb{H}(X),\ Y^\vee = p^\vee \,\big)\,.$$

If X only has one element, or if $x^t = x$ for all x, t, then $\mathbb{H}(X) = \mathbb{F}(X)$; otherwise $\approx$ is not the equality on $\mathbb{F}(X)$, hence an element Y of $\mathbb{H}(X)$ cannot be written in the form $Y = [\mathrm{X}^m]$ for some unique monomial X^m.

1 Derivations

In all that follows, S is a commutative monoid.

1. Multimonoids. In this section, M is a commutative multimonoid over S.

A *multi-M-module over S* is an abelian multigroup A over S together with an action of M on A such that

$$
\begin{aligned}
&(xa)^\vee = x^\vee\, a^\vee\,,\\
&(xa)^t = x\, a^t = x^t\, a\,,\\
&x\,(ya) = (xy)\, a\,, \quad \text{and}\\
&x\,(a+b) = xa + xb \ \text{ if } \ a+b \ \text{ is defined}
\end{aligned}
$$

for all $x, y \in M$, $t \in S$, and $a, b \in A$.

For example, if R is a commutative multiring over S, then every multi-R-module over S remains an multi-R-module over S when R is viewed as a commutative multimonoid over S. Conversely, if A is a multi-M-module over S, then A is a multi-($\mathbb{Z} \dashv M$)-module over S, on which $p = \sum \big(p_x\, x \bigm| x \in M \big) \in \mathbb{Z} \dashv M$ acts by

$$pa \;=\; \textstyle\sum \big(p_x\, xa \bigm| x \in M \big)\,.$$

The categories of multi-M-modules over S and multi-($\mathbb{Z} \dashv M$)-modules over S are therefore (thereby?) isomorphic.

A *prederivation* of a commutative multimonoid M over S into a multi-M-module A over S is a mapping $D\colon M \longrightarrow A$ such that

$$
\begin{aligned}
&D(x)^\vee = x^\vee\,,\\
&D(x^t) = D(x)^t\,, \quad \text{and}\\
&D(xy) = x\, D(y) + y\, D(x) \in A.
\end{aligned}
$$

for all $x, y \in M$ and $t \in S$. In particular, D is a morphism of multisets over S. Also, $D(1) = D(11) = D(1) + D(1)$, so that $D(1) = 0$, whence $D(1^t) = D(1)^t = 0$ for all $t \in S$.

The set $\mathrm{PDer}\,(M, A)$ of all derivations of M into A is an abelian group under pointwise addition, on which M acts (pointwise) by:

$$(xD)(y) \;=\; x\, D(y)$$

for all $x, y \in M$. Then $x\,(D' + D") = xD' + xD")$ and $x\,(yD) = (xy)\, D$ for all $x, y \in M$ and $D, D', D" \in \mathrm{Der}\,(M, A)$. OK, that makes $\mathrm{Der}\,(M, A)$ an M-module, but there are enough new structures already, so I'll skip the formal definition of M-modules.

2. Half-free multimonoids. When X is a multiset over S, recall that $\mathbb{H}(X)$ is the (half)free commutative multimonoid over S over X. In describing $\mathbb{H}(X)$ we dispense with $\overline{X} = \{\, X_x \mid x \in X \,\}$ and write the elements of the free commutative monoid $\mathbb{F}(X)$ on the set X as commutative products of elements of X. As a monoid,

$$\mathbb{H}(X) \;=\; \mathbb{F}(X)/\approx\,,$$

where $\approx$ is the smallest congruence on $\mathbb{F}(X)$ such that

$$x^t\, y \approx x\, y^t$$

for all $x, y \in X$ and $t \in S$. This writes the elements of $\mathbb{H}(X)$ as equivalence classes $[x_1 x_2 \cdots x_n]$ of commutative products of elements of X. Concatenation yields products in $\mathbb{H}(X)$, and, for every $x_1, x_2, \ldots, x_n \in X$ and $t \in S$,

$$[x_1 x_2 \cdots x_n]^\vee \;=\; x_1{}^\vee\, x_2{}^\vee \cdots x_n{}^\vee$$

and

$$[x_1 x_2 \cdots x_n]^t \;=\; x_1 \cdots x_i^t \cdots x_n\,,$$

for any i.

The canonical map $\eta\colon X \longrightarrow \mathbb{H}(X)$ sends $x \in X$ to $[x] \in \mathbb{H}(X)$. It is a morphism of multisets over S.

3. Theorem. For every multiset X over S and multi-$\mathbb{H}(X)$-module A over S there is a natural isomorphism

$$\Theta\colon \mathrm{PDer}\,(\mathbb{H}(X),\, A) \;\cong\; \mathrm{Map}\,(X, A)$$

that sends a prederivation D of $\mathbb{H}(X)$ into A to

$$d \;=\; \Theta(D) \;=\; D \circ \eta\colon X \xrightarrow{\;\eta\;} \mathbb{H}(X) \xrightarrow{\;D\;} A\,,$$

so that $dx = D[x]$.

Proof: First note that $d = D \circ \eta$ is a morphism of multisets over S.

Θ is injective. For every $D \in \mathrm{PDer}\,(\mathbb{H}(X),\, A)$, induction yields

$$D\,[x_1 x_2 \cdots x_n] \;= \textstyle\sum_i \big([x_1 \cdots x_{i-1}\; x_{i+1} \cdots x_n]\; D\,[x_i]\big)$$

for all $n \geq 0$ and $x_1, x_2, \ldots, x_n \in X$ (with $x_1 x_2 \cdots x_n = 1$ if $n = 0$ so that $D\,[x_1 x_2 \cdots x_n,\, 1] = 0$). If $d = \Theta(D) = 0$, then $D\,[x_1 x_2 \cdots x_n]) = 0$ for all $n \geq 0$ and $x_1, x_2, \ldots, x_n \in \overline{X}$ and $D = 0$.

In what follows we denote

$$x_1 \cdots x_{i-1}\; x_{i+1} \cdots x_n \;\text{ by }\; x_1 x_2 \cdots x_n(-i)\,.$$

Θ is surjective. Let $d \in \mathrm{Map}\,(X, A)$, so that $(dx)^\vee = x^\vee$ and $d(x^t) = (dx)^t$ for all $x \in X$ and $t \in S$. We want

$$D\,[x_1 x_2 \cdots x_n] \;= \textstyle\sum_i \big([x_1 x_2 \cdots x_n(-i)]\, d(x_i)\,.$$

Let $j < k$, let $x_1, x_2, \ldots, x_n \in X$, and let $t \in S$, so that

$$x_1 \cdots x_j^t \cdots x_k \cdots x_n \;\approx\; x_1 \cdots x_j \cdots x_k^t \cdots x_n\,.$$

Let

$$\begin{aligned} y_j &= x_j^t\,, \quad y_i \;=\; x_i \;\text{ if }\; i \neq j \;\text{ and} \\ z_k &= x_k^t\,, \quad z_i \;=\; x_i \;\text{ if }\; i \neq k\,, \end{aligned}$$

so that

$$\begin{aligned}
[y_1 y_2 \cdots y_n] &= [x_1 \cdots x_j^t \cdots x_k \cdots x_n]\,, \\
[z_1 z_2 \cdots z_n] &= [x_1 \cdots x_j \cdots x_k^t \cdots x_n]\,, \\
[y_1 y_2 \cdots y_n] &= [z_1 z_2 \cdots z_n]\,, \\
[y_1 y_2 \cdots y_n(-k)] &= [z_1 z_2 \cdots z_n(-k)]^t\,, \quad \text{and} \\
[y_1 y_2 \cdots y_n(-j)]^t &= [z_1 z_2 \cdots z_n(-j)]\,.
\end{aligned}$$

In the multi-$\mathbb{H}(X)$-module A over S, we have $(pw)^t = p^t w = p\, z^t$ for all $p \in \mathbb{Z}[X]$, $w \in M$, and $t \in S$. Hence

$$\begin{aligned}
&\textstyle\sum_i \big([y_1 y_2 \cdots y_n(-i)]\, d(y_i)\big) \\
&= \textstyle\sum_{i \neq j,k} \big([y_1 y_2 \cdots y_n(-i)]\, d(x_i)\big) \\
&\quad + [y_1 y_2 \cdots y_n(-j)]\, d(x_j^t) + [y_1 y_2 \cdots y_n(-k)]\, d(x_k) \\
&= \textstyle\sum_{i \neq j,k} \big([z_1 z_2 \cdots z_n(-i)]\, d(x_i)\big) \\
&\quad + [y_1 y_2 \cdots y_n(-j)]^t\, d(x_j) + [z_1 z_2 \cdots z_n(-k)]^t\, d(x_k) \\
&= \textstyle\sum_{i \neq j,k} \big([z_1 z_2 \cdots z_n(-i)]\, d(z_i)\big) \\
&\quad + [z_1 z_2 \cdots z_n(-j)]\, d(z_j) + [z_1 z_2 \cdots z_n(-k)]\, d(x_k^t) \\
&= \textstyle\sum_i \big([z_1 z_2 \cdots z_n(-i)]\, d(z_i)\big)\,.
\end{aligned}$$

Therefore $y_1 y_2 \cdots y_n \approx z_1 z_2 \cdots z_n$ implies

$$\textstyle\sum_i \big([y_1 y_2 \cdots y_n(-i)]\, d(y_i)\big) = \sum_i \big([z_1 z_2 \cdots z_n(-i)]\, d(z_i)\big)$$

and a mapping $D\colon \mathbb{H}(X) \longrightarrow M$ is well defined by

$$D\,[x_1 x_2 \cdots x_n] = \textstyle\sum_i \big([x_1 x_2 \cdots x_n(-i)]\, d(x_i)\big)\,.$$

In particular, $D[x] = dx$ for all $x \in X$.

D is a prederivation; this will complete the proof. Since d is a morphism of multisets over S we have $(dx)^\vee = x^\vee$ and $d(x^t) = (dx)^t$ for all x, t. Hence $\big([x_1 x_2 \cdots x_n(-i)]\, d(x_i)\big)^\vee = [x_1 x_2 \cdots x_n]^\vee$ and

$$D\,[x_1 x_2 \cdots x_n]\big)^\vee = [x_1 x_2 \cdots x_n]^\vee\,.$$

Moreover,

$$\begin{aligned}
&D\big([x_1 x_2 \cdots x_n]^t\big) = D\,[x_1 x_2 \cdots x_{n-1}\, x_n^t] \\
&= \textstyle\sum_{i<n} \big([x_1 x_2 \cdots x_{n-1}\, x_n^t\,(-i)]\, d(x_i)\big) + [x_1 x_2 \cdots x_{n-1}]\, d\,(x_n^t) \\
&= \textstyle\sum_{i<n} \big([x_1 x_2 \cdots x_{n-1}\, x_n\,(-i)]\, d(x_i)\big)^t + \big([x_1 x_2 \cdots x_{n-1}]\, d\,(x_n)\big)^t\big) \\
&= \big(D\,[x_1 x_2 \cdots x_n]\big)^t\,,
\end{aligned}$$

so that $D(y^t) = (D(y))^t$ for all $y \in \mathbb{H}(X)$, and

$$\begin{aligned} & D\big([x_1 \cdots x_m][y_1 \cdots y_n]\big) \\ &= \textstyle\sum_i \big([x_1 \cdots x_m(-i)][y_1 \cdots y_n]\, d(x_i)\big) \\ &\quad + \textstyle\sum_j \big([x_1 \cdots x_m][y_1 \cdots y_n(-j)]\, d(y_j)\big) \\ &= [x_1 \cdots x_m]\, D\,[y_1 \cdots y_n] \;+\; [y_1 \cdots y_n]\, D\,[x_1 \cdots x_m]\,, \end{aligned}$$

so that $D(yz) = y\,D(z) + z\,D(y)$ for all $y, z \in \mathbb{H}(X)$.

Thus D is a prederivation.

4. Multirings. As before, a *derivation* of a commutative multiring R over S into a multi-R-module M over S is a mapping $D\colon R \longrightarrow M$ such that

$$\begin{aligned} & D(x)^\vee = x^\vee \quad \text{for all} \quad x\,, \\ & D(x^t) = D(x)^t \quad \text{for all} \quad x, t\,, \\ & D(x+y) = D(x) + D(y) \quad \text{whenever} \quad x+y \quad \text{is defined,} \quad \text{and} \\ & D(xy) = x\,D(y) + y\,D(x) \quad \text{for all} \quad x, y \in A\,; \end{aligned}$$

In particular, D is a morphism of abelian multigroups over S, and a prederivation of R, viewed as a commutative multimonoid over S, into M, viewed as a multi-M-module over S. As before, $D(1) = 0$ and $D(1^t) = D(1)^t = 0$, so that $D(r1) = r\,D(1) = 0$, for all $r \in R$ and $t \in S$.

The set $\mathrm{Der}\,(R, M)$ of all derivations of R into M is an R-module under pointwise addition and action of R ($(rD)(x) = r\,D(x)$ for all $x, y \in R$).

Now let X be a multiset over S and let

$$\mathbb{Z}[X] \;=\; \mathbb{Z} \dashv \mathbb{H}(X)$$

denote the free commutative multiring over S on the multiset X, which is also the free commutative multiring over S on the commutative multimonoid $\mathbb{H}(X)$ over S. The typical element of $\mathbb{Z}[X]$ is a homogeneous linear combination

$$p \;=\; \sum \big(\, p_y\, y \;\big|\; y \in \mathbb{H}(X),\; y^\vee = p^\vee \,\big)$$

of elements of $\mathbb{H}(X)$.

The multiring comes with a canonical map

$$\eta\colon X \xrightarrow{\;\eta\;} \mathbb{H}(X) \xrightarrow{\;\eta\;} \mathbb{Z}[X]$$

which is a morphism of multisets over S and sends $x \in X$ to $[x] \in \mathbb{H}(X) \subseteq \mathbb{Z}[X]$.

5. Theorem. For every set X over S and multi-$\mathbb{Z}[X]$-module M over S there is a natural isomorphism

$$\Theta\colon \mathrm{Der}\,(\mathbb{Z}[X],\, M) \;\cong\; \mathrm{Map}\,(X, M)$$

that sends a derivation D of $\mathbb{Z}[X]$ into M to

$$d \;=\; \Theta(D) \;=\; D\circ\eta\colon X \xrightarrow{\;\eta\;} \mathbb{Z}[X] \xrightarrow{\;D\;} M\,,$$

so that $dx = D[x]$.

This is proved by composing the isomorphism

$$\mathrm{PDer}\,(\mathbb{H}(X),\, M) \;\cong\; \mathrm{Map}\,(X, M)$$

in Theorem 1.3 with a similar isomorphism

$$\Psi\colon \mathrm{Der}\,(\mathbb{Z}[X],\, M) \cong \mathrm{PDer}\,(\mathbb{H}(X),\, M)\,.$$

Note that every multi-$\mathbb{Z}[X]$-module M is in particular a multi-$\mathbb{H}(X)$-module.

If $D\colon \mathbb{Z}[X] \longrightarrow M$ is a derivation of $\mathbb{Z}[X]$, then ΨD is the restriction of D to $\mathbb{H}(X) \subseteq \mathbb{Z}[X]$, which is a prederivation of $\mathbb{H}(X)$.

Ψ is injective. Indeed, if $D(y) = 0$ for all $y \in \mathbb{H}(X)$, and $p = \sum \big(p_y\, y \bigm| y \in \mathbb{H}(X),\ y^\vee = p^\vee \big) \in \mathbb{Z}[X]$, then

$$D(p) \;=\; \sum \big(p_y\, D(y) \bigm| y \in \mathbb{H}(X),\ y^\vee = p^\vee \big) \;=\; 0\,.$$

Ψ is surjective. Let M be a multi-$\mathbb{Z}[X]$-module over S and let $d\colon \mathbb{H}(X) \longrightarrow M$ be a prederivation of $\mathbb{H}(X)$. For each $p = \sum \big(p_y\, y \bigm| y \in \mathbb{H}(X),\ y^\vee = p^\vee \big) \in \mathbb{Z}[X]$, define

$$D(p) \;=\; \sum \big(p_y\, d(y) \bigm| y \in \mathbb{H}(X),\ y^\vee = p^\vee \big)\,.$$

In particular, $D[x] = d[x]$ for all $x \in X$. We have $D(p)^\vee = p^\vee$, since $d(y)^\vee = y^\vee$ for all $y \in \mathbb{H}(X)$, and

$$\begin{aligned} D(p^t) \;&=\; \sum \big(p_y\, d(y^t) \bigm| y \in \mathbb{H}(X),\ y^\vee = p^\vee \big) \\ &=\; \sum \big(p_y\, (d(y))^t \bigm| y \in \mathbb{H}(X),\ y^\vee = p^\vee \big) \;=\; (D(p))^t\,. \end{aligned}$$

Moreover, if

$$\begin{aligned} p \;&=\; \sum \big(p_y\, y \bigm| y \in \mathbb{H}(X),\ y^\vee = p^\vee \big) \quad \text{and} \\ q \;&=\; \sum \big(q_z\, z \bigm| z \in \mathbb{H}(X),\ z^\vee = q^\vee \big) \in \mathbb{Z}[X], \end{aligned}$$

then $D(p+q) = D(p) + D(q)$ (if $p^\vee = q^\vee$) and

$$\begin{aligned}
D(pq) &= D\left[\sum \left(p_y\, q_z\, (yz) \;\middle|\; y, z \in \mathbb{H}(X),\ y^\vee = p^\vee,\ z^\vee = q^\vee \right)\right] \\
&= \sum \left(p_y\, q_z\, d(yz) \;\middle|\; y, z \in \mathbb{H}(X),\ y^\vee = p^\vee,\ z^\vee = q^\vee \right) \\
&= \sum \left(p_y\, q_z\, (y\, d(z) + z\, d(y)) \;\middle|\; y, z \in \mathbb{H}(X),\ y^\vee = p^\vee,\ z^\vee = q^\vee \right) \\
&= \sum \left(p_y\, q_z\, y\, d(z) \;\middle|\; y, z \in \mathbb{H}(X),\ y^\vee = p^\vee,\ z^\vee = q^\vee \right) \\
&\quad + \sum \left(p_y\, q_z\, z\, d(y) \;\middle|\; y, z \in \mathbb{H}(X),\ y^\vee = p^\vee,\ z^\vee = q^\vee \right) \\
&= \left[\sum \left(p_y\, y \;\middle|\; y \in \mathbb{H}(X),\ y^\vee = p^\vee \right)\right] \\
&\qquad \left[\sum \left(q_z\, d(z) \;\middle|\; z \in \mathbb{H}(X),\ z^\vee = q^\vee \right)\right] \\
&\quad + \left[\sum \left(q_z\, z \;\middle|\; z \in \mathbb{H}(X),\ z^\vee = q^\vee \right)\right] \\
&\qquad \left[\sum \left(p_y\, d(y) \;\middle|\; y \in \mathbb{H}(X),\ y^\vee = p^\vee \right)\right] \\
&= p\, D(q) \ + \ q\, D(p)\,.
\end{aligned}$$

Thus D is a derivation.

2 Cohomology of multirings

In this section, S is a commutative monoid and R is a commutative multiring over S.

1. The comonad. The adjunction of multisets over S to commutative multirings over S begets a comonad (V, ϵ, ν) in which V sends a commutative multiring R over S to the free commutative multiring $VR = \mathbb{Z}[R] = \mathbb{Z} \dashv \mathbb{H}(R)$ over S on the multiset R over S.

Every morphism $\varphi\colon R \longrightarrow T$ of multisets over S induces a morphism $V\varphi$ of commutative multirings over S from $\mathbb{Z}[R]$ to $\mathbb{Z}[T]$ that sends

$$p \ = \sum \left(p_y\, y \;\middle|\; y \in \mathbb{H}(R),\ y^\vee = p^\vee \right)$$

to

$$(V\varphi)(p) \ = \sum \left(p_y\, (V\varphi)(y) \;\middle|\; y \in \mathbb{H}(R),\ y^\vee = p^\vee \right),$$

where

$$(V\varphi)\, [r_1 r_2 \cdots r_n] \ = \ [(\varphi r_1)(\varphi r_2) \cdots (\varphi r_n)]\,,$$

as calculated in $\mathbb{Z}[T]$, for all $r_1, r_2, \ldots, r_n \in R$. In particular,

$$(V\varphi)\, [r] \ = \ [\varphi r]$$

for all $r \in R$.

The counit $\epsilon\colon VR \longrightarrow R$ is the evaluation morphism of commutative multirings over S that sends $[r]$ to r, sends $y = [r_1 r_2 \cdots r_n]$ to

$$\epsilon\,[r_1 r_2 \cdots r_n] \;=\; r_1 r_2 \cdots r_n$$

as calculated in R, and sends $p = \sum \big(p_y\, y \;\big|\; y \in \mathbb{H}(R),\; y^\vee = p^\vee \big)$ to

$$\epsilon\,(p) \;=\sum \big(p_y\, \epsilon\,(y) \;\big|\; y \in \mathbb{H}(R),\; y^\vee = p^\vee \big)\,.$$

The comultiplication ν assigns to R the morphism $\nu_R = V\eta\colon VR \longrightarrow VVR$ induced by the adjunction unit $\eta\colon R \longrightarrow VR$, viewed as a morphism of multisets over S.

If M is a multi-R-module over S, then the action of R on M induces an action of VR on M in which

$$px \;=\; \epsilon(p)\, x$$

for every $p \in VR$ and $x \in M$, which makes M a multi-VR-module since ϵ is a morphism of commutative multirings over S.

2. The resolution. For every commutative multiring R over S there is now an augmented simplicial commutative multiring R^* with objects $R^0 = R$ and $R^n = V^n R$ (where $n \geq 0$ and $V^n = V \circ V \circ \cdots \circ V$), face maps

$$\epsilon_{n,i} \;=\; V^i\, \epsilon\, V^{n-i}\, R\colon R^{n+1} \longrightarrow R^n \quad (i = 0, 1, \ldots, n),$$

augmentation $\epsilon = \epsilon_{0,0}\colon VR \longrightarrow R$, and degeneracy maps

$$\nu_{n,i} \;=\; V^i\, \nu\, V^{n-i-1} R\colon R^{n-1} \longrightarrow R^n \quad (i = 0, 1, \ldots, n-1),$$

that satisfy the simplicial identities.

Let M be a multi-R-module over S, hence also a multi-VR-module over S and a multi-R^n-module over S for every $n \geq 0$. Applying to R^* the contravariant functor $\mathrm{Der}\,(-, M)$ yields an augmented cosimplicial R-module $\mathrm{Der}\,(R^*,\, M)$ with objects $\mathrm{Der}\,(R^0, M) = \mathrm{Der}\,(R, M)$ and $\mathrm{Der}\,(R^n,\, M) = \mathrm{Der}\,(V^n R,\, M)$ if $n > 0$, face maps

$$d_{n,i} \;=\; \epsilon^*_{n,i} \;=\; \mathrm{Der}\,(\epsilon_{n,i},\, M)\colon \mathrm{Der}\,(R^n,\, M) \longrightarrow \mathrm{Der}\,(R^{n+1},\, M)$$

for $i = 0, 1, \ldots, n$, augmentation

$$d_{0,0} \;=\; \epsilon^* \;=\; \mathrm{Der}\,(\epsilon, M)\colon \mathrm{Der}\,(R, M) \longrightarrow \mathrm{Der}\,(VR, M)\,,$$

and degeneracy maps $\mathrm{Der}\,(\nu_{n,i},\, M)$ $(i = 0, 1, \ldots, n-1)$, that satisfy the cosimplicial identities.

A coboundary homomorphism

$$\delta_n\colon \mathrm{Der}\,(R^n,\, M) \longrightarrow \mathrm{Der}\,(R^{n+1},\, M)$$

is then defined by

$$\delta_n \;=\; d_{n,0} - d_{n,1} + d_{n,2} - \cdots + (-1)^n\, d_{n,n}\,.$$

In particular, $\delta_0 = d_{0,0}$. The simplicial identities imply $\delta_{n+1} \circ \delta_n = 0$ for all $n \geq 0$. This yields an augmented cochain complex of R-modules

$$\begin{aligned} 0 \longrightarrow &\mathrm{Der}\,(R,M) \xrightarrow{\epsilon^*} \mathrm{Der}\,(VR,\,M) \xrightarrow{\delta_1} \;\ldots \\ &\longrightarrow \mathrm{Der}\,(V^nR,\,M) \xrightarrow{\delta_n} \mathrm{Der}\,(V^{n+1}R,\,M)\;\ldots \end{aligned}$$

The *André-Quillen cohomology* of the commutative multiring R over S with coefficients in the multi-R-module M over S assigns to R and M the R-modules $H^0(R,M) = \mathrm{Ker}\,\delta_0\,/\,\mathrm{Im}\,e$ and $H^n(R,M) = \mathrm{Ker}\,\delta_n\,/\,\mathrm{Im}\,\delta_{n-1}$, where $n > 0$.

Since $VR = \mathbb{Z}[R]$, the natural isomorphisms $\mathrm{Der}\,(\mathbb{Z}[X],M) \cong \mathrm{Map}\,(X,M)$ yield an isomorphic complex of R-modules

$$\begin{aligned} 0 \longrightarrow\; &\mathrm{Der}\,(R,M) \xrightarrow{e} \mathrm{Map}\,(R,M) \xrightarrow{d_1} \;\ldots \\ &\longrightarrow \mathrm{Map}\,(V^{n-1}R,\,M) \xrightarrow{d_n} \mathrm{Map}\,(V^nR,\,M)\;\ldots \end{aligned}$$

whose homology modules are natural isomorphic to the André-Quillen cohomology modules of R with coefficients in the multi-R-module M. In particular, $H^1(R,M) \cong \mathrm{Ker}\,d_1\,/\,\mathrm{Im}\,e$. Without the augmentation, $H^1(R,M)$ would be simply $\mathrm{Ker}\,d_1$.

3. Maps. The augmentation $e\colon \mathrm{Der}\,(R,M) \longrightarrow \mathrm{Map}\,(R,M)$ is the composite

$$e\colon \mathrm{Der}\,(R,M) \xrightarrow{\epsilon^*} \mathrm{Der}\,(VR,M) \xrightarrow{\cong} \mathrm{Map}\,(R,M)$$

of $\epsilon^* = \mathrm{Der}\,(\epsilon,M)$, which sends $D \in \mathrm{Der}\,(R,M)$ to $D \circ \epsilon$, and the isomorphism $\mathrm{Der}\,(VR,M) \longrightarrow \mathrm{Map}\,(R,M)$, which sends $D \circ \epsilon$ to the mapping $r \longmapsto (D \circ \epsilon)[r] = D(r)$. Thus $e\,(D)$ is D viewed as simply a mapping of R into M.

Next, $d_{n,i}\colon \mathrm{Map}\,(V^{n-1}R,\,M) \longrightarrow \mathrm{Map}\,(V^nR,\,M)$ is the composite

$$\begin{aligned} d_{n,i}\colon \mathrm{Map}\,(V^{n-1}R,\,M) &\xrightarrow{\cong} \mathrm{Der}\,(V^nR,\,M) \\ &\xrightarrow{\epsilon^*_{n,i}} \mathrm{Der}\,(V^{n+1}R,\,M) \xrightarrow{\cong} \mathrm{Map}\,(V^nR,\,M) \end{aligned}$$

and sends a map u from $V^{n-1}R$ to M to the derivation $\widehat{u}$ of V^nR such that $\widehat{u}[x] = u(x)$ for all $x \in V^{n-1}R$, thence to $\epsilon^*_{n,i}\,(\widehat{u}) = \widehat{u} \circ \epsilon_{n,i}$, thence to the corresponding map from V^nR to M.

If $i = 0$, then $\epsilon_{n,0} = e_{V^nR}$ and

$$(\widehat{u} \circ \epsilon_{V^nR})[x] \;=\; \widehat{u}\,(\epsilon_{V^nR}\,[x]) \;=\; \widehat{u}(x)$$

for every $x \in V^nR$. Hence $d_{n,0}\,u$ sends $x \in VR$ to $\widehat{u}(x)$ and

$$d_{n,0}\,u \;=\; \widehat{u}\,.$$

If $i > 0$, then $\epsilon_{n,i} = V\,\epsilon_{n-1,\,i-1}$ and

$$(\widehat{u} \circ V\epsilon_{n-1,\,i-1})[x] \;=\; \widehat{u}\,(V\epsilon_{n-1,\,i-1}\,[x]) \;=\; \widehat{u}\,[\epsilon_{n-1,\,i-1}(x)] \;=\; u\,(\epsilon_{n-1,\,i-1}(x))$$

for every $x \in V^n R$. Hence $d_{n,i}\,u$ sends $x \in V^n R$ to $u\,(\epsilon_{n-1,\,i-1}(x))$ and

$$d_{n,i}\,u \;=\; u \circ \epsilon_{n-1,\,i-1}\,.$$

In particular, $d_{1,1}\,u \;=\; u \circ \epsilon_R$.

4. Theorem. $H^1_Q(R, M) = 0$.

Indeed let $u \in \mathrm{Map}\,(R, M)$ be an André-Quillen 1-cocycle. By the above, $\delta u \;=\; d_{1,0}\,u - d_{1,1}\,u \;=\; \widehat{u} - (u \circ \epsilon) \;=\; 0$, so that $u \circ \epsilon$ is a derivation. Since $\epsilon\colon VR \longrightarrow R$ is surjective it follows that u is a derivation: for instance, for all $r, s \in R$,

$$\begin{aligned} u\,(rs) \;&=\; u\,(\epsilon[r]\,\epsilon[s]) \;=\; u\,(\epsilon\,[rs]) \\ &=\; r\,u\,(\epsilon[s]) \;+\; s\,u\,(\epsilon[r]) \;=\; r\,u(s) \;+\; s\,u(r)\,. \end{aligned}$$

Therefore u is an André-Quillen 1-coboundary.

5. What's next? We want to get at the cohomology of S (in dimensions $n \geq 2$) by way of the cohomology of $R = \mathbb{Z} \dashv \mathbb{H}(X)$ for some suitable multiset X that depends on S.

There are two obvious candidates for X: one is the multiset S over itself, with projection $a^\vee = a$ and action $a^t = at$. The other is $S^+ = S \times S$, the free multiset over S on S as a set over itself, with projection $(a,t)^\vee = at$ and action $(a,t)^u = (a, tu)$.

Every abelian multigroup G over S can serve as coefficients in the cohomology of $\mathbb{Z} \dashv \mathbb{H}(S)$. Indeed $\mathbb{H}(S)$ acts on every abelian multigroup G over S by

$$px \;=\; x^{(p^\vee)}$$

for every $x \in G$ and $p \in \mathbb{H}(S)$. In particular, $[a]\,x = x^a$ and $1x = x$. Then

$$\begin{aligned} &(px)^\vee \;=\; (x^{(p^\vee)})^\vee \;=\; p^\vee\,x^\vee\,, \\ &(px)^t \;=\; (x^{(p^\vee)})^t \;=\; x^{p^\vee\,t} \;=\; p\,x^t\,, \\ &p^t\,x \;=\; x^{(p^t)^\vee} \;=\; x^{p^\vee\,t} \;=\; p\,x^t\,, \\ &p\,(x+y) \;=\; (x+y)^{(p^\vee)} \;=\; x^{(p^\vee)} + y^{(p^\vee)} \;=\; px + py \\ &\qquad \text{whenever } \;x+y\; \text{ is defined}\,, \;\text{ and} \\ &p\,(qx) \;=\; (x^{(q^\vee)})^{(p^\vee)} \;=\; x^{(pq)^\vee} \;=\; pq \cdot x\,, \end{aligned}$$

for all $p, q \in \mathbb{H}(S)$ and $x, y \in G$.

If now $p \in \mathbb{Z} \dashv \mathbb{H}(S)$,

$$p \;= \sum \left(p_y\,y \;\middle|\; y \in \mathbb{H}(S),\; y^\vee = p^\vee \right),$$

then px is well defined by

$$px = \sum \big(p_y\,(yx) \bigm| y \in \mathbb{H}(S),\ y^\vee = p^\vee\big) = n(p)\, x^{(p^\vee)},$$

where

$$n(p) = \sum \big(p_y \bigm| y \in \mathbb{H}(S),\ y^\vee = p^\vee\big).$$

Then $(p+q)\,x = px + qx$ whenever $p^\vee = q^\vee$, since $n\,(p+q) = n(p) + n(q)$, and the five properties above also hold for all $p, q \in \mathbb{Z}[S]$ and $x, y \in S$.

With this action of $\mathbb{Z}[S]$, the abelian multigroup G becomes a multi-$\mathbb{Z}[S]$-module over S.

For every abelian multigroup G over S there is a natural isomorphism

$$\operatorname{Map}(S, G) \cong G_1.$$

Indeed, if $f\colon S \longrightarrow G$ is a morphism of multisets over S, then $f(a^t) = (f(a))^t$ for all $a, t \in S$; since $a^t = at$ we have

$$f(a) = f(1^a) = (f(1))^a,$$

so that f is uniquely determined by $f(1) \in G_1$. Conversely, if $g \in G_1$, then

$$f\colon a \longmapsto g^a$$

is a morphism of multisets over S from S to G: indeed $f(a)^\vee = a$ and $f(a^t) = g^{at} = (f(a))^t$ for all a, t.

Theorem 1.5 then yields $\operatorname{Der}(\mathbb{Z} \dashv \mathbb{H}(S),\, G) \cong G_1$.

It follows that S is not a suitable choice for X.

6. What's still next? The next choice is $\mathbb{Z} \dashv \mathbb{H}(S^+)$.

Every abelian multigroup G over S can serve as coefficients in the cohomology of $\mathbb{Z} \dashv \mathbb{H}(S^+)$. Indeed $\mathbb{H}(S^+)$ acts on every abelian multigroup G over S by

$$px = x^{(p^\vee)}$$

for every $x \in G$ and $p \in \mathbb{H}(S^+)$. In particular, $[(a,t)]\,x = x^{at}$ and $[(1,1)]\,x = x$. As before, $(px)^\vee = p^\vee\, x^\vee$, $(px)^t = p^t\, x = p\, x^t$, $p\,(qx) = (pq)\,x$, and $p\,(x+y) = px + py$ if $x+y$ is defined, for all $p, q \in \mathbb{H}(S^+)$, $t \in S$, and $x, y \in G$.

If now $p \in \mathbb{Z} \dashv \mathbb{H}(S^+)$,

$$p = \sum \big(p_y\, y \bigm| y \in \mathbb{H}(S),\ y^\vee = p^\vee\big),$$

then px is well defined by

$$px = \sum \big(p_y\,(yx) \bigm| y \in \mathbb{H}(S),\ y^\vee = p^\vee\big) = n(p)\, x^{(p^\vee)},$$

where

$$n(p) = \sum \big(p_y \bigm| y \in \mathbb{H}(S^+),\ y^\vee = p^\vee\big).$$

Then $(p+q)\,x = px + qx$ whenever $p^\vee = q^\vee$, since $n\,(p+q) = n(p) + n(q)$, and the five properties above also hold for all $p, q \in \mathbb{Z}[S]$ and $x, y \in S$.

With this action of $\mathbb{Z}[S]$, the abelian multigroup G over S becomes a multi-$\mathbb{Z} \dashv \mathbb{H}(S^+)$-module over S.

For every abelian multigroup G over S there is a natural isomorphism

$$\mathrm{Map}\,(S^+, G) \;\cong\; C^1_C(S, G)\,.$$

Indeed, if $f\colon S^+ \longrightarrow G$ is a morphism of multisets over S, then $f(y^t) = (f(y))^t$ for all $y \in S^+$ and $t \in S$; since $(a,t)^u = (a, tu)$ we have

$$f(a,t) \;=\; f((a,1)^t) \;=\; (f(a,1))^t\,,$$

so that f is uniquely determined by the (symmetric) 1-cochain $g\colon a \longmapsto f(a,1) \in G_a$. Conversely, given $g \in C^1_C(S,G)$, so that $g(a) \in G_a$ for all $a \in S$, define

$$f(a,t) \;=\; g(a)^t$$

for all $a, t \in S$. In particular, $f(a,1) = g(a)$. Then $f(a,t)^\vee = at = (a,t)^\vee$ and

$$f((a,t)^u) \;=\; f(a,tu) \;=\; g(a)^{tu} \;=\; (f(a,t))^u$$

for all $a, t, u \in S$, so that f is a morphism of multisets over S.

Theorem 1.5 then yields $\mathrm{Der}\,(\mathbb{Z} \dashv \mathbb{H}(S^+),\, G) \cong C^1_C(S,G)$.

It follows that S^+ could be a suitable choice for X. But when I tried it I obtained no result that was specific to S or $\mathbb{Z} \dashv \mathbb{H}(S^+)$, only the results in the next sections that apply to all commutative multirings.

3 Weak 2-cocycles

In this section, S is a commutative monoid, R is a commutative multiring over S, and M is a multi-R-module over S.

1. Cochains. Like André-Quillen n-cochains, which are morphisms of soss $V^{n-1}R \longrightarrow M$, a (regular) *$n$-cochain* u of a set X over S with values in a set G over S, where $n \geq 1$, is a morphism $u\colon X^{(n)} \longrightarrow G$ of sets over S, where

$$X^{(n)} \;=\; X \times X \times \cdots \times X \quad (n \text{ times})$$

and assigns to each $x_1,\, x_2, \dots,\, x_n \in X$ some $u\,(x_1,\, x_2, \dots,\, x_n) \in G$ such that

$$u\,(x_1,\, x_2, \dots,\, x_n)^\vee \;=\; (x_1^\vee)(x_2^\vee)\cdots(x_n^\vee)\,.$$

The n-cochains of X with values in G constitute a set

$$C^n(X,G) \;=\; \mathrm{Map}\,(X^{(n)},\, G)\,.$$

If G is an abelian multigroup over S (and not just a set over S), then n-cochains of X with values in G can be added pointwise: if $u, v \in C^n(X,G)$, then

$$u\,(x_1,\, x_2, \dots,\, x_n)^\vee \;=\; v\,(x_1,\, x_2, \dots,\, x_n)^\vee$$

for all $x_1, x_2, \ldots, x_n \in X$, so that

$$(u+v)(x_1, x_2, \ldots, x_n)^\vee \;=\; u(x_1, x_2, \ldots, x_n)^\vee \;+\; v(x_1, x_2, \ldots, x_n)^\vee$$

is defined in G. Then $C^n(X,G)$ is a abelian group under pointwise addition.

In what follows we do not venture beyond $n = 1$ or $n = 2$.

2. Cocycles. In the André-Quillen cohomology of R we know from Part 2.3 that an André-Quillen 2-cocycle is a 2-cochain $u\colon VR \longrightarrow M$ such that

$$u \circ \epsilon_{VR} \;=\; (u \circ V\epsilon_R) \;+\; \widehat{u}\,,$$

equivalently,

$$u(\epsilon_{VR}A) \;=\; u((V\epsilon_R)(A)) \;+\; \widehat{u}(A) \;\text{ for all }\; A \in V^2R\,, \qquad \text{(Z)}$$

where $V^2R = \mathbb{Z} \dashv \mathbb{H}(VR)$. Like 2-cochains, André-Quillen 2-cocycles can be added pointwise, and constitute an abelian group $Z^2_Q(R,M)$.

A *weak 2-cocycle* is an André-Quillen 2-cochain u such that (Z) only holds for all $A \in \mathbb{H}(VR)$:

$$u(\epsilon_{VR}A) \;=\; u((V\epsilon_R)(A)) \;+\; \widehat{u}(A) \;\text{ for all }\; A \in \mathbb{H}(VR)\,. \qquad \text{(Z–)}$$

Under pointwise addition, weak 2-cocycles constitute a subgroup $Z^2_W(R,M)$ of $Z^2_Q(R,M)$.

We think that the results in this section make weak 2-cocycles more interesting than full-blooded 2-cocycles.

Condition (Z–) can be analyzed as follows. Let $m \geq 1$, $P_1, P_2, \ldots, P_m \in VR$, and $x_i = \epsilon_R P_i \in R$, so that $A = [P_1, P_2, \ldots, P_m]$ is the typical element of $\mathbb{H}(VR)$. We have

$$\begin{aligned}
\epsilon_{VR}[P_1, P_2, \ldots, P_m] &= (\epsilon_{VR}[P_1])(\epsilon_{VR}[P_2]) \cdots (\epsilon_{VR}[P_m]) \\
&= P_1P_2\cdots P_m\,, \\
(V\epsilon_R)[P_1, P_2, \ldots, P_m] &= [\epsilon_R P_1][\epsilon_R P_2] \cdots [\epsilon_R P_m] \\
&= [x_1, x_2, \ldots, x_m]\,, \quad\text{and} \\
\widehat{u}[P_1, P_2, \ldots, P_m] &= \textstyle\sum_i \big([P_1][P_2] \cdots [P_m](-i)\big)\, u(P_i) \\
&= \textstyle\sum_i \big(x_1x_2\cdots x_m(-i)\big)\, u(P_i)\,.
\end{aligned}$$

Hence (Z–) is equivalent to

$$u(P_1P_2\cdots P_m) \;=\; u[x_1, x_2, \ldots, x_m] \;+\; \textstyle\sum_i \big(x_1x_2\cdots x_m(-i)\big)\, u(P_i)\,, \qquad \text{(Z–)}$$

for all $P_1, P_2, \ldots, P_m \in VR$.

With $m = 1$, (Z–) reads $u(P) = u[x] + u(P)$, where $x = \epsilon_R P$; since ϵ_R is surjective, this implies

$$u[x] \;=\; 0 \qquad \text{(Z1)}$$

for all $x \in R$.

For all $n \geq 2$ and $x_1, x_2, \ldots, x_n \in R$, (Z–) implies

$$u[x_1, x_2, \ldots, x_n] \;=\; \textstyle\sum_{2\leq i\leq n} \overrightarrow{x_i}\, u[\overleftarrow{x_i}, x_i], \tag{Y}$$

where $\overleftarrow{x_i} = x_1 \cdots x_{i-1}$, $\overleftarrow{x_i} = 1$ if $i = 1$, $\overrightarrow{x_i} = x_{i+1} \cdots x_n$, and $\overrightarrow{x_i} = 1$ if $i = n$. Indeed (Y) holds, trivially, if $n = 2$. Also, $\epsilon_R [x_1, x_2, \ldots, x_n] = x_1 x_2 \cdots x_n$. If Y holds for $n \geq 2$, then (Z–) and (Z1) imply

$$\begin{aligned}
u[x_1, \ldots, x_n, x_{n+1}] &= u([x_1, \ldots, x_n][x_{n+1}]) \\
&= u[x_1 \cdots x_n, x_{n+1}] + (x_1 \cdots x_n)\, u[x_{n+1}] + x_{n+1}\, u[x_1, \ldots, x_n] \\
&= u[x_1 \cdots x_n, x_{n+1}] + x_{n+1} \left(\textstyle\sum_{2\leq i\leq n} \overrightarrow{x_i}\, u[\overleftarrow{x_i}, x_i]\right) \\
&= \textstyle\sum_{2\leq i\leq n+1} y_i\, u[\overleftarrow{x_i}, x_i],
\end{aligned}$$

where $y_i = x_{i+1} \cdots x_{n+1} = \overrightarrow{x_i}\, x_{n+1}$.

Moreover, (Y) implies

$$\begin{aligned}
u[x,y,z] &= u[xy,z] + z\,u[x,y] \quad \text{and} \\
u[z,y,x] &= u[zy,x] + x\,u[z,y] = u[x,yz] + x\,u[y,z],
\end{aligned}$$

for all $x, y, z \in R$. Since $[x,y,z] = [z,y,x]$ in $\mathbb{H}(R)$, it follows that

$$u[x,yz] + x\,u[y,z] = u[xy,z] + z\,u[x,y] \tag{Z2}$$

for all $x, y, z \in R$, so that on R.

Let $\underline{u}\colon (x,y) \longmapsto u[x,y]$ be the mapping of $R \times R$ into G induced by the weak 2-cocycle u. It follows from (Z1) and (Y) that $\underline{u}$ completely determines all values of u. Moreover, $\underline{u}$ is a symmetric 2-cocycle, by (Z2):

$$x\,\underline{u}(y,z) - \underline{u}(xy,z) + \underline{u}(x,yz) - z\,\underline{u}(x,y) = 0 \quad \text{for all} \quad x,y,z \in R, \tag{Z2}$$

and is *balanced*:

$$(\underline{u}(x,y))^t = \underline{u}(x^t, y) = \underline{u}(x, y^t) \tag{Z3}$$

for all $x, y \in R$ and $t \in S$, since $[x,y]^t = [x^t, y] = [x, y^t]$ in $\mathbb{H}(R)$ and u is a morphism of multisets over S.

Under pointwise addition, balanced symmetric 2-cocycles constitute a subgroup $Z^2_{BC}(R,M)$ of $Z^2_C(R,M)$. The map $u \longmapsto \underline{u}$ is a canonical homomorphism of $Z^2_W(R,M)$ into $Z^2_C(R,M)$.

3. Theorem. The canonical homomorphism $\Theta\colon u \longmapsto \underline{u}$ is a natural isomorphism

$$Z^2_W(R,M) \;\cong\; Z^2_{BC}(R,M).$$

To prove this it remains to show that Θ is surjective. Let $v \in Z^2_{BC}(R,M)$ be a balanced symmetric 2-cocycle on R, so that $v(x,y) \in M_{(xy)^\vee}$ and $v(y,x) = v(x,y)$ for all $x, y \in R$, (Z2):

$$x\,v(y,z) - v(xy,z) + v(x,yz) - z\,v(x,y) = 0 \quad \text{for all} \quad x,y,z \in R, \tag{Z2}$$

holds for all $x, y, z \in R$, and (Z3):

$$(v(x,y))^t \;=\; v\,(x^t,\, y) \;=\; v\,(x,\, y^t) \tag{Z3}$$

holds for all $x, y \in R$ and $t \in S$. Let $u\,(x) = 0$ for all $x \in R$, so that (Z1) holds, and let

$$u\,(x_1,\, x_2, \ldots,\, x_n) \;=\; \textstyle\sum_{2 \le i \le n} \overrightarrow{x_i}\, v\,(\overleftarrow{x_i},\, x_i) \tag{Y–}$$

for all $n \geq 2$ and $x_1,\, x_2, \ldots,\, x_n \in R$. In particular $u\,(x, y) = v\,(x, y)$ for all $x, y \in R$.

We show that u has property (P):

$$u\,(x_{\sigma 1},\, x_{\sigma 2},\, \ldots,\, x_{\sigma n}) \;=\; u\,(x_1,\, x_2, \ldots,\, x_n) \tag{P}$$

for every permutation σ of $1, 2, \ldots, n$. Since every permutation σ of $1, 2, \ldots, n$ is a product of transpositions $(i\;\; i{+}1)$, it suffices to prove (P) when $\sigma = (i\;\; i{+}1)$. If $n = 2$, then (P) holds since $\underline{u}$ is symmetric: $\underline{u}\,(y, x) = \underline{u}\,(x, y)$. Let $n > 2$. If $i + 1 < n$, then (P) follows from the induction hypothesis. Now let $i = n - 1$. Then (Y–) and (Z2) yield

$$\begin{aligned}
&u\,(x_{\sigma 1},\, x_{\sigma 2},\, \ldots,\, x_{\sigma n}) \\
&= u\,(x_1,\, x_2, \ldots,\, x_{n-2},\, x_n,\, x_{n-1}) \\
&= \textstyle\sum_{2 \le i \le n-2} \overrightarrow{x_i}\, v\,(\overleftarrow{x_i},\, x_i) \\
&\quad + x_{n-1}\, v\,(\overleftarrow{x}_{n-1},\, x_n) \;+\; v\,(\overleftarrow{x}_{n-1}\, x_n,\, x_{n-1}) \\
&= \textstyle\sum_{2 \le i \le n-2} \overrightarrow{x_i}\, v\,(\overleftarrow{x_i},\, x_i) \\
&\quad + x_n\, v\,(\overleftarrow{x}_{n-1},\, x_{n-1}) \;+\; v\,(\overleftarrow{x}_{n-1}\, x_{n-1},\, x_n) \\
&= \textstyle\sum_{2 \le i \le n} \overrightarrow{x_i}\, v\,(\overleftarrow{x_i},\, x_i) \\
&= u\,(x_1,\, x_2, \ldots,\, x_n)\,.
\end{aligned}$$

It follows from (P) that $u\,(x_1,\, x_2, \ldots,\, x_n)$ depends only on $[x_1 x_2 \cdots x_n] \in \mathbb{F}(R)$, so that u is well defined on $\mathbb{F}(R)$ by

$$u\,[x_1 x_2 \cdots x_n] \;=\; u\,(x_1,\, x_2, \ldots,\, x_n)\,.$$

Now let $1 \le j < k \le n$, $t \in S$,

$$\begin{aligned}
y_1,\, y_2, \ldots,\, y_n \;&=\; x_1,\, \ldots,\, x_j^t,\, \ldots,\, x_k,\, \ldots,\, x_n\,, \quad \text{and} \\
z_1,\, z_2, \ldots,\, z_n \;&=\; x_1,\, \ldots,\, x_j,\, \ldots,\, x_k^t,\, \ldots,\, x_n\,.
\end{aligned}$$

We show that

$$u\,(y_1,\, y_2, \ldots,\, y_n) \;=\; u\,(z_1,\, z_2, \ldots,\, z_n)\,.$$

Since $u\,(x_1,\, x_2, \ldots,\, x_n)$ depends only on $[x_1 x_2 \cdots x_n] \in \mathbb{F}(R)$ we may assume that $j = 1$ and $k = 2$. Let $a = x_1$ and $b = x_2$, so that

$$\begin{aligned}
y_1,\, y_2, \ldots,\, y_n \;&=\; a^t,\, b,\, x_3,\, \ldots,\, x_n\,, \quad \text{and} \\
z_1,\, z_2, \ldots,\, z_n \;&=\; a,\, b^t,\, x_3,\, \ldots,\, x_n\,.
\end{aligned}$$

Then

$$\overleftarrow{y}_1 = 1, \quad \overleftarrow{y}_2 = a^t, \quad \overleftarrow{y_i} = a^t\, b\, x_3\, \cdots\, x_{i-1} \ \text{ if } \ i > 2,$$
$$\overrightarrow{y}_1 = b\, \overrightarrow{x}_2, \quad \overrightarrow{y_i} = \overrightarrow{x_i} \ \text{ if } \ i \geq 2,$$
$$\overleftarrow{z}_1 = 1, \quad \overleftarrow{z}_2 = a, \quad \overleftarrow{z_i} = a\, b^t\, x_3\, \cdots\, x_{i-1} \ \text{ if } \ i > 2,$$
$$\overrightarrow{z}_1 = b^t\, \overrightarrow{x}_2, \quad \overrightarrow{z_i} = \overrightarrow{x_i} \ \text{ if } \ i \geq 2.$$

Hence

$$\begin{aligned} & u\,(y_1,\, y_2, \ldots,\, y_n) \ = \ \textstyle\sum_{2 \leq i \leq n} \overrightarrow{y_i}\, v\,(\overleftarrow{y_i},\, y_i) \\ & = \ \overrightarrow{x}_2\, v\,(a^t,\, b) \ + \ \textstyle\sum_{2 < i \leq n} \overrightarrow{x_i}\, v\,(a^t\, b\, x_3\, \cdots\, x_{i-1},\, x_i) \ \text{ and} \\ & u\,(z_1,\, z_2, \ldots,\, z_n) \ = \ \textstyle\sum_{2 \leq i \leq n} \overrightarrow{z_i}\, v\,(\overleftarrow{z_i},\, z_i) \\ & = \ \overrightarrow{x}_2\, v\,(a,\, b^t) \ + \ \textstyle\sum_{2 < i \leq n} \overrightarrow{x_i}\, v\,(a\, b^t\, x_3\, \cdots\, x_{i-1},\, x_i)\,. \end{aligned}$$

Since $a^t\, b = a\, b^t$ in R and $v\,(a^t,\, b) = v\,(a,\, b^t)$ by (Z3), this yields

$$u\,(y_1,\, y_2, \ldots,\, y_n) \ = \ u\,(z_1,\, z_2, \ldots,\, z_n)\,.$$

Therefore $[y_1 y_2 \cdots y_n] \approx [z_1 z_2 \cdots z_n]$ in $\mathbb{F}(R)$ implies $u\,(y_1,\, y_2, \ldots,\, y_n) = u\,(z_1,\, z_2, \ldots,\, z_n)$; and $u\,[x_1,\, x_2, \ldots,\, x_n]$ is well defined by

$$u\,[x_1,\, x_2, \ldots,\, x_n] \ = \ u\,(x_1,\, x_2, \ldots,\, x_n) \ = \ \textstyle\sum_{2 \leq i \leq n} \overrightarrow{x_i}\, v\,(\overleftarrow{x_i},\, x_i)\,,$$

for all $n \geq 2$ and $x_1,\, x_2, \ldots,\, x_n \in R$.

Let $P = [x_1,\, x_2, \ldots,\, x_m]$ and $Q = [y_1,\, y_2, \ldots,\, y_n] \in VR$, where $x_i,\, y_j \in R$ for all i, j. Let

$$z_1,\, z_2, \ldots,\, z_{m+n} \ = \ x_1,\, x_2, \ldots,\, x_m,\, y_1,\, y_2, \ldots,\, y_n\,,$$

so that $PQ = [z_1,\, z_2, \ldots,\, z_{m+n}]$. Let

$$x \ = \ x_1 x_2 \cdots x_m = \epsilon_R P \quad \text{ and} y \ = \ y_1 y_2 \cdots y_n = \epsilon_R Q\,.$$

We have

$$\overleftarrow{z_i} = \overleftarrow{x_i} \ \text{ if } \ i \leq m+1, \quad \overleftarrow{z_i} = x\, \overleftarrow{y}_{i-m} \ \text{ if } \ i \geq m+1,$$
$$\overrightarrow{z_i} = \overrightarrow{x_i}\, y \ \text{ if } \ i \leq m, \quad \overrightarrow{z_i} = \overrightarrow{y}_{i-m} \ \text{ if } \ i \geq m\,.$$

Hence (Z2) yields

$$\begin{aligned}
u(PQ) &= \textstyle\sum_{2\le i\le m+n} \overrightarrow{z}_i\, v\,(\overleftarrow{z}_i,\, z_i) \\
&= \textstyle\sum_{2\le i\le m} \overrightarrow{x}_i\, y\, v\,(\overleftarrow{x}_i,\, x_i) \;+\; \sum_{m+1\le i\le m+n} \overrightarrow{y}_{i-m}\, v\,(x\, \overleftarrow{y}_{i-m},\, y_{i-m}) \\
&= y\, u(P) \;+\; \textstyle\sum_{1\le j\le n} \overrightarrow{y}_j\, v\,(x\, \overleftarrow{y}_j,\, y_j) \\
&= y\, u(P) \;+\; \overrightarrow{y}_1\, v\,(x,\, y_1) \;+\; \textstyle\sum_{2\le j\le n} \overrightarrow{y}_j\, v\,(x\, \overleftarrow{y}_j,\, y_j) \\
&= y\, u(P) \;+\; \overrightarrow{y}_1\, v\,(x,\, y_1) \\
&\quad +\; \textstyle\sum_{2\le j\le n} \overrightarrow{y}_j\, \big(x\, v\,(\overleftarrow{y}_j,\, y_j) + v\,(x,\, \overleftarrow{y}_j\, y_j) - y_j\, v\,(x,\, \overleftarrow{y}_j)\big) \\
&= y\, u(P) \;+\; x\, u(Q) \;+\; \overrightarrow{y}_1\, v\,(x,\, y_1) \\
&\quad +\; \textstyle\sum_{2\le j\le n} \big(\, \overrightarrow{y}_j\, v\,(x,\, \overleftarrow{y}_{j+1}) - \overrightarrow{y}_{j-1}\, v\,(x,\, \overleftarrow{y}_j)\big) \\
&= y\, u(P) \;+\; x\, u(Q) \;+\; u\,[x, y]\,,
\end{aligned}$$

since the sum

$$\textstyle\sum_{2\le j\le n} \big(\, \overrightarrow{y}_j\, v\,(x,\, \overleftarrow{y}_{j+1}) - \overrightarrow{y}_{j-1}\, v\,(x,\, \overleftarrow{y}_j)\big)$$

collapses to $v\,(x, y) - \overrightarrow{y}_1\, v\,(x,\, y_1)$.

The equality

$$u(PQ) \;=\; y\, u(P) \;+\; x\, u(Q) \;+\; u\,[x, y] \tag{ZT}$$

is the particular case of (Z–):

$$u\,(P_1 P_2 \cdots P_m) \;=\; u\,[x_1,\, x_2, \ldots,\, x_m] \;+\; \textstyle\sum_i \big(x_1 x_2 \cdots x_m(-i)\big)\, u(P_i) \tag{Z–}$$

when $m = 2$. The general case then follows by induction. First, for all $x_1,\, x_2, \ldots,\, x_m,\, y \in R$, (ZT) and (Z1) yield, with $P = [x_1,\, x_2, \ldots,\, x_m]$ and $Q = [y]$:

$$\begin{aligned}
&u\,[x_1,\, x_2, \ldots,\, x_m,\, y] \;=\; u\,(PQ) \\
&= \; y\, u(P) \;+\; x\, u(Q) \;+\; u\,[x, y] \\
&= \; y\, u\,[x_1,\, x_2, \ldots,\, x_m] \;+\; u\,[x_1 x_2 \cdots x_m,\, y]\,.
\end{aligned} \tag{ZT–}$$

Now let $m \ge 2$, $P_1,\, P_2, \ldots,\, P_{m+1} \in VR$, and $x_i = \epsilon_R P_i \in R$. Then $P_1 P_2 \cdots P_{m+1} = PQ$, where $P = P_1 P_2 \cdots P_m$ and $Q = P_{m+1}$, and the induction hypothesis yields, with $x = \epsilon_R P = x_1 x_2 \cdots x_m$ and $y = \epsilon_R Q = x_{m+1}$:

$$\begin{aligned}
&u\,(P_1 P_2 \cdots P_{m+1}) \;=\; u\,(PQ) \;=\; y\, u(P) \;+\; x\, u(Q) \;+\; u\,[x, y] \\
&= \; y\, u\,[x_1,\, x_2, \ldots,\, x_m] \;+\; \textstyle\sum_{i\le m} \big(\, y\, x_1 x_2 \cdots x_m(-i)\, u(P_i)\big) \\
&\quad +\; x\, u(Q) \;+\; u\,[x_1 x_2 \cdots x_m,\, y] \\
&= \; \textstyle\sum_{i\le m+1} \big(\, x_1 x_2 \cdots x_{m+1}(-i)\, u(P_i)\big) \;+\; u\,[x_1,\, x_2, \ldots,\, x_{m+1}]\,,
\end{aligned}$$

by (ZT–).

Thus u has property (Z–), and u is a weak 2-cocycle. We saw that $u\,(x, y) = v\,(x, y)$ for all $x, y \in R$, so that $\underline{u} = v$. This completes the proof.

Unfortunately, while $\widehat{u}$, ϵ_{VR}, and $V\epsilon_R$ are additive, there is no evidence that u is; as a morphism of multisets over S from VR to G, u is not required to be additive. Hence Theorem 3.3 does not readily extend to $Z^2_Q(R, M)$. These accursed additions are ruining everything.

4. Coboundaries. André-Quillen 2-coboundaries are, in particular, weak 2-cocycles. We note that $u = \delta^Q v$ for some $v \in C^1_Q(R, M)$ implies $\underline{u} = \delta^C v \in cB^2(R, M)$.

Indeed, $\delta v = \widehat{v} - (v \circ \epsilon_R)$. For all $x, y \in R$,

$$\begin{aligned} \widehat{v}\,[x, y] &= \widehat{v}\,([x][y]) = x\,\widehat{v}[y] + y\,\widehat{v}[x] = x\,v(y) + y\,v(x)\,, \quad \text{and} \\ \epsilon_R\,[x, y] &= \epsilon_R\,([x][y]) = (\epsilon_R\,[x])(\epsilon_R\,[y]) = y\,. \end{aligned}$$

Hence

$$(da^Q v)[x, y] = x\,v(y) + y\,v(x) - v\,(xy) = (\delta^C v)(x, y)\,.$$

4 Coextensions.

In this section, S is a commutative monoid.

1. Definition. A *coextension* of a commutative multiring R over S by a multi-R-module M over S is a commutative multimonoid E over S together with a surjection $\pi \colon E \longrightarrow R$ and, for each $a \in S$, a simply transitive action $\cdot$ of M_a on E_a, such that

$$\begin{aligned} (\pi e)^\vee &= e^\vee\,, \\ (\pi e)^t &= \pi\,(e^t)\,, \quad \text{and} \\ \pi\,(ef) &= (\pi e)(\pi f)\,, \end{aligned}$$

for all $e, f \in E$ and $t \in S$, so that π is a morphism of commutative multimonoids over S; there is a morphism $\mu \colon R \longrightarrow E$ of multisets over S (such that $(\mu r)^\vee = r^\vee$ and $\mu\,(r^t) = (\mu r)^t$) which splits π:

$$\pi \circ \mu = 1_R\,;$$

and

$$\begin{aligned} (x \cdot e)^t &= x^t \cdot e^t \ \text{ whenever } \ x^\vee = e^\vee\,, \\ \pi\,(x \cdot e) &= \pi e \ \text{ whenever } \ x^\vee = e^\vee\,, \\ 0 \cdot e &= e\,, \\ x \cdot (y \cdot e) &= (x + y) \cdot e \ \text{ whenever } \ x^\vee = y^\vee = e^\vee\,, \quad \text{and} \\ (x \cdot e)\,f &= ((\pi f)\,x) \cdot ef \ \text{ whenever } \ x^\vee = e^\vee\,, \end{aligned}$$

for all $e, f \in E$, $t \in S$, and $x, y \in M$, so that each action of M_a on E_a is a group action of the additive abelian group M_a on the set E_a. Since M_a acts on E_a we also have

$$(x \cdot e)^\vee \;=\; x^\vee \;=\; e^\vee \;\text{ whenever }\; x \in M,\; e \in E, \;\text{ and }\; x^\vee = e^\vee\,.$$

Commutativity and the condition $(x \cdot e)\, f = ((\pi f)\, x) \cdot ef$ imply

$$e\,(x \cdot f) \;=\; (x \cdot f)\, e \;=\; ((\pi e)\, x) \cdot ef\,.$$

This makes E a kind of group coextension of the commutative monoid R by M. (It could be further seasoned with additions, but this would require additive factor sets.)

Two coextensions E and F of R by M with surjections π and ρ to R are *equivalent* if and only if there exists an isomorphism $\theta\colon E \longrightarrow F$ of commutative multimonoids over S (an *equivalence* of coextensions) such that

$$\begin{aligned} \rho\,\theta\, e \;&=\; \pi\, e \;\text{ for all }\; e \in E \;\text{ and} \\ \theta(x \cdot e) \;&=\; x \cdot \theta e \;\text{ whenever }\; x \in M,\; e \in E, \;\text{ and }\; x^\vee = e^\vee\,. \end{aligned}$$

2. Construction. Coextensions of R by M are constructed up to equivalence by Schreier's method.

Let E be a coextension of R by M. For each $r \in R$ let $p_r = \mu r \in E$, where $\mu\colon R \longrightarrow E$ is the morphism of multisets over S that splits π. Then $\pi p_r = r$, $p_r^\vee = r^\vee$, and $p_{r^t} = p_r^t$. Since M_a acts simply and transitively on E_a, every element e of E can be written in the form

$$e \;=\; x \cdot p_r$$

for some unique $x \in M$ and $r = \pi e \in R$ such that $x^\vee = r^\vee = e^\vee$. In particular, for every $r, s \in R$, $\pi\,(p_r\, p_s) = rs$ and

$$p_r\, p_s \;=\; u\,(r, s) \cdot p_{rs}$$

for some unique $u\,(r, s) \in M$ such that $u\,(r, s)^\vee = (rs)^\vee = r^\vee\, s^\vee$.

The factor set u inherits three properties from the multiplication on R: for all $r, s \in R$,

$$u\,(s, r) \;=\; u\,(r, s)$$

since the multiplication on R is commutative; for all $q, r, s \in R$,

$$\begin{aligned} (p_q\, p_r)\, p_s \;&=\; (u\,(q, r) \cdot p_{qr})\, r_s \;=\; s\, u\,(q, r) \cdot (p_{qr})\, r_s) \\ &=\; s\, u\,(q, r) \cdot (u\,(qr, s) \cdot p_{(qr)s}\,, \\ p_q\, (p_r\, p_s) \;&=\; p_q\, (u\,(r, s) \cdot p_{rs}) \;=\; q\, u\,(r, s) \cdot (p_q\, p_{rs}) \\ &=\; q\, u\,(r, s) \cdot (u\,(q, rs) \cdot p_{q(rs)})\,, \end{aligned}$$

since the multiplication on R is associative; hence

$$s\, u\,(q, r) \;+\; u\,(qr, s) \;=\; q\, u\,(r, s) \;+\; u\,(q, rs)\,.$$

Finally, for all $r, s \in R$ and $t \in S$,

$$\begin{aligned} p_r^t\, p_s \;&=\; p_{r^t}\, p_s \;=\; u\,(r^t,\, s) \cdot p_{r^t s}\,, \\ (p_r\, p_s)^t \;&=\; (u\,(r,s) \cdot p_{rs})^t \;=\; u\,(r,s)^t \cdot p_{(rs)^t}\,, \quad \text{and} \\ p_r\, p_s^t \;&=\; p_r\, p_{s^t} \;=\; u\,(r,\, s^t) \cdot p_{r\,s^t}\,; \end{aligned}$$

since $p_r^t\, p_s = (p_r\, p_s)^t = p_r\, p_s^t$ in E and $r^t\, s = (rs)^t = r\, s^t$ in R, it follows that

$$u\,(x^t,\, y) \;=\; (u(x,y))^t \;=\; u\,(x,\, y^t)\,.$$

Thus u is a balanced symmetric 2-cocycle, $u \in Z^2_{BC}(R, M)$.

The multiplication on E is completely determined by R, M, and the factor set u:

$$\begin{aligned} (x \cdot p_r)(y \cdot p_s) \;&=\; sx \cdot \big(p_r\,(y \cdot p_s)\big) \;=\; sx \cdot \big(ry \cdot (p_r\, p_s)\big) \\ &=\; sx \cdot \big(ry \cdot (u\,(r,s) \cdot p_{rs})\big) \\ &=\; \big(\, sx + u\,(r,s) + ry\,\big) \cdot p_{rs}\,. \end{aligned}$$

This provides a bijection $\theta \colon x \cdot p_r \longmapsto (x,r)$ of E onto the product $\mathrm{E}(R,M)$ of R and M in the category of multisets over S:

$$\mathrm{E}(R,M) \;=\; \{\, (x,r) \mid x \in M,\; r \in R,\; x^\vee = r^\vee \,\}\,,$$

and suggests a multiplication

$$(x,r)(y,s) \;=\; (sx + u\,(r,s) + ry\,,\; rs)$$

on $\mathrm{E}(R,M)$.

Equip $\mathrm{E}(R,M)$ with this multiplication, projection $(x,r)^\vee = x^\vee = r^\vee$ to S, action $(x,r)^t = (x^t,\, r^t$ of S, surjection $\pi \colon (x,r) \longmapsto r$ to R, and, for each $a \in S$, the action of M_a on

$$\mathrm{E}_a \;=\; \{\, (x,r) \in \mathrm{E}(R,M) \mid x^\vee = r^\vee = a \,\}$$

defined by

$$x \cdot (y,r) \;=\; (x + y,\, r)\,.$$

This makes $\mathrm{E}(R,M)$ into what Theorem 4.3 below asserts is a coextension $\mathrm{E}(R,M,u)$ of R by M:

3. Theorem. If u is a balanced symmetric 2-cocycle on R with values in M, then $\mathrm{E}(R,M,u)$ is a coextension $\mathrm{E}(R,M,u)$ of R by M, with factor set u. Moreover every coextension of R by M with factor set u is equivalent to $\mathrm{E}(R,M,u)$.

The proof us straightforward. All the axioms of coextensions are satisfied. Indeed $\mathrm{E}(R,M,u)$ is a commutative multimonoid over S:

$$\begin{aligned}
\big((x,r)^t\big)^\vee &= (x^t,\, r^t)^\vee \;=\; (r^t)^\vee \;=\; (r^\vee)\,t \;=\; \big((x,r)^\vee\big)\,t\,;\\
(x,r)^1 &= (x,r)\,;\\
\big((x,r)^t\big)^u &= \big((x^t)^u,\, (r^t)^u\big) \;=\; (x^{tu},\, r^{tu}) \;=\; (x,r)^{tu}\,;\\
\big((x,r)(y,s)\big)^t &= (sx + u\,(r,s) + ry,\; rs)^t\\
&= ((sx)^t + u\,(r,s)^t + (ry)^t,\; (rs)^t)\\
&= (s\,x^t + u\,(r^t,s) + r^t\,y,\; r^t\,s)\\
&= (x^t,\, r^t)(s,y) \;=\; (x,r)^t(y,s)\\
&= (s^t\,x + u\,(r,s^t) + r\,y^t,\; r\,s^t)\\
&= (x,r)(y^t,\, s^t) \;=\; (x,r)(y,s)^t\,,
\end{aligned}$$

since u is balanced;

$$\big((x,r)(y,s)\big)^\vee \;=\; (sx+u\,(r,s)+ry,\, rs)^\vee \;=\; (rs)^\vee \;=\; r^\vee\,s^\vee \;=\; (x,r)^\vee\,(y,s)^\vee\,;$$

the multiplicationon $\mathrm{E}(R,M,u)$ is commutative:

$$(y,s)(x,r) \;=\; (ry+u\,(s,r)+sx\,,\; sr) \;=\; (sx+u\,(r,s)+ry\,,\; rs) \;=\; (x,r)(y,s)\,,$$

since M_a is abelian, the multiplication on R is commutative, and u is symmetric; and the multiplicationon $\mathrm{E}(R,M,u)$ is associative:

$$\begin{aligned}
\big((x,q)(y,r)\big)(z,s) &= (rx + u\,(q,r) + qy\,,\; qr)(z,s)\\
&= (srx + s\,u(q,r) + sqy + u\,(qr,s) + qrz,\; qrs)\\
&= (rsx + u\,(q,rs) + qsy + qu\,(r,s) + qrz,\; qrs)\\
&= (x,q)(sy + u\,(r,s) + rz,\; rs)\\
&= (x,q)\big((y,r)(z,s)\big)\,,
\end{aligned}$$

since u is a 2-cocycle.

Moreover,

$$\begin{aligned}
(\pi\,(x,r))^\vee &= r^\vee \;=\; (x,r)^\vee\,;\\
(\pi\,(x,r))^t &= r^t \;=\; \pi\,((x,r)^t)\,; \quad \text{and}\\
\pi\,\big((x,r)(y,s)\big) &= rs \;=\; \pi\,(x,r)\,\pi\,(y,s)\,;
\end{aligned}$$

the map $\mu\colon r \longmapsto (0,r)$ is a morphism of multisets over S: $(\mu r)^\vee = (0,r)^\vee = r^\vee$ and $\mu\,(r^t) = (0,\, r^t) = (\mu r)^t$, and splits π: $\pi\,(\mu r) = r$;

$$(x \cdot (y,r))^t \;=\; (x+y,\; r)^t \;=\; (x^t + y^t,\; r^t) \;=\; x^t \cdot (y,r)^t$$

whenever $x^\vee = r^\vee = (y,r)^\vee$;

$$\begin{aligned}
\pi\,(x \cdot (y,r)) &= \pi\,(x+y,\; r) \;=\; \pi\,(y,r)\,;\\
0 \cdot (x,r) &= (0+x,\; r) \;=\; (x,r)\,;\\
x \cdot \big(y \cdot (z,r)\big) &= (x+y+z,\; r) \;=\; (x+y) \cdot (z,r)
\end{aligned}$$

whenever $x^\vee = y^\vee = z^\vee = r^\vee = (z,r)^\vee$; and

$$\begin{aligned}\big(x \cdot (y,r)\big)(z,s) &= (x+y,\ r)(z,s)\\ &= (s\,(x+y) + u\,(r,s) + rz,\ rs)\\ &= (sx + sy + u\,(r,s) + rz,\ rs)\\ &= sx \cdot \big((y,r)(z,s)\big)\end{aligned}$$

whenever $x^\vee = (y,r)^\vee = y^\vee$. Thus $\mathrm{E}(R,M,u)$ is a coextension of R by M. Since $(0,r)(0,s) = (u\,(r,s),\ rs)$, its factor set is u.

Finally let E be a coextension of R by M with surjection ρ to R and with $p_r = \mu r$ for every $r \in R$, where $\mu\colon R \longrightarrow E$ splits ρ, so that $p_r\,p_s = u\,(r,s) \cdot p_{rs}$ for all $r,s \in R$ and u is a factor set of E. We saw that

$$\theta\colon x \cdot p_r \longmapsto (x,r)$$

is a bijection of E onto $\mathrm{E}(R,M,u)$, and that $\theta\,(ef) = (\theta\,e)(\theta\,f)$ for all $e,f \in E$. Moreover

$$\begin{aligned}(\theta\,(x \cdot p_r))^\vee &= r^\vee \;=\; (x \cdot p_r)^\vee\,,\\ \theta\,\big((x \cdot p_r))^t\big) &= \theta\,(x^t \cdot (p_r)^t) \;=\; \theta\,(x^t \cdot p_{r^t})\\ &= (x^t,\ r^t) \;=\; (x,r)^t \;=\; \big(\theta\,(x \cdot p_r)\big)^t\,, \quad\text{and}\\ \theta\big(\,x \cdot (y \cdot p_r)\big) &= \theta\,\big((x+y) \cdot p_r)\big)\\ &= (x+y,\ r) \;=\; x \cdot (y,r) \;=\; x \cdot \theta\,(y \cdot p_r)\,.\end{aligned}$$

Thus θ is an equivalence of coextensions.

4. Proposition. Two coextensions of R by M with factor sets u and v are equivalent if and only if there exists $w(r) \in M_{r^\vee}$ for every $r \in R$ such that

$$\begin{aligned}w(r^t) &= (w(r))^t \quad\text{and}\\ v\,(r,s) - u\,(r,s) &= s\,w(r) - w\,(rs) + r\,w(s)\end{aligned}$$

for all $r,s \in R$ and $t \in S$.

By Theorem 4.3, we may assume in the proof that the two coextensions are $\mathrm{E}(R,M,u)$ and $\mathrm{E}(R,M,v)$.

Let $\theta\colon \mathrm{E}(R,M,u) \longrightarrow \mathrm{E}(R,M,v)$ be an equivalence of coextensions. For each $r \in R$ we have $\pi\,\theta\,(0,r) = \pi\,(0,r) = r$ and

$$\theta\,(0,r) \;=\; (w(r),r)$$

for some unique $w(r) \in M$ such that $w(r)^\vee = r^\vee$. Then

$$\begin{aligned}(w(r^t),\ r^t) &= \theta\,(0,\ r^t) \;=\; \theta\,\big((0,r)^t\big)\\ &= \big(\,\theta\,(0,r)\big)^t \;=\; (w(r)^t,\ r^t)\,,\end{aligned}$$

and

$$w(r^t) \;=\; (w(r))^t\,.$$

Since θ preserves products and $\theta\,(x \cdot e) \;=\; x \cdot \theta e$, we also have

$$\begin{aligned}(u\,(r,s) + w\,(rs),\; rs) \;&=\; u\,(r,s) \cdot \theta\,(0, rs) \;=\; \theta\,\big(\,u(r,s) \cdot (0, rs)\big)\\ &=\; \theta\,(u\,(r,s),\; rs) \;=\; \theta\,\big((0,r)(0,s)\big)\\ &=\; (w(r), r)(w(s), s) \;=\; (r\,w(s) + v\,(r,s) + s\,w(r),\; rs)\,,\end{aligned}$$

whence

$$v\,(r,s) - u\,(r,s) \;=\; s\,w(r) - w\,(rs) + r\,w(s)\,.$$

Conversely, assume that $v\,(r,s) - u\,(r,s) = s\,w(r) - w\,(rs) + r\,w(s)$. Define $\theta\colon \mathrm{E}(R,M,u) \longrightarrow \mathrm{E}(R,M,v)$ by

$$\theta\,(x,r) \;=\; (x + w(r),\; r)\,.$$

Then θ is bijective. Moreover,

$$\begin{aligned}\pi\,\theta\,(x,r) \;&=\; r \;=\; \pi\,(x,r)\,,\\ (\theta\,(x,r))^{\vee} \;&=\; r^{\vee} \;=\; (x,r)^{\vee}\,,\\ \theta\,\big((x,r)^t\big) \;&=\; \theta\,(x^t,\, r^t)\\ &=\; (x^t + w\,(r^t),\; r^t) \;=\; (x^t + w(r)^t,\; r^t)\\ &=\; \big(\,\theta\,(x,r)\big)^t\,,\\ \theta\big(\,x \cdot (y,r)\big) \;&=\; \theta\,(x+y,\; r)\\ &=\; (x + y + w(r),\; r) \;=\; x \cdot (y + w(r),\; r)\\ &=\; x \cdot \theta\,(y,r)\,, \quad \text{and}\\ \theta\,\big((x,r)(y,s)\big) \;&=\; \theta\,(sx + u(r,s) + ry,\; rs)\\ &=\; (sx + u(r,s) + ry + w(rs),\; rs)\\ &=\; (sx + s\,w(r) + v(r,s) + ry + r\,w(s),\; rs)\\ &=\; \theta\,(x,r)\,\theta\,(y,s)\,.\end{aligned}$$

Hence θ is an equivalence of coextensions.

Prop. 4.4 suggests that a *balanced symmetric 1-cochain* on R with values in M is a 1-cochain u (that assigns $u(r) \in M_a$ to $r \in R_a$) such that

$$u(r)^t \;=\; u\,(r^t)$$

for all $r \in R$ and $t \in T$. (It is balanced only to match the balanced symmetric 2-cocycles.) Under pointwise addition, balanced symmetric 1-cochains constitute a subgroup $C^1_{BC}(R,M)$ of $C^1_C(R,M)$.

The coboundary of a balanced symmetric 1-cochain u is a balanced symmetric 2-cocycle:

$$\begin{aligned}(\delta u)(r^t,\, s) \;&=\; s\,u\,(r^t) \;-\; u\,(r^t\,s) \;+\; r^t\,u(s)\\ &=\; s\,(u(r))^t \;-\; u\,(r\,s^t) \;+\; r^t\,u(s)\\ &=\; s^t\,u(r) \;-\; u\,(r\,s^t) \;+\; r^t\,u(s) \;=\; (\delta u)(r,\, s^t)\,.\end{aligned}$$

Under pointwise addition, these *balanced symmetric 2-coboundaries* constitute a subgroup $B^2_{BC}(R,M)$ of $Z^1_{BC}(R,M)$.

It follows from Prop. 4.4 that the quotient group $Z^2_{BC}(R,M)\,/\,B^2_{BC}(R,M)$ classifies coextensions of R by M.