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RESEARCH ARTICLE

Commutative monoid homology

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Abstract
Complexes of symmetric chains are constructed for commutative monoids, whose cohomology is the commutative cohomology.

Keywords Symmetric chain · Commutative cohomology · Commutative homology · Universal coeffcients theorems

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Introduction

Commutative semigroups are tripleable over sets and thus have their own cohomology, whose importance stems from the second cohomology group, which classifies the ways in which finite commutative semigroups can be reconstructed from group-free semigroups and abelian groups. Every finite commutative semigroup S is a group coextension of $T = S/\mathcal{H}$ by the Schützenberger functor $\mathcal{G}$ of S, which is an abelian group valued functor on T. The second cohomology group $H^2(T, \mathcal{G})$ classifies commutative group coextensions of T by $\mathcal{G}$, thereby classifying the ways in which S can be reconstructed from $\mathcal{G}$ and T [2,7,11].

In the cohomology of a commutative semigroup S, coefficients are abelian group valued functors on S, meaning on the Leech category $\mathcal{H}(S)$ [7,11]. Thus, an abelian group valued functor $\mathcal{G} = (G, \gamma)$ on S assigns to each $a \in S$ an abelian group G_a, and to each $a \in S$ and $t \in S^1$ a homomorphism $\gamma_{a,t} : G_a \longrightarrow G_{at}$, in such a way that $\gamma_{a,1}$ is the identity on G_a and $\gamma_{a,tu} = \gamma_{at,u} \circ \gamma_{a,t}$.

Two particular cases of abelian group valued functors on S are of interest. In the above $\mathcal{G}$ is *thin* if $\gamma_{a,t} = \gamma_{a,u}$ whenever $at = au$. Schützenberger functors have this property [2].

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$\mathcal{G}$ is *semiconstant* over a subset B of S if there is an abelian group G such that $G_a = G$ for all $a \in B$, $G_a = 0$ for all $a \notin B$, and $\gamma_{a,t}$ is the identity on G whenever a and $at \in B$. Cohomology with coefficients that are constant over all of S classifies commutative Rédei extensions [13] and is the commutative analogue of Eilenberg–MacLane cohomology [1] (see also [12]). For semigroups with a zero element, cohomology with coefficients that are constant over $S\backslash\{0\}$ is a more appropriate commutative analogue of Eilenberg–MacLane cohomology; if S is a nilmonoid (a nilsemigroup with an identity element adjoined), then this cohomology classifies homogeneous elementary semigroups [2].

As a triple cohomology, $H^n(S, \mathcal{G})$ is defined in all dimensions and classifies commutative group coextensions of S by $\mathcal{G}$, among other good properties, but its cochains have unbounded numbers of variables. The triple cohomology of groups suffers from the same defect but the bar resolution provides an equivalent definition in which n-cochains have only n variables. Commutative semigroups allow an equivalent definition for their cohomology in which n-cochains have only n variables but must satisfy 'symmetry' conditions [7]; unfortunately, the appropriate conditions have only been determined in dimensions $n \leqq 4$ [9].

The triple cohomology of commutative semigroups is also the cohomology of a chain complex of projective functors [7], but it too suffers from overlarge chains. This paper constructs a chain complex of projective objects, similar to the bar resolution, whose cohomology in dimension $n \leqq 3$ is the symmetric cohomology, and in which much simpler n-chains satisfy symmetry conditions. This takes place in the coefficient category and depends on it. Hence we actually have three constructions: one for general coefficients; one for thin coefficients; and one for semiconstant coefficients over a given subset. This last yields a universal coefficients theorem that calculates a commutative semigroup cohomology from its homology, when the coefficient functor is semiconstant.

We consider commutative monoids only, since adjunction of an identity element does not affect commutative cohomology.

The paper is organized as follows. Given a commutative monoid S, Sect. 1 defines semiconstant abelian group valued functors, recalls the definition of symmetric cochains, and gives a general definition and basic properties of symmetric maps.

Section 2 defines symmetric chains and constructs a chain complex of free abelian groups whose cohomology in dimensions $n \leqq 3$ is the commutative cohomology of S when the coefficient functor is semiconstant. This yields commutative homology groups and a universal coefficients theorem with various corollaries. (A different universal coefficients theorem for dimension $n = 2$ was obtained in [10] by another method.) Sect. 4 constructs a similar chain complex of thin projective abelian group valued functors, which serves for thin coefficient functors, and Sect. 5 constructs a similar complex that serves for arbitrary coefficients. These sections bring no universal coefficients theorem, as a crucial hypothesis of the latter fails to hold.

Section 6 studies one example, for which constructions in previous sections are carried out in some detail, also providing a counterexample for Sect. 5. Relegated to Sect. 7 is the rather lengthy proof of some technical properties of symmetric mappings that are used to prove projectivity in previous sections. An index of notations concludes the paper.

1 Preliminaries

1. Coefficients in the commutative cohomology of a commutative monoid S are provided by an abelian group valued functor $\mathcal{G} = (G, \gamma)$ on S (actually, on the Leech category $\mathcal{H}(S)$ [11]), which assigns an abelian group G_a to each $a \in S$ and a homomorphism $\gamma_{a,t} : G_a \longrightarrow G_{at}$ to each pair $(a, t) \in S \times S$, so that $\gamma_{a,1} = 1_{G_a}$ (the identity on G_a) and $\gamma_{at,u} \circ \gamma_{a,t} = \gamma_{a,tu}$, for all $a, t, u \in S$. Subscripts in $\gamma_{a,t}$ will be omitted if they are clear from context.

An abelian group valued functor $\mathcal{G} = (G, \gamma)$ is *thin* if $\gamma_{a,t} = \gamma_{a,u}$ whenever $at = au$ in S. Schützenberger functors have this property [2]. If $\mathcal{G}$ is thin, then γ_b^a is well-defined by $\gamma_b^a = \gamma_{a,at}$ whenever $b = at \in aS$ in S (whenever $a \geqq b$ in the divisibility preorder), and then $\gamma_a^a = 1_{G_a}$ and $\gamma_c^b \circ \gamma_b^a = \gamma_c^a$, whenever defined; this makes $\mathcal{G}$ a functor over S, regarded as a preordered set.

An abelian group valued functor $\mathcal{G} = (G, \gamma)$ is *constant* on S if there is an abelian group G such that $G_a = G$ for all $a \in S$ and $\gamma_{a,t} = 1_G$ for all $a, t \in S$.

More generally, an abelian group valued functor $\mathcal{G} = \mathcal{F}(B, G)$ might be constructed from any abelian group G and suitable subset B as follows: let $G_a = G$ for all $a \in B$, $G_a = 0$ for all $a \notin B$, $\gamma_{a,t} = 1_G$ if $a,\ at \in B$, and $\gamma_{a\,at} = 0$ if $a \notin B$ or $at \notin B$.

Lemma 1.1 *If $G \neq 0$, then $\mathcal{G} = \mathcal{F}(B, G)$ is an abelian group valued functor on S if and only if B has the following property: if $a \in B$ and $abc \in B$, then $ab \in B$.*

Proof If in the above $a \in B$, $abc \in B$, and $ab \notin B$, then $\gamma_{a,bc} = 1_G \neq 0 = \gamma_{ab,c} \circ \gamma_{a,b}$ and $\mathcal{G}$ is not a functor. On the other hand, if $a \in B$, $abc \in B$ implies $ab \in B$, then

$$\begin{aligned} \gamma_{at,atu} \circ \gamma_{a,t} &= 0 = \gamma_{a,atu} \quad \text{if } a \notin B \text{ or if } atu \notin B, \quad \text{and} \\ \gamma_{at,atu} \circ \gamma_{a,t} &= 1_G = \gamma_{a,atu} \quad \text{if } a \in B \text{ and } atu \in B, \end{aligned}$$

for then $at \in B$. □

We call a subset B of S *convex* if $a \in B$, $abc \in B$ implies $ab \in B$. Examples of convex subsets of S include S itself, and, if S has a zero element 0, $S\backslash\{0\}$. More generally, every ideal I of S is convex (since $a \in I$ implies $ab \in I$), and its complement $S\backslash I$ is also convex (since $abc \notin I$ implies $ab \notin I$).

An abelian group valued functor $\mathcal{G}$ is *semiconstant* if $\mathcal{G} = \mathcal{F}(B, G)$ for some abelian group G and convex subset B of S; then $\mathcal{G}$ is semiconstant *on* B *at* G. An abelian group valued functor $\mathcal{G}$ is *constant* if $\mathcal{G} = \mathcal{F}(S, G)$ for some abelian group G; then $\mathcal{G}$ is constant *at* G. If S has a zero element, then an abelian group valued functor $\mathcal{G}$ is *almost constant* if $\mathcal{G} = \mathcal{F}(S\backslash\{0\}, G)$ for some abelian group G; then $\mathcal{G}$ is almost constant *at* G.

2. In dimensions $n \leqq 4$, $H^n(S, \mathcal{G})$ can be calculated from symmetric cochains, cocycles, and coboundaries [6,7,9]. With coefficients in $\mathcal{G}$, a symmetric 2-cochain u on S with values in $\mathcal{G} = (G, \gamma)$ assigns $u\,(a, b) \in G_{ab}$ to each $a, b \in S$ so that the condition

$$u\,(b, a) = u\,(a, b) \tag{S2}$$

holds for every $a, b \in S$. A symmetric 3-cochain u on S with values in $\mathcal{G}$ assigns $u\,(a, b, c) \in G_{abc}$ to each $a, b, c \in S$ so that all three parts of condition (S3):

$$u\,(a, b, a) \;=\; 0, \tag{S3a}$$

$$u\,(c, b, a) \;=\; -\; u\,(a, b, c), \text{ and} \tag{S3b}$$

$$u\,(a, b, c) \;+\; u\,(b, c, a) \;+\; u\,(c, a, b) \;=\; 0, \tag{S3c}$$

hold for all $a, b, c \in S$. A symmetric 4-cochain u on S with values in $\mathcal{G}$ assigns $u\,(a, b, c, d) \in G_{abcd}$ to each $a, b, c, d \in S$ so that all four parts of condition (S4):

$$u\,(a, b, b, a) \;=\; 0, \tag{S4a}$$

$$u\,(d, c, b, a) \;=\; -\; u\,(a, b, c, d), \tag{S4b}$$

$$u\,(a, b, c, d) \;-\; u\,(b, c, d, a) \;+\; u\,(c, d, a, b) \;-\; u\,(d, a, b, c) \;=\; 0, \tag{S4c}$$

$$u\,(a, b, c, d) \;-\; u\,(b, a, c, d) \;+\; u\,(b, c, a, d) \;-\; u\,(b, c, d, a) \;=\; 0. \tag{S4d}$$

hold for all $a, b, c, d \in S$ [9] (in fact, (S4c) follows from (S4b) and (S4d), by Lemma 1.3 below). For the sake of completeness, a symmetric 1-cochain u on S assigns $u\,(a) \in G_a$ to each $a \in S$, and condition (S1) is empty. Under pointwise addition, symmetric n-cochains constitute abelian groups $C^n(S, \mathcal{G})$ ($n \leqq 4$).

Symmetric cochains are not defined in dimensions $n \geqq 5$ for lack of appropriate symmetry conditions. For the sake of completeness we denote by $\widehat{C}^n(S, \mathcal{G})$ the group of all n-cochains on S with values in $\mathcal{G}$ (an n-cochain u assigns $u\,(a_1, \ldots, a_n) \in G_a$ to each $a_1, \ldots, a_n \in S$, where $a = a_1\, a_2 \cdots a_n$).

The coboundary homomorphisms

$$C^1(S, \mathcal{G}) \xrightarrow{\delta} C^2(S, \mathcal{G}) \xrightarrow{\delta} C^3(S, \mathcal{G}) \xrightarrow{\delta} C^4(S, \mathcal{G}) \xrightarrow{\delta} \widehat{C}^5(S, \mathcal{G})$$

are:

$$\begin{aligned}
(\delta u)(a, b) &= \gamma_{b,a}\, u\,(b) \;-\; u\,(ab) \;+\; \gamma_{a,b}\, u\,(a) \;\in G_{ab},\\
(\delta u)(a, b, c) &= \gamma_{bc,a}\, u\,(b, c) \;-\; u\,(ab, c) \;+\; u\,(a, bc) \;-\; \gamma_{ab,c}\, u\,(a, b) \;\in G_{abc},\\
(\delta u)(a, b, c, d) &= \gamma_{bcd,a}\, u\,(b, c, d) \;-\; u\,(ab, c, d) \;+\; u\,(a, bc, d)\\
&\quad - u\,(a, b, cd) \;+\; \gamma_{abc,d}\, u\,(a, b, c) \;\in G_{abcd},\\
(\delta u)(a, b, c, d, e) &= \gamma_{bcde,a}\, u\,(b, c, d, e) \;-\; u\,(ab, c, d, e) \;+\; u\,(a, bc, d, e)\\
&\quad - u\,(a, b, cd, e) \;+\; u\,(a, b, c, de) \;-\; \gamma_{abcd,e}\, u\,(a, b, c, d) \;\in G_{abcde},
\end{aligned}$$

for all $a, b, c, d, e \in S$. Then symmetric n-cocycles and n-coboundaries constitute subgroups

$$
\begin{aligned}
Z^n(S, \mathcal{G}) &= \operatorname{Ker} \delta \ \subseteq \ C^n(S, \mathcal{G}) \quad \text{for} \quad n = 1, 2, 3, 4, \ \text{and} \\
B^n(S, \mathcal{G}) &= \operatorname{Im} \delta \ \subseteq \ Z^n(S, \mathcal{G}) \quad \text{for} \quad n = 2, 3, 4
\end{aligned}
$$

of $C^n(S, \mathcal{G})$ (with $B^1(S, \mathcal{G}) = 0 \subseteq C^1(S, \mathcal{G})$); and

$$H^n(S, \mathcal{G}) \ \cong \ Z^n(S, \mathcal{G}) \,/\, B^n(S, \mathcal{G})$$

when $n = 1, 2, 3, 4$ [6,7,9].

If $\tau : \mathcal{G} \longrightarrow \mathcal{G}'$ is a natural transformation and $u \in C^n(S, \mathcal{G})$ is a symmetric cochain, then the cochain $\tau^* u \in \widehat{C}^n(S, \mathcal{G})$ defined by

$$\tau^* u\,(a_1, \ldots, a_n) \ = \ \tau_a \big(u\,(a_1, \ldots, a_n)\big) \in G'_a, \quad \text{where} \ \ a = a_1 a_2 \cdots a_n,$$

is a symmetric cochain, which inherits all parts of (Cn) from u. Thus $\tau : \mathcal{G} \longrightarrow \mathcal{G}'$ induces a homomorphism $\tau^* = C^n(S, \tau) : C^n(S, \mathcal{G}) \longrightarrow C^n(S, G')$. If every τ_a is injective, then so is τ^*. (Surjectivity also transfers; this is proved in Sect. 4).

3. Symmetry (as considered here) applies more generally to functions $f : X \longrightarrow G$ of $n \leqq 4$ variables, where G is an abelian group and X is a subset of the cartesian product $S^n = S \times \cdots \times S$ of some set S, which is *symmetric* in the sense that

$$(x_1, \ldots, x_n) \in X \ \text{ implies } \ (x_{\sigma 1}, \ldots, x_{\sigma n}) \in X$$

for all $x_1, \ldots, x_n \in S$ and every permutation σ of $1, 2, \ldots, n$. If $n = 1$, then every mapping of X into G is symmetric. If $n = 2$, then a mapping $f : X \longrightarrow G$ is *symmetric* if and only if condition (S2):

$$f\,(b, a) \ = \ f\,(a, b) \tag{S2}$$

holds for all $a, b \in S$. If $n = 3$, then $f : X \longrightarrow G$ is *symmetric* if and only if all three parts of condition (S3):

$$f\,(a, b, a) \ = \ 0, \tag{S3a}$$

$$f\,(c, b, a) \ = \ -\,f\,(a, b, c), \quad \text{and} \tag{S3b}$$

$$f\,(a, b, c) + f\,(b, c, a) + f\,(c, a, b) = 0, \tag{S3c}$$

hold for all $a, b, c \in S$. If $n = 4$, then $f : X \longrightarrow G$ is *symmetric* if and only if all four parts of condition (S4):

$$f\,(a, b, b, a) \ = \ 0, \tag{S4a}$$

$$f\,(d, c, b, a) \ = \ -\,f\,(a, b, c, d), \tag{S4b}$$

$$f(a,b,c,d) - f(b,c,d,a) + f(c,d,a,b) - f(d,a,b,c) \;=\; 0, \quad \text{and} \tag{S4c}$$

$$f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a) \;=\; 0. \tag{S4d}$$

hold for all $a, b, c, d \in S$.

If, for example, S is a commutative monoid, then a symmetric 4-cochain $u \in C^4(S, \mathcal{G})$ consists of symmetric functions $(a,b,c,d) \longmapsto u(a,b,c,d) : X_s \longrightarrow G_s$, one for each $s \in S$, where $X_s = \{(a,b,c,d) \in S \times S \times S \times S \mid abcd = s\}$ is a symmetric set, since S is commutative; and similarly for 2- and 3-cochains.

The general definition of symmetric maps can be shortened if $n = 3$ or $n = 4$.

Lemma 1.2 *Condition (S3) is equivalent to the single condition*

$$f(a,b,c) \;=\; f(b,a,c) - f(b,c,a). \tag{S3d}$$

Proof This generalizes the similar result in [6]. Conditions (S3c) and (S3b) imply (S3d):

$$f(a,b,c) \;=\; -f(c,a,b) - f(b,c,a) \;=\; f(b,a,c) - f(b,c,a).$$

Conversely, if (S3d) holds, then, for all $a, b, c \in S$,

$$\begin{aligned}
&f(a,b,a) = f(b,a,a) - f(b,a,a) \;=\; 0,\\
&f(c,b,a) = f(b,c,a) - f(b,a,c) \;=\; -f(a,b,c), \quad \text{and}\\
&f(a,b,c) + f(b,c,a) + f(c,a,b)\\
&\;=\; f(b,a,c) - f(b,c,a) + f(b,c,a) - f(b,a,c) \;=\; 0.
\end{aligned}$$

□

Lemma 1.3 *Conditions (S4b) and (S4d) imply (S4c).*

Proof For all $a, b, c, d \in S$, conditions (S4b) and (S4d) imply

$$\begin{aligned}
f(a,b,c,d) \;&=\; f(b,a,c,d) - f(b,c,a,d) + f(b,c,d,a),\\
f(a,b,c,d) \;&=\; -f(d,c,b,a)\\
&=\; -f(c,d,b,a) + f(c,b,d,a) - f(c,b,a,d), \quad \text{and}\\
f(a,b,c,d) - f(b,c,d,a) &+ f(c,d,a,b) - f(d,a,b,c)\\
&=\; f(a,b,c,d) + f(a,d,c,b)\\
&\quad - f(a,b,d,c) + f(a,d,b,c) - f(a,d,c,b)\\
&\quad - f(a,d,b,c) + f(a,b,d,c) - f(a,b,c,d) \;=\; 0.
\end{aligned}$$

□

4. With respect to symmetric maps, every symmetric subset $X \subseteq S^n$ has a *basis*: a subset Y of X such that every mapping of Y into an abelian group G extends uniquely to a symmetric mapping of X into G.

If S is a totally ordered set, then X has an explicit basis:

Lemma 1.4 *Let X be a symmetric subset of S^n, where $n \leqq 4$ and S is a totally ordered set.*

If $n = 2$, then the set Y of all $(a, b) \in X$ such that $a \leqq b$ is a basis of X.

If $n = 3$, then the set Y of all $(a, b, c) \in X$ such that $a \leqq b$ and $a < c$ is a basis of X.

If $n = 4$, then the set Y of all $(a, b, c, d) \in X$ such that either $a < b, c, d$, or $a \leqq b, c$ and $b < d$, or both, is a basis of X.

Moreover, if f is a mapping of Y into an abelian group *G and g is the* symmetric *mapping of X into G that extends f, then every value of g is a sum of values of f and opposites of values of f.*

The set Y in Lemma 1.4 is the *standard basis* of X (given the total order on S).

Lemma 1.4 is vital for what follows. Its proof is straightforward but rather lengthy, due to the case $n = 4$, and has been moved to Sect. 7.

2 Symmetric chains

1. First we construct abelian groups $C_n(S/B)$ such that $C^n(S, \mathcal{G}) \cong \mathrm{Hom}\,(C_n(S/B),\ G)$ when $n \leqq 4$ and $\mathcal{G}$ is semiconstant at G on a convex subset B of S.

Let $C_0(S/B) = 0$. For $n = 1, 2, 3, 4$ let

$$X \;=\; \{\, (a_1,\, \ldots,\, a_n) \in S^n = S \times \cdots \times S \;\big|\; a_1\, a_2\, \cdots\, a_n \in B \,\}.$$

If $n = 1, 2, 3, 4$, then $C_n(S/B)$ is generated by X subject to the defining relations given below. We denote by

$$\iota : (a_1,\, \ldots,\, a_n) \longmapsto \langle\, a_1,\, \ldots,\, a_n \,\rangle$$

the canonical map of X into $C_n(S/B)$. If $n = 1$, then $X = B$, the set of defining relations is empty, and $C_1(S/B)$ is free on B.

The defining relations of $C_2(S/B)$ are all

$$\langle\, b, a \,\rangle \;=\; \langle\, a, b \,\rangle, \tag{C2}$$

where $a, b \in S$ and $ab \in B$ (so that $(a, b) \in X$). The defining relations of $C_3(S/B)$, collectively denoted by (C3), are all

$$\langle\, a, b, a \,\rangle \;=\; 0, \tag{C3a}$$

$$\langle\, c, b, a \,\rangle \;=\; -\,\langle\, a, b, c \,\rangle, \quad \text{and} \tag{C3b}$$

$$\langle a, b, c \rangle + \langle b, c, a \rangle + \langle c, a, b \rangle = 0, \tag{C3c}$$

where $a, b, c \in S$ and $abc \in B$ ($aba \in B$ in (C3a)). The defining relations of $C_4(S/B)$, collectively denoted by (C4), are all

$$\langle a, b, b, a \rangle = 0, \tag{C4a}$$

$$\langle d, c, b, a \rangle = - \langle a, b, c, d \rangle, \tag{C4b}$$

$$\langle a, b, c, d \rangle - \langle b, c, d, a \rangle + \langle c, d, a, b \rangle - \langle d, a, b, c \rangle = 0, \tag{C4c}$$

$$\langle a, b, c, d \rangle - \langle b, a, c, d \rangle + \langle b, c, a, d \rangle - \langle b, c, d, a \rangle = 0, \tag{C4d}$$

where $a, b, c, d \in S$ and $abcd \in B$ ($abba \in B$ in (C4a)).

For later use it is convenient to define $\langle a_1, \ldots, a_n \rangle$ for all $a_1, \ldots, a_n$ (where $n \leqq 4$): let

$$\begin{aligned}\langle a \rangle &= 0 \text{ if } a \notin B,\\ \langle a, b \rangle &= 0 \text{ if } ab \notin B,\\ \langle a, b, c \rangle &= 0 \text{ if } abc \notin B, \text{ and}\\ \langle a, b, c, d \rangle &= 0 \text{ if } abcd \notin B.\end{aligned}$$

If $n = 2, 3, 4$, then X is a symmetric set and ι is a symmetric mapping of X into $C_n(S/B)$. In particular, it follows from Lemmas 1.3 and 1.2 that the defining relations (C4c) can be omitted from (C4), and that (C3) can be replaced by

$$\langle a, b, c \rangle = \langle b, a, c \rangle - \langle b, c, a \rangle. \tag{C3d}$$

A *symmetric n-chain on S relative to B* is an element of $C_n(S/B)$. In particular, $C_n(S) = C_n(S/S)$ is the group of *symmetric n-chains* on S.

The use of the arbitrary convex subset B in these definitions has two advantages: it covers the important cases $B = S$ and $B = S\backslash\{0\}$, avoiding duplicate proofs and constructions; and it might allow future proofs by induction, as in [8]. Some of the generality disappears if $S\backslash B$ is an ideal:

Proposition 2.1 *If $S\backslash B = I \neq S$ is an ideal of S, then $C_n(S/B) = C_n(T / T\backslash\{0\})$, where $T = S/I$.*

Proof If $a_1, \ldots, a_n \in S$, then $a_1 \ldots a_n \in B$ in S implies $a_1, \ldots, a_n \in B$ in S and $a_1, \ldots, a_n \in S/I$; hence $a_1 \ldots a_n \in B$ in S if and only if $a_1 \ldots a_n \neq 0$ in $T = S/I$. Thus $C_n(S/B)$ and $C_n(T / T\backslash\{0\})$ have the same generators, and these are subject to the same defining relations. □

Theorem 2.2 *$C_n(S/B)$ is a free abelian group. In particular, $C_n(S)$ is a free abelian group. If $n = 2, 3, 4$, then, relative to any total order on S, $C_n(S/B)$ is free on the standard basis of X.*

Proof The groups $C_0(S/B)$ and $C_1(S/B)$ are already free by definition. Let $n = 2, 3, 4$ and let Y be a standard basis of X from Lemma 1.4, so that every mapping f of X into an abelian group G extends uniquely to a symmetric mapping g of X into G; moreover, every value of g is a sum of values of f and opposites of values of f.

If f is the restriction of ι to Y, then $g = \iota$; hence the subgroup of $C_n(S/B)$ generated by $\iota\,(Y)$ contains all values of ι and is all of $C_n(S/B)$.

Now let f be any mapping of Y into an abelian group G, which extends uniquely to a symmetric mapping g of X into G. The values $g\,(a_1, \ldots, a_n)$ of the symmetric mapping g satisfy all the defining relations (Cn) of $C_n(S/B)$. Hence g factors uniquely through ι: there is a homomorphism $\varphi : C_n(S/B) \longrightarrow G$ such that $\varphi \circ \iota = g$. In particular, $\varphi\,(\iota\,(y)) = f(y)$ for all $y \in Y$, and φ is unique with this property since $C_n(S/B)$ is generated by $\iota\,(Y)$. Thus $C_n(S/B)$ is free on Y (via ι). □

Given the (arbitrary) total order on S, Lemma 1.4 provides an explicit description of Y:

if $n = 2$, then Y is the set of all (a, b) such that $a, b \in S$, $ab \in B$, and $a \leqq b$;

if $n = 3$, then Y is the set of all (a, b, c) such that $a, b, c \in S$, $abc \in B$, $a \leqq b$, and $a < c$;

if $n = 4$, then Y is the set of all (a, b, c, d) such that $a, b, c, d \in S$, $abcd \in B$, and either $a < b, c, d$, or $a \leqq b, c$ and $b < d$, or both.

In the above, Y is the *standard basis* of $C_n(S/B)$ (given the total order on S).

The definition of symmetric chains was cooked up so that the next result would hold:

Proposition 2.3 *If $\mathcal{G} = \mathcal{F}(B, G)$ is semiconstant, then for $n = 1, 2, 3, 4$ there is an isomorphism U : $\mathrm{Hom}\,(C_n(S/B),\ G) \cong C^n(S, \mathcal{G})$ which is natural in G, and assigns to $\varphi : C_n(S/B) \longrightarrow G$ the cochain $u = U\,(\varphi)$ defined by:*

$$u\,(a_1, \ldots, a_n) = \begin{cases} \varphi\,\langle\, a_1, \ldots, a_n \,\rangle \in G & \text{if } a_1 \ldots a_n \in B, \\ = 0 \in G_{a_1 \ldots a_n} & \text{if } a_1 \ldots a_n \notin B, \end{cases}$$

for all $a_1, \ldots, a_n \in S$.

Proof One may assume that $G_a = 0 \subseteq G$ when $a \notin B$, so that $u\,(a_1, \ldots, a_n) = \varphi\,\langle\, a_1, \ldots, a_n \,\rangle$ for all $a_1, \ldots, a_n$ in the above.

Let $\varphi : C_n(S/B) \longrightarrow G$ be a homomorphism. For the elements $\varphi\,\langle\, a_1, \ldots, a_n \,\rangle$ of G such that $a_1\, a_2 \cdots a_n \in B$, property (Sn) follows from (Cn). This is also the case, trivially, if $a_1\, a_2 \cdots a_n \notin B$. Thus $U\,(\varphi)$ is a symmetric n-cochain.

Conversely, let $u \in C^n(S, \mathcal{G})$. The elements $u\,(a_1, \ldots, a_n)$ of G, where $a_1, \ldots, a_n \in S$ and $a_1\, a_2 \cdots a_n \in B$, satisfy all defining relations (Cn) of $C_n(S/B)$. Therefore there is a unique homomorphism $\varphi : C_n(S/B) \longrightarrow G$ such that $\varphi\,\langle\, a_1, \ldots, a_n \,\rangle = u\,(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in S$ such that $a_1\, a_2 \cdots a_n \in B$. This sets up a one-to-one correspondence between $C^n(S, \mathcal{G})$ and $\mathrm{Hom}\,(C_n(S/B),\ G)$, which preserves pointwise addition and is natural in G. □

2. Boundaries are defined as follows (so that Lemma 2.7 will hold).

Proposition 2.4 *Let B be a convex subset of S. For each $n = 1, 2, 3, 4$ there exists a unique homomorphism $\partial_n : C_n(S/B) \longrightarrow C_{n-1}(S/B)$ such that, for all $a, b, c, d \in S$,*

$$\partial_1\langle a \rangle = 0, \tag{B1}$$

$$\partial_2\langle a, b \rangle = \langle b \rangle - \langle ab \rangle + \langle a \rangle \text{ if } ab \in B, \tag{B2}$$

$$\partial_3\langle a, b, c \rangle = \langle b, c \rangle - \langle ab, c \rangle + \langle a, bc \rangle - \langle a, b \rangle \text{ if } abc \in B, \tag{B3}$$

$$\partial_4\langle a, b, c, d \rangle = \langle b, c, d \rangle - \langle ab, c, d \rangle + \langle a, bc, d \rangle - \langle a, b, cd \rangle + \langle a, b, c \rangle \text{ if } abcd \in B. \tag{B4}$$

Moreover, $\partial_{n-1} \circ \partial_n = 0$ ($n = 2, 3, 4$).

Proof First, $\partial_1 = 0$. Next, if $a, b \in S$, then $\langle a \rangle - \langle ba \rangle + \langle b \rangle = \langle b \rangle - \langle ab \rangle + \langle a \rangle$, since S is commutative. Hence the 1-chains $\langle b \rangle - \langle ab \rangle + \langle a \rangle$, where $a, b \in S$ and $ab \in B$, have property (C2). Therefore there is a unique homomorphism $\partial_2 : C_2(S/B) \longrightarrow C_1(S/B)$ such that $\partial_2\langle a, b \rangle = \langle b \rangle - \langle ab \rangle + \langle a \rangle$ for all $a, b \in S$ such that $ab \in B$.

Similarly, if $a, b, c \in S$, then it follows from (C2) and commutativity in S that

$$\begin{aligned}
&\langle b, a \rangle - \langle ab, a \rangle + \langle a, ba \rangle - \langle a, b \rangle = 0,\\
&\ \langle b, a \rangle - \langle cb, a \rangle + \langle c, ba \rangle - \langle c, b \rangle\\
&= -\big(\langle b, c \rangle - \langle ab, c \rangle + \langle a, bc \rangle - \langle a, b \rangle\big), \quad \text{and}\\
&\langle b, c \rangle - \langle ab, c \rangle + \langle a, bc \rangle - \langle a, b \rangle\\
&\quad + \langle c, a \rangle - \langle bc, a \rangle + \langle b, ca \rangle - \langle b, c \rangle\\
&\quad + \langle a, b \rangle - \langle ca, b \rangle + \langle c, ab \rangle - \langle c, a \rangle = 0.
\end{aligned}$$

Hence the 2-chains $\langle b, c \rangle - \langle ab, c \rangle + \langle a, bc \rangle - \langle a, b \rangle$, where $a, b, c \in S$ and $abc \in B$, have properties (C3a), (C3b), and (C3c). Therefore there is a unique homomorphism $\partial_3 : C_3(S/B) \longrightarrow C_2(S/B)$ such that (B3) holds.

Similarly, if $a, b, c, d \in S$, then it follows from (C3a), (C3b), (C3c), (C3d), and commutativity in S that

$$\begin{aligned}
&\langle b, b, a \rangle - \langle ab, b, a \rangle + \langle a, bb, a \rangle - \langle a, b, ba \rangle + \langle a, b, b \rangle = 0,\\
&\langle c, b, a \rangle - \langle dc, b, a \rangle + \langle d, cb, a \rangle - \langle d, c, ba \rangle + \langle d, c, b \rangle\\
&= -\big(\langle b, c, d \rangle - \langle ab, c, d \rangle + \langle a, bc, d \rangle - \langle a, b, cd \rangle + \langle a, b, c \rangle\big),\\
&\big(\langle b, c, d \rangle - \langle ab, c, d \rangle + \langle a, bc, d \rangle - \langle a, b, cd \rangle + \langle a, b, c \rangle\big)\\
&\ - \big(\langle c, d, a \rangle - \langle bc, d, a \rangle + \langle b, cd, a \rangle - \langle b, c, da \rangle + \langle b, c, d \rangle\big)\\
&\ + \big(\langle d, a, b \rangle - \langle cd, a, b \rangle + \langle c, da, b \rangle - \langle c, d, ab \rangle + \langle c, d, a \rangle\big)\\
&\ - \big(\langle a, b, c \rangle - \langle da, b, c \rangle + \langle d, ab, c \rangle - \langle d, a, bc \rangle + \langle d, a, b \rangle\big) = 0,\\
&\quad \text{and}\\
&\big(\langle b, c, d \rangle - \langle ab, c, d \rangle + \langle a, bc, d \rangle - \langle a, b, cd \rangle + \langle a, b, c \rangle\big)\\
&\ - \big(\langle a, c, d \rangle - \langle ba, c, d \rangle + \langle b, ac, d \rangle - \langle b, a, cd \rangle + \langle b, a, c \rangle\big)\\
&\ + \big(\langle c, a, d \rangle - \langle bc, a, d \rangle + \langle b, ca, d \rangle - \langle b, c, ad \rangle + \langle b, c, a \rangle\big)
\end{aligned}$$

$$- \big(\langle c, d, a \rangle - \langle bc, d, a \rangle + \langle b, cd, a \rangle - \langle b, c, da \rangle + \langle b, c, d \rangle \big) = 0.$$

Thus the 3-chains $\langle b, c, d \rangle - \langle ab, c, d \rangle + \langle a, bc, d \rangle - \langle a, b, cd \rangle + \langle a, b, c \rangle$, where $a, b, c, d \in S$ and $abcd \in B$, have properties (C4a), (C4b), (C4c) and (C4d). Therefore there is a unique homomorphism $\partial_4 : C_4(S/B) \longrightarrow C_3(S/B)$ such that (B4) holds.

Finally, we have $\partial_1 \circ \partial_2 = 0$. Let $a, b, c \in S$, $abc \in B$. If $ab, bc \in B$, then

$$\begin{aligned} \partial_2\, \partial_3 \langle a, b, c \rangle &= \partial_2 \big(\langle b, c \rangle - \langle ab, c \rangle + \langle a, bc \rangle - \langle a, b \rangle \big) \\ &= \langle c \rangle - \langle bc \rangle + \langle b \rangle - \langle c \rangle + \langle abc \rangle - \langle ab \rangle \\ &\quad + \langle bc \rangle - \langle abc \rangle + \langle a \rangle - \langle b \rangle + \langle ab \rangle - \langle a \rangle = 0. \end{aligned}$$

If $ab \notin B$ and $bc \in B$, then $a \notin B$, since B is convex; $b \notin B$, since $ba \notin B$ and $bac \in B$; $\partial_2 \langle a, b \rangle = 0$; and

$$\begin{aligned} \partial_2\, \partial_3 \langle a, b, c \rangle &= \partial_2 \big(\langle b, c \rangle - \langle ab, c \rangle + \langle a, bc \rangle - \langle a, b \rangle \big) \\ &= \langle c \rangle - \langle bc \rangle + \langle b \rangle - \langle c \rangle + \langle abc \rangle - \langle ab \rangle \\ &\quad + \langle bc \rangle - \langle abc \rangle + \langle a \rangle = 0. \end{aligned}$$

Exchanging a and c yields $\partial_2\, \partial_3 \langle a, b, c \rangle = 0$ if $ab \in B$ and $bc \notin B$. Finally, if $ab, bc \notin B$, then, as above, $a, b \notin B$ and $\partial_2\, \partial_3 \langle a, b, c \rangle = 0$. Hence $\partial_2 \circ \partial_3 = 0$.

Similarly, let $a, b, c, d \in S$, $abcd \in B$. If $abc, bcd \in B$, then

$$\begin{aligned} &\partial_3\, \partial_4 \langle a, b, c, d \rangle \\ &\quad = \partial_3 \big(\langle b, c, d \rangle - \langle ab, c, d \rangle + \langle a, bc, d \rangle - \langle a, b, cd \rangle + \langle a, b, c \rangle \big) \\ &\quad = \big(\langle c, d \rangle - \langle bc, d \rangle + \langle b, cd \rangle - \langle b, c \rangle \big) \\ &\qquad - \big(\langle c, d \rangle + \langle abc, d \rangle - \langle ab, cd \rangle + \langle ab, c \rangle \big) \\ &\qquad + \big(\langle bc, d \rangle - \langle abc, d \rangle + \langle a, bcd \rangle - \langle a, bc \rangle \big) \\ &\qquad - \big(\langle b, cd \rangle + \langle ab, cd \rangle - \langle a, bcd \rangle + \langle a, b \rangle \big) \\ &\qquad + \big(\langle b, c \rangle - \langle ab, c \rangle + \langle a, bc \rangle - \langle a, b \rangle \big) = 0. \end{aligned}$$

If $abc \notin B$ and $bcd \in B$, then $a \notin B$, since B is convex and $abcd \in B$; $b \notin B$, since $bac \notin B$ and $bacd \in B$; $ab \notin B$, since $abc \notin B$ and $abcd \in B$; $bc \notin B$ since $bca \notin B$ and $bcad \in B$; $\partial_3 \langle a, b, c \rangle = 0$; and

$$\begin{aligned} &\partial_3\, \partial_4 \langle a, b, c, d \rangle \\ &\quad = \partial_3 \big(\langle b, c, d \rangle - \langle ab, c, d \rangle + \langle a, bc, d \rangle - \langle a, b, cd \rangle + \langle a, b, c \rangle \big) \\ &\quad = \big(\langle c, d \rangle - \langle bc, d \rangle + \langle b, cd \rangle - \langle b, c \rangle \big) \\ &\qquad - \big(\langle c, d \rangle - \langle abc, d \rangle + \langle ab, cd \rangle - \langle ab, c \rangle \big) \\ &\qquad + \big(\langle bc, d \rangle - \langle abc, d \rangle + \langle a, bcd \rangle - \langle a, bc \rangle \big) \\ &\qquad - \big(\langle b, cd \rangle - \langle ab, cd \rangle + \langle a, bcd \rangle - \langle a, b \rangle \big) = 0. \end{aligned}$$

Exchanging a and d yields $\partial_2\, \partial_3\langle\, a, b, c, d\,\rangle = 0$ if $abc \in B$ and $bcd \notin B$. Finally, if $abd, bcd \notin B$, then, as above, $a, b, ab, bc \notin B$ and $\partial_3\, \partial_4\langle\, a, b, c, d\,\rangle = 0$. Therefore $\partial_3 \circ \partial_4 = 0$. □

Relative to a convex subset B of S, a *symmetric n-cycle* is an element of

$$Z_n(S/B) \;=\; \operatorname{Ker} \partial_n \;\subseteq\; C_n(S/B)$$

(where $n = 1, 2, 3, 4$); a *symmetric n-boundary* is an element of

$$B_n(S/B) \;=\; \operatorname{Im} \partial_{n+1} \;\subseteq\; C_n(S/B)$$

(where $n = 0, 1, 2, 3$). In Proposition 2.4, $\partial_n \circ \partial_{n+1} = 0$ implies $B_n(S/B) \subseteq Z_n(S/B)$. The *commutative homology groups of S relative to B* are the abelian groups

$$H_n(S/B) \;=\; Z_n(S/B)\, /\, B_n(S/B),$$

where $n = 1, 2, 3$. In particular, the *commutative homology groups of S* are the groups $H_n(S) = H_n(S/S)$ $(n = 1, 2, 3)$.

We take a closer look at $H_1(S/B)$. First we note that every subset P of S can be viewed as a commutative partial semigroup in which the product ab of two elements a, b of P is defined if and only if their product in S lies in P, in which case the two products are equal. In a classic case of adjoint functors, P has a *universal abelian group* $F(P)$ and a partial homomorphism $\iota : P \longrightarrow F(P)$, such that for every partial homomorphism f of P into an abelian group G there is a unique homomorphism φ of $F(P)$ into G such that $\varphi \circ \iota = f$. It is readily seen that $F(P)$ can be described as the abelian group generated by all $[a]$ $(= \iota(a))$ with $a \in P$, subject to all defining relations $[ab] = [a] + [b]$, where $a, b, ab \in P$. Under pointwise addition, the partial homomorphisms of P into an abelian group G constitute an abelian group $\operatorname{PHom}(P, G)$, and the adjunction provides an isomorphism $\varphi \longmapsto \varphi \circ \iota$ of $\operatorname{Hom}(F(P), G)$ onto $\operatorname{PHom}(P, G)$, which is natural in G.

Proposition 2.5 *If $S\backslash B \neq S$ is an ideal of S, then $H_1(S/B)$ is the universal abelian group of the commutative partial monoid B, and $H^1(S, \mathcal{G}) \cong \operatorname{PHom}(B, G)$ whenever $\mathcal{G} = \mathcal{F}(B, G)$ is semiconstant on B. In particular, $H_1(S)$ is the universal abelian group of S and $H^1(S, \mathcal{G}) \cong \operatorname{Hom}(S, G)$ whenever $\mathcal{G}$ is constant at G.*

Proof We saw that $C_0(S/B) = 0$, so that $Z_1(S/B) = C_1(S/B)$ and $H_0(S/B) = 0$, and that $\partial_2 : C_2(S/B) \longrightarrow C_1(S/B)$ sends every generator $\langle\, a, b\,\rangle$ of $C_2(S/B)$, where $ab \in B$, to $\langle\, b\,\rangle - \langle\, ab\,\rangle + \langle\, a\,\rangle$. Since $S\backslash B$ is an ideal, $ab \in B$ implies $a, b \in B$. Hence $H_1(S/B)$ is the abelian group generated by all $\langle\, a\,\rangle$ with $a \in B$, subject to all defining relations $\langle\, ab\,\rangle = \langle\, a\,\rangle + \langle\, b\,\rangle$, where $a, b \in S$ and $ab \in B$; in other words, $H_1(S/B)$ is the universal abelian group of the commutative partial monoid B. The partial homomorphism $\iota : B \longrightarrow H_1(S/B)$ sends each $a \in B$ to the coset of $\langle\, a\,\rangle$ in $H_1(S/B)$. The isomorphism $H^1(S, \mathcal{G}) \cong \operatorname{PHom}(B, G)$ then follows from Proposition 2.3. □

The homology groups $H_n(S)$ (with $B = S$) tell us nothing if S has a zero element (hence $H_n(S\, /\, S\backslash\{0\})$.

Proposition 2.6 *If S has a zero element, then $H_n(S) = 0$ for $n = 1, 2, 3$.*

Proof For every $a \in S$, $\partial_2 \langle a, 0 \rangle = \langle 0 \rangle - \langle a0 \rangle + \langle a \rangle = \langle a \rangle$; hence every 1-chain is a boundary and $H_1(S) = 0$. (This also follows from Proposition 2.5, since 0 is the only homomorphism of S into G.)

Place an arbitrary total order $\leqq$ on S. For every $a, b \in S$, homomorphisms $\sigma : C_2(S) \longrightarrow C_3(S)$ and $\tau : C_2(S) \longrightarrow C_3(S)$ are well-defined by:

$$\sigma \langle a \rangle = \langle a, 0 \rangle \text{ and}$$
$$\tau \langle a, b \rangle = \begin{cases} \langle a, b, 0 \rangle & \text{if } a \leqq b, \\ \langle b, a, 0 \rangle & \text{if } a \geqq b. \end{cases}$$

Indeed the two definitions of $\tau \langle a, b \rangle$ agree if $a = b$, and $\tau \langle b, a \rangle = \tau \langle a, b \rangle$ for all a, b.

We show that $\langle a, b \rangle = \sigma \partial_2 \langle a, b \rangle - \partial_3 \tau \langle a, b \rangle$ for all $a, b \in S$ (so that σ, τ are the beginning of a contracting homotopy). If $a \leqq b$, then

$$\begin{aligned} &\sigma \partial_2 \langle a, b \rangle - \partial_3 \tau \langle a, b \rangle \\ &\quad = \sigma \langle b \rangle - \sigma \langle ab \rangle + \sigma \langle a \rangle - \partial_3 \langle a, b, 0 \rangle \\ &\quad = \langle b, 0 \rangle - \langle ab, 0 \rangle + \langle a, 0 \rangle - \langle b, 0 \rangle + \langle ab, 0 \rangle - \langle a, b0 \rangle + \langle a, b \rangle \\ &\quad = \langle a, b \rangle. \end{aligned}$$

If $a \geqq b$, then

$$\begin{aligned} &\sigma \partial_2 \langle a, b \rangle - \partial_3 \tau \langle a, b \rangle \\ &\quad = \sigma \langle b \rangle - \sigma \langle ab \rangle + \sigma \langle a \rangle - \partial_3 \langle b, a, 0 \rangle \\ &\quad = \langle b, 0 \rangle - \langle ab, 0 \rangle + \langle a, 0 \rangle - \langle a, 0 \rangle + \langle ab, 0 \rangle - \langle b, a0 \rangle + \langle b, a \rangle \\ &\quad = \langle a, b \rangle. \end{aligned}$$

It follows that $\sigma \partial_2 u - \partial_3 \tau u = u$ for every $u \in C_2(S)$. If $\partial_2 u = 0$, then $u = \partial_3 (-\tau u)$. Thus every 2-cycle is a 2-boundary, and $H_2(S) = 0$.

Next we show that, for every $a, b, c \in S$, we have

$$\langle a, b, c \rangle + \tau \partial_3 \langle a, b, c \rangle = \partial_4 v$$

for some $v \in C_4(S)$. First, for all $x, y \in S$,

$$\begin{aligned} \partial_4 \langle x, 0, y, 0 \rangle &= \langle 0, y, 0 \rangle - \langle x0, y, 0 \rangle + \langle x, 0y, 0 \rangle - \langle x, 0, y0 \rangle \\ &\quad + \langle x, 0, y \rangle = \langle x, 0, y \rangle, \end{aligned}$$

so that

$$\langle y, x, 0 \rangle = \langle x, y, 0 \rangle - \langle x, 0, y \rangle = \langle x, y, 0 \rangle - \partial_4 \langle x, 0, y, 0 \rangle$$

by (C3d). Hence

$$\tau\,\langle\, x, y\,\rangle \;=\; \langle\, x, y, 0\,\rangle \;+\; \partial_4\, v,$$

where $v = 0$ if $x \leqq y$ and $v = -\,\langle\, x, 0, y, 0\,\rangle$ if $x \geqq y$. Now

$$\begin{aligned}\tau\,\partial_3\,\langle\, a, b, c\,\rangle &= \tau\,\langle\, b, c\,\rangle \;-\; \tau\,\langle\, ab, c\,\rangle \;+\; \tau\,\langle\, a, bc\,\rangle \;-\; \tau\,\langle\, a, b\,\rangle\\ &= \langle\, b, c, 0\,\rangle \;-\; \langle\, ab, c, 0\,\rangle \;+\; \langle\, a, bc, 0\,\rangle \;-\; \langle\, a, b, 0\,\rangle \;+\; \partial_4 v\\ &= \partial_4\,\langle\, a, b, c, 0\,\rangle \;-\; \langle\, a, b, c\,\rangle \;+\; \partial_4 v\end{aligned}$$

for some $v \in C_4(S)$, whence $\langle\, a, b, c\,\rangle + \tau\,\partial_3\,\langle\, a, b, c\,\rangle \in \operatorname{Im}\partial_4$. Hence $u + \tau\,\partial_3\, u \in \operatorname{Im}\partial_4$ for every $u \in C_3(S)$. If $\partial_3\, u = 0$, then $u \in \operatorname{Im}\partial_4$. Hence $H_3(S) = 0$. □

3. If $\mathcal{G} = \mathcal{F}(B, G)$ is semiconstant, then, in dimensions $n = 1, 2, 3$, the commutative cohomology groups $H^n(S, \mathcal{G})$ are the cohomology groups of the symmetric chain complex

$$C_*(S/B) :\; 0 \xleftarrow{\partial_1} C_1(S/B) \xleftarrow{\partial_2} C_2(S/B) \xleftarrow{\partial_3} C_3(S/B) \xleftarrow{\partial_4} C_4(S/B).$$

This is proved as follows, using the isomorphism U in Proposition 2.3.

Lemma 2.7 *If $n = 1, 2, 3, 4$, $\mathcal{G} = \mathcal{F}(B, G)$, $\varphi : C_{n-1}(S/B) \longrightarrow G$ is a homomorphism, and $u = U\,(\varphi) \in C^{n-1}(S, \mathcal{G})$, then $\delta u = U\,(\varphi \circ \partial_n)$.*

Proof If $n = 1$, then $u = 0$, $\delta u = 0$, $\partial_n = 0$, and $\delta u = U\,(\varphi \circ \partial_n)$. In general, one may assume that $G_a = 0 \subseteq G$ when $a \in N$, so that $U\,(\varphi)(a_1, \ldots, a_n) = \varphi\,\langle\, a_1, \ldots, a_n\,\rangle$ for all $a_1, \ldots, a_n$.

Let $n = 2$. If $a, b \in S$ and $ab \in B$, then

$$(\delta u)(a, b) \;=\; u\,(b) \;-\; u\,(ab) \;+\; u\,(a) \;=\; \varphi\,\big(\langle\, b\,\rangle - \langle\, ab\,\rangle + \langle\, a\,\rangle\big) \;=\; \varphi\,(\partial_2\langle\, a, b\,\rangle).$$

If $ab \notin B$, then $\langle\, a, b\,\rangle = 0$ and $(\delta u)(a, b) = 0$, since $(\delta u)(a, b) \in G_{ab}$. Hence $\delta u = U\,(\varphi \circ \partial_2)$.

If $n = 3$, then $u\,(a, b) = \varphi\,\langle\, a, b\,\rangle$ and, for all $a, b, c \in S$ such that $abc \in B$,

$$\begin{aligned}(\delta u)(a, b, c) &= u\,(b, c) \;-\; u\,(ab, c) \;+\; u\,(a, bc) \;-\; u\,(a, b)\\ &= \varphi\,\big(\langle\, b, c\,\rangle - \langle\, ab, c\,\rangle + \langle\, a, bc\,\rangle - \langle\, a, b\,\rangle\big) \;=\; \varphi\,(\partial_3\langle\, a, b, c\,\rangle).\end{aligned}$$

If $abc \notin B$, then $\langle\, a, b, c\,\rangle = 0$ and $(\delta u)(a, b, c) = 0$, since $(\delta u)(a, b, c) \in G_{abc}$. Hence $\delta u = U\,(\varphi \circ \partial_3)$.

If $n = 4$, then $u\,(a, b, c) = \varphi\,\langle\, a, b, c\,\rangle$ and, for all $a, b, c, d \in S$ such that $abcd \in B$,

$$\begin{aligned}(\delta u)(a, b, c, d) &= u\,(b, c, d) - u\,(ab, c, d) + u\,(a, bc, d) = -\,u\,(a, b, cd) + u\,(a, b, c)\\ &= \varphi\,\big(\langle\, b, c, d\,\rangle - \langle\, ab, c, d\,\rangle + \langle\, a, bc, d\,\rangle - \langle\, a, b, cd\,\rangle + \langle\, a, b, c\,\rangle\big)\\ &= \varphi\,(\partial_4\langle\, a, b, c, d\,\rangle).\end{aligned}$$

If $abcd \notin B$, then $\langle\, a, b, c, d\,\rangle = 0$ and $(\delta u)(a, b, c, d) = 0$, since $(\delta u)(a, b, c, d) \in G_{abcd}$. Hence $\delta u = U\,(\varphi \circ \partial_4)$. □

Theorem 2.8 *Let $n = 1, 2, 3$. If $\mathcal{G}$ is semiconstant at G on B, then U induces an isomorphism $H^n(S, \mathcal{G}) \cong H^n(C_*(S/B),\ G)$ which is natural in G.*

Proof In the cohomology of $C_*(S/B)$ with coefficients in G, an n-cochain is a homomorphism $\varphi : C_n(S/B) \longrightarrow G$; an n-cocycle is a homomorphism $\varphi : C_n(S/B) \longrightarrow G$ such that $\varphi \circ \partial_{n+1} = 0$; an n-coboundary is a homomorphism $\psi \circ \partial_n : C_n(S/B) \longrightarrow G$ for some homomorphism $\psi : C_{n-1}(S/B) \longrightarrow G$. These constitute abelian groups C^n, Z^n, and B^n, such that $B^n \subseteq Z_n$ since $\partial_n \circ \partial_{n+1} = 0$. By Lemma 2.7, the isomorphism U in Proposition 2.3 sends Z^n onto $Z^n(S, \mathcal{G})$, sends B^n onto $B^n(S, \mathcal{G})$, and induces an isomorphism $H^n(C_*(S/B),\ G) \cong H^n(S, \mathcal{G})$, which, like U, is natural in G. □

3 The universal coefficients theorem

Since $C_n(S/B)$ is a free abelian group when $n = 0, 1, 2, 3, 4$, and subgroups of free abelian groups are also free, Theorem 2.8 yields a universal coefficients theorem:

Theorem 3.1 *If $\mathcal{G} = \mathcal{F}(B, G)$ is semiconstant at G, then there is an isomorphism*

$$H^n(S, \mathcal{G}) \ \cong\ \mathrm{Ext}\,(H_{n-1}(S/B),\ G) \oplus \mathrm{Hom}\,(H_n(S/B),\ G) \ \ (n = 1, 2, 3)$$

which is natural in G. In particular, $H^n(S, \mathcal{G}) \cong \mathrm{Ext}\,(H_{n-1}(S),\ G) \oplus \mathrm{Hom}\,(H_n(S),\ G)$ if $\mathcal{G}$ is constant at G.

Proof This follows from more general Universal Coefficient Theorems (e.g. Theorem 3.6.5 of [14]). □

Corollary 3.2 *If S has a zero element and $\mathcal{G}$ is constant, then $H^n(S, \mathcal{G}) = 0$ for $n = 1, 2, 3$.*

Proof By Proposition 2.6, $H_n(S) = 0$ for all $n \leqq 3$. □

This extends the similar result for nilmonoids, obtained in [10] by different methods.

Corollary 3.3 *If S has a zero element, J is an ideal of S, and $\mathcal{G} = \mathcal{F}(G\ /\ S\backslash J)$, then $H^1(S, \mathcal{G}) = 0$, $H^2(S, \mathcal{G}) \cong H^1(S, \mathcal{G}')$ and $H^3(S, \mathcal{G}) \cong H^2(S, \mathcal{G}')$, where $\mathcal{G}' = \mathcal{F}(G/J)$.*

Proof Let $\mathcal{G}'' = \mathcal{F}(G/S)$ be constant at G. The short exact sequences

$$\begin{aligned} &0 \longrightarrow G_a = 0 \longrightarrow\ G''_a = G \xrightarrow{=} G'_a = G \longrightarrow 0 \ \text{ if } \ a \in S\backslash J, \\ &0 \longrightarrow G_a = G \xrightarrow{=} G''_a = G \longrightarrow G'_a = 0 \longrightarrow 0 \ \text{ if } \ a \in J, \end{aligned}$$

yield a short exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow \mathcal{G}' \longrightarrow 0$$

that induces exact sequences

$$\begin{array}{l} 0 \longrightarrow H^1(S, \mathcal{G}) \longrightarrow H^1(S, \mathcal{G}'') \\ H^1(S, \mathcal{G}'') \longrightarrow H^1(S, \mathcal{G}') \longrightarrow H^2(S, \mathcal{G}) \longrightarrow H^2(S, \mathcal{G}'') \\ H^2(S, \mathcal{G}'') \longrightarrow H^2(S, \mathcal{G}') \longrightarrow H^3(S, \mathcal{G}) \longrightarrow H^3(S, \mathcal{G}'') \end{array}$$

in which $H^1(S, \mathcal{G}'') = H^2(S, \mathcal{G}'') = H^3(S, \mathcal{G}'') = 0$ by Corollary 3.2. □

Corollary 3.4 *If S is free, then $H_2(S) = H_3(S) = 0$.*

Proof Let $S = \mathbb{F}$ be free on X. It is a property of commutative semigroup cohomology that $H^n(\mathbb{F}, \mathcal{G}) = 0$ for all $n \geqq 2$ and all $\mathcal{G}$ [7]. Now the universal abelian group of $\mathbb{F}$ is the free abelian group $\mathbb{G}$; hence $H_1(\mathbb{F}) \cong \mathbb{G}$, by Proposition 2.5. If $\mathcal{G}$ is constant on $\mathbb{F}$ at G, then Theorem 3.1 yields

$$\mathrm{Hom}\,(H_2(\mathbb{F}), G) \;\cong\; \mathrm{Ext}\,(\mathbb{G}, G) \oplus \mathrm{Hom}\,(H_2(\mathbb{F}), G) \;\cong\; H^2(\mathbb{F}, \mathcal{G}) \;=\; 0\,;$$

hence $\mathrm{Hom}\,(H_2(\mathbb{F}), G) = 0$ for every abelian group G, and it follows that $H_2(\mathbb{F}) = 0$. Then Theorem 3.1 also yields

$$\mathrm{Hom}\,(H_3(\mathbb{F}), G) \;\cong\; \mathrm{Ext}\,(H_2(\mathbb{F}),\; G) \oplus \mathrm{Hom}\,(H_3(\mathbb{F}), G) \;\cong\; H^3(\mathbb{F}, \mathcal{G}) \;=\; 0\,;$$

hence $\mathrm{Hom}\,(H_3(\mathbb{F}), G) = 0$ for every abelian group G, and it follows that $H_3(\mathbb{F}) = 0$. □

2. We now look at H^1.

Proposition 3.5 *If $S\backslash B \neq S$ is an ideal of S and $\mathcal{G} = \mathcal{F}(B, G)$ is semiconstant, then*

$$H^1(S, \mathcal{G}) \;\cong\; \mathrm{PHom}(B, G),$$

the group of partial homomorphisms of B into G. In particular, $H^1(S, \mathcal{G}) \cong$ $\mathrm{Hom}\,(S, G)$ whenever $\mathcal{G}$ is constant at G.

Proof By Proposition 2.5, $H_1(S/B)$ is the universal abelian group of the commutative partial monoid B; hence Theorem 3.1 yields $H^1(S, \mathcal{G}) \cong \mathrm{Hom}\,(H_1(S/B),\; G) \cong$ $\mathrm{PHom}(B, G)$. In fact, if $S\backslash B \neq S$ is an ideal of S, then 1-cocycles $u \in Z^1(S, \mathcal{G})$ coincide with partial homomorphisms of B into G. In particular, if $B = S$, then $H_1(S) = H_1(S/B) \cong \mathrm{Hom}\,(S, G)$. □

Example 3.6 is the commutative nilmonoid

$$S \;\cong\; \langle\, x, y \,\big|\, x^5 = x^3y = x^2y^2 = y^4 = 0,\; x^4 = x^2y = xy^3 \,\rangle$$

(this is Example 3.3 of [10]). A partial homomorphism $\varphi : S\backslash\{0\} \longrightarrow G$ into an abelian group G is determined by $g = \varphi\,(x)$ and $h = \varphi\,(y)$ such that $4g = 2g + h = g + 3h$, equivalently, $2g = h$ and $g = 2h$; thus φ is determined by $g \in G$ such that $3g = 0$. If $\mathcal{G}$ is almost constant at G, then

$$H^1(S, \mathcal{G}) \;\cong\; \mathrm{PHom}(S\backslash\{0\},\, G) \;\cong\; \{\, g \in G \;\big|\; 3g = 0 \,\} \;\cong\; \mathrm{Hom}\,(\mathbb{Z}_3,\, G)$$

and $H_1(S \,/\, S\backslash\{0\})$ is cyclic of order 3. □

Example 3.7 is the commutative nilmonoid

$$S \;\cong\; \langle\, x, y \;\big|\; x^8 = x^5 y^2 = x^3 y^4 = x y^6 = y^7 = 0,\; x^4 y^2 = x^2 y^4,\; x^3 y^3 = x y^5 \,\rangle.$$

A partial homomorphism $\varphi : S\backslash\{0\} \longrightarrow G$ into an abelian group G is determined by $g = \varphi(x)$ and $h = \varphi(y)$ such that $4g + 2h = 2g + 4h$ and $3g + 3h = g + 5h$, equivalently, $2g = 2h$; thus φ is determined h and t $(= g - h)$ such that $2t = 0$. If $\mathcal{G}$ is almost constant at G, then

$$H^1(S, \mathcal{G}) \;\cong\; \mathrm{PHom}(S\backslash\{0\},\, G) \;\cong\; G \oplus \{\, t \in G \;\big|\; 2t = 0 \,\} \;\cong\; \mathrm{Hom}\,(\mathbb{Z} \oplus \mathbb{Z}_2,\, G)$$

and $H_1(S \,/\, S\backslash\{0\}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. □

An example that is not a nilmonoid is given in Sect. 6.

4 Cohomology with thin coefficients

1. In this section, $\mathbf{T}$ is the category of thin abelian group valued functors on S; we construct a projective chain complex $\mathcal{A}_*(S)$ in the category $\mathbf{T}$ such that $H^n(\mathcal{A}_*(S), \mathcal{G}) \cong H^n(S, \mathcal{G})$ when $\mathcal{G}$ is thin. First we analyze the chain groups $C_n(S)$ as follows.

Lemma 4.1 *Let $n = 1, 2, 3, 4$. For each $s \in S$ let $C_n(s)$ be the subgroup of $C_n(S)$ generated by all $\langle\, a_1,\, \ldots,\, a_n \,\rangle$ such that $a_1\, a_2\, \cdots\, a_n = s$. Then $C_n(s)$ is, up to isomorphism, the abelian group generated by all $(a_1,\, \ldots,\, a_n)$ such that $a_1\, a_2\, \cdots\, a_n = s$, subject to all defining relations (Cn). Moreover, $C_n(S) = \bigoplus_{s \in S} C_n(s)$.*

Proof Let $n = 2, 3, 4$. For each $s \in S$ let $B_n = B_n(s)$ be the abelian group generated by the symmetric set

$$X_s \;=\; \{\, (a_1,\, \ldots,\, a_n) \in S^n \;\big|\; a_1\, a_2\, \cdots\, a_n = s \,\},$$

subject to all defining relations (Cn). As in the proof of Theorem 2.2 we show that B_n is free and that, relative to any total order on S, B_n is free on the standard basis Y_s of X_s. Every mapping f of Y_s into an abelian group G extends uniquely to a symmetric mapping g of X into G; moreover, every value of g is a sum of values of f and opposites of values of f.

Let $\kappa : (a_1,\, \ldots,\, a_n) \longmapsto [a_1,\, \ldots,\, a_n]$ denote the canonical mapping of X_s into B_n. The defining relations of B_n show that κ is a symmetric mapping. If f is the restriction of κ to Y_s, then κ is the symmetric mapping that extends f; hence every

$[a_1, \ldots, a_n]$ is a sum of elements of $\kappa\,(Y_s)$ and their opposites. Hence B_n is generated by $\kappa\,(Y_s)$.

Now let f be any mapping of Y_s into an abelian group G, which extends uniquely to a symmetric mapping $g : X_s \longrightarrow G$. Since g is symmetric, the elements $g\,(a_1, \ldots, a_n)$ with $(a_1, \ldots, a_n) \in X_s$ satisfy all defining relations of B_n; therefore g factors through κ: there is a unique homomorphism $\varphi : B_n \longrightarrow G$ such that $\varphi \circ \kappa = g$. In particular, $\varphi\,[a_1, \ldots, a_n] = f\,(a_1, \ldots, a_n)$ for all $(a_1, \ldots, a_n) \in Y_s$, and φ extends f. Moreover, φ is unique with this property, since B_n is generated by $\kappa\,(Y_s)$. Thus B_n is free on Y_s (via κ).

The set X is a disjoint union $X = \bigcup_{s \in S} X_s$. Relative to the same (arbitrary) total order on S, its standard basis Y is the disjoint union $Y = \bigcup_{s \in S} Y_s$, as shown by the description of standard bases in Lemma 1.4. As above, every $\iota\,(a_1, \ldots, a_n)$ is a sum of elements of $\iota\,(Y_s)$ and their opposites; hence $C_n(s)$ is generated by $\iota\,(Y_s)$. Since $C_n(S)$ is free on Y, it follows that $C_n(s)$ is free on Y_s (via ι). Therefore $C_n(s) \cong B_n$. Also, $C_n(S) = \bigoplus_{s \in S} C_n(s)$, since Y is the disjoint union $Y = \bigcup_{s \in S} Y_s$. □

In what follows, $\leqq$ now denotes the divisibility preorder on S, under which $a \leqq b$ if and only if $a = bt$ for some $t \in S$; equivalently, $aS \subseteq bS$.

Proposition 4.2 *Let $n = 1, 2, 3, 4$. For each $a \in S$ let $A_n(a) = \bigoplus_{s \in S,\, s \geqq a} C_n(s)$, where $C_n(s)$ is the subgroup of $C_n(S)$ generated by all $\langle a_1, \ldots, a_n \rangle$ such that $a_1 a_2 \cdots a_n = s$; equivalently, $A_n(a)$ is the subgroup of $C_n(S)$ generated by all $\langle a_1, \ldots, a_n \rangle$ such that $a_1 a_2 \cdots a_n \geqq a$. For each $a \geqq b$ in S (under divisibility) let $\alpha^a_b : A_n(a) \longrightarrow A_n(b)$ be the inclusion homomorphism. Then $\mathcal{A}_n(S) = (A_n, \alpha)$ is a thin abelian group valued functor on S. Moreover, for every thin abelian group valued functor functor $\mathcal{G}$ on S there is a isomorphism*

$$U : \mathrm{Hom}_{\mathbf{T}}\,(\mathcal{A}_n(S), \mathcal{G}) \cong C^n(S, \mathcal{G})$$

which sends each natural transformation $\tau : \mathcal{A}_n(S) \longrightarrow \mathcal{G} = (G, \gamma)$ to the n-cochain u defined by

$$u\,(a_1, \ldots, a_n) = \tau_a\,\langle a_1, \ldots, a_n \rangle, \quad \textit{where } a = a_1 a_2 \cdots a_n,$$

and is natural in $\mathcal{G}$. (Thus the functor $C^n(S, -)$ of $\mathbf{T}$ to abelian groups is representable.) Then $\tau_a\,\langle a_1, \ldots, a_n \rangle = \gamma^s_a\, u\,(a_1, \ldots, a_n)$ whenever $s = a_1 a_2 \cdots a_n \geqq a$.

Proof If $a \geqq b$ in S, then $a_1 a_2 \cdots a_n \geqq a$ implies $a_1 a_2 \cdots a_n \geqq b$, so that $A_n(a) \subseteq A_n(b)$ and there is an inclusion homomorphism $\alpha^a_b : A_n(a) \longrightarrow A_n(b)$. Then $\alpha^a_a = 1_{A_n(a)}$, $\alpha^b_c \circ \alpha^a_b = \alpha^a_c$ when $a \geqq b \geqq c$, and $\mathcal{A}_n(S)$ is a thin functor.

Let $\mathcal{G} = (G, \gamma)$ be a thin abelian group valued functor on S, and let $\tau : \mathcal{A}_n(S) \longrightarrow \mathcal{G}$ be a natural transformation. If $a_1 a_2 \cdots a_n = a$, then $\langle a_1, \ldots, a_n \rangle \in A_n(a)$ and $\tau_a\,\langle a_1, \ldots, a_n \rangle \in G_a$. Moreover, the elements $\tau_a\,\langle a_1, \ldots, a_n \rangle$ of G_a inherit all properties (Cn) from the chains $\langle a_1, \ldots, a_n \rangle$. Hence $u : (a_1, \ldots, a_n) \longrightarrow \tau_a\,\langle a_1, \ldots, a_n \rangle$ is a symmetric n-cochain and $U\,(\tau) = u \in C^n(S, \mathcal{G})$.

Conversely, let $u \in C^n(S, \mathcal{G})$. For each $s \in S$ the values $u\,(a_1, \ldots, a_n)$ of u such that $a_1\, a_2 \cdots a_n = s$ inherit from the symmetric cochain u all the defining relations (Cn) of $C_n(s)$ in Lemma 4.1. Therefore there is a unique homomorphism $\varphi_s : C_n(s) \longrightarrow G_s$ such that $\varphi_s\langle\, a_1, \ldots, a_n\,\rangle = u\,(a_1, \ldots, a_n)$ whenever $a_1\, a_2 \cdots a_n = s$. Since $A_n(a) = \bigoplus_{s \in S,\, s \geqq a} C_n(s)$ there is a unique homomorphism $\tau_a : A_n(a) \longrightarrow G_a$ such that

$$\tau_a\, x \;=\; \gamma_a^s\, \varphi_s\, x$$

for all $s \geqq a$ and $x \in C_n(s)$. In particular,

$$\tau_a\, \langle\, a_1, \ldots, a_n\,\rangle \;=\; \gamma_a^s\, \varphi_s\, \langle\, a_1, \ldots, a_n\,\rangle \;=\; \gamma_a^s\, u\,(a_1, \ldots, a_n)$$

whenever $s = a_1\, a_2 \cdots a_n \geqq a$. If $a \geqq b$, then

$$\begin{aligned} \gamma_b^a\, \tau_a\, \langle\, a_1, \ldots, a_n\,\rangle &= \gamma_b^a\, \gamma_a^s\, u\,(a_1, \ldots, a_n) \\ = \gamma_b^s\, u\,(a_1, \ldots, a_n) \;&=\; \tau_b\, \langle\, a_1, \ldots, a_n\,\rangle \;=\; \tau_b\, \alpha_b^a\, \langle\, a_1, \ldots, a_n\,\rangle \end{aligned}$$

whenever $a_1\, a_2 \cdots a_n = s \geqq a$. Therefore $T(u) = \tau = (\tau_a)_{a \in S}$ is a natural transformation. Moreover, $U\,(\tau) = u$.

The maps T and U preserve pointwise addition, and are mutually inverse isomorphisms: if $\tau : \mathcal{A}_n(S) \longrightarrow \mathcal{G}$ is a natural transformation, and $u = U\,(\tau)$, then $\tau_a\, \langle\, a_1, \ldots, a_n\,\rangle = \alpha_a^s\, u\,(a_1, \ldots, a_n)$ whenever $a_1\, a_2 \cdots a_n = s$, so that $T(u) = \tau$.

Finally, let $\sigma : \mathcal{G} \longrightarrow \mathcal{G}'$ be a natural transformation. We saw in Sect. 1 that σ induces a homomorphism $\sigma^* = C^n(S, \sigma) : C^n(S, \mathcal{G}) \longrightarrow C^n(S, \mathcal{G}')$; if $u \in C^n(S, \mathcal{G})$, then

$$(\sigma^* u)(a_1, \ldots, a_n) \;=\; \sigma_a\, u\,(a_1, \ldots, a_n), \quad \text{where } \; a = a_1\, a_2 \cdots a_n.$$

If $\tau : \mathcal{A}_n \longrightarrow \mathcal{G}$ is a natural transformation, then so is $\sigma \circ \tau = \mathrm{Hom}_{\mathbf{T}}\,(\mathcal{C}_n(S),\, \sigma)(\tau) : \mathcal{A}_n \longrightarrow \mathcal{G}'$, and

$$U'(\sigma \circ \tau)\,(a_1, \ldots, a_n) \;=\; \sigma_a\, \tau_a\, u\,(a_1, \ldots, a_n) \;=\; (\sigma^*\,(U\tau))(a_1, \ldots, a_n)$$

whenever $a_1\, a_2 \cdots a_n = a$; hence $U' \circ \mathrm{Hom}_{\mathbf{T}}\,(\mathcal{A}_n(S),\, \sigma) = \sigma^* \circ U$. Thus U is natural in $\mathcal{G}$. $\square$

2. We note some properties of $\mathcal{A}_n(S)$.

First, $C_n(S)$ is a directed union $C_n(S) = \bigcup_{s \in S} A_n(s)$. Indeed $\bigcup_{s \in S} A_n(s)$ is a directed union, since $A_n(a) \subseteq A_n(ab)$, $A_n(b) \subseteq A_n(ab)$, for all $a, b \in S$. Then $\bigcup_{s \in S} A_n(s)$ is a subgroup of $C_n(S)$, which contains every generator of $C_n(S)$ and is therefore all of $C_n(S)$. In particular, $C_n(S) = \varinjlim \mathcal{A}_n(S)$.

The abelian groups $A_n(a)$ are subgroups of $C_n(S)$, which is a free abelian group by Theorem 2.2, and are therefore free.

Next we show that the functors $\mathcal{A}_n(S)$ are projective in $\mathbf{T}$.

Lemma 4.3 *Let $n = 1, 2, 3, 4$. If $\sigma : \mathcal{G} \longrightarrow \mathcal{G}'$ is an epimorphism in $\mathbf{T}$, then $\sigma^* = C^n(S, \sigma)$ is surjective.*

Proof Recall that $(\sigma^* u)(a_1, \ldots, a_n) = \sigma_a\, u\,(a_1, \ldots, a_n) \in G'_a$ for all $a_1, \ldots, a_n \in S$, where $u \in C^n(S, \mathcal{G})$ and $a = a_1\, a_2 \cdots a_n$. Let $\sigma : \mathcal{G} \longrightarrow \mathcal{G}'$ be an epimorphism. Since cokernels in $\mathbf{T}$ are pointwise, every $\sigma_a : G_a \longrightarrow G'_a$ is surjective.

Let $u' \in C^n(S, \mathcal{G}')$. If $n = 0$, then $u' = 0 = \sigma^* 0$. If $n = 1$, then for each $a \in S$ we have $u'_a = \sigma_a\, u_a$ for some $u_a \in G_a$, which provides $u \in C^1(S, \mathcal{G})$ such that $u' = \sigma^* u$. Now let $n = 2, 3, 4$.

As a set of ordered pairs, u' is a disjoint union of mappings u'_s $(s \in S)$, where $u'_s : (a_1, \ldots, a_n) \longmapsto u'\,(a_1, \ldots, a_n)$ is a symmetric mapping of

$$X_s \;=\; \{\, (a_1, \ldots, a_n) \;\big|\; a_1, \ldots, a_n \in S,\; a_1\, a_2 \cdots a_n = s \,\}$$

into G'_s. By Lemma 1.4, X_s has a basis $Y_s \subseteq X_s$: every mapping of Y_s into an abelian group G extends uniquely to a symmetric mapping of X_s into G. Since σ_s is surjective there exists for each $(a_1, \ldots, a_n) \in Y_s$ some $v\,(a_1, \ldots, a_n) \in G_s$ such that $\sigma_s\, v\,(a_1, \ldots, a_n) = u'\,(a_1, \ldots, a_n)$. This mapping v of Y_s into G_s extends uniquely to a symmetric mapping u_s of X_s into G_s. Then $\sigma_s \circ u_s$ and u'_s agree on Y_s, whence $\sigma_s \circ u_s = u'_s$. Since each u_s is symmetric, the union u of all u_s is a symmetric n-cochain. For all $a_1, \ldots, a_n \in S$,

$$(\sigma^* u)(a_1, \ldots, a_n) = \sigma_s u(a_1, \ldots, a_n) = \sigma_s u_s(a_1, \ldots, a_n) = u'(a_1, \ldots, a_n),$$

where $s = a_1\, a_2 \cdots a_n$. Thus $\sigma^* u = u'$. □

Theorem 4.4 *$\mathcal{A}_n(S)$ is projective in $\mathbf{T}$, for $n = 1, 2, 3, 4$.*

Proof By Lemma 4.3, $C^n(S, -)$ preserves epimorphisms. Hence $\mathrm{Hom}_{\mathbf{T}}\,(\mathcal{A}_n(S), -)$, which is naturally isomorphic to $C^n(S, -)$ by Proposition 4.2, also preserves epimorphisms. □

3. Boundaries are inherited from $C_n(S)$.

Lemma 4.5 *The boundary homomorphisms $\partial_n : C_n(S) \longrightarrow C_{n-1}(S)$ in Proposition 2.4 induce natural transformations $\partial_n : \mathcal{A}_n(S) \longrightarrow \mathcal{A}_{n-1}(S)$ such that $\partial_{n-1} \circ \partial_n = 0$.*

Proof If $\langle\, a_1, \ldots, a_n \,\rangle \in A_n(a)$, then $\partial_n \langle\, a_1, \ldots, a_n \,\rangle \in A_{n-1}(a)$: indeed if $n = 2$ and $\langle\, x, y \,\rangle \in A_2(a)$, then $xy \geqq a$, $x \geqq xy \geqq a$, $y \geqq xy \geqq a$, and

$$\partial_2 \langle\, x, y \,\rangle \;=\; \langle\, y \,\rangle - \langle\, xy \,\rangle + \langle\, x \,\rangle \;\in A_1(a)\,;$$

if $n = 3$ and $\langle\, x, y, z \,\rangle \in A_3(a)$, then $xyz \geqq a$, $xy \geqq xyz \geqq a$, $yz \geqq xyz \geqq a$, and

$$\partial_3 \langle\, x, y, z \,\rangle \;=\; \langle\, y, z \,\rangle - \langle\, x, yz \,\rangle + \langle\, xy, z \,\rangle - \langle\, x, y \,\rangle \;\in A_2(a)\,;$$

if $n = 4$ and $\langle\, x, y, z, t \,\rangle \in A_4(a)$, then $xyzt \geqq a$, $xyz \geqq xyzt \geqq a$, $yzt \geqq xyzt \geqq a$, and

$$\begin{aligned}\partial_4 \langle x, y, z, t \rangle &= \langle y, z, t \rangle - \langle x, y, zt \rangle \\ &+\langle x, yz, t \rangle - \langle xy, z, t \rangle + \langle x, y, z \rangle \in A_3(a).\end{aligned}$$

In each case, ∂_n induces a homomorphism $(\partial_n)_a : A_n(a) \longrightarrow A_{n-1}(a)$.

Since α_b^a is an inclusion homomorphism, $a \geqq b$ implies

$$\alpha_b^a (\partial_n)_a \langle a_1, \ldots, a_n \rangle = (\partial_n)_b \, \alpha_b^a \langle a_1, \ldots, a_n \rangle$$

whenever $a_1 a_2 \cdots a_n = a$; hence $\partial_n = \big((\partial_n)_a\big)_{a \in S}$ is a natural transformation. Then $\partial_{n-1} \circ \partial_n = 0$ follows from Proposition 2.4. □

Lemma 4.6 *If $n = 1, 2, 3, 4$, $\mathcal{G}$ is thin, and $\tau : \mathcal{A}_{n-1}(S) \longrightarrow \mathcal{G}$ is a natural transformation, then $U(\tau \circ \partial_n) = \delta\big(U(\tau)\big)$.*

Proof U is the isomorphism in Proposition 4.2: if $u = U(\tau)$, then $\tau_a \langle a_1, \ldots, a_{n-1} \rangle = \gamma_a^c \, u(a_1, \ldots, a_{n-1})$ whenever $a_1 a_2 \cdots a_{n-1} = c \geqq a$; in particular, $u(a_1, \ldots, a_{n-1}) = \tau_a \langle a_1, \ldots, a_{n-1} \rangle$ if $a_1 a_2 \cdots a_{n-1} = a$. Let $v = U(\tau \circ \partial_n)$, so that $v(a_1, \ldots, a_n) = \tau_a \, \partial_n \langle a_1, \ldots, a_n \rangle$ if $a_1 a_2 \cdots a_n = a$. We show that $v = \delta u$. This is trivial if $n = 1$.

If $n = 2$, then

$$\begin{aligned} v(x, y) = \tau_{xy} \, \partial_2 \langle x, y \rangle &= \tau_{xy} \langle y \rangle - \tau_{xy} \langle xy \rangle + \tau_{xy} \langle x \rangle \\ &= \gamma_{xy}^{y} \, u(y) - u(xy) + \gamma_{xy}^{x} \, u(x) = (\delta u)(x, y).\end{aligned}$$

If $n = 3$, then

$$\begin{aligned} v(x, y, z) = \tau_{xyz} \, \partial_3 \langle x, y, z \rangle &= \tau_{xyz} \langle y, z \rangle - \tau_{xyz} \langle x, yz \rangle + \tau_{xyz} \langle xy, z \rangle - \tau_{xyz} \langle x, y \rangle \\ &= \gamma_{xyz}^{yz} \, u(y, z) - u(x, yz) + u(xy, z) - \gamma_{xyz}^{xy} \, u(x, y) = (\delta u)(x, y, z).\end{aligned}$$

If $n = 4$, then

$$\begin{aligned} v(x, y, z, t) &= \tau_{xyzt} \, \partial_4 \langle x, y, z, t \rangle \\ &= \tau_{xyzt} \langle y, z, t \rangle - \tau_{xyzt} \langle x, y, zt \rangle + \tau_{xyzt} \langle x, yz, t \rangle \\ &\quad - \tau_{xyzt} \langle xy, z, t \rangle + \tau_{xyzt} \langle x, y, z \rangle \\ &= \gamma_{xyzt}^{yzt} \, u(y, z, t) - u(x, y, zt) + u(x, yz, t) - \gamma_{xyzt}^{xyz} \, u(x, y, z) \\ &= (\delta u)(x, y, z, t). \end{aligned}$$

□

3. Let $\mathcal{A}_*(S)$ be the chain complex

$$\mathcal{A}_*(S) : 0 \longleftarrow \mathcal{A}_1(S) \xleftarrow{\partial_2} \mathcal{A}_2(S) \xleftarrow{\partial_3} \mathcal{A}_3(S) \xleftarrow{\partial_4} \mathcal{A}_4(S) \longleftarrow 0 \longleftarrow \cdots$$

Theorem 4.7 *If $n = 1, 2, 3$ and $\mathcal{G}$ is thin, then there is an isomorphism*

$$H^n(S, \mathcal{G}) \;\cong\; H^n(\mathcal{A}_*(S), \mathcal{G})$$

which is natural in $\mathcal{G}$.

Proof Proposition 4.2 provides an isomorphism $U : \mathrm{Hom}_{\mathbf{T}}(\mathcal{A}_n(S),\ \mathcal{G}) \cong C^n(S, \mathcal{G})$ which is natural in $\mathcal{G}$. In the cohomology of $\mathcal{A}_*$ with coefficients in $\mathcal{G}$, an n-cochain $\tau : \mathcal{A}_n \longrightarrow \mathcal{G}$ is a cocycle if and only if $\tau \circ \partial_{n+1} = 0$, if and only if $\delta\big(U(\tau)\big) = 0$, by Lemma 4.4; hence U sends $Z^n(\mathcal{A},\ \mathcal{G})$ onto $Z^n(S, \mathcal{G})$. Similarly, a cochain $\tau : \mathcal{A}_n \longrightarrow \mathcal{G}$ is a coboundary if and only if $\tau = \sigma \circ \partial_n$ for some cochain $\sigma : \mathcal{C}_{n-1} \longrightarrow \mathcal{G}$, if and only if $U(\tau) = \delta u$ for some $u = U(\sigma) \in C^{n-1}(S, \mathcal{G})$, by Lemma 4.4; hence U sends $B^n(\mathcal{A},\ \mathcal{G})$ onto $B^n(S, \mathcal{G})$. Therefore U induces an isomorphism of $H^n(\mathcal{A}_*(S), \mathcal{G}) = Z^n(\mathcal{A}_*(S), \mathcal{G})\,/\,B^n(\mathcal{A}_*(S), \mathcal{G})$ onto $H^n(S, \mathcal{G}) = Z^n(S, \mathcal{G})\,/\,B^n(S, \mathcal{G})$, which, like U, is natural in $\mathcal{G}$. □

5 Cohomology with arbitrary coefficients

1. We now turn to the general case, when coefficient functors are not necessarily thin, and show that the commutative cohomology of S with thin coefficients is the cohomology of a chain complex in the category $\mathbf{A}$ of all abelian group valued functors on S. This turns out to require slightly longer chains, which are defined as follows.

Let $\mathcal{L}_0(S) = 0$.

Let $\mathcal{L}_1(S)$ assign to each $a \in S$ the free abelian group $L_1(a)$ generated by

$$X_1(a) \;=\; \{\,\langle\, x, t\,\rangle \;\big|\; x, t \in S,\ xt = a\,\}.$$

Let $\mathcal{L}_2(S)$ assign to $a \in S$ the abelian group $L_2(a)$ generated by

$$X_2(a) \;=\; \{\,\langle\, x, y, t\,\rangle \;\big|\; x, y, t \in S,\ xyt = a\,\},$$

subject to all defining relations

$$\langle\, y, x, t\,\rangle \;=\; \langle\, x, y, t\,\rangle. \tag{C2}$$

Let $\mathcal{L}_3(S)$ assign to $a \in S$ the abelian group $L_3(a)$ generated by

$$X_3(a) \;=\; \{\,\langle\, x, y, z, t\,\rangle \;\big|\; x, y, z, t \in S,\ xyzt = a\,\},$$

subject to all defining relations, collectively denoted by (C3):

$$\langle\, x, y, z, t\,\rangle \;=\; 0, \tag{C3a}$$

$$\langle\, z, y, x, t\,\rangle \;=\; -\ \langle\, x, y, z, t\,\rangle, \quad \text{and} \tag{C3b}$$

$$\langle\, x, y, z, t\,\rangle \;+\; \langle\, y, z, x, t\,\rangle \;+\; \langle\, z, x, y, t\,\rangle \;=\; 0. \tag{C3c}$$

Let $\mathcal{L}_4(S)$ assign to $a \in S$ the abelian group $L_4(a)$ generated by

$$X_4(a) \;=\; \{\, \langle\, w, x, y, z, t\,\rangle \;\big|\; w, x, y, z, t \in S,\; wxyzt = a\,\},$$

subject to all defining relations, collectively denoted by (C4):

$$\langle\, w, x, x, w, t\,\rangle \;=\; 0, \tag{C3a}$$

$$\langle\, z, y, x, w, t\,\rangle \;=\; -\;\langle\, w, x, y, z, t\,\rangle, \tag{C3b}$$

$$\langle\, w, x, y, z, t\,\rangle \;-\; \langle\, x, y, z, w, t\,\rangle \;+\; \langle\, y, z, w, x, t\,\rangle \;-\; \langle\, z, w, x, y, t\,\rangle \;=\; 0, \;\text{ and} \tag{C3c}$$

$$\langle\, w, x, y, z, t\,\rangle \;-\; \langle\, x, w, y, z, t\,\rangle \;+\; \langle\, x, y, w, z, t\,\rangle \;-\; \langle\, x, y, z, w, t\,\rangle \;=\; 0. \tag{C3d}$$

Thus the typical generator of $L_n(a)$ consists of a symmetric n-chain plus an extra element t. In fact, for each $t \in S$, $(x_1, \,\ldots,\, x_n) \longmapsto \langle\, x_1, \,\ldots,\, x_n,\; t\,\rangle$ is a symmetric mapping into $L_n(a)$. Hence it follows from Lemma 1.3 that the defining relations (C4c) may be dropped from the definition of $L_4(a)$, and it follows from Lemma 1.2 that (C3) may be replaced by the single defining relation

$$\langle\, x, y, z, t\,\rangle \;=\; \langle\, y, x, z, t\,\rangle \;-\; \langle\, y, z, x, t\,\rangle. \tag{C3d}$$

The groups $L_n(a)$ can be analysed as follows, much as in Lemma 4.1.

Lemma 5.1 *Let $n = 1, 2, 3, 4$. Given $s, t \in S$, let $L_n(s; t)$ be the subgroup of $L_n(st)$ generated by all $\langle\, x_1, \,\ldots,\, x_n,\; t\,\rangle$ such that $x_1\, x_2\, \cdots\, x_n = s$. Then $L_n(s; t)$ is, up to isomorphism, the abelian group generated by all $(x_1, \,\ldots,\, x_n,\; t)$ such that $x_1\, x_2\, \cdots\, x_n = s$, subject to all defining relations* (Cn). *Moreover, $L_n(s; t)$ is a free abelian group, and $L_n(a) = \bigoplus_{s,t \in S,\, st=a} L_n(s; t)$.*

Proof Let B_n be the abelian group generated by all $[x_1, \,\ldots,\, x_n]$, where $x_1, \,\ldots,\, x_n \in S$ and $x_1\, x_2\, \cdots\, x_n = s$, subject to all defining relations (Cn). The elements $\langle\, x_1, \,\ldots,\, x_n,\; t\,\rangle$ of $L_n(s; t)$ satisfy all the defining relations (Cn) of B_n; hence there is a unique homomorphism $\theta : B_n \longrightarrow L_n(s; t)$ such that $\theta\, [x_1, \,\ldots,\, x_n] = \langle\, x_1, \,\ldots,\, x_n,\; t\,\rangle$ for all $x_1, \,\ldots,\, x_n$ such that $x_1\, x_2\, \cdots\, x_n = s$. Similarly the elements $[x_1, \,\ldots,\, x_n]$ of B_n satisfy all the defining relations (Cn) of $L_n(s; t)$; therefore there is a unique homomorphism $\zeta : L_n(s; t) \longrightarrow B_n$ such that $\zeta\, \langle\, x_1, \,\ldots,\, x_n,\; t\,\rangle = [x_1, \,\ldots,\, x_n]$ for all $x_1, \,\ldots,\, x_n$ such that $x_1\, x_2\, \cdots\, x_n = s$. Now θ and ζ are mutually inverse isomorphisms; hence $L_n(s; t) \cong B_n$. By Lemma 4.1, $B_n \cong A_n(s)$ and B_n is free.

For each $s, t \in S$ such that $st = a$, let $\varphi_{s;t}$ be a homomorphism of $L_n(s; t)$ into an abelian group G. The elements $\varphi_{s;t}\, \langle\, x_1, \,\ldots,\, x_n,\; t\,\rangle$ of G, where $x_1\, x_2\, \cdots\, x_n = s$, inherit from each $L_n(s; t)$ all the defining relations (Cn) of $L_n(a)$; hence there is a unique homomorphism $\varphi : L_n(a) \longrightarrow G$ such that $\varphi\, \langle\, x_1, \,\ldots,\, x_n,\; t\,\rangle =$

$\varphi_{s;t}\,\langle\, x_1,\, \ldots,\, x_n,\, t\,\rangle$ whenever $x_1\, x_2\, \cdots\, x_n = s$ and $st = a$. Therefore $L_n(a) = \bigoplus_{s,t\in S,\; st=a} L_n(s;\, t)$. □

Theorem 5.2 *Let $n = 1, 2, 3, 4$. For every $a \in S$, $L_n(a)$ is a free abelian group.*

Proof By Lemma 5.1, $L_n(a)$ is a direct sum of free abelian groups. □

2. Next we arrange the groups $L_n(a)$ into an abelian group valued functor on S.

Lemma 5.3 *For all $n = 1, 2, 3, 4$ and $a, u \in S$, a homomorphism $\lambda_{a,u} : L_n(a) \longrightarrow L_n(au)$ is well-defined by:*

$$\lambda_{a,u}\,\langle\, x_1,\, \ldots,\, x_n,\, t\,\rangle \;=\; \langle\, x_1,\, \ldots,\, x_n,\, tu\,\rangle$$

whenever $x_1\, x_2\, \cdots\, x_n\, t = a$. Moreover, $\lambda_{a,1}$ is the identity on $L_n(a)$ and $\lambda_{au,v} \circ \lambda_{a,u} = \lambda_{a,uv}$, for all $a, u, v \in S$, so that $\mathcal{L}_n(S) = (L_n, \lambda)$ is an abelian group valued functor on S.

Proof If $x_1\, x_2\, \cdots\, x_n\, t = a$, then $x_1\, x_2\, \cdots\, x_n\, tu = au$ and the elements $\langle\, x_1,\, \ldots,\, x_n,\, tu\,\rangle$ of $L_n(a)$ satisfy all conditions (Cn). Therefore there is a unique homomorphism $\lambda_{a,u} : L_n(a) \longrightarrow L_n(au)$ such that $\lambda_{a,u}\,\langle\, x_1,\, \ldots,\, x_n,\, t\,\rangle = \langle\, x_1,\, \ldots,\, x_n,\, tu\,\rangle$ for all $x_1,\, \ldots,\, x_n\, t \in S$ such that $x_1\, x_2\, \cdots\, x_n t = a$. Moreover, $\lambda_{a,1}$ is the identity on $L_n(a)$ and

$$\lambda_{au,v}\, \lambda_{a,u}\,\langle\, x_1,\, \ldots,\, x_n,\, t\,\rangle \;=\; \lambda_{au,v}\,\langle\, x_1,\, \ldots,\, x_n,\, tu\,\rangle \;=\; \langle\, x_1,\, \ldots,\, x_n,\, tuv\,\rangle,$$

so that $\lambda_{au,v} \circ \lambda_{a,u} = \lambda_{a,uv}$. □

Proposition 5.4 *For $n = 1, 2, 3, 4$ there is for every abelian group valued functor $\mathcal{G} = (G, \gamma)$ on S an isomorphism*

$$U : \mathrm{Hom}_{\mathbf{A}}\,(\mathcal{L}_n(S),\, \mathcal{G}) \longrightarrow C^n(S, \mathcal{G})$$

which sends each natural transformation $\tau : \mathcal{L}_n(S) \longrightarrow \mathcal{G}$ to the n-cochain u defined by

$$u\,(x_1,\, \ldots,\, x_n) \;=\; \tau_a\,\langle\, x_1,\, \ldots,\, x_n,\, 1\,\rangle, \;\; \textit{where} \;\; a = x_1\, x_2\, \cdots\, x_n,$$

and is natural in $\mathcal{G}$. Then $\tau_a\,\langle\, x_1,\, \ldots,\, x_n,\, t\,\rangle = \gamma_{c,t}\; u\,(x_1,\, \ldots,\, x_n)$ whenever $c = x_1\, x_2\, \cdots\, x_n$ and $ct = a$.

Proof Let $\tau : \mathcal{L}_n(S) \longrightarrow \mathcal{G}$ be a natural transformation and let $u = U\,(\tau)$, so that $u\,(x_1,\, \ldots,\, x_n) = \tau_a\,\langle\, x_1,\, \ldots,\, x_n,\, 1\,\rangle \in G_a$, where $a = x_1\, x_2\, \cdots\, x_n$, and u is a cochain on S with values in G. The elements $u\,(x_1,\, \ldots,\, x_n)$ of G_a inherit all properties (Cn) from the chains $\langle\, x_1,\, \ldots,\, x_n,\, 1\,\rangle$; hence u is a symmetric n-cochain and $u \in C^n(S, \mathcal{G})$.

If $c = x_1\, x_2\, \cdots\, x_n$ and $ct = a$, then

$$\tau_a\,\langle\, x_1,\, \ldots,\, x_n,\, t\,\rangle = \tau_a\, \lambda_{c,t}\,\langle\, x_1,\, \ldots,\, x_n,\, 1\,\rangle$$

$$= \gamma_{c,t}\, \tau_c \,\langle\, x_1,\, \ldots,\, x_n,\, 1\,\rangle \;=\; \gamma_{c,t}\, u\,(x_1,\, \ldots,\, x_n),$$

since τ is a natural transformation. Therefore $U\,(\tau) = 0$ implies $\tau = 0$, and U is injective.

Conversely, if $u \in C^n(S, \mathcal{G})$ is a symmetric n-cochain, then the elements $\lambda_{c,t}\, u\,(x_1,\, \ldots,\, x_n)$ of G_a (with $x_1\, x_2\, \cdots\, x_n = c$ and $ct = a$) inherit from u all properties (Cn); therefore there is for each $a \in S$ a unique homomorphism $\tau_a : L_n(a) \longrightarrow G_a$ such that

$$\tau_a \,\langle\, x_1,\, \ldots,\, x_n,\, t\,\rangle \;=\; \gamma_{c,t}\, u\,(x_1,\, \ldots,\, x_n)$$

whenever $c = x_1\, x_2\, \cdots\, x_n$ and $ct = a$. If $v \in S$, then

$$\begin{aligned} \gamma_{a,v}\, \tau_a \,\langle\, x_1,\, \ldots,\, x_n,\, t\,\rangle &= \gamma_{a,v}\, \gamma_{c,t}\, u\,(x_1,\, \ldots,\, x_n) \;=\; \gamma_{c,tv}\, u\,(x_1,\, \ldots,\, x_n) \\ &= \tau_{av} \,\langle\, x_1,\, \ldots,\, x_n,\, tv\,\rangle \;=\; \tau_{av}\, \lambda_{a,v} \,\langle\, x_1,\, \ldots,\, x_n,\, t\,\rangle. \end{aligned}$$

hence τ is a natural transformation. Moreover, $U\,(\tau) = u$. Hence U is surjective.

Naturality is equally straightforward. We saw that every natural transformation $\sigma : \mathcal{G} \longrightarrow \mathcal{G}'$ induces a homomorphism $\sigma^* = C^n(S, \sigma) : C^n(S, \mathcal{G}) \longrightarrow C^n(S, \mathcal{G}')$, given by

$$(\sigma^* u)(x_1,\, \ldots,\, x_n) \;=\; \sigma_a\, u\,(x_1,\, \ldots,\, x_n)$$

whenever $x_1\, x_2\, \cdots\, x_n = a$. If $\tau : \mathcal{L}_n(S) \longrightarrow \mathcal{G}$ is a natural transformation, then so is $\sigma \circ \tau : \mathcal{L}_n(S) \longrightarrow \mathcal{G}'$, and $U' : \mathrm{Hom}_{\mathbf{A}}\,(\mathcal{L}_n(S),\, \mathcal{G}') \longrightarrow C^n(S, \mathcal{G}')$ yields

$$U'\,(\sigma \circ \tau)\,(x_1,\, \ldots,\, x_n) \;=\; \sigma_a\, \tau_a \,\langle\, x_1,\, \ldots,\, x_n,\, 1\,\rangle \;=\; \sigma^*\big(U(\tau)\,(x_1,\, \ldots,\, x_n)\big)$$

whenever $x_1\, x_2\, \cdots\, x_n = a$, so that $U' \circ \mathrm{Hom}_{\mathbf{A}}\,(\mathcal{L}_n(S),\, \sigma) = \sigma^* \circ U$. Thus U is natural in $\mathcal{G}$. □

The natural isomorphism $\mathrm{Hom}_{\mathbf{A}}\,(\mathcal{L}_n(S),\, \mathcal{G}) \cong C^n(S, \mathcal{G})$ determines $\mathcal{L}_n(S)$ uniquely up to isomorphism. This justifies the longer symmetric chains in the construction of $\mathcal{L}_n(S)$.

Theorem 5.5 *$\mathcal{L}_n(S)$ is projective in $\mathbf{A}$, for $n = 1, 2, 3, 4$.*

Proof By Lemma 4.3, $C^n(S, -)$ preserves epimorphisms. Hence $\mathrm{Hom}_{\mathbf{A}}\,(\mathcal{L}_n(S), -)$, which is naturally isomorphic to $C^n(S, -)$ by Proposition 5.4, also preserves epimorphisms. □

2. Boundaries of longer chains are defined as follows.

Lemma 5.6 *If $n = 1, 2, 3, 4$, then there is a natural transformation $\partial_n : \mathcal{L}_n(S) \longrightarrow \mathcal{L}_{n-1}(S)$ such that*

$$\begin{aligned}
\partial_1 \langle\, x, t \,\rangle &= 0,\\
\partial_2 \langle\, x, y, t \,\rangle &= \langle\, y, xt \,\rangle - \langle\, xy, t \,\rangle + \langle\, x, yt \,\rangle,\\
\partial_3 \langle\, x, y, z, t \,\rangle &= \langle\, y, z, xt \,\rangle - \langle\, xy, z, t \,\rangle + \langle\, x, yz, t \,\rangle - \langle\, x, y, zt \,\rangle,\\
\partial_4 \langle\, w, x, y, z, t \,\rangle &= \langle\, x, y, z, wt \,\rangle - \langle\, wx, y, z, t \,\rangle\\
&\qquad + \langle\, w, xy, z, t \,\rangle - \langle\, w, x, yz, t \,\rangle + \langle\, w, x, y, zt \,\rangle.
\end{aligned}$$

Proof Let $a \in S$. If $\langle\, x, y, t \,\rangle \in X_2(a)$, then $xyt = a$, $\langle\, y, xt \,\rangle$, $\langle\, xy, t \,\rangle$, $\langle\, x, yt \,\rangle \in X_1(a)$, and $\langle\, y, xt \,\rangle - \langle\, xy, t \,\rangle + \langle\, x, yt \,\rangle \in L_1(a)$. Moreover,

$$\langle\, y, xt \,\rangle - \langle\, xy, t \,\rangle + \langle\, x, yt \,\rangle \;=\; \langle\, x, yt \,\rangle - \langle\, yx, t \,\rangle + \langle\, y, xt \,\rangle$$

for all x, y, t; hence there is a unique homomorphism $\partial_2(a) : L_2(a) \longrightarrow L_1(a)$ such that

$$\partial_2(a)\, \langle\, x, y, t \,\rangle \;=\; \langle\, y, xt \,\rangle - \langle\, xy, t \,\rangle + \langle\, x, yt \,\rangle$$

for all $\langle\, x, y, t \,\rangle \in X_2(a)$.

If $u \in S$, then

$$\begin{aligned}
\lambda_{a,u}\, \partial_2(a)\, \langle\, x, y, t \,\rangle &= \lambda_{a,u}\, \langle\, y, xt \,\rangle \;-\; \lambda_{a,u}\, \langle\, xy, t \,\rangle \;+\; \lambda_{a,u}\, \langle\, x, yt \,\rangle\\
&= \langle\, y, xtu \,\rangle - \langle\, xy, tu \,\rangle + \langle\, x, ytu \,\rangle \;=\; \partial_2(au)\, \lambda_{a,u}\, \langle\, x, y, t \,\rangle\,;
\end{aligned}$$

therefore $\lambda_{a,u} \circ \partial_2(a) = \partial_2(au) \circ \lambda_{a,u}$ and ∂_2 is a natural transformation.

Similarly, if $\langle\, x, y, z, t \,\rangle \in X_3(a)$, then $xyzt = a$; $\langle\, y, z, xt \,\rangle$, $\langle\, xy, z, t \,\rangle$, $\langle\, x, yz, t \,\rangle$, and $\langle\, x, y, zt \,\rangle \in X_2(a)$; and

$$f\,(x, y, z, t) \;=\; \langle\, y, z, xt \,\rangle - \langle\, xy, z, t \,\rangle + \langle\, x, yz, t \,\rangle - \langle\, x, y, zt \,\rangle \in L_2(a).$$

Moreover, the elements $f\,(x, y, z, t)$ of $L_3(a)$ have property (C3d):

$$\begin{aligned}
&f\,(y, x, z, t) - f\,(y, z, x, t)\\
&\quad = \langle\, x, z, yt \,\rangle - \langle\, yx, z, t \,\rangle + \langle\, y, xz, t \,\rangle - \langle\, y, x, zt \,\rangle\\
&\qquad - \langle\, z, x, yt \,\rangle + \langle\, yz, x, t \,\rangle - \langle\, y, zx, t \,\rangle + \langle\, y, z, xt \,\rangle\\
&\quad = \langle\, y, z, xt \,\rangle - \langle\, xy, z, t \,\rangle + \langle\, x, yz, t \,\rangle - \langle\, x, y, zt \,\rangle \;=\; f\,(x, y, z, t),
\end{aligned}$$

due to (C2) and commutativity in S. By Lemma 1.2, all of (C3) holds; therefore there is a unique homomorphism $\partial_3(a) : L_3(a) \longrightarrow L_2(a)$ such that $\partial_3(a)\, \langle\, x, y, z, t \,\rangle = f\,(x, y, z, t)$ for all $\langle\, x, y, z, t \,\rangle \in X_3(a)$.

If $u \in S$, then

$$\begin{aligned}
&\lambda_{a,u}\, \partial_3(a)\, \langle\, x, y, z, t \,\rangle\\
&= \lambda_{a,u}\, \langle\, y, z, xt \,\rangle - \lambda_{a,u}\, \langle\, xy, z, t \,\rangle + \lambda_{a,u}\, \langle\, x, y, zt \,\rangle - \lambda_{a,u}\, \langle\, x, y, zt \,\rangle\\
&= \langle\, y, z, xtu \,\rangle - \langle\, xy, z, tu \,\rangle + \langle\, x, yz, tu \,\rangle - \langle\, x, y, ztu \,\rangle\\
&= \partial_3(au)\, \lambda_{a,u}\, \langle\, x, y, z, t \,\rangle\,;
\end{aligned}$$

therefore $\lambda_{a,u} \circ \partial_3(a) = \partial_3(au) \circ \lambda_{a,u}$ and ∂_3 is a natural transformation.

Finally, if $\langle\, w, x, y, z, t \,\rangle \in X_4(a)$, then $wxyzt = a$ and

$$f\,(w, x, y, z, t) = \langle\, x, y, z, wt \,\rangle - \langle\, wx, y, z, t \,\rangle \\ + \langle\, w, xy, z, t \,\rangle - \langle\, w, x, yz, t \,\rangle + \langle\, w, x, y, zt \,\rangle \in L_3(a).$$

Moreover, the elements $f\,(w, x, y, z, t)$ of $L_3(a)\,\langle\, w, x, y, z, t \,\rangle$ satisfy (C4a), (C4b), and (C4d): if $wxyzt = a$, then (C3a), (C3b), and (C3d) yield

$$\begin{aligned} f\,(w, x, x, w, t) &= \langle\, x, x, w, wt \,\rangle - \langle\, wx, x, w, t \,\rangle + \langle\, w, xx, w, t \,\rangle \\ &\quad - \langle\, w, x, xw, t \,\rangle + \langle\, w, x, x, wt \,\rangle \;=\; 0, \\ f\,(z, y, x, w, t) &= \langle\, y, x, w, zt \,\rangle - \langle\, zy, x, w, t \,\rangle + \langle\, z, yx, w, t \,\rangle \\ &\quad - \langle\, z, y, xw, t \,\rangle + \langle\, z, y, c, wt \,\rangle \;=\; -\,f\,(w, x, y, z, t), \\ f\,(w, x, y, z, t) \;-\; & f\,(x, w, y, z, t) + f\,(x, y, w, z, t) - f\,(x, y, z, w, t) \\ &= \langle\, x, y, z, wt \,\rangle - \langle\, wx, y, z, t \,\rangle + \langle\, w, xy, z, t \,\rangle - \langle\, w, x, yz, t \,\rangle + \langle\, w, x, y, zt \,\rangle \\ &\quad - \langle\, w, y, z, xt \,\rangle + \langle\, xw, y, z, t \,\rangle - \langle\, x, wy, z, t \,\rangle + \langle\, x, w, yz, t \,\rangle - \langle\, x, w, y, zt \,\rangle \\ &\quad + \langle\, y, w, z, xt \,\rangle - \langle\, xy, w, z, t \,\rangle + \langle\, x, yw, z, t \,\rangle - \langle\, x, y, wz, t \,\rangle + \langle\, x, y, w, zt \,\rangle \\ &\quad - \langle\, y, z, w, xt \,\rangle + \langle\, xy, z, w, t \,\rangle - \langle\, x, yz, w, t \,\rangle + \langle\, x, y, zw, t \,\rangle \\ &\quad - \langle\, x, y, z, wt \,\rangle = 0. \end{aligned}$$

Hence there is a unique homomorphism $\partial_4(a) : L_4(a) \longrightarrow L_3(a)$ such that $\partial_4(a)\,\langle\, w, x, y, z, t \,\rangle = f\,(w, x, y, z, t)$ for all $\langle\, w, x, y, z, t \,\rangle \in X_4(a)$.

If $u \in S$, then

$$\begin{aligned} &\lambda_{a,u}\,\partial_4(a)\,\langle\, w, x, y, z, t \,\rangle \\ &\quad = \lambda_{a,u}\,\langle\, x, y, z, xt \,\rangle - \lambda_{a,u}\,\langle\, wx, y, z, t \,\rangle \\ &\qquad + \lambda_{a,u}\,\langle\, w, xy, z, t \,\rangle - \lambda_{a,u}\,\langle\, w, x, yz, t \,\rangle + \lambda_{a,u}\,\langle\, w, x, y, zt \,\rangle \\ &\quad = \langle\, x, y, z, xtu \,\rangle - \langle\, wx, y, z, tu \,\rangle + \langle\, w, xy, z, tu \,\rangle - \langle\, w, x, yz, tu \,\rangle + \langle\, w, x, y, ztu \,\rangle \\ &\quad = \partial_4(au)\,\lambda_{a,u}\,\langle\, w, x, y, z, t \,\rangle\,; \end{aligned}$$

therefore $\lambda_{a,u} \circ \partial_4(a) = \partial_4(au) \circ \lambda_{a,u}$ and ∂_4 is a natural transformation. □

Lemma 5.7 *If $n = 1, 2, 3, 4$ and $\tau : \mathcal{L}_{n-1}(S) \longrightarrow \mathcal{G}$ is a natural transformation, then $U\,(\tau \circ \partial_n) = \delta\,\big(U\,(\tau)\big)$. Hence, if $n = 2, 3, 4$, then $\partial_{n-1} \circ \partial_n = 0$.*

Proof U is the isomorphism in Proposition 5.4: if $u = U\,(\tau)$, $x_1\,x_2 \cdots x_{n-1} = c$, and $a = ct$, then $\tau_a\,\langle\, x_1, \ldots, x_{n-1}, t \,\rangle = \gamma_{c,t}\; u\,(x_1, \ldots, x_{n-1})$; in particular, $u\,(x_1, \ldots, x_{n-1}) = \tau_c\,\langle\, x_1, \ldots, x_{n-1}, 1 \,\rangle$. Let $v = U\,(\tau \circ \partial_n)$, so that $v\,(x_1, \ldots, x_n) = \tau_a\,\partial_n\,\langle\, x_1, \ldots, x_n, 1 \,\rangle$ when $x_1\,x_2 \cdots x_n = a$. We want to show that $v = \delta u$. This is trivial if $n = 1$.

If $n = 2$ and $\langle\, x, y, 1 \,\rangle \in L_2(a)$, where $a = xy$, then $\partial_2\,\langle\, x, y, 1 \,\rangle = \langle\, y, x \,\rangle - \langle\, xy, 1 \,\rangle + \langle\, x, 1 \,\rangle$ and

$$\begin{aligned} v\,(x, y) = \tau_a\,\partial_2\,\langle\, x, y, 1 \,\rangle \;&=\; \tau_a\,\langle\, y, x \,\rangle \;-\; \tau_a\,\langle\, xy, 1 \,\rangle \;+\; \tau_a\,\langle\, x, y \,\rangle \\ &= \gamma_{y,x}\,u\,(y) \;-\; u\,(xy) \;+\; \gamma_{x,y}\,u(x) \;=\; (\delta u)(x, y). \end{aligned}$$

If $n = 3$ and $\langle\, x, y, z, 1\,\rangle \in L_3(a)$, where $a = xyz$, then $\partial_3 \langle\, x, y, z, 1\,\rangle = \langle\, y, z, x\,\rangle - \langle\, xy, z, 1\,\rangle + \langle\, x, yz, 1\,\rangle - \langle\, x, y, z\,\rangle$ and

$$\begin{aligned} & v\,(x, y, z) = \tau_a\, \partial_3 \langle\, x, y, z, 1\,\rangle \\ & = \tau_a \langle\, y, z, x\,\rangle \;-\; \tau_a \langle\, xy, z, 1\,\rangle \;+\; \tau_a \langle\, x, yz, 1\,\rangle \;-\; \tau_a \langle\, x, y, z\,\rangle \\ & = \gamma_{yz,x}\, u\,(y, z) \;-\; u\,(xy, z) \;+\; u\,(x, yz) \;-\; \gamma_{xy,z}\, u\,(x, y) \;=\; (\delta u)\,(x, y, z). \end{aligned}$$

Finally, if $n = 4$ and $\langle\, w, x, y, z, 1\,\rangle \in L_4(a)$, where $a = wxyz$, then

$$\begin{aligned} & \partial_4 \langle\, w, x, y, z, 1\,\rangle = \langle\, x, y, z, w\,\rangle \\ & \quad -\langle\, wx, y, z, 1\,\rangle + \langle\, w, xy, z, 1\,\rangle - \langle\, w, x, yz, 1\,\rangle + \langle\, w, x, y, z\,\rangle \end{aligned}$$

and

$$\begin{aligned} & v\,(w, x, y, z) = \tau_a\, \partial_3 \langle\, w, x, y, z, 1\,\rangle \\ & = \tau_a \langle\, x, y, z, w\,\rangle \;-\; \tau_a \langle\, wx, y, z, 1\,\rangle \\ & \quad +\; \tau_a \langle\, w, xy, z, 1\,\rangle \;-\; \tau_a \langle\, w, x, yz, 1\,\rangle \;+\; \tau_a \langle\, w, x, y, z\,\rangle \\ & = \gamma_{xyz,w}\, u\,(x, y, z) \;-\; u\,(wx, y, z) \\ & \quad +\; u\,(w, xy, z) \;-\; u\,(w, x, yz) \;+\; \gamma_{wxy,z}\, u\,(w, x, y) \\ & = (\delta u)\,(w, x, y, z). \end{aligned}$$

In each case, $U\,(\tau \circ \partial_n) = \delta\, U(\tau)$.

Let $n > 1$, $\mathcal{G} = \mathcal{L}_{n-1}(S)$, and $\tau : \mathcal{L}_{n-1}(S) \longrightarrow \mathcal{G}$ be the identity. We have

$$U\,(\partial_{n-1} \circ \partial_n) \;=\; U\,(\tau \circ \partial_{n-1} \circ \partial_n) \;=\; \delta\, U\,(\tau \circ \partial_{n-1}) \;=\; \delta\, \delta\, U(\tau) = 0.$$

Therefore $\partial_{n-1} \circ \partial_n = 0$. (This can also be shown directly.) □

Let $\mathcal{L}_*(S)$ be the chain complex

$$\mathcal{L}_*(S) : 0 \longleftarrow \mathcal{L}_1(S) \xleftarrow{\partial_2} \mathcal{L}_2(S) \xleftarrow{\partial_3} \mathcal{L}_3(S) \xleftarrow{\partial_3} \mathcal{L}_4(S) \longleftarrow 0 \longleftarrow \cdots$$

Theorem 5.8 *If* $n = 1, 2, 3$, *then there is an isomorphism* $H^n(S, \mathcal{G}) \cong H^n(\mathcal{L}_*(S), \mathcal{G})$ *which is natural in* $\mathcal{G}$.

Proof Proposition 5.4 provides isomorphisms $U : \mathrm{Hom}_{\mathbf{A}}\,(\mathcal{L}_n(S),\, \mathcal{G}) \cong C^n(S, \mathcal{G})$ which are natural in $\mathcal{G}$. In the cohomology of $\mathcal{L}_* = \mathcal{L}_*(S)$ with coefficients in $\mathcal{G}$, an n-cochain $\tau : \mathcal{L}_n(S) \longrightarrow \mathcal{G}$ is a cocycle if and only if $\tau \circ \partial_{n+1} = 0$, if and only if $\delta\,(U\tau) = 0$, by Lemma 5.5; hence U sends $Z^n(\mathcal{L}_*,\, \mathcal{G})$ onto $Z^n(S, \mathcal{G})$. Similarly, a cochain $\tau : \mathcal{L}_n(S) \longrightarrow \mathcal{G}$ is a coboundary if and only if $\tau = \sigma \circ \partial_n$ for some cochain $\sigma : \mathcal{C}_{n-1} \longrightarrow \mathcal{G}$, if and only if $U\tau = \delta u$ for some $u = U\sigma \in C^{n-1}(S, \mathcal{G})$, by Lemma 5.5; hence U sends $B^n(\mathcal{L}_*,\, \mathcal{G})$ onto $B^n(S, \mathcal{G})$. Therefore U induces an isomorphism of $H^n(\mathcal{L}_*, \mathcal{G}) = Z^n(\mathcal{L}_*, \mathcal{G})\,/\,B^n(\mathcal{L}_*(S), \mathcal{G})$ onto $H^n(S, \mathcal{G}) = Z^n(S, \mathcal{G})\,/\,B^n(S, \mathcal{G})$, which, like U, is natural in $\mathcal{G}$. □

Unfortunately, the category $\mathbf{A}$ does not lend itself to universal coefficients theorems; the next section gives a counterexample.

6 An example

In this section, S is the commutative monoid $M_5 = \{\, 1, a, e, b, 0 \,\}$ with multiplication

$$\begin{array}{|ccccc} 1 & a & e & b & 0 \\ a & e & e & b & 0 \\ e & e & e & b & 0 \\ b & b & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

1. We begin by computing the complex $C_*(M_5 \,/\, M_5 \backslash 0)$ and its homology groups: the homology groups $H_1(M_5 \,/\, M_5 \backslash 0)$ and $H_2(M_5 \,/\, M_5 \backslash 0)$ of M_5 relative to $M_5 \backslash 0$. (Since M_5 has a zero element, $H_1(M_5) = H_2(M_5) = 0$, by Proposition 2.6.)

Lemma 6.1 *$H_1(M_5 \,/\, M_5 \backslash 0) \cong \mathbb{Z}$ and $H_2(M_5 \,/\, M_5 \backslash 0) = 0$.*

Proof Order M_5 so that $1 < a < e < b < 0$. By Theorem 2.2, $C_2(M_5 \,/\, M_5 \backslash 0)$ is free on all $\langle\, x, y \,\rangle$ such that $x \leqq y$ and $xy \neq 0$:

$$\begin{aligned} &\langle\, 1, 1 \,\rangle,\ \langle\, 1, a \,\rangle,\ \langle\, 1, e \,\rangle,\ \langle\, 1, b \,\rangle, \\ &\langle\, a, a \,\rangle,\ \langle\, a, e \,\rangle,\ \langle\, a, b \,\rangle,\ \langle\, e, e \,\rangle,\ \langle\, e, b \,\rangle\,; \end{aligned}$$

and $C_3(M_5 \,/\, M_5 \backslash 0)$ is free on all $\langle\, x, y, z \,\rangle$ such that $x \leqq y$, $x < z$, and $xyz \neq 0$:

$$\begin{aligned} &\langle\, 1, 1, a \,\rangle,\ \langle\, 1, 1, e \,\rangle,\ \langle\, 1, 1, b \,\rangle,\ \langle\, 1, a, a \,\rangle,\ \langle\, 1, a, e \,\rangle,\ \langle\, 1, a, b \,\rangle, \\ &\langle\, 1, e, a \,\rangle,\ \langle\, 1, e, e \,\rangle,\ \langle\, 1, e, b \,\rangle,\ \langle\, 1, b, a \,\rangle,\ \langle\, 1, b, e \,\rangle, \\ &\langle\, a, a, e \,\rangle,\ \langle\, a, a, b \,\rangle,\ \langle\, a, e, e \,\rangle,\ \langle\, a, e, b \,\rangle,\ \langle\, a, b, e \,\rangle,\ \langle\, e, e, b \,\rangle. \end{aligned}$$

The 1-boundaries are determined by:

$$\begin{aligned} \partial_2 \langle\, 1, 1 \,\rangle &= \langle\, 1 \,\rangle - \langle\, 11 \,\rangle + \langle\, 1 \,\rangle \ = \ \langle\, 1 \,\rangle\,; \\ \partial_2 \langle\, 1, a \,\rangle &= \langle\, a \,\rangle - \langle\, 1a \,\rangle + \langle\, 1 \,\rangle \ = \ \langle\, 1 \,\rangle\,; \\ \partial_2 \langle\, 1, e \,\rangle &= \langle\, e \,\rangle - \langle\, 1e \,\rangle + \langle\, 1 \,\rangle \ = \ \langle\, 1 \,\rangle\,; \\ \partial_2 \langle\, 1, b \,\rangle &= \langle\, b \,\rangle - \langle\, 1b \,\rangle + \langle\, 1 \,\rangle \ = \ \langle\, 1 \,\rangle\,; \\ \partial_2 \langle\, a, a \,\rangle &= \langle\, a \,\rangle - \langle\, aa \,\rangle + \langle\, a \,\rangle \ = \ 2\langle\, a \,\rangle - \langle\, e \,\rangle\,; \\ \partial_2 \langle\, a, e \,\rangle &= \langle\, e \,\rangle - \langle\, ae \,\rangle + \langle\, a \,\rangle \ = \ \langle\, a \,\rangle\,; \\ \partial_2 \langle\, a, b \,\rangle &= \langle\, b \,\rangle - \langle\, ab \,\rangle + \langle\, a \,\rangle \ = \ \langle\, a \,\rangle\,; \\ \partial_2 \langle\, e, e \,\rangle &= \langle\, e \,\rangle - \langle\, ee \,\rangle + \langle\, e \,\rangle \ = \ \langle\, e \,\rangle\,; \\ \partial_2 \langle\, e, b \,\rangle &= \langle\, b \,\rangle - \langle\, eb \,\rangle + \langle\, e \,\rangle \ = \ \langle\, e \,\rangle. \end{aligned}$$

Hence $\operatorname{Im} \partial_2$ is generated by $\langle\, 1 \,\rangle$, $\langle\, a \,\rangle$, and $\langle\, e \,\rangle$. Since $C_1(M_5 \,/\, M_5 \backslash 0)$ is free on $\langle\, 1 \,\rangle$, $\langle\, a \,\rangle$, $\langle\, e \,\rangle$, and $\langle\, b \,\rangle$, it follows that $H_1(M_5 \,/\, M_5 \backslash 0) = C_1(M_5 \,/\, M_5 \backslash 0) \,/\, \operatorname{Im} \partial_2 \cong \mathbb{Z}$.

The above also yields some obvious 2-cycles:

$$\begin{aligned} &\langle\, 1, a \,\rangle - \langle\, 1, 1 \,\rangle,\ \ \langle\, 1, e \,\rangle - \langle\, 1, 1 \,\rangle,\ \ \langle\, 1, b \,\rangle - \langle\, 1, 1 \,\rangle, \\ &\langle\, a, b \,\rangle - \langle\, a, e \,\rangle,\ \ \langle\, e, b \,\rangle - \langle\, e, e \,\rangle,\ \ \text{and}\ \langle\, a, a \,\rangle - 2\,\langle\, a, e \,\rangle + \langle\, e, e \,\rangle. \end{aligned}$$

It is readily verified that these six 2-cycles constitute a basis of $\operatorname{Ker} \partial_2$.

The 2-boundaries are determined by:

$$
\begin{aligned}
\partial_3 \langle 1, 1, a \rangle &= \langle 1, a \rangle - \langle 1, 1 \rangle a + \langle 1, 1 \rangle a - \langle 1, 1 \rangle = \langle 1, a \rangle - \langle 1, 1 \rangle, \\
\partial_3 \langle 1, 1, e \rangle &= \langle 1, e \rangle - \langle 1, 1 \rangle e + \langle 1, 1 \rangle e - \langle 1, 1 \rangle = \langle 1, e \rangle - \langle 1, 1 \rangle, \\
\partial_3 \langle 1, 1, b \rangle &= \langle 1, b \rangle - \langle 1, 1 \rangle b + \langle 1, 1 \rangle b - \langle 1, 1 \rangle = \langle 1, b \rangle - \langle 1, 1 \rangle, \\
\partial_3 \langle 1, a, a \rangle &= \langle a, a \rangle - \langle 1, a \rangle a + \langle 1, a \rangle a - \langle 1, a \rangle = \langle 1, e \rangle - \langle 1, a \rangle, \\
\partial_3 \langle 1, a, e \rangle &= \langle a, e \rangle - \langle 1, a \rangle e + \langle 1, a \rangle e - \langle 1, a \rangle = \langle 1, e \rangle - \langle 1, a \rangle, \\
\partial_3 \langle 1, a, b \rangle &= \langle a, b \rangle - \langle 1, a \rangle b + \langle 1, a \rangle b - \langle 1, a \rangle = \langle 1, b \rangle - \langle 1, a \rangle, \\
\partial_3 \langle 1, e, a \rangle &= \langle e, a \rangle - \langle 1, e \rangle a + \langle 1, e \rangle a - \langle 1, e \rangle = 0, \\
\partial_3 \langle 1, e, e \rangle &= \langle e, e \rangle - \langle 1, e \rangle e + \langle 1, e \rangle e - \langle 1, e \rangle = 0, \\
\partial_3 \langle 1, e, b \rangle &= \langle e, b \rangle - \langle 1, e \rangle b + \langle 1, e \rangle b - \langle 1, e \rangle = \langle 1, b \rangle - \langle 1, e \rangle, \\
\partial_3 \langle 1, b, a \rangle &= \langle b, a \rangle - \langle 1, b \rangle a + \langle 1, b \rangle a - \langle 1, b \rangle = 0, \\
\partial_3 \langle 1, b, e \rangle &= \langle b, e \rangle - \langle 1, b \rangle e + \langle 1, b \rangle e - \langle 1, b \rangle = 0, \\
\partial_3 \langle a, a, e \rangle &= \langle a, e \rangle - \langle a, a \rangle e + \langle a, a \rangle e - \langle a, a \rangle = 2 \langle a, e \rangle - \langle a, a \rangle - \langle e, e \rangle, \\
\partial_3 \langle a, a, b \rangle &= \langle a, b \rangle - \langle a, a \rangle b + \langle a, a \rangle b - \langle a, a \rangle = 2 \langle a, b \rangle - \langle a, a \rangle - \langle e, b \rangle, \\
\partial_3 \langle a, e, e \rangle &= \langle e, e \rangle - \langle a, e \rangle e + \langle a, e \rangle e - \langle a, e \rangle = 0, \\
\partial_3 \langle a, e, b \rangle &= \langle e, b \rangle - \langle a, e \rangle b + \langle a, e \rangle b - \langle a, e \rangle = \langle a, b \rangle - \langle a, e \rangle, \\
\partial_3 \langle a, b, e \rangle &= \langle b, e \rangle - \langle a, b \rangle e + \langle a, b \rangle e - \langle a, b \rangle = 0, \\
\partial_3 \langle e, e, b \rangle &= \langle e, b \rangle - \langle e, e \rangle b + \langle e, e \rangle b - \langle e, e \rangle = \langle e, b \rangle - \langle e, e \rangle.
\end{aligned}
$$

Hence $\operatorname{Im} \partial_3$ is generated by

$$
\begin{aligned}
&\langle 1, a \rangle - \langle 1, 1 \rangle, \ \ \langle 1, e \rangle - \langle 1, 1 \rangle, \ \ \langle 1, b \rangle - \langle 1, 1 \rangle, \\
&\langle a, b \rangle - \langle a, e \rangle, \ \ \langle e, b \rangle - \langle e, e \rangle, \ \ \langle a, a \rangle - 2 \langle a, e \rangle + \langle e, e \rangle, \\
&\text{and } \ \langle a, a \rangle - 2 \langle a, b \rangle + \langle e, b \rangle.
\end{aligned}
$$

Thus $\operatorname{Im} \partial_3$ contains all the generators of $\operatorname{Ker} \partial_2$. Hence $\operatorname{Im} \partial_3 = \operatorname{Ker} \partial_2$ and $H_2(M_5 \,/\, M_5 \backslash 0) = 0$. The size of this last computation reveals a need for more efficient methods. □

It follows from Theorem 3.1 and Lemma 6.1 that $H^2(M_5,\ \mathcal{G}) \cong \operatorname{Ext}(\mathbb{Z}, G) \oplus \operatorname{Hom}(0, G) = 0$ whenever $\mathcal{G}$ is almost constant at G.

2. Next we look at the homology functors of M_5. First note that an abelian group valued functor $\mathcal{G} = (G, \gamma)$ on M_5 assigns to the elements $1, a, e, b, 0$ of M_5 five abelian groups G_1, G_a, G_e, G_b, and G_0, with maps as follows.

$$
\begin{array}{|ccccc}
\hline
1 & a & e & b & 0 \\
a & e & e & b & 0 \\
e & e & e & b & 0 \\
b & b & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}
$$

Since $1t = 1$ only if $t = 1$ there is only one map $\gamma_{1,1} : G_1 \longrightarrow G_1$.
Since $1t = a$ only if $t = a$ there is only one map $\gamma_{1,a} : G_1 \longrightarrow G_a$.
Since $at = a$ only if $t = 1$ there is only one map $\gamma_{a,1} : G_a \longrightarrow G_a$.

Since $at = e$ if and only if $t = a$ or $t = e$, there are two maps $\gamma_{a,a}, \gamma_{a,e} : G_a \longrightarrow G_e$.

Since $et = e$ if and only if $t = 1$, $t = a$, or $t = e$, there are three maps $\gamma_{e,1}$, $\gamma_{e,a}$, $\gamma_{e,e} : G_e \longrightarrow G_e$.

Since $et = b$ only if $t = b$ there is only one map $\gamma_{e,b} : G_e \longrightarrow G_b$.

Since $bt = b$ if and only if $t = 1$, $t = a$, or $t = e$, there are three maps $\gamma_{b,1}$, $\gamma_{b,a}$, $\gamma_{b,e} : G_b \longrightarrow G_b$.

Since $bt = 0$ if and only if $t = b$ or $t = 0$, there are two maps $\gamma_{b,b}, \gamma_{b,0} : G_b \longrightarrow G_0$.

Since $0t = 0$ for all $t \in S$, there are five maps $\gamma_{0,1}, \gamma_{0,a}, \gamma_{0,e}, \gamma_{0,b}, \gamma_{0,0} : G_0 \longrightarrow G_0$.

All other maps arise from composition $\gamma_{c,tu} = \gamma_{ct,u} \circ \gamma_{c,t}$; for instance there are two maps $\gamma_{e,b} \circ \gamma_{a,a}, \gamma_{e,b} \circ \gamma_{a,e} : G_a \longrightarrow G_b$.

Now $\mathcal{L}_1(M_5) = (\mathcal{L}_1, \lambda)$ assigns to each $s \in M_5$ the free abelian group $L_1(s)$ generated by $\{\, \langle x, t \rangle \bigm| xt = s \,\}$:

$L_1(1)$ is free on $\langle 1, 1 \rangle$;
$L_1(a)$ is free on $\langle 1, a \rangle, \langle a, 1 \rangle$;
$L_1(e)$ is free on $\langle 1, e \rangle, \langle a, a \rangle, \langle a, e \rangle, \langle e, 1 \rangle, \langle e, a \rangle, \langle e, e \rangle$;
$L_1(b)$ is free on $\langle 1, b \rangle, \langle a, b \rangle, \langle e, b \rangle, \langle b, 1 \rangle, \langle b, a \rangle, \langle b, e \rangle$;
$L_1(0)$ is free on $\langle 1, 0 \rangle, \langle a, 0 \rangle, \langle e, 0 \rangle, \langle b, 0 \rangle, \langle 0, 1 \rangle, \langle 0, a \rangle, \langle 0, e \rangle, \langle 0, b \rangle, \langle 0, 0 \rangle, \langle b, b \rangle$;

and $\lambda_{c,u} \langle x, t \rangle = \langle x, tu \rangle$ whenever $xt = c$.

With S ordered so that $1 < a < e < b < 0$, $\mathcal{L}_2(M_5) = (\mathcal{L}_2, \lambda)$ assigns to each $s \in M_5$ the free abelian group $L_2(s)$ on $\{\, \langle x, y, t \rangle \bigm| x \leqq y,\ xyt = s \,\}$:

$L_2(1)$ is free on $\langle 1, 1, 1 \rangle$;
$L_2(a)$ is free on $\langle 1, 1, a \rangle, \langle 1, a, 1 \rangle$;
$L_2(e)$ is free on $\langle 1, 1, e \rangle$, $\langle 1, a, a \rangle$, $\langle 1, a, e \rangle$, $\langle 1, e, 1 \rangle$, $\langle 1, e, a \rangle$, $\langle 1, e, e \rangle$, $\langle a, a, 1 \rangle$,
$\langle a, a, a \rangle, \langle a, a, e \rangle, \langle a, e, 1 \rangle, \langle a, e, a \rangle, \langle a, e, e \rangle, \langle e, e, 1 \rangle, \langle e, e, a \rangle, \langle e, e, e \rangle$;
$L_2(b)$ is free on $\langle 1, 1, b \rangle$, $\langle 1, a, b \rangle$, $\langle 1, e, b \rangle$, $\langle 1, b, 1 \rangle$, $\langle 1, b, a \rangle$, $\langle 1, b, e \rangle$, $\langle a, a, b \rangle, \langle a, e, b \rangle, \langle a, b, 1 \rangle, \langle a, b, a \rangle, \langle a, b, e \rangle, \langle e, e, b \rangle, \langle e, b, 1 \rangle, \langle e, b, a \rangle$, $\langle e, b, e \rangle$;
$L_2(0)$ is free on $\langle 1, 1, 0 \rangle$, $\langle 1, a, 0 \rangle$, $\langle 1, e, 0 \rangle$, $\langle 1, b, b \rangle$, $\langle 1, b, 0 \rangle$, $\langle 1, 0, 1 \rangle$, $\langle 1, 0, a \rangle, \langle 1, 0, e \rangle, \langle 1, 0, b \rangle, \langle 1, 0, 0 \rangle, \langle a, a, 0 \rangle, \langle a, e, 0 \rangle, \langle a, b, b \rangle, \langle a, b, 0 \rangle$, $\langle a, 0, 1 \rangle, \langle a, 0, a \rangle, \langle a, 0, e \rangle, \langle a, 0, b \rangle, \langle a, 0, 0 \rangle, \langle e, e, 0 \rangle, \langle e, b, b \rangle, \langle e, b, 0 \rangle$, $\langle e, 0, 1 \rangle, \langle e, 0, a \rangle, \langle e, 0, e \rangle, \langle e, 0, b \rangle, \langle e, 0, 0 \rangle, \langle b, b, 1 \rangle, \langle b, b, a \rangle, \langle b, b, e \rangle$, $\langle b, b, b \rangle, \langle b, b, 0 \rangle, \langle b, 0, 1 \rangle, \langle b, 0, a \rangle, \langle b, 0, e \rangle, \langle b, 0, b \rangle, \langle b, 0, 0 \rangle, \langle 0, 0, 1 \rangle$, $\langle 0, 0, a \rangle$, $\langle 0, 0, e \rangle$, $\langle 0, 0, b \rangle$, $\langle 0, 0, 0 \rangle$;

and $\lambda_{c,u} \langle x, y \rangle t = \langle x, y \rangle tu$ whenever $xyt = c$.

The functor $\mathcal{Z}_1(M_5) = \operatorname{Ker} \partial_1$ is isomorphism to $\mathcal{L}_1$, since $\partial_1 = 0$.

Lemma 6.2 *$\mathcal{B}_1(M_5) = Im\, \partial_2 = (B, \beta)$ assigns the following groups: $B(1)$ is the free abelian group on $\langle 1, 1 \rangle$; $B(a)$ is the free abelian group on $\langle 1, a \rangle$; $B(e)$ is the free abelian group on $\langle 1, e \rangle$, $\langle a, e \rangle$, $\langle e, a \rangle$, $\langle e, 1 \rangle$, $\langle e, e \rangle$, and $2 \langle e, e \rangle$; $B(b)$ is the free* *abelian group on* $\langle 1, b \rangle$, $\langle a, b \rangle$, $\langle e, b \rangle$, $\langle b, a \rangle - \langle b, 1 \rangle$, *and* $\langle b, e \rangle - \langle b, 1 \rangle$; $B(0)$ *is the free abelian group on* $\langle 1, 0 \rangle$, $\langle a, 0 \rangle$, $\langle e, 0 \rangle$, $\langle b, 0 \rangle$, $\langle 0, 1 \rangle$, $\langle 0, a \rangle$, $\langle 0, e \rangle$, $\langle 0, b \rangle$, $\langle 0, 0 \rangle$, *and* $2 \langle b, b \rangle$.

Proof The group $B(s)$ is generated by all $\partial \langle x, y, t \rangle$, where $x \leqq y$, $xyt = s$, and $\partial \langle x, y, t \rangle = \langle y, xt \rangle - \langle xy, t \rangle + \langle x, yt \rangle$. Note that $\partial \langle 1, x, t \rangle = \langle x, 1t \rangle - \langle 1x, t \rangle + \langle 1, xt \rangle = \langle 1, xt \rangle$. Hence

$B(1)$ is generated by $\partial \langle 1, 1, 1 \rangle = \langle 1, 1 \rangle$;

$B(a)$ is generated by $\partial \langle 1, 1, a \rangle = \langle 1, a \rangle = \partial \langle 1, a, 1 \rangle$;

$B(e)$ is generated by

$$\begin{aligned}
\partial \langle 1, 1, e \rangle = \partial \langle 1, a, a \rangle &= \partial \langle 1, a, e \rangle = \langle 1, e \rangle, \\
\partial \langle 1, e, 1 \rangle = \partial \langle 1, e, a \rangle &= \partial \langle 1, e, e \rangle = \langle 1, e \rangle, \\
\partial \langle a, a, 1 \rangle = \langle a, a1 \rangle - \langle aa, 1 \rangle + \langle a, a1 \rangle &= 2 \langle a, a \rangle - \langle e, 1 \rangle, \\
\partial \langle a, a, a \rangle = \langle a, aa \rangle - \langle aa, a \rangle + \langle a, aa \rangle &= 2 \langle a, e \rangle - \langle e, a \rangle, \\
\partial \langle a, a, e \rangle = \langle a, ae \rangle - \langle aa, e \rangle + \langle a, ae \rangle &= 2 \langle a, e \rangle - \langle e, e \rangle, \\
\partial \langle a, e, 1 \rangle = \langle e, a1 \rangle - \langle ae, 1 \rangle + \langle a, e1 \rangle &= \langle e, a \rangle - \langle e, 1 \rangle + \langle a, e \rangle, \\
\partial \langle a, e, a \rangle = \langle e, aa \rangle - \langle ae, a \rangle + \langle a, ea \rangle &= \langle e, e \rangle - \langle e, a \rangle + \langle a, e \rangle, \\
\partial \langle a, e, e \rangle = \langle e, ae \rangle - \langle ae, e \rangle + \langle a, ee \rangle &= \langle a, e \rangle, \\
\partial \langle e, e, 1 \rangle = \langle e, e1 \rangle - \langle ee, 1 \rangle + \langle e, e1 \rangle &= 2 \langle e, e \rangle - \langle e, 1 \rangle, \\
\partial \langle e, e, a \rangle = \langle e, ea \rangle - \langle ee, a \rangle + \langle e, ea \rangle &= 2 \langle e, e \rangle - \langle e, a \rangle, \\
\partial \langle e, e, e \rangle = \langle e, ee \rangle - \langle ee, e \rangle + \langle e, ee \rangle &= \langle e, e \rangle;
\end{aligned}$$

hence $B(e)$ is generated by $\langle 1, e \rangle, \langle a, e \rangle, \langle e, e \rangle$, and $\langle e, 1 \rangle$ (from $\partial \langle e, e, 1 \rangle$), $\langle e, a \rangle$ (from $\partial \langle e, e, a \rangle$), and $2 \langle a, a \rangle$ (from $\partial \langle a, a, 1 \rangle$).

$B(b)$ is generated by

$$\begin{aligned}
\partial \langle 1, 1, b \rangle &= \partial \langle 1, a, b \rangle = \partial \langle 1, e, b \rangle = \langle 1, b \rangle, \\
\partial \langle 1, b, 1 \rangle &= \partial \langle 1, b, a \rangle = \partial \langle 1, b, e \rangle = \langle 1, b \rangle, \\
\partial \langle a, a, b \rangle &= \langle a, ab \rangle - \langle aa, b \rangle + \langle a, ab \rangle = 2 \langle a, b \rangle - \langle e, b \rangle, \\
\partial \langle a, e, b \rangle &= \langle e, ab \rangle - \langle ae, b \rangle + \langle a, eb \rangle = \langle a, b \rangle, \\
\partial \langle a, b, 1 \rangle &= \langle b, a1 \rangle - \langle ab, 1 \rangle + \langle a, b1 \rangle = \langle b, a \rangle - \langle b, 1 \rangle + \langle a, b \rangle, \\
\partial \langle a, b, a \rangle &= \langle b, aa \rangle - \langle ab, a \rangle + \langle a, ba \rangle = \langle b, e \rangle - \langle b, a \rangle + \langle a, b \rangle, \\
\partial \langle a, b, e \rangle &= \langle b, ae \rangle - \langle ab, e \rangle + \langle a, be \rangle = \langle a, b \rangle, \\
\partial \langle e, e, b \rangle &= \langle e, eb \rangle - \langle ee, b \rangle + \langle e, eb \rangle = \langle e, b \rangle, \\
\partial \langle e, b, 1 \rangle &= \langle b, e1 \rangle - \langle eb, 1 \rangle + \langle e, b1 \rangle = \langle b, e \rangle - \langle b, 1 \rangle + \langle e, b \rangle, \\
\partial \langle e, b, a \rangle &= \langle b, ea \rangle - \langle eb, a \rangle + \langle e, ba \rangle = \langle b, e \rangle - \langle b, a \rangle + \langle e, b \rangle, \\
\partial \langle e, b, e \rangle &= \langle b, ee \rangle - \langle eb, e \rangle + \langle e, be \rangle = \langle e, b \rangle;
\end{aligned}$$

hence $B(b)$ is generated by $\langle 1, b \rangle, \langle a, b \rangle, \langle e, b \rangle, \langle b, a \rangle - \langle b, 1 \rangle$ (from $\partial \langle a, b, 1 \rangle$), and $\langle b, e \rangle - \langle b, 1 \rangle$ (from $\partial \langle e, b, 1 \rangle$; then $\langle b, e \rangle - \langle b, a \rangle = (\langle b, e \rangle - \langle b, 1 \rangle) - (\langle b, a \rangle - \langle b, 1 \rangle))$.

Finally, $B(0)$ is generated by

$$
\begin{aligned}
\partial\langle 1,y,0\rangle &= \langle 1,0\rangle \ \ (y=1,a,e,b,0),\\
\partial\langle 1,0,t\rangle &= \langle 1,0\rangle \ \ (t=1,a,e,b,0),\\
\partial\langle a,a,0\rangle &= \langle a,a0\rangle-\langle aa,0\rangle+\langle a,a0\rangle \;=\; 2\langle a,0\rangle-\langle e,0\rangle,\\
\partial\langle a,e,0\rangle &= \langle e,a0\rangle-\langle ae,0\rangle+\langle a,e0\rangle \;=\; \langle a,0\rangle,\\
\partial\langle a,b,b\rangle &= \langle b,ab\rangle-\langle ab,b\rangle+\langle a,bb\rangle \;=\; \langle a,0\rangle,\\
\partial\langle a,b,0\rangle &= \langle b,a0\rangle-\langle ab,0\rangle+\langle a,b0\rangle \;=\; \langle a,0\rangle,\\
\partial\langle a,0,1\rangle &= \langle 0,a1\rangle-\langle a0,1\rangle+\langle a,01\rangle \;=\; \langle 0,a\rangle-\langle 0,1\rangle+\langle a,0\rangle,\\
\partial\langle a,0,a\rangle &= \langle 0,aa\rangle-\langle a0,a\rangle+\langle a,0a\rangle \;=\; \langle 0,e\rangle-\langle 0,a\rangle+\langle a,0\rangle,\\
\partial\langle a,0,e\rangle &= \langle 0,ae\rangle-\langle a0,e\rangle+\langle a,0e\rangle \;=\; \langle a,0\rangle,\\
\partial\langle a,0,b\rangle &= \langle 0,ab\rangle-\langle a0,b\rangle+\langle a,0b\rangle \;=\; \langle a,0\rangle,\\
\partial\langle a,0,0\rangle &= \langle 0,a0\rangle-\langle a0,0\rangle+\langle a,00\rangle \;=\; \langle a,0\rangle,\\
\partial\langle e,e,0\rangle &= \langle e,e0\rangle-\langle ee,0\rangle+\langle e,e0\rangle \;=\; \langle e,0\rangle,\\
\partial\langle e,b,b\rangle &= \langle b,eb\rangle-\langle eb,b\rangle+\langle e,bb\rangle \;=\; \langle e,0\rangle,\\
\partial\langle e,b,0\rangle &= \langle b,e0\rangle-\langle eb,0\rangle+\langle e,b0\rangle \;=\; \langle e,0\rangle,\\
\partial\langle e,0,1\rangle &= \langle 0,e1\rangle-\langle e0,1\rangle+\langle e,01\rangle \;=\; \langle 0,e\rangle-\langle 0,1\rangle+\langle e,0\rangle,\\
\partial\langle e,0,a\rangle &= \langle 0,ea\rangle-\langle e0,a\rangle+\langle e,0a\rangle \;=\; \langle 0,e\rangle-\langle 0,a\rangle+\langle e,0\rangle,\\
\partial\langle e,0,e\rangle &= \langle 0,ee\rangle-\langle e0,e\rangle+\langle e,0e\rangle \;=\; \langle e,0\rangle,\\
\partial\langle e,0,b\rangle &= \langle 0,eb\rangle-\langle e0,b\rangle+\langle e,0b\rangle \;=\; \langle e,0\rangle,\\
\partial\langle e,0,0\rangle &= \langle 0,e0\rangle-\langle e0,0\rangle+\langle e,00\rangle \;=\; \langle e,0\rangle,\\
\partial\langle b,b,1\rangle &= \langle b,b1\rangle-\langle bb,1\rangle+\langle b,b1\rangle \;=\; 2\langle b,b\rangle-\langle 0,1\rangle,\\
\partial\langle b,b,a\rangle &= \langle b,ba\rangle-\langle bb,a\rangle+\langle b,ba\rangle \;=\; 2\langle b,b\rangle-\langle 0,a\rangle,\\
\partial\langle b,b,e\rangle &= \langle b,be\rangle-\langle bb,e\rangle+\langle b,be\rangle \;=\; 2\langle b,b\rangle-\langle 0,e\rangle,\\
\partial\langle b,b,b\rangle &= \langle b,bb\rangle-\langle bb,b\rangle+\langle b,bb\rangle \;=\; 2\langle b,0\rangle-\langle 0,b\rangle,\\
\partial\langle b,b,0\rangle &= \langle b,b0\rangle-\langle bb,0\rangle+\langle b,b0\rangle \;=\; 2\langle b,0\rangle-\langle 0,0\rangle,\\
\partial\langle b,0,1\rangle &= \langle 0,b1\rangle-\langle b0,1\rangle+\langle b,01\rangle \;=\; \langle 0,b\rangle-\langle 0,1\rangle+\langle b,0\rangle,\\
\partial\langle b,0,a\rangle &= \langle 0,ba\rangle-\langle b0,a\rangle+\langle b,0a\rangle \;=\; \langle 0,b\rangle-\langle 0,a\rangle+\langle b,0\rangle,\\
\partial\langle b,0,e\rangle &= \langle 0,be\rangle-\langle b0,e\rangle+\langle b,0e\rangle \;=\; \langle 0,b\rangle-\langle 0,e\rangle+\langle b,0\rangle,\\
\partial\langle b,0,b\rangle &= \langle 0,bb\rangle-\langle b0,b\rangle+\langle b,0b\rangle \;=\; \langle 0,0\rangle-\langle 0,b\rangle+\langle b,0\rangle,\\
\partial\langle b,0,0\rangle &= \langle 0,b0\rangle-\langle b0,0\rangle+\langle b,00\rangle \;=\; \langle b,0\rangle,\\
\partial\langle 0,0,1\rangle &= \langle 0,01\rangle-\langle 00,1\rangle+\langle 0,01\rangle \;=\; 2\langle 0,0\rangle-\langle 0,1\rangle,\\
\partial\langle 0,0,a\rangle &= \langle 0,0a\rangle-\langle 00,a\rangle+\langle 0,0a\rangle \;=\; 2\langle 0,0\rangle-\langle 0,a\rangle,\\
\partial\langle 0,0,e\rangle &= \langle 0,0e\rangle-\langle 00,e\rangle+\langle 0,0e\rangle \;=\; 2\langle 0,0\rangle-\langle 0,e\rangle,\\
\partial\langle 0,0,b\rangle &= \langle 0,0b\rangle-\langle 00,b\rangle+\langle 0,0b\rangle \;=\; 2\langle 0,0\rangle-\langle 0,b\rangle,\\
\partial\langle 0,0,0\rangle &= \langle 0,00\rangle-\langle 00,0\rangle+\langle 0,00\rangle \;=\; \langle 0,0\rangle;
\end{aligned}
$$

hence $B(0)$ is generated by $\langle 1, 0 \rangle$, $\langle a, 0 \rangle$, $\langle e, 0 \rangle$, $\langle b, 0 \rangle$, $\langle 0, 0 \rangle$; $\langle 0, 1 \rangle$, $\langle 0, a \rangle$, $\langle 0, e \rangle$, $\langle 0, b \rangle$ (from $\partial \langle 0, 0, t \rangle$; and $2 \langle b, b \rangle$ (from $\partial \langle b, b, 1 \rangle$).

These are free abelian subgroups of the free abelian group on all $\langle x, t \rangle$. □

Comparing the generators of $\mathcal{B}_1(M_5)$ to the generators of $\mathcal{L}_1(S)$, we see that $\mathcal{H}_1(M_5) = \mathcal{Z}_1(M_5)/\mathcal{B}_1(M_5) = (\mathcal{H}_1, \eta)$ has

$$H_1(1) = 0;\ \ H_1(a) = 0;\ \ H_1(e) \cong \langle a, a \rangle\mathbb{Z} / 2 \langle a, a \rangle\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z};$$
$$H_1(b) \cong \mathbb{Z};\ \ \text{and}\ \ H_1(0) \cong \langle b, b \rangle\mathbb{Z} / 2 \langle b, b \rangle\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

It is interesting that some torsion appears in the homology functor $\mathcal{H}_1(M_5)$, but not in the homology group $H_1(M_5) \cong \mathbb{Z}$.

Finding $\mathcal{H}_2(M_5)$ from its definition is positively horrendous and will be skipped.

Lemma 6.3 *The maps in $\mathcal{B}_1(M_5) = (B, \beta)$ are determined by the following: $\beta_{1,1} : B(1) \longrightarrow B(1)$ is the identity on $B(1)$; $\beta_{1,a} : B(1) \longrightarrow B(a)$ sends $\langle 1, 1 \rangle$ to $\langle 1, a \rangle$; $\beta_{a,1} : B(a) \longrightarrow B(a)$ is the identity on $B(1)$; $\beta_{a,a} = \beta_{a,e} : B(a) \longrightarrow B(e)$ sends $\langle 1, a \rangle$ to $\langle 1, e \rangle$;*

$B(e) \longrightarrow B(e)$	$\langle 1, e \rangle$	$\langle a, e \rangle$	$\langle e, 1 \rangle$	$\langle e, a \rangle$	$\langle e, e \rangle$	$2 \langle a, a \rangle$
$\beta_{e,1}$	$\langle 1, e \rangle$	$\langle a, e \rangle$	$\langle e, 1 \rangle$	$\langle e, a \rangle$	$\langle e, e \rangle$	$2 \langle a, a \rangle$
$\beta_{e,a}$	$\langle 1, e \rangle$	$\langle a, e \rangle$	$\langle e, a \rangle$	$\langle e, e \rangle$	$\langle e, e \rangle$	$2 \langle a, e \rangle$
$\beta_{e,e}$	$\langle 1, e \rangle$	$\langle a, e \rangle$	$\langle e, e \rangle$	$\langle e, e \rangle$	$\langle e, e \rangle$	$2 \langle a, e \rangle$

$B(e) \longrightarrow B(b)$	$\langle 1, e \rangle$	$\langle e, 1 \rangle$	$\langle a, e \rangle$	$\langle e, a \rangle$	$\langle e, e \rangle$	$2 \langle a, a \rangle$
$\beta_{e,b}$	$\langle 1, b \rangle$	$\langle e, b \rangle$	$\langle a, b \rangle$	$\langle e, b \rangle$	$\langle e, b \rangle$	$2 \langle a, b \rangle$

$B(b) \longrightarrow B(b)$	$\langle 1, b \rangle$	$\langle a, b \rangle$	$\langle e, b \rangle$	$\langle b, a \rangle - \langle b, 1 \rangle$	$\langle b, e \rangle - \langle b, 1 \rangle$
$\beta_{b,1}$	$\langle 1, b \rangle$	$\langle a, b \rangle$	$\langle e, b \rangle$	$\langle b, a \rangle - \langle b, 1 \rangle$	$\langle b, e \rangle - \langle b, 1 \rangle$
$\beta_{b,a}$	$\langle 1, b \rangle$	$\langle a, b \rangle$	$\langle e, b \rangle$	$\langle b, e \rangle - \langle b, a \rangle$	$\langle b, e \rangle - \langle b, a \rangle$
$\beta_{b,e}$	$\langle 1, b \rangle$	$\langle a, b \rangle$	$\langle e, b \rangle$	0	0

$B(b) \longrightarrow B(0)$	$\langle 1, b \rangle$	$\langle a, b \rangle$	$\langle e, b \rangle$	$\langle b, a \rangle - \langle b, 1 \rangle$	$\langle b, e \rangle - \langle b, 1 \rangle$
$\beta_{b,b}$	$\langle 1, 0 \rangle$	$\langle a, 0 \rangle$	$\langle e, 0 \rangle$	0	0
$\beta_{b,0}$	$\langle 1, 0 \rangle$	$\langle a, 0 \rangle$	$\langle e, 0 \rangle$	0	0

$B(0) \longrightarrow B(0)$	$\langle x,0\rangle$	$\langle 0,1\rangle$	$\langle 0,a\rangle$	$\langle 0,e\rangle$	$\langle 0,b\rangle$	$2\langle b,b\rangle$
$\beta_{0,1}$	$\langle x,0\rangle$	$\langle 0,1\rangle$	$\langle 0,a\rangle$	$\langle 0,e\rangle$	$\langle 0,b\rangle$	$2\langle b,b\rangle$
$\beta_{0,a}$	$\langle x,0\rangle$	$\langle 0,a\rangle$	$\langle 0,e\rangle$	$\langle 0,e\rangle$	$\langle 0,b\rangle$	$2\langle b,b\rangle$
$\beta_{0,e}$	$\langle x,0\rangle$	$\langle 0,e\rangle$	$\langle 0,e\rangle$	$\langle 0,e\rangle$	$\langle 0,b\rangle$	$2\langle b,b\rangle$
$\beta_{0,b}$	$\langle x,0\rangle$	$\langle 0,b\rangle$	$\langle 0,b\rangle$	$\langle 0,b\rangle$	$\langle 0,0\rangle$	$2\langle b,0\rangle$
$\beta_{0,0}$	$\langle x,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$2\langle b,0\rangle$

In particular, $\beta_{b,b} = \beta_{b,0}$. *The remaining maps obtain by composition.*

Proof We noted that a similar list of maps applies to every abelian group valued functor on M_5. The particular values in the tables follow from $\beta_{c,u}\,\langle\, x, t\,\rangle = \langle\, x, tu\,\rangle$, whenever $xt = c$. □

3. We went into some detail regarding $\mathcal{B}_1(M_5)$ in order to prove the following result.

Proposition 6.4 $\mathcal{B}_1(M_5)$ *is not projective.*

Proof We construct an epimorphism $\sigma : \mathcal{G} \longrightarrow \mathcal{B} = \mathcal{B}_1(M_5)$ through which the identity on $\mathcal{B}$ does not lift. This works because $\mathcal{G} = (G, \gamma)$ and σ can be rigged so that $\gamma_{b,0}$ is injective, whereas $\beta_{b,0}$ is not.

The groups G_s are free abelian groups on generators denoted by $[xt]$, where $x, t \in S$, plus four generators $[p]$, $[q]$, $[r]$, and $[s]$: G_1 is free on $[11]$; G_a is free on $[1a]$; G_e is free on $[1e]$, $[ae]$, $[ee]$, $[e1]$, $[ea]$, and $[aa]$; G_b is free on $[1b]$, $[ab]$, $[eb]$, $[p]$, and $[q]$; G_0 is free on $[10]$, $[a0]$, $[e0]$, $[b0]$, $[01]$, $[0a]$, $[0e]$, $[0b]$, $[00]$, $[bb]$, $[r]$, and $[s]$.

The maps $\gamma_{c,u}$ are as follows: $\gamma_{c,u}\,[xt] = [xv]$, where $xt = c$ and $v = tu$ (compare with $\beta_{c,u}\,\langle\, x, t\,\rangle = \langle\, x, v\,\rangle$), with one exception: $\gamma_{e,t}\,[aa] = 2\,[ac]$, if $t \neq 1$ and $at = c$. In particular, $\gamma_{a,a} = \gamma_{a,e}$ and $\gamma_{b,b} = \gamma_{b,0}$. In addition, $\gamma_{b,0}\,[p] = [r]$; $\gamma_{b,0}\,[q] = [s]$; $\gamma_{0,t}\,[r] = [r]$ for all $t \in S$; and $\gamma_{0,t}\,[s] = [s]$ for all $t \in S$. The remaining maps are then obtained by composition. Note that $\gamma_{b,0}$ sends the generators of G_b onto distinct generators of G_0 and is therefore injective.

Next, a homomorphism $\sigma_c : G_c \longrightarrow B_c$ is defined for each $c \in S$ by $\sigma_{xt}\,[xt] = \langle\, x, t\,\rangle$ (where $xt = c$), with one exception: $\sigma_e\,[aa] = 2\,\langle\, a, a\,\rangle$; in addition, $\sigma_b\,[p] = \langle\, b, a\,\rangle - \langle\, b, 1\,\rangle$; $\sigma_b\,[q] = \langle\, b, e\,\rangle - \langle\, b, 1\,\rangle$; $\sigma_0\,[r] = \sigma_0\,[s] = 0$.

We show that $\sigma = (\sigma_c)_{c\in S} : \mathcal{G} \longrightarrow \mathcal{B}$ is a natural transformation. Comparing the values of $\beta_{c,u}$ and $\gamma_{c,u}$, where $(c, u) = (1, 1), (1, a), (a, a), (a, e), (e, e), (e, b), (b, b)$, or $(b, 0)$, shows that $\beta_{c,u}\,\sigma_c\,[xt] = \sigma_{cu}\,\gamma_{c,u}\,[xt]$ (where $c = xt$), unless $[xt] = [aa]$; for instance,

$$\beta_{e,b}\,\sigma_e\,[ae] \;=\; \beta_{e,b}\,\langle\, a, e\,\rangle \;=\; \langle\, a, b\,\rangle \;=\; \sigma_b\,[ab] \;=\; \sigma_b\,\gamma_{e,b}\,[ae].$$

In addition,

$$\beta_{e,1}\,\sigma_e\,[aa] \;=\; \beta_{e,1}\,(2\,\langle\, a, a\,\rangle) \;=\; 2\,\langle\, a, a\,\rangle \;=\; \sigma_e\,(2\,[aa]) \;=\; \sigma_e\,\gamma_{e,1}\,[aa]\,;$$

if $t \neq 1$ and $at = c$, then $et = ac$ and

$$\beta_{e,t}\,\sigma_e\,[aa] \;=\; \beta_{e,t}\,(2\,\langle\, a, a\,\rangle) \;=\; 2\,\langle\, a, c\,\rangle \;=\; \sigma_{ac}\,(2\,[ac]) \;=\; \sigma_{et}\,\gamma_{e,t}\,[aa]\,;$$

and

$$\begin{aligned}
\beta_{b,0}\,\sigma_b\,[p] &= \beta_{b,0}\,(\langle\, b, a\,\rangle - \langle\, b, 1\,\rangle) = 0 = \sigma_0\,[r] = \sigma_0\,\gamma_{b,0}\,[p],\\
\beta_{b,0}\,\sigma_b\,[q] &= \beta_{b,0}\,(\langle\, b, e\,\rangle - \langle\, b, 1\,\rangle) = 0 = \sigma_0\,[s] = \sigma_0\,\gamma_{b,0}\,[q],\\
\beta_{0,t}\,\sigma_0\,[r] &= 0 = \sigma_0\,[r] = \sigma_0\,\gamma_{0,t}\,[r] \text{ and}\\
\beta_{0,t}\,\sigma_0\,[s] &= 0 = \sigma_0\,[s] = \sigma_0\,\gamma_{0,t}\,[s], \quad \text{for all } t \in S.
\end{aligned}$$

Thus $\beta_{c,u} \circ \sigma_c = \sigma_{cu} \circ \gamma_{c,u} : G_c \longrightarrow B_{cu}$ when $(c, u) = (1, 1), (1, a), (a, a), (a, e), (e, e), (e, b), (b, b)$, or $(b, 0)$. Since the remaining maps are then obtained by composition, it follows that $\beta_{c,u} \circ \sigma_c = \sigma_{cu} \circ ga_{c,u}$ for all c, u. Thus σ is a natural transformation.

Moreover, all σ_s are isomorphisms, except for σ_0, which is surjective; hence σ is an epimorphism.

Now assume that $\mathcal{B}$ is projective. Then the identity on $\mathcal{B}$ lifts through σ and there is a natural transformation $\mu : \mathcal{B} \longrightarrow \mathcal{G}$ such that $\sigma \circ \mu$ is the identity on $\mathcal{B}$. In particular, μ_s is injective, for every $s \in S$. Since μ is a natural transformation, we have $\mu_0\,\beta_{b,0} = \gamma_{b,0}\,\mu_b$, and $\mu_0\,\beta_{b,0}$ is injective. But $\beta_{b,0}$ is not injective. This contradiction shows that $\mathcal{B}$ is not projective. □

Since Im ∂_2 is not projective, the universal coefficients theorem (e.g. Theorem 3.6.5 of [14]) cannot be applied to $\mathcal{L}_*(S)$. (But we do not have a counterexample for $\mathcal{A}_*(S)$.)

7 Proof of Lemma 1.4

Recall that a basis of a symmetric set X is a subset Y of X such that every mapping of Y into an abelian group G extends uniquely to a symmetric mapping of X into G.

Lemma 1.4 states: *Let X be a* symmetric *subset of S^n, where $n \leqq 4$ and S is a* totally ordered *set.*

If $n = 2$, then the set Y of all $(a, b) \in X$ such that $a \leqq b$ is a basis of X.

If $n = 3$, then the set Y of all $(a, b, c) \in X$ such that $a \leqq b$ and $a < c$ is a basis of X.

If $n = 4$, then the set Y of all $(a, b, c, d) \in X$ such that either $a < b, c, d$, or $a \leqq b, c$ and $b < d$, or both, is a basis of X.

Moreover, if f is a mapping of Y into an abelian group G and g is the symmetric mapping of X into G that extends g, then every value of g is a sum of values of f and opposites of values of f.

1. If $n = 2$, then a mapping $f : X \longrightarrow G$ is symmetric if and only if $f\,(b, a) = f\,(a, b)$ whenever $b > a$ in S. Consequently, f is uniquely determined by its values on $Y = \{\,(a, b) \in X \mid a \leqq b\,\}$.

Now let f be a mapping of Y into an abelian group G. Extend f to a mapping g of X into G, namely:

$$g\,(x, y) = \begin{cases} f\,(x, y) & \text{if } x \leqq y,\\ f\,(y, x) & \text{if } x > y. \end{cases}$$

Then g is symmetric. Moreover, every value of g is a value of f. This proves Lemma 1.4 if $n = 2$.

2. Now let $n = 3$; the symmetry conditions are:

$$f(a, b, a) = 0, \tag{S3a}$$

$$f(c, b, a) = -f(a, b, c), \tag{S3b}$$

$$f(a, b, c) + f(b, c, a) + f(c, a, b) = 0, \quad \text{and} \tag{S3c}$$

$$f(a, b, c) = f(b, a, c) - f(b, c, a). \tag{S3d}$$

Lemma 7.1 *If S is totally ordered, then for any given $x, y, z \in S$ exactly one of the following holds:*

$$\begin{array}{ll}
(1) & x = z, \\
(2) & x = y < z, \\
(3) & x = y > z, \\
(4) & x < y = z, \\
(5) & x > y = z, \\
(6) & x < y < z \text{ or } x < z < y, \\
(7) & y < x < z \text{ or } y < z < x, \\
(8) & z < x < y \text{ or } z < y < x.
\end{array}$$

Proof This is clear. □

Lemma 7.2 *If S is totally ordered and $n = 3$, then a mapping $f : X \longrightarrow G$ is symmetric if and only if, for all $(x, y, z) \in X$,*

$$\textit{if } x = z, \textit{ then } f(x, y, z) = 0, \tag{P1}$$

$$\textit{if } x = y > z, \textit{ then } f(x, y, z) = -f(z, y, x), \tag{P3}$$

$$\textit{if } x > y = z, \textit{ then } f(x, y, z) = -f(y, z, x), \tag{P5}$$

$$\textit{if } y < x < z \textit{ or if } y < z < x, \textit{ then } f(x, y, z) = f(y, x, z) - f(y, z, x), \tag{P7}$$

$$\textit{if } z < x < y \textit{ or if } z < y < x, \textit{ then } f(x, y, z) = -f(z, y, x). \tag{P8}$$

Moreover,

$$Y = \{ (a, b, c) \in X \mid a \leqq b \textit{ and } a < c \}$$

contains every (a, b, c) that appears in the right hand side of (P1), ..., (P8).

Proof Any given $x, y, z \in S$ fall in exactly one of the cases (1) to (8) in Lemma 7.1. In case (1), (S3a) yields $f(x, y, z) = 0$. In cases (3) and (8), (S3b) yields (P3) and (P8): $f(x, y, z) = -f(z, y, x)$; moreover, (z, y, x) is in case (4) or (6). In case (5), (S3d) and (S3a) yield (P5):

$$f(x, y, z) \;=\; f(y, x, z) - f(y, z, x) \;=\; -f(y, z, x)\,;$$

moreover, (y, z, x) is in case (2). In case (7), (S3d) yields (P7): $f(x, y, z) = f(y, x, z) - f(y, z, x)$; moreover, (y, x, z) and (y, z, x) are in case (6). Thus a symmetric mapping has properties (P1) through (P8). Moreover, $(a, b, c) \in Y$ ($a \leqq b$ and $a < c$) if and only if (a, b, c) is in case (2), (4), or (6); this includes every (a, b, c) that appears in the right hand side of (P1) through (P8).

Conversely, assume that f has properties (P1) through (P8). We show that f has property (S3d).

If $a = c$, then $f(b, a, c) - f(b, c, a) = 0 = f(a, b, c)$, by (P1), and (S3d) holds.

If $a = b < c$, or if $a = b > c$, then $f(b, c, a) = 0$ by (P1), $f(b, a, c) = f(a, b, c)$, and (S3d) holds.

If $a < b = c$, then $f(b, a, c) = 0$ by (P1), $f(b, c, a) = -f(a, c, b) = -f(a, b, c)$ by (P3), and (S3d) holds.

If $a > b = c$, then $f(b, a, c) = 0$ by (P1), $f(a, b, c) = -f(b, c, a)$ by (P5), and (S3d) holds.

If $a < b < c$, or if $a < c < b$, then $f(b, a, c) = f(a, b, c) - f(a, c, b)$ by (P7), $f(b, c, a) = -f(a, c, b)$ by (P8), and (S3d) holds.

If $b < a < c$, or if $b < c < a$, then $f(a, b, c) = f(b, a, c) - f(b, c, a)$ by (P7), and (S3d) holds.

If $c < a < b$, or if $c < b < a$, then $f(b, a, c) = -f(c, a, b)$ by (P8), $f(b, c, a) = f(c, b, a) - f(c, a, b)$ by (P7), $f(a, b, c) = -f(c, b, a)$ by (P8), and (S3d) holds.

In every case (S3d) holds; hence f is symmetric, by Lemma 1.2. □

We now prove Lemma 1.4 in case $n = 3$:

Lemma 7.3 *($n = 3$) Let S be a totally ordered set, let X be a symmetric subset of $S \times S \times S$, and let G be an abelian group. Every mapping f of*

$$Y \;=\; \{\, (a, b, c) \in X \;\big|\; a \leqq b \;\text{ and }\; a < c \,\}$$

into G extends uniquely to a symmetric mapping g of X into G. Moreover, every value of g is a sum of values of f and opposites of values of f.

Proof Given $f : Y \longrightarrow G$, define $\widehat{f}$ as follows:

$$\widehat{f}(a,b,c) \;=\; \begin{cases} (1)\;\; 0 & \text{if } a = c,\\ (2)\;\; f(a,b,c) & \text{if } a = b < c,\\ (3)\;\; -f(c,b,a) & \text{if } a = b > c,\\ (4)\;\; f(a,b,c) & \text{if } a < b = c,\\ (5)\;\; -f(b,c,a) & \text{if } a > b = c,\\ (6)\;\; f(a,b,c) & \text{if } a < b < c \text{ or if } a < c < b,\\ (7)\;\; f(b,a,c) - f(b,c,a) & \text{if } b < a < c \text{ or if } b < c < a,\\ (8)\;\; -f(c,b,a) & \text{if } c < a < b \text{ or if } c < b < a. \end{cases}$$

First $(a,b,c) \in Y$ ($a \leqq b$ and $a < c$) if and only if a, b, c are in cases (2), (4), or (6) in Lemma 7.1. By Lemma 7.2, a symmetric mapping g that extends f must have properties (P1) through (P8), hence must coincide with $\widehat{f}$. Conversely, $\widehat{f}$ extends f and has properties (P1) through (P8), hence is symmetric, by Lemma 7.2. □

Note that the set Y in the above is not unique: it depends on the total order on S; also $Y' = \{\, (x,y,z) \in X \mid z \leqq y \text{ and } z < x \,\}$ would also serve.

3. Symmetric mappings of four variables are more complex. First the symmetry conditions

$$f(a,b,b,a) \;=\; 0, \tag{S4a}$$

$$f(d,c,b,a) \;=\; -f(a,b,c,d), \tag{S4b}$$

$$f(a,b,c,d) - f(b,c,d,a) + f(c,d,a,b) - f(d,a,b,c) \;=\; 0, \quad \text{and} \tag{S4c}$$

$$f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a) \;=\; 0, \tag{S4d}$$

imply additional properties (some of which were noted in [9]). By (S4b), $f(a,b,c,d) = -f(d,c,b,a)$, whence (S4d) yields

$$f(a,b,c,d) \;=\; -f(c,d,b,a) + f(c,b,d,a) - f(c,b,a,d). \tag{S4e}$$

By (S4d), $f(a,b,c,a) \;=\; f(b,a,c,a) - f(b,c,a,a) + f(b,c,a,a)$, so that $f(a,b,c,a) = f(b,a,c,a)$, in particular

$$f(b,a,b,a) \;=\; 0, \tag{S4f}$$

by (S4a); and then (S4b) yields $f(a,b,a,c) = -f(c,a,b,a) = -f(a,c,b,a)$ and

$$f(a,b,a,c) \;=\; f(a,b,c,a) \;=\; f(b,a,c,a). \tag{S4g}$$

By (S4e), $f(a,a,c,a) = -f(c,a,a,a) + f(c,a,a,a) - f(c,a,a,a)$ and

$$f(a,a,c,a) \;=\; -f(c,a,a,a). \tag{S4h}$$

Properties (S4g) and (S4b) also imply

$$f(a,b,a,a) = f(b,a,a,a) = -f(a,a,a,b), \tag{S4i}$$

$$f(a,b,a,c) = f(a,b,c,a) = -f(a,c,a,b), \tag{S4j}$$

$$f(a,b,c,a) = f(b,a,c,a) = -f(c,a,b,a), \tag{S4k}$$

since $f(a,b,c,a) = f(b,a,c,a)$ and $f(a,b,c,a) = f(a,b,a,c)$, by (S4g).

Lemma 7.4 *If S is totally ordered, then for any given $x, y, z, t \in S$ exactly one of the following holds:*

(1) $x = y = z = t$;

(2) $x = y = z < t$; (3) $x = y = z > t$;

(4) $x = y = t < z$; (5) $x = y = t > z$;

(6) $x = z = t < y$; (7) $x = z = t > y$;

(8) $y = z = t < x$; (9) $y = z = t > x$;

(10) $x = y < z = t$; (11) $x = y > z = t$;

(12) $x = z < y = t$; (13) $x = z > y = t$;

(14) $x = t < y = z$; (15) $x = t > y = z$;

(16) $x = y < z < t$; (17) $x = y < t < z$;

(18) $z < x = y < t$; (19) $t < x = y < z$;

(20) $z < t < x = y$; (21) $t < z < x = y$;

(22) $x = z < y < t$; (23) $x = z < t < y$;

(24) $y < x = z < t$; (25) $t < x = z < y$;

(26) $y < t < x = z$; (27) $t < y < x = z$;

(28) $x = t < y < z$; (29) $x = t < z < y$;

(30) $y < x = t < z$; (31) $z < x = t < y$;

(32) $y < z < x = t$; (33) $z < y < x = t$;

(34) $y = z < x < t$; (35) $y = z < t < x$;

(36) $x < y = z < t$; (37) $t < y = z < x$;

(38) $x < t < y = z$; (39) $t < x < y = z$;

(40) $y = t < x < z$; (41) $y = t < z < x$;

(42) $x < y = t < z$; (43) $z < y = t < x$;

(44) $x < z < y = t$; (45) $z < x < y = t$;

(46) $z = t < x < y$; (47) $z = t < y < x$;

(48) $x < z = t < y$; (49) $y < z = t < x$;

(50) $x < y < z = t$; (51) $y < x < z = t$;

(52) $x < y, z, t$ and y, z, t are distinct;

(53) $y < x, z, t$ and x, z, t are distinct;

(54) $z < x, y, t$ and x, y, t are distinct;

(55) $t < x, y, z$ and x, y, z are distinct.

Proof The 55 cases in this Lemma are arranged in decreasing numbers of equalities between x, y, z, and t. Some of these cases could be merged, but it would not be as clear that they are disjoint and cover all possibilities. □

Lemma 7.5 *Let $n = 4$ and let X be a symmetric subset of $S \times S \times S \times S$. Let Y be the set of all $(a, b, c, d) \in X$ such that either $a < b, c, d$, or $a \leqq b, c$ and $b < d$, or both. Then $(a, b, c, d) \in Y$ if and only if (a, b, c, d) is in case (2), (9), (10), (16), (17), (22), (36), (38), (42), (44), (48), (50), or (52).*

Proof. If (a, b, c, d) is in case (9), (22), (36), (38), (42), (44), (48), (50), or (52), then $a < b, c, d$; if in case (2) ($a = b = c < d$), (10) ($a = b < c = d$), (16) ($a = b < c < d$), (17) ($a = b < d < c$), or (22) ($a = c < b < d$), then $a \leqq b, c$ and $b < d$.

Conversely, if $a < b, c, d$, then either $b = c$ and (a, b, c, d) is in case (9), (36), or (38); or $b = d$ and (a, b, c, d) is in case (9), (42), or (44); or $c = d$ and (a, b, c, d) is in case (9), (10), (48), or (50); or b, c, d are all distinct and (a, b, c, d) is in case (52). If $a \leqq b, c$ and $b < d$, but not $a < b, c$, then either $a = b = c < d$ and (a, b, c, d) is in case (2); or $a = b < c, d$ and (a, b, c, d) is in case (10), (16), or (17); or $a = c < b < d$ and (a, b, c, d) is in case (22). □

Lemma 7.6 *Let S be a totally ordered set, let X be a symmetric subset of $S \times S \times S \times S$, and let G be an abelian group. Let Y be the set of all $(a, b, c, d) \in X$ in case (2), (9), (10), (16), (17), (22), (36), (38), (42), (44), (48), (50), or (52). A mapping $f : X \longrightarrow G$ is symmetric if and only if it has the following properties:*

(P1) *if* $x = y = z = t$, *then* $f(x, y, z, t) = 0$;

(P3) *if* $x = y = z > t$, *then* $f(x, y, z, t) = -f(t, z, y, x)$;

(P4) *if* $x = y = t < z$, *then* $f(x, y, z, t) = f(x, y, t, z)$;

(P5) *if* $x = y = t > z$, *then* $f(x, y, z, t) = -f(z, x, y, t)$;

(P6) *if* $x = z = t < y$, *then* $f(x, y, z, t) = -f(x, z, t, y)$;

(P7) *if* $x = z = t > y$, *then* $f(x, y, z, t) = f(y, x, z, t)$;

(P8) *if* $y = z = t < x$, *then* $f(x, y, z, t) = -f(y, z, t, x)$;

(P11) *if* $x = y > z = t$, *then* $f(x, y, z, t) = -f(t, z, y, x)$;

(P12) *if* $x = z < y = t$, *then* $f(x, y, z, t) = 0$;

(P13) *if* $x = z > y = t$, *then* $f(x, y, z, t) = 0$;

(P14) *if* $x = t < y = z$, *then* $f(x, y, z, t) = 0$;

(P15) *if* $x = t > y = z$, *then* $f(x, y, z, t) = 0$;

(P18) *if* $z < x = y < t$, *then* $f(x, y, z, t) = -f(z, t, y, x) + f(z, y, t, x) - f(z, y, x, t)$;

(P19) *if* $t < x = y < z$, *then* $f(x, y, z, t) = -f(t, z, y, x)$;

(P20) *if* $z < t < x = y$, *then* $f(x, y, z, t) = -f(z, t, y, x) + f(z, y, t, x) - f(z, y, x, t)$;

(P21) *if* $t < z < x = y$, *then* $f(x, y, z, t) = -f(t, z, y, x)$;

(P23) *if* $x = z < t < y$, *then* $f(x, y, z, t) = -f(x, t, z, y)$;
(P24) *if* $y < x = z < t$, *then* $f(x, y, z, t) = f(y, z, t, x)$;
(P25) *if* $t < x = z < y$, *then* $f(x, y, z, t) = -f(t, z, y, x)$;
(P26) *if* $y < t < x = z$, *then* $f(x, y, z, t) = f(y, z, t, x)$;
(P27) *if* $t < y < x = z$, *then* $f(x, y, z, t) = -f(t, z, y, x)$;
(P28) *if* $x = t < y < z$, *then* $f(x, y, z, t) = f(x, y, t, z)$;
(P29) *if* $x = t < z < y$, *then* $f(x, y, z, t) = -f(t, z, x, y)$;
(P30) *if* $y < x = t < z$, *then* $f(x, y, z, t) = f(y, x, z, t)$;
(P31) *if* $z < x = t < y$, *then* $f(x, y, z, t) = -f(z, t, y, x)$;
(P32) *if* $y < z < x = t$, *then* $f(x, y, z, t) = f(y, x, z, t)$;
(P33) *if* $z < y < x = t$, *then* $f(x, y, z, t) = -f(z, t, y, x)$;
(P34) *if* $y = z < x < t$, *then* $f(x, y, z, t) = f(y, x, z, t) - f(y, z, x, t) + f(y, z, t, x)$;
(P35) *if* $y = z < t < x$, *then* $f(x, y, z, t) = -f(z, t, y, x)$
$' + f(z, y, t, x) - f(z, y, x, t)$;
(P37) *if* $t < y = z < x$, *then* $f(x, y, z, t) = -f(t, z, y, x)$;
(P39) *if* $t < x < y = z$, *then* $f(x, y, z, t) = -f(t, z, y, x)$;
(P40) *if* $y = t < x < z$, *then* $f(x, y, z, t) = f(t, x, y, z)$;
(P41) *if* $y = t < z < x$, *then* $f(x, y, z, t) = -f(t, z, y, x)$;
(P43) *if* $z < y = t < x$, *then* $f(x, y, z, t) = -f(z, y, x, t)$;
(P45) *if* $z < x < y = t$, *then* $f(x, y, z, t) = -f(z, y, x, t)$;
(P46) *if* $z = t < x < y$, *then* $f(x, y, z, t) = -f(t, z, y, x)$;
(P47) *if* $z = t < y < x$, *then* $f(x, y, z, t) = -f(t, z, y, x)$;
(P49) *if* $y < z = t < x$, *then* $f(x, y, z, t) = f(y, x, z, t) - f(y, z, x, t) + f(y, z, t, x)$;
(P51) *if* $y < x < z = t$, *then* $f(x, y, z, t) = f(y, x, z, t) - f(y, z, x, t) + f(y, z, t, x)$;
(P53) *if* $y < x, z, t$ *and* x, z, t *are distinct, then*
$f(x, y, z, t) = f(y, x, z, t) - f(y, z, x, t) + f(y, z, t, x)$;
(P54) *if* $z < x, y, t$ *and* x, y, t *are distinct, then*
$f(x, y, z, t) = -f(z, t, y, x) + f(z, y, t, x) - f(z, y, x, t)$;
(P55) *if* $t < x, y, z$ *and* x, y, z *are distinct, then* $f(x, y, z, t) = -f(t, z, y, x)$;

Moreover, Y contains every (a, b, c, d) that appears in the right hand side of (P1), ..., (P55).

Proof First, assume that f is symmetric. Let $a, b, c, d \in S$.

(P1): if $a = b = c = d$, then $f(a, b, c, d) = 0$ by (S4a).
(P3): if $a = b = c > d$, then (d, c, b, a) is in case (9) and $f(a, b, c, d) = -f(d, c, b, a)$ by (S4b).
(P4): if $a = b = d < c$, then (a, b, d, c) is in case (2) and $f(a, b, c, d) = f(a, b, d, c)$ by (S4g).
(P5): if $a = b = d > c$, then (c, a, b, d) is in case (9) and $f(a, b, c, d) = -f(c, a, b, d)$ by (S4h).

(P6): if $a = c = d < b$, then (a, c, d, b) is in case (2) and $f(a, b, c, d) = -f(a, c, d, b)$ by (S4i).

(P7): if $a = c = d > b$, then (b, a, c, d) is in case (9) and $f(a, b, c, d) = f(b, a, c, d)$ by (S4i).

(P8): if $b = c = d < a$, then (b, c, d, a) is in case (2) and $f(a, b, c, d) = -f(b, c, d, a)$ by (S4b).

(P11): if $a = b > c = d$, then (d, c, b, a) is in case (10) and $f(a, b, c, d) = -f(d, c, b, a)$ by (S4b).

(P12): if $a = c < b = d$, then $f(a, b, c, d) = 0$ by (S4f).

(P13): if $a = c > b = d$, then $f(a, b, c, d) = 0$ by (S4f).

(P14): if $a = d < b = c$, then $f(a, b, c, d) = 0$ by (S4a).

(P15): if $a = d > b = c$, then $f(a, b, c, d) = 0$ by (S4a).

(P18): if $c < a = b < d$, then (c, d, b, a) and (c, d, b, a) are in case (50), (c, b, a, d) is in case (36), and $f(a, b, c, d) = -f(c, d, b, a) + f(c, b, d, a) - f(c, b, a, d)$ by (S4e).

(P19): if $d < a = b < c$, then (d, c, b, a) is in case (48) and $f(a, b, c, d) = -f(d, c, b, a)$ by (S4b).

(P20): if $c < d < a = b$, then (c, d, b, a) is in case (50), (c, b, d, a) is in case (44), (c, b, a, d) is in case (36), and $f(a, b, c, d) = -f(c, d, b, a) + f(c, b, d, a) - f(c, b, a, d)$ by (S4e).

(P21): if $d < c < a = b$, then (d, c, b, a) is in case (50) and $f(a, b, c, d) = -f(d, c, b, a)$ by (S4b).

(P23): if $a = c < d < b$, then (a, d, c, b) is in case (22) and $f(a, b, c, d) = -f(a, d, c, b)$ by (S4j).

(P24): if $b < a = c < d$, then (b, c, d, a) is in case (42) and $f(a, b, c, d) = f(b, c, d, a)$ by (S4g).

(P25): if $d < a = c < b$, then (d, c, b, a) is in case (42) and $f(a, b, c, d) = -f(d, c, b, a)$ by (S4b).

(P26): if $b < d < a = c$, then (b, c, d, a) is in case (44) and $f(a, b, c, d) = f(b, c, d, a)$ by (S4g).

(P27): if $d < b < a = c$, then (d, c, b, a) is in case (44) and $f(a, b, c, d) = -f(d, c, b, a)$ by (S4b).

(P28): if $a = d < b < c$, then (a, b, d, c) is in case (22) and $f(a, b, c, d) = f(a, b, d, c)$ by (S4g).

(P29): if $a = d < c < b$, then (d, c, a, b) is in case (22) and $f(a, b, c, d) = -f(d, c, a, b)$ by (S4j).

(P30): if $b < a = d < c$, then (b, a, c, d) is in case (42) and $f(a, b, c, d) = f(b, a, c, d)$ by (S4g).

(P31): if $c < a = d < b$, then (c, d, b, a) is in case (42) and $f(a, b, c, d) = -f(c, d, b, a)$ by (S4k).

(P32): if $b < c < a = d$, then (b, a, c, d) is in case (44) and $f(a, b, c, d) = f(b, a, c, d)$ by (S4g).

(P33): if $c < b < a = d$, then (c, d, b, a) is in case (44) and $f(a, b, c, d) = -f(c, d, b, a)$ by (S4k).

(P34): if $b = c < a < d$, then (b, a, c, d) is in case (22), (b, c, a, d) is in case (16), (b, c, d, a) is in case (17), and $f(a, b, c, d) = f(b, a, c, d) - f(b, c, a, d) + f(b, c, d, a)$ by (S4d).

(P35): if $b = c < d < a$, then (c, d, b, a) is in case (22), (c, b, d, a) is in case (16), (c, b, a, d) is in case (17), and $f(a, b, c, d) = -f(c, d, b, a) + f(c, b, d, a) - f(c, b, a, d)$ by (S4e).

(P37): if $d < b = c < a$, then (d, c, b, a) is in case (36) and $f(a, b, c, d) = -f(d, c, b, a)$ by (S4b).

(P39): if $d < a < b = c$, then (d, c, b, a) is in case (38) and $f(a, b, c, d) = -f(d, c, b, a)$ by (S4b).

(P40): if $b = d < a < c$, then (d, a, b, c) is in case (22) and $f(a, b, c, d) = f(d, a, b, c)$ by (S4g).

(P41): if $b = d < c < a$, then (d, c, b, a) is in case (22) and $f(a, b, c, d) = -f(d, c, b, a)$ by (S4b).

(P43): if $c < b = d < a$, then (c, d, a, b) is in case (42) and $f(a, b, c, d) = -f(c, b, a, d)$ by (S4k).

(P45): if $c < a < b = d$, then (c, b, a, d) is in case (44) and $f(a, b, c, d) = -f(c, b, a, d)$ by (S4k).

(P46): if $c = d < a < b$, then (d, c, b, a) is in case (17) and $f(a, b, c, d) = -f(d, c, b, a)$ by (S4b).

(P47): if $c = d < b < a$, then (d, c, b, a) is in case (17) and $f(a, b, c, d) = -f(d, c, b, a)$ by (S4b).

(P49): if $b < c = d < a$, then (b, a, c, d) is in case (48), (b, c, a, d) is in case (42), (b, c, d, a) is in case (36), and $f(a, b, c, d) = f(b, a, c, d) - f(b, c, a, d) + f(b, c, d, a)$ by (S4d).

(P51): if $b < a < c = d$, then (b, a, c, d) is in case (50), (b, c, a, d) is in case (42), (b, c, d, a) is in case (36), and $f(a, b, c, d) = f(b, a, c, d) - f(b, c, a, d) + f(b, c, d, a)$ by (S4d).

(P53): if $b < a, c, d$ and a, c, d are distinct, then (b, a, c, d), (b, c, a, d), and (b, c, d, a) are in case (52), and $f(a, b, c, d) = f(b, a, c, d) - f(b, c, a, d) + f(b, c, d, a)$ by (S4d).

(P54): if $c < a, b, d$ and a, b, d are distinct, then (c, d, b, a), (c, b, d, a), and (c, b, a, d) are in case (52), and $f(a, b, c, d) = -f(c, d, b, a) + f(c, b, d, a) - f(c, b, a, d)$ by (S4e).

(P55): if $d < a, b, c$ and a, b, c are distinct, then (d, c, b, a) is in case (52) and $f(a, b, c, d) = -f(d, c, b, a)$ by (S4b).

Thus a symmetric mapping f has properties (P1) through (P55). In each case, every (x, y, z, t) in the right hand side falls in case (2), (9), (10), (16), (17), (22), (36), (38), (42), (44), (48), (50), or (52). □

Conversely, let f be a mapping with properties (P1) through (P55).

Property (S4a): $f(a, b, b, a) = 0$ follows from (P1), (P14), and (P15).

Property (S4b): $f(d, c, b, a) = -f(a, b, c, d)$ is proved by considering all possible cases. Some cases follow from others, since (S4b) is not affected by reversing a, b, c, d into d, c, b, a.

(1). If $a = b = c = d$, then $f(a, b, c, d) = f(d, c, b, a) = 0$ by (P1).

(2). If $a = b = c < d$, then $f(d, c, b, a) = - f(c, b, a, d)$ by (P8).
(3). If $a = b = c > d$, then $f(a, b, c, d) = - f(d, c, b, a)$ by (P3).
(4). If $a = b = d < c$, then $f(a, b, c, d) = f(a, b, d, c)$ by (P4) and $f(d, c, b, a) = - f(d, b, a, c)$ by (P6).
(5). If $a = b = d > c$, then $f(a, b, c, d) = - f(c, a, b, d)$ by (P5) and $f(d, c, b, a) = f(c, d, b, a)$ by (P7).

Reversing a, b, c, d into d, c, b, a yields (S4b) in cases (8): $b = c = d < a$, (9): $b = c = d > a$, (6): $a = c = d < b$, and (7): $a = c = d > b$. (10). If $a = b < c = d$, then $f(d, c, b, a) = - f(a, b, c, d)$ by (P11).

Reversal yields (S4b) in case (11): $a = b > c = d$.

In cases (12): $a = c < b = d$, (13): $a = c > b = d$, (14): $a = d < b = c$, and (15): $a = d > b = c$, $f(a, b, c, d) = f(d, c, b, a) = 0$ and (S4b) holds.

(16). If $a = b < c < d$, then $f(d, c, b, a) = -f(a, b, c, d)$ by (P47).
(17). If $a = b < d < c$, then $f(d, c, b, a) = -f(a, b, c, d)$ by (P46).
(18). If $c < a = b < d$, then $f(a, b, c, d) = -f(c, d, b, a) + f(c, b, d, a) - f(c, b, a, d)$ by (P18) and $f(d, c, b, a) = f(c, d, b, a) - f(c, b, d, a) + f(c, b, a, d)$ by (P49).
(19). If $d < a = b < c$, then $f(a, b, c, d) = -f(d, c, b, a)$ by (P19).
(20). If $c < d < a = b$, then $f(a, b, c, d) = -f(c, d, b, a) + f(c, b, d, a) - f(c, b, a, d)$ by (P20) and $f(d, c, b, a) = f(c, d, b, a) - f(c, b, d, a) + f(c, b, a, d)$ by (P51).
(21). If $d < c < a = b$, then $f(a, b, c, d) = -f(d, c, b, a)$ by (P21).

Reversal yields (S4b) in cases (47): $c = d < b < a$, (46): $c = d < a < b$, (49): $b < c = d < a$, (48): $a < c = d < b$, (51): $b < a < c = d$, and (50): $a < b < c = d$.

(22). If $a = c < b < d$, then $f(d, c, b, a) = -f(a, b, c, d)$ by (P41).
(23). If $a = c < d < b$, then $f(a, b, c, d) = -f(a, d, c, b)$ by (P23) and $f(d, c, b, a) = f(a, d, c, b)$ by (P40).
(24). If $b < a = c < d$, then $f(a, b, c, d) = f(b, c, d, a)$ by (P24) and $f(d, c, b, a) = -f(b, c, d, a)$ by (P43).

(25). If $d < a = c < b$, then $f(a, b, c, d) = -f(d, c, b, a)$ by (P25).
(26). If $b < d < a = c$, then $f(a, b, c, d) = f(b, c, d, a)$ by (P26) and $f(d, c, b, a) = -f(b, c, d, a)$ by (P45).
(27). If $d < b < a = c$, then $f(a, b, c, d) = -f(d, c, b, a)$ by (P27).

Reversal yields (S4b) in cases (41): $b = d < c < a$, (40): $b = d < a < c$, (43): $c < b = d < a$, (42): $a < b = d < c$, (45): $c < a < b = d$, and (44): $a < c < b = d$.

(28). If $a = d < b < c$, then $f(a, b, c, d) = f(a, b, d, c)$ by (P28) and $f(d, c, b, a) = -f(a, b, d, c)$ by (P29).
(30). If $b < a = d < c$, then $f(a, b, c, d) = f(b, a, c, d)$ by (P30) and $f(d, c, b, a) = -f(b, a, c, d)$ by (P31).
(32). If $b < c < a = d$, then $f(a, b, c, d) = f(b, a, c, d)$ by (P32) and $f(d, c, b, a) = -f(b, a, c, d)$ by (P33).

(34). If $b = c < a < d$, then $f(a, b, c, d) = f(b, a, c, d) - f(b, c, a, d) + f(b, c, d, a)$ by (P34)
and $f(d, c, b, a) = -f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a)$ by (P35).
(36). If $a < b = c < d$, then $f(d, c, b, a) = -f(a, b, c, d)$ by (P37).
(38). If $a < d < b = c$, then $f(d, c, b, a) = -f(a, b, c, d)$ by (P39).

Reversal yields (S4b) in cases (29): $a = d < c < b$, (31): $c < a = d < b$, (33): $c < b < a = d$, (35): $b = c < d = a$, (37): $d < b = c < a$, and (39): $d < a < b = c$.

This covers cases (1) through (51). In the remaining cases, a, b, c, d are all distinct:

(52). If $a < b, c, d$, but not $a < d < c < b$, then $f(d, c, b, a) = -f(a, b, c, d)$ by (P55).
(53). If $b < a, c, d$, then $f(a, b, c, d) = f(b, a, c, d) - f(b, c, a, d) + f(b, c, d, a)$ by (P53) and $f(d, c, b, a) = -f(b, c, d, a) + f(b, c, a, d) - f(b, a, c, d)$ by (P54).

Reversal yields (S4b) in cases (55): $d < a, b, c$ and (54): $c < a, b, d$. Thus (S4b) holds in all cases.

In view of Lemma 1.3 it remains to prove that f has property (S4d). Using (S4b) transforms (S4d): $f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) = 0$ into $-f(d, c, b, a) + f(d, c, a, b) - f(d, a, c, b) + f(a, d, c, b) = 0$, which is (S4d) applied to a, d, c, b, so that the latter follows from (S4b) and from (S4d) applied to a, b, c, d. Exchanging b and d will thus make some of the cases below follow from other cases.

(1). If $a = b = c = d$, then $f(a, b, c, d) = f(b, a, c, d) = f(b, c, a, d) = f(b, c, d, a) = 0$.

(2). If $a = b = c < d$, then

$$f(a, b, c, d) = f(b, a, c, d) = f(b, c, a, d) = f(a, a, a, d),$$
$$f(b, c, d, a) = f(a, a, a, d) \text{ by (P4), and}$$
$$\begin{aligned} &f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) \\ &= f(a, a, a, d) - f(a, a, a, d) + f(a, a, a, d) - f(a, a, a, d) = 0. \end{aligned}$$

(3). If $a = b = c > d$, then

$$f(a, b, c, d) = f(b, a, c, d) = f(b, c, a, d) = -f(d, a, a, a) \text{ by (P3),}$$
$$f(b, c, d, a) = -f(d, a, a, a) \text{ by (P5), and}$$
$$\begin{aligned} &f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) \\ &= -f(d, a, a, a) + f(d, a, a, a) - f(d, a, a, a) + f(d, a, a, a) = 0. \end{aligned}$$

Exchanging b and d yields cases (6): $a = d = c < b$ and (7): $a = d = c > b$.

(4). If $a = b = d < c$, then

$$f(a, b, c, d) = f(b, a, c, d) = f(a, a, a, c) \text{ by (P4),}$$
$$f(b, c, a, d) = f(b, c, d, a) = -f(a, a, a, c) \text{ by (P6), and}$$
$$\begin{aligned} &f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) \\ &= f(a, a, a, c) - f(a, a, a, c) - f(a, a, a, c) + f(a, a, a, c) = 0. \end{aligned}$$

(5). If $a = b = d > c$, then

$$\begin{aligned}
&f(a,b,c,d) = f(b,a,c,d) = -f(c,a,a,a) \text{ by (P5)},\\
&f(b,c,a,d) = f(c,a,a,a) \text{ by (P7)}, \text{ and}\\
&f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a)\\
&= -f(c,a,a,a) + f(d,a,a,a) + f(c,a,a,a) - f(c,a,a,a) = 0.
\end{aligned}$$

(8). If $b = c = d < a$, then

$$\begin{aligned}
&f(a,b,c,d) = -f(b,b,b,a) \text{ by (P8)},\\
&f(b,a,c,d) = -f(b,b,b,a) \text{ by (P6)},\\
&f(b,c,a,d) = f(b,b,b,a) \text{ by (P4)}, \text{ and}\\
&f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a)\\
&= -f(b,b,b,a) + f(b,b,b,a) + f(b,b,b,a) - f(b,b,b,a) = 0.
\end{aligned}$$

(9). If $b = c = d > a$, then

$$\begin{aligned}
&f(b,a,c,d) = f(a,b,b,b) \text{ by (P7)},\\
&f(b,c,a,d) = -f(a,b,b,b) \text{ by (P5)},\\
&f(b,c,d,a) = -f(a,b,b,b) \text{ by (P3)}, \text{ and}\\
&f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a)\\
&= f(a,b,b,b) + f(a,b,b,b) - f(a,b,b,b) - f(a,b,b,b) = 0.
\end{aligned}$$

(10). If $a = b < c = d$, then

$$\begin{aligned}
&f(b,c,a,d) = 0 \text{ by (P12)}, \quad f(b,c,d,a) = 0 \text{ by (P14)}, \text{ and}\\
&f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a) = 0.
\end{aligned}$$

(11). If $a = b > c = d$, then

$$\begin{aligned}
&f(a,b,c,d) = f(b,a,c,d) = -f(c,c,a,a) \text{ by (P11)},\\
&f(b,c,a,d) = 0 \text{ by (P13)},\\
&f(b,c,d,a) = 0 \text{ by (P15)}, \text{ and}\\
&f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a) = 0.
\end{aligned}$$

Exchanging b and d yields cases (14): $a = d < b = c$ and (15): $a = d > b = c$.
(12). If $a = c < b = d$, then $f(a,b,c,d) = 0$ by (P12), $f(b,a,c,d) = f(b,c,a,d) = 0$ by (P15), and $f(b,c,d,a) = 0$ by (P13).
(13). If $a = c > b = d$, then $f(a,b,c,d) = 0$ by (P13), $f(b,a,c,d) = f(b,c,a,d) = 0$ by (P14), and $f(b,c,d,a) = 0$ by (P12).

(16). If $a = b < c < d$, then $f(a, b, c, d) = f(b, a, c, d)$,

$$\begin{aligned} f(b, c, d, a) &= f(b, c, a, d) \text{ by (P28), and} \\ f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) &= 0. \end{aligned}$$

(17). If $a = b < d < c$, then $f(a, b, c, d) = f(b, a, c, d)$,

$$\begin{aligned} f(b, c, a, d) &= -f(b, d, a, c) \text{ by (P23), and} \\ f(b, c, d, a) &= -f(a, d, b, c) = -f(b, d, a, c) \text{ by (P29), and} \\ f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) &= 0. \end{aligned}$$

Exchanging b and d yields cases (29): $a = d < c < b$ and (28): $a = d < c < b$.
(18). If $c < a = b < d$, then $f(a, b, c, d) = f(b, a, c, d)$,

$$\begin{aligned} f(b, c, a, d) &= f(c, a, d, b) \text{ by (P24), and} \\ f(b, c, d, a) &= f(c, b, d, a) = f(c, a, d, b) \text{ by (P30), and} \\ f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) &= 0. \end{aligned}$$

(19). If $d < a = b < c$, then $f(a, b, c, d) = f(b, a, c, d)$,

$$\begin{aligned} f(b, c, a, d) &= -f(d, a, c, b) \text{ by (P25), and} \\ f(b, c, d, a) &= -f(d, a, c, b) \text{ by (P31), and} \\ f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) &= 0. \end{aligned}$$

Exchanging b and d yields cases (31): $c < a = d < b$ and (30): $c < a = d < b$.
(20). If $c < d < a = b$, then $f(a, b, c, d) = f(b, a, c, d)$,

$$\begin{aligned} f(b, c, a, d) &= f(c, a, d, b) \text{ by (P26), and} \\ f(b, c, d, a) &= f(c, b, d, a) = f(c, a, d, b) \text{ by (P32), and} \\ f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) &= 0. \end{aligned}$$

(21). If $d < c < a = c$, then $f(a, b, c, d) = f(b, a, c, d)$,

$$\begin{aligned} f(b, c, a, d) &= -f(d, a, c, b) \text{ by (P27), and} \\ f(b, c, d, a) &= -f(d, a, c, b) \text{ by (P33), and} \\ f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) &= 0. \end{aligned}$$

Exchanging b and d yields cases (33): $c < d < a = b$ and (32): $d < c < a = b$.
(22). If $a = c < b < d$, then $f(b, a, c, d) = f(b, c, a, d)$,

$$\begin{aligned} f(b, c, d, a) &= f(a, b, c, d) \text{ by (P40), and} \\ f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) &= 0. \end{aligned}$$

(24). If $b < a = c < d$, then $f(b, a, c, d) = f(b, c, a, d)$,

$$\begin{aligned} f(a, b, c, d) = f(b, c, d, a) \ \text{ by (P24), and} \\ f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) \ = \ 0. \end{aligned}$$

Exchanging b and d yields cases (23): $b < c < a = d$ and (25): $d < c < a = b$.
(26). If $b < d < a = c$, then $f(b, a, c, d) = f(b, c, a, d)$,

$$\begin{aligned} f(a, b, c, d) = f(b, c, d, a) \ \text{ by (P26), and} \\ f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) \ = \ 0. \end{aligned}$$

Exchanging b and d yields case (27): $d < b < a = c$.
(34). If $b = c < a < d$, then

$$\begin{aligned} f(a, b, c, d) \ = \ f(b, a, c, d) - f(b, c, a, d) + f(b, c, d, a) \ \text{ by (P34), and} \\ f(a, b, c, d) - f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) \ = \ 0. \end{aligned}$$

(35). If $b = c < d < a$, then

$$\begin{aligned} f(a, b, c, d) \ &= \ -f(c, d, b, a) + f(c, b, d, a) - f(c, b, a, d) \ \text{ by (P35),} \\ f(b, a, c, d) \ &= \ -f(b, d, c, a) \ \text{ by (P23), and} \\ f(a, b, c, d) &- f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) \\ = \ &-f(c, d, b, a) + f(c, b, d, a) \\ &- f(c, b, a, d) + f(b, d, c, a) + f(b, c, a, d) - f(b, c, d, a) \ = \ 0, \end{aligned}$$

since $b = c$. Exchanging b and d yields cases (46): $c = d < a < b$ and (47): $c = d < b < a$.
(36). If $a < b = c < d$, then

$$\begin{aligned} f(b, a, c, d) \ &= \ f(a, c, d, b) \ \text{ by (P24), and} \\ f(b, c, a, d) \ &= \ -f(a, d, c, b) + f(a, c, d, b) - f(a, c, b, d) \ \text{ by (P18),} \\ f(b, c, d, a) \ &= \ -f(a, d, c, b) \ \text{ by (P19), and} \\ f(a, b, c, d) &- f(b, a, c, d) + f(b, c, a, d) - f(b, c, d, a) \\ = \ &f(a, b, c, d) - f(a, c, d, b) - f(a, d, c, b) \\ &+ f(a, c, d, b) - f(a, c, b, d) + f(a, d, c, b) \ = \ 0, \end{aligned}$$

since $b = c$.

(37). If $d < b = c < a$, then

$$\begin{aligned} f(a,b,c,d) &= -f(d,c,b,a) \text{ by (P37)},\\ f(b,a,c,d) &= -f(d,c,a,b) \text{ by (P25)},\\ f(b,c,a,d) &= -f(d,a,c,b) \text{ by (P19)},\\ f(b,c,d,a) &= -f(d,a,c,b)+f(d,c,a,b)-f(d,c,b,a) \text{ by (P18)}, \quad \text{and}\\ &f(a,b,c,d)-f(b,a,c,d)+f(b,c,a,d)-f(b,c,d,a)\\ &= -f(d,c,b,a)+f(d,c,a,b)-f(d,a,c,b)\\ &\quad +f(d,a,c,b)-f(d,c,a,b)+f(d,c,b,a) = 0. \end{aligned}$$

Exchanging b and d yields case (48): $a < c = d < b$ and (49): $b < c = d < a$.
(38). If $a < d < b = c$, then

$$\begin{aligned} f(b,a,c,d) &= f(a,c,d,b) \text{ by (P26)},\\ f(b,c,a,d) &= -f(a,d,c,b)+f(a,c,d,b)-f(a,c,b,d) \text{ by (P20)},\\ f(b,c,d,a) &= -f(a,d,c,b) \text{ by (P21)}, \quad \text{and}\\ &f(a,b,c,d)-f(b,a,c,d)+f(b,c,a,d)-f(b,c,d,a)\\ &= f(a,b,c,d)-f(a,c,d,b)-f(a,d,c,b)\\ &\quad +f(a,c,d,b)-f(a,c,b,d)+f(a,d,c,b) = 0, \end{aligned}$$

since $b = c$.
(39). If $d < a < b = c$, then

$$\begin{aligned} f(a,b,c,d) &= -f(d,c,b,a) \text{ by (P39)},\\ f(b,a,c,d) &= -f(d,c,a,b) \text{ by (P27)},\\ f(b,c,a,d) &= -f(d,a,c,b) \text{ by (P21)},\\ f(b,c,d,a) &= -f(d,a,c,b)+f(d,c,a,b)-f(d,c,b,a) \text{ by (P20)}, \quad \text{and}\\ &f(a,b,c,d)-f(b,a,c,d)+f(b,c,a,d)-f(b,c,d,a)\\ &= -f(d,c,b,a)+f(d,c,a,b)-f(d,a,c,b)\\ &\quad +f(d,a,c,b)-f(d,c,a,b)+f(d,c,b,a) = 0. \end{aligned}$$

Exchanging b and d yields cases (50): $a < c = d < b$ and (51): $b < c = d < a$.
(40). If $b = d < a < c$, then

$$\begin{aligned} f(a,b,c,d) &= f(d,a,b,c) \text{ by (P40)},\\ f(b,a,c,d) &= f(b,a,d,c) \text{ by (P28)},\\ f(b,c,a,d) &= -f(d,a,b,c) \text{ by (P29)},\\ f(b,c,d,a) &= -f(b,a,d,c) \text{ by (P23)}, \quad \text{and}\\ &f(a,b,c,d)-f(b,a,c,d)+f(b,c,a,d)-f(b,c,d,a)\\ &= f(d,a,b,c)-f(b,a,d,c)-f(d,a,b,c)+f(b,a,d,c) = 0. \end{aligned}$$

(41). If $b = d < c < a$, then

$$\begin{aligned} f(a,b,c,d) &= -f(d,c,b,a) \text{ by (P41)},\\ f(b,a,c,d) &= -f(d,c,b,a) \text{ by (P29)},\\ f(b,c,a,d) &= f(b,c,d,a) \text{ by (P28), and} \end{aligned}$$

$$\begin{aligned} &f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a)\\ &= f(d,c,b,a) - f(d,c,b,a) + f(b,c,d,a) - f(b,c,d,a) = 0. \end{aligned}$$

(42). If $a < b = d < c$, then

$$\begin{aligned} f(b,a,c,d) &= f(a,b,c,d) \text{ by (P30)},\\ f(b,c,a,d) &= -f(a,d,c,b) \text{ by (P31)},\\ f(b,c,d,a) &= -f(a,d,c,b) \text{ by (P25), and} \end{aligned}$$

$$\begin{aligned} &f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a)\\ &= f(a,b,c,d) - f(a,b,c,d) - f(a,d,c,b) + f(a,d,c,b) = 0. \end{aligned}$$

(43). If $b = d < c < a$, then

$$\begin{aligned} f(a,b,c,d) &= -f(c,b,a,d) \text{ by (P43)},\\ f(b,a,c,d) &= -f(c,d,a,b) \text{ by (P31)},\\ f(b,c,a,d) &= f(c,b,a,d) \text{ by (P30)},\\ f(b,c,d,a) &= f(c,d,a,b) \text{ by (P24), and} \end{aligned}$$

$$\begin{aligned} &f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a)\\ &= f(c,b,a,d) - f(c,d,a,b) + f(c,b,a,d) - f(c,d,a,b) = 0, \end{aligned}$$

since $b = d$.

(44). If $a < b = d < c$, then

$$\begin{aligned} f(b,a,c,d) &= f(a,b,c,d) \text{ by (P32)},\\ f(b,c,a,d) &= -f(a,d,c,b) \text{ by (P33)},\\ f(b,c,d,a) &= -f(a,d,c,b) \text{ by (P27), and} \end{aligned}$$

$$\begin{aligned} &f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a)\\ &= f(a,b,c,d) - f(a,b,c,d) - f(a,d,c,b) + f(a,d,c,b) = 0. \end{aligned}$$

(45). If $b = d < c < a$, then

$$\begin{aligned} f(a,b,c,d) &= -f(c,b,a,d) \text{ by (P45)},\\ f(b,a,c,d) &= -f(c,d,a,b) \text{ by (P33)},\\ f(b,c,a,d) &= f(c,b,a,d) \text{ by (P32)},\\ f(b,c,d,a) &= f(c,d,a,b) \text{ by (P26), and} \end{aligned}$$

$$\begin{aligned} &f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a)\\ &= f(c,b,a,d) - f(c,d,a,b) + f(c,b,a,d) - f(c,d,a,b) = 0, \end{aligned}$$

since $b = d$.

(52). If $a < b, c, d$ and b, c, d are distinct, then

$$\begin{aligned}
f(b,a,c,d) &= f(a,b,c,d) - f(a,c,b,d) + f(a,c,d,b) \text{ by (P53)},\\
f(b,c,a,d) &= -f(a,c,b,d) + f(a,c,d,b) - f(a,d,c,b) \text{ by (P54)},\\
f(b,c,d,a) &= -f(a,d,c,b) \text{ by (P55), and}
\end{aligned}$$

$$\begin{aligned}
&f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a)\\
&= f(a,b,c,d) - f(a,b,c,d) + f(a,c,b,d) - f(a,c,d,b)\\
&\quad - f(a,c,b,d) + f(a,c,d,b) - f(a,d,c,b) + f(a,d,c,b) = 0.
\end{aligned}$$

(53). If $b < c, d, a$ and a, c, d are distinct, then

$$f(a,b,c,d) = f(b,a,c,d) - f(b,c,a,d) + f(b,c,d,a) \text{ by (P53), and}$$

$$\begin{aligned}
&f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a)\\
&= f(b,a,c,d) - f(b,c,a,d)\\
&\quad + f(b,c,d,a) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a) = 0.
\end{aligned}$$

Exchanging b and d yields case (55): $d < a, b, c$ and a, b, c are distinct.
(54). If $c < a, b, d$ and a, b, d are distinct, then

$$\begin{aligned}
f(a,b,c,d) &= -f(c,b,a,d) + f(c,b,d,a) - f(c,d,b,a) \text{ by (P54)},\\
f(b,a,c,d) &= -f(c,a,b,d) + f(c,a,d,b) - f(c,d,a,b) \text{ by (P54)},\\
f(b,c,a,d) &= f(c,b,a,d) - f(c,a,b,d) + f(c,a,d,b) \text{ by (P53)},\\
f(b,c,d,a) &= f(c,b,d,a) - f(c,d,b,a) + f(c,d,a,b) \text{ by (P53), and}
\end{aligned}$$

$$\begin{aligned}
&f(a,b,c,d) - f(b,a,c,d) + f(b,c,a,d) - f(b,c,d,a)\\
&= -f(c,b,a,d) + f(c,b,d,a) - f(c,d,b,a)\\
&\quad + f(c,a,b,d) - f(c,a,d,b) + f(c,d,a,b)\\
&\quad + f(c,b,a,d) - f(c,a,b,d) + f(c,a,d,b)\\
&\quad - f(c,b,d,a) + f(c,d,b,a) - f(c,d,a,b) = 0.
\end{aligned}$$

Thus (S4d) holds in all cases. □
We now prove Lemma 1.4 in case $n = 4$:

Lemma 7.7 *($n = 4$) Let S be a totally ordered set, let X be a symmetric subset of $S \times S \times S \times S$, and let G be an abelian group. Let Y be the set of all $(a,b,c,d) \in X$ such that either $a < b, c, d$, or $a \leqq b, c$ and $b < d$, or both. Every mapping f of Y into G extends uniquely to a symmetric mapping g of X into G. Moreover, every value of g is a sum of values of f and opposites of values of f.*

Proof Given $f : Y \longrightarrow G$, define $\widehat{f}$ as follows. If $(a,b,c,d) \in Y$, then $\widehat{f}(a,b,c,d) = f(a,b,c,d)$. If $(a,b,c,d) \in X \backslash Y$, then a, b, c, d is in case (N) $\neq$(2), (9), (10), (16), (17), (22), (36), (38), (42), (44), (48), (50), (52), by Lemma 7.5; all (x,y,z,t) in the right hand side of (PN) are in Y, by Lemma 7.6; define $\widehat{f}(a,b,c,d)$ so that (PN) holds. For example, if (a,b,c,d) is in case (N) = (54), then (c, b, a, d), (c, b, d, a), $(c, d, b, a) \in Y$; let $\widehat{f}(a, b, c, d) = -f(c, b, a, d) + f(c, b, d, a) - f(c, d, b, a)$.

By Lemma 7.6, a symmetric mapping g that extends f must have properties (P1) through (P55), hence must coincide with $\widehat{f}$. Conversely, $\widehat{f}$ extends f and has properties (P1) through (P55), hence is symmetric, by Lemma 7.6; and every value of $\widehat{f}$ is a sum of values of f and opposites of values of f. □

8 Index of notations

Lowercase Roman: elements of groups or sets; mappings

∂: boundary homomorphism (Proposition 2.4, Lemmas 4.5, 5.6)

Uppercase Roman: sets and groups, mostly.

$A_n(a)$: a group of symmetric chains (Proposition 4.2)
B: a convex subset of S (Introduction, Sect. 1)
B^n: a group of symmetric n-coboundaries
B_n: a group of symmetric n-boundaries
B_n: a certain abelian group
$B(a)$: a value of $\mathcal{B}$ (Sect. 6)
C^n: group of symmetric n-cochains
$\widehat{C}^n$: group of all n-cochains (symmetric or not) (Sect. 1)
C_n: a group of symmetric n-chains
$C_n(S/B)$: the group of symmetric n-cochains relative to B (Sect. 2)
$C_n(s)$: a subgroup of $C_n(S)$ (Lemma 4.1)
(Cn): a symmetry condition for chains (Sect. 2)
C_*: a chain complex of free symmetric chain groups (Sect. 2)
G_a: a value of $\mathcal{G}$
H^n: a cohomology group
H_n: a homology group
$H_n(S/B)$: the homology group of S relative to B (Sect. 2)
L_n: a group of 'long' symmetric n-chains
$L_n(a)$ (Sect. 5)
$L_n(s;t)$ (Lemma 5.1)
M_5: a certain commutative monoid (Sect. 6)
PHom: a group of partial homomorphisms
S: your typical long suffering commutative monoid; a set, perhaps totally ordered
T, U: canonical isomorphisms (Propositions 2.3, 4.2, 5.4)
X: a symmetric set
$X_n(a)$: a certain symmetric set (Sect. 5)
Y: a basis of X
Z^n: a group of symmetric n-cocycles
Z_n: a group of symmetric n-cycles

Boldface uppercase Roman: categories.

A: the category of all abelian group valued functors on S
T: the category of thin, abelian group valued functors on S

Script uppercase: functors and sequences thereof.

$\mathcal{A}_n$: a projective, thin symmetric n-chain functor (Proposition 4.2)
$\mathcal{A}_*$: a chain complex of projective, thin symmetric chain functors
$\mathcal{B}$: a boundary functor
$\mathcal{F}(B, G)$: a semiconstant abelian group valued functor (Sect. 1)
$\mathcal{G}$: your typical idle abelian group valued functor
$\mathcal{H}$: one of Green's relations (Introduction)
$\mathcal{H}_n$: a homology functor
$\mathcal{L}_n$: a projective symmetric, 'long' n-chain functor (Sect. 5)
$\mathcal{L}_*$: a chain complex of projective symmetric, 'long' chain functors (Sect. 5)
$\mathcal{Z}$: a cycle functor

Lowercase Greek: homomorphisms, natural transformations.

α: a value of $\mathcal{A}_n$
β: a value of $\mathcal{B}$
γ: a value of $\mathcal{G}$
δ: coboundary homomorphism (Sect. 1)
ι: a canonical homomorphism or partial homomorphism
λ: a value of $\mathcal{L}_n$ (Lemma 5.3)
τ: your typical undistinguished natural transformation
τ^*: a cochain homomorphism induced by τ (Sect. 1)

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