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RESEARCH ARTICLE

The inheritance of symmetry conditions in commutative semigroup cohomology

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Abstract
Cochains in commutative semigroup cohomology satisfy symmetry conditions that have only been ascertained in dimensions $n \leq 4$. This article studies how symmetry conditions are passed from one dimension to the next, and proposes two conjectures about the appropriate conditions for dimensions 5 and 6.

Keywords Commutative semigroup cohomology · Symmetry · Symmetric mapping · Symmetry property · Symmetry condition · Inherited symmetry property.

Mathematics Subject Classification 20M14, 20M50, 18G35

1 Introduction

1. When S is a commutative semigroup, Green's relation $\mathcal{H}$ is a congruence and the Schützenberger groups of S arrange themselves into an abelian group valued functor $\mathcal{G} = (G, \gamma)$ on the quotient $T = S/\mathcal{H}$: thus $\mathcal{G}$ assigns an abelian group G_a to every $a \in T$ and a homomorphism $\gamma_{a,t} \colon G_a \longrightarrow G_{at}$ to every $a \in S$ and every $t \in S^1$, such that $\gamma_{a,1}$ is the identity on G_a and $\gamma_{at,u} \circ \gamma_{a,t} = \gamma_{a,tu}$ for every a, t, u. We call S a group coextension of T by $\mathcal{G}$. As in a group extension, the multiplication on S is completely determined by T, $\mathcal{G}$, and a factor set s, which assigns $s(a, b) \in G_{ab}$ to each $a, b \in T$ so that (in the additive notation)

$$s(b, a) = s(a, b) \text{ (for commutativity) and}$$
$$\gamma_{bc,a}\, s(b, c) + s(a, bc) = s(ab, c) + \gamma_{ab,c}\, s(a, b) \text{ (for associativity),}$$

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for all $a, b, c \in T$. Moreover, the projection $S \longrightarrow T$ splits if and only if there exists $u(a) \in G_a$ to each $a \in T$ such that

$$s(a,b) = \gamma_{b,a}\, u(b) - u(ab) + \gamma_{a,b}\, u(a), \text{ for all } a, b \in T;$$

and two group coextensions of T by $\mathcal{G}$ are equivalent if and only if their factor sets differ by some $\gamma_{b,a}\, u(b) - u(ab) + \gamma_{a,b}\, u(a)$ [5].

Thus begins the cohomology of commutative semigroups.

One way to define commutative semigroup cohomology was devised by Beck [2] (see also [1]) for any monadic (tripleable) category. (Calvo-Cervera and Cegarra [3], Kurdiani and Pirashvili [10] also have good alternative definitions.) Beck cohomology has excellent properties but, in the case of commutative semigroups, suffers from overweight cochains, so that $H^2(S, \mathcal{G})$ is not computable even if S is finite, and how $H^2(S, \mathcal{G})$ relates to group coextensions is not immediately obvious [7].

It seems more natural to have a 1-cochain u assign $u\,(a) \in G_a$ to each $a \in S$, each 2-cochain s assign $s\,(a, b) \in G_{ab}$ to each $a, b \in S$, with coboundaries

$$(\delta u)(a,b) = \gamma_{b,a}\, u(b) - u(ab) + \gamma_{a,b}\, u(a) \text{ and}$$
$$(\delta s)(a,b,c) = \gamma_{bc,a}\, s\,(b,c) - s\,(ab,c) + s\,(a,bc) - \gamma_{ab,c}\, s\,(a,b)\,;$$

commutativity also requires that $s\,(b, a) = s\,(a, b)$, for all $a, b \in S$. This approach yields an equivalent cohomology in dimensions $n \leq 4$, in which n-cochains are functions of n variables [7], but in dimensions $n \geq 5$ runs into difficulties that are caused by the subject of this article.

When a 2-cochain s satisfies the commutativity property (S2): $s\,(b, a) = s\,(a, b)$, its coboundary $t = \delta s$ is readily seen to have the following properties, collectively denoted by (S3):

$$t\,(c,b,a) = -t\,(a,b,c) \text{ and} \tag{S3a}$$

$$t\,(a,b,c) = t\,(b,a,c) - t\,(b,c,a)\,, \tag{S3b}$$

for all a, b, c. We call (S2), (S3a), and (S3b) symmetry conditions; conditions (S3a) and (S3b) can then be used to define symmetric 3-cochains [6], [7]. In turn the coboundaries of symmetric 3-cochains have symmetry properties, collectively denoted by (S4):

$$u\,(d,c,b,a) = -u\,(a,b,c,d)\,, \tag{S4a}$$

$$u\,(a,b,c,d) = u\,(b,a,c,d) - u\,(b,c,a,d) + u\,(b,c,d,a)\,, \text{ and} \tag{S4b}$$

$$u\,(a,b,b,a) = 0\,, \tag{S4c}$$

that are used to define symmetric 4-cochains [8].

We call *inheritance* the process by which symmetry conditions propagate from one dimension to the next higher.

2. To continue this process beyond dimension 4, one must find the appropriate symmetry conditions for dimension 5, and one must prove that they yield the right group $H^5(S, \mathcal{G})$, either by a homeric calculation in the style of [6] and [8], or by inventing a smarter method of proof. (Kurdiani and Pirashvili [10] write that they have solved the author's 'cocycle problem' but do not provide symmetry conditions.)

This article finds what seems to be appropriate conditions but does not prove that they are. We have a miraculous method that yields *all* the symmetry properties that symmetric n-cochains bequeath their coboundaries. Thus we found that the symmetry properties that are inherited from (S2) are precisely all the consequences of (S3); the symmetry properties that are inherited from (S3) are precisely all the consequences of (S4); and the symmetry properties that are inherited from (S4) are precisely all the consequences of

$$u\,(e, d, c, b, a) = u\,(a, b, c, d, e), \tag{S5a}$$

$$\begin{aligned} u\,(a, b, c, d, e) = u\,(b, a, c, d, e) - u\,(b, c, a, d, e) \\ + u\,(b, c, d, a, e) - u\,(b, c, d, e, a)\,. \end{aligned} \tag{S5b}$$

Hence we conjecture that (S5a) and (S5b) are the appropriate symmetry conditions for dimension 5. In turn, the symmetry properties that are inherited from (S5a) and (S5b) are precisely all the consequences of

$$u\,(a, b, c, d, e, f) = u\,(f, e, d, c, b, a)\,, \tag{S6a}$$

$$u\,(a, b, c, d, e, f) = u\,(b, a, c, d, e, f) - u\,(b, c, a, d, e, f) \tag{S6b}$$

$$+ u\,(b, c, d, a, e, f) - u\,(b, c, d, e, a, f) - u\,(b, c, d, e, f, a)$$

$$\begin{aligned} u\,(a, b, c, d, e, f) = -u\,(c, b, a, d, e, f) + u\,(c, b, d, a, e, f) \\ - u\,(c, b, d, e, a, f) + u\,(c, b, d, e, f, a) \\ - u\,(c, d, b, a, e, f) + u\,(c, d, b, e, a, f) \\ - u\,(c, d, b, e, f, a) - u\,(c, d, e, b, a, f) \\ + u\,(c, d, e, b, f, a) - u\,(c, d, e, f, b, a)\,, \quad \text{and} \end{aligned} \tag{S6c}$$

$$u\,(a, b, a, b, a, b) = 0\,. \tag{S6d}$$

Again we conjecture that (S6a), (S6b), (S6c), and (S6d) constitute an appropriate definition of symmetric 6-cochains.

Due to extensive use of computers, these results are to a large extent experimental and will require independent confirmation.

3. Section 2 recalls the basic definitions and properties of symmetric cochains [7], symmetric mappings (as defined in [9]), and generalizes the symmetric chains in [9].

Section 3 constructs a marvelous universal coboundary $\mathfrak{d}$ that maps to the coboundary of every symmetric 4-cochain, and thereby transmits its own symmetry properties to it. Conversely, $\mathfrak{d}$ is itself the coboundary of a symmetric 4-cochain, but its variables are distinct by construction; hence every symmetry property inherited from (S4) is a symmetry property of $\mathfrak{d}$, as long as it does not require equalities between variables.

Condition (S4c) above shows that equalities between variables can result in additional symmetry properties. Section 4 adjusts the arguments in Sect. 3 to find all the symmetry properties that are inherited from (S4) and require exactly one equality between variables.

Section 5 provides, without details (which have been sent to the referee), further adjustments that suit every possible pattern of equalities between variables, together with the elementary linear algebra that, in each case, finds all the symmetry properties inherited from (S4) and shows them to be consequences of (S5).

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2 Symmetry

1. Cochains on a commutative semigroup S take their values in an abelian group valued functor $\mathcal{G} = (G, \gamma)$ on S (actually, on the Leech category $\mathcal{H}(S)$ [11]), which assigns to each $a \in S$ an abelian group G_a, and to each $a \in S$ and $t \in S^1$ a homomorphism $\gamma_{a,t} \colon G_a \longrightarrow G_{at}$, conveniently denoted by $\gamma_{a,t}\, g = g^t$, in such a way that $\gamma_{a,1}$ is the identity on G_a and $\gamma_{at,u} \circ \gamma_{a,t} = \gamma_{a,tu}$; equivalently, $g^1 = g$ and $(g^t)^u = g^{tu}$.

An n-cochain on S with values in $\mathcal{G}$ assigns to each $a_1, \ldots, a_n \in S$ an element $u\,(a_1, \ldots, a_n)$ of G_s, where $s = a_1 a_2 \cdots a_n$.

A 4-cochain u on S is symmetric if and only if it has the following properties, collectively denoted by (S4):

$$u\,(d, c, b, a) = -u\,(a, b, c, d)\,, \tag{S4a}$$

$$u\,(a, b, c, d) = u\,(b, a, c, d) - u\,(b, c, a, d) + u\,(b, c, d, a)\,, \quad \text{and} \tag{S4b}$$

$$u\,(a, b, b, a) = 0\,, \tag{S4c}$$

for all $a, b, c, d \in S$. These properties are numbered differently here than in [8] and [7], in order to show the analogy between (S3) and (S4). The original definition in [8] also included a fourth condition

$$u\,(a, b, c, d) - u\,(b, c, d, a) + u\,(c, d, a, b) - u\,(d, a, b, c) = 0,$$

which was shown in [9] to follow from (S4a) and (S4b).

Symmetric 2- and 3-cochains are defined similarly by (S2) and (S3) in [5], [7], and [6].

The coboundary of a 4-cochain u is the 5-cochain δu defined by:

$$\begin{aligned}(\delta u)(a, b, c, d, e) = u\,(b, c, d, e)^a - u\,(ab, c, d, e) + u\,(a, bc, d, e)\\ - u\,(a, b, cd, e) + u\,(a, b, c, de) - u\,(a, b, c, d)^e \ \in G_{abcde}\end{aligned}$$

for all $a, b, c, d, e \in S$.

2. Symmetry (in this article) can be defined more generally, following [7]. Given a set S, a symmetric set on S of order n is a subset X of the cartesian product

$S^n = S \times \ldots \times S$ such that

$$(x_1,\, x_2, \ldots,\, x_n) \in X \text{ implies } (x_{\sigma 1}\,,\, x_{\sigma 2}\,, \ldots,\, x_{\sigma n}) \in X$$

for every $x_1,\, x_2, \ldots,\, x_n \in S$ and every permutation σ of $1,\ 2,\ \ldots,\ n$. If $n \le 4$, then a mapping f of X into an abelian group G is symmetric of order n if and only if it satisfies (Sn), where (S2) is the condition

$$f(b,a) \;=\; f(a,b) \tag{S2}$$

for all $(a,b) \in X$; (S3) consists of

$$f\,(c,b,a) \;=\; -\; f\,(a,b,c) \text{ and} \tag{S3a}$$

$$f\,(a,b,c) \;=\; f\,(b,a,c) \;-\; f\,(b,c,a)\,, \tag{S3b}$$

for all $(a,b,c) \in X$; and (S4) consists of

$$f\,(d,c,b,a) \;=\; -\; f\,(a,b,c,d)\,, \tag{S4a}$$

$$f\,(a,b,c,d) \;=\; f\,(b,a,c,d) \;-\; f\,(b,c,a,d) \;+\; f\,(b,c,d,a)\,, \text{ and} \tag{S4b}$$

$$f\,(a,b,b,a) \;=\; 0\,, \tag{S4c}$$

for all $(a,b,c,d) \in S$ [7].

For example, if S is a commutative semigroup and $s \in S$, then

$$X_s \;=\; \{\,(a,b,c,d) \in S \times S \times S \times S \mid abcd = s\,\}$$

is a symmetric set of order 4, and a symmetric 4-cochain u on S consists of symmetric mappings $(a,b,c,d) \longmapsto u\,(a,b,c,d)\colon\ X_s \longrightarrow G_s$, one for each $s \in S$.

It is clear from (Sn) that some values of a symmetric mapping are determined by other values. Formally, a *basis* of X is a subset Y of X such that every mapping of Y into an abelian group G extends uniquely to a symmetric mapping of X into G (so that the values of a symmetric mapping on Y determine all its other values).

Theorem 2.1 *Every symmetric set X of order $n \le 4$ has a basis.*

Theorem 2.1 is proved by placing an arbitrary total order on the set S. An explicit basis, the standard basis of X, can then be constructed in the following fashion:

Lemma 2.2 *Let X be a symmetric subset of order $n = 1, 2, 3, 4$ on a totally ordered set S. Define Y as follows: if $n = 1$, then $Y = X$;*

if $n = 2$, then $Y = \{(a,b) \in X \mid a \le b\}$;

if $n = 3$, then $Y = \{(a,b,c) \in X \mid a \le b \text{ and } a < c\}$;

if $n = 4$, then Y is the set of all $(a,b,c,d) \in X$ such that either $a < b, c, d$, or $a \le b, c$ and $b < d$, or both.

Then Y is a basis of X: every mapping f of Y into an abelian group G extends uniquely to a symmetric mapping $\widehat{f}$ of X into G; moreover, every value of $\widehat{f}$ is a sum of values of f and opposites of values of f.

The reader is referred to [9] for the proof of Lemma 2.2, which is straightforward but rather long if $n = 4$.

3. The definition of symmetric chains in [7] extends readily to every symmetric set. Let X be a symmetric set of order $n \leq 4$ on a set S (perhaps S^n itself). A symmetric n-chain on X is an element of the abelian group $C(X)$ generated by all $\langle a_1, \dots, a_n \rangle$ such that $(a_1, \dots, a_n) \in X$, subject to all defining relations (Cn), where (C2) is the condition

$$\langle b, a \rangle = \langle a, b \rangle \tag{C2}$$

for all $(a, b) \in X$; (C3) consists of

$$\langle c, b, a \rangle = -\langle a, b, c \rangle \text{ and} \tag{C3a}$$

$$\langle a, b, c \rangle = \langle b, a, c \rangle - \langle b, c, a \rangle, \tag{C3b}$$

for all $(a, b, c) \in X$; and (C4) consists of

$$\langle d, c, b, a \rangle = -\langle a, b, c, d \rangle, \tag{C4a}$$

$$\langle a, b, c, d \rangle = \langle b, a, c, d \rangle - \langle b, c, a, d \rangle + \langle b, c, d, a \rangle, \text{ and} \tag{C4b}$$

$$\langle a, b, b, a \rangle = 0, \tag{C4c}$$

for all $(a, b, c, d) \in X$.

A symmetric n-chain on a commutative semigroup S is an element of the abelian group $C_n(S) = C(S^n)$.

In general, the group $C(X)$ comes with a mapping $\iota \colon X \longrightarrow C(X)$ which sends $(a_1, \dots, a_n)$ to $\langle a_1, \dots, a_n \rangle$. The defining relations of $C(X)$ show that ι is a symmetric mapping. In fact, ι is a universal symmetric mapping:

Proposition 2.3 *Every symmetric mapping of X extends uniquely (via ι) to a homomorphism of $C(X)$.*

Proof Let G be an abelian group and let $f \colon X \longrightarrow G$ be a symmetric mapping. Since f is symmetric, the values $f(a_1, \dots, a_n) \in G$ such that $(a_1, \dots, a_n) \in X$ satisfy all the defining relations of $C(X)$. Therefore there is a unique homomorphism $\varphi \colon C(X) \longrightarrow G$ such that $\varphi \langle a_1, \dots, a_n \rangle = f(a_1, \dots, a_n)$ for all $(a_1, \dots, a_n) \in X$, that is, such that $\varphi \circ \iota = f$. □

Combining Proposition 2.3 and Theorem 2.1 yields

Theorem 2.4 *If X is a symmetric set of order $n = 1, 2, 3, 4$, then $C(X)$ is a free abelian group. Moreover, if Y is a basis of X, then $\iota(Y)$ is a basis of $C(X)$, and ι is injective on Y (if $x, y \in Y$ and $\iota(x) = \iota(y)$, then $x = y$).*

Proof Every mapping of Y into an abelian group G extends uniquely to X and thence extends uniquely (via ι) to $C(X)$. □

The next result shows that $C(X)$ and ι are faithfully inherited by subsets. Let $X \subseteq X'$ be symmetric sets of order n on the same set S, with standard bases Y and Y' relative to the same total order on S.

If $a = (a_1, \ldots, a_n) \in X$, then a satisfies the inequalities that define Y in Lemma 2.2 if and only if it satisfies the same inequalities that define Y'. Hence $Y = Y' \cap X$. In particular, $Y \subseteq Y'$.

The restriction of $\iota' \colon X' \longrightarrow C(X')$ to X is a symmetric mapping of X into $C(X')$; by Proposition 2.3 there is a unique homomorphism $\kappa \colon C(X) \longrightarrow C(X')$ such that $\kappa \circ \iota = \iota'$; κ sends $\langle a_1, \ldots, a_n \rangle \in C(X)$ to $\langle a_1, \ldots, a_n \rangle \in C(X')$. Now $C(X)$ is generated by $\iota(Y)$, hence $\kappa(C(X))$ is generated by $\kappa(\iota(Y)) = \iota'(Y)$, which is a subset of the basis $\iota'(Y')$ of $C(X')$. Therefore $\kappa(C(X))$ is a free abelian group and $\kappa(\iota(Y))$ is a basis of $\kappa(C(X))$. Moreover, κ is injective on $\iota(Y)$, since $\kappa \circ \iota = \iota'$ is injective on Y'. Hence κ is an isomorphism of $C(X)$ onto $\kappa(C(X))$. In particular, κ is injective.

In what follows we treat the canonical injection κ as an inclusion homomorphism, so that $C(X) \subseteq C(X')$ and $\langle a_1, \ldots, a_n \rangle$ is the same in $C(X)$ and $C(X')$ when $(a_1, \ldots, a_n) \in X$. This amounts to defining $C(X)$ as a subgroup of $C(\mathbb{F}^n)$, with ι inherited from $C(\mathbb{F}^n)$. The above can then be stated as follows:

Proposition 2.5 *Let X and X' be symmetric sets of order $n \leq 4$ on the same set S. If $X \subseteq X'$, then $C(X) \subseteq C(X')$; the canonical mappings $\iota \colon X \longrightarrow C(X)$ and $\iota' \colon X' \longrightarrow C(X')$ agree on X; if Y and Y' are the standard bases of X and X' relative to the same total order on S, then $Y = Y' \cap X$.* □

4. Let $X \subseteq S^n$ be a symmetric set and f any mapping of X into an abelian group G. A *symmetry property* P of f at $(x_1, \ldots, x_n) \in X$ is an equality

$$\sum_{\sigma \in S_n} p_\sigma \, f(x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma n}) = 0,$$

between permuted values of f, with integer coefficients p_σ. More abstractly, a *symmetry property* of order n is a mapping $P \colon S_n \longrightarrow \mathbb{Z}$, $\sigma \longmapsto p_\sigma$; and P is a symmetry property of a mapping $f \colon X \longrightarrow G$ at $(x_1, \ldots, x_n)$, when

$$\sum_{\sigma \in S_n} p_\sigma \, f(x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma n}) = 0.$$

Thus an n-cochain u on a commutative semigroup S has property P at $(a_1, a_2, \ldots, a_n)$, where $a_1, a_2, \ldots, a_n \in S$, if and only if

$$\sum_{\sigma \in S_n} p_\sigma \, u(a_{\sigma 1}, a_{\sigma 2}, \ldots, a_{\sigma n}) = 0.$$

This definition allows the same symmetry property to hold for several mappings or cochains. Moreover, symmetry properties of order n constitute a free $\mathbb{Z}$-module $\mathbb{P}$.

To each symmetry property $P\colon S_n \longrightarrow \mathbb{Z}$, $\sigma \longmapsto p_\sigma$ corresponds a *symmetry condition* that states that $\sum_{\sigma\in S_n} p_\sigma\ f\,(x_{\sigma 1}, x_{\sigma 2}, \dots, x_{\sigma n}) = 0$ for *all* $(x_1, x_2, \dots, x_n) \in X$; if u is an n-cochain on a commutative semigroup S, that $\sum_{\sigma\in S_n} p_\sigma\ u\,(a_{\sigma 1}, a_{\sigma 2}, \dots, a_{\sigma n}) = 0$ for all $a_1, a_2, \dots, a_n \in S$. For example, the conditions (S4a), (S4b), and (S4c) that define symmetric mappings of order 4 are symmetry conditions of order 4.

A symmetry property P of order n is *inherited* from a set (S) of symmetry conditions of order $n-1$ if and only if it is a property of all coboundaries of $(n-1)$-cochains that satisfy (S).

Our precise, somewhat pedantic definitions of symmetry properties and conditions will be useful in the next sections when we determine all symmetry properties that are inherited from (S4).

3 The universal coboundary

1. The universal coboundary is constructed as follows. Let $\mathbb{F}$ be the free abelian group on a set $\{X_1, \dots, X_5\}$ of five distinct indeterminates. Let $\mathfrak{i}$ denote the canonical ι mapping of $\mathbb{F}^4$ into $C_4(\mathbb{F})$:

$$\mathfrak{i}(A, B, C, D) = \langle A, B, C, D\rangle.$$

The symmetric mapping $\mathfrak{i}$ is a symmetric 4-cochain on $\mathbb{F}$ with values in $C_4(\mathbb{F})$ (more precisely, in the constant abelian group valued functor on $\mathbb{F}$ that assigns $C_4(\mathbb{F})$ to every $A \in \mathbb{F}$.) The universal coboundary is $\mathfrak{d} = \delta\,\mathfrak{i}$.

By definition,

$$\begin{aligned}\mathfrak{d}(A, B, C, D, E) = \langle B, C, D, E\rangle - \langle AB, C, D, E\rangle + \langle A, BC, D, E\rangle \\ - \langle A, B, CD, E\rangle + \langle A, B, C, DE\rangle - \langle A, B, C, D\rangle,\end{aligned}$$

for all $A, B, C, D, E \in \mathbb{F}$. Properties (C4b) and (C4a) then yield the longer expansion which is used thereafter:

$$\begin{aligned}\mathfrak{d}(A, B, C, D, E) &= \langle B, C, D, E\rangle - \langle AB, C, D, E\rangle \\ &+ \langle BC, A, D, E\rangle - \langle BC, D, A, E\rangle + \langle BC, D, E, A\rangle \\ &+ \langle CD, E, B, A\rangle - \langle CD, B, E, A\rangle + \langle CD, B, A, E\rangle \\ &- \langle DE, C, B, A\rangle - \langle A, B, C, D\rangle.\end{aligned}$$

2. To prove the main property of $\mathfrak{d}$ we look at the permuted values

$$\mathfrak{d}^\sigma\,(X_1, X_2, \dots, X_v) = \mathfrak{d}\,(X_{\sigma 1}, X_{\sigma 2}, \dots, X_{\sigma 5})$$

of $\mathfrak{d}$ at $X_1, X_2, \dots, X_5$, one for every permutation σ of 1,2,...,5, and at the subgroup $\mathbb{D}$ of $C_4(\mathbb{F})$ defined below.

Let $T' = \{X_1, \dots, X_5\}$ and let $T'' = \{X_i X_j \mid 1 \le i < j \le 5\} \subseteq \mathbb{F}$. Totally order $\mathbb{F}$ so that

$$AB < X_1 < X_2 < \cdots < X_5$$

for all $AB \in T''$. Let Y be the standard basis of $\mathbb{F}^4$, which consists of all (A, B, C, D) such that $A, B, C, D \in \mathbb{F}$ and either $A < B, C, D$, or $A \le B, C$ and $B < D$, or both.

Let Q' be the set of all $(A, B, C, D) \in Y$ such that A, B, C, D are distinct elements of $X_1, X_2, \dots, X_5$; equivalently, either $A = X_1$, or $A = X_2$ and B, C, D is a permutation of X_3, X_4, X_5.

Let Q'' be the set of all $(AB, C, D, E) \in C_4(\mathbb{F})$ such that $AB \in T''$ and A, B, C, D, E are distinct elements of T' (i.e. constitute a permutation of $X_1, X_2, \dots, X_5$).

Let $Q = Q' \cup Q''$, and let $\mathbb{D}$ be the subgroup of $C_4(\mathbb{F})$ generated by $\mathfrak{i}(Q)$.

Lemma 3.1 *$\mathbb{D}$ is a free abelian group; $Q \subseteq Y$; $\mathfrak{i}(Q)$ is a basis of $\mathbb{D}$; and $\mathbb{D}$ contains all terms of every permuted value of $\mathfrak{d}$ at $X_1, X_2, \dots, X_5$.*

Proof The total order on $\mathbb{F}$ ensures that Q' and Q'' are contained in its standard basis Y of $\mathbb{F}^4$. Since $\mathfrak{i}(Y)$ is a basis of $C_4(\mathbb{F})$, it follows that $\mathbb{D}$ is a free abelian group and that $\mathfrak{i}(Q)$ is a basis of $\mathbb{D}$.

Next we show that $\mathbb{D}$ contains all $\langle A, B, C, D\rangle$ such that A, B, C, D are distinct elements of $X_1, X_2, \dots, X_5$. Indeed let X be the subset of $\mathbb{F}^4$ that consists of all (A, B, C, D) such that A, B, C, D are distinct elements of $X_1, X_2, \dots, X_5$. Then X is a symmetric set. By Proposition 2.5, its standard basis is $Y \cap X$. But $Y \cap X = Q'$. Hence $\mathfrak{i}(Q')$ is a basis of $C(X)$ and $C(X) \subseteq \mathbb{D}$. In particular, $\langle A, B, C, D\rangle \in \mathbb{D}$ for every $(A, B, C, D) \in X$. (This can also be proved directly, using (C4a) and (C4b).)

Let σ be a permutation of 1,2,...,5 and let $A_i = X_{\sigma i}$ for all i, so that $\mathfrak{d}^\sigma(X_1, \dots, X_5) = \mathfrak{d}(A_1, \dots, A_5)$. In the expansion of $\mathfrak{d}(A_1, \dots, A_5)$, all terms $\langle A_i A_j, A_k, A_l, A_m\rangle$ belong to $\mathfrak{i}(Q'') \subseteq \mathbb{D}$. By the above, the terms $\langle A_2, A_3, A_4, A_5\rangle$ and $-\langle A_1, A_2, A_3, A_4\rangle$ also belong to $\mathbb{D}$. Thus, all terms in the expansion of $\mathfrak{d}^\sigma(X_1, X_2, \dots, X_5)$ belong to $\mathbb{D}$. □

3. The next result maps $\mathfrak{d}$ onto the coboundary of any symmetric 4-cochain.

Lemma 3.2 *Let u be a symmetric 4-cochain on a commutative semigroup S with values in an abelian group valued functor $\mathcal{G} = (G, \gamma)$. Let $a_1, \dots, a_5 \in S$ and let $s = a_1 a_2 \cdots a_5$. There exists a homomorphism $\varphi\colon \mathbb{D} \longrightarrow G_s$ such that*

$$\varphi\left(\mathfrak{d}(X_{\sigma 1}, X_{\sigma 2}, \dots, X_{\sigma 5})\right) = (\delta u)(a_{\sigma 1}, a_{\sigma 2}, \dots, a_{\sigma 5})$$

for every permutation σ of 1,2,...,5.

Proof Let u be a 4-cochain on S and let $a_1, a_2, \dots, a_5 \in S$ and $s = a_1 a_2 \cdots a_5$. Since $X_1, X_2, \dots, X_5$ are distinct there exists a mapping $f\colon T' \longrightarrow S$ such that $f(X_i) = a_i$ for all i.

In the free commutative semigroup $\mathbb{F}$, $X_i X_j = X_k X_\ell$ implies $a_i a_j = a_k a_\ell$: indeed, $X_i X_j = X_k X_\ell$ implies either $X_i = X_k$ and $X_j = X_\ell$, or $X_i = X_\ell$ and

$X_j = X_k$; then either $a_i = a_k$ and $a_j = a_\ell$, or $a_i = a_\ell$ and $a_j = a_k$; in either case $a_i\,a_j = a_k\,a_\ell$. Hence a mapping $g\colon T'' \longrightarrow S$ is well defined by $g\,(X_i\,X_j) = a_i\,a_j = f(X_i)f(X_j)$ for all $X_i\,X_j \in T''$.

By Theorem 2.4, $\mathfrak{i}$ is injective on Q: if $(A_1,\ A_2,\ A_3,\ A_4), (B_1,\ B_2,\ B_3,\ B_4) \in Q$ and $\langle A_1, A_2, A_3, A_4\rangle = \langle B_1, B_2, B_3, B_4\rangle$, then

$$(A_1,\ A_2,\ A_3,\ A_4)\ =\ (B_1,\ B_2,\ B_3,\ B_4)$$

and $A_i = B_i$ for all i. Therefore f and g induce a mapping h of $\mathfrak{i}(Q)$ into G_s: if $\langle A, B, C, D\rangle \in Q'$, then A, B, C, D are all but one (say, X_i) of $X_1,\ \dots,\ X_5$, arranged in a different order, and

$$h\,\langle A, B, C, D\rangle\ =\ u\left((f(A),\ f(B),\ f(C),\ f(D)\right)^{f(X_i)} \in G_s\,;$$

if $\langle A, B, C, D, E\rangle \in Q''$, then A, B, C, D, E is a permutation of $X_1,\ \dots,\ X_5$, $g\,(AB) = f(A)\,f(B)$, and

$$h\,\langle AB, C, D, E\rangle\ =\ u\left(f(A)\,f(B),\ f(C), f(D), f(E)\right) \in G_s\,.$$

Since $\mathbb{D}$ is free on $\mathfrak{i}(Q)$, h extends to a homomorphism $\varphi\colon \mathbb{D} \longrightarrow G_s$.

Now let σ be a permutation of 1,2,...,5. Let $X_{\sigma 1},\ X_{\sigma 2},\ X_{\sigma 3}, X_{\sigma 4}, X_{\sigma 5} = A, B, C, D, E$ and $a_{\sigma 1},\ a_{\sigma 2},\ a_{\sigma 3},\ a_{\sigma 4},\ a_{\sigma 5} = a, b, c, d, e$, so that $f\,(A) = a$, $f\,(B) = b$, $f\,(C) = c$, $f\,(D) = d$, $f\,(E) = e$, and $s = abcde$. We have

$$\begin{aligned}
\varphi\,\langle B, C, D, E\rangle\ &=\ h\,\langle B, C, D, E\rangle\ =\ u\,(b, c, d, e)^a \in G_s\,,\\
\varphi\,\langle AB, C, D, E\rangle\ &=\ h\,\langle AB, C, D, E\rangle\ =\ u\,(ab,\ c, d, e)\,,\\
\varphi\,\langle BC, A, D, E\rangle\ &=\ h\,\langle BC, A, D, E\rangle\ =\ u\,(bc,\ a, d, e)\,,\\
&\ \dots\\
\varphi\,\langle DECBA\rangle\ &=\ h\,\langle DECBA\rangle\ =\ u\,(de,\ c, b, a)\,,\quad \text{and}\\
\varphi\,\langle ABCD\rangle\ &=\ h\,\langle ABCD\rangle\ =\ u\,(a, b, c, d)^e \in G_s\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&\varphi\left(\mathfrak{d}\,(A, B, C, D, E)\right)\\
&\quad = \varphi\,\langle B, C, D, E\rangle\ -\ \varphi\,\langle AB, C, D, E\rangle\\
&\qquad +\ \varphi\,\langle BC, A, D, E\rangle\ -\ \varphi\,\langle BC, D, A, E\rangle\ +\ \varphi\,\langle BCDEA\rangle\\
&\qquad +\ \varphi\,\langle CD, E, B, A\rangle\ -\ \varphi\,\langle CD, B, E, A\rangle\ +\ \varphi\,\langle CDBAE\rangle\\
&\qquad -\ \varphi\,\langle DE, C, B, A\rangle\ -\ \varphi\,\langle A, B, C, D\rangle
\end{aligned}$$

$$
\begin{aligned}
&= u(b,c,d,e)^a - u(ab,\ c,d,e) \\
&\quad + u(bc,\ a,d,e) - u(bc,\ d,a,e) + u(bc,\ d,e,a) \\
&\quad + u(cd,\ e,b,a) - u(cd,\ b,e,a) + u(cd,\ b,a,e) \\
&\quad - u(de,\ c,b,a) - u(a,b,c,d)^e \\
&= (\delta u)(a,b,c,d,e)\,,
\end{aligned}
$$

by (S4b) and (S4a). Thus $\varphi\left(\mathfrak{d}\,(X_{\sigma 1}, \ldots, X_{\sigma 5})\right) = (\delta u)(a_{\sigma 1}, \ldots, a_{\sigma 5})$. □

4. We can now prove:

Theorem 3.3 *A symmetry property that does not require an equality between variables is inherited from (S4) if and only if it is a property of $\mathfrak{d}$ at $X_1, \ldots, X_5$.*

Proof If a symmetry property P is inherited from (S4), then P holds at every $a_1, a_2, \ldots, a_5$ for the coboundary of every symmetric 4-cochain on any commutative semigroup (without requiring any two of $a_1, a_2, \ldots, a_5$ to be equal), and in particular holds for $\delta\,\mathfrak{i} = \mathfrak{d}$ at $X_1, X_2, \ldots, X_5$. (On the other hand, a symmetry property that requires an equality between its variables cannot apply to $\mathfrak{d}$ at $X_1, X_2, \ldots, X_5$, since $X_1, X_2, \ldots, X_5$ are distinct.)

Conversely, let $P : \sigma \longmapsto p_\sigma$ hold for $\mathfrak{d}$ at $X_1, X_2, \ldots, X_5$, so that

$$\sum_{\sigma \in S_5} p_\sigma\, \mathfrak{d}\,(X_{\sigma 1}, X_{\sigma 2}, \ldots, X_{\sigma 5}) = 0\,.$$

Let u be a symmetric 4-cochain on a commutative semigroup S with values in some abelian group valued functor $\mathcal{G} = (G, \gamma)$. Let $a_1, a_2, \ldots, a_5 \in S$ and let $s = a_1 a_2 \cdots a_5$. By Lemma 3.2 there is a homomorphism $\varphi : \mathbb{D} \longrightarrow G_s$ such that

$$\varphi\left(\mathfrak{d}\,(X_{\sigma 1}, X_{\sigma 2}, \ldots, X_{\sigma 5})\right) = (\delta u)(a_{\sigma 1}, a_{\sigma 2}, \ldots, a_{\sigma 5})$$

for every permutation σ of 1,2,...,5. Then

$$\sum_{\sigma \in S_5} p_\sigma\, (\delta u)(a_{\sigma 1}, \ldots, a_{\sigma 5}) = \varphi\Big(\sum_{\sigma \in S_5} p_\sigma\, \mathfrak{d}\,(X_{\sigma 1}, \ldots, X_{\sigma 5})\Big) = 0$$

and P holds for δu at $a_1, a_2, \ldots, a_5$. □

4 One equality between variables

Theorem 3.3 specifically excludes symmetry properties that require one or more equalities between variables, like the symmetry condition (S4c) which requires two equalities between its four variables. If some of $a_1, a_2, \ldots, a_n$ are equal, then there are fewer permutations of $a_1, a_2, \ldots, a_n$ and fewer permuted values $f(a_{\sigma 1}, a_{\sigma 2}, \ldots, a_{\sigma n})$; it is natural to expect different relationships between permuted values, and different symmetry properties.

In this Section we consider symmetry properties that require exactly one equality between variables. In this case a simple tweak of the variables of $\mathfrak{d}$ yields a result similar to Theorem 3.3. It should be clear from the proof that symmetry properties that require more than one equality between variables can be handled with similar adjustments.

1. Let $a_1, a_2, \ldots, a_5$ be the five variables of 5-coboundaries and symmetry properties of order 5. Let the required equality be $a_p = a_q$, where $1 \leq p < q \leq 5$. A symmetry property P that requires $a_p = a_q$ is a property of f at $a_1, a_2, \ldots, a_n$ if and only if $\sum_{\sigma \in S_5} p_\sigma \ f(a_{\sigma 1}, ..., a_{\sigma 5}) = 0$ and $a_p = a_q$.

Using X_1, X_2, X_3, X_4 make a copy of the equality $a_p = a_q$: define $Y_1, \ldots, Y_5$ so that $\{Y_1, \ldots, Y_5\} = \{X_1, X_2, X_3, X_4\}$ and $Y_p = Y_q = X_1$; since $X_1, \ldots, X_5$ are distinct, $Y_p = Y_q$ is the only equality between $Y_1, \ldots, Y_5$. For example, if the required equality is $a_2 = a_4$, then X_2, X_1, X_3, X_1, X_4 serve as $Y_1, \ldots, Y_5$.

Construct an abelian group $\mathbb{D}$ that is different from the similar group in Sect. 3, using different sets T', T'', Q', Q'', and Q.

Let $T' = \{Y_1, \ldots, Y_5\}$ and let $T'' = \{Y_i Y_j \mid 1 \leq i < j \leq 5\} \subseteq \mathbb{F}$. Totally order $\mathbb{F}$ so that

$$AB < X_1 < X_2 < X_3 < X_4$$

for all $AB \in T''$. Let Y be the standard basis of $\mathbb{F}^4$, which as before consists of all (A, B, C, D) such that $A, B, C, D \in \mathbb{F}$ and either $A < B, C, D$, or $A \leq B, C$ and $B < D$, or both.

Let Q' be the set of all $(A, B, C, D) \in Y$ such that A, B, C, D is a partial permutation of $Y_1, Y_2, \ldots, Y_5$; hence there is at most one equality between A, B, C, D.

Let Q'' be the set of all (AB, C, D, E) such that $AB \in T''$ and A, B, C, D, E is a permutation of $Y_1, Y_2, \ldots, Y_5$.

Let $Q = Q' \cup Q''$, and let $\mathbb{D}$ be the subgroup of $C_4(\mathbb{F})$ generated by $\mathfrak{i}(Q)$. This new group $\mathbb{D}$ and its cousin in Sect. 3 have very similar properties.

Lemma 4.1 *If $A, B, C, D \in T'$ are distinct except perhaps for one equality, then $\langle A, B, C, D \rangle \in \mathbb{D}$.*

Proof Let X be the subset of $\mathbb{F}^4$ that consists of all (A, B, C, D) such that A, B, C, D are distinct elements of T' except perhaps for one equality. Then X is a symmetric set. By Proposition 2.5, its standard basis is $Y \cap X$. But $Q' = Y \cap X$. Hence $\mathfrak{i}(Q')$ is a basis of $C(X)$ and $C(X) \subseteq \mathbb{D}$. In particular, $\langle A, B, C, D \rangle \in \mathbb{D}$ for every $(A, B, C, D) \in X$. □

Lemma 4.2 *$\mathbb{D}$ is a free abelian group; $Q \subseteq Y$; $\mathfrak{i}(Q)$ is a basis of $\mathbb{D}$; and $\mathbb{D}$ contains all terms of every permuted value of $\mathfrak{d}$ at $Y_1, Y_2, \ldots, Y_5$.*

Like Lemma 3.1 this follows from the choice of the total order on $\mathbb{F}$ and Lemma 4.1. □

Lemma 4.3 *Let u be a symmetric 4-cochain on a commutative semigroup S with values in an abelian group valued functor $\mathcal{G} = (G, \gamma)$, and let $1 \leq p < q \leq 5$. Let*

$a_1, \ldots, a_5 \in S$ and let $s = a_1 a_2 \cdots a_5$. If $a_p = a_q$, then there exists a homomorphism $\varphi\colon \mathbb{D} \longrightarrow G_s$ such that

$$\varphi\left(\mathfrak{d}\,(Y_{\sigma 1}\,,\, Y_{\sigma 2}\,, \ldots,\, Y_{\sigma 5})\right) \;=\; (\delta u)(a_{\sigma 1}\,,\, a_{\sigma 2}\,, \ldots,\, a_{\sigma 5})$$

for every permutation σ of 1,2,...,5.

This is proved like Lemma 3.2: since $Y_p = Y_q$ and $Y_1, \ldots, Y_5$ are otherwise distinct, there is a mapping $f : T' \longrightarrow S$ such that $f\,(Y_i) = a_i$ for all i; as before, f induces φ.

Theorem 4.4 *A symmetry property that requires the equality $a_p = a_q$ between its variables, and requires no other equality, is inherited from (S4) if and only if it is a property of $\mathfrak{d}$ at $Y_1, \ldots, Y_5$.*

This is proved like Theorem 3.3. A symmetry property that is a property of the coboundary of every symmetric 4-cochain on any commutative semigroup S at every $a_1, \ldots, a_5 \in S$ such that $a_p = a_q$, without requiring any other equality between $a_1, \ldots, a_5$, must in particular hold for $\delta\,\mathfrak{i} = \mathfrak{d}$ at $Y_1, \ldots, Y_5$. The converse follows from Lemma 4.3.

To find all the symmetry properties inherited from (S4) that require one equalities between variables, Theorem 4.4 would have us find the symmetry properties of $\mathfrak{d}$ at $Y_1, \ldots, Y_5$ ten times over, one for every choice of $1 \le p < q \le 5$. Fortunately, only the simplest choice needs to be considered, $p = 1$ and $q = 2$, in which case $Y_1, \ldots, Y_5 = X_1, X_1, X_2, X_3, X_4$: indeed $Y_1, \ldots, Y_5$ is a permutation of X_1, X_1, X_2, X_3, X_4; hence $\mathfrak{d}\,(Y_1, \ldots, Y_5)$ and $\mathfrak{d}\,(X_1, X_1, X_2, X_3, X_4)$ have the same permuted values (though in a different order); the same relationships exist between these permuted values; and $\mathfrak{d}$ has the same set of symmetry properties at $Y_1, \ldots, Y_5$ as at X_1, X_1, X_2, X_3, X_4. Hence Theorem 4.4 yields

Theorem 4.5 *A symmetry property that requires exactly one equalities between its variables is inherited from (S4) if and only if it is a property of $\mathfrak{d}$ at X_1, X_1, X_2, X_3, X_4.*

5 Results

1. Based on Theorems 3.3 and 4.5, a symmetry property of order $n + 1$ that requires certain equalities between its variables (perhaps none) is inherited from (Sn) if and only if it is a symmetry property of $\mathfrak{d}$ at some suitable sequence (Y) made from $X_1, X_2, \ldots, X_{n+1}$. The permuted values of $\mathfrak{d}$ at (Y) all belong to some free $\mathbb{Z}$-module $\mathbb{D}$ with a basis $\mathfrak{i}(Q)$. Hence linear relationships between permuted values are found from their coordinate matrix in the basis $\mathfrak{i}(Q)$.

Let $\mathbb{V}$ be the subgroup of $\mathbb{D}$ generated by the permuted values. It turns our that some of the permuted values, the *basic permuted values*, constitute a basis of $\mathbb{V}$. Gauss–Jordan reduction of the coordinate matrix reveals these basic permuted values. Writing the remaining permuted values as linear combinations of the basic permuted values provides *basic relationships*, and *basic symmetry properties*. Every linear relationship between permuted values is a consequence of these basic relationships;

in other words, every symmetry property of $\mathfrak{d}$ at the sequence (Y) is a consequence of the basic symmetry properties.

Gauss–Jordan reduction also reveals if a symmetry property follows from other symmetry properties, or from symmetry conditions. In the $\mathbb{Z}$-module $\mathbb{P}$ of symmetry properties of order n, if a symmetry property is a linear combination of other symmetry properties, then it is a consequence (as a property) of these other symmetry properties. A set (S) of symmetry condition of order n consists of symmetry properties (one for each symmetry condition and each permutation of the n variables); a symmetry property which is a linear combination of these symmetry properties is a consequence of (S). Linear combinations of symmetry properties are revealed by Gauss–Jordan reduction of their coordinate matrix in a basis of $\mathbb{P}$. Thus it can be established that a symmetry property is a consequence of (S).

2. We apply these methods to the inheritance of (S4).

We begin with symmetry properties that require no equalities between variables, which, by Theorem 3.3, are the symmetry properties of $\mathfrak{d}$ at $X_1, \ldots, X_5$. Arranging the coordinates of the permuted values of $\mathfrak{d}(X_1, \ldots, X_5)$ in rows, rather than columns, eases displays and outputs. The resulting matrix has 120 rows, one for each permuted value, and 90 columns, one for each element of Q. Computer column reduction found its rank to be 24 and yielded 96 basic symmetry properties, including

$$\begin{aligned}\mathfrak{d}(X_5, X_4, X_3, X_2, X_1) &= \mathfrak{d}(X_1, X_2, X_3, X_4, X_5) \text{ and}\\ \mathfrak{d}(X_2, X_1, X_3, X_4, X_5) &= \mathfrak{d}(X_1, X_2, X_3, X_4, X_5) - \mathfrak{d}(X_1, X_3, X_2, X_4, X_5)\\ &\quad + \mathfrak{d}(X_1, X_3, X_4, X_2, X_5) - \mathfrak{d}(X_1, X_3, X_4, X_5, X_2).\end{aligned}$$

Every coboundary u of a symmetric 4-cochain satisfies the corresponding symmetry conditions, which constitute a set (S5):

$$u(e, d, c, b, a) = u(a, b, c, d, e) \text{ and} \tag{S5a}$$

$$\begin{aligned}u(b, a, c, d, e) &= u(a, b, c, d, e) - u(a, c, b, d, e)\\ &\quad + u(a, c, d, b, e) - u(a, c, d, e, b).\end{aligned} \tag{S5b}$$

Then another Gauss–Jordan reduction established that all basic symmetry properties (that require no equality) are consequences of (S5).

Between its five variables a, b, c, d, e a symmetry property of order 5 can require either no equality, or one equality (e.g. $a = b$), or two equalities (e.g. $a = b$ and $c = d$), or three equalities (e.g. $a = b = c$), or four equalities (e.g. $a = b = c, d = e$), or six equalities (e.g. $a = b = c = d$), or ten equalities ($a = b = c = d = e$). These cases must be considered separately.

By Theorem 4.5 the symmetry properties that require exactly one equality are the symmetry properties of $\mathfrak{d}$ at X_1, X_1, X_2, X_3, X_4. The coordinate matrix now has 60 rows, 48 columns, rank 12, and provides 48 basic symmetry properties, all of which are consequences of (S5).

The remaining five cases are treated by the same method using similar theorems, with the same results. Thus we obtain (or rather, the author's computer obtained):

Theorem 5.1 *A symmetry property of order 5 is inherited from (S4) if and only if it is a consequence of (S5).*

3. The same method and similar theorems yield:

Theorem 5.2 *A symmetry property of order 3 is inherited from (S2) if and only if it is a consequence of (S3).*

Theorem 5.3 *A symmetry property of order 4 is inherited from (S3) if and only if it is a consequence of (S4).*

In view of this it seems reasonable to conjecture that (S5) is an appropriate set of symmetry conditions for 5-cochains, meaning that it yields a symmetric cohomology group H^5 that is naturally isomorphic to the fifth Beck cohomology group. But this is easier stated than proved.

However, (S5) lends itself to standard bases for symmetric sets of order 5, so that the process can be continued, resulting in:

Theorem 5.4 *A symmetry property of order 6 is inherited from (S5) if and only if it is a consequence of (S6), which consists of four conditions:*

$$u\,(a, b, a, b, a, b) \;=\; 0\,, \tag{S6a}$$

$$u\,(a, b, c, d, e, f) \;=\; u\,(f, e, d, c, b, a)\,, \tag{S6b}$$

$$\begin{aligned} u\,(a, b, c, d, e, f) \;=\; & u\,(b, a, c, d, e, f) \;-\; u\,(b, c, a, d, e, f) \\ & +\; u\,(b, c, d, a, e, f) \;-\; u\,(b, c, d, e, a, f) \\ & -\; u\,(b, c, d, e, f, a)\,, \quad \textit{and} \end{aligned} \tag{S6c}$$

$$\begin{aligned} u\,(a, b, c, d, e, f) = & -u\,(c, b, a, d, e, f) + u\,(c, b, d, a, e, f) \\ & - u\,(c, b, d, e, a, f) + u\,(c, b, d, e, f, a) \\ & - u\,(c, d, b, a, e, f) + u\,(c, d, b, e, a, f) \\ & - u\,(c, d, b, e, f, a) - u\,(c, d, e, b, a, f) \\ & + u\,(c, d, e, b, f, a) - u\,(c, d, e, f, b, a)\,. \end{aligned} \tag{S6d}$$

It seems reasonable to conjecture that (S6) is an appropriate definition of symmetric 6-cochains.s

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