COMMUNICATIONS IN ALGEBRA, 23(10), 3573-3587 (1995)

COMMUTATIVE SEMIGROUP COHOMOLOGY

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Abstract. Triple cohomology for commutative semigroups is described in concrete terms and related to commutative group coextensions.

Introduction.

Semigroup cohomology is justified by coextensions. Every finite commutative semigroup S is a group coextension (called a coset extension in [4]) of the group-free semigroup $T = S/\mathcal{H}$ by the abelian group valued Schützenberger functor G of S. Conversely, every coextension S of T by G has $S/\mathcal{H} \cong T$. Commutative semigroup cohomology classifies these coextensions; hence it also classifies finite commutative semigroups S which have a given group-free quotient $S/\mathcal{H}$ and a given Schützenberger functor. In this respect, commutative cohomology is more interesting than general semigroup cohomology (Leech [9]), which classifies coextensions but not semigroups.

Semigroup cohomology can be defined "concretely", by cocycles and coboundaries arising from coextensions, or "abstractly" as triple cohomology. For semigroups in general, Wells [13] showed that these two approaches yield the same cohomology. Here we do the same for commutative semigroups.

The lack of a satisfactory preexisting cohomology theory beyond dimension 2 makes the commutative case more difficult. We show, as in [13], that the coefficient objects in triple cohomology are equivalent to abelian group valued functors, and that Beck extensions coincide with the group coextensions in [4]. Hence, in dimension 2, triple cohomology coincides with the cohomology in [4]. In higher dimensions, we obtain more concrete definitions of triple cohomology by cocycles and coboundaries, and as the cohomology of a suitable projective complex.

All these results were announced in [5]. Computation techniques and examples are given in [6], [7], and in forthcoming papers.

The reader is referred to [10], [11], [12] for general results on cohomology, categories, and abelian functor categories, and to [1], [2], [3] for triples and their cohomology.

The paper is divided into nine parts. Parts 1 and 2 recall basic facts on commutative group coextensions and on commutative semigroups over a fixed commutative semigroup S. Part 3 determines abelian group objects over S and part 4 shows that Beck extensions of S are equivalent to commutative group coextensions of S. Part 5 recalls basic facts on triple cohomology. In parts 6, 7, and 8, we prove our main result, which describes the triple cohomology of commutative semigroups by cocycles and coboundaries, in all dimensions. (Simpler descriptions in dimensions 2 and 3 are given in [4], [5], and [8].) Part 9 gives an equivalent description, as the cohomology of a projective complex of abelian group valued functors.

1. In all that follows, S is a fixed commutative semigroup. Commutativity makes the Leech categories $\mathcal{L}(S)$ and $\mathcal{R}(S)$ isomorphic [9]. We denote the resulting category by $\mathcal{H}(S)$. The objects of $\mathcal{H}(S)$ are the elements of S. The morphisms of $\mathcal{H}(S)$ are the elements of $S \times S^1$; when $s \in S$ and $t \in S^1$, then (x,t) is a morphism from x to xt. The composition of $(x,t) : x \longrightarrow xt$ and $(xt,u) : xt \longrightarrow xtu$ is $(x,tu) : x \longrightarrow xtu$; the identity on $x \in S$ is $(x,1)$.

An abelian group valued functor $A = (A,\alpha)$ on $\mathcal{H}(S)$ thus assigns to each $x \in S$ an abelian group A_x and to each pair $(x,t) \in S \times S^1$ a homomorphism $\alpha_{x,t} : A_x \longrightarrow A_{xt}$ (written on the left), so that $\alpha_{x,1}$ is the identity on A_x and $\alpha_{xt,u}\,\alpha_{x,t} = \alpha_{x,tu}$ for all x,t,u. Abelian group valued functors on $\mathcal{H}(S)$ form an abelian category $\mathcal{A} = \mathcal{H}(S)^{\mathcal{A}b}$.

A **commutative coextension** of S by an abelian group valued functor A on $\mathcal{H}(S)$, also called a **commutative group coextension** of S, consists of a commutative semigroup C, a surjective homomorphism $p : C \longrightarrow S$, and a family of left group actions, one for each $x \in S$, under which each A_x acts simply and transitively on $C_x = p^{-1}(x)$, so that

$$(a\,.\,p)\,q \;=\; (\alpha_{x,y}a)\,.\,pq$$

for all $x,y \in S$, $a \in A_x$, $p \in C_x$, $q \in C_y$.

For example, the **split** coextension of S by A is the set of all pairs (a,x) with $x \in S$, $a \in A_x$, with multiplication

$$(a,x)(b,y) = (\alpha_{x,y}a + \alpha_{y,x}b,\ xy),$$

projection $(a,x) \longmapsto x$, and group actions $a\,.\,(b,x) = (a+b,x)$.

The coextensions considered in [4] are the particular case where the functor S is **thin** in the sense that $\alpha_{x,t} = \alpha_{x,u}$ whenever $xt = xu$. Schützenberger functors have this property.

A morphism (also called **equivalence**) of commutative coextensions C, C' of S by A is a necessarily bijective homomorphism $e : C \longrightarrow C'$ which preserves projection to S ($p'e = p$) and group actions ($e(g\,.\,a) = g\,.\,ea$ for all $x \in S$, $g \in A_x$, $a \in C_x$).

2. Triple cohomology happens in the (comma) category $\mathcal{C} = \mathcal{C}sg \downarrow S$ of **commutative semigroups over** S. A commutative semigroup over S is a pair (T,p) of a commutative semigroup T and a homomorphism $p : T \longrightarrow S$; a homomorphism $f : (T,p) \longrightarrow (R,q)$ is a homomorphism $f : T \longrightarrow R$ such that $qf = p$.

The category $\mathcal{C}$ has a terminal object, namely $(S,1)$ (where $1 = 1_S$ is the identity on S). The product in $\mathcal{C}$ of two commutative semigroups (T,p) and (R,q) over S is given by the pullback

$$\begin{array}{ccc} P & \longrightarrow & T \\ \big\downarrow & & \big\downarrow p \\ R & \xrightarrow{q} & S \end{array}$$

where $P = \{\,(t,r) \in T \times R;\ pt = qr\,\}$.

When (T,p) is a commutative semigroup over S, the sets $T_x = p^{-1}x$ ($x \in S$) can be viewed as Hom sets in $\mathcal{C}$. The additive semigroup $\mathbb{N}$ of all positive integers is free on $\{1\}$. For each element x of any semigroup T there is one homomorphism $\overline{x} : \mathbb{N} \longrightarrow T$ such that $\overline{x}1 = x$ (namely, $\overline{x}n = x^n$). This provides a bijection Hom $(\mathbb{N},T) \longrightarrow T$. If (T,p) is a commutative semigroup over S and $a \in T$, $pa = x \in S$, then $p\overline{a} = \overline{x}$, and $\overline{x} : (\mathbb{N},\overline{x}) \longrightarrow (T,p)$ is a homomorphism. This provides a bijection $T_x = p^{-1}x \longrightarrow \mathrm{Hom}_{\mathcal{C}}((\mathbb{N},\overline{x}),(T,p))$.

3. An **abelian group object** in a category $\mathcal{C}$ is an object K of $\mathcal{C}$ together with abelian group operations on all $\mathrm{Hom}_{\mathcal{C}}(X,K)$, such that $\mathrm{Hom}_{\mathcal{C}}(-,K)$ is a (contravariant) abelian group valued functor on $\mathcal{C}$; that is, $(f+g)k = fk + gk$ for all $f,g : X \longrightarrow K$ and $k : Y \longrightarrow X$. A morphism $h : K \longrightarrow L$ of abelian group objects is a morphism such that

$\mathrm{Hom}_{\mathcal{C}}(-,h) : \mathrm{Hom}_{\mathcal{C}}(-,K) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(-,L)$ is a natural transformation of abelian group valued functors; that is, $h(f+g) = hf + hg$ for all X and $f, g : X \longrightarrow K$. This yields a category $\mathcal{O} = \mathcal{A}b(\mathcal{C})$. Alternately, abelian group objects in $\mathcal{C}$ can be defined by suitable identity and multiplication morphisms [11].

PROPOSITION 1. *Let $\mathcal{C} = \mathcal{C}sg \downarrow S$ be the category of commutative semigroups over S. The category $\mathcal{O} = \mathcal{A}b(\mathcal{C})$ of abelian group objects of $\mathcal{C}$ is equivalent to the category $\mathcal{A} = \mathcal{H}(S)^{\mathcal{A}b}$ of abelian group valued functors on $\mathcal{H}(S)$.*

PROOF. To construct a functor $\mathbb{O} : \mathcal{A} \longrightarrow \mathcal{O}$ we show that split coextensions are abelian group objects over S. Let A be an abelian group valued functor on $\mathcal{H}(S)$. As a commutative semigroup over S, $\mathbb{O}A = E = (E,p)$ is the split coextension of S by A. Thus $E = \{ (a,x);\ x \in S, a \in A_x \}$ with multiplication

$$(a,x)(b,y) \ = \ (\alpha_{x,y}a + \alpha_{y,x},\ xy).$$

When $T = (T,r)$ is a commutative semigroup over S, $f : T \longrightarrow E$ is a homomorphism over S, and $t \in T$, then $pf = r$ implies $ft = (a, rt)$ for some $a \in A_{rt}$. Homomorphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(T,E)$ are added by

$$(1) \qquad (f+g)t \ = \ (a+b, rt), \quad \text{where} \quad ft = (a,rt),\ gt = (b,rt).$$

This makes $\mathrm{Hom}_{\mathcal{C}}(T,E)$ an abelian group: the identity element sends t to $(0,rt)$; if f sends t to (a,rt), then $-f$ sends t to $(-a,rt)$. Also $(f+g)k = fk + gk$ for all $k : (U,s) \longrightarrow (T,r)$. Thus $\mathrm{Hom}_{\mathcal{C}}(-,E)$ is an abelian group valued functor on $\mathcal{C}$, and $\mathbb{O}A$ is an abelian group object of $\mathcal{C}$.

When B is another abelian group valued functor on $\mathcal{H}(S)$, every natural transformation $\varphi : A \longrightarrow B$ yields a semigroup homomorphism $h : \mathbb{O}A \longrightarrow \mathbb{O}B$, $(a,x) \longmapsto (\varphi_x a, x)$. Since each φ_x is a group homomorphism, we have $h(f+g) = hf + hg$ for all $f, g : (T,r) \longrightarrow (E,p)$. Thus $\mathbb{O}\varphi = h : \mathbb{O}A \longrightarrow \mathbb{O}B$ is a morphism of abelian group objects. This constructs a functor $\mathbb{O} : \mathcal{A} \longrightarrow \mathcal{O}$.

We now show that, conversely, every abelian group object $K = (K,p)$ of $\mathcal{C}$ is (up to isomorphism) a split coextension of S. As in [11], [13] we begin by constructing identity and multiplication homomorphisms $e : (S,1) \longrightarrow K$ and $m : K \times K \longrightarrow K$ (which would serve in the alternate definition of abelian group objects). First, $(S,1) = (S,1_S)$ is the terminal object of $\mathcal{C}$, and e is the identity element of the abelian group $\mathrm{Hom}_{\mathcal{C}}((S,1),K)$. Since

$ep = 1$, p is surjective, and $K = \bigcup_{x \in S} K_x$, where $K_x = p^{-1}x \neq \emptyset$. For each $x \in S$ the bijection $a \longmapsto \overline{a}$, $K_x \longrightarrow \mathrm{Hom}_{\mathcal{C}}((\mathbb{N}, \overline{x}), K)$ yields an abelian group operation on K_x, defined by

$$\overline{a+b} = \overline{a} + \overline{b}. \tag{2}$$

Since e is the identity element of $\mathrm{Hom}_{\mathcal{C}}((S,1),(K,p))$, $e\overline{x} = \overline{ex}$ is the identity element of $\mathrm{Hom}_{\mathcal{C}}((\mathbb{N}, \overline{x}), K)$, and the identity element of K_x is ex.

The direct product $P = (P, r) = K \times K$ in $\mathcal{C}$ and its projections $p_1, p_2 : P \longrightarrow K$ are given by the pullback:

$$\begin{array}{ccc} P & \xrightarrow{p_1} & K \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p} \\ K & \xrightarrow{p} & S \end{array}$$

with $P = \{(a,b) \in K \times K;\ pa = pb\} = \bigcup_{x \in S} K_x \times K_x$, $p_1(a,b) = a$, $p_2(a,b) = b$, and $r = pp_1 = pp_2$. Addition in $\mathrm{Hom}_{\mathcal{C}}(P, K)$ yields a morphism $m = p_1 + p_2 : P \longrightarrow K$. For all $(a,b) \in P$, $(\overline{a}, \overline{b}) : \mathbb{N} \longrightarrow K \times K$ sends 1 to (a,b);

$$\overline{a} + \overline{b} = p_1(\overline{a}, \overline{b}) + p_2(\overline{a}, \overline{b}) = (p_1 + p_2)(\overline{a}, \overline{b}) = m(\overline{a}, \overline{b}),$$

since $\mathrm{Hom}_{\mathcal{C}}(-, K)$ is a functor; and

$$a + b = (\overline{a} + \overline{b})1 = m(\overline{a}, \overline{b})1 = m(a,b).$$

Since $m : P \longrightarrow K$ is a morphism, it follows that

$$(a+b)(c+d) = (m(a,b))(m(c,d)) = m(ac, bc) = ac + bd \tag{3}$$

for all $x, y \in S$, $a, b \in K_x$, $c, d \in K_y$. We use this to define a homomorphism $\alpha_{x,t} : K_x \longrightarrow K_{xt}$ for each $x \in S$, $t \in S^1$. If $t = 1$ then $\alpha_{x,t}$ is the identity on K_x; otherwise $t \in S$ and

$$\alpha_{x,t} a = a(et) \in K_{xt}$$

for all $a \in K_x$. Since $et + et = et$ it follows from (3) that $\alpha_{x,t}$ is a group homomorphism. Since e is a homomorphism, $\alpha_{xt,u}\, \alpha_{x,t} = \alpha_{x,tu}$, and $\mathbb{F}(K,p) = (K, \alpha)$ is an abelian group valued functor on $\mathcal{H}(S)$.

Let $h : K \longrightarrow L = (L, q)$ be a morphism of abelian group objects (so that $h(f+g) = hf + hg$ whenever $f, g : T \longrightarrow K$). Since $hp = q$ we have $hK_x \subseteq L_x$ for each $x \in S$. By the choice of h, the induced mapping $h_x : K_x \longrightarrow L_x$ is a group homomorphism, and $h_{xt}\alpha_{x,t} = \beta_{x,t} h_x$

(where $\mathbb{F}(L,q) = (L,\beta)$). Hence $\mathbb{F}h = (h_x)_{x\in S} : \mathbb{F}K \longrightarrow \mathbb{F}L$ is a natural transformation. We now have a functor $\mathbb{F}$ from $\mathcal{O}$ to $\mathcal{A}$.

By (3), $\alpha_{x,y}a+\alpha_{y,x}b = a(ey)+(ex)b = (a+ex)(ey+b) = ab$ whenever $a \in K_x$, $b \in K_y$. Consequently (K,p) is isomorphic (in $\mathcal{C}$) to the split coextension of S by $\mathbb{F}K$. Furthermore let $f,g : T = (T,r) \longrightarrow K$, $t \in T$, $ft = a \in K_{rt}$, and $gt = b \in K_{rt}$. Then $f\bar{t} = \bar{a}$, $g\bar{t} = \bar{b}$, and

$$(f+g)t \;=\; (f+g)\bar{t}\,1 \;=\; (f\bar{t}+g\bar{t})1 \;=\; (\bar{a}+\bar{b})1 \;=\; a+b$$

by (2). By (1) this implies that $\mathrm{Hom}_{\mathcal{C}}(T,K)$ and $\mathrm{Hom}_{\mathcal{C}}(T,\mathbb{O}\mathbb{F}K)$ are naturally isomorphic abelian groups, so that K and $\mathbb{O}\mathbb{F}K$ are naturally isomorphic in $\mathcal{O}$.

If on the other hand A is an abelian group valued functor on $\mathcal{H}(S)$, then $\mathbb{O}A$ is already a split coextension and it is immediate that $\mathbb{F}\mathbb{O}A \cong A$. ∎

In particular, $\mathrm{Hom}_{\mathcal{C}}((\mathbb{N},\bar{x}),\mathbb{O}A) \cong A_x$ for each $x \in S$; the isomorphism takes $a \in A_x$ to the homomorphism $\bar{a} : \mathbb{N} \longrightarrow E$ such that $\bar{a}1 = (a,x)$.

4. Let $K = (K,p)$ be an abelian group object of the category $\mathcal{C} = \mathcal{C}sg \downarrow S$ of commutative semigroups over S. A **Beck extension** of S by K consists of commutative semigroup $C = (C,q)$ over S, where q is surjective, and of a simply transitive abelian group action of $\mathrm{Hom}_{\mathcal{C}}(T,K)$ on the set $\mathrm{Hom}_{\mathcal{C}}(T,C)$ for each $T \in \mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}}(T,K) \times \mathrm{Hom}_{\mathcal{C}}(T,C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(T,C), \quad (a,c) \longmapsto a\,.\,c$$

which is compatible with q ($(a.c)\,q = cq$) and natural in T ($ak\,.\,ck = (a\,.\,c)\,k$ for all $k \in \mathrm{Hom}_{\mathcal{C}}(U,T)$). (A general definition of Beck extensions is given in [3], Definition 6.) A morphism (also called equivalence) of Beck extensions is a morphism $e : C \longrightarrow C'$ in $\mathcal{C}$ which preserves the action of each $\mathrm{Hom}_{\mathcal{C}}(T,K)$ ($e(a\,.\,c) = a\,.\,ec$).

PROPOSITION 2. *Let A be an abelian group valued functor on $\mathcal{H}(S)$. The category of Beck extensions of S by $\mathbb{O}A$ is isomorphic to the category of commutative group coextensions of S by A.*

PROOF. Let $C = (C,q)$ be a commutative coextension of S by A. Construct a Beck extension $\mathbb{B}C$ of S by $\mathbb{O}A = E = (E,p)$ as follows. As a commutative semigroup over S, $\mathbb{B}C = (C,q)$. When $T = (T,r) \in \mathcal{C}$, $f \in \mathrm{Hom}_{\mathcal{C}}(T,E)$, and $k \in \mathrm{Hom}_{\mathcal{C}}(T,C)$, then $f\,.\,k \in \mathrm{Hom}_{\mathcal{C}}(T,C)$ is defined by

$$(4) \qquad (f.k)\,t \;=\; a.kt, \quad \text{where} \quad ft = (a, rt).$$

This action is compatible with q, and is natural in T since $(f.k)\,l = fl.kl$ for all $l \in \mathrm{Hom}_{\mathcal{C}}(U,T)$. Also $\mathrm{Hom}_{\mathcal{C}}(T,E)$ acts simply on $\mathrm{Hom}_{\mathcal{C}}(T,C)$, since each A_x acts simply on $C_x = q^{-1}x$. Transitivity is shown as follows. Let $k, l \in \mathrm{Hom}_{\mathcal{C}}(T,C)$. For each $t \in T$ there is one $a \in A_{rt}$ such that $a.kt = lt$. Define $ft = (a, rt) \in E$. If $t, u \in T$, $ft = (a, rt)$, and $fu = (b, ru)$, then in the coextension C

$$\begin{aligned}(lt)(lu) &= (a.kt)(b.ku) \\ &= \alpha_{rt,ru}a.(kt)(b.ku) \;=\; (\alpha_{rt,ru}a + \alpha_{ru,rt}b).(kt)(ku),\end{aligned}$$

whereas $(ft)(fu) = (\alpha_{rt,ru}a + \alpha_{ru,rt}b,\ (rt)(ru))$ by definition of E. Thus f is a homomorphism. By definition, $f.k = l$.

We now have a Beck extension $\mathbb{B}C$ of S by E. An equivalence of group coextensions $e : C \longrightarrow C'$ preserves group actions and therefore is an equivalence of Beck extensions. Thus we have a functor $\mathbb{B}$ from group coextensions to Beck extensions.

Conversely, let $B = (B, q)$ be a Beck extension of S by E. Since q is surjective, $B = \bigcup_{x \in S} B_x$, where $B_x = q^{-1}x$. For each $x \in S$ there is a bijection $B_x \longrightarrow \mathrm{Hom}_{\mathcal{C}}((\mathbb{N}, \overline{x}), (B, q))$, $c \longmapsto \overline{c}$, and an abelian group isomorphism $A_x \longrightarrow \mathrm{Hom}_{\mathcal{C}}((\mathbb{N}, \overline{x}), (E, p))$, $a \longmapsto \overline{a}$, where $\overline{a}1 = (a, x)$. Hence the action of $\mathrm{Hom}_{\mathcal{C}}((\mathbb{N}, \overline{x}), (E, p))$ on $\mathrm{Hom}_{\mathcal{C}}((\mathbb{N}, \overline{x}), (B, q))$ yields a simply transitive group action of A_x on B_x, namely:

$$(5) \qquad a.c \;=\; (\overline{a}.\overline{c})\,1$$

for all $x \in S$, $a \in A_x$, $c \in B_x$.

To show that B has the coextension property $(a.c)\,d = \alpha a.cd$ we construct as in [13] a group action morphism $m : E \times B \longrightarrow B$. The direct product $P = E \times B$ in $\mathcal{C}$ and its projections r, s are given by the pullback

$$\begin{array}{ccc} P & \xrightarrow{\ s\ } & E \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\ q\ } & S \end{array}$$

with $P = \{(a,c);\ a \in A_x,\ c \in B_x,\ x \in S\}$, $s(a,c) = (a,x)$, and $r(a,c) = c$. The action of $s \in \mathrm{Hom}_{\mathcal{C}}(P,E)$ on $r \in \mathrm{Hom}_{\mathcal{C}}(P,B)$ yields a homomorphism $m = s.r : P \longrightarrow B$. For each $u = (a,c) \in P_x$ we have $s\overline{u} = \overline{a}$, $r\overline{u} = \overline{c}$, $m\overline{u} = (s.r)\overline{u} = s\overline{u}.r\overline{u} = \overline{a}.\overline{c}$ by naturality, and

$$a.c = (\overline{a}.\overline{c})1 = m\overline{u}1 = mu = m(a,c).$$

When $x, y \in S$, $a \in A_x$, $b \in A_y$, $c \in B_x$, and $d \in B_y$, we have $(a,c)(b,d) = (\alpha_{x,y}a + \alpha_{y,x}b,\ cd)$ in P; since m is a homomorphism,

$$(a.c)(b.d) = m(a,c)\,m(b,d) = (\alpha_{x,y}a + \alpha_{y,x}b).cd.$$

With $b = 0 \in A_y$ we obtain $(a.c)\,d = \alpha_{x,y}a.cd$. Thus B is a coextension of S by A.

If $e : B \longrightarrow B'$ is an equivalence of Beck extensions, then

$$e(a.c) = e(\overline{a}.\overline{c})1 = (\overline{a}.e\overline{c})1 = (\overline{a}.\overline{ec})1 = a.ec$$

for all $x \in S$, $a \in A_x$, $c \in B_x$. Hence e is an equivalence of coextensions. We now have a functor $\mathbb{C}$ from Beck extensions to group coextensions.

When $f : T \longrightarrow E$, $k : T \longrightarrow B$, and $t \in T$, we have $ft = (a, rt)$ for some $a \in A_{rt}$, $f\overline{t} = \overline{a}$, and

$$(f.k)t = (f.k)\overline{t}1 = (f\overline{t}.k\overline{t})1 = (\overline{a}.\overline{kt})1 = a.kt;$$

by (4), $\mathbb{BC}B = B$. If conversely C is a coextension of S by E and $x \in S$, $a \in A_x$, $c \in C_x$, then $\overline{a}1 = (a,x)$ and (4) yields $(\overline{a}.\overline{c})1 = a.\overline{c}1 = a.c$; hence $\mathbb{CB}C = C$. ∎

5. Triple cohomology for any category $\mathcal{C}$ as defined in [1] requires a cotriple (a comonad) $(\mathbb{G}, \epsilon, \nu)$ in $\mathcal{C}$ and a contravariant abelian group valued functor $\mathbb{E}$ on $\mathcal{C}$. We mostly follow the notation in [1], but write functors on the left (as in [11]) and renumber cohomology groups to avoid the dimension shift in [1] and [3].

The counit $\epsilon : \mathbb{G} \longrightarrow 1$ yields natural transformations $\epsilon_i^n = \mathbb{G}^{n-i}\epsilon\,\mathbb{G}^i : \mathbb{G}^{n+1} \longrightarrow \mathbb{G}^n$ $(0 \le i \le n)$. In particular, $\epsilon_n^n = \epsilon\mathbb{G}^n$ and $\epsilon_i^n = \mathbb{G}\epsilon_i^{n-1}$ for all $i < n$. For each object X of $\mathcal{C}$ this yields a complex of abelian groups

$$0 \longrightarrow \mathbb{EG}X \longrightarrow \dots \longrightarrow \mathbb{EG}^nX \xrightarrow{\delta_n} \mathbb{EG}^{n+1}X \longrightarrow \dots$$

where $\delta_n = \sum_{i=0}^{n} (-1)^i\ \mathbb{E}\epsilon_i^n X : \mathbb{EG}^nX \longrightarrow \mathbb{EG}^{n+1}X$. For each $n \geq 1$ the n-th cohomology group of X with coefficients in $\mathbb{E}$ is the abelian group $H^n(X,\mathbb{E}) = \text{Ker } \delta_n/\text{Im } \delta_{n-1}$ (with $\delta_0 = 0$). (In [1] and [3], $H^n(X,\mathbb{E}) = \text{Ker } \delta_{n+1}/\text{Im } \delta_n$; hence H^n in [1] is H^{n+1} here.)

Triple cohomology has the following properties (Propositions 2.1 and 3.2 in [1]):

PROPERTY A (acyclicity). *$H^1(\mathbb{G}X, \mathbb{E}) \cong \mathbb{E}\mathbb{G}X$ and $H^n(\mathbb{G}X, \mathbb{E}) = 0$ for all X and all $n \geq 2$.*

A sequence $\mathbb{E}' \longrightarrow \mathbb{E} \longrightarrow \mathbb{E}''$ of contravariant abelian group valued functors on $\mathcal{C}$ is **short $\mathbb{G}$-exact** in case $0 \longrightarrow \mathbb{E}'\mathbb{G}X \longrightarrow \mathbb{E}\mathbb{G}X \longrightarrow \mathbb{E}''\mathbb{G}X \longrightarrow 0$ is a short exact sequence of abelian groups for every object X of $\mathcal{C}$.

PROPERTY B (exact cohomology sequence). *Every short $\mathbb{G}$-exact sequence*

$$\mathcal{E} : \mathbb{E}' \longrightarrow \mathbb{E} \longrightarrow \mathbb{E}''$$

induces an exact sequence

$$\ldots\ H^n(X, \mathbb{E}') \longrightarrow H^n(X, \mathbb{E}) \longrightarrow H^n(X, \mathbb{E}'') \longrightarrow H^{n+1}(X, \mathbb{E}')\ \ldots$$

which is natural in X and $\mathcal{E}$.

6. Now we specialize triple cohomology to commutative semigroups, following the procedure used by Beck [3] for any tripleable category (see also [13]). In what follows $\mathcal{C} = \mathcal{C}sg \downarrow S$ is the category of commutative semigroups over the given commutative semigroup S.

Every element of the free commutative semigroup FX on a set X is a nonempty product of elements of X, the factors of which are unique up to order. However, X will later have its own multiplication, and it would then be very confusing to write the elements of FX as products of elements of X. Instead we regard the elements of FX as nonempty unordered sequences $[x_1, \ldots, x_m]$ of elements of X (so that $m \geq 1$ and $[x_{\sigma 1}, \ldots, x_{\sigma m}] = [x_1, \ldots, x_m]$ for every permutation σ of $1, 2, \ldots, m$). Multiplication in FX is then given by

$$[x_1, \ldots, x_m][y_1, \ldots, y_k] = [x_1, \ldots, x_m, y_1, \ldots, y_k];$$

$i : X \longrightarrow FX$ sends $x \in X$ to $[x] \in FX$.

The free commutative semigroup functor and the forgetful functor to sets induce a pair of adjoint functors between $\mathcal{C}$ and the similar category of sets over S. The resulting cotriple $(\mathbb{G}, \epsilon, \nu)$ in $\mathcal{C}$ is as follows. For each commutative semigroup $T = (T, p)$ over S, $\mathbb{G}T = (FT, q)$, where FT is the free commutative semigroup on the set T, with $i : T \longrightarrow FT$, and $q : FT \longrightarrow S$ is the homomorphism such that $qi = p$. Since $q[x] = px$,

$$q[x_1, \ldots, x_m] = (px_1) \ldots (px_m) = p(x_1 \ldots x_m)$$

for all $x_1, \ldots, x_m \in T$. If $f : T = (T, p) \longrightarrow U = (U, r)$ is a homomorphism over S then $\mathbb{G}f : \mathbb{G}T \longrightarrow \mathbb{G}U$ is given by

$$(\mathbb{G}f)[x_1, \ldots, x_m] = [fx_1, \ldots, fx_m].$$

The counit $\epsilon : \mathbb{G} \longrightarrow \mathbf{1}$ sends $T = (T, p)$ to the product homomorphism $\pi : FT \longrightarrow T$,

$$\pi [x_1, \ldots, x_m] = x_1 \ldots x_m.$$

The comultiplication $\nu : \mathbb{G} \longrightarrow \mathbb{G}\mathbb{G}$ (not used thereafter) sends T to the homomorphism $[x_1, \ldots, x_m] \longmapsto [[x_1], \ldots, [x_m]]$.

For each abelian group object K in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(-, K)$ is a contravariant abelian group valued functor on $\mathcal{C}$ and provides coefficients for the cohomology. Thus the **triple cohomology groups** $H^n(S, K)$ are the cohomology groups $H^n(X, \mathbb{E})$ where $X = (S, 1)$ and $\mathbb{E} = \mathrm{Hom}_{\mathcal{C}}(-, K)$.

In dimension 2, Theorem 6 of [3] implies:

PROPERTY C. *There is a bijection between $H^2(S, K)$ and the set of equivalence classes of Beck extensions of S by K.*

In general, $H^n(S, K)$ is the homology group $\mathrm{Ker}\ \delta_n / \mathrm{Im}\ \delta_{n-1}$ of the complex of abelian groups

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathbb{G}X, K) \ \ldots \xrightarrow{\delta_{n-1}} \mathrm{Hom}_{\mathcal{C}}(\mathbb{G}^n X, K) \xrightarrow{\delta_n} \mathrm{Hom}_{\mathcal{C}}(\mathbb{G}^{n+1} X, K) \ \ldots$$

where $X = (S, 1)$ and

$$\delta_n = \textstyle\sum_{i=0}^{n} (-1)^i \, \mathrm{Hom}_{\mathcal{C}}(\epsilon_i^n X, K).$$

We denote $\mathbb{G}^n X$ by $T_n = (T_n, p)$, so that $T_{n+1} = FT_n$, and $\epsilon_i^n X$ by π_i^n, so that $\delta_n u = \sum_{i=0}^{n} (-1)^i \, \pi_i^n u$ for all $u : \mathbb{G}^n X \longrightarrow K$. Let $t_1, \ldots, t_m \in T_n$. Since $\epsilon_n^n X : \mathbb{G}^{n+1} X \longrightarrow \mathbb{G}^n X$ is the counit,

$$\pi_n^n [t_1, \ldots, t_m] = t_1 \ldots t_m,$$

in particular $\pi_n^n [t] = t$ for all $t \in T_n$, whereas

$$\pi_i^n [t_1, \ldots, t_m] = [\pi_i^{n-1} t_1, \ldots, \pi_i^{n-1} t_m]$$

for all $i < n$, since $\epsilon_i^n = \mathbb{G}\epsilon_i^{n-1}$; this determines $\pi_i^n [t_1, \ldots, t_m]$ by induction.

7. We now give a more concrete description of $H^n(S, K)$. For each commutative semigroup $T = (T, r)$ over S and abelian group valued functor $A = (A, \alpha)$ on $\mathcal{H}(S)$, let

$$C(T, A) = \textstyle\bigoplus_{t \in T} A_{rt}.$$

The elements of $C(T, A)$ are cochains $u = (u_t)_{t \in T}$, with $u_t \in A_{rt}$.

LEMMA 3. $\mathrm{Hom}_{\mathcal{C}}(\mathbb{G}T, \mathbb{O}A)$ *and* $C(T, A)$ *are naturally isomorphic.*

PROOF. Recall that $\mathbb{G}(T,r) = (FT, q)$, where FT is free on the set T and $q\,[a_1, \dots, a_m] = r\,(a_1 \dots a_m)$ for all $a_1, \dots, a_m \in T$, and that $\mathbb{O}A = (E, p)$ is the split coextension of S by A as in the proof of Proposition 1.

Let $f : FT \longrightarrow E \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{G}T, \mathbb{O}A)$. Since $pf = r$ there exists for each $t \in T$ some $u_t \in A_{rt}$ such that $f[t] = (u_t, rt)$. Since conversely FT is free on T there is for each $u = (u_t)_{t \in T} \in C(T, A)$ a homomorphism $f : FT \longrightarrow E$ over S unique such that $f[t] = (u_t, rt)$ for all $t \in T$. Then

(6) $f\,[a_1, \dots, a_m] \;=\; (u_{a_1}, ra_1) \dots (u_{a_m}, ra_m) \;=\; \left(\sum_{j=1}^m \alpha_{ra_j, r\widehat{a}_j} u_{a_j}\,,\; ra\right),$

where $\widehat{a}_j = a_1 \dots a_{j-1} a_{j+1} \dots a_m$ and $a = a_1 \dots a_m$. This provides a bijection $\mathrm{Hom}_{\mathcal{C}}(\mathbb{G}T, \mathbb{O}A) \longrightarrow C(T, A)$ which is readily seen to be natural in T and A. If $g \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{G}T, \mathbb{O}A)$ and $g[t] = (v_t, rt)$ then $(f + g)[t] = (u_t + v_t, rt)$, by equation (1) (in the proof of Proposition 1). Thus $\mathrm{Hom}_{\mathcal{C}}(\mathbb{G}T, \mathbb{O}A) \cong C(T, A)$ as abelian groups. ∎

8. Let $X = (S, 1)$ and K be an abelian group object over S. By Proposition 1, $K \cong \mathbb{O}A$ for some abelian group valued functor A on $\mathcal{H}(S)$. By Lemma 3, the complexes

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathbb{G}X, K) \;\dots\; \mathrm{Hom}_{\mathcal{C}}(\mathbb{G}^n X, K) \xrightarrow{\delta_n} \mathrm{Hom}_{\mathcal{C}}(\mathbb{G}^{n+1} X, K) \;\dots$$
$$0 \longrightarrow C(X, A) \longrightarrow \;\dots\; \xrightarrow{\delta_{n-1}} C(\mathbb{G}^{n-1} X, A) \xrightarrow{\delta_n} C(\mathbb{G}^n X, A) \;\dots$$

have naturally isomorphic homology groups. We need $\delta_n : C(\mathbb{G}^{n-1} X, A) \longrightarrow C(\mathbb{G}^n X, A)$ for $n \geq 1$. Let $\mathbb{G}^n X = T_n$. We denote all the projections $T_n \longrightarrow S$ by p. Then $p[t_1, \dots, t_m] = p\,(t_1 \dots t_m)$; this determines $p[t_1, \dots, t_m]$ by induction.

Let $u \in C(\mathbb{G}^{n-1} X, A)$. By (6), the corresponding homomorphism $f \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{G}^n X, \mathbb{O}A)$ is given for each $t = [a_1, \dots, a_m] \in T_n$ by

$$ft \;=\; f\,[a_1, \dots, a_m] \;=\; \left(\sum_{j=1}^m \alpha_{pa_j, p\widehat{a}_j} u_{a_j}\,,\; pt\right),$$

where $\widehat{a}_j = a_1 \dots a_{j-1} a_{j+1} \dots a_m$ (note that $pt = p\,(a_1 \dots a_m)$). In particular, $f[a] = (u_a, pa)$ for all $a \in T_{n-1}$. Let

$$g \;=\; \delta_n f \;=\; \sum_{i=0}^n (-1)^i\, f\pi_i^n \;:\; \mathbb{G}^{n+1} X \;\longrightarrow\; \mathbb{O}A$$

and $v \in C(\mathbb{G}^n X, A)$ be the corresponding cochain. For each $t = [a_1, \dots, a_m] \in T_n$ we have

$$
\begin{aligned}
(v_t, pt) &= g[t] = \textstyle\sum_{i=0}^{n} (-1)^i f\pi_i^n[t] \\
&= \textstyle\sum_{i=0}^{n-1} (-1)^i f\pi_i^n[t] + (-1)^n f\pi_n^n[t] \\
&= \textstyle\sum_{i=0}^{n-1} (-1)^i f[\pi_i^{n-1}t] + (-1)^n ft \\
&= \left(\textstyle\sum_{i=0}^{n-1} (-1)^i f\pi_i^{n-1} + (-1)^n f\right) t \\
&= \left(\textstyle\sum_{i=0}^{n-1} (-1)^i u_{t_i'} + (-1)^n \sum_{j=1}^{m} \alpha_{pa_j, p\widehat{a}_j} u_{a_j} , pt\right)
\end{aligned}
$$

by (1), where $t_i' = \pi_i^{n-1}t$ and $\widehat{a}_j = a_1 \dots a_{j-1}a_{j+1} \cdots a_m$ as before. Hence

$$v_t = \textstyle\sum_{i=0}^{n-1} (-1)^i u_{t_i'} + (-1)^n \sum_{j=1}^{m} \alpha_{pa_j, p\widehat{a}_j} u_{a_j} .$$

and we have proved:

MAIN THEOREM. *Let S be a commutative semigroup. Up to natural isomorphisms, $H^n(S,A)$ has an abelian group valued functor A on $\mathcal{H}(S)$ for coefficients, and is the homology group* Ker δ_n/Im δ_{n-1} *of the complex*

$$0 \longrightarrow C^1(S,A) \longrightarrow \dots \xrightarrow{\delta_{n-1}} C^n(S,A) \xrightarrow{\delta_n} C^{n+1}(S,A) \dots,$$

where $C^n(S,A) = \bigoplus_{t \in T_{n-1}} A_{pt}$, $(T_n, p) = \mathbb{G}^n(S,1)$, and

$$(\delta_n u)_t = \textstyle\sum_{i=0}^{n-1} (-1)^i u_{t_i'} + (-1)^n \sum_{j=1}^{m} \alpha_{pa_j, p\widehat{a}_j} u_{a_j} ,$$

for all $u \in C^n(S,A)$ and $t = [a_1, \dots, a_m] \in T_n$, with $t_i' = \pi_i^{n-1}t$ and $\widehat{a}_j = a_1 \dots a_{j-1}a_{j+1} \cdots a_m$.

Properties A, B, and C then read as follows.

PROPERTY A. *If S is a free commutative semigroup then $H^1(S,A) \cong C^1(S,A)$ and $H^n(S,A) = 0$ for all $n \geq 2$.* ∎

If $\mathcal{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence of abelian group valued functors, then $0 \longrightarrow C(T,A) \longrightarrow C(T,B) \longrightarrow C(T,C) \longrightarrow 0$ is short exact. Hence $\text{Hom}_{\mathcal{C}}(-, \mathbb{O}A) \longrightarrow \text{Hom}_{\mathcal{C}}(-, \mathbb{O}B) \longrightarrow \text{Hom}_{\mathcal{C}}(-, \mathbb{O}C)$ a short $\mathbb{G}$-exact sequence, by Lemma 3, and we obtain

PROPERTY B. *For each short exact sequence*

$$\mathcal{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of abelian group valued functors, there is an exact sequence

$$\dots H^n(S,A) \longrightarrow H^n(S,B) \longrightarrow H^n(S,C) \longrightarrow H^{n+1}(S,A) \dots$$

which is natural in $\mathcal{E}$. ∎

This also follows from Proposition 5 below.

PROPERTY C. *$H^2(S, A)$ classifies commutative group coextensions of S by A.* ∎

The description of $H^n(S, A)$ in the Main Theorem does not lend itself to the computation of examples. When $n \geq 2$, n-cochains $u \in C^n(S, A)$ are families indexed by free commutative semigroups, and have infinitely many components. This makes the computation of $H^n(S, A)$ an infinite task, even when S is finite.

Simpler descriptions exist in low dimensions. Thus $H^2(S, A)$, which classifies group coextensions, can be described as in [4]. Let $SZ^2(S, A)$ be the group of all **symmetric 2-cocycles**, which are families $s = (s_{a,b})_{a,b \in S}$ such that $s_{b,a} = s_{a,b} \in A_{ab}$ and

$$\alpha_{ab,c} s_{a,b} + s_{ab,c} = s_{a,bc} + \alpha_{bc,a} s_{b,c}$$

for all $a, b, c \in S$. Let $SB^2(S, A)$ be the group of all **symmetric 2-coboundaries**, which are 2-cocycles $s \in SZ^2(S, A)$ of the form

$$s_{a,b} = \alpha_{a,b} u_a + \alpha_{b,a} u_b - u_{ab}$$

for some $u \in C^1(S, A)$. As in [4], $SZ^2(S, A)/SB^2(S, A)$ classifies commutative group coextensions of S by A; therefore

$$H^2(S, A) \cong SZ^2(S, A)/SB^2(S, A).$$

A direct proof that this is equivalent to the description in the Main Theorem has been submitted in [8], together with a similar description of $H^3(S, A)$.

9. We complete the Main Theorem by constructing a projective complex in $\mathcal{A} = \mathcal{H}(S)^{\mathcal{A}b}$ of which $H^n(S, A)$ is the cohomology.

For each commutative semigroup $T = (T, p)$ over S we construct an abelian group valued functor $\mathbb{C}T = C$ on $\mathcal{H}(S)$ ($\mathbb{C}$ is not the same as in the proof of Proposition 2). For each $x \in S$, C_x is the free abelian group generated by

$$G_x = \{ (a, u) \in T \times S^1;\ (pa)u = x \}.$$

When $t \in S^1$, $(a, u) \in G_x$ implies $(a, ut) \in G_{xt}$; $\gamma_{x,t} : C_x \longrightarrow C_{xt}$ is the group homomorphism such that $\gamma_{x,t}(a, u) = (a, ut)$ for all $(a, u) \in G_x$.

LEMMA 4. *For each $T \in \mathcal{C}$ and $A \in \mathcal{A}$ there is a natural isomorphism $C(T, A) \cong \mathrm{Hom}_{\mathcal{A}}(\mathbb{C}T, A)$. In particular, $\mathbb{C}T$ is projective in $\mathcal{A}$.*

PROOF. Let $T = (T, p)$ and $C = \mathbb{C}T$. Let $\varphi : C \longrightarrow A$ be a natural transformation. For each $t \in T$, $(t, 1) \in G_{pt}$ and $c_t = \varphi_{pt}(t, 1) \in A_{pt}$. Since φ is natural we have

$$(7) \qquad \varphi_x(a, u) = \varphi_x \gamma_{pa,u}(a, 1) = \alpha_{pa,u} \varphi_{pa}(a, 1) = \alpha_{pa,u} c_a$$

for all $x \in S$ and $(a, u) \in G_x$. Thus φ is uniquely determined by $c = (c_t)_{t \in T} \in C(T, A)$. Conversely, for each $c \in C(T, A)$, equation (7) determines a homomorphism $\varphi_x : C_x \longrightarrow A_x$, and it is immediate that $\varphi : C \longrightarrow A$ is a natural transformation. Thus $\mathrm{Hom}_{\mathcal{A}}(C, A) \cong C(T, A)$.

When $\psi : A \longrightarrow B$ is an epimorphism in $\mathcal{A}$, every ψ_x is surjective, and $C(T, A) \longrightarrow C(T, B)$ is surjective. It follows that $\mathbb{C}T$ is projective. (This also follows from [12], Corollary IX.7.3.) ∎

Let $T_n = \mathbb{G}^n(S, 1)$ and $c \in C^n(S, A) = C(T_{n-1}, A)$. For all $t = [a_1, \dots, a_m] \in T_n$,

$$(\delta_n c)_t = \sum_{i=0}^{n-1} (-1)^i c_{t'_i} + (-1)^n \sum_{j=1}^{m} \alpha_{pa_j, p\widehat{a}_j} c_{a_j},$$

where $t'_i = \pi_i^{n-1} t$ and $\widehat{a}_j = a_1 \dots a_{j-1} a_{j+1} \dots a_m$. By Lemma 4 there are natural transformations $\varphi : \mathbb{C}T_{n-1} \longrightarrow A$ and $\psi : \mathbb{C}T_n \longrightarrow A$ which correspond to c and δc. For all $t = [a_1, \dots, a_m] \in T_n$ and $u \in S^1$, equation (7) yields, with $x = (pt)u \in S$:

$$\begin{aligned} \psi_x(t, u) &= \alpha_{pt,u} (\delta c)_t \\ &= \sum_{i=0}^{n-1} (-1)^i \alpha_{pt,u}\, c_{t'_i} + (-1)^n \sum_{j=1}^{m} \alpha_{pt,u}\, \alpha_{pa_j, p\widehat{a}_j} c_{a_j} \\ &= \sum_{i=0}^{n-1} (-1)^i \varphi_x(t'_i, u) + (-1)^n \sum_{j=1}^{m} \varphi_x(a_j, (p\widehat{a}_j)u) \\ &= \varphi_x \left(\sum_{i=0}^{n-1} (-1)^i (t'_i, u) + (-1)^n \sum_{j=1}^{m} (a_j, (p\widehat{a}_j)u) \right). \end{aligned}$$

Thus $\psi = \varphi\partial$, where $\partial : \mathbb{C}T_n \longrightarrow \mathbb{C}T_{n-1}$ is given by

$$(8) \qquad \partial_x(t, u) = \sum_{i=0}^{n-1} (-1)^i (\pi_i^{n-1} t, u) + (-1)^n \sum_{j=1}^{m} (a_j, (p\widehat{a}_j)u)$$

for all $t = [a_1, \dots, a_m] \in T_n$ and $u \in S^1$, with $x = (pt)u \in S$ and $\widehat{a}_j = a_1 \dots a_{j-1} a_{j+1} \dots a_m$. Thus we obtain:

PROPOSITION 5. *For each abelian group valued functor A on $\mathcal{H}(S)$, $H^n(S, A)$ is the cohomology with coefficients in A of the projective chain complex*

$$0 \longleftarrow C_1(S) \longleftarrow \ldots \longleftarrow C_n(S) \overset{\partial}{\longleftarrow} C_{n+1}(S) \longleftarrow \ldots,$$

where $C_n(S) = \mathbb{CG}^{n-1}(S,1)$ *and* ∂ *is given by equation* (8). ∎

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Received: October 1992

Revised: February 1995