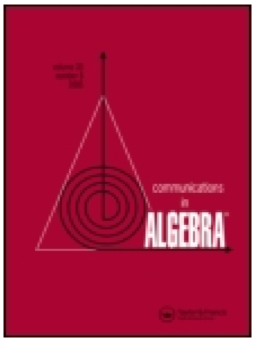

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COCYCLES IN COMMUTATIVE SEMIGROUP COHOMOLOGY

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Abstract. An alternate description of triple cohomology for commutative semigroups is given in dimensions 1, 2, and 3.

Introduction.

1. Commutative semigroup cohomology refers to triple cohomology in the variety of commutative semigroups (Beck [2]; see also [1]). In [4] we gave a concrete description of this cohomology and showed that it coincides with the cohomology in [3] in dimension 2; the second cohomology group $H^2(S, \mathcal{A})$ thus classifies commutative group coextensions of S by $\mathcal{A}$.

The description of commutative cohomology in [4] is derived from its definition by triples and does not lend itself to the computation of examples. Cochains in dimensions $n \geq 2$ are indexed by an unbounded number of elements of S; this makes the computation of cohomology groups an infinite task, even for a finite semigroup.

In dimension 2 one can use the equivalent computable description in [3], in which cochains are indexed by pairs of elements of S. In section 1 we prove a stronger result: the cocycle and coboundary groups for triple cohomology coincide with the groups of symmetric cocycles and coboundaries in [3]. (A sharper description is given in [6].)

In Section 2 we prove a similar but more difficult result for dimension 3, which describes $H^3(S, \mathcal{A})$ using symmetric cochains indexed (as in Leech cohomology) by three elements of S. It is an open question whether these results extend to higher dimensions; if so, the main result in Section 2 might be proved as in [2] or [8].

Sections 1 and 2 also contain normalization results for symmetric 2- and 3-cocycles.

The major results in this article were announced in [5].

2. We keep the notation in [4]. In what follows S is a commutative semigroup. The Leech category $\mathcal{H}(S)$ is defined after [7] as follows. The objects of $\mathcal{H}(S)$ are the elements of S. The morphisms of $\mathcal{H}(S)$ are the elements of $S \times S^1$; when $x \in S$, $t \in S^1$, then (x,t) is a morphism from x to xt. The composition of $(x,t) : x \longrightarrow xt$ and $(xt,u) : xt \longrightarrow xtu$ is $(x,tu) : x \longrightarrow xtu$; the identity on $x \in S$ is $(x,1)$. An abelian group valued functor $\mathcal{A}$ on $\mathcal{H}(S)$ thus assigns to each $x \in S$ an abelian group $\mathcal{A}_x$, and to each pair $(x,t) \in S \times S^1$ a homomorphism $\alpha_{x,t} : \mathcal{A}_x \longrightarrow \mathcal{A}_x t$ (written on the left), so that $\alpha_{x,1}$ is the identity on $\mathcal{A}_x$ and $\alpha_{xt,u}\alpha_{x,t} = \alpha_{x,tu}$ for all x, t, u.

In longer calculations it is convenient to write

$$\alpha_{x,t} g \;=\; g^t \;\in\; \mathcal{A}_{xt} \quad \text{when} \quad g \in \mathcal{A}_x\,;$$

then

$$g^1 = g\,, \quad (g^t)^u = g^{tu}$$

whenever $x \in S$, $a \in \mathcal{A}_x$, $t, u \in S^1$.

Define semigroups T_n by induction as follows: $T_0 = S$; T_{n+1} is the free commutative semigroup on the set T_n. An element of T_{n+1} is a nonempty product of elements of T_n, the factors of which are unique up to order. In what follows it would be very confusing to write the elements of T_{n+1} as the usual products of generators; hence we shall write the elements of T_{n+1} as nonempty unordered sequences $t = [\,x_1, \dots, x_m\,]$ of elements of T_n (so that $m \geq 1$ and $t^\sigma = [\,x_{\sigma 1}, \dots, x_{\sigma m}\,] = [\,x_1, \dots, x_m\,] = t$ for every permutation $\sigma \in S_m$ of $1, 2, \dots, m$). Multiplication in T_{n+1} is given by concatenation:

$$[\,x_1, \dots, x_m\,]\,[\,y_1, \dots, y_n\,] \;=\; [\,x_1, \dots, x_m, y_1, \dots, y_n\,]\,.$$

A homomorphism $p : T_n \longrightarrow S$ is defined by induction by

$$p\,[\,x_1, x_2, \dots, x_m\,] \;=\; (px_1)(px_2)\cdots(px_m),$$

starting with $px = x$ for all $x \in S$; in general, $p[x_1, \dots, x_m]$ is the product of all the elements of S which appear as components of $[x_1, \dots, x_m]$. Similarly, homomorphisms $\pi_i^n : T_{n+1} \longrightarrow T_n$ are defined by induction by

$$\pi_n^n[x_1, x_2, \dots, x_m] = x_1 x_2 \cdots x_m$$
$$\pi_i^n[x_1, x_2, \dots, x_m] = [\pi_i^{n-1}x_1,\ \pi_i^{n-1}x_2, \dots, \pi_i^{n-1}x_m] \text{ if } i < n$$

for all $x_1, \dots, x_m \in T_n$. This implies $p(\pi_i^n t) = pt$ for all $t \in T_{n+1}$. (Commutative semigroups are tripleable over sets; in the corresponding cotriple, $GS = T_1$, $\epsilon = \pi_0^0$, and $\pi_i^n = G^{n-i}\epsilon G^i$.)

Let $\mathcal{A}$ be an abelian group valued functor on $\mathscr{H}(S)$. For each $n \geq 1$, a *long n-cochain* on S with coefficients in $\mathcal{A}$ is a family $c = (c_t)_{t \in T_{n-1}}$ such that $c_t \in \mathcal{A}_{pt}$ for all $t \in T_{n-1}$. Under pointwise addition, long n-cochains form an abelian group $C^n(S, \mathcal{A}) = \prod_{t \in T_{n-1}} \mathcal{A}_{pt}$. Coboundary homomorphisms $\delta_n : C^n(S, \mathcal{A}) \longrightarrow C^{n+1}(S, \mathcal{A})$ such that $\delta_n \delta_{n-1} = 0$ are defined by

$$(C) \qquad (\delta_n c)_t = \sum_{i=0}^{n-1} (-1)^i \, c_{\pi_i^{n-1} t} + (-1)^n \sum_{j=1}^{m} c_{x_j}^{pt_j^\wedge}$$

for all $c \in C^n(S, \mathcal{A})$ and $t = [x_1, \dots, x_m] \in T_n$, with

$$t_j^\wedge = [x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m]$$

(so that $px_j \, pt_j^\wedge = pt$). A *long n-cocycle* is an element of $Z^n(S, \mathcal{A}) = \operatorname{Ker} \delta_n \subseteq C^n(S, \mathcal{A})$. A *long n-coboundary* is an element of $B^n(S, \mathcal{A}) = \operatorname{Im} \delta_{n-1} \subseteq Z^n(S, \mathcal{A})$ (with $B^1(S, \mathcal{A}) = 0$).

It is shown in [4] that the triple cohomology group $H^n(S, \mathcal{A})$ (called $H^{n-1}(S, \mathcal{A})$ in [1], [2]) is naturally isomorphic to $Z^n(S, \mathcal{A}) / B^n(S, \mathcal{A})$.

3. In dimension 1, $H^1(S, \mathcal{A}) \cong Z^1(S, \mathcal{A})$. A long 1-cochain is a family $c = (c_x)_{x \in S} \in \prod_{x \in S} \mathcal{A}_x$, with coboundary

$$(C1) \qquad (\delta c)_t = c_{x_1 \cdots x_m} - \sum_{j=1}^{m} c_{x_j}^{pt_j^\wedge}$$

for all $t = [x_1, \dots, x_m] \in T_1$ (since $\pi_0^0 t = pt = a_1 \cdots a_m$). Hence c is a long 1-cocycle if and only if

$$(Z1) \qquad c_{x_1 \cdots x_m} = \sum_{j=1}^{m} c_{x_j}^{x_j^\wedge}$$

for all $x_1, \dots, x_m \in S$, $m \geq 1$, with $x_j^\wedge = x_1 \cdots x_{j-1} x_{j+1} \cdots x_m$. By induction on m, $(Z1)$ is equivalent to

$$(A1) \qquad c_{xy} = c_x^y + c_y^x$$

for all $x, y \in S$. Thus $Z^1(S, \mathcal{A})$ and $H^1(S, \mathcal{A})$ are the same as in [3].

Condition $(A1)$ implies that 1-cocycles are *normalized* ($c_e = 0$ whenever $e^2 = e$ in S).

Section 1. Triple cohomology in dimension 2.

1. We call the 2-cochains defined in [3] *symmetric* 2-cochains to distinguish them from long 2-cochains. In detail, a *short* 2-*cochain* is a family $c = (c_{x,y})_{x,y \in S}$ such that $c_{x,y} \in \mathcal{A}_{xy}$ for all $x, y \in S$. Under pointwise addition, short 2-cochains form an abelian group $\prod_{x,y \in S} \mathcal{A}_{xy}$. A *symmetric* 2-*cochain* is a short 2-cochain $c = (c_{x,y})_{x,y \in S} \in \prod_{x,y \in S} \mathcal{A}_{xy}$ such that

$$(S2) \qquad c_{y,x} = c_{x,y}$$

for all $x, y \in S$. For example, the coboundary of a 1-cochain u yields a short 2-cochain, also denoted by δu:

$$(\delta u)_{x,y} = u_{xy} - u_x^y - u_y^x,$$

which is symmetric.

A *symmetric* 2-*cocycle* or *factor set* is a symmetric 2-cochain s such that

$$(A2) \qquad s_{x,y}^z + s_{xy,z} = s_{x,yz} + s_{y,z}^x$$

for all $x, y, z \in S$. A *symmetric* 2-*coboundary* is a symmetric 2-cochain (necessarily a cocycle) s for which there exists a 1-cochain $u = (u_x)_{x \in S}$ (with $u_x \in \mathcal{A}_x$) such that $s = \delta u$, that is,

$$(B2) \qquad s_{x,y} = u_{xy} - u_x^y - u_y^x$$

for all $x, y \in S$. Under pointwise addition these form groups $SC^2(S, \mathcal{A}) \subseteq \prod_{x,y \in S} \mathcal{A}_{xy}$, $SZ^2(S, \mathcal{A})$, and $SB^2(S, \mathcal{A})$. In [3] these groups are denoted by $C^2(S, \mathcal{A})$, $Z^2(S, \mathcal{A})$, $B^2(S, \mathcal{A})$, and defined only when $\mathcal{A}$ is *thin* ($\alpha_{x,t} = \alpha_{x,u}$ whenever $xt = xu$).

It is shown in [3] that $SZ^2(S,\mathcal{A})/\,SB^2(S,\mathcal{A})$ classifies commutative group coextensions of S by $\mathcal{A}$; therefore $SZ^2(S,\mathcal{A})/\,SB^2(S,\mathcal{A}) \cong H^2(S,\mathcal{A})$. We now prove (Theorem 1.3) that in fact $Z^2(S,\mathcal{A}) \cong SZ^2(S,\mathcal{A})$, with $B^2(S,\mathcal{A}) \cong SB^2(S,\mathcal{A})$.

2. We denote the typical element of T_1 by $X = [x_1,\dots,x_\ell]$; ℓ is the length $\ell = |X|$ of the commutative word X. By definition,

$$X^\sigma = [x_{\sigma 1},\dots,x_{\sigma\ell}] = [x_1,\dots,x_\ell] = X$$

for every permutation $\sigma \in S_\ell$ of $1,2,\dots,\ell$. We also let

$$x = pX = x_1 \cdots x_\ell, \quad x_i' = x_1 \cdots x_i, \quad x_i'' = x_{i+1} \cdots x_\ell,$$

and $x_i^\wedge = x_1 \cdots x_{i-1}\, x_{i+1} \cdots x_\ell$; in these formulas, any empty product is read as $1 \in S^1$. When $c \in S^2(S,\mathcal{A})$ we write $c_X = c_{x_1,\dots,x_\ell}$ (without brackets). Since c depends only on X, we have $c_{x_{\sigma 1},\dots,x_{\sigma\ell}} = c_{x_1,\dots,x_\ell}$ for every $\sigma \in S_\ell$; we write this property as $c_{X^\sigma} = c_X$.

For every $\mathbf{X} = [X_1,\dots,X_m] \in T_2$ we have

$$\pi_1^1\mathbf{X} = X_1 \cdots X_m, \quad \pi_0^1\mathbf{X} = [pX_1,\dots,pX_m] = [x_1,\dots,x_m],$$

and

$$(C2) \qquad (\delta c)_{\mathbf{X}} = c_{x_1,\dots,x_m} - c_{X_1 \cdots X_m} + \sum_{j=1}^{m} c_{X_j}^{x_j^\wedge}$$

for every $c \in C^2(S,\mathcal{A})$ (with $x_j^\wedge = x_1 \cdots x_{j-1}\, x_{j+1} \cdots x_m$). Thus long 2-cocycles are families $s = (s_X)_{X \in T_1} \in \prod_{X \in T_1} \mathcal{A}_{pX}$ such that

$$(P2) \qquad s_{X^\sigma} = s_X$$

for all $X \in T_1$, $\sigma \in S_{|X|}$, and

$$(Z2) \qquad s_{X_1 \cdots X_m} = s_{x_1,\dots,x_m} + \sum_{j=1}^{m} c_{X_j}^{x_j^\wedge}$$

for all $m \geq 1$ and $X_1,\dots,X_m \in T_1$.

3. Lemma 1.1. *When s is a long 2-cocycle, $s_x = 0$ for all $x \in S$, and*

$$(Z2') \qquad s_X = \sum_{i=1}^{\ell-1} s_{x_i', x_{i+1}}^{x_{i+1}''}$$

for all $X \in T_1$ of length ℓ.

PROOF. Let $x \in S$. With $m = 1$ and $X_1 = [x]$, $(Z2)$ yields $s_x = 0$. Hence $(Z2')$ holds when $\ell = 1$. Let $\ell \geq 2$.

With $m = 2$, $X_1 = [x_1, \dots, x_{\ell-1}]$, and $X_2 = [x_\ell]$, (Z2) reads

$$(X2) \qquad s_X = s_{x'_{\ell-1}, x_\ell} + s^{x_\ell}_{x_1, \dots, x_{\ell-1}}$$

(since $s_{X_2} = 0$). Hence $(Z2')$ holds if $\ell = 2$ or $\ell = 3$. If $\ell > 3$ and $(Z2')$ holds for $\ell - 1$, then with $y = x_{i+1} \cdots x_{\ell-1}$ we have $yx_\ell = x''_{i+1}$ and $(X2)$ yields

$$\begin{aligned} s_X &= s^{x_\ell}_{x_1, \dots, x_{\ell-1}} + s_{x'_{\ell-1}, x_\ell} \\ &= \left(\sum_{i=1}^{\ell-2} s^{y}_{x'_i, x_{i+1}}\right)^{x_\ell} + s_{x'_{\ell-1}, x_\ell} \\ &= \sum_{i=1}^{\ell-2} s^{x''_{i+1}}_{x'_i, x_{i+1}} + s_{x'_{\ell-1}, x_\ell} \\ &= \sum_{i=1}^{\ell-1} s^{x''_{i+1}}_{x'_i, x_{i+1}} ; \end{aligned}$$

thus $(Z2')$ holds for ℓ. ∎

4. By 1.1, a long 2-cocycle is uniquely determined by its values on commutative words of length 2. More precisely, let $\Gamma : Z^2(S, \mathcal{A}) \longrightarrow SC^2(S, \mathcal{A})$ be the *trimming homomorphism* defined by $(\Gamma s)_{x,y} = s_{x,y} \in \mathcal{A}_{xy}$ for all $x, y \in S$ (note that $s_{x,y} = s_{y,x}$ by $(P2)$). Lemma 1.1 implies that Γ is injective.

LEMMA 1.2. $\operatorname{Im} \Gamma = SZ^2(S, \mathcal{A})$.

PROOF. Let $s \in Z^2$, $x, y, z \in S$. With $m = 2$, $X_1 = [x]$, and $X_2 = [y, z]$, $(Z2)$ reads: $s_{x,y,z} = s_{x,yz} + s^x_{y,z}$ (since $s_x = 0$). With $X_1 = [x, y]$ and $X_2 = [z]$, $(Z2)$ reads: $s_{x,y,z} = s_{xy,z} + s^z_{x,y}$ (since $s_z = 0$). Hence $s^z_{x,y} + s_{xy,z} = s_{x,yz} + s^x_{y,z}$ and $\Gamma s \in SZ^2$.

Conversely let $s \in SZ^2$. We use $(Z2')$ to define s_X for all $X \in T_1$. In detail, let

$$t_{x_1, \dots, x_\ell} = \sum_{i=1}^{\ell-1} s^{x''_{i+1}}_{x'_i, x_{i+1}}$$

for all $\ell \geq 1$ and $x_1, \dots, x_\ell \in S$. If $\ell = 1$, then the right hand side is empty, and $t_x = 0$ for all $x \in S$. If $\ell = 2$ we obtain $t_{x,y} = s_{x,y}$, so that $\Gamma t = s$. It remains to prove $(P2)$ and $(Z2)$, so that $t \in Z^2$.

First we note that

$$\begin{aligned} t_{x_1,\ldots,x_\ell,y} &= \sum_{i=1}^{\ell-1} s_{x'_i,x_{i+1}}^{x''_{i+1}y} + s_{x'_\ell,y} \\ &= \left(\sum_{i=1}^{\ell-1} s_{x'_i,x_{i+1}}^{x''_{i+1}}\right)^y + s_{x'_\ell,y} = t^y_{x_1,\ldots,x_\ell} + s_{x'_\ell,y} \end{aligned}$$

so that $(X2)$ holds for t.

We prove $(P2)$: $t_{X\sigma} = t_X$ for all $X = [x_1,\ldots,x_\ell]$ by induction on ℓ. For $\ell \leq 2$, $(P2)$ follows from $(S2)$. For $\ell > 2$ it suffices to show that $t_{X\tau} = t_X$ for every transposition $\tau = (i \;\; i{+}1)$ with $i < \ell$. For $i < \ell - 1$, $t_{X\tau} = t_X$ follows from the induction hypothesis, since

$$t_{x_1,\ldots,x_\ell} = t^{x_\ell}_{x_1,\ldots,x_{\ell-1}} + s_{x'_{\ell-1},x_\ell}\,.$$

by $(X2)$. For $i = \ell - 1$ we have, with $x'_{\ell-2} = b$, $x_{\ell-1} = c$, $x_\ell = d$:

$$\begin{aligned} t_X &= \sum_{i=1}^{\ell-3} s_{x'_i,x_{i+1}}^{x''_{i+1}} + s^d_{b,c} + s_{bc,d} \\ t_{X\tau} &= \sum_{i=1}^{\ell-3} s_{x'_i,x_{i+1}}^{x''_{i+1}} + s^c_{b,d} + s_{bd,c} \end{aligned}$$

and it follows from $(A2)$ and $(S2)$ that

$$s^d_{b,c} + s_{bc,d} = s^d_{c,b} + s_{cb,d} = s_{c,bd} + s^c_{b,d} = s^c_{b,d} + s_{bd,c}\,.$$

Therefore $(P2)$ holds.

$(Z2)$ holds when $m = 1$; for $m > 1$ we proceed by induction on m. Assume that $(Z2)$ holds for m and let $Y_1,\ldots,Y_m, Z \in T_1$, $pY_j = y_j$, $pZ = z$. Let $Y_1 \cdots Y_m = X = [x_1,\ldots,x_q] \in T_1$ and $Z = [z_1,\ldots,z_r]$. By the induction hypothesis,

$$t_X = t_{y_1,\cdots,y_m} + \sum_{k=1}^m t^{y_k^\wedge}_{Y_k}\,,$$

where $y_k^\wedge = y_1 \cdots y_{k-1}\, y_{k+1} \cdots y_m$; we want to prove that

$$t_{XZ} = t_{y_1,\cdots,y_m,z} + \sum_{k=1}^m t^{y_k^\wedge z}_{Y_k} + t^z_Z\,.$$

By definition, $t_{XZ} = t_{x_1,\ldots,x_q,z_1,\ldots,z_r}$ equals

$$\begin{aligned}
t_{XZ} &= \textstyle\sum_{i=1}^{q-1} s_{x'_i,x_{i+1}}^{x''_{i+1}b} + \sum_{j=0}^{r-1} s_{xz'_j,z_{j+1}}^{z''_{j+1}} \\
&= t_X^z + s_{x,z_1}^{z''_1} + \textstyle\sum_{j=1}^{r-1} s_{xz'_j,z_{j+1}}^{z''_{j+1}} \\
&= t_{y_1,\cdots,y_m}^z + \textstyle\sum_{k=1}^{m} t_{Y_k}^{y_k^\wedge z} + s_{x,z_1}^{z''_1} \\
&\quad + \textstyle\sum_{j=1}^{r-1} \left(- s_{x,z'_j}^{z_{j+1}} + s_{x,z'_j z_{j+1}} + s_{z'_j,z_{j+1}}^{x} \right)^{z''_{j+1}}
\end{aligned}$$

by the induction hypothesis and $(A2)$,

$$\begin{aligned}
&= t_{y_1,\cdots,y_m}^z + \textstyle\sum_{k=1}^{m} t_{Y_k}^{y_k^\wedge z} + s_{x,z'_1}^{z''_1} \\
&\quad - \textstyle\sum_{j=1}^{r-1} s_{x,z'_j}^{z''_j} + \sum_{j=2}^{r} s_{x,z'_j}^{z''_j} + \left(\sum_{j=1}^{r-1} s_{z'_j,z_{j+1}}^{z''_{j+1}} \right)^x \\
&= t_{y_1,\cdots,y_m}^z + \textstyle\sum_{k=1}^{m} t_{Y_k}^{y_k^\wedge z} + s_{x,z} + t_Z^x \\
&= t_{y_1,\cdots,y_m,z} + \textstyle\sum_{k=1}^{m} t_{Y_k}^{y_k^\wedge z} + t_Z^x
\end{aligned}$$

by $(X2)$, and $(Z2)$ is proved. ∎

THEOREM 1.3. *For every commutative semigroup S and abelian group valued functor $\mathcal{A}$ on $\mathscr{H}(S)$: $Z^2(S,\mathcal{A}) \cong SZ^2(S,\mathcal{A})$; $B^2(S,\mathcal{A}) \cong SB^2(S,\mathcal{A})$; and $H^2(S,\mathcal{A}) \cong SZ^2(S,\mathcal{A})/\,SB^2(S,\mathcal{A})$.*

PROOF. By 1.1, 1.2, Γ is an isomorphism $Z^2 \longrightarrow SZ^2$. When $c \in C^1$, $(C1)$ implies $(\delta c)_{x,y} = c_{xy} - c_x^y - c_y^x$; hence $\Gamma(B^2) = SB^2$. ∎

5. If $\mathcal{A}$ is *thin* (if $\alpha_{x,t} = \alpha_{x,u}$ whenever $xt = xu$ in S), normalization can be used to sharpen Theorem 1.3. A symmetric 2-cochain c is *normalized* when $c_{e,x} = 0$ whenever $e^2 = e$ and $ex = x$ in S. These cochains form a subgroup $NSC^2(S,\mathcal{A})$ of $SC^2(S,\mathcal{A})$. Normalized symmetric 2-cocycles and 2-coboundaries form abelian groups $NSZ^2(S,\mathcal{A}) = SZ^2(S,\mathcal{A}) \cap NSC^2(S,\mathcal{A})$ and $NSB^2(S,\mathcal{A}) = SB^2(S,\mathcal{A}) \cap NSC^2(S,\mathcal{A})$. If $\mathcal{A}$ is thin, it is readily verified that a symmetric 2-coboundary is normalized if and only if it is the coboundary of a normalized 1-cochain.

PROPOSITION 1.4. *If $\mathcal{A}$ is thin, $H^2(S,\mathcal{A}) \cong NSZ^2(S,\mathcal{A})/NSB^2(S,\mathcal{A})$.*

PROOF. We show that $SZ^2 = NSZ^2 + SB^2$; then $H^2 \cong NSZ^2/NSB^2$ follows from $H^2 \cong SZ^2/SB^2$ and $SB^2 \cap NSZ^2 = NSB^2$.

Let $s \in SZ^2$. Take any $u \in C^1(S, \mathcal{A})$ such that $u_e = s_{e,e}$ whenever $e^2 = e$ in S. Since $\mathcal{A}$ is thin, $\alpha_{e,e} = \alpha_{e,1}$ is the identity on $\mathcal{A}_e$ and $(\delta u)_{e,e} = -u_e$. Hence $t = s + \delta u \in SZ^2$ satisfies $t_{e,e} = 0$ whenever $e^2 = e$. It follows from $(A2)$ that t is normalized: if $e^2 = e$ and $ex = x$, then $\alpha_{x,e} = \alpha_{x,1}$ is the identity on $\mathcal{A}_x$ and

$$\alpha_{e,x}\, t_{e,e} \;+\; t_{ee,x} \;=\; t_{e,ex} \;+\; \alpha_{ex,e}\, t_{e,x}$$

yields $t_{e,x} = 0$. Thus $s = t - \delta u \in NSZ^2 + SB^2$. ∎

Section 2. Cocycles in dimension 3.

1. A *short 3-cochain* on S with coefficients in $\mathcal{A}$ is a family $c = (c_{x,y,z})_{x,y,z \in S}$ such that $c_{x,y,z} \in \mathcal{A}_{xyz}$ for all $x, y, z \in S$. Under pointwise addition, short 3-cochains form an abelian group $\prod_{x,y,z \in S} \mathcal{A}_{xyz}$. A *symmetric 3-cochain* on S with coefficients in $\mathcal{A}$ is a short 3-cochain $c = (c_{x,y,z})_{x,y,z \in S}$ such that

$$c_{z,y,x} \;=\; -\,c_{x,y,z}\,, \quad \text{and} \quad c_{x,y,z} + c_{y,z,x} + c_{z,x,y} \;=\; 0$$

for all $x, y, z \in S$. For example, the coboundary of a *symmetric* 2-cochain u, defined by

$$(\delta u)_{x,y,z} \;=\; u^x_{y,z} \;-\; u_{xy,z} \;+\; u_{x,yz} \;-\; u^z_{x,y}\,,$$

is a symmetric 3-cochain.

A *symmetric 3-cocycle* is a symmetric 3-cochain t such that

$$t^x_{y,z,w} \;-\; t_{xy,z,w} \;+\; t_{x,yz,w} \;-\; t_{x,y,zw} \;+\; t^w_{x,y,z} \;=\; 0$$

for all $x, y, z, w \in S$. A *symmetric 3-coboundary* is a symmetric 3-cochain t (necessarily a 3-cocycle) for which there exists a symmetric 2-cochain u such that $t = \delta u$. Under pointwise addition, symmetric 3-cochains, 3-cocycles, and 3-coboundaries form abelian groups $SC^3(S, \mathcal{A}) \subseteq \prod_{x,y,z \in S} \mathcal{A}_{xyz}$, $SZ^3(S, \mathcal{A})$, and $SB^3(S, \mathcal{A})$. The main result in this section (Theorem 2.11) is that $H^3(S, \mathcal{A}) \cong SZ^3(S, \mathcal{A}) / SB^3(S, \mathcal{A})$.

2. The first step in the proof is to state the definition of long 3-cocycles in usable form. We denote the typical element of T_2 by $\mathbf{X} = [X_1, X_2, \ldots, X_m]$; by definition,

$$\mathbf{X}^\sigma = [X_{\sigma 1}, \ldots, X_{\sigma m}] = \{X_1, \ldots, X_m] = \mathbf{X}$$

for every permutation $\sigma \in S_m$ of $1, 2, \ldots, m$. We denote $p\mathbf{X}$ by x, $\pi_1^1\mathbf{X}$ by $\mathbb{X}$, and $\pi_0^1\mathbf{X}$ by X. Then $x = p\mathbf{X} = p\mathbb{X} = pX$. If $\mathbf{X} = [X_1, X_2, \ldots, X_m]$ and $x_j = pX_j$, then

$$\mathbb{X} = \pi_1^1\mathbf{X} = X_1 X_2 \cdots X_m,$$
$$X = \pi_0^1\mathbf{X} = [x_1, x_2, \cdots, x_m].$$

When $c \in C^3(S, \mathcal{A})$, we write $c_{\mathbf{X}} = c_{X_1; X_2; \ldots; X_m}$ (with semicolons), separating the components of each X_j with commas if necessary:

$$c_{\mathbf{X}} = c_{x_{11}, \ldots, x_{1m_1}; x_{21}, \ldots, x_{2m_2}; \ldots; x_{n1}, \ldots, x_{nm_n}}.$$

By definition, $c_{\mathbf{X}^\sigma} = c_{X_{\sigma 1}; \ldots; X_{\sigma m}} = c_{X_1; \ldots; X_m} = c_{\mathbf{X}}$ for every permutation $\sigma \in S_m$, and $c_{X_1^{\sigma_1}; \ldots; X_m^{\sigma_m}} = c_{X_1; \ldots; X_m}$ for all suitable permutations $\sigma_1, \ldots, \sigma_m$.

For all $[\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n] \in T_3$ we have

$$\pi_2^2[\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n] = \mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_n,$$
$$\pi_1^2[\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n] = [\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_n],$$
$$\pi_0^2[\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n] = [X_1, X_2, \ldots, X_n],$$

(with $X_i = \pi_0^1\mathbf{X}_i$); hence

$$(C3) \qquad (\delta c)_{[\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n]} = c_{X_1; X_2; \ldots; X_n} - c_{\mathbb{X}_1; \mathbb{X}_2; \ldots; \mathbb{X}_n} + c_{\mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_n} - \sum_{k=1}^n c_{\mathbf{X}_k}^{x_k^\wedge},$$

where $x_k^\wedge = x_1 \cdots x_{k-1} x_{k+1} \cdots x_n$, $x_k = pX_k = p\mathbb{X}_k = p\mathbf{X}_k$. Thus a long 3-cocycle is a family $s = (s_{\mathbf{X}})_{\mathbf{X} \in T_2}$ such that $s_{\mathbf{X}} \in \mathcal{A}_x$ and the following conditions hold:

$$(P3') \qquad s_{X_{\sigma 1}; \ldots; X_{\sigma m}} = s_{X_1; \ldots; X_m}$$

for all $m \geq 1$, $\mathbf{X} \in T_2$ of length m, and $\sigma \in S_m$;

$$(P3') \qquad s_{X_1^{\sigma_1};\dots;X_m^{\sigma_m}} = s_{X_1;\dots;X_m}$$

for all $m \geq 1$, $\mathbf{X} \in T_2$ of length m, and suitable permutations $\sigma_1, \dots, \sigma_m$; and

$$(Z3) \qquad s_{\mathbf{X}_1\mathbf{X}_2\cdots\mathbf{X}_n} = s_{\mathbb{X}_1;\mathbb{X}_2;\dots;\mathbb{X}_n} - s_{X_1;X_2;\dots;X_n} + \sum_{k=1}^{n} s_{\mathbf{X}_k}^{x_k^\wedge}$$

for all $\mathbf{X}_1, \dots, \mathbf{X}_n \in T_2$, where, as before, $X_i = \pi_0^1 \mathbf{X}_i$, $x_i = p\mathbf{X}_i$, and $x_k^\wedge = x_1 \cdots x_{k-1} x_{k+1} \cdots x_n$.

3. Condition $(Z3)$ implies that long 3-cocycles can be trimmed (as we trimmed long 2-cocycles in Section 1). This will be done in three stages.

When $\mathbf{X} = [X_1, \dots, X_m] \in T_1$, we let $x_i = pX_i$ and

$$x_j' = x_1 x_2 \cdots x_j, \qquad x_j'' = x_{j+1} \cdots x_m, \qquad X_j' = X_1 X_2 \cdots X_j.$$

LEMMA 2.1. *Every long 3-cocycle s satisfies*

$$(Z') \qquad s_X = s_x$$

for all $X \in T_1$, and

(Z'')

$$s_{X_1;\dots;X_m} = \sum_{j=1}^{m} s_{x_j}^{x_j^\wedge} + \sum_{j=1}^{m-1} s_{X_j';X_{j+1}}^{x_{j+1}''} - \sum_{j=1}^{m-1} s_{x_1,\dots,x_j;x_{j+1}}^{x_{j+1}''}$$

for all $X_1, \dots, X_m \in T_1$.

PROOF. Let $X \in T_1$. With $n = 1$ and $\mathbf{Y}_1 = [X]$, we have $\mathbb{Y}_1 = X$, $Y_1 = [x]$, $y_1 = x$, and $(Z3)$ reduces to (Z').

Now let $X_1, \dots, X_m \in T_1$. If $m = 1$, then (Z'') follows from (Z'). Let $m \geq 2$. With $\mathbf{Y}_1 = [X_1, \dots, X_{m-1}]$ and $\mathbf{Y}_2 = [X_m]$, we have $\mathbf{Y}_1\mathbf{Y}_2 = [X_1, \dots, X_m]$, $\mathbb{Y}_1 = X_1 \cdots X_{m-1} = X_{m-1}'$, $\mathbb{Y}_2 = X_m$, $Y_1 = [x_1, \cdots, x_{m-1}]$, $Y_2 = [x_m]$, and $(Z3)$ yields

$$(X3) \qquad \begin{aligned} s_{X_1;\dots;X_m} &= s_{X_1\cdots X_{m-1};X_m} - s_{x_1,\dots,x_{m-1};x_m} \\ &\quad + s_{X_1;\dots;X_{m-1}}^{x_m} + s_{X_m}^{x_{m-1}'}. \end{aligned}$$

This proves (Z'') if $m = 2$. For $m > 2$ we proceed by induction on m. If $m \geq 2$ and (Z'') holds for m, then:

$$\begin{aligned}
&s_{X_1;\dots;X_m;X_{m+1}} \\
&= s^{x_{m+1}}_{X_1;\dots;X_m} + s_{X_1\cdots X_m;X_{m+1}} - s_{x_1,\dots,x_{m-1};x_m} + s^{x'_{m-1}}_{X_m} \\
&\qquad \text{by } (X3) \\
&= \left(\sum\nolimits_{j=1}^{m} s^{x_j^\wedge}_{x_j} + \sum\nolimits_{j=1}^{m-1} s^{x''_{j+1}}_{X'_j;X_{j+1}} - \sum\nolimits_{j=1}^{m-1} s^{x''_{j+1}}_{x_1,\dots,x_j;x_{j+1}}\right)^{x_{m+1}} \\
&\quad + s_{X_1\cdots X_m;X_{m+1}} - s_{x_1,\dots,x_{m-1};x_m} + s^{x'_{m-1}}_{x_m} \\
&\qquad \text{by the induction hypothesis and } (Z') \\
&= \sum\nolimits_{j=1}^{m} s^{x_j^\wedge x_{m+1}}_{x_j} + s^{x'_{m-1}}_{x_m} + \sum\nolimits_{j=1}^{m-1} s^{x''_{j+1}}_{X'_j;X_{j+1}} + s_{X'_m;X_{m+1}} \\
&\quad - \sum\nolimits_{j=1}^{m-1} s^{x''_{j+1}x_{m+1}}_{x_1,\dots,x_j;x_{j+1}} - s_{x_1,\dots,x_{m-1};x_m}
\end{aligned}$$

and thus (Z'') holds for $m+1$. ∎

4. Lemma 2.1 shows that a long 3-cocycle is determined by its values on commutative words of length at most 2. In detail, let

$$C_1(S,\mathcal{A}) = \left(\prod\nolimits_{x\in S}\mathcal{A}_x\right) \times \left(\prod\nolimits_{X,Y\in T_1}\mathcal{A}_{xy}\right)$$

be the abelian group of all families

$$c = ((c_x)_{x\in S}), (c_{X;Y})_{X,Y\in T_1}))$$

such that $c_x \in \mathcal{A}_x$ for all $x \in S$ and $c_{X,Y} \in \mathcal{A}_{xy}$ for all $X, Y \in T_1$. The trimming homomorphism $\Gamma_1 : Z^3(S,\mathcal{A}) \longrightarrow C_1(S,\mathcal{A})$ is defined by $(\Gamma_1 s)_x = s_x$, $(\Gamma_1 s)_{X;Y} = s_{X;Y}$ for all $x \in S$, $X, Y \in T_1$. Lemma 2.1 implies that Γ_1 is injective.

LEMMA 2.2. *Let $s \in C_1(S,\mathcal{A})$. Then $s \in \operatorname{Im}\Gamma_1$ if and only if:*

(P_1') $\qquad s_{B;A} = s_{A;B}$ *for all* $A, B \in T_1$;

(P_1'') $\qquad s_{A^\sigma;B^\tau} = s_{A;B}$ *for all* $A, B \in T_1$ *and suitable* σ, τ;

(Z_1') $\qquad s_{a;b} = s_a^b + s_b^a$ *for all* $a, b \in S$; *and*

(Z_1'') $\qquad s_{A;BC} + s^a_{B;C} + s^{bc}_a - s_{a;b,c} = s_{AB;C} + s^c_{A;B} + s^{ab}_c - s_{a,b;c}$

for all $A, B, C \in T_1$.

PROOF. Let $s \in SZ^3$. Properties (P_1') and (P_1'') follow from $(P3')$ and $(P3'')$. Let $a, b \in S$. With $n = 2$, $\mathbf{X}_1 = [\,[a]\,]$, and $\mathbf{X}_2 = [\,[b]\,]$, we have $\mathbb{X}_1 = X_1 = [a]$, $\mathbb{X}_2 = X_2 = [b]$, and $(Z3)$ reduces to (Z_1'). Next let $A, B, C \in T_1$. With $n = 2$, $\mathbf{X}_1 = [A, B]$, and $\mathbf{X}_2 = [C]$, we have $\mathbb{X}_1 = AB$, $X_1 = [a, b]$, $\mathbb{X}_2 = C$, $X_2 = [c]$, and $(Z3)$ reads

$$s_{A;B;C} \;=\; s_{AB;C} \;-\; s_{a,b;\,c} \;+\; s^{c}_{A;B} \;+\; s^{ab}_{c}$$

(using (Z')). Similarly, with $n = 2$, $\mathbf{X}_1 = [A]$, and $\mathbf{X}_2 = [B, C]$, $(Z3)$ reads

$$s_{A;B;C} \;=\; s_{A;BC} \;-\; s_{a;\,b,c} \;+\; s^{bc}_{a} \;+\; s^{a}_{B;C}.$$

This proves (Z_1'').

For the converse, let $c \in C_1$ have properties (P_1'), (P_1''), (Z_1'), and (Z_1''). Define $s_{X_1;\dots;X_m} \in \mathcal{A}_x$ for all $X_1, \dots, X_m \in T_1$ by

$$s_{X_1;\dots;X_m} \;=\; \sum\nolimits_{j=1}^{m} c^{x_j^\wedge}_{x_j} \;+\; \sum\nolimits_{j=1}^{m-1} c^{x''_{j+1}}_{X'_j;X_{j+1}} \;-\; \sum\nolimits_{j=1}^{m-1} c^{x''_{j+1}}_{x_1,\dots,x_j;\,x_{j+1}}\,.$$

In particular, $s_{X_1} = c_{x_1} = c_x$, so that (Z') holds for s and $s_x = c_x$ for all $x \in S$. Also

$$s_{A;B} \;=\; c^b_a + c^a_b + c_{A;B} - c_{a,b} \;=\; c_{A;B}$$

by (Z_1'); therefore $\Gamma_1 s = c$ and (Z'') holds for s. Property $(P3'')$ follows from (P_1''). It remains to show that $(P3')$ and $(Z3)$ hold for s.

First we show that s has property $(X3)$ in the proof of Lemma 2.1:

$$(X3)\qquad \begin{aligned} s_{X_1;\dots;X_m} \;&=\; s_{X_1\cdots X_{m-1};X_m} \;-\; s_{x_1,\dots,x_{m-1};\,x_m} \\ &\quad+\; s^{x_m}_{X_1;\dots;X_{m-1}} \;+\; s^{x'_{m-1}}_{X_m}\,. \end{aligned}$$

This property is trivial if $m = 1$ and follows from (Z') and (Z_1') if $m = 2$. For $m > 2$, let $y''_j = x_{j+1} \cdots x_{m-1}$ (with $y''_{m-1} = 1 \in S^1$) and $y^\wedge_j = x_1 \cdots x_{j-1}\, x_{j+1} \cdots x_{m-1}$. Then $x''_j = y''_j x_m$ and $x^\wedge_j = y^\wedge_j\, x_m$ for all $j \le m - 1$, and (Z'') yields

$$\begin{aligned}
s_{X_1;\ldots;X_m} &= \sum_{j=1}^{m} s_{x_j}^{x_j^{\wedge}} + \sum_{j=1}^{m-1} s_{X_j';X_{j+1}}^{x_{j+1}''} - \sum_{j=1}^{m-1} s_{x_1,\ldots,x_j;x_{j+1}}^{x_{j+1}''} \\
&= \left(\sum_{j=1}^{m-1} s_{x_j}^{y_j^{\wedge}}\right)^{x_m} + s_{X_m}^{x_{m-1}'} \\
&\quad + \left(\sum_{j=1}^{m-2} s_{X_j';X_{j+1}}^{y_{j+1}''}\right)^{x_m} + s_{X_{m-1}';X_m} \\
&\quad - \left(\sum_{j=1}^{m-2} s_{x_1,\ldots,x_j;x_{j+1}}^{y_{j+1}''}\right)^{x_m} - s_{x_1,\ldots,x_{m-1};x_m} \\
&= s_{X_1;\ldots;X_{m-1}}^{x_m} + s_{X_m}^{x_{m-1}'} + s_{X_{m-1}';X_m} - s_{x_1,\ldots,x_{m-1};x_m}.
\end{aligned}$$

Thus $(X3)$ holds for s.

We use induction on m to prove $(P3')$: $s_{\mathbf{X}\sigma} = s_{\mathbf{X}}$, for all $m \geq 1$, $\mathbf{X} = [X_1, \ldots, X_m]$, and $\sigma \in S_m$. By (P_1'), s has this property for $m \leq 2$. If $m > 2$ it suffices to prove that $s_{\mathbf{X}\sigma} = s_{\mathbf{X}}$ when $\sigma = (i \;\; i{+}1)$, $i < m$. If $i < m - 1$, then $\sigma m = m$ and $s_{\mathbf{X}\sigma} = s_{\mathbf{X}}$ follows from $(X3)$ and the induction hypothesis. Let $i = m - 1$. Let

$$\mathbf{B} = [X_1, \ldots, X_{m-2}], \quad A = X_{m-1}, \quad C = X_m,$$

so that $\mathbb{B} = X_1 \cdots X_{m-2}$. By (Z'') we have

$$\begin{aligned}
s_{\mathbf{X}} &= s_{B_1;\ldots;B_{m-2};A;C} \\
&= \sum_{j=1}^{m-2} s_{b_j}^{b_j^{\wedge}ac} + s_a^{bc} + s_c^{ba} \\
&\quad + \sum_{j=1}^{m-3} s_{B_j';B_{j+1}}^{b_{j+1}''ac} + s_{\mathbb{B};A}^{c} + s_{\mathbb{B}A;C} \\
&\quad - \sum_{j=1}^{m-3} s_{b_1,\ldots,b_j;b_{j+1}}^{b_{j+1}''ac} - s_{b_1,\ldots,b_{m-2};a}^{c} - s_{b_1,\ldots,b_{m-2},a;c},
\end{aligned}$$

$$\begin{aligned}
s_{\mathbf{X}\sigma} &= s_{B_1;\ldots;B_{m-2};C;A} \\
&= \sum_{j=1}^{m-2} s_{b_j}^{b_j^{\wedge}ca} + s_c^{ba} + s_a^{bc} \\
&\quad + \sum_{j=1}^{m-3} s_{B_j';B_{j+1}}^{b_{j+1}''ca} + s_{\mathbb{B};C}^{a} + s_{\mathbb{B}C;A} \\
&\quad - \sum_{j=1}^{m-3} s_{b_1,\ldots,b_j;b_{j+1}}^{b_{j+1}''ca} - s_{b_1,\ldots,b_{m-2};c}^{a} - s_{b_1,\ldots,b_{m-2},c;a}.
\end{aligned}$$

Hence we need to show that

$$s^c_{\mathbb{B};A} + s_{\mathbb{B}A;C} - s^c_{b_1,\ldots,b_{m-2};a} - s_{b_1,\ldots,b_{m-2},a;c}$$
$$= s^a_{\mathbb{B};C} + s_{\mathbb{B}C;A} - s^a_{b_1,\ldots,b_{m-2};c} - s_{b_1,\ldots,b_{m-2},c;a}\,;$$

this follows from:

$$\begin{aligned} & s_{b_1,\ldots,b_{m-2},a;c} + s^c_{b_1,\ldots,b_{m-2};a} \\ &= s_{a,b_1,\ldots,b_{m-2};c} + s^c_{a;b_1,\ldots,b_{m-2}} \quad \text{by } (P_1'),(P_1'') \\ &= s_{a;b_1,\ldots,b_{m-2},c} + s^a_{b_1,\ldots,b_{m-2};c} + s^{bc}_a - s_{a;b,c} - s^{ab}_c + s_{a,b;c} \quad \text{by } (Z_1'') \\ &= s_{b_1,\ldots,b_{m-2},c;a} + s^a_{b_1,\ldots,b_{m-2};c} + s_{A\mathbb{B};C} + s^c_{A;\mathbb{B}} - s_{A;\mathbb{B}C} - s^a_{\mathbb{B};C} \text{ by } (Z_1'') \\ &= s_{b_1,\ldots,b_{m-2},c;a} + s^a_{b_1,\ldots,b_{m-2};c} + s_{\mathbb{B}A;C} + s^c_{\mathbb{B};A} - s_{\mathbb{B}C;A} - s^a_{\mathbb{B};C}\,. \end{aligned}$$

This proves $(P3')$.

We now turn to $(Z3)$. First we prove

$$(Z_1^*)\quad \begin{aligned} s_{X;Y_1\cdots Y_\ell} &= \sum\nolimits_{i=0}^{\ell-1} s^{y''_{i+1}}_{XY'_i;Y_{i+1}} + \sum\nolimits_{i=2}^{\ell} s^{xy^\wedge_i}_{y_i} - \sum\nolimits_{i=1}^{\ell-1} s^{y''_{i+1}}_{x,y'_i;y_{i+1}} \\ &\quad - (\ell-1)s^y_x - \sum\nolimits_{i=1}^{\ell-1} s^{xy''_{i+1}}_{Y'_i;Y_{i+1}} + \sum\nolimits_{i=1}^{\ell-1} s^{y''_{i+1}}_{x;y'_i,y_{i+1}} \end{aligned}$$

for all $X, Y_1, \ldots, Y_\ell \in T_1$. This is trivial if $\ell = 1$ and reduces to (Z_1'') if $\ell = 2$. For $\ell > 2$ we proceed by induction on ℓ. Let $\mathbf{B} = [Y_1, \ldots, Y_{\ell-1}]$, so that $b''_i = y_{i+1} \cdots y_{\ell-1}$ and $b^\wedge_i = y_1 \cdots y_{i-1} y_{i+1} \cdots y_{\ell-1}$, $y''_i = b''_i y_\ell$, and $y^\wedge_i = b^\wedge_i y_\ell$, for all $i < \ell$. With $A = X$, $B = Y_1 \cdots Y_{\ell-1}$, and $C = Y_\ell$, (Z_1'') yields

$$\begin{aligned} s_{X;Y_1\cdots Y_\ell} &= s^{y_\ell}_{X;B} + s_{XB;Y_\ell} + s^{xb}_{y_\ell} - s_{x,b;y_\ell} - s^x_{B;Y_\ell} - s^{by_\ell}_x + s_{x;b,y_\ell} \\ &= \Big(\sum\nolimits_{i=0}^{\ell-2} s^{b''_{i+1}}_{XY'_i;Y_{i+1}} + \sum\nolimits_{i=2}^{\ell-1} s^{xb^\wedge_i}_{y_i} - \sum\nolimits_{i=1}^{\ell-2} s^{b''_{i+1}}_{x,y'_i;y_{i+1}} \\ &\quad - (\ell-2)s^b_x - \sum\nolimits_{i=1}^{\ell-2} s^{xb''_{i+1}}_{Y'_i;Y_{i+1}} + \sum\nolimits_{i=1}^{\ell-2} s^{b''_{i+1}}_{x;y'_i,y_{i+1}}\Big)^{y_\ell} \\ &\quad + s_{XB;Y_\ell} + s^{xb}_{y_\ell} - s_{x,b;y_\ell} - s^x_{B;Y_\ell} - s^{by_\ell}_x + s_{x;b,y_\ell} \\ &\quad \text{by the induction hypothesis} \end{aligned}$$

$$= \sum_{i=0}^{\ell-2} s_{XY_i';Y_{i+1}}^{y_{i+1}''} + \sum_{i=2}^{\ell-1} s_{y_i}^{xy_i^\wedge} - \sum_{i=1}^{\ell-2} s_{x,y_i';y_{i+1}}^{y_{i+1}''}$$
$$- (\ell-2)s_x^y - \sum_{i=1}^{\ell-2} s_{Y_i';Y_{i+1}}^{xy_{i+1}''} + \sum_{i=1}^{\ell-2} s_{x;y_i',y_{i+1}}^{y_{i+1}''}$$
$$+ s_{XY_{\ell-1}';Y_\ell} + s_{y_\ell}^{xy_{\ell-1}'} - s_{x,y_{\ell-1}';y_\ell} - s_{Y_{\ell-1}';Y_\ell}^x - s_x^{y_\ell'} + s_{x;y_{\ell-1}',y_\ell}$$
$$= \sum_{i=0}^{\ell-1} s_{XY_i';Y_{i+1}}^{y_{i+1}''} + \sum_{i=2}^{\ell} s_{y_i}^{xy_i^\wedge} - \sum_{i=1}^{\ell-1} s_{x,y_i';y_{i+1}}^{y_{i+1}''}$$
$$- (\ell-1)s_x^y - \sum_{i=1}^{\ell-1} s_{Y_i';Y_{i+1}}^{xy_{i+1}''} + \sum_{i=1}^{\ell-1} s_{x;y_i',y_{i+1}}^{y_{i+1}''},$$

and (Z_1^*) holds for ℓ.

We now prove $(Z3)$. With $n=1$ and $\mathbf{X}_1 = \mathbf{X}$, $(Z3)$ reads $s_{\mathbb{X}} = s_X$; this follows from (Z') since $p\mathbb{X} = pX$.

For $n > 1$ we proceed by induction on n. Let $\mathbf{X}_k = [X_{k1}, \dots, X_{km_k}]$, so that $\mathbb{X}_k = X_{k1} \cdots X_{km_k}$ and $X_k = [x_{k1}, \dots, x_{km_k}]$ (with $x_{kj} = pX_{kj}$). The left hand side of $(Z3)$ is

$$LHS(n) = s_{\mathbf{X}_1 \cdots \mathbf{X}_n} = s_{X_{11};\dots;X_{1m_1};\dots;X_{n1};\dots;X_{nm_n}};$$

the right hand side is

$$RHS(n) = s_{\mathbb{X}_1;\dots;\mathbb{X}_n} - s_{X_1;\dots;X_n} + \sum_{k=1}^{n} s_{\mathbf{X}_k}^{x_k^\wedge}$$
$$= s_{X_{11}\cdots X_{1m_1};\dots;X_{n1}\cdots X_{nm_n}}$$
$$- s_{x_{11}\cdots x_{1m_1};\dots;x_{n1}\cdots x_{nm_n}}$$
$$+ \sum_{k=1}^{n} s_{X_{k1};\dots;X_{km_k}}^{x_k^\wedge}.$$

We use (Z''), then separate the terms which contain n:

$$LHS(n) = \sum_{k=1}^{n} \sum_{j=1}^{m_k} s_{x_{kj}}^{x_k^\wedge (x_k)_j^\wedge}$$
$$+ \sum_{j=1}^{m_1-1} s_{(X_1)_j';X_{1,j+1}}^{(x_1)_{j+1}'' x_1''} + \sum_{k=2}^{n} \sum_{j=0}^{m_k-1} s_{\mathbb{X}_{k-1}'(X_k)_j';X_{k,j+1}}^{(x_k)_{j+1}'' x_k''}$$
$$- \sum_{j=1}^{m_1-1} s_{x_{11},\dots,x_{1j};x_{1,j+1}}^{(x_1)_{j+1}'' x_1''} - \sum_{k=2}^{n} \sum_{j=0}^{m_k-1} s_{x_{11},\dots,x_{kj};x_{k,j+1}}^{(x_k)_{j+1}'' x_k''}$$

$$= \sum_{k=1}^{n-1} \sum_{j=1}^{m_k} s_{x_{kj}}^{x_k^\wedge (x_k)_j^\wedge}$$

$$+ \sum_{j=1}^{m_1-1} s_{(X_1)_j';X_{1,j+1}}^{(x_1)_{j+1}'' x_1''} + \sum_{k=2}^{n-1} \sum_{j=0}^{m_k-1} s_{\mathbb{X}_{k-1}'(X_k)_j';X_{k,j+1}}^{(x_k)_{j+1}'' x_k''}$$

$$- \sum_{j=1}^{m_1-1} s_{x_{11},\dots,x_{1j};x_{1,j+1}}^{(x_1)_{j+1}'' x_1''} - \sum_{k=2}^{n-1} \sum_{j=0}^{m_k-1} s_{x_{11},\dots,x_{kj};x_{k,j+1}}^{(x_k)_{j+1}'' x_k''}$$

$$+ \sum_{j=1}^{m_n} s_{x_{nj}}^{x_{n-1}'(x_n)_j^\wedge}$$

$$+ \sum_{j=0}^{m_n-1} s_{\mathbb{X}_{n-1}'(X_n)_j';X_{n,j+1}}^{(x_n)_{j+1}''} - \sum_{j=0}^{m_n-1} s_{x_{11},\dots,x_{nj};x_{n,j+1}}^{(x_n)_{j+1}''}$$

$$= LHS(n-1)^{x_n} + \sum_{j=1}^{m_n} s_{x_{nj}}^{x_{n-1}'(x_n)_j^\wedge} \qquad (\mathbf{1})$$

$$+ \sum_{j=0}^{m_n-1} s_{\mathbb{X}_{n-1}'(X_n)_j';X_{n,j+1}}^{(x_n)_{j+1}''} - \sum_{j=0}^{m_n-1} s_{x_{11},\dots,x_{nj};x_{n,j+1}}^{(x_n)_{j+1}''} ,(\mathbf{2}),(\mathbf{3})$$

$$RHS(n) = \sum_{k=1}^{n} s_{\mathbb{X}_k}^{x_k^\wedge} + \sum_{k=1}^{n-1} s_{\mathbb{X}_k';\mathbb{X}_{k+1}}^{x_{k+1}''} - \sum_{k=1}^{n-1} s_{x_1,\dots,x_k;x_{k+1}}^{x_{k+1}''}$$

$$- \sum_{k=1}^{n} s_{X_k}^{x_k^\wedge} - \sum_{k=1}^{n-1} s_{X_k';X_{k+1}}^{x_{k+1}''} + \sum_{k=1}^{n-1} s_{x_1,\dots,x_k;x_{k+1}}^{x_{k+1}''}$$

$$+ \sum_{k=1}^{n}\Big(\sum_{j=1}^{m_k} s_{x_{kj}}^{(x_k)_j^\wedge}\Big)^{x_k^\wedge} + \sum_{k=1}^{n}\Big(\sum_{j=1}^{m_k-1} s_{(X_k)_j';X_{k,j+1}}^{(x_k)_{j+1}''}\Big)^{x_k^\wedge}$$

$$- \sum_{k=1}^{n}\Big(\sum_{j=1}^{m_k-1} s_{x_{k1},\dots,x_{kj};x_{k,j+1}}^{(x_k)_{j+1}''}\Big)^{x_k^\wedge}$$

$$= \sum_{k=1}^{n-1} s_{\mathbb{X}_k}^{x_k^\wedge} + \sum_{k=1}^{n-2} s_{\mathbb{X}_k';\mathbb{X}_{k+1}}^{x_{k+1}''}$$

$$- \sum_{k=1}^{n-1} s_{X_k}^{x_k^\wedge} - \sum_{k=1}^{n-2} s_{X_k';X_{k+1}}^{x_{k+1}''}$$

$$+ \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{m_k} s_{x_{kj}}^{(x_k)_j^\wedge}\Big)^{x_k^\wedge} + \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{m_k-1} s_{(X_k)_j';X_{k,j+1}}^{(x_k)_{j+1}''}\Big)^{x_k^\wedge}$$

$$- \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{m_k-1} s_{x_{k1},\dots,x_{kj};x_{k,j+1}}^{(x_k)_{j+1}''}\Big)^{x_k^\wedge}$$

$$+ s_{\mathbb{X}_n}^{x_{n-1}'} + s_{\mathbb{X}_{n-1}';\mathbb{X}_n} - s_{X_n}^{x_{n-1}'} - s_{X_{n-1}';X_n}$$

$$+ \sum_{j=1}^{m_n} s_{x_{nj}}^{(x_n)_j^\wedge x_{n-1}'} + \sum_{j=1}^{m_n-1} s_{(X_n)_j';X_{n,j+1}}^{(x_n)_{j+1}'' x_{n-1}'}$$

$$- \sum_{j=1}^{m_n-1} s_{x_{n1},\dots,x_{nj};x_{n,j+1}}^{(x_n)_{j+1}'' x_{n-1}'}$$

$$\begin{aligned} &= RHS(n-1)^{x_n} \\ &\quad + s_{\mathbb{X}_n}^{x'_{n-1}} + s_{\mathbb{X}'_{n-1};\mathbb{X}_n} - s_{X_n}^{x'_{n-1}} - s_{X'_{n-1};X_n} \qquad \mathbf{(A)},\mathbf{(B)},\mathbf{(C)},\mathbf{(D)} \\ &\quad + \sum_{j=1}^{m_n} s_{x_{nj}}^{(x_n)_j^\wedge\, x'_{n-1}} + \sum_{j=1}^{m_n-1} s_{(X_n)'_j;X_{n,j+1}}^{(x_n)''_{j+1}\, x'_{n-1}} \qquad \mathbf{(1)},\mathbf{(E)} \\ &\quad - \sum_{j=1}^{m_n-1} s_{x_{n1},\dots,x_{nj};x_{n,j+1}}^{(x_n)''_{j+1}\, x'_{n-1}}. \qquad \mathbf{(F)} \end{aligned}$$

Since $LHS(n-1) = RHS(n-1)$ by the induction hypothesis, it remains to show that

$$\mathbf{(2)} - \mathbf{(3)} = \mathbf{(A)} + \mathbf{(B)} - \mathbf{(C)} - \mathbf{(D)} + \mathbf{(E)} - \mathbf{(F)}.$$

By (Z'), $s_{\mathbb{X}_n} = s_{x_n} = s_{X_n}$; hence $\mathbf{(A)} = \mathbf{(C)}$. By $(Z*_1)$,

$$\begin{aligned} & s_{\mathbb{X}'_{n-1};\mathbb{X}_n} - s_{X'_{n-1};X_n} \\ &= s_{\mathbb{X}'_{n-1};X_{n1}\cdots X_{nm_n}} - s_{x_{11},\dots,x_{n-1,m_{n-1}};[x_{n1}]\cdots[x_{nm_n}]} \\ &= \sum_{j=0}^{m_n-1} s_{\mathbb{X}'_{n-1}(X_n)'_j;X_{n,j+1}}^{(x_n)''_{j+1}} + \sum_{j=2}^{m_n} s_{x_{nj}}^{x'_{n-1}(x_n)_j^\wedge} \\ &\quad - \sum_{j=1}^{m_n-1} s_{x'_{n-1},(x_n)'_j;x_{n,j+1}}^{(x_n)''_{j+1}} - (m_n-1)\, s_{x'_{n-1}}^{x_n} \\ &\quad - \sum_{j=1}^{m_n-1} s_{(X_n)'_j;X_{n,j+1}}^{x'_{n-1}\,(x_n)''_{j+1}} + \sum_{j=1}^{m_n-1} s_{x'_{n-1};(x_n)'_j,x_{n,j+1}}^{(x_n)''_{j+1}} \\ &\quad - \sum_{j=0}^{m_n-1} s_{x_{11},\dots,x_{nj};x_{n,j+1}}^{(x_n)''_{j+1}} - \sum_{j=2}^{m_n} s_{x_{nj}}^{x'_{n-1}(x_n)_j^\wedge} \\ &\quad + \sum_{j=1}^{m_n-1} s_{x'_{n-1},(x_n)'_j;x_{n,j+1}}^{(x_n)''_{j+1}} + (m_n-1)\, s_{x'_{n-1}}^{x_n} \\ &\quad + \sum_{j=1}^{m_n-1} s_{x_{n1},\dots,x_{nj};x_{n,j+1}}^{x'_{n-1}\,(x_n)''_{j+1}} - \sum_{j=1}^{m_n-1} s_{x'_{n-1};(x_n)'_j,x_{n,j+1}}^{(x_n)''_{j+1}} \\ &= \sum_{j=0}^{m_n-1} s_{\mathbb{X}'_{n-1}(X_n)'_j;X_{n,j+1}}^{(x_n)''_{j+1}} - \sum_{j=1}^{m_n-1} s_{(X_n)'_j;X_{n,j+1}}^{x'_{n-1}\,(x_n)''_{j+1}} \\ &\quad - \sum_{j=0}^{m_n-1} s_{x_{11},\dots,x_{nj};x_{n,j+1}}^{(x_n)''_{j+1}} + \sum_{j=1}^{m_n-1} s_{x_{n1},\dots,x_{nj};x_{n,j+1}}^{x'_{n-1}\,(x_n)''_{j+1}}. \end{aligned}$$

Thus $\mathbf{(B)} - \mathbf{(D)} = \mathbf{(2)} - \mathbf{(E)} - \mathbf{(3)} + \mathbf{(F)}$; therefore $\mathbf{(2)} - \mathbf{(3)} = \mathbf{(A)} + \mathbf{(B)} - \mathbf{(C)} - \mathbf{(D)} + \mathbf{(E)} - \mathbf{(F)}$ and $(Z3)$ is proved. ■

5. It follows from Lemmas 2.1 and 2.2 that $Z^3(S,\mathcal{A})$ is isomorphic to the group $Z_1(S,\mathcal{A})$ of all $s \in C_1(S,\mathcal{A})$ which satisfy (P_1'), (P_1''), (Z_1'), and (Z_1''); the proof of 2.2 shows that these properties imply (Z_1^*).

LEMMA 2.3. *Every* $s \in Z_1(S,\mathcal{A})$ *satisfies*

$$(Z_1^{**})\quad \begin{aligned} s_{X;Y} \;=\; &\sum\nolimits_{i=0}^{\ell-1} s^{y_{i+1}''}_{XY_{|i};\,y_{i+1}} \;+\; \sum\nolimits_{i=2}^{\ell} s^{xy_i^\wedge}_{y_i} \;-\; \sum\nolimits_{i=1}^{\ell-1} s^{y_{i+1}''}_{x,y_i';\,y_{i+1}} \\ &-\;(\ell-1)\,s^{y}_{x} \;-\; \sum\nolimits_{i=1}^{\ell-1} s^{xy_{i+1}''}_{Y_{|i};\,y_{i+1}} \;+\; \sum\nolimits_{i=1}^{\ell-1} s^{y_{i+1}''}_{x;\,y_i',y_{i+1}} \end{aligned}$$

for all $X, Y = [y_1,\dots,y_\ell] \in T_1$, *with* $Y_{|i} = [y_1,\dots,y_i]$.

PROOF. Property (Z_1^{**}) is the particular case of (Z_1^*) where $Y_i = [y_i]$ for all i. ∎

This permits further trimming. Since $s_{x;\,y,z} = s_{y,z;\,x}$, (Z_1^{**}) shows that each $s \in Z_1$ is uniquely determined by its values s_x with $x \in S$ and $s_{X;\,y}$ with $X \in T_1$, $y \in S$. Let

$$C_2(S,\mathcal{A}) \;=\; (\textstyle\prod_{x\in S}\mathcal{A}_x)\times(\prod_{X\in T_1,\,y\in S}\mathcal{A}_{(pX)y})$$

be the abelian group of all families

$$c \;=\; ((c_x)_{x\in S}),\,(c_{X;\,y})_{X\in T_1,\,y\in S})$$

such that $c_x \in \mathcal{A}_x$ and $c_{X;\,y} \in \mathcal{A}_{(pX)y}$ for all $x,y \in S$, $X \in T_1$. Let $\Gamma_2 : Z_1(S,\mathcal{A}) \longrightarrow C_2(S,\mathcal{A})$ be the trimming homomorphism defined by $(\Gamma_2 s)_x = s_x$, $(\Gamma_2 s)_{X;\,y} = s_{X;\,y}$ for all $x,y \in S$, $X \in T_1$. Lemma 2.3 implies that Γ_2 is injective.

LEMMA 2.4. *Let* $s \in C_2(S,\mathcal{A})$. *Then* $s \in \operatorname{Im}\Gamma_2$ *if and only if it has properties*

$$(Z_1')\qquad s_{x;y} \;=\; s^y_x + s^x_y$$

$$(P_2')\qquad s_{[x]Y;\,z} \;-\; s_{x,y;\,z} \;-\; s^x_{Y;\,z} \;+\; s^{xy}_z \;=\; s_{[z]Y;\,x} \;-\; s_{z,y;\,x} \;-\; s^z_{Y;\,x} \;+\; s^{zy}_x$$

$$(P_2'')\qquad s_{X\sigma;\,y} = s_{X;\,y}$$

$$(Z_2)\qquad \begin{aligned} &s_{wx,y;\,z} + s_{xy,z;\,w} + s_{w,x;\,yz} + s^{wxz}_y + s^{wxy}_z + s^z_{x,y;\,w} + s^w_{y,z;\,x} + s^{yz}_{wx} \\ &= s_{y,z;\,wx} + s_{w,xy;\,z} + s_{x,yz;\,w} + s^{xyz}_w + s^{wyz}_x + s^z_{w,x;\,y} + s^w_{x,y;\,z} + s^{wx}_{yz} \end{aligned}$$

for all $w,x,y,z \in S$, $X, Y \in T_1$, *and suitable* σ.

PROOF. First we show that every long 3-cocycle $s \in Z^3$ has properties (Z_2) and (P_2').

Let $w, x, y, z \in S$. With $A = [w]$, $B = [x, y]$, $C = [z]$, (Z_1'') reads (with sides exchanged)

(1) $s_{w,x,y;\,z} + s^{z}_{w;\,x,y} + s^{wxy}_{z} - s_{w,xy;\,z} = s_{w;\,x,y,z} + s^{w}_{x,y;\,z} + s^{xyz}_{w} - s_{w;\,xy,z}$.

With $A = [w]$, $B = [x]$, $C = [y, z]$, (Z_1'') reads

(2) $s_{w;\,x,y,z} + s^{w}_{x;\,y,z} + s^{xyz}_{w} - s_{w;\,x,yz} = s_{w,x;\,y,z} + s^{yz}_{w;\,x} + s^{wx}_{yz} - s_{w,x;\,yz}$.

With $A = [w, x]$, $B = [y]$, $C = [z]$, (Z_1'') reads

(3) $s_{w,x;\,y,z} + s^{wx}_{y;\,z} + s^{yz}_{wx} - s_{wx;\,y,z} = s_{w,x,y;\,z} + s^{z}_{w,x;\,y} + s^{wxy}_{z} - s_{wx,y;\,z}$.

Adding these equalities yields

$$\begin{aligned}
&\underline{s_{w,x,y;\,z}} + s^{z}_{w;\,x,y} + \underline{s^{wxy}_{z}} - s_{w,xy;\,z}\\
&\quad + \underline{s_{w;\,x,y,z}} + s^{w}_{x;\,y,z} + \underline{s^{xyz}_{w}} - s_{w;\,x,yz}\\
&\quad + \underline{s_{w,x;\,y,z}} + s^{wx}_{y;\,z} + s^{yz}_{wx} - s_{wx;\,y,z}\\
&= \underline{s_{w;\,x,y,z}} + s^{w}_{x,y;\,z} + \underline{s^{xyz}_{w}} - s_{w;\,xy,z}\\
&\quad + \underline{s_{w,x;\,y,z}} + s^{yz}_{w;\,x} + s^{wx}_{yz} - s_{w,x;\,yz}\\
&\quad + \underline{s_{w,x,y;\,z}} + s^{z}_{w,x;\,y} + \underline{s^{wxy}_{z}} - s_{wx,y;\,z}\,;
\end{aligned}$$

cancelling the underlined terms, and applying (Z_1') to $s_{w;\,x}$ and $s_{y;\,z}$, yields

$$\begin{aligned}
&s^{z}_{w;\,x,y} - s_{w,xy;\,z} + s^{w}_{x;\,y,z} - s_{w;\,x,yz} + s^{wxz}_{y} + s^{wxy}_{z} + s^{yz}_{wx} - s_{wx;\,y,z}\\
&= s^{w}_{x,y;\,z} - s_{w;\,xy,z} + s^{xyz}_{w} + s^{wyz}_{x} + s^{wx}_{yz} - s_{w,x;\,yz} + s^{z}_{w,x;\,y} - s_{wx,y;\,z}\,;
\end{aligned}$$

since $s_{A;B} = s_{B;A}$, this yields (Z_2).

Now let $x, z \in S$, $Y \in T_1$. With $m = 3$, $X_1 = [x]$, $X_2 = Y$, and $X_3 = [z]$, (Z'') reads

$$\begin{aligned}
s_{x;\,Y;\,z} &= s^{yz}_{x} + s^{xz}_{y} + s^{xy}_{z} + s^{z}_{x;\,Y} + s_{[x]Y;\,z} - s^{z}_{x;\,y} - s_{x,y;\,z}\\
&= s^{xy}_{z} + s^{z}_{Y;\,x} + s_{[x]Y;\,z} - s_{x,y;\,z}\,,
\end{aligned}$$

since $s_x^{yz} + s_y^{xz} = s_{x;y}^z$ and $s_{x;Y} = s_{Y;x}$. Exchanging x and z yields

$$s_{z;Y;x} \;=\; s_x^{yz} + s_{Y;z}^x + s_{[z]Y;x} - s_{z,y;x}\,.$$

Since $s_{x;Y;z} = s_{z;Y;x}$, we obtain

$$s_z^{xy} + s_{Y;x}^z + s_{[x]Y;z} - s_{x,y;z} \;=\; s_x^{yz} + s_{Y;z}^x + s_{[z]Y;x} - s_{z,y;x}$$

and (P_2').

Thus every long 3-cocycle $s \in Z^3$ has properties (Z_2), (P_2'), (Z_1') (which was proved before), and (P_2'') (which follows from (P_1'') and ultimately from $(P3'')$). Hence every $t = \Gamma_1 s \in Z_1$ has these properties, and so does every $\Gamma_2 t$ with $t \in Z_1$.

Conversely let $c \in C_2(S, \mathcal{A})$ have properties (P_2'), (P_2''), (Z_1'), and (Z_2). Define $s_x = c_x$ for all $x \in S$ and

$$\begin{aligned} s_{X;Y} \;=\; & \sum_{i=0}^{\ell-1} c_{XY_{|i};\,y_{i+1}}^{y_{i+1}''} \;+\; \sum_{i=2}^{\ell} c_{y_i}^{xy_i^\wedge} \;-\; \sum_{i=1}^{\ell-1} c_{x,y_i';\,y_{i+1}}^{y_{i+1}''} \\ & -\; (\ell-1)\, c_x^y \;-\; \sum_{i=1}^{\ell-1} c_{Y_{|i};\,y_{i+1}}^{xy_{i+1}''} \;+\; \sum_{i=1}^{\ell-1} c_{y_i',y_{i+1};\,x}^{y_{i+1}''} \end{aligned}$$

for all $X, Y = [\,y_1, \dots, y_\ell] \in T_1$ (where $Y_{|i} = [\,y_1, \dots, y_i\,]$). This is according to (Z_1^{**}), except for the last term. If $\ell = 1$, then $s_{X;y} = c_{X;y}$; therefore s has properties (P_2'), (P_2''), (Z_1'), and (Z_2), and will satisfy $\Gamma_2 s = c$.

By (Z_1'), we also have

$$\begin{aligned} s_{x;y,z} \;&=\; c_{x;y}^z + c_{x,y;z} + c_z^{xy} - c_{x,y;z} - c_x^{yz} - c_{y;z}^x + c_{y,z;x} \\ &=\; c_x^{yz} + c_y^{xz} + c_z^{xy} - c_x^{yz} - c_y^{xz} - c_z^{xy} + c_{y,z;x} \\ &=\; c_{y,z;x} \end{aligned}$$

for all $x, y, z \in S$. In particular,

$$s_{x;y,z} \;=\; s_{y,z;x}$$

for all $x, y, z \in S$. The definition of s then shows that it has property (Z_1^{**}). Since c satisfies (P_2'') we have $s_{X\sigma;Y} = s_{X;Y}$ for all X, Y, σ; (P_1'') follows from this property and (P_1'). It remains to prove that s satisfies (P_1') and (Z_1'').

We begin with (Z''_1). Let $A, B = [b_1, \dots, b_\ell], C = [c_1, \dots, c_m] \in T_1$. By (Z_1^{**}), the left hand side and right hand side of (Z''_1) are:

$$
\begin{aligned}
LHS \;=\;& s_{A;BC} + s^{a}_{B;C} + s^{bc}_{a} - s_{a;b,c} \\
=\;& \sum_{i=0}^{\ell-1} s^{b''_{i+1}c}_{AB_{|i};b_{i+1}} + \sum_{j=0}^{m-1} s^{c''_{j+1}}_{ABC_{|j};c_{j+1}} && \mathbf{(1),(2)} \\
&+ \sum_{i=2}^{\ell} s^{ab^{\wedge}_{i}c}_{b_i} + s^{abc^{\wedge}_{1}}_{c_1} + \sum_{j=2}^{m} s^{abc^{\wedge}_{j}}_{c_j} && \mathbf{(3),(a),(4)} \\
&- \sum_{i=1}^{\ell-1} s^{b''_{i+1}c}_{a,b'_i;b_{i+1}} - s^{c''_1}_{a,b;c_1} - \sum_{j=1}^{m-1} s^{c''_{j+1}}_{a,bc'_j;c_{j+1}} && \mathbf{(5),(b),(c)} \\
&- (\ell-1)\, s^{bc}_{a} - s^{bc}_{a} - (m-1)\, s^{bc}_{a} && \mathbf{(6),(7),(d)} \\
&- \sum_{i=1}^{\ell-1} s^{ab''_{i+1}c}_{B_{|i};b_{i+1}} - \sum_{j=0}^{m-1} s^{abc''_{j+1}}_{BC_{|j};c_{j+1}} && \mathbf{(8),(9)} \\
&+ \sum_{i=1}^{\ell-1} s^{b''_{i+1}c}_{a;b'_i,b_{i+1}} + s^{c''_1}_{a;b,c_1} + \sum_{j=1}^{m-1} s^{c''_{j+1}}_{a;bc'_j,c_{j+1}} && \mathbf{(10),(g),(h)} \\
&+ \sum_{j=0}^{m-1} s^{ac''_{i+1}}_{BC_{|j};c_{i+1}} + \sum_{j=2}^{m} s^{abc^{\wedge}_{j}}_{c_j} - \sum_{j=1}^{m-1} s^{ac''_{j+1}}_{b,c'_j;c_{j+1}} && \mathbf{(9),(j),(k)} \\
&- (m-1)\, s^{ac}_{b} - \sum_{j=1}^{m-1} s^{abc''_{j+1}}_{C_{|j};c_{j+1}} + \sum_{j=1}^{m-1} s^{ac''_{j+1}}_{b;c'_j,c_{j+1}} && \mathbf{(\ell),(11),(m)} \\
&+ s^{bc}_{a} - s_{a;b,c}\,, && \mathbf{(7),(n)} \\
RHS \;=\;& s_{AB;C} + s^{c}_{A;B} + s^{ab}_{c} - s_{a,b;c} \\
=\;& \sum_{j=0}^{m-1} s^{c''_{j+1}}_{ABC_{|j};c_{j+1}} + \sum_{j=2}^{m} s^{abc^{\wedge}_{j}}_{c_j} - \sum_{j=1}^{m-1} s^{c''_{j+1}}_{ab,c'_j;c_{j+1}} && \mathbf{(2),(4),(p)} \\
&- (m-1)\, s^{c}_{ab} - \sum_{j=1}^{m-1} s^{abc''_{j+1}}_{C_{|j};c_{j+1}} + \sum_{j=1}^{m-1} s^{c''_{j+1}}_{ab;c'_j,c_{j+1}} && \mathbf{(q),(11),(r)} \\
&+ \sum_{i=0}^{\ell-1} s^{b''_{i+1}c}_{AB_{|i};b_{i+1}} + \sum_{i=2}^{\ell} s^{ab^{\wedge}_{i}c}_{b_i} - \sum_{i=1}^{\ell-1} s^{b''_{i+1}c}_{a,b'_i;b_{i+1}} && \mathbf{(1),(3),(5)} \\
&- (\ell-1)\, s^{bc}_{a} - \sum_{i=1}^{\ell-1} s^{ab''_{i+1}c}_{B_{|i};b_{i+1}} + \sum_{i=1}^{\ell-1} s^{b''_{i+1}c}_{a;b'_i,b_{i+1}} && \mathbf{(6),(8),(10)} \\
&+ s^{ab}_{c} - s_{a,b;c}\,. && \mathbf{(s),(t)}
\end{aligned}
$$

As indicated, 13 terms of LHS cancel with each other or with 9 terms of RHS, leaving the equality

$$
\begin{aligned}
&(\mathbf{a}) - (\mathbf{b}) - (\mathbf{c}) - (\mathbf{d}) + (\mathbf{g}) + (\mathbf{h}) + (\mathbf{j}) - (\mathbf{k}) - (\mathbf{l}) + (\mathbf{m}) - (\mathbf{n}) \\
&\quad = -(\mathbf{p}) - (\mathbf{q}) + (\mathbf{r}) + (\mathbf{s}) - (\mathbf{t})\,;
\end{aligned}
$$

equivalently,

$$
\begin{aligned}
&(\mathbf{a}) + (\mathbf{g}) + (\mathbf{h}) + (\mathbf{j}) + (\mathbf{m}) + (\mathbf{p}) + (\mathbf{q}) + (\mathbf{t}) \\
&\quad = (\mathbf{b}) + (\mathbf{c}) + (\mathbf{d}) + (\mathbf{k}) + (\mathbf{l}) + (\mathbf{n}) + (\mathbf{r}) + (\mathbf{s})\,.
\end{aligned}
$$

With $w = a$, $x = b$, $y = c'_j$, and $z = c_{j+1}$, (Z_2) reads

$$
\begin{aligned}
&s_{ab,c'_j;c_{j+1}} + s_{bc'_j,c_{j+1};a} + s_{a,b;c'_jc_{j+1}} + s_{c'_j}^{abc_{j+1}} \\
&\quad + s_{c_{j+1}}^{abc'_j} + s_{b,c'_j;a}^{c_{j+1}} + s_{c'_j,c_{j+1};b}^{a} + s_{ab}^{c'_jc_{j+1}} \\
&= s_{c'_j,c_{j+1};ab} + s_{a,bc'_j;c_{j+1}} + s_{b,c'_jc_{j+1};a} + s_a^{bc'_jc_{j+1}} \\
&\quad + s_b^{ac'_jc_{j+1}} + s_{a,b;c'_j}^{c_{j+1}} + s_{b,c'_j;c_{j+1}}^{a} + s_{c'_jc_{j+1}}^{ab}\,.
\end{aligned}
$$

Since $c'_jc_{j+1} = c'_{j+1}$, applying $\alpha_{abc'_{j+1},c''_{j+1}}$ and adding from $j = 1$ to $j = m-1$ yields

$$
\begin{aligned}
&\sum_{j=1}^{m-1} s_{ab,c'_j;c_{j+1}}^{c''_{j+1}} + \sum_{j=1}^{m-1} s_{bc'_j,c_{j+1};a}^{c''_{j+1}} + \sum_{j=1}^{m-2} s_{a,b;c'_{j+1}}^{c''_{j+1}} && (\mathbf{p}),(\mathbf{h}),(\mathbf{A}) \\
&\quad + s_{a,b;c} + s_{c_1}^{abc''_1} + \sum_{j=2}^{m-1} s_{c'_j}^{abc''_j} + \sum_{j=1}^{m-1} s_{c_{j+1}}^{abc^{\wedge}_{j+1}} && (\mathbf{t}),(\mathbf{a}),(\mathbf{B}),(\mathbf{j}) \\
&\quad + s_{b,c_1;a}^{c''_1} + \sum_{j=2}^{m-1} s_{b,c'_j;a}^{c''_j} + \sum_{j=1}^{m-1} s_{c'_j,c_{j+1};b}^{ac''_{j+1}} && (\mathbf{g}),(\mathbf{C}),(\mathbf{m}) \\
&\quad + (m-1)\, s_{ab}^{c} && (\mathbf{q}) \\
&= \sum_{j=1}^{m-1} s_{c'_j,c_{j+1};ab}^{c''_{j+1}} + \sum_{j=1}^{m-1} s_{a,bc'_j;c_{j+1}}^{c''_{j+1}} + \sum_{j=1}^{m-2} s_{b,c'_{j+1};a}^{c''_{j+1}} && (\mathbf{r}),(\mathbf{c}),(\mathbf{C}) \\
&\quad + s_{b,c;a} + (m-1)\, s_a^{bc} + (m-1)\, s_b^{ac} + s_{a,b;c_1}^{c''_1} && (\mathbf{n}),(\mathbf{d}),(\ell),(\mathbf{b}) \\
&\quad + \sum_{j=2}^{m-1} s_{a,b;c'_j}^{c''_j} + \sum_{j=1}^{m-1} s_{b,c'_j;c_{j+1}}^{ac''_{j+1}} && (\mathbf{A}),(\mathbf{k}) \\
&\quad + \sum_{j=1}^{m-2} s_{c'_{j+1}}^{abc''_{j+1}} + s_c^{ab}\,. && (\mathbf{B}),(\mathbf{s})
\end{aligned}
$$

Since $s_{x;y,z} = s_{y,z;x}$ we obtain, after cancellations,

$$\begin{aligned}&(\mathbf{a}) + (\mathbf{g}) + (\mathbf{h}) + (\mathbf{j}) + (\mathbf{m}) + (\mathbf{p}) + (\mathbf{q}) + (\mathbf{t}) \\ &= (\mathbf{b}) + (\mathbf{c}) + (\mathbf{d}) + (\mathbf{k}) + (\boldsymbol{\ell}) + (\mathbf{n}) + (\mathbf{r}) + (\mathbf{s})\end{aligned}$$

and (Z_1'') is proved.

Next we prove

$$s_{A;b} = s_{b;A}$$

for all $A = [a_1, \dots, a_\ell] \in T_1$, $b \in S$. This follows from (Z_1') if $\ell = 1$ and was shown above if $\ell = 2$. For $\ell > 2$, we proceed by induction on ℓ. Let $C = A[t] = [a_1, \dots, a_\ell, t]$. We use (Z_1^{**}) and separate the terms containing t:

$$\begin{aligned}
s_{b;C} &= \sum_{i=0}^{\ell} s^{c''_{i+1}}_{[b]C_{|i};c_{i+1}} + \sum_{i=2}^{\ell+1} s^{bc_i^\wedge}_{c_i} - \sum_{i=1}^{\ell} s^{c''_{i+1}}_{b,c'_i;c_{i+1}} \\
&\quad - \ell\, s^c_b - \sum_{i=1}^{\ell} s^{bc''_{i+1}}_{C_{|i};c_{i+1}} + \sum_{i=1}^{\ell} s^{c''_{i+1}}_{b;c'_i,c_{i+1}} \\
&= \sum_{i=0}^{\ell-1} s^{a''_{i+1}t}_{[b]A_{|i};a_{i+1}} + \sum_{i=2}^{\ell} s^{ba_i^\wedge t}_{a_i} - \sum_{i=1}^{\ell-1} s^{a''_{i+1}t}_{b,a'_i;a_{i+1}} \\
&\quad - (\ell-1)\, s^{at}_b - \sum_{i=1}^{\ell-1} s^{ba''_{i+1}t}_{A_{|i};a_{i+1}} + \sum_{i=1}^{\ell-1} s^{a''_{i+1}t}_{b;a'_i,a_{i+1}} \\
&\quad + s_{[b]A;t} + s^{ba}_t - s_{b,a;t} - s^{at}_b - s^b_{A;t} + s_{b;a,t} \\
&= s^t_{b;A} + s_{[b]A;t} + s^{ba}_t - s_{b,a;t} - s^{at}_b - s^b_{A;t} + s_{b;a,t} \\
&= s_{[b]A;t} - s_{b,a;t} - s^b_{A;t} + s^{ba}_t + s_{t,a;b} + s^t_{A;b} - s^{at}_b \\
&\qquad \text{by the induction hypothesis and } (P_2'') \\
&= s_{[t]A;b} = s_{C;b}
\end{aligned}$$

by (P_2') (with $x = b$, $Y = A$, $z = t$) and (P_2'').

We can now prove $(P_1') : s_{B;A} = s_{A;B}$ for all $A, B = [b_1, \dots, b_m] \in T_1$ by induction on m. Assume $s_{B;A} = s_{A;B}$ and let $C = B[t] = [b_1, \dots, b_m, t]$. We use (Z_1^{**}) and separate the terms containing t:

$$s_{A;C} = \sum_{i=0}^{m} s^{c''_{i+1}}_{AC_{|i};c_{i+1}} + \sum_{i=2}^{m+1} s^{ac_i^\wedge}_{c_i} - \sum_{i=1}^{m} s^{c''_{i+1}}_{a,c'_i;c_{i+1}}$$

$$
\begin{aligned}
&- m\, s_a^c \;-\; \textstyle\sum_{i=1}^{m} s^{ac''_{i+1}}_{C_{|i};c_{i+1}} \;+\; \textstyle\sum_{i=1}^{m} s^{c''_{i+1}}_{a;c'_i,c_{i+1}} \\
&= \textstyle\sum_{i=0}^{m-1} s^{b''_{i+1}t}_{AB_{|i};b_{i+1}} \;+\; \textstyle\sum_{i=2}^{m} s^{ab^\wedge_i t}_{b_i} \;-\; \textstyle\sum_{i=1}^{m-1} s^{b''_{i+1}t}_{a,b'_i;b_{i+1}} \\
&\quad - (m-1)\, s_a^{bt} \;-\; \textstyle\sum_{i=1}^{m-1} s^{ab''_{i+1}t}_{B_{|i};b_{i+1}} \;+\; \textstyle\sum_{i=1}^{m-1} s^{b''_{i+1}t}_{a;b'_i,b_{i+1}} \\
&\quad + s_{AB;t} \;+\; s_t^{ab} \;-\; s_{a,b;t} \;-\; s_a^{bt} \;-\; s^a_{B;t} \;+\; s_{a;b,t} \\
&= s^t_{A;B} \;+\; s_{AB;t} \;+\; s_t^{ab} \;-\; s_{a,b;t} \;-\; s_a^{bt} \;-\; s^a_{B;t} \;+\; s_{a;b,t}\,.
\end{aligned}
$$

Also,

$$
\begin{aligned}
s_{C;A} &= s_{B[t];A} \;=\; s_{[t]B;A} \quad \text{since } s_{X\sigma;Y} = s_{X;Y} \\
&= s_{t;BA} \;+\; s^t_{B;A} \;+\; s_t^{ba} \;-\; s_{t;b,a} \;-\; s^a_{t;B} \;-\; s_a^{tb} \;+\; s_{t,b;a} \quad \text{by } (Z''_1) \\
&= s^t_{A;B} \;+\; s_{AB;t} \;+\; s_t^{ab} \;-\; s_{a,b;t} \;-\; s_a^{bt} \;-\; s^a_{B;t} \;+\; s_{a;b,t}
\end{aligned}
$$

by the induction hypothesis and the case $m = 1$. This proves (P'_1). ∎

6. It follows from Lemmas 2.3 and 2.4 that $Z^3(S, \mathcal{A})$ is isomorphic to the group $Z_2(S, \mathcal{A})$ of all $s \in C_2(S, \mathcal{A})$ with properties (P'_2), (P''_2), (Z'_1), and (Z_2).

Property (P'_2) implies that $Z_2(S, \mathcal{A})$ can be trimmed further. For this we use an arbitrary total order relation $<$ on S (which need not be compatible with the multiplication). Let R be the set of all *restricted* sequences $r = (x_1, x_2, \dots, x_\ell,\, y)$ of elements of S such that $\ell \geq 2$ and $y \leq x_1, \dots, x_\ell$ whenever $\ell \geq 3$ (there is no restriction if $\ell = 2$). (One could require $x_1 \geq x_2 \geq \cdots \geq x_\ell \geq y$; but this would complicate the notation and the proofs.) As before, $pr = x_1 \cdots x_\ell\, y$. Let

$$
C_3(S, \mathcal{A}) \;=\; \left(\textstyle\prod_{x \in S} \mathcal{A}_x\right) \times \left(\textstyle\prod_{r \in R} \mathcal{A}_{pr}\right).
$$

The elements of $C_3(S, \mathcal{A})$ are families c consisting of a family $c_x \in \mathcal{A}_x$ ($x \in S$) and a family $c_{X;y} = c_{x_1, \dots, x_\ell; y} \in \mathcal{A}_{xy}$ ($\ell \geq 2$, $(x_1, \dots, x_\ell,\, y) \in R$). The trimming homomorphism $\Gamma_3 : Z_2(S, \mathcal{A}) \longrightarrow C_3(S, \mathcal{A})$ is defined for each $s \in Z_2$ by: $(\Gamma_3 s)_x = s_x$ for all $x \in S$, and $(\Gamma_3 s)_{x_1, \dots, x_\ell; y} = s_{x_1, \dots, x_\ell; y}$ for all $(x_1, \dots, x_\ell,\, y) \in R$.

LEMMA 2.5. *Γ_3 is injective.*

Proof. Assume $\Gamma_3 s = 0$, where $s \in Z_2$. Then $s_x = 0$ for all $x \in S$ and we want to show that $s_{x_1,\ldots,x_\ell;y} = 0$ for all $x_1, \ldots, x_\ell$, $y \in S$. This follows from (Z_1') if $\ell = 1$ and from $\Gamma_3 s = 0$ if $\ell = 2$, or if $\ell \geq 3$ and $y \leq x_1, \ldots, x_\ell$. For $\ell \geq 3$ we proceed by induction on ℓ. Let $x_i = \min(x_1, \ldots, x_\ell)$, $X = [x_1, \ldots, x_i, \ldots, x_\ell]$, and $T = [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_\ell]$. If $y \leq x_i$, then $y \leq x_1, \ldots, x_\ell$ and $s_{x_1,\ldots,x_\ell;y} = 0$. If $y > x_i$, then $s_{[y]T;x_i} = s_{y,x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_\ell;x_i} = 0$ and

$$\begin{aligned} s_{X;y} &= s_{[x_i]T;y} \quad \text{by } (P_2'') \\ &= s_{[y]T;x_i} - s_{y,t;x_i} - s^{y}_{T;x_i} + s^{ty}_{x_i} + s_{x_i,t;y} + s^{x_i}_{T;y} - s^{x_i t}_{y} \quad \text{by } (P_2') \\ &= 0 \end{aligned}$$

by the induction hypothesis. ∎

7. The last link in the chain from long 3-cocycles to symmetric 3-cocycles is the following homomorphism.

Lemma 2.6. *A homomorphism $\Delta : Z_2(S, \mathcal{A}) \longrightarrow SZ^3(S, \mathcal{A})$ is defined by:*

$$(\Delta) \qquad (\Delta s)_{x,y,z} = s_{x,y;z} - s_{z,y;x} + s^{yz}_{x} - s^{xy}_{z}.$$

More generally, if $s \in C_2(S, \mathcal{A})$ has properties (P_2'') and (Z_2), then $\Delta s \in SZ^3(S, \mathcal{A})$.

Recall that symmetric 3-cocycles are characterized by

$$(S3) \qquad t_{z,y,x} = -t_{x,y,z}, \quad t_{x,y,z} + t_{y,z,x} + t_{z,x,y} = 0,$$

$$(A3) \qquad t^{w}_{x,y,z} - t_{wx,y,z} + t_{w,xy,z} - t_{w,x,yz} + t^{z}_{w,x,y} = 0.$$

Proof. Let s satisfy (P_2'') and (Z_2) (for instance, let $s \in Z_2$) and $t = \Delta s$. It is clear from (Δ) that $t_{z,y,x} = -t_{x,y,z}$. Also

$$\begin{aligned} t_{x,y,z} + t_{y,z,x} + t_{z,x,y} &= s_{x,y;z} - s_{z,y;x} + s^{yz}_{x} - s^{xy}_{z} \\ &\quad + s_{y,z;x} - s_{x,z;y} + s^{xz}_{y} - s^{yz}_{x} \\ &\quad + s_{z,x;y} - s_{y,x;z} + s^{xy}_{z} - s^{xz}_{y} \\ &= 0. \end{aligned}$$

Thus t satisfies $(S3)$. Finally,

$$\begin{aligned}
& t^w_{x,y,z} - t_{wx,y,z} + t_{w,xy,z} - t_{w,x,yz} + t^z_{w,x,y} \\
&= s^w_{x,y;\,z} - s^w_{z,y;\,x} + s^{wyz}_x - s^{wxy}_z && \mathbf{(1),(2),(x),(z)} \\
&\quad - s_{wx,y;\,z} + s_{z,y;\,wx} - s_w x^{yz} + s^{wxy}_z && \mathbf{(3),(4),(wx),(z)} \\
&\quad + s_{w,xy;\,z} - s_{z,xy;\,w} + s^{xyz}_w - s^{wxy}_z && \mathbf{(5),(6),(w),(z)} \\
&\quad - s_{w,x;\,yz} + s_{yz,x;\,w} - s^{xyz}_w + s^{wx}_{yz} && \mathbf{(7),(8),(w),(yz)} \\
&\quad + s^z_{w,x;\,y} - s^z_{y,x;\,w} + s^{xyz}_w - s^{wxz}_y && \mathbf{(9),(10),(w),(y)} \\
&= -s_{wx,y;\,z} - s_{xy,z;\,w} - s_{w,x;\,yz} - s^{wxz}_y && \mathbf{(3),(6),(7),(y)} \\
&\quad - s^{wxy}_z - s^z_{x,y;\,w} - s^w_{y,z;\,x} - s^{yz}_{wx} && \mathbf{(z),(10),(2),(wx)} \\
&\quad + s_{y,z;\,wx} + s_{w,xy;\,z} + s_{x,yz;\,w} + s^{xyz}_w && \mathbf{(4),(5),(8),(w)} \\
&\quad + s^{wyz}_x + s^z_{w,x;\,y} + s^w_{x,y;\,z} + s^{wx}_{yz} && \mathbf{(x),(9),(1),(yz)} \\
&= 0
\end{aligned}$$

by (P''_2) and (Z_2), and $(A3)$ holds. ∎

With $t = \Delta s$, property $(P'2)$ can be restated as:

$$(P'_2) \qquad s_{[x]Y;\,z} - s_{[z]Y;\,x} = t_{x,y,z} + s^x_{Y;z} - s^z_{Y;x}\,.$$

LEMMA 2.7. *Let $c \in C_3(S,\mathcal{A})$. Then $c \in \operatorname{Im}\Gamma_3$ if and only if c satisfies (P''_2) and (Z_2).*

PROOF. These conditions are necessary by Lemma 2.4. Conversely let $c \in C_3$ have properties (P''_2) and (Z_2). By Lemma 2.6, $t = \Delta c \in SZ^3$ (t satisfies $(S3)$ and $(A3)$). Let $A = [a_1, \dots, a_\ell] \in T_1$. If $\ell \geq 3$, let $m = \min(a_1, \dots, a_\ell)$ and $A = [m]D$ (actually, $A^\sigma = [m]D$ for some σ). Define $s_x = c_x$, $s_{x;y} = c^y_x + c^x_y$, $s_{x,y;\,z} = c_{x,y;\,z}$, and $s_{A;b}$ by induction on ℓ:

$$s_{A;b} = \begin{cases} c_{A;b} & \text{if } b \leq m, \\ c_{[b]D;\,m} + t_{m,d,b} + s^m_{D;b} - s^b_{D;m} & \text{if } b \geq m\,; \end{cases}$$

There is no ambiguity if $b = m$, since $t_{m,d,m} = -t_{d,m,m} - t_{m,m,d} = 0$ by $(S3)$, $(A3)$. We see that s satisfies (Z'_1), inherits (P''_2) and (Z_2) from c, and will satisfy $\Gamma_3 s = c$. It remains to prove that s satisfies (P'_2).

First we show that every symmetric 3-cocycle satisfies

$$(T)\qquad t_{x,zw,y} + t_{y,xw,z} + t_{z,yw,x} = t^{z}_{x,w,y} + t^{x}_{y,w,z} + t^{y}_{z,w,x}$$

for all $x, y, z, w \in S$. By $(A3)$,

$$\begin{aligned} t^{x}_{z,w,y} - t_{xz,w,y} + t_{x,zw,y} - t_{x,z,wy} + t^{y}_{x,z,w} &= 0 \\ = t^{z}_{x,w,y} - t_{xz,w,y} + t_{z,xw,y} - t_{z,x,wy} + t^{y}_{z,x,w}, \end{aligned}$$

so that

$$\begin{aligned} t_{x,zw,y} - t_{x,z,wy} - t_{z,xw,y} + t_{z,x,wy} \\ = t^{z}_{x,w,y} + t^{y}_{z,x,w} - t^{y}_{x,z,w} - t^{x}_{z,w,y}\,. \end{aligned}$$

By $(S3)$, $-t_{z,xw,y} = t_{y,xw,z}$, $-t_{z,w,y} = t_{y,w,z}$, and

$$\begin{aligned} t_{z,x,wy} - t_{x,z,wy} &= -t_{wy,x,z} - t_{x,z,wy} = t_{z,wy,x}\,, \\ t_{z,x,w} - t_{x,z,w} &= -t_{w,x,z} - t_{x,z,w} = t_{z,w,x}\,; \end{aligned}$$

this yields (T).

Let $x, z \in S$ and $Y = [\,y_1, \dots, y_k\,] \in T$. We prove

$$(P_2')\qquad s_{[x]Y;\,z} - s_{[z]Y;\,x} = t_{x,y,z} + s^{x}_{Y;z} - s^{z}_{Y;x}\,.$$

by induction on k. If $k = 1$, then (P_2') reads

$$s_{x,y;\,z} - s_{z,y;\,x} = t_{x,y,z} + s^{x}_{y;z} - s^{z}_{y;x}\,;$$

this follows from the definitions of s and t. Now let $k \ge 2$, $m = \min Y = \min(y_1, \dots, y_\ell)$, and $Y = [\,m\,]W$. We consider several cases, based on the possible order arrangements of m, y, and z.

Case 1: $x \le m \le z$. Then $\min([\,x]Y) = x \le z$, $\min([\,z\,]Y) = m \ge x$,

$$\begin{aligned} s_{[x]Y;\,z} &= c_{[z]Y;\,x} + t_{x,y,z} + s^{x}_{Y;z} - s^{z}_{Y;x}\,, \\ s_{[z]Y;\,x} &= c_{[z]Y;\,x}\,, \quad \text{and} \\ s_{[x]Y;\,z} - s_{[z]Y;\,x} &= t_{x,y,z} + s^{x}_{Y;z} - s^{z}_{Y;x}\,. \end{aligned}$$

Case 2: $z \le m \le x$ follows from Case 1 by exchanging x and z.

Case 3: $x \le z \le m$. Then $\min([\,x\,]Y) = x \le z$, $\min([\,z\,]Y) = z \ge x$, and

$$\begin{aligned} s_{[x]Y;\,z} &= c_{[z]Y;\,x} + t_{x,y,z} + s^{x}_{Y;z} - s^{z}_{Y;x}\,, \\ s_{[z]Y;\,x} &= c_{[z]Y;\,x}\,, \end{aligned}$$

as in Case 1.

Case 4: $z \leq x \leq m$ follows from Case 3 by exchanging x and z.

Case 5: $m \leq x, z$. Then

$$\begin{aligned} \min([x]Y) = m \leq z, \quad [x]Y = [m][x]W, \\ \min([z]Y) = m \leq x, \quad [z]Y = [m][z]W, \\ \min([x]W) \geq m, \quad \min([z]W) \geq m, \end{aligned}$$

and

$$\begin{aligned} s_{Y;x} &= c_{[x]W;m} + t_{m,w,x} + s^m_{W;x} - s^x_{W;m}, \\ s_{Y;z} &= c_{[z]W;m} + t_{m,w,z} + s^m_{W;z} - s^z_{W;m}, \\ s_{[x]Y;z} &= c_{[z][x]W;m} + t_{m,xw,z} + s^m_{[x]W;z} - s^z_{[x]W;m}, \\ s_{[z]Y;x} &= c_{[x][z]W;m} + t_{m,zw,x} + s^m_{[z]W;x} - s^x_{[z]W;m}, \end{aligned}$$

$$s_{[x]W;m} = c_{[x]W;m}, \quad s_{[z]W;m} = c_{[z]W;m},$$

so that

$$\begin{aligned} s_{[x]Y;z} - s_{[z]Y;x} &= t_{m,xw,z} - t_{m,zw,x} \\ &\quad + s^m_{[x]W;z} - s^m_{[z]W;x} - s^z_{[x]W;m} + s^x_{[z]W;m} \\ &= t_{m,xw,z} - t_{m,zw,x} \\ &\quad + t^m_{x,w,z} + s^{xm}_{W;z} - s^{zm}_{W;x} - c^z_{[x]W;m} + c^x_{[z]W;m} \\ &= t_{m,xw,z} + t_{x,zw,m} - t^m_{z,w,x} \\ &\quad + s^{mx}_{W;z} - s^{mz}_{W;x} - c^z_{[x]W;m} + c^x_{[z]W;m} \end{aligned}$$

by the induction hypothesis and $(S3)$, whereas

$$\begin{aligned} t_{x,y,z} + s^x_{Y;z} - s^z_{Y;x} &= t_{x,mw,z} + c^x_{[z]W;m} + t^x_{m,w,z} + s^{mx}_{W;z} \\ &\quad - c^z_{[x]W;m} - t^z_{m,w,x} - s^{mz}_{W;x} \\ &= -t_{z,mw,x} + c^x_{[z]W;m} - c^z_{[x]W;m} \\ &\quad + t^x_{m,w,z} + t^z_{x,w,m} + s^{mx}_{W;z} - s^{mz}_{W;x} \end{aligned}$$

by $(S3)$; then (P_2') follows from (T). ∎

8. By Lemmas 2.5 and 2.7, $Z^3(S,\mathcal{A})$ is isomorphic to the group $Z_3(S,\mathcal{A})$ of all families $s \in C_3(S,\mathcal{A})$ with properties (Z_2) and (P_2'').

The next trimming reduces $Z_3(S,\mathcal{A})$ to its direct summand $Z_4'(S,\mathcal{A})$ whose elements are all $s \in Z_3$ such that $s_x = 0$ for all $x \in S$ (hence $s_{x;y} = 0$ for all $x, y \in S$ by (Z_1')) and $s_{X;y} = 0$ when X has length 3 or more (and $y \leq \min X$). In Z_4', (P_2'') reduces to

$$(P_4'') \qquad s_{y,x;\,z} \;=\; s_{x,y;\,z}$$

for all $x, y, z \in S$, and (Z_2) reduces to

$$(Z_4) \qquad \begin{aligned} & s_{wx,y;\,z} + s_{xy,z;\,w} + s_{w,x;\,yz} + s^{z}_{x,y;\,w} + s^{w}_{y,z;\,x} \\ &= s_{y,z;\,wx} + s_{w,xy;\,z} + s_{x,yz;\,w} + s^{z}_{w,x;\,y} + s^{w}_{x,y;\,z}\,. \end{aligned}$$

$Z_4'(S,\mathcal{A})$ is isomorphic to the group $Z_4(S,\mathcal{A}) \subseteq \prod_{x,y,z\in S} \mathcal{A}_{xyz}$ of all families $s = (s_{x,y;\,z})_{x,y,z\in S}$ such that $s_{x,y;\,z} \in \mathcal{A}_{xyz}$ for all $x, y, z \in S$ and (P_4''), (Z_4) hold.

Z_4 is not isomorphic to Z_3; rather, we prove that the remaining elements of Z_3 contribute nothing to the cohomology.

The trimming homomorphisms Γ_1, Γ_2, Γ_3 provide an isomorphism $\Gamma : Z^3(S,\mathcal{A}) \longrightarrow Z_3(S,\mathcal{A})$ which affects coboundaries as follows. Recall that $\delta c \in Z^3(S,\mathcal{A})$ is defined by

$$(C2) \qquad (\delta c)_{X_1;\,\dots;\,X_n} \;=\; c_{x_1,\,\dots,\,x_n} \;-\; c_{X_1\cdots X_n} \;+\; \sum\nolimits_{k=1}^{n} c^{\hat{x_k}}_{X_k}$$

for all $X_1, \dots, X_n \in T_1$ and $c \in C^2$. In particular,

$$\begin{aligned} (\delta c)_x &= c_x\,, \\ (\delta c)_{x,y;\,z} &= c_{xy,z} \;-\; c_{x,y,z} \;+\; c^{z}_{x,y} \;+\; c^{xy}_{z}\,, \\ (\delta c)_{X;y} &= c_{x,y} \;-\; c_{X[y]} + c^{y}_{X} + c^{x}_{y}\,. \end{aligned}$$

This describes the subgroup $B_3(S,\mathcal{A}) = \Gamma\,(B^3(S,\mathcal{A}))$ of $Z_3(S,\mathcal{A})$.

LEMMA 2.8. $Z_3(S,\mathcal{A}) = Z_4'(S,\mathcal{A}) + B_3(S,\mathcal{A})$.

PROOF. Given $s \in Z_3$, define $c_X \in \mathcal{A}_x$ for all $X = [x_1, \dots, x_\ell] \in T_1$ by induction on ℓ as follows:

$$c_X = \begin{cases} s_x & \text{if } \ell = 1, \\ 0 & \text{if } \ell = 2, 3, \\ -s_{W;m} + c_W^m + c_m^w & \text{if } \ell \geq 4, \end{cases}$$

where $m = \min X$ and $X = W[m]$. We see that $(\delta c)_x = c_x = s_x$ for all $x \in S$ and

$$(\delta c)_{X;y} = c_{x,y} - c_{X[y]} + c_X^y + c_y^x = s_{X;y}$$

whenever X has length 3 or more and $y \leq \min X$. Hence $s - \Gamma\delta c \in Z_4'$ and $s = (s - \Gamma\delta c) + \Gamma\delta c \in Z_4' + B_3$. ∎

Since Γ is an isomorphism, $H^3 = Z^3/B^3 \cong Z_3/B_3 \cong Z_4'/B_4'$, where

$$B_4'(S, \mathcal{A}) + Z_4'(S, \mathcal{A}) \cap B_3(S, \mathcal{A}).$$

In other words, $B_4'(S, \mathcal{A})$ is the group of all $s \in Z_4'(S, \mathcal{A})$ such that $s = \Gamma\delta c$ for some $c \in C^2(S, \mathcal{A})$. Then $c_x = s_x = 0$ for all $x \in S$, and $s = \Gamma\delta c$ reduces to

$$(C_4) \qquad s_{x,y;z} = c_{xy,z} - c_{x,y,z} + c_{x,y}^z .$$

$B_4'(S, \mathcal{A})$ is isomorphic to the group $B_4(S, \mathcal{A}) \subseteq Z_4(S, \mathcal{A})$ of all families $s = (s_{x,y;z})_{x,y,z \in S}$ such that $s_{x,y;z} = c_{xy,z} - c_{x,y,z} + c_{x,y}^z \in \mathcal{A}_{xyz}$ for some $c \in C^2(S, \mathcal{A})$. We now have

LEMMA 2.9. $H^3(S, \mathcal{A}) \cong Z_4(S, \mathcal{A}) / B_4(S, \mathcal{A})$. ∎

9. Recall that a symmetric 3-coboundary is a symmetric 3-cochain t (necessarily a symmetric 3-cocycle) for which there exists $u = (u_{x,y})_{x,y \in S}$ such that $u_{x,y} \in \mathcal{A}_{xy}$,

$$(S2) \qquad u_{y,x} = u_{x,y}, \quad \text{and}$$

$$(B3) \qquad t_{x,y,z} = u_{y,z}^x - u_{xy,z} + u_{x,yz} - u_{x,y}^z$$

for all $x, y, z \in S$. Under pointwise addition, symmetric 3-coboundaries form an abelian group $SB^3(S, \mathcal{A})$.

The homomorphism Δ in Lemma 2.6 induces a homomorphism $D : Z_4(S, \mathcal{A}) \longrightarrow SZ^3(S, \mathcal{A})$ given by:

$$(Ds)_{x,y,z} = s_{x,y;z} - s_{z,y;x}$$

for all $x, y, z \in S$. We show that D is surjective. For this we again use an arbitrary total order $\leq$ on S.

LEMMA 2.10. *A homomorphism* $E : SZ^3(S, \mathcal{A}) \longrightarrow Z_4(S, \mathcal{A})$ *is defined by:*

$$(Et)_{x,y;\,z} = \begin{cases} t_{x,y,z} & \text{if } x \leq y, z, \\ t_{y,x,z} & \text{if } y \leq x, z, \\ 0 & \text{if } z \leq x, y. \end{cases}$$

Moreover $DE = 1$; $\operatorname{Im}(1 - ED) \subseteq B_4$; $D(B_4) \subseteq SB^3$; *and* $E(SB^3) \subseteq B_4$.

PROOF. The three cases in the definition of Et are consistent with each other: if $x \leq y, z$ and $y \leq x, z$, then $x = y$ and $t_{x,y,z} = t_{y,x,z}$; if $x \leq y, z$ and $z \leq x, y$, then $x = z$ and $(S3)$ implies $t_{x,y,z} = t_{x,y,x} = t_{x,y,x} + t_{y,x,x} + t_{x,x,y} = 0$; if $y \leq x, z$ and $z \leq x, y$, then $y = z$ and $(S3)$ implies $t_{y,x,z} = 0$.

Let $t \in SZ^3$ and $s = Et$. First we show that

$$s_{x,y;\,z} - s_{z,y\,x} = t_{x,y,z}$$

for all $x, y, z \in S$. If $x \leq y, z$, then

$$s_{x,y;\,z} - s_{z,y\,x} = t_{x,y,z} - 0 = t_{x,y,z}\,.$$

If $y \leq x, z$, then

$$s_{x,y;\,z} - s_{z,y\,x} = t_{y,x,z} - t_{y,z,x} = -t_{y,z,x} - t_{z,x,y} = t_{x,y,z}$$

by $(S3)$. If finally $z \leq x, y$, then

$$s_{x,y;\,z} - s_{z,y\,x} = 0 - t_{z,y,x} = t_{x,y,z}$$

by $(S3)$.

This implies $s \in Z_4$: indeed $(P_4'') : s_{x,y;\,z} = s_{y,x;\,z}$ holds by definition, and (Z_4) holds since

$$\begin{aligned} & s_{wx,y;\,z} - s_{y,z;\,wx} + s_{z,xy;\,w} - s_{w,xy;\,z} + s_{w,x;\,yz} - s_{yz,x;\,w} \\ & \quad + s^z_{y,x;\,w} - s^z_{w,x;\,y} + s^w_{z,y;\,x} - s^w_{x,y;\,z} \\ & = t_{wx,y,z} + t_{z,xy,w} + t_{x,w,yz} + t^z_{y,x,w} + t^w_{z,y,x} \\ & = -t^w_{x,y,z} + t_{wx,y,z} - t_{w,xy,z} + t_{w,x,yz} - t^z_{w,x,y} = 0 \end{aligned}$$

by $(S3)$ and $(A3)$. Hence E sends SZ^3 into Z_4. Then $s_{x,y;\,z} - s_{z,y\,x} = t_{x,y,z}$ shows that $Ds = t$, so that DE is the identity on SZ^3.

Next we show that $u = s - EDs$ is given for each $s \in Z_4$ by:

$$u_{x,y;z} = \begin{cases} s_{y,z;x} & \text{if } x \leq y,z, \\ s_{x,z;y} & \text{if } y \leq x,z, \\ s_{x,y;z} & \text{if } z \leq x,y. \end{cases}$$

If $x \leq y, z$, then

$$(s - EDs)_{x,y;z} = s_{x,y;z} - (Ds)_{x,y,z} = s_{x,y;z} - s_{x,y;z} + s_{z,y;x} = s_{y,z;x}$$

by (P_4''). If $y \leq x, z$, then

$$(s - EDs)_{x,y;z} = s_{x,y;z} - (Ds)_{y,x,z} = s_{x,y;z} - s_{y,x;z} + s_{z,x;y} = s_{x,z;y}.$$

If $z \leq x, y$, then $(EDs)_{x,y;z} = 0$ and $(s - EDs)_{x,y;z} = s_{x,y;z}$.

It follows that

$$u_{x,y;z} = u_{x,z;y} = u_{y,x;z} = u_{y,z;x} = u_{z,x;y} = u_{z,y;x} :$$

for instance, $u_{x,y;z} = u_{y,x;z}$ holds by (P_4'') if $z \leq x,y$; if $x \leq y,z$, then $u_{x,y;z} = s_{y,z;x} = u_{y,z;x}$; if $y \leq x,z$, then $u_{x,y;z} = s_{x,z;y} = u_{y,x;z}$. Therefore a long 2-cochain c is well defined by:

$$\begin{cases} c_{x,y,z} = -u_{x,y;z} & \text{for all } x,y,z \in S, \\ c_X = 0 & \text{whenever } X \text{ does not have length 3.} \end{cases}$$

Then $(s - EDs)_{x,y;z} = c_{xy,z} - c_{x,y,z} + c^z_{x,y}$ for all $x,y,z \in S$. Thus $\operatorname{Im}(1 - ED) \subseteq B_4$.

Next, let $s \in B_4$, so that

$$s_{x,y;z} = c_{xy,z} - c_{x,y,z} + c^z_{x,y},$$

for some $c \in C^2$. Then

$$(Ds)_{x,y,z} = s_{x,y;z} - s_{z,y;x} = -c^x_{y,z} + c_{xy,z} - c_{zy,x} + c^z_{x,y}.$$

Thus $D(B_4) \subseteq SB^3$.

Finally let $t \in SB^3$, so that

$$t_{x,y,z} = u^x_{y,z} - u_{xy,z} + u_{x,yz} - u^z_{x,y}$$

where u is symmetric ($u_{b,a} = u_{a,b}$ for all a,b). Let $c_{x,y} = u_{x,y}$ and

$$c_{x,y,z} = \begin{cases} u^x_{y,z} + u_{x,yz} & \text{if } x \leq y, z, \\ u^y_{x,z} + u_{y,xz} & \text{if } y \leq x, z, \\ u^z_{x,y} + u_{z,xy} & \text{if } z \leq x, y. \end{cases}$$

These three cases are consistent with each other: if, say, $x \leq y, z$ and $y \leq x, z$, then $x = y$ and $u^x_{y,z} + u_{x,yz} = u^y_{x,z} + u_{y,xz}$. We see that $c_{x,y,z} = c_{x,z,y} = c_{y,x,z}$ etc. If $x \leq y, z$, then

$$(Et)_{x,y;\,z} = t_{x,y,z} = u^x_{y,z} - u_{xy,z} + u_{x,yz} - u^z_{x,y} = c_{x,y,z} - c_{xy,z} - c^z_{x,y}\,.$$

If $y \leq x, z$, then

$$(Et)_{x,y;\,z} = t_{y,x,z} = u^y_{x,z} - u_{yx,z} + u_{y,xz} - u^z_{y,x} = c_{x,y,z} - c_{xy,z} - c^z_{x,y}\,.$$

If $y \leq x, z$, then

$$(Et)_{x,y;\,z} = 0 = u^z_{x,y} - u_{xy,z} + u_{z,xy} - u^z_{x,y} = c_{x,y,z} - c_{xy,z} - c^z_{x,y}\,.$$

Thus $(Et)_{x,y;\,z} = c_{x,y,z} - c_{xy,z} - c^z_{x,y}$ for all x, y, z. Hence $E(SB^3) \subseteq B_4$. ∎

By Lemma 2.10, $D : Z_4 \longrightarrow SZ^3$ satisfies $D(B_4) \subseteq SB^3$ and induces a homomorphism $D^* : H^3 \longrightarrow SZ^3/\,SB^3$. Since $DE = 1$, D is surjective and so is D^*. Moreover, $Ds \in SB^3$ implies $EDs \in B_4$ and $s = (s - EDs) + EDs \in B_4$; therefore D^* is injective and we have proved

THEOREM 2.11. *For every commutative semigroup S and abelian group valued functor $\mathcal{A}$ on $\mathscr{H}(S)$,*

$$H^3(S, \mathcal{A}) \cong SZ^3(S, \mathcal{A})/\,SB^3(S, \mathcal{A})\,. \blacksquare$$

10. Normalization can be used to sharpen Theorem 2.11. A symmetric 3-cochain c is *normalized* when

$$c_{e,x,y} = 0 \quad \text{whenever} \quad e^2 = e,\ ex = x\,.$$

By $(S3)$, this condition implies

$$\begin{cases} c_{x,e,y} = 0 & \text{whenever } e^2 = e,\ ex = x,\ ey = y, \\ c_{x,y,e} = 0 & \text{whenever } e^2 = e,\ ey = y. \end{cases}$$

Normalized symmetric 3-cochains, cocycles and coboundaries form groups $NSC^3(S,\mathcal{A}) \subseteq SC^3(S,\mathcal{A})$, $NSZ^3(S,\mathcal{A}) = SZ^3(S,\mathcal{A}) \cap NSC^3(S,\mathcal{A})$, and $NSB^3(S,\mathcal{A}) = SB^3(S,\mathcal{A}) \cap NSC^3(S,\mathcal{A})$. We note:

LEMMA 2.12. *If $\mathcal{A}$ is thin, then $NSB^3(S,\mathcal{A}) = \delta(NSC^2(S,\mathcal{A}))$.*

PROOF. Let $\mathcal{A}$ be thin. If $e^2 = e$ and $ex = x$, then $exy = xy$, $\alpha_{xy,e} = \alpha_{xy,1}$, $u^e_{x,y} = u_{x,y}$, and

$$(\delta u)_{e,x,y} \;=\; u^e_{x,y} - u_{ex,y} + u_{e,xy} - u^y_{e,x} \;=\; u_{e,xy} - u^y_{e,x}\,.$$

In particular, if u is normalized, then δu is normalized.

Conversely assume that $t = \delta u$ is normalized. Let $w \in C^1(S,\mathcal{A})$ satisfy $w_e = u_{e,e}$ whenever $e^2 = e$. Since $\alpha_{e,e} = \alpha_{e,1}$ we have $(\delta w)_{e,e} = w^e_e - w_e + w^e_e = w_e = u_{e,e}$ for all $e^2 = e$. Let $v = u - \delta w \in SC^2$. Then $\delta v = t$ and $v_{e,e} = 0$ for all $e^2 = e$. Since t is normalized we have $v_{e,xy} - v^y_{e,x} = (\delta v)_{e,x,y} = 0$ whenever $e^2 = e$, $ex = x$. In particular $v_{e,ey} = v^y_{e,e} = 0$, so that $v_{e,x} = 0$ whenever $e^2 = e$, $ex = x$, and $t = \delta v$ with v normalized. ∎

PROPOSITION 2.13. *If $\mathcal{A}$ is thin, then*

$$H^3(S,\mathcal{A}) \;\cong\; NSZ^3(S,\mathcal{A})/\,NSB^3(S,\mathcal{A})\,.$$

PROOF. We show that $SZ^3 = NSZ^3 + SB^3$; then $H^3 \cong NSZ^3/\,NSB^3$ follows from $H^3 \cong SZ^3/\,SB^3$ and $NSZ^3 \cap SB^3 = NSB^3$.

Let $t \in SZ^3$. Define

$$\begin{cases} u_{e,x} = u_{x,e} = t_{e,e,x} & \text{if } e^2 = e,\ ex = x, \\ u_{x,y} = 0 & \text{if neither } x^2 = x,\ xy = y \text{ nor } y^2 = y,\ yx = x. \end{cases}$$

If $e = x$, then $t_{e,e,x} = t_{x,x,e}$, so that u is well defined. We see that $u \in SC^2$ and that $u_{e,e} = t_{e,e,e} = 0$ whenever $e^2 = e$, by $(S3)$. Let $s = t - \delta u \in SZ^3$. When $e^2 = e$, $ex = x$, then $\alpha_{x,e} = \alpha_{x,1}$,

$$(\delta u)_{e,e,x} \;=\; u^e_{e,x} - u_{ee,x} + u_{e,ex} - u^x_{e,e} \;=\; u_{e,x} \;=\; t_{e,e,x}\,,$$

and $s_{e,e,x} = 0$; hence $(A3)$ yields

$$s_{e,x,y} \;=\; s^e_{e,x,y} - s_{ee,x,y} + s_{e,ex,y} - s_{e,e,xy} + s^y_{e,e,x} \;=\; 0$$

for all $y \in S$. Thus s is normalized, and $t = s + \delta u \in NSZ^3 + SB^3$. ∎

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