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RELATIONS AMONG HOMOTOPY OPERATIONS FOR SIMPLICIAL COMMUTATIVE ALGEBRAS

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ABSTRACT. The homotopy groups of a simplicial commutative algebra over the field with two elements support natural operations. Here we write the relations among these operations in an admissible form.

Let A be a simplicial commutative algebra over the field $\mathbb{F}_2$. Then A is, among other things, a simplicial set and, as such, has homotopy groups. In fact,

$$\pi_n A \cong H_n(A, \partial)$$

where $\partial = \sum d_i\colon A_n \to A_{n-1}$ is the sum (or alternating sum) of the face operators. By the Eilenberg-Zilber Theorem, $\pi_* A$ is a graded commutative $\mathbb{F}_2$ algebra. Cartan [2], Bousfield [1], and Dwyer [3] have pointed out the existence of natural operations on these homotopy groups. Indeed, Dwyer proved the following theorem.

Theorem 1. *Let A be a simplicial commutative $\mathbb{F}_2$ algebra. Then there are natural operations*

$$\delta_i\colon \pi_n A \to \pi_{n+i} A, \qquad 2 \le i \le n,$$

so that

(1) $\delta_i(x+y) = \delta_i(x) + \delta_i(y)$ *if* $i < n$ *and* $\delta_n(x+y) = \delta_n(x) + \delta_n(y) + xy$;

(2) $\delta_i(xy) = 0$ *unless* $x \in \pi_0 A$ *or* $y \in \pi_0 A$. *If* $x \in \pi_0 A$, *then* $\delta_i(xy) = x^2\delta_i(y)$; *if* $y \in \pi_0 A$, *then* $\delta_i(xy) = \delta_i(x)y^2$;

(3) *if* $j < 2i$, *then there is a relation*

$$\delta_j\delta_i(x) = \sum_{(j+1)/2 \le k < i} \binom{i-j+k-1}{i-k} \delta_{i+j-k}\delta_k(x).$$

Dwyer further points out that if $x \in \pi_n A$, $n \ge 1$, then $x^2 = 0$.

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The purpose of this note is to discuss the relations of Theorem 1, part (3). It is clear that any composition $\delta_j\delta_i$ can be rewritten as a linear combination of the compositions $\delta_q\delta_p$ which are admissible in the sense that $q \geq 2p$. However, the relations as written do not immediately return such a sum but have to be applied recursively. Our main point is to get a closed formula, which is supplied in the next result.

Proposition 2. *Let A be a simplicial commutative $\mathbb{F}_2$ algebra and $x \in \pi_n A$. Then if $j < 2i$ there is a relation*

$$\delta_j\delta_i(x) = \sum_{(j+1)/2 \leq k \leq (i+j)/3} \binom{i-j+k-1}{i-k} \delta_{i+j-k}\delta_k(x).$$

Notice that the binomial coefficients haven't changed (an amusing curiosity) but the limits on k have; indeed, since $k \leq \frac{i+j}{3}$ one has $i+j-k \geq 2k$ and the relations now return a sum of admissible compositions.

This result has been known to the second author for some time (see [5]) and was proved independently, but not published, by the first author for the work in [4].

The motivation for studying these relations was this: in the work of the first author on the cohomology of commutative $\mathbb{F}_2$ algebras [4], it is fairly simple to construct certain primary cohomology operations analogous to Steenrod operations. Then one wants to know the "Adem relations" among them. These are obtained by a "Koszul duality" argument (cf. [6]) once one has the relations among the operations δ_i in closed form, as in Proposition 2. And, in fact, it was low-degree calculations in the cohomology of commutative $\mathbb{F}_2$ algebras that suggested the truth of Proposition 2.

The proof of Proposition 2 is modeled on one of the standard ways of producing the Adem relations in the Steenrod algebra. If $\mathscr{A}$ is the Steenrod algebra at the prime 2, then there is a derivation $\partial : \mathscr{A} \to \mathscr{A}$ defined by

$$\partial(\mathrm{Sq}^i) = \mathrm{Sq}^{i-1}.$$

The Adem relations, then, are consequences of the existence of this derivation and the relations

$$\mathrm{Sq}^{2n-1}\mathrm{Sq}^n = 0, \qquad n \geq 1.$$

A similar argument will apply here.

We form an algebra analogous to the Steenrod algebra out of the operations δ_i. Let $\mathscr{D}$ be the graded $\mathbb{F}_2$ algebra formed by taking the quotient of the tensor algebra on symbols δ_i of degree i, $i \geq 2$, by the ideal generated by the relations, for $j < 2i$

$$R(j, i) = \sum_{(j+1)/2 \leq k \leq i} \binom{i-j+k-1}{i-k} \delta_{i+j-k}\delta_k.$$

These are the relations implicit in Theorem 1, part (3).

The algebra $\mathscr{D}$ comes equipped with a degree-raising derivation $\partial : \mathscr{D} \to \mathscr{D}$. In fact, ∂ is defined by the requirement that it be a derivation over $\mathbb{F}_2$ and the formula

$$\partial(\delta_i) = \delta_{i+1}.$$

That this function is well defined is implied by the following result.

Lemma 3. *We have that*

$$\partial R(j, i) = R(j+1, i) + R(j, i+1)$$

where, if $j+1 = 2i$, $R(j+1, i) = 0$.

Proof. If $j+1 < 2i$, this is a routine, if tedious, exercise in binomial coefficients. Because of the denominator in the limits of the summation, it is easiest to break the calculation into two cases: one where j is even and one where j is odd. We omit the details. If $j+1 = 2i$, one simply checks that

$$R(2i-1, i) = \delta_{2i-1}\delta_i,$$
$$R(2i-1, i+1) = \delta_{2i}\delta_i + \delta_{2i-1}\delta_{i+1}$$

as required. □

We next restate Proposition 2 in a form suitable for induction.

Proposition 4. *Let* $t > 0$ *and* $2b - t \geq 2$. *Then in the algebra* $\mathscr{D}$

$$\delta_{2b-t}\delta_b = \sum_{t/3 \leq s \leq (t-1)/2} \binom{t-s-1}{s} \delta_{2b-t+s}\delta_{b-s}.$$

Notice that Proposition 4 implies Proposition 2 by setting $b = i$, $t = 2i - j$, and $s = b - k = i - k$.

The rest of this note is devoted to the proof of Proposition 4. This is accomplished by induction on t. The only point obscuring the argument is that because of the 2 and 3 in the denominators of the summation limits in the formula, there are six cases, one for each congruence class of t modulo 6.

To begin with, note that if $t = 1$ or $t = 2$, the formula in Proposition 5 merely restates $R(2b-1, b)$ or $R(2b-2, b)$ respectively. The cases $t = 3, 4, 5$ and 6 are covered using the derivation ∂ as follows.

The case $t = 3$: One has

$$0 = \partial(\delta_{2a-1}\delta_a) = \delta_{2a}\delta_a + \delta_{2a-1}\delta_{a+1}.$$

Setting $a = b - 1$ yields the desired formula.

The case $t = 4$: One has

$$0 = \partial(\delta_{2a-2}\delta_a) = \delta_{2a-1}\delta_a + \delta_{2a-2}\delta_{a+1} = \delta_{2a-2}\delta_{a+1}.$$

Setting $a = b - 1$ yields the desired formula.

The case $t = 5$: One has, by the case $t = 3$,

$$\partial(\delta_{2a-3}\delta_a) = \partial(\delta_{2a-2}\delta_{a-1})$$

which yields

$$\delta_{2a-3}\delta_{a+1} = \delta_{2a-1}\delta_{a-1}.$$

Again setting $a = b - 1$ yields the desired result.

The case $t = 6$ is similar.

By now the pattern is clear. Fix $t \geq 7$. Apply ∂ to

$$\delta_{2b-t+2}\delta_b = \sum_{(t-2)/3 \leq s \leq (t-3)/2} \binom{t-s-3}{s} \delta_{2b-t+2+s}\delta_{b-s}$$

and use the fact that

$$\delta_{2b-t+3}\delta_b = \sum_{(t-3)/3\le s\le (t-4)/2} \binom{t-s-4}{s} \delta_{2b-t+3+s}\delta_{b-s}$$

to obtain a formula

$$\begin{aligned}\delta_{2b-t+2}\delta_{b+1} = &\sum_{(t-3)/3\le s\le (t-4)/2} \binom{t-s-4}{s} \delta_{2b-t+3+s}\delta_{b-s}\\ &+ \sum_{(t-2)/3\le s\le (t-3)/2} \binom{t-s-3}{s} \delta_{2b-t+3+s}\delta_{b-s}\\ &+ \sum_{(t-5)/3\le s\le (t-5)/2} \binom{t-s-4}{s+1} \delta_{2b-t+3+s}\delta_{b-s}.\end{aligned}$$

The difficulty comes in simplifying this formula, mostly because of the end-points of the three summations are all different. We break this into six cases, one for each congruence class of t modulo 6. We do one case; the other cases are similar. So assume that $t = 6a+2$ and $a \ge 1$. Then

$$\begin{aligned}\delta_{2b-t+2}\delta_{b+1} = &\sum_{2a\le s\le 3a-1} \binom{t-s-4}{s} \delta_{2b-t+3+s}\delta_{b-s}\\ &+ \sum_{2a\le s\le 3a-1} \binom{t-s-3}{s} \delta_{2b-t+3+s}\delta_{b-s}\\ &+ \sum_{2a-1\le s\le 3a-2} \binom{t-s-4}{s+1} \delta_{2b-t+3+s}\delta_{b-s}.\end{aligned}$$

This can be rewritten as

$$\begin{aligned}\delta_{2b-t+2}\delta_{b+1} = &\binom{t-2a-3}{2a} \delta_{2b-4a}\delta_{b-2a+1}\\ &+ \sum_{2a\le s\le 3a-2} \binom{t-s-2}{s+1} \delta_{2b-t+s+3}\delta_{b-s}\\ &+ \binom{3a-1}{3a-2} \delta_{2b-3a}\delta_{b-3a+1}.\end{aligned}$$

Since, by the case $t=2$, $\delta_{2b-4a}\delta_{b-2a+1} = 0$ and $\binom{3a+1}{3a} = \binom{3a-1}{3a-2}$, we have

$$\delta_{2b-t+2}\delta_{b+1} = \sum_{2a\le s\le 3a-1} \binom{t-s-2}{s+1} \delta_{2b-t+s+3}\delta_{b-s}$$

or

$$\delta_{2b-t}\delta_b = \sum_{2a+1\le s\le 3a} \binom{t-s-1}{s} \delta_{2b-t+s}\delta_{b-s}$$

as predicted.

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