SIMPLICIAL METHODS IN ALGEBRA AND ALGEBRAIC GEOMETRY

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Abstract. This is an introduction to / survey of simplicial techniques in algebra and algebraic geometry. We begin with the basic notions of simplicial objects and model categories. We then give a complete, elementary treatment of the model category structure on the category of simplicial (commutative) rings. As a sort of interlude, we also discuss differential graded rings (DGAs) and the functor from simplicial rings to DGAs, as well as some additional structures (divided powers) possessed by the DGAs in the essential image of this functor. Finally, we give an introduction to derived algebraic geometry.

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1. Model categories

In this section we recall the definition of a *model category*, following [Hov], with the modified definition of "functorial factorization" from the Errata to [Hov]. We review the basic notions of model categories: *lifting properties*, *cofibrant generation*, etc. We also discuss the notion of a *weak model category*, which I have introduced as a convenient formalism for *derived functors*, Ken Brown's Lemma, etc. We will use the formalism of model categories throughout the text, though our use of the actual *theory / machinery* of model categories will be rather limited, so one can largely take this whole section as a list of definitions.

1.1. **Definitions.** For a category $\mathbf{C}$, we let

$$\mathbf{Map}\,\mathbf{C} := \mathrm{Hom}_{\mathbf{Cat}}(\bullet \to \bullet, \mathbf{C})$$

denote the category of morphisms in $\mathbf{C}$, whose morphisms are commutative squares in $\mathbf{C}$. The category $\mathbf{Map}\,\mathbf{C}$ comes with two obvious functors

$$\mathrm{Dom} : \mathbf{Map}\,\mathbf{C} \to \mathbf{C}$$
$$\mathrm{Cod} : \mathbf{Map}\,\mathbf{C} \to \mathbf{C},$$

taking a morphism to its domain and codomain.

Definition 1.1.1. We say that a $\mathbf{C}$ morphism $f' : X' \to Y'$ is a *retract* of a $\mathbf{C}$ morphism $f : X \to Y$ iff f' is a retract of f as objects of $\mathbf{Map}\,\mathbf{C}$.

In other words, f' is a retract of f iff there is a commutative $\mathbf{C}$ diagram

$$\begin{tikzcd} X' \arrow[r, "i"] \arrow[d, "f'"'] & X \arrow[r, "p"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ X' \arrow[r, "j"] & X \arrow[r, "q"] & X' \end{tikzcd}$$

with $pi = \mathrm{Id}$ and $qj = \mathrm{Id}$. In such a diagram note that i, j are monic and p, q are epic.

Definition 1.1.2. A *functorial factorization* on a category $\mathbf{C}$ is an ordered pair (α, β) of endofunctors of $\mathbf{Map}\,\mathbf{C}$ with $\mathrm{Cod}\,\alpha = \mathrm{Dom}\,\beta$ satisfying $\beta(f)\alpha(f) = f$ for every $\mathbf{C}$ morphism f.

Definition 1.1.3. We say that a $\mathbf{C}$ morphism $f : X \to Y$ has the *left lifting property* (*LLP* for short) with respect to a $\mathbf{C}$ morphism $g : U \to V$ and that g has the *right lifting property* (*RLP*) with respect to f iff there is a completion as indicated in any solid (commutative) $\mathbf{C}$ diagram as below.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & U \arrow[d, "g"] \\ Y \arrow[r] \arrow[ur, dotted] & V \end{tikzcd}$$

Definition 1.1.4. A *model structure* on a category $\mathbf{C}$ (typically assumed to have all direct and inverse limits, or at least all finite direct and inverse limits) consists of two functorial factorizations (α, β), (γ, δ) and three subcategories called *weak equivalences*, *fibrations*, and *cofibrations* satisfying the following axioms:

(2-out-of-3) If f, g are composable $\mathbf{C}$-morphisms and two out of three of f, g, gf are weak equivalences, then so is the third.
(Retracts) Weak equivalences, fibrations, and cofibrations are closed under retracts.
(Lifting) Call a morphism a *trivial fibration* (resp. *trivial cofibration*) iff it is both a fibration (resp. cofibration) and a weak equivalence. Then trivial cofibrations have the LLP with respect to all fibrations and cofibrations have the LLP with respect to all trivial fibrations.
(Factorization) For every $\mathbf{C}$-morphism f, $\alpha(f)$ is a trivial cofibration, $\beta(f)$ is a fibration, $\gamma(f)$ is a cofibration, and $\delta(f)$ is a trivial fibration.

A *model category* is a category equipped with a model structure.

Definition 1.1.5. An object X of a model category $\mathbf{C}$ is called *fibrant* (resp. *cofibrant*) iff the map to the terminal object (resp. from the initial object) is a fibration (resp. cofibration).

Lemma 1.1.6. *Let $\mathbf{C}$ be a model category. A $\mathbf{C}$-morphism is a cofibration (resp. a trivial cofibration) iff it has the LLP property with respect to all trivial fibrations (resp. all fibrations). Similarly, a map is a fibration (resp. a trivial fibration) iff it has the RLP with respect to all trivial cofibrations (resp. all cofibrations).*

Proof. [Hov, 1.1.10] □

1.2. **Homotopy category.** Given a category $\mathbf{C}$ equipped with a subcategory of weak equivalences satisfying 2-out-of-3, one can form the *homotopy category* $\operatorname{Ho}\mathbf{C}$, modulo set-theoretic issues, characterized by the property: there is a functor $i_{\mathbf{C}} : \mathbf{C} \to \operatorname{Ho}\mathbf{C}$ taking weak equivalences to isomorphisms which is initial in the 2-category of categories under $\mathbf{C}$ where weak equivalences are taken to isomorphisms. In the case of a model category $\mathbf{C}$, one can show that the homotopy category $\operatorname{Ho}\mathbf{C}$ is equivalent to the category whose objects are objects of $\mathbf{C}$ which are both fibrant and cofibrant, where a morphism between such objects is an equivalence class of $\mathbf{C}$ morphisms under the equivalence relation of *homotopy* (which we will not define here). The relation of homotopy for maps of fibrant and cofibrant objects of $\mathbf{C}$ is stable under composition.

1.3. **Basic constructions.** One can make new model categories from old as in the following examples.

Example 1.3.1. If $\mathbf{C}$ is a model category, then $\mathbf{C}^{\mathrm{op}}$ has a natural model structure where a $\mathbf{C}^{\mathrm{op}}$-morphism $f : X \to Y$ is a cofibration (resp. fibration, weak equivalence) iff the corresponding $\mathbf{C}$-morphism $f : Y \to X$ is a fibration (resp. cofibration, weak equivalence). This is the sense in which the model category axioms are "self-dual" so that any theorem about model categories has a corresponding dual theorem obtained by applying the original theorem to the opposite model category.

Example 1.3.2. If $\mathbf{C}$ is a model category and X is an object of $\mathbf{C}$, then category $X/\mathbf{C}$ of objects of $\mathbf{C}$ under X admits an obvious forgetful functor

$$\begin{aligned} F : X/\mathbf{C} &\to \mathbf{C} \\ (X \to Y) &\mapsto Y. \end{aligned}$$

Declare an $X/\mathbf{C}$ morphism f to be a fibration (resp. cofibration, weak equivalence) iff Ff is a fibration (resp. cofibration, weak equivalence) in $\mathbf{C}$. Then $X/\mathbf{C}$ is a model category.

In [Hov, 1.1.8] this is stated and proved in the case where X is the terminal object, but there is no need to restrict to that case. Dually, the category $\mathbf{C}/X = X/\mathbf{C}^{\mathrm{op}}$ inherits a model structure from $\mathbf{C}$ in a similar manner.

1.4. **Properness.** The various concepts of *properness* are certain stability properties for weak equivalences which do not follow from the model category axioms, but are never-the-less enjoyed by many model categories and which play a role in the theory of homotopy limits (§1.8).

Definition 1.4.1. A model category $\mathbf{C}$ is called *right proper* iff weak equivalences are closed under base change along fibrations. That is, in any cartesian diagram

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "q"'] & X \arrow[d, "p"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

where p is a fibration and g is a weak equivalence, f is a weak equivalence. Dually, $\mathbf{C}$ is called *left proper* iff weak equivalences are closed under pushouts along cofibrations. We say $\mathbf{C}$ is *proper* iff it is both left and right proper.

One can often prove that a model category is left or right proper by using the following:

Lemma 1.4.2. (Reedy's Lemma) *In any model category, a pushout of a weak equivalence between cofibrant objects along a cofibration is a weak equivalence. Dually, a pullback of a weak equivalence between fibrant objects along a fibration is a weak equivalence.*

Proof. [Hir, 13.1.2] or [Red]. The proof uses, in an essential way, the "abstract homotopy theory of model categories," (c.f. [Hov, 1.2], [Hir, 7.3-7.9], or [Red]) which I do not want to explain in the present notes. An unfairly dismissive, though roughly correct, point of view would be that this result is just a collection of formal diagram lemmas motivated by the basic constructions of classical algebraic topology. □

Proposition 1.4.3. (Reedy) *If* $\mathbf{C}$ *is a model category where every object is cofibrant, then* $\mathbf{C}$ *is left proper. Dually, if every object of* $\mathbf{C}$ *is fibrant, then* $\mathbf{C}$ *is right proper.*

Proof. [Hir, 13.1.3]. This is immediate from Reedy's Lemma. □

For more on proper model categories, see [Hir, Ch. 13].

1.5. **Cofibrantly generated model categories.** Most model categories used in practice are constructed through the same "general nonsense" surrounding the Small Object Argument. Here we give a brief summary of the basic notions and results (originally due to Kan) following [Hov, 2.1].

Definition 1.5.1. Let $\mathbf{C}$ be a category, S a set of morphisms in $\mathbf{C}$. Let λ be an ordinal (isomorphism class of well-ordered sets), regarded as a category where there is a unique morphism $\alpha \to \beta$ iff $\alpha \leq \beta$. A λ-*sequence* in $\mathbf{C}$ (resp. in S) is a direct limit preserving functor $\{X_\alpha : \alpha \in \lambda\}$ from λ to $\mathbf{C}$ (resp. such that for each $\alpha \in \lambda$, the transition function $X_\alpha \to X_{\alpha+1}$ is in S). Given a λ-sequence $\{X_\alpha\}$ in S with direct limit X, the structure maps $X_\alpha \to X$ are called *transfinite compositions* of maps in S. Dually, if $\{X_\alpha : \alpha \in \lambda\}$ is an inverse limit preserving functor from λ^{op} to $\mathbf{C}$ with inverse limit X and structure maps

$X_{\alpha+1} \to X_\alpha$ in S, the structure maps $X \to X_\alpha$ are called *inverse transfinite compositions* of maps in S.

In particular, note that a composition of morphisms in S is both a transfinite composition and an inverse transfinite composition of morphisms in S (take $\lambda = \{0, 1, 2\}$). The basic reason for introducing the notion of a "transfinite composition" is the following:

Lemma 1.5.2. *Fix a* $\mathbf{C}$*-morphism* f*. Then the set of* $\mathbf{C}$*-morphisms with the LLP w.r.t.* f *is closed under retracts, pushouts, and transfinite compositions. Dually, the set of* $\mathbf{C}$*-morphisms with the RLP w.r.t.* f *is closed under retracts, pullbacks, and inverse transfinite compositions.*

Proof. This is a straightforward diagram exercise. □

Definition 1.5.3. Fix a cardinal κ. An object Y of $\mathbf{C}$ is called *κ-small relative to S* iff the natural map

$$\varinjlim \operatorname{Hom}_{\mathbf{C}}(Y, X_\alpha) = \operatorname{Hom}_{\mathbf{C}}(Y, \varinjlim X_\alpha)$$

is an isomorphism for each λ-sequence $\{X_\alpha\}$ in S where the cofinality of λ is larger than κ. The object Y is called *small relative to* S (or just *small* when S is the set of all $\mathbf{C}$-morphisms) iff Y is κ-small relative to S for some cardinal κ.

Definition 1.5.4. Let $\mathbf{C}$ be a category. A class of morphisms in $\mathbf{C}$ is called *saturated* iff it is closed under pushouts, transfinite compositions, and retracts. Let I be a class of morphisms of $\mathbf{C}$. Let *I-cell* denote the class of $\mathbf{C}$-morphisms expressible as a transfinite composition of pushouts of maps in I. Let *I-sat* denote the smallest saturated class of $\mathbf{C}$-morphisms containing I. Let *I-inj* (resp. *I-proj*) denote the class of $\mathbf{C}$-morphisms with the RLP (resp. LLP) w.r.t. every map in I. Set *I-cof* := (I-inj)-proj.

Fix a class of $\mathbf{C}$-morphisms I. From the definitions it is clear that $I \subseteq I$-cof. By Lemma 1.5.2, I-proj is closed under pushouts, retracts, and transfinite compositions. In particular, I-cof is closed under pushouts, retracts, and transfinite compositions, so we have I-cell $\subseteq I$-sat $\subseteq I$-cof.

Remark 1.5.5. The dual concepts "I-cocell" and "I-cosat" are, in practice, not particularly useful. See the discussion after [Hov, 2.1.18].

In the language of Definition 1.5.4, Lemma 1.1.6 says that

$$\begin{gathered}
\{ \text{ cofibrations } \} = \{ \text{ trivial fibrations } \}\text{-proj} \\
\{ \text{ trivial cofibrations } \} = \{ \text{ fibrations } \}\text{-proj} \\
\{ \text{ fibrations } \} = \{ \text{ trivial cofibrations } \}\text{-inj} \\
\{ \text{ trivial fibrations } \} = \{ \text{ cofibrations } \}\text{-inj}
\end{gathered}$$

and hence:

$$\begin{gathered}
\{ \text{ cofibrations } \} = \{ \text{ cofibrations } \}\text{-cof} \\
\{ \text{ trivial cofibrations } \} = \{ \text{ trivial cofibrations } \}\text{-cof.}
\end{gathered}$$

Proposition 1.5.6. *Let* $\mathbf{C}$ *be a model category. The cofibrations (resp. trivial cofibrations) in* $\mathbf{C}$ *are saturated—i.e. closed under retracts, pushouts, and transfinite compositions. Dually, the fibrations (resp. trivial fibrations) are cosaturated—i.e. closed under retracts, pullbacks, and inverse transfinite compositions.*

Proof. Combine Lemmas 1.5.2 and 1.1.6. □

Definition 1.5.7. Let $\mathbf{C}$ be a model category with all direct and inverse limits, I (resp. J) a set of cofibrations (resp. trivial cofibrations) in $\mathbf{C}$. We say that $\mathbf{C}$ is *cofibrantly generated* by I, J (or *with generating cofibrations* I and *generating trivial cofibrations* J) iff the following hold:

1. The domains of the maps in I are small relative to I-cell.
2. The domains of the maps in J are small relative to J-cell.
3. The class of fibrations is J-inj.
4. The class of trivial fibrations is I-inj.

Lemma 1.5.8. (Small Object Argument) *Suppose* $\mathbf{C}$ *is a category with all direct limits and* I *is a set*[1] *of* $\mathbf{C}$*-morphisms whose domains are small relative to* I*-cell. Then any* $\mathbf{C}$*-morphism* f *can be functorially factored as* $f = ip$ *where* $i \in I$*-cell and* $p \in I$*-inj.*

Proof. [Hov, 2.1.14] or [Hir, 10.5.16] □

Lemma 1.5.9. *Let* $\mathbf{C}$ *be a model category cofibrantly generated by the set* I *(resp.* J*) of generating cofibrations (resp. generating trivial cofibrations). Let* $f : X \to Y$ *be a cofibration (resp. trivial cofibration) in* $\mathbf{C}$*. Then we can find a* $\mathbf{C}$*-diagram*

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "f"] \\ Y \arrow[r, "s"] & Z \arrow[r, "p"] & Y \end{tikzcd}$$

with $ps = \mathrm{Id}$ *and* $i \in I$*-cell (resp.* $i \in J$*-cell). It follows that:*

$$\{ \text{ cofibrations } \} = I\text{-sat}$$
$$\{ \text{ trivial cofibrations } \} = J\text{-sat.}$$

Proof. By the Small Object Argument, we can factor f as $i : X \to Z$ followed by $p : Z \to Y$ where i is in I-cell (resp. J-cell) and p is in I-inj (resp. J-inj). In other words, p is a trivial fibration (resp. fibration) in our cofibrantly generated model category. By the Lifting axiom for our model category we can lift as indicated in

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, dotted, "s"] & Y \end{tikzcd}$$

to build the desired retract diagram. For the "It follows that:" We have the containments $\supseteq$ because the sets on the left are saturated by Proposition 1.5.6, and we have the containments $\subseteq$ because the first part of the lemma says any cofibration (resp. trivial cofibration) is a retract of a map in I-cell (resp. J-cell), hence is in I-sat (resp. J-sat). □

The introduction of all this convoluted terminology is justified by the following:

[1]We really mean "set" here and not "class" because the proof involves various direct limit constructions "over I."

Theorem 1.5.10. *Suppose* $\mathbf{C}$ *is a category with all direct and inverse limits,* W *is a subcategory of* $\mathbf{C}$*, and* I *and* J *are sets of maps in* $\mathbf{C}$*. Then* $\mathbf{C}$ *admits a model category structure cofibrantly generated by* I*,* J *with* W *as the subcategory of weak equivalences iff the following hold:*

(CG1) W *satisfies "2-out-of-3" and is closed under retracts.*
(CG2) The domains of I *are small relative to* I*-cell.*
(CG3) The domains of J *are small relative to* J*-cell.*
(CG4) J*-sat* $\subseteq W \cap I$*-cof.*
(CG5) I*-inj* $\subseteq W \cap J$*-inj.*
(CG6) Either $W \cap I$*-cof* $\subseteq J$*-cof or* $W \cap J$*-inj* $\subseteq I$*-inj.*

Proof. [Hov, 2.1.19] or [Hir, 11.3.1] □

Remark 1.5.11. In most model categories the "smallness" hypotheses (CG2) and (CG3) hold trivially because all objects are (absolutely) small. The one notable exception is the category of topological spaces (§2), where not every object is small.

1.6. **Weak model categories.** If $\mathbf{C}$ is a model category and I is a category, then the category of functors $\mathbf{C}^I := \mathrm{Hom}_{\mathbf{Cat}}(I, \mathbf{C})$ may not have any natural model category structure.

Remark 1.6.1. Actually, in [Hir, 11.6.1], Hirschhorn constructs a model category structure on $\mathbf{C}^I$ under the assumption that the model category $\mathbf{C}$ is cofibrantly generated. The discussion in the present section is basically designed to circumvent the need to discuss the latter.

In any case, $\mathbf{C}^I$ inherits *most* of the model category axioms from $\mathbf{C}$ when we declare $f : X \to Y$ to be a weak equivalence (resp. fibration, cofibration) iff $f_i : X_i \to Y_i$ is a weak equivalence (resp. fibration, cofibration) in $\mathbf{C}$ for each object i of I. The only model category property that will be lost is the lifting property, because even if one can lift "at each object of i of I," there is no reason to expect these lifts to be natural in i. To formalize the amount of structure we retain, we introduce the following

Definition 1.6.2. A *weak model category* is a category $\mathbf{C}$ with all finite direct and inverse limits equipped with three subcategories called weak equivalences, cofibrations, and fibrations, and two functorial factorizations (α, β), (γ, δ) satisfying the following axioms:

(2-out-of-3) If f and g are composable $\mathbf{C}$ morphisms and two out of three of f, g, gf are weak equivalences, then so is the third.
(Retracts) Weak equivalences, cofibrations, and fibrations are closed under retracts.
(Factorization) For any morphism f, $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, and $\delta(f)$ is a fibration.
(Push-pull) Fibrations are closed under pullback and cofibrations are closed under pushout.

We can define *left proper* and *right proper* for weak model categories as in Definition 1.4.1.

Example 1.6.3. Any model category can be regarded as a weak model category. Proposition 1.5.6 yields the Push-pull axiom.

Example 1.6.4. If $\mathbf{C}$ is a weak model category and I is an arbitrary category, then $\mathbf{C}^I$ becomes a weak model category as discussed in the beginning of the section. This is straightforward to check. In particular, $\mathbf{Map}\,\mathbf{C}$ becomes a weak model category.

1.7. **Derived functors.** Here we briefly recall the construction of derived functors between homotopy categories of weak model categories.

Lemma 1.7.1. (Ken Brown) *Suppose* $\mathbf{C}$ *is a weak model category and* $\mathbf{D}$ *is a category with a subcategory of weak equivalences satisfying 2-out-of-3. Then any functor* $F : \mathbf{C} \to \mathbf{D}$ *that takes trivial cofibrations between cofibrant objects to weak equivalences takes* all *weak equivalences between cofibrant objects to weak equivalences.*

Proof. The usual proof [Hov, Lemma 1.1.12] of Ken Brown's Lemma when $\mathbf{C}$ is a model category goes through verbatim when $\mathbf{C}$ is merely a weak model category. $\square$

The reader should formulate the dual form of Ken Brown as an exercise.

Definition 1.7.2. A functor $F : \mathbf{C} \to \mathbf{D}$ between weak model categories is called a *left Quillen functor* iff F preserves finite direct limits, cofibrations, and trivial cofibrations. Similarly, F is called a *right Quillen functor* iff F preserves finite inverse limits, fibrations, and trivial fibrations.

Suppose now that $\mathbf{C}$ is a weak model category and $\mathbf{D}$ is a category equipped with a subcategory of *weak equivalences* satisfying 2-out-of-3. Suppose $F : \mathbf{C} \to \mathbf{D}$ is a functor taking weak equivalences between cofibrant objects to weak equivalences. For example, a left Quillen functor between weak model categories has this property in light of Ken Brown. Then, by the universal property of the homotopy category, F descends to a functor

$$\operatorname{Ho} F : \operatorname{Ho} \mathbf{C}_c \to \operatorname{Ho} \mathbf{D},$$

where $\mathbf{C}_c$ denotes the full subcategory of $\mathbf{C}$ consisting of cofibrant objects. Let $Q : \mathbf{C} \to \mathbf{C}_c$ denote the cofibrant replacement functor for $\mathbf{C}$. It follows easily from 2-out-of-3 that Q takes weak equivalences to weak equivalences, so Q descends to

$$\operatorname{Ho} Q : \operatorname{Ho} \mathbf{C} \to \operatorname{Ho} \mathbf{C}_c.$$

We define the *left derived functor*

$$\mathsf{L}F : \operatorname{Ho} \mathbf{C} \to \operatorname{Ho} \mathbf{D}$$

to be the composition: $\mathsf{L}F := (\operatorname{Ho} F)(\operatorname{Ho} Q)$. That is, we set $\mathsf{L}F(X) := FQ(X)$ for $X \in \mathbf{C}$.

The diagram

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "F"] \arrow[d, "i_{\mathbf{C}}"'] & \mathbf{D} \arrow[d, "i_{\mathbf{D}}"] \\
\operatorname{Ho} \mathbf{C} \arrow[r, "\mathsf{L}F"] & \operatorname{Ho} \mathbf{D}
\end{tikzcd}$$

does not commute, but there is a natural transformation $\eta : \mathsf{L}F i_{\mathbf{C}} \to i_{\mathbf{D}} F$ obtained by applying F to the natural trivial fibrations $QX \to X$. The pair $(\mathsf{L}F, \eta)$ is characterized up to unique isomorphism by noting that $(\mathsf{L}F, \eta)$ is a terminal object in the category whose objects are pairs (G, α) consisting of a functor $G : \operatorname{Ho} \mathbf{C} \to \operatorname{Ho} D$ and a natural transformation $\alpha : Gi_{\mathbf{C}} \to i_{\mathbf{D}}F$ and whose morphisms $(G, \alpha) \to (G', \alpha')$ are natural transformations $\zeta : G \to G'$ such that

$$\begin{tikzcd} Gi_{\mathbf{C}}X \arrow[rr, "\zeta_{i_{\mathbf{C}}X}"] \arrow[dr, "\alpha_X"'] & & G'i_{\mathbf{C}}X \arrow[dl, "\alpha'_X"] \\ & i_{\mathbf{D}}FX & \end{tikzcd}$$

commutes for every object X of $\mathbf{C}$. This is straightforward to prove. The upshot is that, up to unique isomorphism, the pair $(\mathsf{L}F, \eta)$ does not depend on the particular choice of a cofibrant replacement functor.

1.8. **Homotopy limits.** Let $\mathbf{C}$ be a category with pushouts, $f : X \to Y$ a $\mathbf{C}$-morphism. The pushout functor

$$Y \coprod_X \_ : X/\mathbf{C} \to Y/\mathbf{C} \tag{1.8.1}$$

is left adjoint to the "forgetful" functor

$$\begin{gathered} Y/\mathbf{C} \to X/\mathbf{C} \\ (g : Y \to Z) \mapsto (gf : X \to Z). \end{gathered}$$

Dually, if $\mathbf{C}$ has pullbacks, then the base change

$$X \times_Y \_ : \mathbf{C}/Y \to \mathbf{C}/X \tag{1.8.2}$$

is right adjoint to

$$\begin{gathered} \mathbf{C}/X \to \mathbf{C}/Y \\ (g : Z \to X) \mapsto (fg : Z \to Y). \end{gathered}$$

Now assume that $\mathbf{C}$ is a weak model category, so that $X/\mathbf{C}$, $\mathbf{C}/X$, etc. become weak model categories much as in Example 1.3.2. We would like to describe some circumstances in which we can form the left derived pushout and right derived pullback—we will concentrate on the former and let the reader formulate the dual statements. As we saw in the last section, to form the left derivative of (1.8.1), we need to know that whenever $a : X \to Z$ and $a' : X \to Z'$ are cofibrations in $\mathbf{C}$ and $g : Z \to Z'$ is a weak equivalence in $\mathbf{C}$ with $ga = a'$, the induced map

$$Y \coprod_X g : Y \coprod_X Z \to Y \coprod_X Z'$$

is a weak equivalence. In practice, we typically check the following condition (which implies the latter condition by Ken Brown) instead:

Lemma 1.8.1. *For a weak model category* $\mathbf{C}$*, the pushout functor* (1.8.1) *takes cofibrations to cofibrations. If trivial cofibrations in* $\mathbf{C}$ *are stable under pushout*[2]*, then* (1.8.1) *is a left Quillen functor.*

[2]Note that cofibrations are always stable under pushout by definition of a weak model category. This condition is a variant of "left proper" (Definition 1.4.1).

Proof. Suppose $a : X \to Z$ and $a' : X \to Z'$ are objects of $X/\mathbf{C}$ and $g : Z \to Z'$ is a cofibration in $X/\mathbf{C}$—i.e. a cofibration in $\mathbf{C}$ with $ga = a'$. The key point is that the $\mathbf{C}$ diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d] & Z' \arrow[d] \\ Y \coprod_X Z \arrow[r, "Y \coprod_X g"] & Y \coprod_X Z' \end{tikzcd}$$

is also a pushout, so $Y \coprod_X g$ is also a cofibration in $\mathbf{C}$ (hence in $Y/\mathbf{C}$) because cofibrations in a weak model category are stable under pushout. Similarly, if g is a trivial cofibration in $X/\mathbf{C}$ then the second hypothesis implies that $Y \coprod_X g$ is a trivial cofibration in $\mathbf{C}$ (hence in $Y/\mathbf{C}$). $\square$

When it can be formed, the left derived pushout will be denoted

$$Y \coprod_X^{\mathsf{L}} \_ : \mathrm{Ho}(X/\mathbf{C}) \to \mathrm{Ho}(Y/\mathbf{C}).$$

The right derived pullback will be denoted

$$X \times_Y^{\mathrm{R}} \_ : \mathrm{Ho}(\mathbf{C}/Y) \to \mathrm{Ho}(\mathbf{C}/X).$$

2. Topological spaces

Perhaps the best example of a model category is “the” model category of topological spaces **Top**. Actually, there are several different model category structures on **Top**, but only the so-called *Quillen model structure* of §2.1 will be used in the present notes. After giving the basic definitions and results concerning this model structure, we will discuss path spaces and cylinders (§2.3) and prove Whitehead’s Theorem (§2.4). Many of the formal arguments of these later sections will be recycled in our treatment of simplicial sets (§4) and, by extension, the entirety of these notes.

2.1. **Model structure.** Throughout, we let S^n, D^n, and I denote the n-dimensional sphere, the n-dimensional disk, and the closed interval $[0,1]$, respectively. Whenever we refer to a map of topological spaces $S^{n-1} \to D^n$, the map is understood to be the usual inclusion of the boundary. Similarly, $D^n \to D^n \times I$ is always understood to refer to the map $x \mapsto (x, 0)$.

Definition 2.1.1. Call a map $f : X \to Y$ of topological spaces a *weak equivalence* iff f induces a bijection $\pi_0(f)$ on path components and

$$\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism for all $x \in X$ and all $n > 0$.

Theorem 2.1.2. *The category* **Top** *of topological spaces admits a cofibrantly generated model category structure with the above weak equivalences and where*

$$I := \{S^{n-1} \hookrightarrow D^n : n \in \mathbb{N}\}$$
$$J := \{D^n \hookrightarrow D^n \times I : n \in \mathbb{N}\}$$

are the generating cofibrations and trivial cofibrations. Every topological space is fibrant.

(Apologies for the double usage of the notation “I.” Hovey avoids this mistake.)

Proof. [Hov, 2.4.19] (c.f. [Q1, II.3]). This is proved by appeal to the Recognition Theorem (1.5.10). Most of the effort goes into checking the containment $W \cap J\text{-inj} \subseteq I\text{-inj}$ (every Serre fibration which is also a weak equivalence has the RLP w.r.t. the maps $S^{n-1} \hookrightarrow D^n$) in (CG6) of that theorem. For the smallness hypotheses (CG2) and (CG3), we first note that the maps in I and J are injective, so by the second part of Lemma 2.1.3 below, I-sat and J-sat are contained in the class of injective maps. By the first part of Lemma 2.1.3, all topological spaces (in particular the domains of the maps in I and J) are small relative to injective maps. To see that every topological space is fibrant, one checks directly that the map to a point is in J-inj using the fact that each map in J has a retract. □

Lemma 2.1.3. *Every topological space is small relative to injective maps. The set of injective maps of topological spaces is saturated.*

Proof. The first part is [Hov, 2.4.1]. For the second statement, we reduce to the corresponding statement in **Sets** since all direct and inverse limits in **Top** commute with the forgetful functor to **Sets**. The statement in **Sets** is easy. □

Note that the fibrations (i.e. the maps J-inj) in this model structure on **Top** are the *Serre fibrations*: maps with the RLP with respect to the maps $D^n \hookrightarrow I \times D^n$.

Theorem 2.1.4. *The model category* **Top** *is proper (Definition 1.4.1).*

Proof. [Hir, 13.1.11]. Right proper is also [Hov, 2.4.18]. Both Hovey and Hirschhorn give a direct proof of the right properness, though it follows formally from Proposition 1.4.3 because every topological space is fibrant. For the left properness one uses the structure of cofibrations in **Top** from Lemma 1.5.9, to reduce to proving stability of weak equivalences under pushouts along $S^{n-1} \hookrightarrow D^n$. □

2.2. **Function spaces.** Recall that for topological spaces X and Y, the set of continuous maps $f : X \to Y$ can itself be given the structure of a topological space Y^X, called the *function space*, by giving it the compact-open topology. For a fixed X, the formation of the topological space Y^X is covariantly functorial in Y and we have:

Lemma 2.2.1. *If X is locally compact Hausdorff, then $(\_)^X$ is right adjoint to the product functor $\_ \times X$ so that we have a natural bijection*

$$\mathrm{Hom}_{\mathbf{Top}}(U \times X, Y) = \mathrm{Hom}_{\mathbf{Top}}(U, Y^X)$$

for any topological spaces U, Y. In particular, the "product with X" functor $\_ \times X$ preserves direct limits.

Proof. [Mun, 7.5.4] or [Hat, A.14] □

In other words, when X is locally compact Hausdorff, the topological space Y^X represents the functor

$$\begin{aligned} \mathbf{Top}^{\mathrm{op}} &\to \mathbf{Sets} \\ U &\mapsto \mathrm{Hom}_{\mathbf{Top}}(U \times X, Y). \end{aligned}$$

The right adjoint to a product functor in any category is often called the *function space*, *internal Hom* or *Weil restriction*, and is of interest in many different categories.

Later, in §4.6, we will discuss the right adjoint to the product functor in the category of simplicial sets—the so called *function complex*. We will also give the analogs, for simplicial sets, of the topological results of the next few sections. The reader can take this as motivation for the "topological digression" which constitutes the remainder of §2.

2.3. **Path spaces and cylinders.** Since the unit interval $I = [0, 1]$ is compact Hausdorff, Lemma 2.2.1 says that for any topological space Y, the *path space* Y^I of Y has the property that there is a natural bijection

$$\mathrm{Hom}_{\mathbf{Top}}(X, Y^I) = \mathrm{Hom}_{\mathbf{Top}}(X \times I, Y) \tag{2.3.1}$$

for any topological space X.

We have a natural map $i_Y : Y \to Y^I$ given by the *constant path*; i_Y corresponds to $\pi_1 : Y \times I \to Y$ under (2.3.1). The inclusions of the endpoints $0, 1$ of I induce two *evaluation morphisms* $e_0, e_1 : Y^I \to Y$ (we have made the canonical identifications $Y^{\{0\}} = Y^{\{1\}} = Y$) with

$$e_0 i_Y = e_1 i_Y = \mathrm{Id}_Y . \tag{2.3.2}$$

If $f, g : X \rightrightarrows Y$ are two maps of topological spaces, then a homotopy $H : X \times I \to Y$ from f to g is the same thing (via the bijection (2.3.1)) as a map of topological spaces $H : X \to Y^I$ such that $e_0 H = f$ and $e_1 H = g$.

The maps e_0 and e_1 are both deformation retracts of $i_Y : Y \to Y^I$. For example, to see that e_0 is a deformation retract of i_Y, consider the map

$$\begin{aligned} H : Y^I \times I &\to Y^I \\ H(\gamma, t)(s) &:= \begin{cases} \gamma(0), & s \leq t \\ \gamma(s-t), & s \geq t. \end{cases} \end{aligned}$$

This map satisfies

$$\begin{aligned} H|(Y^I \times \{0\}) &= \mathrm{Id} \\ H|(Y^I \times \{1\}) &= i_Y e_0 \end{aligned}$$

and sits in a commutative diagram as below.

(2.3.3)
$$\begin{tikzcd} Y \times I \arrow[r, "\pi_1"] \arrow[d, "i_Y \times I"'] & Y \arrow[d, "i_Y"] \\ Y^I \times I \arrow[r, "H"] \arrow[d, "e_0 \times I"'] & Y^I \arrow[d, "e_0"] \\ Y \times I \arrow[r, "\pi_1"] & Y \end{tikzcd}$$

In particular, i_X, e_0, and e_1 are all homotopy equivalences.

For a topological space X, we refer to the maps

$$\begin{aligned} e_0 \times e_1 : X^I &\to X \times X \\ (x \mapsto (x,0)) \coprod (x \mapsto (x,1)) : X \coprod X &\to X \times I \end{aligned}$$

as the *evaluation at the endpoints* and the the *boundary inclusion*, respectively.

Lemma 2.3.1. *For any topological space X, the constant path $i_X : X \to X^I$ followed by the evaluation at the endpoints $e_0 \times e_1 : X^I \to X \times X$ factors the diagonal map $\Delta : X \to X \times X$ as a homotopy equivalence followed by a (Serre) fibration. (In fact, one can lift as indicated in any solid diagram*

(2.3.4)
$$\begin{tikzcd} Y \times \{0\} \arrow[r, "F"] \arrow[d] & X^I \arrow[d, "e_0 \times e_1"] \\ Y \times I \arrow[r, "H"] \arrow[ur, dotted, "l"] & X \times X \end{tikzcd}$$

where Y is locally compact Hausdorff, hence in particular when $Y = D^n$ is a disk.)

Dually,[3] *for any cofibrant topological space X, the boundary inclusion $X \coprod X \hookrightarrow X \times I$ followed by the projection $\pi_1 : X \times I \to X$ factors the codiagonal (fold) map $X \coprod X \to X$ as a cofibration followed by a homotopy equivalence (which is also a fibration).*

[3]To think of this as dual to the first statement, note that all topological spaces are fibrant (Theorem 2.1.2). We wish to emphasize the analogy with the corresponding statement for simplicial sets (Lemma 4.7.5).

Proof. Define the topological space J as the pushout

$$\begin{tikzcd} \{0\} \times \{0,1\} \arrow[r] \arrow[d] & I \times \{0,1\} \arrow[d] \\ \{0\} \times I \arrow[r] & J \end{tikzcd}$$

so J consists of three sides of the unit square $I \times I$. The inclusion $J \hookrightarrow I \times I$ obviously has a retract $r : I \times I \to J$ (in fact a deformation retract).

Since Y is locally compact Hausdorff, taking products with Y preserves direct limits (Lemma 2.2.1), hence

$$\begin{tikzcd} Y \times \{0\} \times \{0,1\} \arrow[r] \arrow[d] & Y \times I \times \{0,1\} \arrow[d] \\ Y \times \{0\} \times I \arrow[r] & Y \times J \end{tikzcd}$$

is a pushout diagram in **Top**. If, in (2.3.4), we view F as a map $F : Y \times \{0\} \times I \to X$ and H as a map $H : Y \times I \times \{0,1\} \to X$, then the commutativity of the solid square says that F and H agree on $Y \times \{0\} \times \{0,1\}$, so they correspond via the pushout square above to a map

$$F \cup H : Y \times J \to X$$

and the lift l is the same thing as a map $l : Y \times I \times I \to X$ restricting to $F \cup H$ on $Y \times J$. We can easily find such an l using our retract r by setting

$$\begin{aligned} l : Y \times I \times I &\to X \\ (y, s, t) &\mapsto (F \cup H)(r(s,t)). \end{aligned}$$

The dual statement requires a little more work. Since X is cofibrant and the model category structure on **Top** is cofibrantly generated with

$$I = \{S^{n-1} \hookrightarrow D^n : n \in \mathbb{N}\}$$

as the set of generating cofibrations (Theorem 2.1.2), we can find a diagram of spaces

$$X \xrightarrow{s} Z \xrightarrow{p} X$$

with $ps = \mathrm{Id}_X$ where $\emptyset \to Z$ is in I-cell (Lemma 1.5.9). This diagram yields a diagram

$$\begin{tikzcd} X \coprod X \arrow[r, "s \coprod s"] \arrow[d] & Z \coprod Z \arrow[r, "p \coprod p"] \arrow[d] & X \coprod \arrow[d] \\ X \times I \arrow[r, "s \times \mathrm{Id}"] & Z \times I \arrow[r, "p \times \mathrm{Id}"] & X \times I \end{tikzcd}$$

displaying the boundary inclusion map for X as a retract of the boundary inclusion map for Z. Since cofibrations are closed under retracts, we thus reduce to proving the second statement under the assumption that $\emptyset \hookrightarrow X$ is in I-cell. We begin with the following:

Claim: Suppose we have a pushout diagram of topological spaces as below.

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "a"'] & D^n \arrow[d, "b"] \\ X \arrow[r] & Y \end{tikzcd}$$

Then, given any solid commutative diagram of topological spaces

$$\begin{tikzcd} Y \coprod Y \arrow[r, "f \coprod g"] \arrow[d] & S \arrow[d] \\ Y \times I \arrow[r, "H"] \arrow[ur, dotted, "L'"] & T \end{tikzcd} \tag{2.3.5}$$

where $S \to T$ is a trivial fibration and any commutative diagram

$$\begin{tikzcd} X \coprod X \arrow[r, "f|X \coprod g|X"] \arrow[d] & S \arrow[d] \\ X \times I \arrow[r, "H|X\times I"] \arrow[ur, "L"] & T, \end{tikzcd}$$

we can extend L to a lift L' as indicated in (2.3.5).

To prove the claim, consider the diagram

$$\begin{tikzcd} (D^n \coprod D^n) \cup (S^{n-1} \times I) \arrow[r] \arrow[d, "j"'] & S \arrow[d] \\ D^n \times I \arrow[r, "H(b\times I)"] \arrow[ur, dotted, "K"] & T \end{tikzcd}$$

where j is the obvious inclusion and the top horizontal arrow is

$$(fb \coprod gb) \cup H(a \times I).$$

This map j is nothing but the inclusion $S^n \hookrightarrow D^{n+1}$, so we can lift as indicated. Since product with I preserves pushouts, we have

$$\begin{aligned} Y \times I &= (X \coprod_{S^{n-1}} D^n) \times I \\ &= (X \times I) \coprod_{S^{n-1}\times I} (D^n \times I). \end{aligned}$$

Using this description of $Y \times I$, the map $L' := L \cup K$ will do the job. The claim is proven.

Now suppose λ is an ordinal and $\{X_\alpha : \alpha \in \lambda\}$ is a λ-sequence in **Top** such that $X_0 = \emptyset$ and each map $X_\alpha \to X_{\alpha+1}$ ($\alpha \in \lambda$) is a pushout of a map in I. We need to show that the direct limit X_λ has the property that

$$X_\lambda \coprod X_\lambda \to X_\lambda \times I$$

is a cofibration. By transfinite induction, it will suffice to prove the following statement: Given any $\gamma \in \lambda$, any commutative diagram of topological spaces

$$\begin{tikzcd} X_\gamma \coprod X_\gamma \arrow[r, "f \coprod g"] \arrow[d] & S \arrow[d] \\ X_\gamma \times I \arrow[r, "H"] \arrow[ur, dotted, "L_\gamma"] & T \end{tikzcd} \tag{2.3.6}$$

with $S \to T$ a trivial fibration, and any set of maps

$$\{L_\alpha : X_\alpha \times I \to S : \alpha < \gamma\}$$

compatible with the structure maps $X_\alpha \to X_\beta$ for $\alpha < \beta < \gamma$ and making the diagrams

$$\begin{tikzcd} X_\alpha \coprod X_\alpha \arrow[r, "f|X_\alpha \coprod g|X_\alpha"] \arrow[d] & S \arrow[d] \\ X_\alpha \times I \arrow[r, "H|X_\alpha \times I"] \arrow[ur, dotted, "L_\alpha"] & T, \end{tikzcd}$$

commute for $\alpha < \gamma$, there is a lift L_γ as indicated in (2.3.6) compatible with the L_α and the structure maps $X_\alpha \to X_\gamma$. This statement is trivial when $\gamma = 0$ since $X_0 = \emptyset$. When γ is a limit ordinal, we can simply take

$$L_\gamma := \varinjlim \{L_\alpha : \alpha < \gamma\}$$

using the fact that

$$L_\gamma \times I = \varinjlim \{L_\alpha \times I : \alpha < \gamma\}$$

because $\{X_\alpha\}$ is a λ-sequence (direct limit preserving functor) and product with I preserves direct limits. When $\gamma = \alpha + 1$ is a successor, the result follows from the Claim applied with $X \to Y$ given by $X_\alpha \to X_\gamma = X_{\alpha+1}$.

The projection map $\pi_1 : X \times I \to X$ is a fibration since all topological spaces are fibrant and fibrations are stable under base change. $\square$

Lemma 2.3.2. *Suppose $f : X \to Y$ is a map of topological spaces. Let $i_Y : Y \to Y^I$ denote the constant path and $e_0, e_1 : Y^I \to Y$ the evaluations. Then we have a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "(\mathrm{Id}_X, i_Y f)"] \arrow[dr, "f"'] & & X \times_Y^{e_0} Y^I \arrow[dl, "e_1 \pi_2"] \\ & Y & \end{tikzcd}$$

where $(\mathrm{Id}_X, i_Y f)$ is a homotopy equivalence (in fact it admits a deformation retract) and $e_1\pi_2$ is a (Serre) fibration.

Dually, if $f : X \to Y$ is a map of cofibrant *topological spaces, then we have a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "x \mapsto (x,1)"] \arrow[dr, "f"'] & & Y \coprod_X (X \times I) \arrow[dl, "\mathrm{Id} \cup f\pi_1"] \\ & Y & \end{tikzcd}$$

where $x \mapsto (x,1)$ is a cofibration and $\mathrm{Id} \cup f\pi_1$ is a homotopy equivalence. (The pushout over X is defined using the maps f and $x \mapsto (x,0)$.)

Proof. The diagram commutes because $e_1 i_Y = \mathrm{Id}_Y$. To see that $(\mathrm{Id}_X, i_Y f)$ admits a deformation retract, take the base change of (2.3.3) along f. To see that $e_1\pi_2$ is a fibration, notice that we have a cartesian diagram:

$$\begin{tikzcd} X \times^{e_0}_Y Y^I \arrow[r] \arrow[d, "\mathrm{Id} \times e_1"'] & Y^I \arrow[d, "e_0 \times e_1"] \\ X \times Y \arrow[r, "f \times \mathrm{Id}"] & Y \times Y \end{tikzcd}$$

The left vertical arrow $\mathrm{Id} \times e_1$ is a fibration because $e_0 \times e_1$ is a fibration (Lemma 2.3.1) and fibrations are stable under base change. The map $e_1\pi_2$ is the composition of $\mathrm{Id} \times e_1$ and the projection $p_2 : X \times Y \to Y$. Since a composition of fibrations is a fibration, $e_1\pi_2$ is a fibration provided p_2 is a fibration; this is the case because p_2 is a base change of the map from X to a point, which is a fibration (all topological spaces are fibrant).

For the dual statement, we have a pushout diagram:

$$\begin{tikzcd} X \coprod X \arrow[r, "f \times \mathrm{Id}"] \arrow[d] & Y \coprod X \arrow[d, "j"] \\ X \times I \arrow[r] & Y \coprod_X (X \times I) \end{tikzcd}$$

Since X is cofibrant, the left vertical arrow (the boundary inclusion for X) is a cofibration by Lemma 2.3.1, hence the map labelled j is a cofibration. The map $x \mapsto (x,1)$ is the composition of the structure map $X \to Y \coprod X$ (which is a cofibration since Y is cofibrant) and j. □

2.4. **Whitehead's Theorem.**

There are many theorems in algebraic topology that might be called "Whitehead's Theorem." Here is a typical one:

Theorem 2.4.1. (Whitehead) *In the model category* **Top***:*

(1) Any trivial cofibration admits a deformation retract.
(2) Any trivial fibration $p : Z \to Y$ between cofibrant spaces Z, Y is a homotopy equivalence. More precisely, any such p has a section s such that sp and Id_Z are homotopic via a homotopy $J : Z \times I \to Z$ sitting in a commutative diagram as below.

$$\begin{tikzcd} Z \times I \arrow[r, "J"] \arrow[d, "\pi_1"'] & Z \arrow[d, "p"] \\ Z \arrow[r, "p"] & Y \end{tikzcd}$$

(3) Any weak equivalence between cofibrant spaces is a homotopy equivalence.

Proof. (1): Suppose $j : X \to Z$ is a trivial cofibration. By Lemma 2.3.2 the map $e_1\pi_2$ in the diagram

$$\begin{tikzcd} X \arrow[r, "(\mathrm{Id}, i_Z j)"] \arrow[d, "j"'] & X \times_Z^{e_0} Z^I \arrow[d, "e_1\pi_2"] \\ Z \arrow[r, equal] \arrow[ur, dotted, "(r,H)"] & Z \end{tikzcd}$$

is a fibration, so we can lift as indicated by the Lifting axiom. This (r, H) can be viewed as the desired deformation retract, with $rj = \mathrm{Id}_X$ and the abusively denoted map $H : Z \times I \to Z$ corresponding to H under the path space adjunction (2.3.1) providing the homotopy rel X between jr and the identity of Z.

(2): Since Y and Z are cofibrant, the map $z \mapsto (z, 1)$ in the diagram

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, "z \mapsto (z,1)"'] & Z \arrow[d, "p"] \\ Y \coprod_Z (Z \times I) \arrow[r, "\mathrm{Id} \cup p\pi_1"] \arrow[ur, dotted, "s \cup J"] & Y \end{tikzcd}$$

is a cofibration by Lemma 2.3.2, so we can lift as indicated. This s and J can be interpreted as the desired section and homotopy.

(3): By either part of the Factorization axiom and 2-out-of-3 for weak equivalences, we can factor an arbitrary weak equivalence $f : X \to Y$ of cofibrant objects as a trivial cofibration $j : X \to Z$ followed by a trivial fibration $p : Z \to Y$. Since j is a cofibration and X is cofibrant, Z is also cofibrant. Now we combine (1) (applied to j) and (2) (applied to p) for the desired result: The homotopy $p*H*s$ provides a homotopy between $pjrs = frs$ and $\mathrm{Id}_Y = ps$. The homotopy $r * J * j$ provides a homotopy between $rspj = rsf$ and $\mathrm{Id}_X = rj$. We conclude that f is a homotopy equivalence with homotopy inverse $rs : Y \to X$. $\square$

3. Simplicial objects

3.1. **The simplicial category Δ.** Fora nonnegative integer n, let $[n] := \{0, \dots, n\}$. Let Δ denote the category whose objects are the finite sets $[0], [1], \dots$ and where a Δ-morphism $[m] \to [n]$ is a nondecreasing function. For each $i \in [n]$, let $\partial_n^i : [n-1] \to [n]$ be the unique Δ-morphism whose image is $[n] \setminus \{i\}$. That is:

$$\begin{aligned} \partial_n^i : [n-1] &\to [n] \\ j &\mapsto \begin{cases} j, & j < i \\ j+1, & j \geq i. \end{cases} \end{aligned}$$

For each $i \in [n]$, let $\sigma_n^i : [n+1] \to [n]$ be the unique surjective Δ-morphism with $(\sigma_n^i)^{-1}(i) = \{i, i+1\}$. That is:

$$\begin{aligned} \sigma_n^i : [n+1] &\to [n] \\ j &\mapsto \begin{cases} j, & j \leq i \\ j-1, & j > i. \end{cases} \end{aligned}$$

We will denote morphisms in Δ by the Greek letters $\rho, \sigma, \tau, \dots$.

Lemma 3.1.1. *The maps $\partial_n^i : [n-1] \to [n]$ and $\sigma_n^i : [n+1] \to [n]$ defined above satisfy the following relations:*

$$\begin{aligned} \partial_{n+1}^j \partial_n^i &= \partial_{n+1}^i \partial_n^{j-1} \ i < j \\ \sigma_{n-1}^j \sigma_n^i &= \sigma_{n-1}^i \sigma_n^{j+1} \ i \leq j \\ \sigma_{n-1}^j \partial_n^i &= \partial_{n-1}^i \sigma_{n-2}^{j-1} \ i < j \\ &= \mathrm{Id}_{n-1} \qquad j \leq i \leq j+1 \\ &= \partial_{n-1}^{i-1} \sigma_{n-2}^j \ j+1 < i \end{aligned}$$

Every morphism $\sigma : [n] \to [m]$ in Δ factors uniquely as a composition

$$\sigma = \partial_m^{i_1} \cdots \partial_{m-s+1}^{i_s} \sigma_{n-t+1}^{j_1} \cdots \sigma_{n-2}^{j_{t-1}} \sigma_{n-1}^{j_t}$$

with $0 \leq i_s < \dots < i_1 \leq m$ and $0 \leq j_1 < \dots < j_t < n$. The morphisms ∂_n^i and σ_n^i generate the category Δ and the relations above generate all relations between them.

Proof. It is straightforward to check the claimed relations. For the second part, take the i_k to be the elements of $[m]$ not in the image of σ, listed in decreasing order. Take $j_1 < \dots < j_t$ to be the elements j of $[n]$ with $\sigma(j) = \sigma(j+1)$. The final statement can be obtained from the second, or see [GZ, Page 24]. □

3.2. **Simplicial objects.** Let $\mathbf{C}$ be a category. A *simplicial object* in $\mathbf{C}$ is a contravariant functor from the simplicial category Δ to $\mathbf{C}$. Simplicial objects in $\mathbf{C}$ form a category

$$\mathbf{sC} := \mathrm{Hom}_{\mathbf{Cat}}(\Delta^{\mathrm{op}}, \mathbf{C}),$$

which evidently contains $\mathbf{C}$ as a full subcategory by identifying $\mathbf{C}$ with the constant simplicial objects in $\mathbf{sC}$. For $X \in \mathbf{sC}$, we will write X_n for $X([n])$, $d_n^i : X_n \to X_{n-1}$ for $X(\partial_n^i)$. The maps d_n^i are called the (basic) *boundary maps* of X. Similarly, we will write $s_n^i : X_n \to X_{n+1}$ as abuse of notation for $X(\sigma_n^i)$. The maps s_n^i are called (basic) *degeneracy maps* of X.

One motivation for considering simplicial objects stems from Čech theory in topology; this is thoroughly discussed in [Con, §1-2]. Another motivation is that, as we will see later (§5), simplicial objects in an abelian category are essentially the same thing as chain complexes, so we can use simplicial objects in an arbitrary category as a "non-abelian" analog of chain complexes which will allow us to do "homological algebra." Quillen [Q1] calls this "homotopical algebra." For example, in §7, we will be particularly interested in simplicial rings and various non-abelian analogs of the usual homological concepts: injective/projective objects, resolutions, quasi-isomorphisms, derived functors, and so forth.

Example 3.2.1. Any object X of $\mathbf{C}$ determines a *constant simplicial object* in $\mathbf{sC}$ (also denoted X) via the constant functor $[n] \mapsto \mathbf{C}$ which takes every Δ^{op} morphism to Id_X.

More generally, for any surjective Δ morphism $\sigma : [m] \to [n]$, the map $X(\sigma) : X_n \to X_m$ is often called a degeneracy map. Notice that every surjective Δ morphism σ has a section (any set-theoretic section of a Δ morphism is again a Δ morphism), so every degeneracy map $X(\sigma)$ of a simplicial object has a retract; in particular, the degeneracy maps are monomorphisms in $\mathbf{C}$. Similarly, for any injective Δ morphism σ, we will often call $X(\sigma)$ a boundary map. Any injective Δ morphism has a retract, hence any boundary map has a section in $\mathbf{C}$, hence is, in particular, an epimorphism.

Lemma 3.2.2. *To give a simplicial object X in $\mathbf{C}$ it is equivalent to give objects X_n of $\mathbf{C}$, together with boundary and degeneracy maps*

$$\begin{aligned} d_n^i : X_n &\to X_{n-1}, \quad i = 0, \dots, n \\ s_n^i : X_n &\to X_{n+1}, \quad i = 0, \dots, n \end{aligned}$$

satisfying the simplicial relations

$$\begin{aligned} d_{n-1}^i d_n^j &= d_{n-1}^{j-1} d_n^i, & i < j \\ s_{n+1}^i s_n^j &= s_{n+1}^{j+1} s_n^i, & i \leq j \\ d_n^i s_{n-1}^j &= s_{n-2}^{j-1} d_{n-1}^i, & i < j \\ &= \mathrm{Id}_{n-1}, & j \leq i \leq j+1 \\ &= s_{n-2}^j d_{n-1}^{i-1}, & j+1 < i. \end{aligned}$$

Proof. This follows from Lemma 3.1.1. □

Example 3.2.3. Suppressing subscripts, let us spell out explicitly all the simplicial relations among the face and boundary maps between X_2, X_1, and X_0. First we have the following equalities of maps $X_2 \to X_0$:

$$d^0 d^0 = d^0 d^1 \tag{3.2.1}$$

$$d^1 d^0 = d^0 d^2 \tag{3.2.2}$$

$$d^1 d^1 = d^1 d^2 \tag{3.2.3}$$

We next have

$$s^0 s^0 = s^1 s^0 : X_0 \to X_2 \tag{3.2.4}$$

and finally the relations between the face and boundary maps are:

$$d^1 s^0 = \mathrm{Id} : X_0 \to X_0 \tag{3.2.5}$$
$$d^0 s^0 = \mathrm{Id} : X_0 \to X_0 \tag{3.2.6}$$
$$d^2 s^0 = s^0 d^1 : X_1 \to X_1 \tag{3.2.7}$$
$$d^2 s^1 = \mathrm{Id} : X_1 \to X_1 \tag{3.2.8}$$
$$d^1 s^1 = \mathrm{Id} : X_1 \to X_1 \tag{3.2.9}$$
$$d^0 s^1 = s^0 d^0 : X_1 \to X_1. \tag{3.2.10}$$

The category **sC**, like any functor category, inherits various properties from **C**. For instance, if **C** has all (inverse, say) limits indexed by a category **D**, then so does **sC**, and the inverse limit of a functor $F : \mathbf{D} \to \mathbf{sC}$ is given by in the obvious way:

$$(\varprojlim F)_n = \varprojlim_{D \in \mathbf{D}} F(D)_n.$$

We can also define *cosimplicial objects* in a category **C** to be functors $X : \Delta \to \mathbf{C}$. The category **csC** of cosimplicial objects in **C** arises less frequently than **sC**, but plays an important role in *geometric realization* (§4.3).

Example 3.2.4. Let $X \times G \to X$ be a right action of a group object G (in some category **C**—really a monoid object will suffice) on an object X of **C**. To this action, one can functorially associate a simplicial object $X_\bullet$ of **C** with

$$\begin{aligned} X_0 &= X \\ X_1 &= X \times G \\ X_2 &= X \times G \times G \\ &\vdots \\ X_n &= X \times G^n. \end{aligned}$$

The boundary maps $d_n^i : X_n \to X_{n-1}$ for $X_\bullet$ are given by

$$\begin{aligned} d_n^0(x, g_1, \dots, g_n) &:= (xg_1, g_2, \dots, g_n) \\ d_n^1(x, g_1, \dots, g_n) &:= (x, g_1 g_2, g_3, \dots, g_n) \\ &\vdots \\ d_n^{n-1}(x, g_1, \dots, g_n) &:= (x, g_1, \dots, g_{n-2}, g_{n-1} g_n) \\ d_n^n(x, g_1, \dots, g_n) &:= (x, g_1, \dots, g_{n-1}) \end{aligned}$$

and the face maps $s_n^i : X_n \to X_{n+1}$ are given by

$$\begin{aligned} s_n^0(x, g_1, \dots, g_n) &:= (x, 1, g_1, \dots, g_n) \\ s_n^1(x, g_1, \dots, g_n) &:= (x, g_1, 1, g_2, \dots, g_n) \\ &\vdots \\ s_n^n(x, g_1, \dots, g_n) &:= (x, g_1, \dots, g_n, 1). \end{aligned}$$

Supressing subscripts, these maps are given for small n by:

$$\begin{aligned} d^0(x,g) &= xg \\ d^1(x,g) &= x \\ d^0(x,g,h) &= (xg,h) \\ d^1(x,g,h) &= (x,gh) \\ d^2(x,g,h) &= (x,g) \\ s^0(x) &= (x,1) \\ s^0(x,g) &= (x,1,g) \\ s^1(x,g) &= (x,g,1). \end{aligned}$$

The relations among the low order boundary maps as in in Example 3.2.3 are the following:

$$d^0d^0 = d^0d^1 : (x,g,h) \mapsto xgh \tag{3.2.11}$$

$$d^1d^0 = d^0d^2 : (x,g,h) \mapsto xg \tag{3.2.12}$$

$$d^1d^1 = d^1d^2 : (x,g,h) \mapsto x \tag{3.2.13}$$

The low order degeneracy map relations are

$$s^0s^0 = s^1s^0 : x \mapsto (x,1,1) \tag{3.2.14}$$

and finally the relations between the face and boundary maps are:

$$d^1s^0 = \mathrm{Id} : X \to X \tag{3.2.15}$$

$$d^0s^0 = \mathrm{Id} : X \to X \tag{3.2.16}$$

$$d^2s^0 = s^0d^1 : (x,g) \mapsto (x,1) \tag{3.2.17}$$

$$d^2s^1 = \mathrm{Id} : X_1 \to X_1 \tag{3.2.18}$$

$$d^1s^1 = \mathrm{Id} : X_1 \to X_1 \tag{3.2.19}$$

$$d^0s^1 = s^0d^0 : (x,g) \mapsto (xg,1). \tag{3.2.20}$$

The equality (3.2.11) uses the compatibility of the action with multiplication for G and the equality (3.2.16) uses the fact that “the identity of G acts trivially on X”. All of the other equalities above use only the fact that G is a group object and have nothing to do with the fact that $X \times G \to X$ is an action.

3.3. **Examples of cosimplicial objects.** In this section we give some examples of cosimplicial objects.

For a non-negative integer n, let $\Delta_n^\top \subseteq \mathbb{R}^{n+1}$ denote the convex hull of the standard basis vectors $e_0, \dots, e_n$:

$$\Delta_n^\top := \{t_0e_0 + \cdots + t_ne_n \in \mathbb{R}^{n+1} : t_i \in [0,1], \sum t_i = 1\}.$$

A morphism $\sigma : [m] \to [n]$ in the simplicial category Δ induces a morphism of topological spaces

$$\begin{aligned} \Delta^\top(\sigma) : \Delta_m^\top &\to \Delta_n^\top \\ (t_0e_0 + \cdots + t_me_m) &\mapsto (\sum_{i\in\sigma^{-1}(0)} t_ie_0 + \cdots + \sum_{i\in\sigma^{-1}(n)} t_ie_n), \end{aligned}$$

so we can regard $\Delta^\top$ as a cosimplicial topological space called the *standard (topological) simplex*.

The algebraic analog of the standard topological simplex is the following: For a nonnegative integer n, let A_n denote the ring

$$A_n := \mathbb{Z}[t_0, \dots, t_n]/\langle 1 - t_0 - t_1 - \cdots - t_n \rangle.$$

A Δ-morphism $\sigma : [m] \to [n]$ induces a ring homomorphism

$$\begin{aligned} A(\sigma) : A_n &\to A_m \\ t_i &\mapsto \sum_{j \in \sigma^{-1}(i)} t_j, \end{aligned}$$

so that the A_n form a simplicial ring A, and hence the $\Delta_n^{\mathrm{alg}} := \operatorname{Spec} A_n$ form a cosimplicial scheme Δ^{alg} called the *standard (algebraic) simplex.*

3.4. **Bisimplicial objects.** We can also consider "simplicial simplicial objects" in a category $\mathbf{C}$. We have natural isomorphisms

$$\begin{aligned} \mathbf{ssC} &= \operatorname{Hom}(\Delta^{\mathrm{op}}, \operatorname{Hom}(\Delta^{\mathrm{op}}, \mathbf{C})) \\ &= \operatorname{Hom}(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathbf{C}). \end{aligned}$$

We will always view the category of "simplicial simplicial objects" as the category

$$\mathbf{ssC} = \operatorname{Hom}(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathbf{C}),$$

which we will also call the category of *bisimplicial objects of* $\mathbf{C}$. The diagonal inclusion

$$\begin{aligned} \Delta^{\mathrm{op}} &\to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \\ [n] &\mapsto ([n], [n]) \end{aligned}$$

induces, via restriction of functors, a functor

$$\Delta : \mathbf{ssC} \to \mathbf{sC}.$$

For a bisimplicial object $X \in \mathbf{ssC}$, the simplicial object $\Delta(X) \in \mathbf{sC}$ is called the *diagonal* of X. The diagonal construction will make an appearance in §5.8.

3.5. **Truncations and (co)skeleta.** For a nonnegative integer n, let Δ_n denote the full subcategory of Δ whose objects are $[0], \dots, [n]$. An *n-truncated simplicial object* in $\mathbf{C}$ is a functor $\Delta_n^{\mathrm{op}} \to \mathbf{C}$. Let

$$\mathbf{s}_n\mathbf{C} := \operatorname{Hom}_{\mathbf{Cat}}(\Delta_n^{\mathrm{op}}, \mathbf{C})$$

be the category of n-truncated simplicial objects in $\mathbf{C}$. The inclusion $\Delta_n^{\mathrm{op}} \hookrightarrow \Delta^{\mathrm{op}}$ induces a restriction functor

$$\begin{aligned} \operatorname{tr}_n : \mathbf{sC} &\to \mathbf{s}_n\mathbf{C} \\ X &\mapsto X|_{\Delta_n^{\mathrm{op}}}, \end{aligned}$$

which we call the *n-truncation.* For any $m \geq n$ we have analogous functor

$$\operatorname{tr}_n : \mathbf{s}_m\mathbf{C} \to \mathbf{s}_n\mathbf{C}.$$

Given an n-truncated simplicial object $Y \in \mathbf{s}_n\mathbf{C}$ and a simplicial object $X \in \mathbf{sC}$ (or a truncated simplicial object $X \in \mathbf{s}_m\mathbf{C}$ for some $m \geq n$) we say that X is a *lift of* Y if $\operatorname{tr}_n X = Y$. We have similar notions of *lifting* morphisms of n-truncated simplicial objects.

Let Δ^{mon} (resp. Δ_n^{mon}) denote the subcategory of Δ (resp. Δ_n) with the same objects, but where a morphism $[m] \to [k]$ is also required to be monic. Define Δ^{epi} and Δ_n^{epi} similarly.

Assume $\mathbf{C}$ has finite inverse limits. By the Kan Extension Theorem [Mac, X.3], the truncation $\operatorname{tr}_n : \mathbf{sC} \to \mathbf{s}_n\mathbf{C}$ admits a right adjoint

$$\operatorname{Cosk}_n : \mathbf{s}_n\mathbf{C} \to \mathbf{sC}$$

called the n^{th} *coskeleton* functor, defined by

$$\begin{aligned}(\operatorname{Cosk}_n X)_m &:= \varprojlim \{X_k : ([m] \to [k]) \in [m] \downarrow \Delta_n^{\text{op}}\} \\ &= \varprojlim \{X_k : ([k] \to [m]) \in \Delta_n \downarrow [m]\} \\ &= \varprojlim \{X_k : ([k] \to [m]) \in \Delta_n^{\text{mon}} \downarrow [m]\}.\end{aligned}$$

The first equality is the usual "formula" for Kan extension [Mac, X.3.1]; the second is a translation of it. To be clear, we emphasize that the inverse limit is the inverse limit of the functor

$$\begin{aligned}(\Delta_n \downarrow [m])^{\text{op}} &\to \mathbf{C} \\ ([k] \to [m]) &\mapsto X_k.\end{aligned}$$

The third equality is by cofinality of $\Delta_n^{\text{mon}} \downarrow [m]$ in $\Delta_n \downarrow [m]$: every object $([k] \to [m])$ in $\Delta_n \downarrow [m]$ admits a map to an object $([l] \to [m])$ where $[l] \to [m]$ is monic.

In particular, notice that $X_m = (\operatorname{Cosk}_n X)_m$ for $m \leq n$. Indeed, when $m \leq n$, $\operatorname{Id} : [m] \to [m]$ is the terminal object of $\Delta_n \downarrow m$.

The adjunction isomorphism

$$\operatorname{Hom}_{\mathbf{s}_n\mathbf{C}}(\operatorname{tr}_n X, Y) \to \operatorname{Hom}_{\mathbf{sC}}(X, \operatorname{Cosk}_n Y)$$

takes a map $f : \operatorname{tr}_n X \to Y$ to the map $X \to \operatorname{Cosk}_n Y$ given in degree m by the map $X_m \to (\operatorname{Cosk}_n Y)_m$ obtained as the inverse limit of the maps

$$f_k X(\sigma) : X_m \to Y_k$$

over all $\sigma : [k] \to [m]$ in $\Delta_n \downarrow [m]$.

For $X \in \mathbf{sC}$, we use the abusive notation

$$\operatorname{Cosk}_n X := \operatorname{Cosk}_n(\operatorname{tr}_n X).$$

For $m \leq n$, we have a natural map

$$\operatorname{Cosk}_m \to \operatorname{Cosk}_n \tag{3.5.1}$$

corresponding to the natural isomorphism

$$\operatorname{tr}_n \operatorname{Cosk}_m X = \operatorname{tr}_n X$$

under the adjunction isomorphism

$$\operatorname{Hom}_{\mathbf{sC}}(\operatorname{Cosk}_m X, \operatorname{Cosk}_n X) = \operatorname{Hom}_{\mathbf{s}_n\mathbf{C}}(\operatorname{tr}_n \operatorname{Cosk}_m X, \operatorname{tr}_n X).$$

Since the maps (3.5.1) are isomorphisms in degrees $\leq n$, we have

$$X = \varprojlim \{\operatorname{Cosk}_n X : n \in \mathbb{N}\} \tag{3.5.2}$$

(the inverse limit is computed degreewise).

Dually, if $\mathbf{C}$ has finite direct limits, then the Kan Extension Theorem implies that truncation tr_n admits a left adjoint

$$\operatorname{Sk}_n : \mathbf{s}_n\mathbf{C} \to \mathbf{sC}$$

called the n^{th} *skeleton* functor, given on $X \in \mathbf{s}_n\mathbf{C}$ by the formula

$$\begin{aligned}(\operatorname{Sk}_n X)_m &= \varinjlim \{X_k : ([m] \to [k]) \in [m] \downarrow \Delta_n\} \\ &= \varinjlim \{X_k : ([m] \to [k]) \in [m] \downarrow \Delta_n^{\text{epi}}\}.\end{aligned}$$

The structure map

$$(\operatorname{Sk}_n X)(\sigma) : (\operatorname{Sk}_n X)_m \to (\operatorname{Sk}_n X)_l$$

induced by a Δ morphism $\sigma : [l] \to [m]$ is the map on direct limits induced by the following commutative diagram of functors:

$$\begin{tikzcd} ([m] \downarrow \Delta_n)^{\text{op}} \arrow[rr, "\tau \mapsto \sigma^* \tau"] \arrow[dr, "(\tau:[m]\to[k])\mapsto X_k"'] & & ([l] \downarrow \Delta_n)^{\text{op}} \arrow[dl, "([l]\to[k])\mapsto X_k"] \\ & \mathbf{C} & \end{tikzcd}$$

As with the coskeleton functor, we use the abusive notation

$$\operatorname{Sk}_n X := \operatorname{Sk}_n \operatorname{tr}_n X$$

for $X \in \mathbf{sC}$. Reasoning much as with the coskeleton functor, we see that there are natural maps

$$\operatorname{Sk}_0 X \to \operatorname{Sk}_1 X \to \cdots \to X$$

and that

$$X = \varinjlim \{\operatorname{Sk}_n X : n \in \mathbb{N}\}. \tag{3.5.3}$$

Example 3.5.1. Note that $\mathbf{s}_0\mathbf{C}$ is naturally isomorphic to $\mathbf{C}$, so we can regard Sk_0 and Cosk_0 as functors $\mathbf{C} \to \mathbf{sC}$. With this understanding, $\operatorname{Sk}_0 X$ is the constant simplicial object and $\operatorname{Cosk}_0 X$ is the simplicial object with $(\operatorname{Cosk}_0 X)_n = X^{[n]}$ (the $n+1$ fold product of X in $\mathbf{C}$) whose boundary maps are the natural projections $X^{[n]} \to X^{[n-1]}$ "forgetting one coordinate" and whose degeneracy maps $X^{[n]} \to X^{[n+1]}$ are the diagonals given by "repeating one coordinate".

3.6. **Latching and matching objects.** The latching and matching objects we are about to define arise naturally in many inductive constructions with simplicial objects. The matching objects also arise in the theory of fibrations of simplicial sets (§**??**), and hence in many other model categories where fibrations are defined in terms of an underlying simplicial set.

Fix $k, n \in \mathbb{N}$ with $k \leq n$ We call the categories

$$\begin{aligned}\mathsf{L}_n^k &:= [n] \downarrow \Delta_k^{\text{epi}} \\ \mathsf{M}_n &:= \Delta_k^{\text{mono}} \downarrow [n]\end{aligned}$$

the (k,n)-*latching* and -*matching* categories, respectively. Explicitly, an object of L_n^k is a surjective Δ-morphism $\sigma : [n] \to [m]$ with $m \leq k$ and a morphism τ in L_n^k is a commutative triangle

$$\begin{tikzcd} & {[n]} \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\ {[m]} \arrow[rr, "\tau"] & & {[m']} \end{tikzcd}$$

of surjective Δ-morphisms. Similarly, an object of M_n^k is an injective Δ-morphism $\sigma : [m] \hookrightarrow [n]$ with $m \leq k$ and a morphism τ in M_n^k is a commutative triangle

$$\begin{tikzcd} {[m]} \arrow[rr, "\tau"] \arrow[dr, "\sigma"'] & & {[m']} \arrow[dl, "\sigma'"] \\ & {[n]} & \end{tikzcd}$$

of injective Δ-morphisms. The case where $k = n-1$ is especially common/useful; we will simply call

$$\begin{aligned} \mathsf{L}_n &:= \mathsf{L}_n^{n-1} \\ \mathsf{M}_n &:= \mathsf{M}_n^{n-1} \end{aligned}$$

the n^{th} *latching* and *matching* categories.

If X is a simplicial object in a category $\mathbf{C}$ with finite direct limits, then the (k, n) *latching object* object of X is defined by

$$\begin{aligned} \mathsf{L}_n^k(X) &:= \varinjlim \{X_m : (\sigma : [n] \to [m]) \in \mathsf{L}_n^k\} \\ &= (\operatorname{Sk}_k X)_n. \end{aligned}$$

Notice that $\mathsf{L}_n^n(X) = X_n$ (the identity $[n] \to [n]$ is the initial object in L_n^n) and $\mathsf{L}_n^0(X) = \coprod_{[n]} X_0$ (the category L_n^0 is just the set $\operatorname{Hom}_\Delta([0], [n]) = [n]$). When $k = n-1$, we simply call

$$\mathsf{L}_n(X) := \mathsf{L}_n^{n-1}(X)$$

the n^{th} *latching object* of X. Notice that $\mathsf{L}_n^k(X)$ depends only on the restriction of X to Δ_k^{epi}—that is, it depends only on the degeneracies of $\operatorname{tr}_k X$. By restriction, we obtain natural maps

$$\coprod_{[n]} X_0 = \mathsf{L}_n^0(X) \to \mathsf{L}_n^1(X) \to \cdots \to \mathsf{L}_n^{n-1}(X) \to \mathsf{L}_n^n(X) = X_n$$

which one may think of as successively better approximations of X_n

Similarly, if $\mathbf{C}$ has finite inverse limits, the (k, n) *matching object* of X is the object of $\mathbf{C}$ defined by

$$\begin{aligned} \mathsf{M}_n^k(X) &:= \varprojlim \{X_m : (\sigma : [m] \to [n]) \in \mathsf{M}_n^k\} \\ &= (\operatorname{Cosk}_k X)_n. \end{aligned}$$

We have $\mathsf{M}_n^k(X) = X_n$ and $\mathsf{M}_n^0(X) = X_0^{[n]}$. Notice that $\mathsf{M}_n^k(X)$ depends only on the boundary maps for $\operatorname{tr}_k X$. When $k = n-1$, we simply call

$$\mathsf{M}_n(X) := \mathsf{M}_n^{n-1}(X)$$

the n^{th} *matching object* of X. Restriction yields maps

$$X_n = \mathsf{M}_n^n(X) \to \mathsf{M}_n^{n-1}(X) \to \cdots \to \mathsf{M}_n^1(X) \to \mathsf{M}_n^0(X) = X_0^{[n]}.$$

In practice it isn't really so important that the limits used to define latching and matching objects actually *exist* in $\mathbf{C}$ (be representable), since we really just use them as shorthand for their universal properties, so to speak. They just provide a way of packaging certain finite diagrams of $\mathbf{C}$-morphisms.

The morphisms

$$\mathsf{L}_n(X) \to X_n \to \mathsf{M}_n(X) \tag{3.6.1}$$

can often be described quite explicitly.

For example, we can use the simplicial relations (Lemma 3.2.2) to give the following explicit descriptions of the latching and matching objects associated to a simplicial set X:

$$\begin{aligned} \mathsf{L}_n(X) &= \left(\coprod_{i \in [n-1]} (X_{n-1})_i \right) / \sim \\ \mathsf{M}_n(X) &= \{(x_0, \dots, x_n) \in X_{n-1}^{[n]} : d^i_{n-1} x_j = d^{j-1}_{n-1} x_i \, \forall \, i < j\}. \end{aligned} \tag{3.6.2}$$

In the formula for $\mathsf{L}_n(X)$, the subscript i in $(X_{n-1})_i$ is used to distinguish between the different copies of of X_{n-1} and $\sim$ is the smallest equivalence relation such that $s^j_{n-2}(y) \in (X_{n-1})_i$ is equivalent to $s^i_{n-2}(y) \in (X_{n-1})_{j+1}$ for each $y \in X_{n-2}$ and each $0 \leq i \leq j \leq n-2$.

In terms of the formulas (3.6.2), the morphism $\mathsf{L}_n(X) \to X_n$ in (3.6.1) is given by taking the equivalence class of $x \in (X_{n-1})_i$ to $s^i_{n-1}(x) \in X_n$. The morphism $X_n \to \mathsf{M}_n(X)$ is given by

$$x \mapsto (d^0_n x, \dots, d^n_n x).$$

Notice that these maps are well-defined in light of the simplicial relations (Lemma 3.2.2).

The formulas (3.6.2) (or appropriate variations) will of course be valid in many other categories where there is some notion of "underlying set" so that, say, the set underlying a finite inverse limit is the inverse limit of the underlying sets. For example, the formula (3.6.2) for $\mathsf{L}_n(X)$ is equally valid in simplicial groups, simplicial rings, etc. For an abelian category $\mathbf{A}$ and a simplicial object $A \in \mathbf{sA}$, the object $\mathsf{L}_n(A)$ of $\mathbf{A}$ is given by the cokernel of an appropriate map

$$\bigoplus_{0 \leq i \leq j \leq n-2} (A_{j-2})_{ij} \to \bigoplus_{i \in [n-1]} (A_{n-1})_i,$$

though in practice it is generally easier to work directly with the direct limit description.

The adjunction morphism $\mathrm{Sk}_n(X) \to X$ need not be injective:

Example 3.6.1. Consider the simplicial set X with $X_0 = \{x\}$ and $X_n = \{x, y\}$ for all $n > 0$. The map $s^0_0 : X_0 \to X_1$ takes x to x and for any Δ-morphism $\sigma : [m] \to [n]$ with $m, n \geq 1$, the map $X(\sigma)$ is the identity. The map $\mathrm{Sk}_1(X) \to X$ will not be an isomorphism in degree 2. Note that $\mathrm{Sk}_1(X)_2 = \mathsf{L}_2(X)$ is the second latching object of X, and the degree two part of $\mathrm{Sk}_1(X) \to X$ is the natural map $\mathsf{L}_2(X) \to X_2$ mentioned in (3.6.1). Using the explicit description of $\mathsf{L}_2(X)$ in (3.6.2), we see that $\mathsf{L}_2(X) = (X_1 \coprod X_1)/\sim$, where $\sim$ simply identifies x in the first copy of X_1 with x in the second copy of X_1. In particular, $\mathsf{L}_2(X)$ has three elements, while X_2 only has two elements.

Example 3.6.2. In Example 4.1.2 we will consider, for a fixed $N \in \mathbb{N}$, the simplicial set $\Delta[N]$ with

$$\Delta[N]_n = \mathrm{Hom}_\Delta([n], [N]).$$

For a Δ-morphism $\sigma : [m] \to [n]$, the structure map $\Delta[N](\sigma) : \Delta[N]_n \to \Delta[N]_m$ is σ^*. Using the factorization statements in Lemma 3.2.2, one can show that

$$\mathsf{L}^k_n(\Delta[N]) \to \Delta[N]_n$$

is the inclusion of the subset of those $\tau \in \Delta[N]_n = \operatorname{Hom}_\Delta([n],[N])$ where $|\operatorname{Im}\tau| \leq k$. Indeed, $\operatorname{Sk}_k \Delta[N]$ is the sub simplicial set of $\Delta[N]$ consisting of the Δ-morphisms to $[N]$ with image of cardinality at most k.

3.7. **Simplicial homotopy.** Let $f, g : X \to Y$ be $\mathbf{sC}$ morphisms. A *homotopy* h from f to g consists of a $\mathbf{C}$ morphism $h(\phi) : X_n \to Y_n$, defined for each morphism $\phi : [n] \to [1]$ of Δ, satisfying the conditions:

(1) For any commutative triangle
$$\begin{tikzcd} {[m]} \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & {[n]} \arrow[dl, "\theta"] \\ & {[1]} & \end{tikzcd}$$
in Δ, the square
$$\begin{tikzcd} X_n \arrow[r, "h(\theta)"] \arrow[d, "X(\psi)"'] & Y_n \arrow[d, "Y(\psi)"] \\ X_m \arrow[r, "h(\phi)"] & Y_m \end{tikzcd}$$
commutes in $\mathbf{C}$.

(2) For any n, if $\phi : [n] \to [1]$ is the constant map to $0 \in [1]$, then $h(\phi) = f_n$ and if ϕ is the constant map to $1 \in [1]$, then $h(\phi) = g_n$.

The relation consisting of pairs
$$(f, g) \in \operatorname{Hom}_{\mathbf{sC}}(C, D) \times \operatorname{Hom}_{\mathbf{sC}}(C, D)$$
defined by "there is a homotopy from f to g" is not generally an equivalence relation. It is clearly reflexive, but not generally symmetric or transitive. However, if there is a homotopy h from f to g, and $k, l : D \to E$ are $\mathbf{sC}$ morphisms with a homotopy h' from k to l, then there is a homotopy from kf to lg, obtained by composing h and h':
$$(\phi : [n] \to [1]) \mapsto (h'(\phi)h(\phi) : C_n \to E_n).$$

We let $\sim$ be the equivalence relation generated by this relation and we say f is *homotopic* to g if $f \sim g$. In other words, $f \sim g$ means there is a finite string of morphisms $f = f_0, f_1, \dots, f_n = g$ such that, for every i, either there is a homotopy from f_i to f_{i+1} or there is a homotopy from f_{i+1} to f_i.

An $\mathbf{sC}$ morphism $f : C \to D$ is a *homotopy equivalence* if there is an $\mathbf{sC}$ morphism $g : D \to C$ such that $fg \sim \operatorname{Id}_D$ and $gf \sim \operatorname{Id}_C$.

Remark 3.7.1. More generally, for any $X, Y \in \mathbf{sC}$, there is a naturally associated simplicial set $\underline{\operatorname{Hom}}(X, Y)$ with $\underline{\operatorname{Hom}}(X_\bullet, Y_\bullet)_n$ given by the set of functions h assigning to every Δ-morphism $\sigma : [m] \to [n]$, a morphism $h(\sigma) : X_m \to Y_m$ in such a way that for any commutative triangle
$$\begin{tikzcd} {[m]} \arrow[rr, "\tau"] \arrow[dr, "\sigma"'] & & {[m']} \arrow[dl, "\sigma'"] \\ & {[n]} & \end{tikzcd}$$

in Δ (i.e. any morphism in $\Delta/[n]$), the square

$$\begin{array}{ccc} X_{m'} & \xrightarrow{h(\sigma')} & Y_{m'} \\ {\scriptstyle X(\tau)}\downarrow & & \downarrow{\scriptstyle Y(\tau)} \\ X_m & \xrightarrow{h(\sigma)} & Y_m \end{array}$$

commutes in $\mathbf{C}$. For a map $\phi : [n] \to [n']$ the simplicial set structure map

$$\underline{\mathrm{Hom}}(X,Y)(\phi) : \underline{\mathrm{Hom}}(X,Y)_{n'} \to \underline{\mathrm{Hom}}(X,Y)_n$$

is defined by taking $h \in \underline{\mathrm{Hom}}(X,Y)_{n'}$ to the function $\phi^* h \in \underline{\mathrm{Hom}}(X,Y)_n$ assigning $h(\phi\psi) : X_m \to Y_m$ to a Δ-morphism $\psi : [m] \to [n']$.

Notice that $\Delta/[0] = \Delta$ and $\underline{\mathrm{Hom}}(X,Y)_0 = \mathrm{Hom}_{\mathbf{sC}}(X,Y)$ (a moment's thought shows that the commutativity condition above amounts to the assertion that $h_n := h([n] \to [0])$ is a natural transformation).

In this language, a homotopy from f to g is an element of $\underline{\mathrm{Hom}}(X,Y)_1$ mapping to

$$f, g \in \underline{\mathrm{Hom}}(X,Y)_0 = \mathrm{Hom}_{\mathbf{sC}}(X,Y)$$

under the maps

$$\underline{\mathrm{Hom}}(X,Y)_1 \rightrightarrows \underline{\mathrm{Hom}}(X,Y)_0$$

induced by the two injective Δ morphisms $[0] \rightrightarrows [1]$. We will revisit this again in §**??** for simplicial sets, where it is a little more concrete.

Remark 3.7.2. Sometimes one encounters the following definition of a homotopy from f to g: There are morphisms $h^i_n : X_n \to Y_{n+1}$, for $i = 0, \dots, n$, satisfying the following conditions:

(1) $d^0_{n+1} h^0_n = f_n$ and $d^{n+1}_{n+1} h^n_n = g_n$

(2) $d^i_{n+1} h^j_n = \begin{cases} h^{j-1}_{n+1} d^i_n & i < j \\ d^i_{n+1} h^{i-1}_n & i = j \neq 0 \\ h^j_{n+1} d^{i-1}_n & i > j+1 \end{cases}$

(3) $s^i_{n+1} h^j_n = \begin{cases} h^{j+1}_{n+1} s^i_n & i \leq j \\ h^j_{n+1} s^{i-1}_n & i > j \end{cases}$

To see the equivalence of the two definitions, observe that the maps $[n] \to [1]$ in Δ are the unique maps α^n_i ($i = 0, \dots, n+1$) with $\alpha_i^{-1}(0) = \{0, \dots, i-1\}$. To go from the first definition of homotopy to the second, set

$$h^i_n := u_{\alpha^{n+1}_i} s^i_n : X_n \to Y_{n+1} \quad i = 0, \dots, n.$$

To go from the second definition to the first, set

$$u_{\alpha^n_i} := d^i_{n+1} h^i_n : X_n \to Y_n \quad i = 0, \dots, n+1.$$

One can check by drawing diagrams that the two sets of conditions are exchanged.

Once we have a notion of (simplicial) homotopy, we can define many of the usual homotopy notions from topology. For example, if $i : X \to Y$ is an $\mathbf{sC}$ morphism with a retract $r : Y \to X$, then we say that r is a *deformation retract* of i iff there is a homotopy from $\mathrm{Id} : Y \to Y$ to $ir : Y \to Y$ restricting to the identity on X.

The following example of a simplicial homotopy arises "in nature" and is useful in topology.

Lemma 3.7.3. *Suppose* $\mathbf{C}$ *has products and* $X \in \mathbf{C}$ *is such that the unique map* $f : X \to \mathbf{1}$ *to the terminal object (empty product) admits section* s *(i.e. there is some map* $\mathbf{1} \to X$*). Then*

$$\operatorname{Cosk}_0 f : \operatorname{Cosk}_0 X \to \mathbf{1}$$

is a homotopy equivalence in $\mathbf{sC}$ *with homotopy inverse*

$$\operatorname{Cosk}_0 s : \mathbf{1} \to \operatorname{Cosk}_0 X.$$

Proof. Certainly $(\operatorname{Cosk}_0 f)(\operatorname{Cosk}_0 s) = \operatorname{Cosk}_0(fs)$ is the identity of $\operatorname{Cosk}_0 \mathbf{1} = \mathbf{1}$ because Cosk_0 is a functor and $fs = \operatorname{Id}_{\mathbf{1}}$ by definition of "section". We need to show that

$$F := (\operatorname{Cosk}_0 s)(\operatorname{Cosk}_0 f) = \operatorname{Cosk}_0(sf)$$

is homotopic to the identity of $\operatorname{Cosk}_0 X$. The map F is given by

$$\begin{aligned} F_n : X^{n+1} &\to X^{n+1} \\ (x_0, \dots, x_n) &\mapsto (sf(x_0), \dots, sf(x_n)). \end{aligned}$$

We define a homotopy to the identity by associating to $\phi : [n] \to [1]$ the "straight line homotopy" map

$$\begin{aligned} h(\phi) : X^{n+1} &\to X^{n+1} \\ h(\phi)(x_0, \dots, x_n)_i &:= \begin{cases} sf(x_i), & \phi(i) = 0 \\ x_i, & \phi(i) = 1. \end{cases} \end{aligned}$$

When ϕ is the constant map onto $0 \in [1]$, obviously $h(\phi) = F_n$ and when ϕ is the constant map to $1 \in [1]$ obviously $h(\phi)$ is the identity. To finish the proof that h is the desired homotopy, we must show that, for any commutative square

$$\begin{tikzcd} {[m]} \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & {[n]} \arrow[dl, "\theta"] \\ & {[1]} & \end{tikzcd}$$

in Δ, the square

$$\begin{tikzcd} X^{n+1} \arrow[r, "h(\theta)"] \arrow[d, "\psi^*"'] & X^{n+1} \arrow[d, "\psi^*"] \\ X^{m+1} \arrow[r, "h(\phi)"] & X^{m+1} \end{tikzcd}$$

commutes in $\mathbf{C}$.

Here ψ^* is short for $(\operatorname{Cosk}_0 X)(\psi)$. It is given by $\psi^*(x_0,\dots,x_n)_i = x_{\psi(i)}$. The commutativity we want is now a simple computation:

$$\begin{aligned}
(\psi^* h(\theta))(x_0,\dots,x_n)_i &= \begin{cases} sf(x_{\psi(i)}), & \theta(\psi(i)) = 0 \\ x_{\psi(i)}, & \theta(\psi(i)) = 1 \end{cases} \\
&= \begin{cases} sf(x_{\psi(i)}), & \phi(i) = 0 \\ x_{\psi(i)}, & \phi(i) = 1 \end{cases} \\
&= \begin{cases} sf(\psi^*(x_0,\dots,x_n))_i, & \phi(i) = 0 \\ \psi^*(x_0,\dots,x_n)_i, & \phi(i) = 1 \end{cases} \\
&= (h(\phi)\psi^*)(x_0,\dots,x_n)_i.
\end{aligned}$$

$\square$

4. Simplicial sets

In this section we study the category **sSets** of simplicial sets. Much of the theory of simplicial objects in an arbitrary category is based on the theory of simplicial sets.

For a simplicial set X, we refer to the elements of X_n as *simplicies* of X of *dimension* n. Fix a simplex $x \in X_n$. For any injective Δ morphism $\sigma : [k] \to [n]$, the simplex $X(\sigma)(x) \in X_k$ is called a *face* of x. For any surjective Δ morphism $\sigma : [k] \to [n]$, the simplex $X(\sigma)(x) \in X_k$ is called a *degeneracy* of x. Recall that the degeneracy map $X(\sigma)$ corresponding to a Δ surjection σ always has a retract, so in particular it is injective. The simplex x is called *non-degenerate* if it is a degeneracy only of itself. It is easy to see that any simplex $x \in X_m$ is a degeneracy of a unique non-degenerate simplex y: just choose l minimal such that there is a Δ surjection $\sigma : [m] \to [l]$ and a $y \in X_l$ with $X(\sigma)(y) = x$.

4.1. **Examples.** Here we mention several examples of simplicial sets. The n-simplex of Example 4.1.2 and its cousins in the examples that follow will be used throughout our study of simplicial sets.

Example 4.1.1. (Simplicial complexes) An easy way to describe simplicial sets is in terms of the simplicial complexes one may be familiar with from topology (c.f. [Hat], for example). Recall that a *simplicial complex* consists of a set X of finite sets such that for any $A \in X$ and any $B \subseteq A$, we have $B \in X$. Any element of any set belonging to X is called a *vertex* of X. An *ordered simplicial set* is a simplicial set X together with a total ordering of the set of vertices of X.

Given an ordered simplicial complex X, we define a simplicial set $X^{\mathrm{ss}} \in \mathbf{sSets}$ by letting X^{ss}_n be the set of nondecreasing sequences $(x_0, \dots, x_n)$ of vertices, such that $\{x_0, \dots, x_n\} \in X$. Note that there may be repeats in the sequence. For a Δ-morphism $\sigma : [m] \to [n]$, we define

$$\begin{aligned} X^{\mathrm{ss}}(\sigma) : X^{\mathrm{ss}}_n &\to X^{\mathrm{ss}}_m \\ (x_0, \dots, x_n) &\mapsto (x_{\sigma(0)}, \dots, x_{\sigma(m)}). \end{aligned}$$

This is well-defined because σ is non-decreasing and because X is closed under passage to subsets. The *nondegenerate* n simplices of X^{ss} are the strictly increasing sequences $(x_0, \dots, x_n)$; such simplices are in bijective correspondence with the sets in X of cardinality $n+1$.

Example 4.1.2. (Standard simplices) For each $n \in \mathbb{N}$ we have the *n-simplex* $\Delta[n] \in$ **sSets** defined by setting

$$\Delta[n]_m := \{(i_0, \dots, i_m) : 0 \le i_0 \le \cdots \le i_m \le n\}.$$

The boundary maps for $\Delta[n]$ are given by

$$\begin{aligned} d^j_m : \Delta[n]_m &\to \Delta[n]_{m-1} \\ (i_0, \dots, i_m) &\mapsto (i_0, \dots, i_{j-1}, i_{j+1}, \dots, i_m) \end{aligned}$$

and the basic degeneracy maps for $\Delta[n]$ are given by

$$\begin{aligned} s^j_m : \Delta[n]_m &\to \Delta[n]_{m+1} \\ (i_0, \dots, i_m) &\mapsto (i_0, \dots, i_{j-1}, i_j, i_j, i_{j+1}, \dots, i_m). \end{aligned}$$

It is easy to check that for any $X \in \mathbf{sSets}$, the map

$$\begin{aligned} \mathrm{Hom}_{\mathbf{sSets}}(\Delta[n], X) &\mapsto X_n \\ f &\mapsto f_n((0, 1, \ldots, n)) \end{aligned}$$

is bijective.

In the language of Example 4.1.1, the n-simplex $\Delta[n]$ is the simplicial set associated to the ordered simplicial complex X consisting of all subsets of $[n]$, with the obvious ordering on $X^0 = [n]$.

Notice that a sequence $(i_0, \ldots, i_m) \in \Delta[n]_m$ may be viewed as a Δ morphism

$$\begin{aligned} [m] &\to [n] \\ j &\mapsto i_j. \end{aligned}$$

Indeed, we could *alternatively define* $\Delta[n]$ by setting

$$\Delta[n]_m := \mathrm{Hom}_\Delta([m], [n]).$$

For a Δ-morphism $\sigma : [m] \to [m']$, we can define the corresponding structure map for $\Delta[n]$ by the formula

$$\begin{aligned} \Delta[n](\sigma) : \mathrm{Hom}_\Delta([m'], [m]) &\to \mathrm{Hom}_\Delta([m], [n]) \\ f &\mapsto f\sigma. \end{aligned}$$

It is clear from this description that a Δ-morphism $\sigma : [m] \to [n]$ gives rise to map of simplicial sets $\Delta[\sigma] : \Delta[m] \to \Delta[n]$ in a functorial manner, so that $\Delta[\bullet]$ forms a cosimplicial simplicial set, called the *standard cosimplicial simplicial set.*

It is worth emphasizing that the formula

$$\begin{aligned} \mathrm{Hom}_{\mathbf{sSets}}(\Delta[m], \Delta[n]) &= \Delta[n]_m \\ &= \mathrm{Hom}_\Delta([m], [n]) \end{aligned} \tag{4.1.1}$$

shows that the standard cosimplicial simplicial set

$$\Delta[\bullet] : \Delta \to \mathbf{sSets}$$

is a fully faithful functor, so that we can regard the simplicial category Δ as a full subcategory of $\mathbf{sSets}$ via the standard cosimplicial simplicial set.

One sees immediately from the description of maps out of $\Delta[n]$ that the functor

$$\begin{aligned} \mathbf{Sets} &\to \mathbf{sSets} \\ X &\mapsto \coprod_X \Delta[n] \end{aligned}$$

is left adjoint to the forgetful functor

$$\begin{aligned} \mathbf{sSets} &\to \mathbf{Sets} \\ Y &\mapsto Y_n. \end{aligned}$$

That is, we have

$$\mathrm{Hom}_{\mathbf{sSets}}(\coprod_W \Delta[n], X) = \mathrm{Hom}_{\mathbf{Sets}}(W, X_n). \tag{4.1.2}$$

Example 4.1.3. (Boundary of the n-simplex) For each $n \in \mathbb{N}$ we also have the *boundary* $\partial\Delta[n]$ of $\Delta[n]$, which is the sub-simplicial set of $\Delta[n]$ generated by

$$(1, 2, \dots, n), (0, 2, \dots, n), \dots, (0, 1, \dots, n-1) \in \Delta[n]_{n-1}.$$

Denote the natural inclusion by $i_n : \partial\Delta[n] \hookrightarrow \Delta[n]$. The map

$$\begin{aligned} \operatorname{Hom}_{\mathbf{sSets}}(\partial\Delta[n], X) &\to \operatorname{Hom}_{\mathbf{Sets}}([n], X_{n-1}) \\ f &\mapsto (i \mapsto f_{n-1}(0, \dots, i-1, i+1, \dots, n)) \end{aligned}$$

induced by i_n is a bijection onto the set of $f \in \operatorname{Hom}_{\mathbf{Sets}}([n], X_{n-1})$ satisfying

$$d^i_{n-1} f(j) = d^{j-1}_{n-1} f(i) \quad \text{for all } 0 \leq i < j \leq n.$$

This set is nothing but the n^{th} matching object $\mathsf{M}_n(X)$ (§3.6) of X, so we have a bijection

$$\operatorname{Hom}_{\mathbf{sSets}}(\partial\Delta[n], X) = \mathsf{M}_n(X)$$

natural in X.

In terms of the alternative definition of $\Delta[n]$, $\partial\Delta[n]$ can be viewed as the subobject where

$$(\partial\Delta[n])_m \subseteq (\Delta[n])_m = \operatorname{Hom}_\Delta([m], [n])$$

is the subset consisting of Δ-morphisms $\sigma : [m] \to [n]$ which are *not surjective*.

In the language of Example 4.1.1, $\partial\Delta[n]$ is the simplicial set associated to the ordered simplicial complex X consisting of all *proper* subsets of $[n]$, with the obvious ordering on $X^0 = [n]$.

Example 4.1.4. (Horns) For each $n \in \mathbb{N}$ and each $k \in [n]$, the *k-horn* $\Lambda^k[n]$ is the sub-simplicial set of $\partial\Delta[n]$ generated by the $n-1$ simplices

$$(1, 2, \dots, n), \dots, (0, \dots, k-2, k, \dots, n), (0, \dots, k, k+2, \dots, n), (0, 1, \dots, n-1).$$

The map

$$\begin{aligned} \operatorname{Hom}_{\mathbf{sSets}}(\Lambda^k[n], X) &\to \operatorname{Hom}_{\mathbf{Sets}}([n], X_{n-1}) \\ f &\mapsto (i \mapsto f_{n-1}(0, \dots, i-1, i+1, \dots, n)) \end{aligned}$$

is a bijection onto the set of $f \in \operatorname{Hom}_{\mathbf{Sets}}([n] \setminus \{k\}, X_{n-1})$ satisfying

$$d^i_{n-1} f(j) = d^{j-1}_{n-1} f(i) \quad \text{for all } 0 \leq i < j \leq n, \ i, j \neq k.$$

Denote the natural inclusion by $i^k_n : \Lambda^k[n] \hookrightarrow \Delta[n]$.

In terms of the alternative description of $\Delta[n]$ in Example 4.1.2, the subobject $\Lambda^k[n]$ can be viewed as the subobject where

$$(\Lambda^k[n])_m \subseteq (\Delta[n])_m = \operatorname{Hom}_\Delta([m], [n])$$

is the subset consisting of Δ-morphisms $\sigma : [m] \to [n]$ where $k \notin \operatorname{Im}(\sigma)$.

4.2. **Presentations.** The basic properties of the simplicial sets introduced in §4.1 allow us to give various descriptions of arbitrary simplicial sets in terms of direct limits, typically of simplicial sets of the form $\Delta[n]$ for various n. We will refer to these direct limit decriptions as "presentations."

The disjoint union, over $n \in \mathbb{N}$, of the maps corresponding to $\mathrm{Id} : X_n \to X_n$ under the adjunction (4.1.2) is a surjection of simplicial sets

$$\coprod_{n \in \mathbb{N}} \coprod_{X_n} \Delta[n] \to X.$$

Call the domain of this surjection X' and note that the formation of the surjection $X' \to X$ is functorial in X. Notice also that the two projections define a coequalizer diagram

$$(X' \times_X X') \rightrightarrows X' \to X$$

in **sSets** (since direct limits in **sSets** are formed degree-wise, the fact that this is a coequalizer diagram is immediate from the fact that this diagram would be a coequalizer diagram in **Sets** when $X' \to X$ is a surjection of sets). Since the natural surjection

$$X'' := (X' \times_X X')' \to X' \times_X X'$$

is an epimorphism in **sSets**, the induced diagram

$$X'' \rightrightarrows X' \to X \tag{4.2.1}$$

is also a coequalizer diagram, which we will call the *first presentation of* X. It is functorial in X. Notice that X' and X'' are disjoint unions of copies of various $\Delta[n]$ and that the maps $X'' \rightrightarrows X'$ are constructed by mapping summands $\Delta[m]$ of the coproduct X'' to summands $\Delta[n]$ of the coproduct X' via a map of the form $\Delta[\sigma]$ for some Δ-morphism $\sigma : [m] \to [n]$. We can thus write the first presentation in the form

$$X = \varinjlim \left(\coprod_{j \in J} \Delta[n_j] \rightrightarrows \coprod_{i \in I} \Delta[n_i] \right), \tag{4.2.2}$$

where I and J are certain index sets depending functorially on X and the parallel arrows are specified by maps $\phi, \psi : J \rightrightarrows I$ of the index sets together with Δ-morphisms

$$\begin{aligned} \phi(j) &: [n_j] \to [n_{\phi(j)}] \\ \psi(j) &: [n_j] \to [n_{\psi(j)}]. \end{aligned}$$

The upshot is that we can express an arbitrary simplicial set X as a direct limit of copies of standard n-simplices $\Delta[n]$ for various $n \in \mathbb{N}$.

Another way to do this is as follows. For $X \in$ **sSets**, let $\mathrm{Hom}(\Delta[\bullet], X)$ denote the category whose objects are maps of simplicial sets $x : \Delta[n] \to X$ (for some $n \in \mathbb{N}$) and whose morphisms are the obvious commutative triangles

$$\begin{tikzcd} \Delta[n] \arrow[rr] \arrow[dr, "x"'] & & \Delta[m] \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

in **sSets**. Observe: Such a map x is the same thing an element of X_n (Example 4.1.2), and, in such a commutative triangle, the map $\Delta[n] \to \Delta[m]$ is necessarily of the form $\Delta[\sigma]$ for some Δ-morphism $\sigma : [n] \to [m]$ (c.f. (4.1.1)), so the commutativity of this triangle is equivalent to saying that $x = X(\sigma)(y) \in X_n$. It follows easily from these observations that we have

$$X = \varinjlim \{\Delta[n] : (x : \Delta[n] \to X) \in \operatorname{Hom}(\Delta[\bullet], X)\}. \tag{4.2.3}$$

We will call (4.2.3) the *second presentation* of X. It is also clearly functorial in X.

Recall the skeleton functors $\operatorname{Sk}_n : \mathbf{sSets} \to \mathbf{sSets}$ from §3.5. For $X \in \mathbf{sSets}$, we have natural maps of simplicial sets

$$\emptyset =: \operatorname{Sk}_{-1} X \to \operatorname{Sk}_0 X \to \operatorname{Sk}_1 X \to \cdots \to X$$

and

$$X = \varinjlim \{\operatorname{Sk}_n X : n \in \mathbb{N}\}. \tag{4.2.4}$$

We also have a natural map

$$X_n \to \mathsf{M}_n(X) = (\operatorname{Sk}_{n-1} X)_n, \tag{4.2.5}$$

where the set $\mathsf{M}_n X$ is the n^{th} matching object of X discussed in §3.6. By the description of maps out of $\partial\Delta[n]$ in Example 4.1.3, we can view each element of $\mathsf{M}_n(X)$ as a map of simplicial sets $\partial\Delta[n] \to X$. Recall (§3.6) that $\mathsf{M}_n(X)$ depends only on $\operatorname{tr}_{n-1}(X)$. The natural maps

$$\operatorname{Sk}_{n-1}(X) \to \operatorname{Sk}_n(X) \to X$$

induces isomorphisms

$$\operatorname{tr}_{n-1} \operatorname{Sk}_{n-1} X = \operatorname{tr}_{n-1} \operatorname{Sk}_n X = \operatorname{tr}_{n-1} X$$

(§3.5) and hence also isomorphisms

$$\mathsf{M}_n(\operatorname{Sk}_{n-1} X) = \mathsf{M}_n(\operatorname{Sk}_n X) = \mathsf{M}_n(X).$$

Each element $x \in X_n$ hence determines a commutative diagram of simplicial sets

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "i_n"] \arrow[d, "\partial x"'] & \Delta[n] \arrow[d, "x"] \\ \operatorname{Sk}_{n-1} X \arrow[r] & \operatorname{Sk}_n X \end{tikzcd} \tag{4.2.6}$$

where ∂x denotes the image of x (viewing x as an element of X_n) under (4.2.5). The map ∂x is called the *attaching map for* X.

Taking the coproduct of the top parts of (4.2.6) over the *non-degenerate* elements X_n^{nd} of X_n, we obtain a commutative diagram of simplicial sets as below.

$$\begin{tikzcd} \coprod_{x \in X_n^{\text{nd}}} \partial\Delta[n] \arrow[r] \arrow[d] & \coprod_{x \in X_n^{\text{nd}}} \Delta[n] \arrow[d] \\ \operatorname{Sk}_{n-1} X \arrow[r] & \operatorname{Sk}_n X \end{tikzcd} \tag{4.2.7}$$

Lemma 4.2.1. *For any* $X \in \mathbf{sSets}$, $n \in \mathbb{N}$, (4.2.7) *is a pushout diagram in* $\mathbf{sSets}$.

Proof. The n-skeleton Sk_n is characterized as the left adjoint to the truncation $Y \mapsto \operatorname{tr}_n Y$ (§3.5), so a map $f : \operatorname{Sk}_{n-1} X \to Y$ is the same thing as a map of $(n-1)$-truncated simplicial sets $f : \mathrm{tr}_{n-1} X \to \mathrm{tr}_{n-1} Y$. By the universal property of coproducts and the description of maps out of the $\Delta[n]$ in Example 4.1.2 and maps out of $\partial\Delta[n]$ in Example 4.1.3, a map

$$\coprod_{x \in X^{\mathrm{nd}}} \Delta[n] \to Y \tag{4.2.8}$$

is the same thing as a map of sets $f_n : X_n^{\mathrm{nd}} \to Y_n$ and the condition that the map f and the map (4.2.8) agree on $\coprod_{x \in X_n^{\mathrm{nd}}} \partial\Delta[n]$ is equivalent to saying that the diagram of sets

$$\begin{tikzcd} X_n^{\mathrm{nd}} \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] \\ \mathsf{M}_n(X) \arrow[r, "f"] & \mathsf{M}_n(Y) \end{tikzcd} \tag{4.2.9}$$

commutes—both vertical arrows in (4.2.9) are given by

$$z \mapsto (d_n^0 z, \dots, d_n^n z)$$

(§3.6). The lemma can thus be reinterpreted as the following statement: Suppose X and Y are simplicial sets and $f : \mathrm{tr}_{n-1} X \to \mathrm{tr}_{n-1} Y$ is a map of $(n-1)$-truncated simplicial sets. Then

$$f \mapsto (f_n : X_n^{\mathrm{nd}} \to Y_n)$$

establishes a bijection between the set of liftings of f to a(n abusively denoted) map $f : \mathrm{tr}_n X \to \mathrm{tr}_n Y$ of n-truncated simplicial sets and the set of maps of sets $f_n : X_n^{\mathrm{nd}} \to Y_n$ making (4.2.9) commute. (We will prove a similar, but fancier, statement in Lemma 7.9.2).

The basic point is that, given the $(n-1)$-truncated map f, there is only one possible way to define $f_n : X_n \to Y_n$ on the subset $X_n^{\mathrm{deg}} \subseteq X_n$ of *degenerate* elements in a manner compatible with the degeneracies for $\mathrm{tr}_n X$ and $\mathrm{tr}_n Y$ and, furthermore, when $f_n : X_n^{\mathrm{deg}} \to Y_n$ is defined in this manner, the diagram

$$\begin{tikzcd} X_n^{\mathrm{deg}} \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] \\ \mathsf{M}_n(X) \arrow[r, "f"] & \mathsf{M}_n(Y) \end{tikzcd} \tag{4.2.10}$$

will commute. To see this, suppose $x \in X^{\mathrm{deg}}$, so that we can write $x = X(\sigma)(x')$ for some surjective Δ-morphism $\sigma : [n] \to [m]$, with $m < n$ maximal w.r.t. to this property, so that $x' \in X_m$ is non-degenerate. Since we need

$$\begin{tikzcd} X_m \arrow[r, "f_m"] \arrow[d, "X(\sigma)"'] & Y_m \arrow[d, "Y(\sigma)"] \\ X_n \arrow[r, "f_n"] & Y_n \end{tikzcd}$$

to commute, there is no choice but to define $f_n(x) := Y(\sigma) f_m(x')$. The resulting map $f_n : X_n^{\mathrm{deg}} \to Y_n$ is compatible with the boundary maps d_n^i for X and Y (i.e. makes (4.2.10) commute) because we compute

$$\begin{aligned} f_{n-1} d_n^i x &= f_{n-1} d_n^i X(\sigma) x' \\ &= f_{n-1} X(\sigma \partial_n^i)(x') \\ &= Y(\sigma \partial_n^i)(f_m(x')) \\ &= d_n^i Y(\sigma)(f_m(x')) \\ &= d_n^i f_n x \end{aligned}$$

using the fact that our f was a well-defined map of $(n-1)$-truncated simplicial sets for the key third equality. Now if we have a map $f_n : X_n^{\mathrm{nd}} \to Y_n$ making (4.2.9) commute, we define $f_n : X_n \to Y_n$ by defining it as above on the degenerate subset. By writing every element as a degeneracy of a non-degenerate element, it is straightforward to check that this f_n is compatible with all the boundaries and degeneracies for $\operatorname{tr}_n X$ and $\operatorname{tr}_n Y$, hence yields the desired lift. $\square$

4.3. **Geometric realization.** The fundamental geometric realization construction associates, to a cosimplicial object $Z \in \mathbf{csC}$ in a category $\mathbf{C}$, a functor

$$(\_)^Z : \mathbf{C} \to \mathbf{sSets},$$

called the *singular simplex (with respect to Z)* and, assuming $\mathbf{C}$ has direct limits, a left adjoint

$$| \_ |_Z : \mathbf{sSets} \to \mathbf{C}$$

to the singular simplex functor, called the *geometric realization (with respect to Z)*.

For $X \in \mathbf{C}$, the singular simplex $X^Z \in \mathbf{sSets}$ is defined by

$$X_n^Z := \operatorname{Hom}_{\mathbf{C}}(Z_n, X).$$

For a Δ-morphism $\sigma : [m] \to [n]$, the structure map

$$X^Z(\sigma) : \operatorname{Hom}_{\mathbf{C}}(Z_n, X) \to \operatorname{Hom}_{\mathbf{C}}(Z_m, X)$$

for the simplicial set X^Z is defined by $f \mapsto Z(\sigma)(f)$, where $Z(\sigma) : Z_m \to Z_n$ is the structure map for Z.

Proposition 4.3.1. *Fix $Z \in \mathbf{csC}$. Assume $\mathbf{C}$ has direct limits. Then the singular simplex functor $X \mapsto X^Z$ admits a left adjoint $X \mapsto |X|_Z$. Aside from being characterized as the left adjoint to the singular simplex functor, the geometric realization is also characterized (up to unique isomorphism of functors) by the following properties:*

1. *Geometric realization preserves direct limits.*
2. *The composition of the standard cosimplicial simplicial set $\Delta[\bullet] : \Delta \to \mathbf{sSets}$ and the geometric realization $| \_ |_Z : \mathbf{sSets} \to \mathbf{C}$ is (isomorphic to) $Z : \Delta \to \mathbf{C}$.*

Proof. From the calculation

$$\begin{aligned} \operatorname{Hom}_{\mathbf{sSets}}(\Delta[n], X^Z) &= X_n^Z \\ &= \operatorname{Hom}_{\mathbf{C}}(Z_n, X) \end{aligned}$$

we see that $|\Delta[\bullet]| := Z$ will indeed define a left adjoint to the singular simplex on the full subcategory $\Delta[\bullet](\Delta)$ of $\mathbf{sSets}$ (and indeed, up to unique isomorphism, this is the only possible way to define the geometric realization of the standard simplices and the maps between them). Using the fact that $\mathbf{C}$ has direct limits, we can promote this recipe for the geometric realization of the standard simplices (and the maps $\Delta[\sigma]$ between them) to a recipe for the geometric realization of an arbitrary $X \in \mathbf{sSets}$ by making use of, say, one of the functorial presentations of X as a direct limit of standard simplicies from §4.2. The point is that we can write

$$X = \varinjlim \{\Delta[n_i] : i \in I_X\}$$

for some direct limit system of simplicial sets $\{\Delta[n_i] : i \in I_X\}$ functorial in X. We can then define $|X|_Z$ functorially in X by the formula

$$|X|_Z := \varinjlim \{Z_{n_i} : i \in I_X\}.$$

Notice that the structure maps for the direct limit on the RHS are determined by the cosimplicial structure of Z because each map $\Delta[n_i] \to \Delta[n_j]$ in the direct limit system defining X is necessarily of the form $\Delta[\sigma]$ for some Δ-morphism $\sigma : [n_i] \to [n_j]$ by (4.1.1). The fact that the functor $X \mapsto |X|_Z$, thus defined, actually yields a left adjoint to the singular simplex is just a formal calculation with the universal property of direct limits using the fact that we know this to be the case for the standard simplices. $\square$

For example, the geometric realization $\mathbf{sSets} \to \mathbf{sSets}$ with respect to the standard cosimplicial simplicial set $\Delta[\bullet]$ is the identity functor.

Recall the standard topological simplex $\Delta^\top \in \mathbf{csTop}$ from §3.3. The singular simplex with respect to $\Delta^\top$ is a functor usually denoted

$$\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSets}. \tag{4.3.1}$$

The corresponding geometric realization is a functor

$$|\_| : \mathbf{sSets} \to \mathbf{Top} \tag{4.3.2}$$

usually called, simply, the *geometric realization functor* (or, if there is any possible chance of confusion, the *standard topological geometric realization functor*).

Let us make a few general remarks about this particular geometric realization.

Lemma 4.3.2. *For any simplicial set X, the geometric realization $|X| \in$ **Top** is a CW complex whose n-cells correspond to non-degenerate elements of X_n.*

Proof. First note that

$$\begin{aligned} |\Delta[n]| &\cong D^n \\ |\partial\Delta[n]| &\cong S^{n-1}, \end{aligned}$$

where D^n is the n-dimensional disk and S^n is the n-dimensional sphere, and the isomorphisms are such that

$$|i_n : \partial\Delta[n] \to \Delta[n]|$$

is the usual inclusion $S^{n-1} \to D^n$ of the boundary of the disk. Since geometric realization commutes with direct limits, the result we want follows from the (geometric realization of the) formula (4.2.4) and the fact that (4.2.7) (hence also its geometric realization) is a pushout diagram (Lemma 4.2.1). $\square$

Definition 4.3.3. Let X be a topological space. A subset $U \subseteq X$ is called *compactly open* iff $f^{-1}(U)$ is open in K for each compact (Hausdorff) space K and each continuous map $f : K \to X$. A topological space X is called a *Kelley space* iff every compactly open subset of X is open. The full subcategory of **Top** consisting of Kelley spaces is denoted **K**.

The inclusion $\mathbf{K} \to \mathbf{Top}$ has a right adjoint, left inverse obtained by endowing an arbitrary space X with the "Kelley topology" where a subset $U \subseteq X$ is declared open iff it is compactly open.

From the construction of $|X|$ in the proof of Proposition 4.3.1, we see that $|X|$ can be written as a coequalizer

$$X = \varinjlim (X'' \rightrightarrows X'),$$

in **Top**, where X' and X'' are disjoint unions of copies of standard topological n-simplices $\Delta_n^\top$, for various n. Certainly such X' and X'' are Kelley spaces and it is a general fact that a direct limit of Kelley spaces (taken in **Top**) is again a Kelley space [Hov, 2.4.22(3)], so $|X| \in \mathbf{K}$. Similarly, if X is a *finite* simplicial set (i.e. X has only finitely many non-degenerate simplices), then we can present X as a coequalizer

$$X = \varinjlim \left(\coprod_{j \in J} \Delta[n_j] \rightrightarrows \coprod_{i \in I} \Delta[n_i] \right),$$

where I is the (finite) set of *non-degenerate* simplices in X. (Compared to the standard presentations of §4.2, the latter presentation has the slight disadvantage that it is not functorial because the set of non-degenerate simplices of X isn't functorial in X.) Geometric realization preserves coequalizers, so $|X|$ is a quotient of a finite disjoint union of standard topological simplices, so $|X|$ is compact because a quotient of a compact space is (quasi-)compact. (The geometric realization $|X|$ of any simplicial set is Hausdorff.)

We can view the geometric realization $|\_|$ as a functor

$$|\_| : \mathbf{sSets} \to \mathbf{K}. \tag{4.3.3}$$

The category **K** has inverse limits, obtained by taking the inverse limit in **Top**, then endowing the result with the Kelley topology.

Lemma 4.3.4. *The functor* (4.3.3) *preserves finite inverse limits.*

Proof. [Hov, 3.2.4] □

Since we will require a certain complement of this result, let us recall how Hovey proves this. (We will just discuss the preservation of finite *products*.) First [Hov, 3.1.8] he proves:

Lemma 4.3.5. *For any $m, n \in \mathbb{N}$, the natural map*

$$|\Delta[m] \times \Delta[n]| \to |\Delta[m]| \times |\Delta[n]|$$

is an isomorphism. (On the RHS it doesn't matter whether the product is taken in **K** *or* **Top** *since the spaces $|\Delta[m]| = \Delta_m^\top$ and $|\Delta[n]| = \Delta_n^\top$ are both compact.)*

Definition 4.3.6. Let **C** be a category with products and direct limits. We say that *products commute with direct limits* in **C** iff, for all direct limit systems $i \mapsto X_i$ and $j \mapsto X_j$ in **C**, the natural *comparison map*

$$\varinjlim (X_i \times X_j) \to (\varinjlim X_i) \times (\varinjlim X_j) \tag{4.3.4}$$

is an isomorphism.

For example, it is elementary to see that products commute with direct limits in **Sets**, and hence also in **sSets**, since this can be checked "degree-wise." In **Top**, however, products *do not* commute with general direct limits, though it is elementary to see that they commute with coproducts (direct sums). However, the natural comparison map (4.3.4) is at least bijective, because the underlying set of a direct or inverse limit in **Top** coincides with the direct or inverse limits of the underlying sets and we have already mentioned that products commute with direct limits in **Sets**. Isomorphy for (4.3.4) in **Top** is hence equivalent to saying that a subset of the codomain is open whenever its preimage in the domain is open. The situation is better in **K**:

Proposition 4.3.7. *Products commute with direct limits in* **K**.

Proof. I assume Hovey found this in [Lew, Appendix A], which is the standard reference for basic facts about **K** and related categories. In any case, this is fairly elementary. □

With these observations in hand, the fact that geometric realization commutes with finite products is a formal computation, as follows. Suppose X and Y are simplicial sets. Using, say, the standard presentation, we can write $X = \varinjlim \Delta[m_i]$, $Y = \varinjlim \Delta[n_j]$ for some appropriate direct limits systems of simplicial sets. Then we compute

$$
\begin{aligned}
|X| \times |Y| &= |\varinjlim \Delta[m_i]| \times |\varinjlim \Delta[n_j]| \\
&= \left(\varinjlim |\Delta[m_i]|\right) \times \left(\varinjlim |\Delta[n_j]|\right) \\
&= \varinjlim \left(|\Delta[m_i]| \times |\Delta[n_j]|\right) \\
&= \varinjlim |\Delta[m_i] \times \Delta[n_j]| \\
&= |\varinjlim (\Delta[m_i] \times \Delta[n_j])| \\
&= |X \times Y|
\end{aligned}
\tag{4.3.5}
$$

using the fact that geometric realization commutes with direct limits, Lemma 4.3.5, and the fact that products commute with direct limits in **sSets** and **K** (the products of Kelley spaces here are taken in **K**).

Proposition 4.3.8. *For a fixed locally compact space Y, the "product with Y" functor* **Top** → **Top** *commutes with direct limits.*

Proof. This is Lemma 2.2.1, but we can also argue directly as follows. As mentioned above, it is easy to see that "product with Y" commutes with coproducts (even without hypotheses on Y), so we reduce to treating coequalizers. We need to show that when

$$X'' \rightrightarrows X' \to X$$

is a coequalizer diagram, so is

$$X'' \times Y \rightrightarrows X' \times Y \to X \times Y,$$

when the products are given the product topology. Let $f : X' \to X$ denote the quotient map for the original coequalizer diagram. Since we know the product with Y commutes with coequalizers on the level of underlying sets, the issue is to show that the product topology on $X \times Y$ actually coincides with the quotient topology on $X \times Y$ induced by the continuous surjection

$$f \times \mathrm{Id} : X' \times Y \to X \times Y.$$

That is: we need to show that any subset $U \subseteq X \times Y$ with $(f \times \mathrm{Id})^{-1}(U)$ open in $X' \times Y$ is open in $X \times Y$. It is enough to show that for any topological space W and any map of sets $g : X \times Y \to W$ for which the composition $h := g(f \times \mathrm{Id}) : X' \times Y \to W$ is continuous, the map g is continuous. (In fact, the property we want for $f \times \mathrm{Id}$ is *equivalent* to this latter assertion and both are equivalent to the special case where W is the Sierpinski space, which represents the "open subsets" functor.) Anyway, the result we want is [Bre, 13.19]. I will repeat the argument here for the reader's convenience: Consider an open subset $U \subseteq W$ and suppose $g(x_0, y_0) \in U$ for some $(x_0, y_0) \in X \times Y$. To show that g is continuous, we want to show that there is a neighborhood A of x_0 in X and a neighborhood V of y_0 in Y such that $g(A \times V) \subseteq U$. Since the quotient map f is surjective, we can pick a point $x_0' \in X'$ with $f(x_0') = x_0$. Then $h(x_0', y_0) = g(x_0, y_0) \in U$, so since h is continuous and Y is locally compact, there is a compact neighborhood K of y_0 in Y such that $h(x_0' \times K) \subseteq U$. Set

$$A := \{x \in X : g(x \times K) \subseteq U\}.$$

Then $x_0 \in A$, so A and $V := K$ (or the interior of K if V is to be open) will do the job provided we can show A is open in X. Since f is a quotient map, it suffices to show $f^{-1}(A)$ is open in X'. Notice that

$$f^{-1}(A) = \{x' \in X' : h(x' \times N) = g(f(x') \times K) \subseteq U\},$$

so that

$$X' - f^{-1}(A) = \pi_1(h^{-1}(W - U) \cap (X' \times K)).$$

This set is closed in X' because the projection $\pi_1 : X' \times K \to X'$ is proper since K is compact. $\square$

Corollary 4.3.9. *If X is a finite simplicial set (finitely many non-degenerate simplices), then the "product with X" functor* **sSets** $\to$ **sSets** *commutes with the geometric realization functor*

$$|\_| : \mathbf{sSets} \to \mathbf{Top}.$$

That is, the natural map

$$|X \times Y| \to |X| \times |Y|$$

is an isomorphism for any simplicial set Y, where the product on the RHS is the usual product of topological spaces.

Proof. Repeat the formal computation (4.3.5) using the fact that $|X|$ is compact, so we can commute the product with $|X|$ and the direct limit spaces in (4.3.5) (in **Top**, not just in **K**) using Proposition 4.3.8. $\square$

Remark 4.3.10. In the above corollary, we could replace the condition that X is finite with the condition that $|X|$ is locally compact. However, a simplicial set with infinitely many non-degenerate simplices will rarely have a locally compact geometric realization.

For more on geometric realization, see [Mil], [GJ, I.2].

4.4. **Model structure.** In this section we recall "the" model structure on **sSets**.

Definition 4.4.1. A map $f : X \to Y$ of simplicial sets is called a ...

- ... *weak equivalence* iff its geometric realization $|f| : |X| \to |Y|$ is a weak equivalence of topological spaces (Definition 2.1.1).
- ... *fibration* iff it has the RLP w.r.t. the inclusions $i_n^k : \Lambda^k[n] \hookrightarrow \Delta[n]$ of k-horns (c.f. Example 4.1.4) for all $n > 0$ and all $k \in [n]$.
- ... *cofibration* iff it is injective.

Theorem 4.4.2. *The category* **sSets** *of simplicial sets is a model category with the weak equivalences, fibrations, and cofibrations defined above. This model structure is cofibrantly generated (Definition 1.5.7) by the set I of cofibrations and the set J of trivial cofibrations below.*

$$\begin{aligned} I &:= \{\partial\Delta[n] \hookrightarrow \Delta[n] : n \in \mathbb{N}\} \\ J &:= \{i_n^k : \Lambda^k[n] \hookrightarrow \Delta[n] : n > 0,\ k \in [n]\} \end{aligned}$$

Proof. [Q1, II.3], [Hov, 3.2]. The fact that a map of simplicial sets is a cofibration iff it is injective is [Hov, 3.2.2]. □

Theorem 4.4.3. *Simplicial sets, with the above model category structure, is proper (Definition 1.4.1).*

Proof. [Hir, 13.1.13] Since all simplicial sets are cofibrant, left properness follows from Proposition 1.4.3. Right properness follows formally from the following facts: 1) The geometric realization functor (§4.3) takes fibrations of simplicial sets to fibrations of topological spaces (i.e. Serre fibrations) [Hov, 3.6.2]. 2) Geometric realization preserves fibered products, at least when viewed as a functor to the category **K** of Kelley spaces (Lemma 4.3.4). 3) The model category **Top** (§2) is right proper (Theorem 2.1.4). We use 2-out-of-3 for weak equivalences in **Top** and the obvious fact that the map from the fibered product in **K** to the fibered product in **Top** is always a weak equivalence (both spaces receive the same maps from spheres). □

Let us make a few remarks about fibrations of simplicial sets. By definition, $f : X \to Y$ is a fibration iff there is a lift as indicated in any solid diagram

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, "i_n^k"'] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

of simplicial sets ($n > 0$, $k \in [n]$). According to the description of maps out of $\Lambda^k[n]$ and $\Delta[n]$ given in Examples 4.1.2 and 4.1.4, the solid part of this diagram corresponds to elements

$$x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in X_{n-1},\ y \in Y_n$$

satisfying:

$$\begin{aligned} d_{n-1}^i x_j &= d_{n-1}^{j-1} x_i, \quad 0 \le i < j \le n,\ i,j \ne k \\ d_n^i y &= f_{n-1} x_i, \quad 0 \le i \le n,\ i \ne k. \end{aligned} \tag{4.4.1}$$

Similarly, a lift as indicated corresponds to an element $x \in X_n$ satisfying $d_n^i x = x_i$ for $i = 0, \dots, k-1, k+1, \dots, n$ and $f_n x = y$. In other words, if we let

$$\mathsf{M}_n^k(X/Y) \subseteq X_{n-1}^n \times Y$$

be the set of all

$$(x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_n, y) \in X_{n-1}^n \times Y$$

satisfying the relations (4.4.1), then we have a natural map

$$\begin{aligned} X_n &\to \mathsf{M}_n^k(X/Y) \\ x &\mapsto (d_n^0 x, \dots, d_n^{k-1} x, d_n^{k+1} x, \dots, d_n^n x, f_n x). \end{aligned} \tag{4.4.2}$$

The map f is a fibration iff the maps (4.4.2) are surjective for all $n > 0$, $k \in [n]$.

Remark 4.4.4. When $n = 0$, $\Lambda^k[n] = \emptyset$ is the initial object; the RLP with respect to $\emptyset \hookrightarrow \Delta[0]$ is equivalent to surjectivity of $f_0 : X_0 \to Y_0$, but we emphasize that this surjectivity is *not* required for f to be a fibration.

The following "corollary of Theorem 4.4.2" is often useful (actually one usually proves some result to this effect *before* proving Theorem 4.4.2):

Proposition 4.4.5. *For a map $f : X \to Y$ of simplicial sets, the following are equivalent:*

(1) f is a fibration (Definition 4.4.1).
(2) f has the RLP w.r.t. the inclusions

$$(\partial\Delta[n] \times \Delta[1]) \cup (\Delta[n] \times \{e\}) \hookrightarrow \Delta[n] \times \Delta[1]$$

for all $n \in \mathbb{N}$, $e \in \{0, 1\}$.
(3) f has the RLP w.r.t. the inclusion

$$(K \times \Delta[1]) \cup (L \times \{e\}) \hookrightarrow L \times \Delta[1]$$

for any injective map of simplicial sets $K \hookrightarrow L$ and any $e \in \{0, 1\}$.
(4) f has the RLP w.r.t. $K \hookrightarrow L$ whenever $K \hookrightarrow L$ is an injective map of simplicial sets for which $|K| \to |L|$ is a weak equivalence of topological spaces.

Proof. The equivalence of the first three statements is [GZ, IV.2.1] (one proves this "from first principles"). The equivalence of the first and last statements is the statement that fibrations in the model category **sSets** have the RLP w.r.t. trivial cofibrations, so it is part of the assertion in Theorem 4.4.2 that **sSets** is a model category with the definitions of Definition 4.4.1. □

Proposition 4.4.6. *For a map $f : X \to Y$ of simplicial sets, the following are equivalent:*

(1) f is a trivial fibration (Definition 4.4.1).
(2) $f : X \to Y$ has the RLP w.r.t. the inclusion $i_n : \partial\Delta[n] \hookrightarrow \Delta[n]$ for any $n \in \mathbb{N}$.
(3) $f : X \to Y$ has the RLP w.r.t. all injective maps of simplicial sets.

Proof. (C.f. [Q1, II.3.2.1]) The equivalence of the first two statements is part of the cofibrant generation assertion in Theorem 4.4.2—namely the assertion that I-inj is the set of trivial fibrations. The equivalence of the first and third statements is the assertion that trivial fibrations have the RLP w.r.t. cofibrations, which is part of the assertion in Theorem 4.4.2 that **sSets** is a model category with the definitions of Definition 4.4.1. □

Recall (Definition 1.1.5) that a simplicial set is called *fibrant* iff the map to the terminal simplicial set is a fibration. (All simplicial sets are cofibrant since the map from the empty simplicial set is certainly injective.) Let us unravel this a bit. Let $\mathsf{M}_n(X/Y)$ denote the set of

$$(x_0, \ldots, x_n, y) \in X_{n-1}^{[n]} \times Y_n$$

satisfying the relations

$$\begin{aligned} d^i_{n-1} x_j &= d^{j-1}_{n-1} x_i, \quad 0 \le i < j \le n \\ d^i_n y &= f_{n-1} x_i, \quad i \in [n]. \end{aligned} \tag{4.4.3}$$

Then a solid diagram

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, "i_n"'] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

is the same thing as an element of $\mathsf{M}_n(X/Y)$ and a lift as indicated is the same thing as an element $x \in X_n$ with $d^i_n x = x_i$ for all $i \in [n]$ and $f_n x = y$. Evidently then, f is a trivial fibration iff the natural map

$$\begin{aligned} X_n &\to \mathsf{M}_n(X/Y) \\ x &\mapsto (d^0_n x, \ldots, d^n_n x, f_n x) \end{aligned} \tag{4.4.4}$$

is surjective.

Notice that $\mathsf{M}_n(X/Y)$ is a relative version of the n^{th} matching object $\mathsf{M}_n(X)$ of §3.6. Indeed, when Y is the terminal object, $\mathsf{M}_n(X/Y) = \mathsf{M}_n(X)$ *is* the n^{th} matching object of §3.6.

4.5. **Homotopy groups.** In this section we define the homotopy sets $\pi_n(X, \mathbf{1})$ of a *fibrant*, based simplicial set X.

A *basepoint* of a simplicial set X is a map from the terminal simplicial set $\mathbf{1}$ to X. The terminal simplicial set is the constant simplicial set associated to a one-element set, so

$$\begin{aligned} \mathrm{Hom}_{\mathbf{sSets}}(\mathbf{1}, X) &= \mathrm{Hom}_{\mathbf{Sets}}(\mathbf{1}, \mathrm{tr}_0 X) \\ &= X_0 \end{aligned}$$

(c.f. Example 3.5.1). That is, a basepoint of X is the same thing as an element $\mathbf{1} \in X_0$. A *based simplicial set* is a simplicial set X equipped with a basepoint. By slight abuse of notation, we write $\mathbf{1} \in X_n$ for the image of the base point $\mathbf{1} \in X_0$ under the map $X_0 \to X_n$ induced by the unique Δ morphism $[n] \to [0]$. Note that for any Δ-morphism $\sigma : [m] \to [n]$, the structure map $X(\sigma) : X_n \to X_m$ takes $\mathbf{1}$ to $\mathbf{1}$, so this abuse of notation should be harmless.

For a based simplicial set X and an $n \in \mathbb{N}$, we let

$$\mathsf{Z}_n(X) := \{x \in X_n : d^i_n x = \mathbf{1} \text{ for } i = 0, \ldots, n\}$$

(note $\mathsf{Z}_0(X) := X_0$). Define a relation $\sim$ on $\mathsf{Z}_n(X)$ by declaring $x \sim y$ iff there is some $z \in X_{n+1}$ such that $d^i_{n+1} z = \mathbf{1}$ for $i = 0, \ldots, n-1$ and $d^n_{n+1} = x$, $d^{n+1}_{n+1} z = y$.

Recall from the previous section that for $n > 0$, we let

$$\mathsf{M}^k_n(X) := \{(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in X^n_{n-1} : d^i_{n-1} x_j = d^{j-1}_{n-1} x_i \, \forall\, i < j\},$$

so we have a natural map

$$
\begin{aligned}
X_n &\to \mathsf{M}^k_n(X) \\
x &\mapsto (d^0_n x, \dots, d^{k-1}_n x, d^{k+1}_n x, \dots, d^n_n x).
\end{aligned}
\tag{4.5.1}
$$

The simplicial set X is fibrant iff (4.5.1) is surjective for all $n > 0$.

Proposition 4.5.1. *If X is fibrant, then the relation $\sim$ is an equivalence relation.*

Proof. **Reflexive:** Perhaps surprisingly, this is the most difficult property to establish. Given $x \in \mathsf{Z}_n(X) \subseteq X_n$, we seek a $z \in X_{n+1}$ such that

$$d^i_{n+1} z = \begin{cases} \mathbf{1}, & i = 0, \dots, n-1 \\ x, & i = n, n+1 \end{cases}$$

Since we can see easily that $(\mathbf{1}, \dots, \mathbf{1}, x) \in \mathsf{M}^n_{n+1}(X)$, by fibrancy there is a $y \in X_{n+1}$ such that $d^i_{n+1} y = \mathbf{1}$ for $i \in [n-1]$ and $d^{n+1}_{n+1} y = x$ (we know nothing about $d^n_{n+1} y$). Next observe that $(\mathbf{1}, \dots, \mathbf{1}, y, y) \in \mathsf{M}^{n+2}_{n+2}(X)$, so by fibrancy there is a $w \in X_{n+2}$ such that

$$d^i_{n+2} w = \begin{cases} \mathbf{1}, & i = 0, \dots, n-1 \\ y, & i = n, n+1 \end{cases}$$

(we know nothing about $d^{n+2}_{n+2} w$). I claim $z := d^{n+2}_{n+2} w \in X_{n+1}$ is as desired. Indeed, for $i \in [n-1]$, we compute

$$
\begin{aligned}
d^i_{n+1} z &= d^i_{n+1} d^{n+2}_{n+2} w \\
&= d^{n+1}_{n+1} d^i_{n+2} w \\
&= d^{n+1}_{n+1} \mathbf{1} \\
&= \mathbf{1},
\end{aligned}
$$

and for $i \in \{n, n+1\}$, we compute

$$
\begin{aligned}
d^i_{n+1} z &= d^i_{n+1} d^{n+2}_{n+2} w \\
&= d^{n+1}_{n+1} d^i_{n+2} w \\
&= d^{n+1}_{n+1} y \\
&= x.
\end{aligned}
$$

Symmetric: Suppose $x, y \in \mathsf{Z}_n(X)$ are such that there is some $z \in X_{n+1}$ with

$$d^i_{n+1} z = \begin{cases} \mathbf{1}, & i = 0, \dots, n-1 \\ x, & i = n \\ y, & i = n+1. \end{cases}$$

By reflexivity proved above, there is $r \in X_{n+1}$ such that

$$d^i_{n+1} r = \begin{cases} \mathbf{1}, & i = 0, \dots, n-1 \\ y, & i = n, n+1. \end{cases}$$

Since $d^{n+1}_{n+1} r = d^{n+1}_{n+1} z$ it follows that $(\mathbf{1}, \dots, \mathbf{1}, z, r) \in \mathsf{M}^{n+2}_{n+2}(X)$, so by fibrancy there is some $u \in X_{n+2}$ with

$$d^i_{n+2} u = \begin{cases} \mathbf{1}, & i = 0, \dots, n-1 \\ z, & i = n \\ r, & i = n+1. \end{cases}$$

Now we check, just as above, that $w := d^{n+2}_{n+2} u$ satisfies

$$d^i_{n+1} w = \begin{cases} \mathbf{1}, & i = 0, \dots, n-1 \\ y, & i = n \\ x, & i = n+1. \end{cases}$$

Transitive: Suppose $x, y, z \in \mathsf{Z}_n(X)$ have $x \sim y$ and $y \sim z$. We know $\sim$ is symmetric so, since $y \sim x$, there is some $u \in X_{n+1}$ with

$$d^i_{n+1} u = \begin{cases} \mathbf{1}, & i = 0, \dots, n-1 \\ y, & i = n \\ x, & i = n+1 \end{cases}$$

and since $y \sim z$ there is some $v \in X_{n+1}$ with

$$d^i_{n+1} v = \begin{cases} \mathbf{1}, & i = 0, \dots, n-1 \\ y, & i = n \\ w, & i = n+1 \end{cases}$$

From the equality $d^n_{n+1} u = d^n_{n+1} v$ it follows that $(\mathbf{1}, \dots, \mathbf{1}, u, v) \in \mathsf{M}^{n+2}_{n+2}(X)$, so by fibrancy, there is some $w \in X_{n+2}$ with

$$d^i_{n+2} w = \begin{cases} \mathbf{1}, & i = 0, \dots, n-1 \\ u, & i = n \\ v, & i = n+1 \end{cases}$$

and one checks easily that $d^{n+2}_{n+2} w \in X_{n+1}$ witnesses $x \sim z$. □

Definition 4.5.2. The quotient sets

$$\pi_n(X) := \mathsf{Z}_n(X)/\sim$$

are called the *homotopy sets* of the based, fibrant simplicial set X.

Remark 4.5.3. The set $\pi_0(X)$ of *path components* of X does not depend on the choice of basepoint, and can be defined without requiring X to be fibrant by the formula

$$\pi_0(X) = \varinjlim (d^0_1, d^1_1 : X_1 \rightrightarrows X_0).$$

Theorem 4.5.4. *Let $f, g : X \rightrightarrows Y$ be homotopic maps of fibrant simplicial sets. Then the maps*

$$\pi_0(f), \pi_0(g) : \pi_0(X) \rightrightarrows \pi_0(Y)$$

coincide, as do the maps

$$\pi_n(f), \pi_n(g) : \pi_n(X, x) \rightrightarrows \pi_n(Y, f(x))$$

for any base point x of X and any $n > 0$.

Proof. Exercise. □

4.6. **Function complexes.** Our first task is to describe the analog of the function space (i.e. the space of continuous maps $Y \to Z$ between topological spaces Y and Z discussed in §**??**) in the category of simplicial sets. Given simplicial sets Y, Z, we will define a new simplicial set Z^Y called the *function complex* of Y and Z characterized (up to unique isomorphism) by the existence of a bijection

$$\operatorname{Hom}_{\mathbf{sSets}}(X, Z^Y) = \operatorname{Hom}_{\mathbf{sSets}}(X \times Y, Z), \tag{4.6.1}$$

natural in $X \in \mathbf{sSets}$. The construction will be covariantly functorial in Z and contravariantly functorial in Y:

$$\begin{aligned} \mathbf{sSets} \times \mathbf{sSets}^{\mathrm{op}} &\to \mathbf{sSets} \\ (Z, Y) &\mapsto Z^Y. \end{aligned} \tag{4.6.2}$$

Notice that the situation in simplicial sets is slightly nicer than the one in **Top** since we do not need any hypotheses on X.

To construct Z^Y, we make use of the standard simplices $\Delta[n]$ of Example 4.1.2. Recall that

$$\Delta[n]_m = \operatorname{Hom}_{\Delta}([m], [n]).$$

We let

$$Z^Y_n := \operatorname{Hom}_{\mathbf{sSets}}(Y \times \Delta[n], Z).$$

For a Δ-morphism $\sigma : [n] \to [m]$, recall (Example 4.1.2) that we have a natural map $\Delta[\sigma] : \Delta[n] \to \Delta[m]$ of simplicial sets. We define the structure map

$$Z^Y(\sigma) : Z^Y_m \to Z^Y_n \tag{4.6.3}$$

by taking an **sSets** morphism $f : Y \times \Delta[m] \to Z$ to the **sSets** morphism

$$f(\mathrm{Id}_Y \times \Delta[\sigma]) : Y \times \Delta[n] \to Z.$$

Proposition 4.6.1. *For simplicial sets X, Y, Z, we have a natural bijection* (4.6.1).

Proof. Define

$$\begin{aligned} \Psi : \operatorname{Hom}_{\mathbf{sSets}}(X, Z^Y) &\to \operatorname{Hom}_{\mathbf{sSets}}(X \times Y, Z) \\ \Phi : \operatorname{Hom}_{\mathbf{sSets}}(X \times Y, Z) &\to \operatorname{Hom}_{\mathbf{sSets}}(X, Z^Y) \end{aligned}$$

by the following formulas:

$$\begin{aligned} \Psi(g)_n(x, y) &:= g_n(x)(y, \mathrm{Id} : [n] \to [n]) \\ \Phi(f)_n(x)_m(y, \sigma : [m] \to [n]) &:= f_m(X(\sigma)(x), y). \end{aligned}$$

We leave it to the reader to check that:

(1) For every $n \in \mathbb{N}$ and every $x \in X_n$, $\Phi(f)_n(x) : Y \times \Delta[n] \to Z$ is a map of simplicial sets.
(2) $\Phi(f) : X \to Z^Y$ is a map of simplicial sets.
(3) $\Psi(g) : X \times Y \to Z$ is a map of simplicial sets.

The fact that $\Psi\Phi(f) = f$ for all f amounts to the following:

$$\begin{aligned}\Psi\Phi(f)_n(x,y) &= \Phi(f)_n(x)(y, \mathrm{Id} : [n] \to [n]) \\ &= f_n(X(\mathrm{Id})(x), y) \\ &= f(x,y)\end{aligned}$$

and the fact that $\Phi\Psi(g) = g$ for all g amounts to the following:

$$\begin{aligned}\Phi\Psi(g)_n(x)_m(y, \sigma : [m] \to [n]) &= \Psi(g)_n(X(\sigma)(x), y) \\ &= g_m(X(\sigma)(x))_n(y, \mathrm{Id} : [n] \to [n]) \\ &= Z^Y(\sigma)(g_n(x))_m(y, \mathrm{Id} : [n] \to [n]) \\ &= g_n(x)_m(y, \mathrm{Id}\,\sigma : [m] \to [n]) \\ &= g_n(x)_m(y, \sigma : [m] \to [n]).\end{aligned}$$

In the third equality we use that g is a map of simplicial sets, so it is compatible with the structure maps associated by X and by Z^Y to σ and in the fourth equality we use the definition of the maps (4.6.3). $\square$

4.7. **Simplicial homotopy revisited.** For simplicity, we introduce the notation $I := \Delta[1]$ for the standard 1-simplex (Example 4.1.2). Recall that $I_n = \mathrm{Hom}_\Delta([n],[1])$. For a simplicial set Y, the function complex (§4.6) $Y^I \in$ **sSets** is called the *path space* of Y, in analogy with the topological situation (§**??**). Via the functoriality of function complexes (4.6.2), the two maps $i_0, i_1 : \Delta[0] \rightrightarrows \Delta[1]$ induce two evaluation maps $e_0, e_1 : Y^I \rightrightarrows Y$ (note $\Delta[0]$ is the terminal simplicial set with $|\Delta[0]_n| = 1$ for every n, so $Y^{\Delta[0]} = Y$). The common retract $r : I \to \Delta[0]$ of the i_j induces a map $i_Y : Y \to Y^I$ called the *constant path map*. We have

$$e_0 i_Y = e_1 i_Y = \mathrm{Id}_Y .$$

Proposition 4.7.1. *Let $f, g : X \rightrightarrows Y$ be maps of simplicial sets. There are natural bijections between the following sets:*

(1) the set of simplicial homotopies h from f to g in the sense of §3.7
(2) the set of morphisms of simplicial sets $H : X \times I \to Y$ with $Hi_0 = f$, $Hi_1 = g$
(3) the set of morphisms of simplicial sets $H : X \to Y^I$ with $e_0 H = f$, $e_1 H = g$.

Proof. Recall (§3.7) that such an h consists of maps $h(\phi) : X_n \to Y_n$ defined for each Δ-morphism $\phi : [n] \to [1]$ (i.e. for each element $\phi \in \mathrm{Hom}_\Delta([n],[1]) = I_n$, satisfying various conditions. It is straightforward to check that those conditions are equivalent to saying that the maps

$$\begin{aligned}X_n \times I_n &\to Y_n \\ (x,\phi) &\mapsto h(\phi)(x)\end{aligned}$$

define a map of simplicial sets $H : X \times I \to Y$ and it is straightforward to see that such an H gives maps $h(\phi)$ by the formula $h(\phi)(x) := H_n(x,\phi)$. This gives the bijection between the first two sets. The bijection between the second and third sets follows from the adjointness (4.6.1). $\square$

Similarly, we can reformulate the definition of *deformation retract* from §3.7 as follows: If $f : X \to Y$ is a map of simplicial sets we say that r is a *deformation retract* of f iff

$rf = \mathrm{Id}_X$ and there is a commutative diagram of simplicial sets

(4.7.1)
$$\begin{tikzcd} X \times I \arrow[r, "\pi_1"] \arrow[d, "f \times \mathrm{Id}"'] & X \arrow[d, "f"] \\ Y \times I \arrow[r, "H"] & Y \end{tikzcd}$$

with $Hi_0 = fr$, $Hi_1 = \mathrm{Id}_Y$. (This H is the same thing as a homotopy rel X from fr to the identity of Y in the sense of §3.7.) Using (4.6.1) we see that such an H is the same thing as a map of simplicial sets $H : Y \to Y^I$ making

(4.7.2)
$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f"'] & X^I \arrow[d, "f^I"] \\ Y \arrow[r, "H"] & Y^I \end{tikzcd}$$

commute and satisfying $e_0 H = fr$, $e_1 H = \mathrm{Id}_Y$.

Starting only with f, we see similarly that a pair (r, H) consisting of a retract r of f and a homotopy H rel X from fr to the identity of Y is the same thing as a lift as indicated in the commutative square of simplicial sets below.

(4.7.3)
$$\begin{tikzcd} X \arrow[r, "(\mathrm{Id}_X, i_Y f)"] \arrow[d, "f"'] & X \times_Y^{f, e_0} Y^I \arrow[d, "\pi_2 e_1"] \\ Y \arrow[ur, dotted, "(r,H)"] \arrow[r, equal] & Y \end{tikzcd}$$

(Note that $f^I i_X = i_Y f$.)

Proposition 4.7.2. *If $f, g : X \rightrightarrows Y$ are simplicially homotopic maps (§3.7) of simplicial sets, then the geometric realizations $|f|, |g| : |X| \rightrightarrows |Y|$ are homotopic maps of topological spaces in the usual sense. Similarly, if $r : Y \to X$ is a deformation retract of $f : X \to Y$ in the "simplicial" sense of §3.7, then $|r|$ is a deformation retract of $|f|$ in the usual topological sense.*

Proof. The geometric realization of the simplicial set $I = \Delta[1]$ is the standard topological 1-simples $\Delta_1^{\top}$ (i.e. the usual closed interval I from topology). Similarly, the geometric realizations of the maps $i_e : \Delta[0] \to \Delta[1]$ ($e = 0, 1$) are the usual inclusions of the endpoints of the interval I. According to Proposition 4.7.1, we can view a simplicial homotopy from f to g as a map of simplicial sets $H : X \times I \to Y$ with $Hi_0 = f$, $Hi_1 = g$. Since $I = \Delta[1]$ is a finite simplicial set, Corollary 4.3.9 says that the geometric realization of H can be viewed as a map of spaces $|H| : |X| \times I \to |Y|$ (where $I = |\Delta[1]|$ is the usual topological interval) with $|Hi_0| = |H||i_0| = |f|$, $|Hi_1| = |H||i_1| = |g|$, which is a homotopy from f to g in the usual sense. The statement about deformation retracts is obtained similarly by considering the geometric realization of (4.7.1). □

Lemma 4.7.3. *The retract $r : \Delta[1] \to \Delta[0]$ is a deformation retract of both inclusions $i_0, i_1 : \Delta[0] \to \Delta[1]$.*

Proof. On symmetry grounds, it will be enough to treat the case of i_1. We already have $ri_1 = \mathrm{Id}$, so it remains only to produce a strict homotopy $H : \Delta[1] \times \Delta[1] \to \Delta[1]$ from the identity of $\Delta[1]$ to the map $i_1 r : \Delta[1] \to \Delta[1]$. An n simplex of $\Delta[1] \times \Delta[1]$ is a pair of

Δ morphisms $\sigma, \tau : [n] \to [1]$, while an n simplex of $\Delta[1]$ is just a Δ morphism $[n] \to [1]$. Observe that

$$\begin{aligned} h : [1] \times [1] &\to [1] \\ (0,0) &\mapsto 0 \\ (1,0) &\mapsto 1 \\ (0,1) &\mapsto 1 \\ (1,1) &\mapsto 1 \end{aligned}$$

is a map of ordered sets, so

$$(\sigma, \tau) \mapsto h(\sigma \times \tau)$$

defines a map

$$(\Delta[1] \times \Delta[1])_n \to \Delta[1]_n$$

which is clearly contravariantly functorial in n for Δ morphisms, hence we can view this as a map of simplicial sets H as above. From the fact that $h|[1] \times \{0\} : [1] \to [1]$ is the identity, and $h|[1] \times \{1\} : [1] \to [1]$ is the constant function 1, we see that H is as desired. $\square$

Lemma 4.7.4. *For any simplicial set Y, the evaluation maps $e_0, e_1 : Y^I \to Y$ are both deformation retracts of the constant path map $i_Y : Y \to Y^I$.*

Proof. We can view the deformation retract of i_0 constructed in the previous lemma as a commutative diagram of simplicial sets

$$\begin{tikzcd} \{0\} \times I \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ I \times I \arrow[r, "H"] & I \end{tikzcd}$$

with $Hi_0 = i_0 r$ and $Hi_1 = \mathrm{Id}$, as in (4.7.1). If we apply the contravariant functor Y^{-} to this diagram and note that $Y^{I \times I} = (Y^I)^I$, the resulting diagram is of the form (4.7.2) with $K := Y^H$ satisfying $e_0 K = i_Y e_0$ and $e_1 K = \mathrm{Id}$. $\square$

We would like to establish analogs of the topological results of §**??** for simplicial sets. The argument used in the proof of Lemma 2.3.1 does *not* carry over verbatim to simplicial sets: the analogous inclusion $J \hookrightarrow \Delta[1] \times \Delta[1]$ does *not* have a retract. However, the analogous result will hold in simplicial sets under the additional assumption that X is fibrant (§4.4):

Lemma 4.7.5. *If X is a fibrant simplicial set, then the product of the evaluation maps $e_0 \times e_1 : X^I \to X \times X$ is a fibration of simplicial sets (§4.4).*

Proof. The map $X^I \to X \times X$ in question is the one induced by the inclusion of the boundary $i : \{0, 1\} \hookrightarrow I$. The lemma is then a special case of [Hov, 3.3.1] where p is the map from X to the terminal object. The result we want is not hard to prove directly using basic facts about the model category of simplicial sets from §4.4. Consider an injective map of simplicial sets $K \hookrightarrow L$ whose geometric realization $|K| \hookrightarrow |L|$ is a weak equivalence of topological spaces (so $K \hookrightarrow L$ is an arbitrary trivial cofibration of simplicial sets) and a commutative diagram

$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d] & X^I \arrow[d] \\ L \arrow[r, "b"] \arrow[ur, dotted, "l"] & X \times X \end{tikzcd}$$

of simplicial sets. We must show that there is a lift l as indicated. By the adjointness (4.6.1), the map a can be viewed as a map $a : K \times I \to X$ and the map b can be viewed as a map $L \times \{0,1\} \to X$ and the commutativity of the solid square can be viewed as saying that a and b agree on $K \times \{0,1\}$. Let W be the union of $K \times I$ and $L \times \{0,1\}$ inside of $L \times I$, so we have a pushout diagram

$$\begin{tikzcd} K \times \{0,1\} \arrow[r] \arrow[d] & L \times \{0,1\} \arrow[d] \\ K \times I \arrow[r] & W \end{tikzcd}$$

of simplicial sets, an inclusion $j : W \hookrightarrow L \times I$, and a map $f = (a,b) : W \to X$. Since weak equivalences of simplicial sets are closed under products,[4] the top horizontal arrow is a weak equivalence. Since the left vertical arrow is a cofibration (i.e. is injective) and **sSets** is left proper (Theorem 4.4.3), the bottom horizontal arrow is a weak equivalence. The map j is hence a trivial cofibration because it is injective and is a weak equivalence by 2-out-of-3. Using the adjointness (4.6.1) we see that a lift l in the original diagram is the same thing as a lift in the diagram below.

$$\begin{tikzcd} W \arrow[r, "f=(a,b)"] \arrow[d, "j"'] & X \\ L \times I \arrow[ur, dotted] & \end{tikzcd}$$

Such a lift exists because X is fibrant and j is a trivial cofibration. □

Lemma 4.7.6. *Suppose $f : X \to Y$ is an arbitrary map between fibrant simplicial sets. Let $i_Y : Y \to Y^I$ denote the constant path and $e_0, e_1 : Y^I \to Y$ the evaluations. Then we have a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "(\mathrm{Id}_X, i_Y f)"] \arrow[dr, "f"'] & & X \times_Y^{e_0} Y^I \arrow[dl, "e_1 \pi_2"] \\ & Y & \end{tikzcd}$$

where $(\mathrm{Id}_X, i_Y f)$ is a homotopy equivalence and $e_1\pi_2$ is a fibration.

Proof. Argue exactly as in the proof of the topological analog Lemma 2.3.2, replacing the use of Lemma 2.3.1 there with the use of Lemma 4.7.5 (this is where we need Y fibrant). Note that $i_Y : Y \to Y^I$ is a homotopy equivalence by Lemma 4.7.4. One uses the fact that X is fibrant to know that the projection $p_2 : X \times Y \to Y$ is a fibration. □

[4]This follow from the following facts: 1) geometric realization (to **K**) preserves products (Lemma 4.3.5) and 2) a product of weak equivalences in **Top** is a weak equivalence between formation of homotopy groups commutes with products.

4.8. **Whitehead's Theorem.** There are various results in algebraic topology that go by the name "Whitehead's Theorem." In this section, we recall some of these results.

Theorem 4.8.1. (Whitehead) *Any trivial cofibration between between fibrant simplicial sets admits a deformation retract. Any weak equivalence between fibrant simplicial sets is a homotopy equivalence.*

Proof. For the first statement: Suppose $j : X \to Z$ is a trivial cofibration and X and Z are fibrant. By Lemma 4.7.6 the map $e_1\pi_2$ in the diagram

$$\begin{tikzcd} X \arrow[r, "(\mathrm{Id}, i_Z j)"] \arrow[d, "j"'] & X \times_Z^{e_0} Z^I \arrow[d, "e_1\pi_2"] \\ Z \arrow[r, equal] \arrow[ur, dotted, "(r,H)"] & Z \end{tikzcd}$$

is a fibration, so we can lift as indicated. As discussed above, this (r, H) can be viewed as the desired deformation retract, with $rj = \mathrm{Id}_X$ and the abusively denoted map $H : Z \times I \to Z$ corresponding to H under the adjunction (2.3.1) providing the homotopy between jr and the identity of Z.

For the second statement: Suppose $f : X \to Y$ is a weak equivalence of fibrant simplicial sets. Factor f as a trivial cofibration $j : X \to Z$ followed by a fibration $p : Z \to Y$. Note that p is in fact a trivial fibration, by 2-out-of-3 for weak equivalences. Notice also that Z is fibrant since Y is fibrant and fibrations are closed under composition. Since Y (like any simplicial set) is cofibrant, we can find a section $s : Y \to Z$ of p by lifting in the diagram below.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Z \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, dotted, "s"] & Y \end{tikzcd}$$

The section s is a trivial cofibration (it is injective since it has a retract p and it is a weak equivalence by 2-out-of-3) so we can lift as indicated in the diagram

$$\begin{tikzcd} Y \arrow[r, "(i_Z s)"] \arrow[d, "s"'] & Z^I \arrow[d, "(e_0, e_1)"] \\ Z \arrow[r, "(\mathrm{Id}, sp)"] \arrow[ur, dotted, "J"] & Z \times Z \end{tikzcd}$$

because (e_0, e_1) is a fibration by Lemma 4.7.5 since Z is fibrant. Note that the solid square commutes since

$$\begin{aligned} (e_0, e_1) i_Z s &= (s, s) \\ &= (s, sps) \\ &= (\mathrm{Id}, sp)s. \end{aligned}$$

This lift J can be viewed as a homotopy between sp and the identity of Z.

To finish the proof of the second statement, we combine the result of the first statement with what we did immediately above. The homotopy $p * H * s$ provides a homotopy between $pjrs = frs$ and $\mathrm{Id}_Y = ps$. The homotopy $r * J * j$ provides a homotopy between $rspj = rsf$ and $\mathrm{Id}_X = rj$. We conclude that f is a homotopy equivalence with homotopy inverse $rs : Y \to X$. □

Remark 4.8.2. The most general variant of "Whitehead's Theorem" is perhaps the statement that, in any model category, a map between fibrant and cofibrant objects is a weak equivalence iff it is a "homotopy equivalence" in the sense of the abstract homotopy theory of model categories ([Hov, 1.2.8] or [Hir, 7.5.10]). In any given model category one can usually unravel the abstract homotopy equivalence notion into something recognizable by the "man on the street" (ha!) who has no knowledge of model categories. I prefer to give the unravelled result in these notes whenever possible.

Theorem 4.8.3. (Whitehead) *If $f : X \to Y$ is a trivial cofibration of simplicial sets, then $|f| : |X| \to |Y|$ admits a deformation retract. If $f : X \to Y$ is any weak equivalence of simplicial sets then $|f| : |X| \to |Y|$ is a homotopy equivalence of topological spaces.*

Proof. Since $|X|$ and $|Y|$ are CW complexes by Lemma 4.3.2, this follows by applying the "usual formulation" (c.f. [Hat, 4.5]) of Whitehead's theorem to each connected component of $|X|$ (the connected componenents of $|X|$ are bijective with those of $|Y|$ via the weak equivalence $|f|$). $\square$

Corollary 4.8.4. *Suppose $f : X \to Y$ is a weak equivalence of simplicial sets. Then the induced map of degree-wise free simplicial abelian groups $\oplus_X \mathbb{Z} \to \oplus_Y \mathbb{Z}$ is a weak equivalence of simplicial abelian groups.*

Proof. By construction of simplicial homology, the homology of the chain complex $\mathsf{C}(\oplus_X \mathbb{Z})$ of free abelian groups computes the simplicial homology of $|X|$. (Depending on your particular definition of singular homology, this might be more easily recognizable as the homology of the normalized chain complex $\mathsf{N}(\oplus_X \mathbb{Z})$.) By the basic relationship between simplicial and singular homology (c.f. [Hat, 2.1]), this simplicial homology coincides with singular homology, which is homotopy invariant [Hat, 2.10]. Since $|f|$ is a homotopy equivalence by Theorem 4.8.3, the map $|f|$ induces isomorphisms on singular homology groups, hence the maps of simplicial homology groups must also be isomorphisms, whence the result. $\square$

4.9. **Quillen's Theorem.** Quillen's theorem on simplicial sets and topological spaces says that the model categories of simplicial sets (§4.4) and topological spaces (§2) are very closely related via the singular simplex functor Sing in (4.3.1) and the geometric realization functor $|\_|$ in (4.3.2).

We first mention an easy result on the singular simplex and geometric realization constructions of §4.3:

Lemma 4.9.1. *Let $\mathbf{C}$ be a category with all direct limits, $Z \in \mathbf{csC}$, $X^Z : \mathbf{C} \to \mathbf{sSets}$ the singular simplex functor, $|\_|_Z : \mathbf{sSets} \to \mathbf{C}$ its left adjoint geometric realization functor (Proposition 4.3.1). Suppose that for each $n > 0$ and each $k \in [n]$, the geometric realization*

$$|i_n^k|_Z : |\Lambda^k[n]|_Z \to |\Delta[n]|_Z$$

of the inclusion of the k-horn has a retract in $\mathbf{C}$. (This is the case for the standard topological geometric realization.) Then the singular simplex $X^Z \in \mathbf{sSets}$ is fibrant for every $X \in \mathbf{C}$.

Proof. For the parenthetical remark: The standard topological geometric realization of i_n^k is the inclusion of all but one of the codimension one faces of the standard simplex $\Delta_n^\top$.

This map clearly admits a retract, even a deformation retract. For the main statement of the lemma: By adjointness, a lift in the diagram of simplicial sets

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, "i_n^k"'] & X^Z \\ \Delta[n] \arrow[ur, dotted] & \end{tikzcd}$$

is the same thing as a lift in the **C**-diagram

$$\begin{tikzcd} |\Lambda^k[n]|_Z \arrow[r] \arrow[d, "|i_n^k|_Z"'] & X \\ |\Delta[n]|_Z \arrow[ur, dotted] & \end{tikzcd}$$

which can certainly be found if $|i_n^k|_Z$ has a retract. □

Theorem 4.9.2. (Quillen) *The singular simplex functor*

$$\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSets}$$

and its left adjoint, the geometric realization functor

$$|\_| : \mathbf{sSets} \to \mathbf{Top}$$

satisfy the following properties:

(1) A map f in **sSets** *is a weak equivalence iff $|f|$ is a weak equivalence in* **Top**.
(2) $|\_|$ takes cofibrations to cofibrations and trivial cofibrations to trivial cofibrations. That is: $|\_|$ is a left Quillen functor.
(3) Sing *takes fibrations to fibrations and trivial fibrations to trivial fibrations. That is:* Sing *is a right Quillen functor.*
(4) For a fibrant simplicial set X, and a base point $\mathbf{1} \in X_0$, the homotopy sets $\pi_n(X, \mathbf{1})$ of Definition 4.5.2 are naturally isomorphic to the homotopy sets of $|X|$ with the corresponding base point.
(5) For any $X \in \mathbf{Top}$, $\mathrm{Sing}\, X \in \mathbf{sSets}$ is fibrant and for any $x \in X$, the usual homotopy set $\pi_n(X, x)$ is naturally bijective with $\pi_n(\mathrm{Sing}\, X, x)$, the latter defined by Definition 4.5.2.
(6) If f is a fibration in **sSets**, *then $|f|$ is a fibration in* **Top**.
(7) A map f in **Top** *is a weak equivalence iff $\mathrm{Sing}\, f$ is a weak equivalence in* **sSets**.
(8) For any simplicial set X, the adjunction morphism $X \to \mathrm{Sing}\, |X|$ is a weak equivalence of simplicial sets.
(9) For any topological space Y, the adjunction morphism $|\mathrm{Sing}\, Y| \to Y$ is a weak equivalence in **Top**.
(10) The functors

$$\mathrm{Ho}\,\mathrm{Sing} : \mathrm{Ho}\,\mathbf{Top} \to \mathrm{Ho}\,\mathbf{sSets}$$
$$\mathrm{Ho}\,|\_| : \mathrm{Ho}\,\mathbf{sSets} \to \mathrm{Ho}\,\mathbf{Top}$$

yield an equivalence of homotopy categories $\mathrm{Ho}\,\mathbf{Top} \cong \mathrm{Ho}\,\mathbf{sSets}$.
(11) For every $X \in \mathbf{sSets}$ and every $Y \in \mathbf{Top}$, a map $|X| \to Y$ in **Top** *is a weak equivalence iff the adjoint map $X \to \mathrm{Sing}\, Y$ is a weak equivalence of simplicial sets.*

Proof. (1) holds by definition of a weak equivalence of simplicial sets (Definition 4.4.1).

For (2), the only issue, in light of (1), is to show that $| \_ |$ takes cofibrations to cofibrations. Recall (Theorem 4.4.2, Lemma 1.5.9) that the cofibrations in **sSets** are the smallest saturated class of **sSets** morphisms containing the set

$$I = \{\partial\Delta[n] \hookrightarrow \Delta[n] : n \in \mathbb{N}\}.$$

Since the left adjoint $| \_ |$ preserves direct limits and the cofibrations in **Top** are a saturated class of maps (as is the case in any model category by Proposition 1.5.6), it suffices to note that

$$|\partial\Delta[n] \hookrightarrow \Delta[n]| \cong |S^{n-1} \hookrightarrow D^n|$$

is a cofibration in **Top** (c.f. Theorem 2.1.2).

(3) follows formally from (2) by adjointness and the characterization of fibrations (resp. trivial fibrations) as maps with the RLP w.r.t. trivial cofibrations (resp. cofibrations).

(4) is [Hov, 3.6.3].

(5) We established fibrancy of $\operatorname{Sing} X$ in Lemma 4.9.1. The other part is just a matter of staring at the definition of $\pi_n(\operatorname{Sing} X, \mathbf{1})$ until you agree that it coincides with whatever you think is the "usual definition" of homotopy sets. (In a sense there is really nothing to prove since we can take this as the "usual definition.") First note that

$$\begin{aligned}(\operatorname{Sing} X)_0 &= \operatorname{Hom}_{\mathbf{Top}}(\Delta_0^{\top}, X) \\ &= X,\end{aligned}$$

so the chosen basepoint $x \in X$ is the same thing as a choice of base point $\mathbf{1}$ for $\operatorname{Sing} X$. An element of $\mathsf{Z}_n(\operatorname{Sing} X)$ is the same thing as an element $f \in (\operatorname{Sing} X)_n$ with all boundaries equal to $\mathbf{1}$, which is the same thing as a map of topological spaces $f : \Delta_n^{\mathrm{top}} \to X$ contracting the boundary of Δ_n^{top} to the basepoint x. The equivalence relation $\sim$ on $\mathsf{Z}_n(\operatorname{Sing} X)$ is given by declaring $f \sin f'$ iff there is a map $g : \Delta_{n+1}^{\mathrm{top}} \to X$ which contracts all but the last two codimension one facts to x and is given on these last two faces by f and f'. This should clearly be equivalent to any other definition of homotopy sets.

(6) is [Hov, 3.6.2]. This fact won't be necessary to establish the other parts of Quillen's theorem, though it is an important result originally due to Quillen. Hovey proves this en route to proving that **sSets** is a model category.

(7) By (5) and the definition of a weak equivalence in **Top**, $f : X \to Y$ is a weak equivalence in **Top** iff

$$\pi_n(\operatorname{Sing} f) : \pi_n(\operatorname{Sing} X) \to \pi_n(\operatorname{Sing} Y) \tag{4.9.1}$$

is bijective for all n (and all choices of basepoint when $n > 0$). But $\operatorname{Sing} X$ and $\operatorname{Sing} Y$ are fibrant, so the map (4.9.1) is naturally identified, via (4), with the map

$$\pi_n(|\operatorname{Sing} f|) : \pi_n(|\operatorname{Sing} X|) \to \pi_n(|\operatorname{Sing} Y|) \tag{4.9.2}$$

whose isomorphy (for all basepoints when $n > 0$) is, by definition, equivalent to $\operatorname{Sing} f$ being a weak equivalence in **sSets**.

For (8), we need to show that

$$\pi_n(|X|) \to \pi_n(|\operatorname{Sing}|X||)$$

is bijective (for all choices of basepoint when $n > 0$). But the LHS is naturally bijective with $\pi_n(\operatorname{Sing}|X|)$ by (5) and the RHS is naturally bijective with $\pi_n(\operatorname{Sing}|X|)$ by (4), since $\operatorname{Sing}|X|$ is fibrant.

(9) is similar to (8)

For (10), first note that the functors Sing and $|\_|$ are well-defined on the homotopy categories since they preserve weak equivalences by (1) and (7). The fact that the resulting functors are equivalences is immediate from (8) and (9), since the adjunctions become isomorphisms in the homotopy categories.

(11) is a formal consequence of (10).

□

Corollary 4.9.3. *A map $f : X \to Y$ of fibrant simplicial sets is a weak equivalence iff*

$$\pi_n(f) : \pi_n(X) \to \pi_n(Y)$$

is an isomorphism for all n (and all basepoints when $n > 0$).

Proof. Use (4) and the definition of weak equivalence of simplicial sets. □

Remark 4.9.4. Hovey replaces **Top** with **K** in his version of this statement [Hov, 3.6.7], but his proof makes perfect sense for arbitrary spaces, as far as I can tell. (He does a much better job of actually keeping track of how various isomorphisms are given rather than just mumbling "A is naturally identified with B" as I did.) Also, if you want to reconcile his "Quillen equivalence" statement with the statement above, you will want to keep track of the fact that this particular Quillen equivalence is much nicer than a typical Quillen equivalence, in that the functors involved both already reflect *arbitrary* weak equivalences. In fact, Hovey makes an implicit use of this fact in establishing his Quillen equivalence via the criterion of [Hov, 1.3.16(b)]: instead of checking that $FQUY \to Y$ is a weak equivalence (as he should in that criterion), he actually checks that $FUY \to Y$ (i.e. the map $|\operatorname{Sing} Y| \to Y$ for a space Y) is a weak equivalence. The two statements are equivalent because $FQUY \to Y$ factors as $FQUY \to FUY \to Y$ and the first map is F applied to the trivial fibration $QUX \to UX$, which is part of the "cofibrant replacement of UX" (the functorial factorization of the map from the initial object $\emptyset$ to UX as a cofibration $\emptyset \to QUX$ followed by a trivial fibration $QUX \to UX$) and we already know F will take the weak equivalence $QUX \to UX$ to a weak equivalence $FQUX \to FUX$ (this is all trivial because F is the geometric realization functor, so it takes weak equivalences to weak equivalences by definition).

Also note that there is a typo in [Hov, 1.3.13(b)] (the FQX should be FUX) and a confusing double usage of R in [Hov, 1.3.6.2] both for the "right-derived" and the "fibrant replacement," so in a sense he defines $RU := UR$, where the R's on the two sides have the two different meanings.

5. The abelian setting

Here we briefly review the theory of simplicial objects in an abelian category $\mathbf{A}$. The basic theorem of Dold-Kan-Puppe (Theorem 5.3.1) asserts that there is an equivalence of categories

$$\mathsf{N} : \mathbf{sA} \to \mathbf{Ch}_{\geq 0}\mathbf{A}$$

between simplicial objects in $\mathbf{A}$ and the category of chain complexes in $\mathbf{A}$ supported in non-negative degrees. Furthermore, if $\mathbf{A}$ is the category modules over a ring (or any other abelian category with a reasonable notion of underlying sets), then the equivalence N identifies the homotopy groups of the simplicial set $|M|$ underlying a simplicial module M with the homology of the corresponding complex (§5.4):

$$\pi_n(|M|) = \mathrm{H}_n(\mathsf{N}(M)).$$

Furthermore, the equivalence N takes homotopies in the sense of chain complexes to homotopies of simplicial objects in the sense of §3.7. The upshot is that the theory of simplicial objects in an abelian category $\mathbf{A}$ is essentially the same as the theory of chain complexes in $\mathbf{A}$ supported in non-negative degrees.

5.1. **Associated chain complexes.** Given an $A \in \mathbf{sA}$ and an $n \in \mathbb{N}$, we define a chain complex $\mathsf{C}(A) \in \mathbf{Ch}_{\geq 0}(\mathbf{A})$ by setting $\mathsf{C}(A)_n := A_n$, with boundary map

$$d_n : \mathsf{C}(A)_n \to \mathsf{C}(A)_{n-1}$$

defined by

$$d_n := \sum_{i=0}^{n} (-1)^i d_n^i : A_n \to A_{n-1}.$$

Lemma 5.1.1. $d_{n-1}d_n = 0$

Proof. Fix i, j with $0 \leq i < j \leq n$. Let $\partial : [n-2] \hookrightarrow [n]$ be the unique monic Δ morphism whose image contains neither i nor j. Then ∂ can be factored as $\partial = \partial_n^j \partial_{n-1}^i$ or as $\partial = \partial_n^i \partial_{n-1}^{j-1}$ (Lemma 3.1.1). In the formal sum $d_{n-1}d_n$ of maps from $A_n \to A_{n-2}$, the map $A(\partial)$ appears once for each of these two factorizations, with opposite signs. $\square$

The complex $\mathbf{C}(A)$ is called *unnormalized chain complex* associated to A. We define a subcomplex $\mathsf{N}(A) \subseteq \mathsf{C}(A)$ called the *normalized chain complex* (c.f. [DP, 3.1]) by setting

$$\mathsf{N}_n(A) := \bigcap_{i=1}^{n} \mathrm{Ker}(d_n^i : A_n \to A_{n-1}).$$

(Note that the intersection is over the kernels of all *but* the zeroth boundary map $d_n^0 : A_n \to A_{n-1}$.) By convention, $\mathsf{N}_0(A) := A_0$. It is easy to check that $d_n|\mathsf{N}_n(A) = d_n^0$ takes $\mathsf{N}_n(A)$ into $\mathsf{N}_{n-1}(A)$, so that $\mathsf{N}(A) \subseteq \mathsf{C}(A)$ is a subcomplex. Formation of $\mathsf{C}(A)$ and $\mathsf{N}(A)$ is clearly functorial in A, so we have functors

$$\mathsf{C} : \mathbf{sA} \to \mathbf{Ch}_{\geq 0}\mathbf{A}$$
$$\mathsf{N} : \mathbf{sA} \to \mathbf{Ch}_{\geq 0}\mathbf{A}.$$

We can compose either of these with the homology functors

$$\mathrm{H}_n : \mathbf{Ch}_{\geq 0}\mathbf{A} \to \mathbf{A}$$

to define functors abusively denoted

$$\mathrm{H}_n : \mathbf{sA} \to \mathbf{A}.$$

As the notation suggests, it does not matter which chain complex we use to compute the homology:

Lemma 5.1.2. *The inclusion $i : \mathsf{N}(A) \hookrightarrow \mathsf{C}(A)$ admits a retract $p : \mathsf{C}(A) \to \mathsf{N}(A)$ such that ip is homotopic to the identity. In particular, i is a homotopy equivalence and hence*

$$\mathrm{H}_n(\mathsf{N}(A)) = \mathrm{H}_n(\mathsf{C}(A))$$

for all n.

Proof. It is easiest to see this if we pass from $\mathsf{C}(A)$ to $\mathsf{N}(A)$ "one step at a time" instead of "all at once." To do this, we define a decreasing filtration

$$\cdots \subseteq \mathsf{N}^2(A) \subseteq \mathsf{N}^1(A) \subseteq \mathsf{N}^0(A) = \mathsf{C}(A)$$

by setting

$$\mathsf{N}^k(A)_n := \bigcap_{i=1}^{\min(k,n)} \operatorname{Ker}(d_n^{n+1-k} : A_n \to A_{n-1})$$

so that

$$\begin{aligned}
\mathsf{N}^0(A)_n &:= A_n \\
\mathsf{N}^1(A)_n &:= \begin{cases} \operatorname{Ker} d_n^n, & n > 0 \\ A_0, & n = 0 \end{cases} \\
\mathsf{N}^2(A)_n &:= \begin{cases} (\operatorname{Ker} d_n^n) \cap (\operatorname{Ker} d_n^{n-1}), & n > 1 \\ \operatorname{Ker} d_1^1, & n = 1 \\ A_0, & n = 0, \end{cases}
\end{aligned}$$

and so forth. It is easy to see that this is a filtration by subcomplexes, finite in each degree, and that $\cap_k \mathsf{N}^k(A) = \mathsf{N}(A)$.

Consider the map $p_n^1 := (\mathrm{Id} - s_{n-1}^{n-1} d_n^n) : A_n \to A_n$. (We adopt the convention that $s_n^i = 0$ when $i < 0$.) Using the simplicial identities from Lemma 3.2.2 one sees that p_1 takes values in $\mathsf{N}^1(A)$ and in fact defines a map of chain complexes $\mathsf{N}^0(A) \to \mathsf{N}^1(A)$ which clearly retracts the inclusion $\mathsf{N}^1(A) \hookrightarrow \mathsf{N}^0(A)$. Define a homotopy operator H^1 on $\mathsf{N}^0(A)$ by setting

$$H_n^1 := (-1)^n s_n^n : A_n \to A_{n+1}.$$

Using the simplicial identities

$$d_{n+1}^i s_n^n = \begin{cases} s_{n-1}^{n-1} d_n^i, & i < n \\ \mathrm{Id}, & i = n, n+1 \end{cases}$$

we see that

$$\begin{aligned}
d_{n+1} H_n^1 + H_{n-1}^1 d_n &= -s_{n-1}^{n-1} d_n^n \\
&= p_1 - \mathrm{Id},
\end{aligned}$$

so the homotopy H^1 shows that the inclusion $\mathsf{N}^1(A) \subseteq \mathsf{N}^0(A)$ is a homotopy equivalence with homotopy inverse p_1. Similarly, one sees that the map

$$p_n^2 := (\mathrm{Id} - s_{n-1}^{n-2} d_n^{n-1}) : \mathsf{N}^1(A)_n \to A_n$$

in fact defines a chain complex map $p_2 : \mathsf{N}^1(A) \to \mathsf{N}^2(A)$ retracting the inclusion $\mathsf{N}^2(A) \hookrightarrow \mathsf{N}^1(A)$. Furthermore, the homotopy operator H^2 on $\mathsf{N}^1(A)$ defined by

$$H_n^2 := (-1)^{n-1} s_n^{n-1} : \mathsf{N}^1(A)_n \to \mathsf{N}^1(A)_{n+1}$$

satisfies

$$d_{n+1} H_n^2 + H_{n-1}^2 d_n = p_2 - \mathrm{Id},$$

so $\mathsf{N}^2(A) \subseteq \mathsf{N}^1(A)$ is a homotopy equivalence. Continuing in this manner, we see that $i : \mathsf{N}(A) \subseteq \mathsf{C}(A)$ is a homotopy equivalence with homotopy inverse $p : \mathsf{C}(A) \to \mathsf{N}(A)$ defined by

$$p_n := (\mathrm{Id} - s_{n-1}^0 d_n^1) \cdots (\mathrm{Id} - s_{n-1}^{n-2} d_n^{n-1})(\mathrm{Id} - s_{n-1}^{n-1} d_n^n) : A_n \to \mathsf{N}(A)_n.$$

□

Example 5.1.3. For example, consider the constant simplicial object (Example 3.2.1) $\underline{A} \in \mathbf{sA}$ associated to an object $A \in \mathbf{A}$. Then $\mathsf{C}(\underline{A})_n = A$ for every $n \in \mathbb{N}$, and $\mathsf{C}(\underline{A})$ has boundary map $\mathsf{C}(A)_n \to \mathsf{C}(A)_{n-1}$ given by the alternating sum of $n+1$ copies of the identity of A, which is zero when n is odd and the indentity of A when n is even, so

$$\mathsf{C}(\underline{A}) = [\, \cdots \longrightarrow A \xrightarrow{=} A \xrightarrow{0} A \xrightarrow{=} A \xrightarrow{0} A \,],$$

which clearly has homology

$$\mathrm{H}_n(\underline{A}) = \begin{cases} A, & n = 0 \\ 0, & n > 0. \end{cases}$$

On the other hand, $\mathsf{N}(\underline{A})$ is just the subcomplex of $\mathsf{C}(\underline{A})$ consisting solely of the A in degree zero—this obviously has the same homology as $\mathsf{C}(\underline{A})$ and it is not particularly hard in this example to see that $\mathsf{N}(\underline{A}) \hookrightarrow \mathsf{C}(\underline{A})$ is a homotopy equivalence.

Remark 5.1.4. Some authors (e.g. [Wei, 8.3.6]) define the normalized chain complex[5] by

$$\mathsf{N}(A)_n := \bigcap_{i=0}^{n-1} \mathrm{Ker}(d_n^i : A_n \to A_{n-1}).$$

The homology of this complex is the same as the homology of the normalized complex as we have defined it here. For one thing, trivial modifications of the proof of Lemma 5.1.2 show that this version of the normalized complex is also homotopy equivalent to $\mathsf{C}(A)$. Alternatively, one can use the "front-to-back" duality in [Wei, 8.2.10].

[5]Weibel also calls the normalized chain complex the *Moore complex* whereas Goerss and Jardine [GJ] call the *un*normalized chain complex the Moore complex.

5.2. **Degenerate subcomplex.** For $A \in \mathbf{sA}$, the *degenerate subcomplex* $\mathsf{D}(A)$ of the unnormalized chain complex $\mathsf{C}(A)$ is defined by

$$\mathsf{D}(A)_n := \sum_{\sigma:[n]\to[m]} \mathrm{Im}(A(\sigma) : A_m \to A_n),$$

where the (non-direct!) sum is over surjective Δ-morphisms $\sigma : [n] \to [m]$ with $n > m$. Since all such Δ morphisms factor though one of the Δ-morphisms $\sigma^i_{n-1} : [n] \to [n-1]$ (Lemma 3.1.1) whose images under A are the degeneracies $s^i_{n-1} : A_{n-1} \to A_n$, we could alternatively define the degenerate subcomplex by

$$\mathsf{D}(A)_n := \sum_{i=0}^{n-1} \mathrm{Im}(s^i_{n-1} : A_{n-1} \to A_n).$$

To see that $\mathsf{D}(A)$ is a subcomplex (i.e. to see that $d\mathsf{D}(A)_n \subseteq \mathsf{D}(A)_{n-1}$), it suffices to prove that, for any fixed $j \in [n-1]$,

$$\left(\sum_{i=0}^{n}(-1)^i d^i_n\right)(\mathrm{Im}\, s^j_n) \subseteq \mathsf{D}(A)_{n-1}.$$

This follows immediately from the simplicial relations of Lemma 3.2.2—the terms in the sum with $i = j$ and $i = j+1$ cancel out and the others are degeneracies.

Lemma 5.2.1. *The unnormalized complex $\mathsf{C}(A)$ splits naturally as a direct sum of the normalized chain complex $\mathsf{N}(A)$ and the degenerate subcomplex $\mathsf{D}(A)$.*

Proof. [6] First we show that $\mathsf{D}(A) \cap \mathsf{N}(A) = 0$. The key is to make use of the map

$$p = (\mathrm{Id} - s^0_{n-1}d^1_n)\cdots(\mathrm{Id} - s^{n-2}_{n-1}d^{n-1}_n)(\mathrm{Id} - s^{n-1}_{n-1}d^n_n) : A_n \to A_n$$

of Lemma 5.1.2. It is clear that p is the identity on $\mathsf{N}(A)_n$, so it suffices to show that $\mathsf{D}(A)_n \subseteq \mathrm{Ker}\, p$. From the simplicial relations concerning $d^n_n s^i_{n-1}$ and $s^{n-1}_{n-1}s^i_{n-2}$ (Lemma 3.2.2) we see that the image of any s^i_{n-1} under the map $p_1 = (\mathrm{Id} - s^{n-1}_{n-1}d^n_n)$ is again in the image of s^i_{n-1} and we see that the image of s^{n-1}_{n-1} is in the kernel of p_1. Using this, and the simplicial identities concerning $d^{n-1}_n s^i_{n-1}$ and $s^{n-2}_{n-1}s^i_{n-2}$ we then see that the image of any s^i_{n-1} under the map

$$p_2 = (\mathrm{Id} - s^{n-2}_{n-1}d^{n-1}_n)p_1$$

is again in the image of s^i_{n-1} and the images of both s^{n-1}_{n-1} and s^{n-2}_{n-1} are in the kernel of p_2. Continuing in this manner we find that the image of any s^i_{n-1} is in the kernel of p.

Next we show that $\mathsf{D}(A) + \mathsf{N}(A) = \mathsf{C}(A)$. This is already clear from the fact that p takes values in $\mathsf{N}(A)_n$ as we saw in Lemma 5.1.2 because it is clear from the formula for p that p is equal to the identity modulo morphisms factoring through one of the s^i_{n-1}. □

The proof shows that the projection $\mathsf{C}(A) \to \mathsf{N}(A)$ corresponding to the splitting of Lemma 5.2.1 coincides with the map p of Lemma 5.1.2.

Proposition 5.2.2. *The degenerate subcomplex $\mathsf{D}(A)$ is contractible.*

Proof. [DP, 3.22] □

[6] Weibel's proof of this in [Wei, 8.3.7] seems to be wrong. Besides the typo with the subscripted j (should be an i), the third sentence contains the false assertion $d^i_n y = x_i$.

5.3. **Dold-Kan-Puppe.** In this section we will prove that the normalized chain complex functor N of §5.1 is an equivalence of categories. We begin by defining the inverse functor

$$\mathsf{K} : \mathbf{Ch}_{\geq 0}\mathbf{A} \to \mathbf{sA}$$
$$\mathsf{K}(B)_n := \bigoplus_{\sigma:[n]\to[k]} B_k,$$

where the direct sum is over all surjective Δ morphisms $\sigma : [n] \to [k]$. For such a σ, we let $\overline{\sigma} : B_k \to \mathsf{K}(B)_n$ denote the structure map to the direct sum. The degeneracy map

$$\mathsf{K}(B)(\tau) : \mathsf{K}(B)_m \to \mathsf{K}(B)_n$$

associated to a surjective Δ morphism $\tau : [n] \to [m]$ is defined tautologically by requiring

$$\begin{tikzcd} \mathsf{K}(B)_m \arrow[r, "\mathsf{K}(B)(\tau)"] & \mathsf{K}(B)_n \\ B_k \arrow[u, "\overline{\sigma}"] \arrow[ur, "\overline{\sigma\tau}"'] & \end{tikzcd}$$

to commute for each surjective Δ morphism $\sigma : [m] \to [k]$. The boundary map

$$d^i_n : \mathsf{K}(B)_n \to \mathsf{K}(B)_{n-1}$$

associated to the Δ morphism $\partial^i_n : [n-1] \to [n]$ is defined as follows: For any surjective Δ morphism $\sigma : [n] \to [k]$, if the composition $\sigma\partial^i_n$ is also surjective, then we require $d^i_n\overline{\sigma} = \overline{\sigma\partial^i_n}$. Otherwise, we can write $\sigma\partial^i_n = \partial^j_k\tau$ (uniquely) for a surjective Δ morphism $\tau : [n-1] \to [k-1]$ and an injective Δ morphism $\partial^j_k : [k-1] \to [k]$. In this case we let $d^i_n\overline{\sigma}$ be the boundary map $B_k \to B_{k-1}$ followed by $\overline{\tau} : B_{k-1} \to \mathsf{K}(B)_{n-1}$ if $j = 0$, and we let $d^i_n\overline{\sigma} := 0$ if $j > 0$.

Theorem 5.3.1. *The functor* $\mathsf{N} : \mathbf{sA} \to \mathbf{Ch}_{\geq 0}(\mathbf{A})$ *is an equivalence of categories with inverse* K*.*

Proof. See [DP, 3.6] or [Wei, 8.4.1]. □

Lemma 5.3.2. *The functor* N *takes simplicial homotopies to homotopies of chain complexes and the functor* K *takes homotopies of chain complexes to simplicial homotopies.*

Proof. This is a straightforward exercise with the definitions and the formulas for N and K. See [Wei, 8.3.13], for example. □

5.4. **Homology and homotopy.** We now fix a ring A and specialize to the case where our abelian category $\mathbf{A} = \mathbf{Mod}(A)$ is the category of A modules. Then we can pass to underlying sets to define a functor

$$|\ | : \mathbf{Mod}(A) \to \mathbf{Sets},$$

and hence also a functor

$$|\ | : \mathbf{sMod}(A) \to \mathbf{sSets}.$$

Lemma 5.4.1. *For any* $M \in \mathbf{sMod}(A)$ *we have* $\pi_n(|M|) = \mathrm{H}_n(M)$ *for any base point of* $|M|$*.*

Proof. First note that M has a simplicial abelian group structure, so $|M|$ is a fibrant simplicial set (Corollary 5.6.6) and it makes sense to speak of the simplicial homotopy sets of $|M|$. Next we claim, quite generally, that the homotopy sets of a simplicial set $|G|$ underlying any simplicial group G have no dependence on the choice of basepoint (§4.5). The point is that for any choice of basepoint $\mathbf{1}$ (i.e. for any choice of an element $g_0 \in G_0$), multiplication by g_0^{-1} and its degeneracies yields an isomorphism of simplicial sets $|G| \to |G|$ taking the base point g_0 to the basepoint $1 \in G_0$. That is, we have an isomorphism $(|G|, g_0) \cong (|G|, 1)$ of based simplicial sets and hence an isomorphism of the corresponding homotopy groups. Now we compute the homotopy sets of $|M|$ using the basepoint $0 \in M_0$ (whose degeneracies are just the zeros of the other M_n). It is clear that the subset $\mathsf{Z}_n(|M|)$ defined in §4.5 is nothing but

$$\{m \in M_n : d_n^i(m) = 0 \,\text{for}\, i = 0, \ldots, n\},$$

which is nothing but the group of cycles $\mathsf{Z}_n(\mathsf{N}(M))$ in the normalized chain complex $\mathsf{N}(M)$. We will denote this subset of M_n by $\mathsf{Z}_n(M)$ in the rest of the proof.

It remains only to prove that the equivalence relation $\sim$ on $\mathsf{Z}_n(M)$ defined in §4.5 is equivalent to the relation of "differing by a coboundary" used to define homology as a quotient of $\mathsf{Z}_n(M)$. For $x, y \in \mathsf{Z}_n(M)$, if $x \sim y$ then there is $z \in M_{n+1}$ with

$$d_{n+1}^i z = \begin{cases} 0, & i = 0, \ldots, n-1 \\ x, & i = n \\ y, & i = n+1. \end{cases}$$

Using the simplicial relations

$$d_{n+1}^i s_n^n = \begin{cases} s_{n-1}^{n-1} d_n^i, & i = 0, \ldots, n-1 \\ \mathrm{Id}, & i = n, n+1 \end{cases}$$

(Lemma 3.2.2), we then see that $z' := s_n^n x - z$ satisfies

$$d_{n+1}^i z' = \begin{cases} 0, & i = 0, \ldots, n \\ x - y, & i = n+1, \end{cases}$$

hence z' is an element of $\mathsf{N}(M)_{n+1}$ (in its "front-to-back" incarnation as in Remark 5.1.4) whose coboundary therein is $x - y$ (up to a sign). This process can clearly be reversed to go from z' to z so that differing by a coboundary implies $\sim$ equivalence. □

5.5. **Model structure on connective complexes.** Throughout this section we let $\mathbf{A}$ be an abelian category with *functorial projective resolutions*: i.e. for each object $A \in \mathbf{A}$, there is a projective object P_A of $\mathbf{A}$ and a surjection $P_A \to A$, which is functorial in A in the evident sense (any $\mathbf{A}$-morphism $A \to B$ lifts functorially to an $\mathbf{A}$-morphism $P_A \to P_B$ making the obvious square commute). For example, if $\mathbf{A} = \mathbf{Mod}(R)$ for a ring R, one obtains such a functorial projective resolution by letting P_A be the free R-module on the underlying set of A.

In practice, most abelian categories "encountered in nature" with enough projectives will have functorial projective resolutions because Grothendieck's general criterion [T, 1.10.1] for the existence of enough projectives in a nice abelian category also yields the existence of functorial projective resolutions.

We will now discuss the *projective model structure* on the category $\mathbf{Ch}_{\geq 0}\mathbf{A}$ of non-negatively graded (aka *connective*) chain complexes in $\mathbf{A}$.

Definition 5.5.1. A morphism $h : B \to C$ in $\mathbf{Ch}_{\geq 0}\mathbf{A}$ is called a ...

... *weak equivalence* (*quasi-isomorphism*) iff $\mathrm{H}_n(h)$ is an isomorphism for all $n \in \mathbb{N}$.
... *fibration* iff $h_n : B_n \to C_n$ is surjective for all $n > 0$.
... *cofibration* iff $h_n : B_n \to C_n$ is injective with projective cokernel for all $n \in \mathbb{N}$.
... *trivial fibration* iff it is both a fibration and a weak equivalence.
... *trivial cofibration* iff it is both a cofibration and a weak equivalence.

An object A of $\mathbf{Ch}_{\geq 0}\mathbf{A}$ is called *cofibrant* (resp. *acyclic*) iff $0 \to A$ is a cofibration (resp. a weak equivalence—i.e. $\mathrm{H}_n(A) = 0$ for all $n \in \mathbb{N}$).

Theorem 5.5.2. *Suppose* $\mathbf{A}$ *is an abelian category with functorial projective resolutions. Then* $\mathbf{Ch}_{\geq 0}\mathbf{A}$ *is a model category with the weak equivalences, fibrations, and cofibrations in Definition 5.5.1.*

Proof. The details will occupy the remainder of the section. It is obvious that weak equivalences satisfy 2-out-of-3. It is also easy to see that weak equivalences, fibrations, and cofibrations are closed under retracts (exercise). We will establish the necessary lifting properties in Lemmas 5.5.6 and 5.5.11 and the factorizations in Theorem 5.5.8. □

Remark 5.5.3. If, in the definition of model category, one drops the requirement that the factorizations be functorial, then the above theorem continues to hold under the weaker assumption of the existence of projective resolutions which are not necessarily functorial. For example, we can take $\mathbf{A}$ to be the category $\mathbf{Ab}^{\mathrm{fg}}$ of *finitely generated* abelian groups.

Definition 5.5.4. For $A \in \mathbf{A}$ and $n \in \mathbb{N}$, we let $D^n(A) \in \mathbf{Ch}_{\geq 0}\mathbf{A}$ denote the complex which is A in degrees n and $n-1$, zero in other degrees, and with boundary map in degree n given by the identity $A \to A$. To be clear, our convention is that $D^0(A)$ is the complex which is A in degree zero and zero in all other degrees. The complex $D^n(A)$ is called the n-dimensional *disk* on A. Similarly, we let $S^n(A)$ denote the complex—called the n-dimensional *sphere* on A—which is A in degree n and zero in all other degrees. The convention $S^{-1}(A) = 0$ is convenient.

Notice that, for $n > 0$, $D^n(A)$ is acyclic for any A. We have a natural bijection

$$\mathrm{Hom}_{\mathbf{Ch}_{\geq 0}\mathbf{A}}(D^n(A), B) = \mathrm{Hom}_{\mathbf{A}}(A, B_n) \tag{5.5.1}$$

(given by $f \mapsto f_n$) expressing the fact that the n-disk functor D^n is left adjoint to the "degree n" functor $B \mapsto B_n$. We also have a natural bijection

$$\mathrm{Hom}_{\mathbf{Ch}_{\geq 0}\mathbf{A}}(S^n(A), B) = \mathrm{Hom}_{\mathbf{A}}(A, \mathsf{Z}_n(B)), \tag{5.5.2}$$

where $\mathsf{Z}_n(B) = \mathrm{Ker}(d_n : B_n \to B_{n-1})$ denotes the object of n-dimensional cycles in the complex B. For each $n \in \mathbb{N}$, we have a natural map of chain complexes $S^{n-1}(A) \to D^n(A)$ given by the identity in degree $n-1$ and zero in other degrees.

We will often be interested in the case where $\mathbf{A} = \mathbf{Mod}(R)$ for a ring R. The case $R = \mathbb{Z}$ is of particular interest, for then $\mathbf{Ch}_{\geq 0}\mathbf{A} = \mathbf{Ch}_{\geq 0}\mathbf{Ab}$ is the category of non-negatively graded chain complexes of abelian groups. We set $D^n := D^n(R)$ and $S^n := S^n(R)$. The bijections (5.5.1) and (5.5.2) specialize to

$$\operatorname{Hom}_{\mathbf{Ch}_{\geq 0}\mathbf{Mod}(R)}(D^n, B) = B_n \tag{5.5.3}$$

$$\operatorname{Hom}_{\mathbf{Ch}_{\geq 0}\mathbf{Mod}(R)}(S^n, B) = \mathsf{Z}_n(B), \tag{5.5.4}$$

(given by $f \mapsto f_n(1)$) for each $B \in \mathbf{Ch}_{\geq}\mathbf{Mod}(R)$ and each $n \in \mathbb{N}$.

Lemma 5.5.5. *For a map $h : B \to C$ in $\mathbf{Ch}_{\geq 0}\mathbf{A}$, the following are equivalent:*

(1) h is a trivial fibration (Definition 5.5.1).
(2) The map $h_n : B_n \to C_n$ is surjective for every $n \in \mathbb{N}$ and h is a quasi-isomorphism.
(3) The map $(d, h_n) : B_n \to \mathsf{Z}_{n-1}(B) \times_{\mathsf{Z}_{n-1}(C)} C_n$ is surjective for every $n \in \mathbb{N}$. (When $n = 0$, this map is just the map $h_0 : B_0 \to C_0$.)

When $\mathbf{A} = \mathbf{Mod}(R)$ for a ring R, these conditions are equivalent to:

(4) The map h has the RLP with respect to the maps $S^{n-1} \hookrightarrow D^n$ for every $n \in \mathbb{N}$.

Proof. (2) implies (1) is trivial. For the converse, consider some $c \in C_0 = \mathsf{Z}_0(C)$. Since $\mathrm{H}_0(h)$ is surjective, there is a $b \in B_0$ and a $c' \in C_1$ with $c = h_0 b + dc'$. Since h_1 is surjective, $c' = h_1 b'$ for some $b' \in B_1$ and then we see that $c = h_0(b + db')$.

We will give an "element theoretic" proof of the remaining implications; the reader can translate it into a more abstract argument if desired.

For (2) implies (3), consider some

$$(b, c) \in \mathsf{Z}_{n-1}(B) \times_{\mathsf{Z}_{n-1}(C)} C_n,$$

so $h_{n-1}b = dc$. Since h_n is surjective, there is a $b' \in B_n$ such that $h_n b' = c$. Then $db' - b \in \mathsf{Z}_{n-1}(B)$ (because $b \in \mathsf{Z}_{n-1}(B)$) and we have

$$\begin{aligned} h_{n-1}(db' - b) &= dh_n b' - h_{n-1} b \\ &= dc - dc \\ &= 0. \end{aligned}$$

Since h is a surjective quasi-isomorphism, the complex $\operatorname{Ker} h$ is acyclic so there is some $b'' \in \operatorname{Ker} h_n \subseteq B_n$ with $db'' = db' - b$. The image of $b' - b''$ under the map in (3) is (b, c).

For (3) implies (2), first argue that each map $\mathsf{Z}_n(B) \to \mathsf{Z}_n(C)$ is surjective by considering, for $c \in \mathsf{Z}_n(C)$, the element $(0, c) \in \mathsf{Z}_{n-1}(B) \times_{\mathsf{Z}_{n-1}(C)} C_n$ and using surjectivity of the map in (3). Once it is known that the maps $\mathsf{Z}_n(B) \to \mathsf{Z}_n(C)$ are surjective, surjectivity of $B_n \to C_n$ is immediate from surjectivity of the maps in (3). Since each $\mathsf{Z}_n(B) \to \mathsf{Z}_n(C)$ is surjective, so is each $\mathrm{H}_n(h)$, so it remains only to prove the injectivity of the maps $\mathrm{H}_{n-1}(h)$. To see this, suppose $b \in \mathsf{Z}_{n-1}(B)$ has $h_{n-1}(b) = dc$ for some $c \in C_n$. Then $(b, c) \in \mathsf{Z}_{n-1}(B) \times_{\mathsf{Z}_{n-1}(C)} C_n$ and surjecitivity of the map in (3) says that b is a boundary in B.

For the equivalence of (3) and (4), use the natural bijections (5.5.3) and (5.5.4) to see that a solid commutative diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & B \arrow[d, "h"] \\ D^n \arrow[r] \arrow[ur, dotted] & C \end{tikzcd}$$

in $\mathbf{Ch}_{\geq 0}\mathbf{Mod}(R)$ is the same thing as an element of $\mathsf{Z}_{n-1}(B) \times_{\mathsf{Z}_{n-1}(C)} C_n$, and a lift as indicated is a preimage of this element under the map in (3). $\square$

Lemma 5.5.6. *Cofibrations in* $\mathbf{Ch}_{\geq 0}\mathbf{A}$ *have the LLP w.r.t. trivial fibrations.*

Proof. Consider a commutative diagram

(5.5.5)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "h"] \\ P \arrow[r, "g"'] \arrow[ur, dotted, "l"] & C \end{tikzcd}$$

where h is a trivial fibration and i is a cofibration. We must construct the lift l as indicated. We do this inductively. Assume we have constructed maps $l_k : P_k \to B_k$ for $k < n$ which are compatible with the boundary maps and make the diagrams

(5.5.6)
$$\begin{tikzcd} A_k \arrow[r, "f_k"] \arrow[d, "i_k"'] & B_k \arrow[d, "h_k"] \\ P_k \arrow[r, "g_k"'] \arrow[ur, "l_k"] & C_k \end{tikzcd}$$

commute. We can certainly get started since our complexes are supported in non-negative degrees, so we need only construct a map $l_n : P_n \to B_n$ compatible with the boundary maps and making

(5.5.7)
$$\begin{tikzcd} A_n \arrow[r, "f_n"] \arrow[d, "i_n"'] & B_n \arrow[d, "h_n"] \\ P_n \arrow[r, "g_n"'] \arrow[ur, "l_n"] & C_n \end{tikzcd}$$

commute. Since i is a cofibration, we can find a projective subobject $E \subseteq P_n$ so that the inclusion $i_n : A_n \to P_n$ induces a splitting $P_n = A_n \oplus E$. We already have a unique way (namely f_n) of defining l_n on the summand A_n compatibly with the boundary maps, so we need only produce an $\mathbf{A}$-morphism $l : E \to B_n$ so that the following diagram commutes:

(5.5.8)
$$\begin{tikzcd} E \arrow[r, "l"] \arrow[d, "d"'] \arrow[rr, bend left, "g_n"] & B_n \arrow[r, "h_n"] \arrow[d, "d"] & C_n \arrow[d, "d"] \\ P_{n-1} \arrow[r, "l_{n-1}"] & B_{n-1} \arrow[r, "h_{n-1}"] & C_{n-1} \end{tikzcd}$$

The issue is only to make the left square and upper triangle commute; in particular, the "big square" already commutes and a moment's thought shows that finding such an l is equivalent to producing a lift in the following $\mathbf{A}$-diagram:

$$\begin{tikzcd} & B_n \arrow[d, "{d,h_n}"] \\ E \arrow[r, "{dl_{n-1},g_n}"] \arrow[ur, dotted, bend left, "l"] & \mathsf{Z}_{n-1}(B) \times_{\mathsf{Z}_{n-1}(C)} C_n \end{tikzcd}$$

This can be done because E is projective and the vertical arrow is surjective since h is a trivial fibration (Lemma 5.5.5). $\square$

Lemma 5.5.7. *For any $B \in \mathbf{Ch}_{\geq 0}\mathbf{A}$, there is a map $q : E \to B$ (functorial in B) where:*

(1) The complex E is acyclic.
(2) $E_n \in \mathbf{A}$ is projective for all $n \in \mathbb{N}$.
(3) $q_n : E_n \to B_n$ is surjective for all $n > 0$ (i.e. q is a fibration).

Proof. For each $n > 0$ choose (functorially in B) a surjection $p_n : P_n \to B_n$ with P_n projective. This surjection corresponds, via the adjunction (5.5.1), to a $\mathbf{Ch}_{\geq 0}\mathbf{A}$-morphism $D^n(p_n) : D^n(P_n) \to B$ given by p_n (hence surjective) in degree n. The sum over $n > 0$ of the $D^n(p_n)$ will do the job since each $D^n(P_n)$ with $n > 0$ is acyclic. $\square$

Theorem 5.5.8. *Suppose $\mathbf{A}$ is an abelian category with functorial projective resolutions. Then each morphism $f : A \to B$ in $\mathbf{Ch}_{\geq 0}\mathbf{A}$ admits two functorial factorizations:*

(1) $f = ip$ where i is a trivial cofibration and p is a fibration.
(2) $f = ip$ where i is a cofibration and p is a trivial fibration.

Proof. For the first factorization, pick $q : E \to B$ as in Lemma 5.5.7 and factor f as the inclusion $i : A \to A \oplus E$ followed by $p = (f, q) : A \oplus E \to B$. The map i a trivial cofibration because each E_n is projective and the complex E is acyclic. The maps p_n are surjective when $n > 0$ because the maps q_n are surjective when $n > 0$.

For the second factorization, we build up the factorization inductively. Assume we have constructed a factorization of f in degrees $0, \ldots, n$ as $i : A \to P$ followed by $p : P \to B$ with the following properties:

(1) For each $k \in \{0, \ldots, n\}$ there is a projective subobject $E_k \subseteq P_k$ so that i_k induces a splitting $P_k = A_k \oplus E_k$.
(2) For each $k \in \{0, \ldots, n\}$, $p_k : P_k \to B_k$ is surjective.
(3) For each $k \in \{0, \ldots, n-1\}$, the map $\mathrm{H}_k(p)$ is an isomorphism.
(4) The map $p_n : \mathsf{Z}_n(P) \to \mathsf{Z}_n(B)$ is surjective.

Functorially choose the following:

(1) a surjection $a : X'' \to \operatorname{Ker}(\mathsf{Z}_n(P) \to \mathrm{H}_n(B))$ with X'' projective. Define X' by the cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B_{n+1} \arrow[d, "d"] \\ X'' \arrow[r, "p_n a"] & \mathsf{B}_n(B) \end{tikzcd}$$

and choose a surjection $X \to X'$ with X projective. The map d in this diagram is surjective, so π_1 is also surjective, as is the induced map

$$\overline{a} : X \to \operatorname{Ker}(\mathsf{Z}_n(P) \to \mathrm{H}_n(B)).$$

We will abusively write $\pi_2 : X \to B_{n+1}$ for the composition of our chosen surjection $X \to X'$ and the map π_2 in the diagram.
(2) a surjection $b : Y \to \mathsf{Z}_{n+1}(B)$ with Y projective.

(3) a surjection $c : Z'' \to B_{n+1}$ with Z'' projective. Define Z' by the cartesian diagram

$$\begin{tikzcd} Z' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & P_n \arrow[d, "p_n"] \\ Z'' \arrow[r, "dc"] & B_n \end{tikzcd}$$

and choose a surjection $Z \to Z'$ with Z projective. The map p_n is surjective, so π_1 is also surjective and the induced map $\overline{c} : Z \to B_{n+1}$ is also surjective. We will abusively write $\pi_2 : Z \to P_n$ for the composition of our chosen surjection $Z \to Z'$ and the map π_2 in the diagram.

Set $E_{n+1} := X \oplus Y \oplus Z$, $P_{n+1} := A_{n+1} \oplus E_{n+1}$ and let $i_{n+1} : A_{n+1} \to P_{n+1}$ be the natural map. Our constructions of X, Y, Z ensure that the diagram

$$\begin{tikzcd} A_{n+1} \oplus X \oplus Y \oplus Z \arrow[rr, "p_{n+1}:=(f_{n+1},\pi_2,b,\overline{c})"] \arrow[d, "(d,\overline{a},0,\pi_2)"'] & & B_{n+1} \arrow[d, "d"] \\ P_n \arrow[rr, "p_n"] & & B_n \end{tikzcd}$$

commutes (the left vertical arrow here is the differential $P_{n+1} \to P_n$ for our lifted complex P) and it is straightforward to see that our choices of X, Y, Z yield a factorization of f in degrees $0, \dots, n+1$ with all the properties above (replacing n by $n+1$). $\square$

Lemma 5.5.9. *An object E of $\mathbf{Ch}_{\geq 0}\mathbf{A}$ is acyclic and cofibrant iff E is a direct sum of projective disks of positive dimension—i.e. we can write*

$$E = \bigoplus_{n=0}^{\infty} D^{n+1}(P_n)$$

for projective objects $P_0, P_1, \dots$ of $\mathbf{A}$.

Proof. It is clear that such a direct sum of projective disks is acyclic and cofibrant. Suppose E is acyclic and cofibrant. Since E is cofibrant, each E_n is projective. Since $\mathrm{H}_0(E) = 0$, the boundary map $E_1 \to E_0$ must be surjective, so it has a section $s : E_0 \to E_1$ since E_0 is projective. Via the formula (5.5.1), this s determines a map $D^1(E_0) \to E$ which is given by s in degree 1 and is an isomorphism in degree zero (because $ds = \mathrm{Id}$); so $D^1(E_0) \to E$ is injective. Note that, since E_1 is also projective, the quotient $E_1/s(E_0)$ is also projective, so the quotient $F := E/D^1(E_0)$ is also cofibrant, and has $F_0 = 0$. This F is also acyclic since $D^1(E_0)$ and E are acyclic. The quotient map $E \to F$ is surjective and both E and F are acyclic, so $E \to F$ is a trivial fibration, hence it has a section because F is cofibrant (apply Lemma 5.5.6 to the diagram below).

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & E \arrow[d] \\ F \arrow[r, equal] & F \end{tikzcd}$$

Since $\mathbf{Ch}_{\geq 0}\mathbf{A}$ is an abelian category, this section yields a splitting $E = D^1(E_0) \oplus F$. This F has $F_0 = 0$ and is acyclic, so the boundary map $F_2 \to F_1$ is surjective and since F_1 is projective, we can split off another projective disk by the same process: $F = D^2(F_1) \oplus G$. Continuing in this manner yields the desired direct sum decomposition of E into a sum of projective disks (only finitely many steps are needed to get the desired direct sum decomposition in any given degree, so there is no issue in "taking the limit"). □

We eventually want to prove that trivial cofibrations have the LLP w.r.t. fibrations. We first prove this in a special case:

Lemma 5.5.10. *If $0 \to E$ is a trivial cofibration, then $0 \to E$ has the LLP w.r.t. fibrations.*

Proof. We have to construct a lift in any diagram

$$\begin{tikzcd} & B \arrow[d, "h"] \\ E \arrow[r, "g"] \arrow[ur, dotted] & C \end{tikzcd}$$

where h is a fibration (i.e. h_n is surjective for all $n > 0$). By the previous lemma, $E = \oplus_{n=0}^{\infty} D^{n+1}(P_n)$ is a sum of positive dimensional projective disks. By the universal property of direct sums and maps out of disks (5.5.1), it is equivalent to lift in each $\mathbf{A}$-diagram

$$\begin{tikzcd} & B_{n+1} \arrow[d, "h_{n+1}"] \\ P_n \arrow[r, "g"] \arrow[ur, dotted] & C_{n+1} \end{tikzcd}$$

($n \in \mathbb{N}$), which we can do because h_{n+1} is surjective and P_n is projective. □

Now we give the structure of trivial cofibrations in $\mathbf{Ch}_{\geq 0}\mathbf{A}$ and prove that they have the LLP w.r.t. fibrations.

Lemma 5.5.11. *For any trivial cofibration $i : A \to P$ there is a splitting $P = A \oplus E$ where $E = \operatorname{Cok} i$ is acyclic and cofibrant. Trivial cofibrations have the LLP w.r.t. fibrations.*

Proof. For the first statement: Since i is a trivial cofibration, $E = \operatorname{Cok} i$ is acylic and cofibrant. To obtain the splitting, we need to find a section of the projection $P \to E$, which we can do by the previous lemma because $P \to E$ is certainly a fibration. The second statement follows easily from the first using the universal property of direct sums and Lemma 5.5.10. □

Theorem 5.5.12. *Let R be a ring. Then the model structure on $\mathbf{Ch}_{\geq 0}\mathbf{Mod}(R)$ from Theorem 5.5.2 is cofibrantly generated (Definition 1.5.7) by the set I of cofibrations and the set J of trivial cofibrations below.*

$$I := \{S^{n-1} \hookrightarrow D^n : n \in \mathbb{N}\}$$
$$J := \{0 \hookrightarrow D^n : n > 0\}$$

Proof. First: It is clear from the definition of "cofibration" (Definition 5.5.1), the definitions of the complexes S^n and D^n (Definition 5.5.4), and the acyclicity of the D^n with $n > 0$ that the maps in I are cofibrations and the maps in J are trivial cofibrations. The smallness conditions in Definition 1.5.7 hold trivially since one can easily show that every object of $\mathbf{Ch}_{\geq 0}\mathbf{Mod}(R)$ is small (c.f. [Hov, 2.3.2]). It is immediate from the definitions of "fibration" (Definition 5.5.1) and "J-inj" (Definition 1.5.4) and the description (5.5.3) of a map out of D^n that J-inj is the set of fibrations. The equivalence of the first and last conditions in Lemma 5.5.5 says that I-inj is the set of trivial fibrations. □

Remark 5.5.13. One could also prove Theorem 5.5.12 using the Recognition Theorem (1.5.10). This is a good exercise for the reader. One can follow Hovey's proof of the analog for unbounded complexes in [Hov, 2.3].

Remark 5.5.14. One could also obtain a cofibrant generation statement similar to Theorem 5.5.12 for any abelian category $\mathbf{A}$ with all direct limits, functorial projective resolutions, and a *generator* in the sense of [T, 1.9].

5.6. **Inherited model structure on simplicial objects.** Let $\mathbf{A}$ be an abelian category with functorial projective resolutions (§5.5). Using the inverse equivalences

$$\mathsf{N} : \mathbf{sA} \to \mathbf{Ch}_{\geq 0}\mathbf{A}$$
$$\mathsf{K} : \mathbf{Ch}_{\geq 0}\mathbf{A} \to \mathbf{sA}$$

of Theorem 5.3.1 and the model structure on $\mathbf{Ch}_{\geq 0}\mathbf{A}$ of Theorem 5.5.2, we formally obtain a model structure on $\mathbf{sA}$. It is not so obvious *a priori* that this model structure on $\mathbf{sA}$ has anything much to do with the simplicial nature of $\mathbf{sA}$. It turns out, however, that when $\mathbf{A} = \mathbf{Mod}(R)$ for a ring $\mathbf{Mod}(R)$, a map $f : A \to B$ is a fibration (resp. weak equivalence) in $\mathbf{sMod}(R)$ for this "formally obtained" model structure iff the underlying map of simplicial sets $f : A \to B$ is a fibration (resp. weak equivalence) in the model category $\mathbf{sSets}$ of simplicial sets (§4.4).

Definition 5.6.1. A morphism $f : A \to B$ in $\mathbf{sA}$ is called a *weak equivalence* (resp. *fibration*, *cofibration*) iff the map $\mathsf{N}(f) : \mathsf{N}(A) \to \mathsf{N}(B)$ is a weak equivalence (resp. *fibration*, *cofibration*) in the model structure on $\mathbf{Ch}_{\geq 0}\mathbf{A}$ of Theorem 5.5.2.

Proposition 5.6.2. *The category* $\mathbf{sA}$ *is a model category with the definitions above. If* $\mathbf{A} = \mathbf{Mod}(R)$ *for a ring* R*, this model category is cofibrantly generated by the set* I *of cofibrations and the set* J *of trivial cofibrations below.*

$$I := \{\mathsf{K}(S^{n-1} \hookrightarrow D^n) : n \in \mathbb{N}\}$$
$$J := \{\mathsf{K}(0 \hookrightarrow D^n) : n > 0\}$$

Proof. This is immediate from Theorems 5.5.2 and Theorem 5.5.12 because N and $\mathbf{K}$ are inverse equivalences by Theorem 5.3.1. □

Recall from §4.4 that a map of simplicial sets f is called a *fibration* iff it has the RLP with respect to the inclusions of k-horns $i_k : \Lambda^k[n] \hookrightarrow \Delta[n]$ for all $n > 0$, $k \in [n]$.

Proposition 5.6.3. *For a map* $f : A \to B$ *of simplicial abelian groups, the following are equivalent:*

1. *The underlying map* $f : A \to B$ *of simplicial sets is a fibration.*
2. *The map* $\mathsf{N}_n(f) : \mathsf{N}_n(A) \to \mathsf{N}_n(B)$ *is surjective for* $n > 0$.
3. *The map* $A_n \to B_n \times_{\mathrm{H}_0(B)} \mathrm{H}_0(A)$ *is surjective for* $n \geq 0$.

Proof. This is [Q1, Proposition II.3.8.1]. □

Corollary 5.6.4. *A map* $f : A \to B$ *of simplicial groups is surjective in each dimension iff it is a fibration and* $\mathrm{H}_0(f)$ *is surjective.*

Corollary 5.6.5. *A weak equivalence* $f : A \to B$ *of simplicial abelian groups is a (trivial) fibration iff it is surjective (in each degree).*

Corollary 5.6.6. *Any simplicial set underlying a simplicial abelian group is fibrant.*

Corollary 5.6.7. *Any map between constant simplicial abelian groups is a fibration.*

Remark 5.6.8. The word "abelian" isn't necessary in the results of this section, but we will only use this abelian case in what follows.

5.7. **Shuffles.** Fix non-negative integers $m, n \in \mathbb{N}$. We begin by observing that there is a bijection between surjective Δ-morphisms $[m+n] \to [m]$ and increasing sequences

$$1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_m \leq m+n$$

given as follows: To the sequence $\sigma_1, \ldots, \sigma_m$, we associate the surjective Δ-morphism

$$\begin{aligned} [m+n] &\to [m] \\ 0, \ldots, \sigma_1 - 1 &\to 0 \\ \sigma_1, \ldots, \sigma_2 - 1 &\to 1 \\ &\vdots \\ \sigma_{m-1}, \ldots, \sigma_m - 1 &\to m-1 \\ \sigma_m, \ldots, m+n &\mapsto m, \end{aligned}$$

and to a surjective Δ-morphism $\sigma : [m+n] \to [m]$ we associate the increasing sequence

$$1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_m \leq m+n$$

defined by $\sigma_i := \min \sigma^{-1}(i)$. Our convention when $m = 0$ is that the unique Δ-morphism $[m+n] \to [0]$ corresponds to the "empty sequence".

An (m,n)-*shuffle* (σ, τ) is a permutation

$$(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n)$$

of $\{1, 2, \ldots, m+n\}$ such that $\sigma_1 < \sigma_2 < \cdots < \sigma_m$ and $\tau_1 < \cdots < \tau_n$. Write $\mathbf{Shuff}(m,n)$ for the set of (m,n)-shuffles. The *sign* of an (m,n) shuffle

$$\operatorname{sign}(\sigma, \tau) \in \{1, -1\}$$

is, by definition, the sign of the corresponding permutation. An (m,n)-shuffle (σ,τ) gives rise to surjective Δ-morphisms

$$\begin{aligned} \sigma : [m+n] &\to [m] \\ \tau : [m+n] &\to [n] \end{aligned}$$

via the aforementioned bijection. Note that an (m,n)-shuffle is uniquely recovered from the set of σ_i's (just list everything in increasing order), so (m,n)-shuffles are in bijective correspondence with subsets

$$\Sigma \subseteq \{1, \ldots, m+n\}$$

of cardinality m.

There is an obvious bijection

$$\begin{aligned} \mathbf{Shuff}(m,n) &\to \mathbf{Shuff}(n,m) \\ (\sigma,\tau) &\mapsto (\tau,\sigma) \end{aligned}$$

and we have

$$\operatorname{sign}(\sigma,\tau) = (-1)^{mn} \operatorname{sign}(\tau,\sigma).$$

The rest of this section will be devoted to establishing similar, but slightly more elaborate, bijections with similar "sign rules."

If $p \in \mathbb{N}$ is a third nonnegative integer, then we can also consider (m, n, p)-*shuffles*, which are of course permutations

$$(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_p)$$

of $\{1, \ldots, m+n+p\}$ where the α_i are increasing with i, the β_i are increasing with i, and the γ_i are increasing with i. Such a shuffle gives rise to three surjective Δ-morphisms

$$\begin{aligned} \alpha &: [m+n+p] \to [m] \\ \beta &: [m+n+p] \to [n] \\ \gamma &: [m+n+p] \to [p] \end{aligned}$$

using the correspondence discussed at the beginning of the section, and a sign

$$\operatorname{sign}(\alpha, \beta, \gamma) \in \{1, -1\}$$

(the sign of the corresponding permuation). Quite generally, we can consider the set $\mathbf{Shuff}(m_1, \ldots, m_n)$ of $(m_1, \ldots, m_n)$-shuffles for any $m_1, \ldots, m_n \in \mathbb{N}$.

Lemma 5.7.1. *For $m, n, p \in \mathbb{N}$, there is a bijection*

$$\begin{aligned} \mathbf{Shuff}(m, n) \times \mathbf{Shuff}(m+n, p) &\to \mathbf{Shuff}(m, n, p) \\ ((\sigma, \tau), (\rho, \theta)) &\mapsto (\sigma\rho, \tau\rho, \theta) \end{aligned}$$

with

$$\operatorname{sign}(\sigma, \tau) \operatorname{sign}(\rho, \theta) = \operatorname{sign}(\sigma\rho, \tau\rho, \theta)$$

and a bijection

$$\begin{aligned} \mathbf{Shuff}(m, n+p) \times \mathbf{Shuff}(n, p) &\to \mathbf{Shuff}(m, n, p) \\ ((\sigma, \tau), (\rho, \theta)) &\mapsto (\sigma, \rho\tau, \theta\tau) \end{aligned}$$

with

$$\operatorname{sign}(\sigma, \tau) \operatorname{sign}(\rho, \theta) = \operatorname{sign}(\sigma, \rho\tau, \theta\tau).$$

Proof. We will just discuss the first bijection; the second is similar. First of all, if we write out the first map in terms of permuatations, I claim that it is given by

$$((\sigma, \tau), (\rho, \theta)) \mapsto (\rho_{\sigma_1}, \ldots, \rho_{\sigma_m}, \rho_{\tau_1}, \ldots, \rho_{\tau_n}, \theta_1, \ldots, \theta_p).$$

This latter map is clearly bijective with the indicated sign relationship. To prove the claim, note that, since σ, ρ are non-decreasing, we have

$$\begin{aligned} (\sigma\rho)_i &= \min(\sigma\rho)^{-1}(i) \\ &= \min \rho^{-1}(\sigma^{-1}(i)) \\ &= \min \rho^{-1}(\min \sigma^{-1}(i)) \\ &= \min \rho^{-1}(\sigma_i) \\ &= \rho_{\sigma_i} \end{aligned}$$

and similarly with σ replaced by τ. $\square$

Lemma 5.7.2. *Fix $m, n \in \mathbb{N}$ with $m+n > 0$ and $i \in [m+n]$. Let*

$$\delta^i := \delta^i_{m+n} : [m+n-1] \hookrightarrow [m+n]$$

denote the unique injective Δ-morphism whose image misses i. For every (m,n)-shuffle (σ, τ) exactly one of the following occurs:

(1) Both $\sigma\delta^i$ and $\tau\delta^i$ are surjective, $i \in \{1, \dots, m+n-1\}$, and exactly one of the following occurs:
(a) $i = \sigma_p$ for some $p \in \{1, \dots, m\}$ and $i+1 = \tau_q$ for some $q \in \{1, \dots, n\}$
(b) $i+1 = \sigma_p$ for some $p \in \{1, \dots, m\}$ and $i = \tau_q$ for some $q \in \{1, \dots, n\}$.
In either case, if (σ', τ') is obtained from (σ, τ) by interchanging i and $i+1$, then

$$\sigma'\delta^i = \sigma\delta^i$$
$$\tau'\delta^i = \tau\delta^i.$$

(2) There is a unique $j \in [m]$ and a unique $(m-1, n)$-shuffle (σ', τ') such that

$$\begin{tikzcd}
{[m-1]} \arrow[d, "\delta^j_m"] & {[m+n-1]} \arrow[l, "\sigma'"'] \arrow[d, "\delta^i"] \arrow[dr, "\tau'"] & \\
{[m]} & {[m+n]} \arrow[l, "\sigma"] \arrow[r, "\tau"'] & {[n]}
\end{tikzcd}$$

commutes and we have

$$(-1)^i \operatorname{sign}(\sigma, \tau) = (-1)^j \operatorname{sign}(\sigma', \tau').$$

Every pair

$$((\sigma', \tau'), j) \in \mathbf{Shuff}(m-1, n) \times [m]$$

arises in this manner from a unique (m,n)-shuffle (σ, τ) for which j is not in the image of $\sigma\delta^i$.

(3) There is a unique $j \in [n]$ and a unique $(m, n-1)$ shuffle (σ', τ') such that

$$\begin{tikzcd}
& {[m+n-1]} \arrow[r, "\tau'"'] \arrow[dl, "\sigma'"'] \arrow[d, "\delta^i"] & {[n-1]} \arrow[d, "\delta^j_n"] \\
{[m]} & {[m+n]} \arrow[l, "\sigma"] \arrow[r, "\tau"'] & {[n]}
\end{tikzcd}$$

commutes and we have

$$(-1)^i \operatorname{sign}(\sigma, \tau) = (-1)^{m+j} \operatorname{sign}(\sigma', \tau').$$

Every pair

$$((\sigma', \tau'), j) \in \mathbf{Shuff}(m, n-1) \times [n]$$

arises in this manner from a unique (m,n)-shuffle (σ, τ) for which j is not in the image of $\tau\delta^i$.

Proof. When $i = 0$, the number 1 appears somewhere in the permuatation

$$(\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_n),$$

so it must be that $1 = \sigma_1$ or $1 = \tau_1$. If, say, $1 = \sigma_1$, then $\sigma^{-1}(0) = \{0\}$, so $\sigma\delta^0$ isn't surjective (but $\tau\delta^0$ *is* surjective since $\tau_1 > 1$) and it is straightforward to check that

$$(\sigma', \tau') := (\sigma_2 - 1, \sigma_3 - 1, \dots, \sigma_m - 1, \tau_1 - 1, \dots, \tau_n - 1)$$

is as desired. When $i = m+n$, we either have $m+n = \sigma_m$ or $m+n = \tau_n$. In, say, the former case, $\sigma^{-1}(m) = \{m+n\}$, so $\sigma\delta^{m+n}$ isn't surjective and it is straightforward to check that

$$(\sigma', \tau') := (\sigma_1, \ldots, \sigma_{m-1}, \tau_1, \ldots, \tau_n)$$

is as desired.

Suppose $i \in \{1, \ldots, m+n-1\}$ and, say, $\sigma\delta^i$ isn't surjective. Since σ is surjective and δ^i only misses i, this means that there is some $j \in [m]$ for which $\sigma^{-1}(j) = \{i\}$, hence $\sigma_j = i$ and $\sigma_{j+1} = i+1$. Then we set $\sigma' := (\sigma_1, \ldots, \sigma_{j-1}, \sigma_{j+1} - 1, \sigma_{j+2} - 1, \ldots, \sigma_m - 1)$, and we obtain τ' from τ by subtracting 1 from each τ_k which is greater than i. It is straightforward to check that this (σ', τ') is as desired and that the indicated sign relationship holds. It is also clear that the recipe we are describing for producing $((\sigma', \tau'), j)$ from (σ, τ) with $j \notin \operatorname{Im} \sigma\delta^i$ can be inverted.

On the other hand, suppose $i \in \{1, \ldots, m+n-1\}$, but $\sigma\delta^i$ and $\tau\delta^i$ are surjective. The number i appears somewhere in the permutation $(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n)$. Say $i = \sigma_p$ for some p. The number $i+1$ also appears somewhere in this partition, but it cannot be one of the σ_j, for then it would have to be σ_{p+1}, but then we would have $\sigma^{-1}(p) = \{i\}$, contradicting surjectivity of $\sigma\delta^i$. We conclude that $i+1 = \tau_q$ for some q. If (σ', τ') is obtained from (σ, τ) by permuting $i, i+1$, then the claimed equalities $\sigma\delta^i = \sigma'\delta^i$ and $\tau\delta^i = \tau'\delta^i$ are straightforward to check. □

5.8. **Eilenberg-Zilber.** One use of shuffles (§5.7) occurs in the Eilenberg-Zilber Theorem, which we will now describe.

Let $\mathbf{A}$ be an abelian category and let $V \in \mathbf{ssA}$ be a bisimplicial object of $\mathbf{A}$ (§3.4). By applying the (unnormalized) chain complex functor (§5.1) twice we obtain a functor abusively denoted

$$\mathsf{C} : \mathbf{ssA} \to \mathbf{Ch}_{\geq 0}\mathbf{Ch}_{\geq 0}\mathbf{A}$$

from bisimplicial objects in $\mathbf{A}$ to first-quadrant double complexes in $\mathbf{A}$. We have $\mathsf{C}(V)_{p,q} = V_{p,q}$. We can compose this with the total complex functor

$$\mathrm{Tot} : \mathbf{Ch}_{\geq 0}\mathbf{Ch}_{\geq 0}\mathbf{A} \to \mathbf{Ch}_{\geq 0}\mathbf{A}$$

to associate a chain complex $\mathrm{Tot}\,\mathsf{C}(V)$ to our bisimplicial object V with

$$(\mathrm{Tot}\,\mathsf{C}(V))_n = \bigoplus_{p+q=n} V_{p,q}.$$

We can also apply the diagonal functor (§3.4), followed by the unnormalized chain complex functor to obtain another chain complex $\mathsf{C}(\Delta(V))$ from V with $\mathsf{C}(\Delta(V))_n = V_{n,n}$. The Eilenberg-Zilber Theorem asserts that the chain complexes $\mathrm{Tot}\,\mathsf{C}(V)$ and $\mathsf{C}(\Delta(V))$ are homotopy equivalent via natural maps which we now describe.

First we have the *shuffle map*

$$\mathrm{Tot}\,\mathsf{C}(V) \to \mathsf{C}(\Delta(V)) \tag{5.8.1}$$

given in degree n by the sum, over $p, q \in \mathbb{N}$ with $p+q = n$, of the maps

$$\sum_{(\sigma,\tau)} \mathrm{sign}(\sigma, \tau) V(\sigma, \tau) : V_{p,q} \to V_{n,n},$$

where the sum here is over all (p,q)-shuffles (σ,τ) and $V(\sigma,\tau)$ is of course the structure map that V associates to the $\Delta\times\Delta$ morphism

$$\sigma\times\tau : [n]\times[n] \to [p]\times[q].$$

Next we have the Alexander-Whitney map

$$\mathsf{C}(\Delta(V)) \to \operatorname{Tot}\mathsf{C}(V) \tag{5.8.2}$$

given in degree n by the product,[7] over $p,q\in\mathbb{N}$ with $p+q=n$, of the maps

$$V(\sigma_{n,p},\tau_{n,q}) : V_{n,n} \to V_{p,q},$$

where $\sigma_{n,p}$ and $\tau_{n,q}$ are the Δ morphisms defined below.

$$\begin{aligned} \sigma_{n,p} : [n] &\to [p] & \tau_{n,p} : [n] &\to [q] \\ 0 &\mapsto 0 & 0,\dots,n-q &\mapsto 0 \\ &\vdots & n-q+1 &\mapsto 1 \\ p-1 &\mapsto p-1 & &\vdots \\ p,\dots,n &\mapsto p & n=n-q+q &\mapsto q. \end{aligned} \tag{5.8.3}$$

Theorem 5.8.1. (Eilenberg-Zilber) *The shuffle map is a homotopy equivalence of complexes with the Alexander Whitney map as its homotopy inverse.*

Proof. This is [DP, 2.15]. □

In practice, one is often interested in knowing something about the homology of $\mathsf{C}(\Delta(V))$, which is identified, via Eilenberg-Zilber with the homology of $\operatorname{Tot}\mathsf{C}(V)$. The good thing about the *latter* homology is that it is calculated as the homology of the total complex associated to a double complex, hence it comes with two spectral sequences:

$$'E_1^{p,q} = \mathrm{H}_p(V_{\bullet,q}) \implies \mathrm{H}_{p+q}(\operatorname{Tot}\mathsf{C}(V)) \tag{5.8.4}$$

$$''E_1^{p,q} = \mathrm{H}_q(V_{p,\bullet}) \implies \mathrm{H}_{p+q}(\operatorname{Tot}\mathsf{C}(V)). \tag{5.8.5}$$

We will be most interested in this general result in the following situation. Let A be a ring and let $M,N\in\mathbf{sMod}(A)$ be two simplicial A modules. There are various possible meanings of the tensor product $M\otimes N$. For one thing, we can form a bisimplicial (§3.4) A module $M\otimes N\in\mathbf{ssMod}(A)$ with

$$(M\otimes N)_{m,n} := M_m\otimes N_n.$$

For a $\Delta\times\Delta$ morphism

$$\sigma\times\tau : ([k],[l]) \to ([m],[n]),$$

the structure map

$$(M\otimes N)(\sigma,\tau) : M_m\otimes N_n \to M_k\otimes N_l$$

is of course given by $M(\sigma)\otimes N(\tau)$, where

$$\begin{aligned} M(\sigma) &: M_m \to M_l \\ N(\tau) &: N_n \to N_l \end{aligned}$$

[7] The product in question is finite, so it is also the sum, but I prefer to call it the product since we are defining the map using the categorical property of the product.

are the structure maps for M and N. From the bisimplicial A-module $M \otimes N$, we can form the complexes $\operatorname{Tot} \mathsf{C}(M \otimes N)$ and $\mathsf{C}(\Delta(M \otimes N))$ with

$$(\operatorname{Tot} \mathsf{C}(M \otimes N))_n := \bigoplus_{p+q=n} M_p \otimes N_q$$
$$\mathsf{C}(\Delta(M \otimes N))_n := M_n \otimes N_n.$$

Using Eilenberg-Zilber, we can view the spectral sequences (5.8.4) and (5.8.5) as spectral sequences

$$'E_1^{p,q} = \mathrm{H}_p(M_\bullet \otimes N_q) \implies \mathrm{H}_{p+q}(\mathsf{C}(\Delta(M \otimes N))) \tag{5.8.6}$$
$$''E_1^{p,q} = \mathrm{H}_q(M_p \otimes N_\bullet) \implies \mathrm{H}_{p+q}(\mathsf{C}(\Delta(M \otimes N))). \tag{5.8.7}$$

With these spectral sequences in hand, we can prove the following lemma, which will be useful later.

Lemma 5.8.2. *Let A be a ring, L a simplicial A module, $f : M \to N$ a quasi-isomorphism of simplicial A modules. If L (or both M and N) is (are) degree-wise flat, then*

$$\Delta(f \otimes L) : \Delta(M \otimes L) \to \Delta(N \otimes L)$$

is also a quasi-isomorphism of simplicial A modules.

Proof. We want to prove that

$$\mathrm{H}_n(\mathsf{C}(\Delta(f \otimes L))) : \mathrm{H}_n(\mathsf{C}(\Delta(M \otimes L))) \to \mathrm{H}_n(\mathsf{C}(\Delta(N \otimes L)))$$

is an isomorphism of A modules for all n. As discussed above, by Eilenberg-Zilber, we can view this map as the abutment of two different maps of spectral sequences induced by the map of double complexes

$$\mathsf{C}(f \otimes L) : \mathsf{C}(M \otimes L) \to \mathsf{C}(N \otimes L).$$

On $E_1^{p,q}$ terms the first map of spectral sequences is the natural map

$$\mathrm{H}_p(M_\bullet \otimes L_q) \to \mathrm{H}_p(N_\bullet \otimes L_q).$$

When each L_q is flat, the homology commutes with the tensor product so these maps are just the maps

$$\mathrm{H}_p(f) \otimes L_q : \mathrm{H}_p(M_\bullet) \otimes L_q \to \mathrm{H}_p(N_\bullet) \otimes L_q,$$

which are isomorphisms by the assumption that f is a quasi-isomorphism, hence the map on abutments is also an isomorphism and we're done. When both M and N are degree-wise flat, we instead look at the other map of spectral sequences and argue similarly that it is given on $E_1^{p,q}$ by

$$f_p \otimes \mathrm{H}_q(L_\bullet) : M_p \otimes \mathrm{H}_q(L_\bullet) \to N_p \otimes \mathrm{H}_q(L_\bullet).$$

Of course these maps need not be isomorphisms, but if we now look at the map on $E_2^{p,q}$ terms (that is, we take homology in the p direction), using that all the M_p and N_p are flat, we find that this map is nothing but the map

$$\mathrm{H}_p(f) \otimes \mathrm{H}_q(L_\bullet) : \mathrm{H}_p(M) \otimes \mathrm{H}_q(L_\bullet) \to \mathrm{H}_p(N) \otimes \mathrm{H}_q(L_\bullet),$$

which is an isomorphism since f is a quasi-isomorphism, hence the map on abutments is also a quasi-isomorphism. $\square$

6. Homotopy rings survey

In §7 and §8, we will spend a considerable amount of time discussing the categories of simplicial rings and differential graded rings. These categories are both used to accomplish the same basic goal: We seek some category of "ring complexes" that will play roughly the same role that the category of chain complexes plays in the usual homological algebra of abelian categories. There should then be some corresponding "homotopy category" ("the" homotopy category of rings) playing the role played by the derived category in the abelian setting.

For abelian categories there really is only "one" homotopy category, *the* derived category. It doesn't matter whether we think about it in terms of simplicial objects or (non-negatively graded) complexes. (Throughout the rest of this discussion, all "complexes" are understood to be chain complexes with non-negative grading and all "differential graded objects" are understood to be "connective," meaning "graded by $\mathbb{N}$".) This is perhaps an extreme example, because the category of simplicial objects is already equivalent to the category of complexes; it is not merely the case that the two homotopy categories are equivalent. Perhaps a better example to keep in mind would be simplicial sets versus topological spaces, where the two model categories in question are not equivalent categories, but are merely "Quillen equivalent" model categories, so they have equivalent homotopy categories.

The situation for "ring complexes" is much more complicated. The basic point is that there are several possible choices for the category of "ring complexes" with a notion of "weak equivalences." It is not entirely clear which of these categories can be endowed with a model structure, and it is not clear which of the corresponding homotopy categories are equivalent.

The two "classical" candidates for the category of "ring complexes" are simplicial rings and differential graded rings, to be discussed in the next sections. For *not necessarily commutative rings* the situation is, perhaps surprisingly, simpler:

Theorem 6.0.1. (Schwede-Shipley) [SS] *There are model category structures on the category of simplicial (not necessarily commutative) rings and on the category of (not necessarily commutative) differential graded rings, and a Quillen equivalence between these two model categories.*

In fact, in the above theorem the fibrations and weak-equivalences are "as expected," meaning that they the ones inherited from the underlying (additive) simplicial abelian group (c.f. Proposition 5.6.3) and the underlying "differential graded abelian group" (complex of abelian groups), respectively. Even the functors yielding this equivalence are "as expected" on the level of underlying simplicial/differential abelian groups.

Theorem 6.0.2. (Bousfield-Gugenheim) [BG] *There is a model category structure on the category of (commutative) differential graded $\mathbb{Q}$-algebras with the "expected" fibrations and trivial fibrations.*

Theorem 6.0.3. (Stanley) [Sta] *There is a model category structure on the category of (commutative) differential graded rings where the weak equivalences and the cofibrations are "as expected."*

To say that the cofibrations are "as expected" here means that they are retracts of "free morphisms." (C.f. §**??** and §7.12 for various kinds of "free morphisms.") It turns out that in Stanley's model category, the fibrations are *not* as expected: a map can be a fibration without the corresponding map of normalized chain complexes being surjective in each positive degree (c.f. Proposition 5.6.3).

There is yet another alternative category of "ring complexes," namely the category of so-called E_∞ rings, the standard example being the complex of singular cochains in a topological space X. An E_∞-ring doesn't "commute on the nose," but it "commutes up to homotopy" in an appropriate sense.

Theorem 6.0.4. (Mandell) [Man] *There is a Quillen equivalence between the model category of simplicial E_∞ rings and the model category of differential graded E_∞ rings.*

There are several more possibilities for the category of "ring complexes." One can consider topological rings, or A_∞-rings. I am not aware of any results concerning model structures or homotopy categories of these.

For further discussion of these issues, I recommend [DAG, 2.6] and the introductions to [Ric2] and [SS].

7. Simplicial rings

Let **An** denote the category of commutative rings with unit. A simplicial object in **An** is called a *simplicial ring*. Simplicial rings form a category

$$\mathbf{sAn} := \operatorname{Hom}_{\mathbf{Cat}}(\Delta^{\mathrm{op}}, \mathbf{An})$$

(§3.2) which we will study in detail in this section.

We begin in §7.2 with the basic theory of *modules* over a simplicial ring. In §7.4 we explain how to view the homology of a simplicial ring as a graded-commutative ring—we will give a more sophisticated treatment of this point in §8, but the treatment in §7.4 is simpler and sufficient for our current purposes. We next describe "the" model category structure on the category of simplicial rings (§7.5) and the category of modules over a fixed simplicial ring (§7.12). This requires various factorization results established in §7.8.

7.1. Symmetric products.

For lack of a better place, we will collect in this section some basic facts about symmetric products of topological spaces.

Definition 7.1.1. Let X be a topological space, $n \in \mathbb{N}$. Then n^{th} *symmetric product* of X, denoted $\operatorname{Sym}^n X$, is the quotient of X^n by the action of the symmetric group $\mathscr{S}_n$ permuting the coordinates:

$$\sigma \cdot (x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}) \qquad (\sigma \in \mathscr{S}_n).$$

Formally, this quotient is the coequalizer

$$\operatorname{Sym}^n X = \varinjlim (a, \pi_2 : \mathscr{S}_n \times X^n \rightrightarrows X^n)$$

of the action and projection.

From the universal property of the quotient, one see that $X \mapsto \operatorname{Sym}^n X$ is a functor $\mathbf{Top} \to \mathbf{Top}$.

Lemma 7.1.2. *If $f, g : X \rightrightarrows Y$ are homotopic maps of topological spaces, then*
$$\mathrm{Sym}^n f, \mathrm{Sym}^n g : \mathrm{Sym}^n X \rightrightarrows \mathrm{Sym}^n Y$$
are homotopic maps of topological spaces.

Proof. Say $H : X \times I \to Y$ is a homotopy from f to g. The map
$$X^n \times I \to Y^n$$
$$(x_1, \dots, x_n, t) \mapsto (H(x_1, t), \dots, (x_n, t))$$
is clearly continuous and $\mathscr{S}_n$ equivariant (with $\mathscr{S}_n$ acting on $X^n \times I$ by acting on the first factor). The induced map of quotients by $\mathscr{S}_n$ will provide the desired homotopy. To be precise, one uses Proposition 4.3.8 and the local compactness of I to ensure that
$$(X^n \times I)/\mathscr{S}_n = (\mathrm{Sym}^n X) \times I.$$
□

7.2. **Modules.** Let A be a a simplicial ring. An A-*module* is a simplicial abelian group M such that each M_n is equipped with the structure of an A_n module compatibly with the simplicial structures: that is, for each Δ-morphism $\sigma : [m] \to [n]$, the map of abelian groups $M(\sigma) : M_n \to M_m$ should be $A(\sigma) : A_n \to A_m$ linear:
$$M(\sigma)(a \cdot m) = A(\sigma) \cdot M(\sigma)(m)$$
for all $a \in A_n$, $m \in M_n$. A *morphism* of A-modules $f : M \to N$ consists of an A_n module homomorphism $f_n : M_n \to N_n$ for each $n \in \mathbb{N}$ such that $N(\sigma)f_n = f_m M(\sigma)$ for each Δ-morphism $\sigma : [m] \to [n]$.

Modules over A form an abelian category $\mathbf{Mod}(A)$ with all direct and inverse limits constructed "degree-wise." We will make use of the usual notations of projective and injective objects in the abelian category $\mathbf{Mod}(A)$. Exactness of a sequence of A-modules is equivalent to exactness in each degree. The category $\mathbf{Mod}(A)$ also has tensor products, also constructed degree-wise.

Example 7.2.1. If $A \to B$ is a morphism of simplicial rings, then B becomes an A module in an obvious manner and the modules of Kähler differentials $\Omega_{B_n/A_n} \in \mathbf{Mod}(B_n)$ fit together to form a B module $\Omega_{B/A} \in \mathbf{Mod}(B)$.

If $\underline{A}$ is the constant simplicial ring associated to a ring A (Example 3.2.1), then it is clear from the definitions that we have a natural isomorphism of categories
$$\mathbf{Mod}(\underline{A}) = \mathbf{sMod}(A). \tag{7.2.1}$$
This is one situation where the abuse of notation given by writing $A \in \mathbf{sAn}$ instead of $\underline{A} \in \mathbf{sAn}$ could cause some confusion, because $\mathbf{Mod}(A)$ has ambiguous meaning. For a constant simplicial ring $\underline{A}$, the tensor product of $\underline{A}$ modules
$$\otimes : \mathbf{Mod}(\underline{A}) \times \mathbf{Mod}(\underline{A}) \to \mathbf{Mod}(\underline{A})$$
is identified under the isomorphism (7.2.1) with the functor
$$\Delta(\_ \otimes \_) : \mathbf{sMod}(A) \times \mathbf{sMod}(A) \to \mathbf{sMod}(A)$$
discussed in §5.8.

Definition 7.2.2. For an A-module M, we say that M is *degree-wise* flat (resp. finitely generated, ...) iff M_n is a flat (resp. finitely generated, ...) A_n module for every $n \in \mathbb{N}$.

The forgetful functor

$$\mathbf{Mod}(A) \to \mathbf{sSets}$$

taking an A-module to the underlying simplicial set has a left adjoint, the *free module functor*

$$\oplus_{\_} A : \mathbf{sSets} \to \mathbf{Mod}(A),$$

which associates to each simplicial set X, the A module $\oplus_X A$ given by $\oplus_{X_n} A_n \in \mathbf{Mod}(A_n)$ in degree n. For a Δ-morphism $\sigma : [n] \to [m]$, the structure map $\oplus_{X_m} A_m \to \oplus_{X_n} A_n$ for $\oplus_X A$ takes the summand indexed by $x \in X_m$ into the summand indexed by $X(\sigma)(x) \in X_n$ via the map $A(\sigma)$. An A-module M in the essential image of this functor will be called *free* and a chosen isomorphism $\oplus_X A \to M$ will be called a *basis* for M. Clearly a free A-module is degree-wise flat.

Fix $n \in \mathbb{N}$. The forgetful functor

$$\begin{aligned} \mathbf{Mod}(A) &\to \mathbf{Mod}(A_n) \\ N &\mapsto N_n \end{aligned}$$

is exact and has a left adjoint

$$\begin{aligned} \mathsf{F}_n : \mathbf{Mod}(A_n) &\to \mathbf{Mod}(A) \\ M &\mapsto \mathsf{F}_n M. \end{aligned} \tag{7.2.2}$$

To construct $\mathsf{F}_n M$, we set

$$(\mathsf{F}_n M)_k := \bigoplus_{\tau:[k]\to[n]} M \otimes_{A_n} A_k.$$

The direct sum runs over the (finite!) set of Δ-morphisms $\tau : [k] \to [n]$ and the tensor product is of course defined using $A(\tau) : A_n \to A_k$. Write $\overline{\tau} : M \otimes_{A_n} A_k \to (\mathsf{F}_n M)_k$ for the structure map to the direct sum. In particular we have the structure map

$$\overline{\mathrm{Id}} : M \to (\mathsf{F}_n M)_n. \tag{7.2.3}$$

The adjunction isomorphism

$$\mathrm{Hom}_A(\mathsf{F}_n M, N) = \mathrm{Hom}_{A_n}(M, N_n)$$

takes $g : \mathsf{F}_n M \to N$ to $g_n \overline{\mathrm{Id}} : M \to N_n$. The inverse takes $f : M \to N_n$ to the map $\Phi(f) : \mathsf{F}_n M \to N$ given in degree k using the universal property of the direct sum by

$$\begin{aligned} \Phi(f)_k \overline{\tau} := N(\tau) f \otimes_{A_n} A_k : M \otimes_{A_n} A_k &\to N_k \\ m \otimes a &\mapsto a \cdot N(\tau) f(m). \end{aligned}$$

We leave it to the reader to verify that $\Phi(f)$ is a well-defined map of A-modules and that Φ is inverse to $g \mapsto g_n \overline{\mathrm{Id}}$.

Lemma 7.2.3. *Suppose M is a projective A_n-module. Then $\mathsf{F}_n M$ is a projective A-module.*

Proof. This follows formally from the fact that $M \mapsto \mathsf{F}_n M$ is left adjoint to the exact functor $N \mapsto N_n$. □

7.3. **Associated chain complexes.** For a simplicial ring A, and an A module M, we set

$$\begin{aligned} \mathsf{N}_n(M) &:= \cap_{i=1}^n \operatorname{Ker}(d_n^i : M_n \to M_{n-1}) \\ \mathsf{Z}_n(M) &:= \cap_{i=0}^n \operatorname{Ker}(d_n^i : M_n \to M_{n-1}) \\ \mathrm{H}_n(M) &:= \mathsf{Z}_n(M)/d_{n+1}^0 \mathsf{N}_{n+1}(M). \end{aligned}$$

Note that $\mathsf{N}(M)$ is just the normalized chain complex (§5.1) associated to the underlying simplicial abelian group of M, $\mathsf{Z}_n(M)$ is the group of n-cycles in $\mathsf{N}(M)$, and $\mathrm{H}_*(M)$ is the homology of $\mathsf{N}(M)$. By Lemma 5.1.2, we can alternatively compute $\mathrm{H}_n(M)$ as the homology of the unnormalized chain complex $\mathsf{C}(M) = \oplus_{n=0}^\infty M_n$ with differential

$$d_{n+1} := \sum_{i=0}^{n+1} (-1)^i d_{n+1}^i : M_{n+1} \to M_n.$$

It is clear that all of the above constructions are natural in the A module M, so that a map of A modules $f : M \to N$ in particular induces maps

$$\mathrm{H}_n(f) : \mathrm{H}_n(M) \to \mathrm{H}_n(N).$$

Notice that $\mathsf{N}_n(M)$ and $\mathsf{Z}_n(M)$ are A_n submodules of M_n. The subset $d_{n+1}^0 \mathsf{N}_{n+1}(M) \subseteq M_n$ is also an A_n submodule because $d_{n+1}^0 : M_{n+1} \to M_n$ has a section $s_n^0 : M_n \to M_{n+1}$ (§3.2) so for $a \in A_n$ and $m \in \mathsf{N}_{n+1}(M)$ we have

$$\begin{aligned} a d_{n+1}^0(m) &= d_{n+1}^0(s_n^0(a)) d_{n+1}^0(m) \\ &= d_{n+1}^0(s_n^0(a)m) \end{aligned}$$

and $s_0^n(a)m \in \mathsf{N}_{n+1}(M)$ because $\mathsf{N}_{n+1}(M) \subseteq M_{n+1}$ is a sub A_{n+1} module. (In fact this statement requires only the surjectivity of $d_{n+1}^0 : A_{n+1} \to A_n$, not the existence of a section.) Consequently, $\mathrm{H}_n(M)$, being a quotient of two A_n modules, is also an A_n module.

Remark 7.3.1. We defined $\mathrm{H}_n(M)$ to be the quotient of the A_n-module $\mathsf{Z}_n(M)$ by the the image of the $(d_{n+1}^0 : A_{n+1} \to A_n)$-linear map

$$d_{n+1}^0 : \mathsf{N}_{n+1}(M) \to \mathsf{Z}_n(M). \tag{7.3.1}$$

As discussed above, the image of (7.3.1) is in fact an A_n-submodule of $\mathsf{Z}_n(M)$. It follows that the image of (7.3.1) is the same as the image of the corresponding A_n-module map

$$\mathsf{N}_{n+1}(M) \otimes_{A_{n+1}} A_n \to \mathsf{Z}_n(M) \tag{7.3.2}$$

(the tensor product here is of course formed using d_{n+1}^0). Evidently then, $\mathrm{H}_n(M)$ is nothing but the cokernel of the A_n-module map (7.3.2).

The A_n-modules $\mathrm{H}_n(M)$ do not assemble into an A-module in any reasonable way. We will see in §7.4 that the $\mathrm{H}_n(M)$ carry some additional structures, but for the time being we will view the H_n as functors

$$\mathrm{H}_n : \mathbf{Mod}(A) \to \mathbf{Mod}(A_n),$$

which are "unrelated" for different n.

If each A_n is a noetherian ring and each M_n is a finitely generated A_n module, then the A_n modules $\mathsf{Z}_n(M), d_{n+1}^{n+1}\mathsf{Z}_{n+1}(M), \mathrm{H}_n(M)$ are all finitely generated.

We will often be interested in the case $M = A$. Note that $\mathrm{H}_0(A) = A_0/d_1^1\mathsf{Z}_1(A)$ is the quotient of A_0 by the ideal $d_1^1\mathsf{Z}_1(A)$, so it carries a natural ring structure making $A_0 \to \mathrm{H}_0(A)$ a surjective ring homomorphism.

We will also often be interested in the case of a constant simplicial ring $\underline{A}$ associated to a ring A, in which case the homology functors

$$\mathrm{H}_n : \mathbf{Mod}(\underline{A}) \to \mathbf{Mod}(\underline{A}_n) = \mathbf{Mod}(A)$$

are clearly identified under the isomorphism (7.2.1) of §7.2 with the usual homology functors

$$\mathrm{H}_n : \mathbf{sMod}(A) \to \mathbf{Mod}(A)$$

defined for simplicial objects in the abelian category $\mathbf{Mod}(A)$, as discussed in §5.1. In particular, we have

$$\mathrm{H}_n(\underline{A}) = \begin{cases} A, & n = 0 \\ 0, & n > 0. \end{cases}$$

as in Example 5.1.3.

A map $f : M \to N$ of A modules will be called a *quasi-isomorphism* iff $\mathrm{H}_n(f)$ is an isomorphism for every $n \in \mathbb{N}$. We are already in a position to prove the following:

Lemma 7.3.2. *Let A be a simplicial ring, $f : M \to N$ a quasi-isomorphism of A modules, L an A module. Suppose that at least one of the following holds:*

(1) M and N are degree-wise flat
(2) L is degree-wise flat

Then $f \otimes L : M \otimes L \to N \otimes L$ is a quasi-isomorphism of A modules.

Proof. This is [Ill, 3.3.2.1]. The proof there is terse and a bit tricky to follow without a certain amount of familiarity with the techniques and machinery, so I will give a bit of an explanation here. We first treat the *special case* where the simplicial ring in question is a constant simplicial ring $\underline{A}$ associated to a ring A. In this case, the "coincidence" $\mathbf{Mod}(\underline{A}) = \mathbf{sMod}(A)$ and the description of the tensor product of modules and homology of modules under this coincidence (immediately above and §7.2) reduces us to a statement purely about simplicial modules over a ring, which is nothing but Lemma 5.8.2 (Eilenberg-Zilber).

In the general case, we will again end up working with *simplicial* A modules, even though we are only trying to prove a statement about A modules. A simplicial A module V carries the structure of a bisimplicial abelian group $V = (V_{\bullet\bullet})$ such that for each $q \in \mathbb{N}$, the "row" $V_{\bullet q}$ is an A module, and for each $p \in \mathbb{N}$, the "column" $V_{p\bullet}$ is a simplicial A_p module. In particular, we have a functor

$$\mathsf{C} : \mathbf{sMod}(A) \to \mathbf{Ch}_{\geq 0}\mathbf{Ch}_{\geq 0}\mathbf{Ab}$$

from simplicial A modules to double complexes of abelian groups; we can take the total complex of the double complex $\mathsf{C}(V)$ and take its homology to attach to every simplicial A module V homology groups $\operatorname{Tot}\mathrm{H}_n(V)$ arising as the abutment of two different spectral sequences. The constant simplicial A modules $\underline{L}$, $\underline{M}$, $\underline{N}$ of course have $\underline{L}_{\bullet q} = L$, $\underline{M}_{\bullet q} = M$

and $\underline{N}_{\bullet q} = N$ for all $q \in \mathbb{N}$. The spectral sequence where we first take homology in the vertical (i.e. the q) direction degenerates to yield

$$\operatorname{Tot} \mathrm{H}_n(\underline{E}) = \mathrm{H}_n(E) \tag{7.3.3}$$

for any $E \in \mathbf{Mod}(A)$, where $\mathrm{H}_*(E)$ is the "usual" homology of E defined above. By general nonsense, there exists a quasi-isomorphism of simplicial A modules $F \to \underline{L}$ such that $F_{\bullet q}$ is a free A module (§7.2) for every $q \in \mathbb{N}$, say with basis $X_q \in \mathbf{sSets}$, so that

$$F_{\bullet q} \otimes_A E = (\oplus_{X_q} \mathbb{Z}) \otimes_{\mathbb{Z}} E$$

for every A module E. The key point is that $F_{\bullet q} \otimes_A E$ is naturally identified with the result of tensoring a free (hence degree-wise flat) simplicial $\mathbb{Z}$ module (namely $\oplus_{X_q} \mathbb{Z}$) with another simplicial $\mathbb{Z}$ module (namely, the simplicial abelian group underlying E) so we can use the *special case* proved above (in the very special case where the ring is $\mathbb{Z}$ and hypothesis (2) holds) to conclude that, for each $q \in \mathbb{N}$, the functor

$$F_{\bullet q} \otimes_A \_ : \mathbf{Mod}(A) \to \mathbf{Mod}(A) \tag{7.3.4}$$

preserves quasi-isomorphisms.

Now we consider the following commutative diagram of simplicial A modules:

$$\begin{array}{ccc} \underline{M} \otimes F & \longrightarrow & \underline{N} \otimes F \\ \downarrow & & \downarrow \\ \underline{M} \otimes \underline{L} & \longrightarrow & \underline{N} \otimes \underline{L} \end{array} \tag{7.3.5}$$

Here the tensor products $\otimes$ are really the composition of the "bisimplicial tensor product"

$$\_ \otimes \_ : \mathbf{sMod}(A) \times \mathbf{sMod}(A) \to \mathbf{ssMod}(A)$$

(c.f. §5.8) and the diagonal functor

$$\Delta : \mathbf{ssMod}(A) \to \mathbf{sMod}(A),$$

though we have suppressed the Δ to ease notation. Note that $\underline{M} \otimes \underline{L} = \underline{M \otimes L}$ and similarly with M replaced by N. Now, for each fixed $q \in \mathbb{N}$, the top horizontal arrow in (7.3.5) is just

$$F_{\bullet q} \otimes_A f : F_{\bullet q} \otimes_A M \to F_{\bullet q} \otimes_A N,$$

which is the image of the quasi-isomorphism f under (7.3.4), hence is a quasi-isomorphism, hence this top vertical arrow induces an isomorphism on $\operatorname{Tot} \mathrm{H}_n$ for all n by degeneration of spectral sequences.

For each fixed $p \in \mathbb{N}$, the left vertical arrow in (7.3.5) is the map of simplicial A_p modules

$$\underline{M_p} \otimes_{A_p} F_{p\bullet} \to \underline{M_p} \otimes_{A_p} \underline{L_p}. \tag{7.3.6}$$

Since $F \to \underline{L}$ is a quasi-isomorphism of simplicial A modules, $F_{p\bullet} \to \underline{L_p}$ is a quasi-isomorphism of simplicial A_p modules. Under hypothesis (1), the map (7.3.6) is hence a quasi-isomorphism by the *special case* (and similarly with M replaced by N). On the other hand, note that each F_{pq} is a flat (even free) A_p module, so under hypothesis (2), the map (7.3.6) is also a quasi-isomorphism by the special case (applied using hypothesis (1)!). Either way, we conclude that the maps (7.3.6) (and their variants with M replaced by N) are quasi-isomorphisms, hence the vertical arrows in (7.3.5) induce isomorphisms on total homology. Since the top horizontal arrow also induces an isomorphism on total homology, so does the bottom horizontal arrow. But the map on total homology induced by the bottom horizontal arrow is identified with the map on usual homology induced by $f \otimes L$ via (7.3.3), hence we conclude that $f \otimes L$ is a quasi-isomorphism, as desired. $\square$

7.4. **Homology ring.** The homology groups $\mathrm{H}_n(A)$ fit together into a graded-commutative ring

$$\mathrm{H}_*(A) := \oplus_{n=0}^{\infty} \mathrm{H}_n(A),$$

which we will call the *homology ring* (or just the *homology*) of A. Similarly, for any A module M, the homology groups $\mathrm{H}_n(M)$ have a natural graded $\mathrm{H}_*(A)$ module structure.

The fastest way to define the multiplication maps

$$\mathrm{H}_m(A) \otimes \mathrm{H}_n(A) \to \mathrm{H}_{m+n}(A)$$

is via geometric realization: we have

$$\begin{aligned} \mathrm{H}_m(A) &= \pi_m(A) \\ &= \pi_m(|A|), \end{aligned}$$

where $|A| \in \mathbf{K}$ is the Kelley (compactly generated weak Hausdorff) space obtained as the geometric realization (§4.3) of the simplicial set $|A|$ underlying the simplicial ring A. The space $|A|$ is of course pointed by $0 \in |A|$. Since A is a ring object in simplicial sets and geometric realization (viewed as a functor to $\mathbf{K}$) preserves finite inverse limits, $|A|$ is a ring object in $\mathbf{K}$. An element $[f]$ of $\pi_m(|A|)$ is represented by a map $f : I^m \to A$ taking the boundary of the cube I^m to $0 \in |A|$ and an element $[g]$ of $\pi_n(A)$ is represented by a map $g : I^n \to A$ taking the boundary of I^n to 0. Using the ring structure on A, we can define a map

$$\begin{aligned} fg : I^{m+n} &\to |A| \\ (x, y) &\mapsto f(x)g(y) \end{aligned}$$

which also clearly takes the boundary of I^{m+n} to $0 \in |A|$, hence represents a class $[fg] \in \pi_{m+n}(|A|)$. The multiplication $[f][g] := [fg]$ is clearly well-defined; it is graded commutative because the automorphism $(x, y) \mapsto (y, x)$ of I^{m+n} acts by $(-1)^{mn}$ on the orientation of I^{m+n}. For any A module M, one can define the structure maps

$$\mathrm{H}_m(A) \otimes \mathrm{H}_n(M) \to \mathrm{H}_{m+n}(M)$$

in a similar manner.

Although this is the most expedient way to define the ring structure on $\mathrm{H}_*(A)$ and the $\mathrm{H}_*(A)$ module structure on $\mathrm{H}_*(M)$, we will see in §8 that $\mathrm{H}_*(A)$ arises as the homology ring of a differential graded ring, and, as such, is a graded-commutative ring. We will also see that $\mathrm{H}_*(M)$ is then the homology of a module over this differential graded ring.

7.5. **Model structure.** In this section, we describe "the" model category structure on simplicial rings. The results of this section can be obtained from general results of Quillen [Q1, II.4]. Our aim here is mostly to spell out some of the constructions in [Q1] a bit more explicitly. The finiteness results of Theorem 7.8.6 probably do not follow from Quillen's approach; they will be crucial in our applications to derived algebraic geometry (§9).

Definition 7.5.1. Let $f : A \to B$ be a morphism of simplicial rings. We say that f is a *weak equivalence* iff $\mathrm{H}_*(f) : \mathrm{H}_*(A) \to \mathrm{H}_*(B)$ is an isomorphism. We say that f is a *fibration* iff it satisfies the following equivalent conditions:

(1) The underlying map of simplicial sets is a fibration.
(2) The map $A \to B \times_{\mathrm{H}_0(B)} \mathrm{H}_0(A)$ is surjective.

(Proposition 5.6.3). We say that f is a *cofibration* iff it has the LLP with respect to all trivial fibrations.

The following facts are easy to see directly from the definitions:

(1) Weak equivalences satisfy 2-out-of-3.
(2) Fibrations and cofibrations form subcategories closed under retracts.

Theorem 7.5.2. *The category of simplicial rings forms a model category (§1) with the indicated weak equivalences, fibrations, and cofibrations.*

This theorem will be proved in §7.9 using the results of the next two sections. In light of the trivial facts mentioned above, it remains only to construct factorizations and establish the lifting axioms. To do this, we first need a supply of cofibrations.

7.6. **Supply of cofibrations.** In this section we prove that various types of maps of simplicial rings are cofibrations (i.e. have the LLP w.r.t. trivial fibrations). Unfortunately our cofibrations will "come in two flavors." These "two flavors" are enough to perform all the necessary factorizations to prove Theorem 7.5.2. I wasn't able to find a sufficiently flexible class of cofibrations that would include "both flavors."

Let A be a simplicial ring. The forgetful functor

$$A/\mathbf{sAn} \to \mathbf{Mod}(A)$$

has a left adjoint given by taking an A-module M to the symmetric algebra $\mathrm{Sym}^*_A M$. This symmetric algebra is formed degree-wise and the fact that one has the purported adjointness follows, say, from the analogous adjointess for usual rings. The forgetful functor

$$A/\mathbf{sAn} \to \mathbf{sSets}$$

also has a left adjoint taking a simplicial set X to the *free A-algebra* $A[X]$.

These constructions yield some "obvious" cofibrations:

Theorem 7.6.1. *Let A be a simplicial ring, M a projective A-module, $P := \mathrm{Sym}^*_A M$. Then the structure map $i : A \to P$ is a cofibration.*

Proof. We need to produce a lift in a diagram of simplicial rings

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "h"] \\ P \arrow[r, "g"] \arrow[ur, dotted, "l"] & C \end{tikzcd}$$

where h is a trivial fibration. By the universal property of the symmetric algebra such a lift is the same thing as a lift in the following diagram of A-modules:

$$\begin{tikzcd} & B \arrow[d, "h"] \\ M \arrow[r, "g"] \arrow[ur, dotted, "l"] & C \end{tikzcd}$$

Since h is a fibration, $h : B \to C \times_{\mathrm{H}_0(C)} \mathrm{H}_0(B)$ is surjective and since h is a *trivial* fibration (so $\mathrm{H}_0(B) \to \mathrm{H}_0(C)$ is an isomorphism), we see that h is surjective, so we can find the lift above since M is projective. $\square$

Corollary 7.6.2. *Let A be a simplicial ring, $n \in \mathbb{N}$, M a projective A_n-module, $\mathsf{F}_n M$ the associated A-module (§7.2), $P := \mathsf{F}_n M$. Then the structure map $i : A \to P$ is a cofibration.*

Proof. $\mathsf{F}_n M$ is a projective A-module by Lemma 7.2.3. $\square$

Corollary 7.6.3. *Let A be a simplicial ring, X a simplicial set, $P := A[X]$ the free A-algebra on X. Then the structure map $i : A \to P$ is a cofibration.*

Proof. One would like to say: The free A-algebra is the symmetric algebra on the corresponding free module, so this "follows from" the theorem. The problem is that it is not so clear that the free A-module on an arbitrary simplicial set X is actually a projective object in $\mathbf{Mod}(A)$. It is better to directly establish the existence of a lift in a diagram as in the proof of the theorem, as follows. By the universal property of the free algebra $P = A[X]$, it is equivalent to find a lift in the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & B \arrow[d, "h"] \\ X \arrow[r, "g|_X"'] \arrow[ur, dotted, "l"] & C \end{tikzcd}$$

of simplicial sets. All simplicial sets are cofibrant, so $\emptyset \to X$ is a cofibration of simplicial sets. By our definition of (trivial) fibrations, the forgetful functor $\mathbf{sAn} \to \mathbf{sSets}$ takes fibrations to fibrations and trivial fibrations to trivial fibrations, so h is also a trivial fibration of simplicial sets, hence such a lift exists. $\square$

The next flavor of cofibration is more subtle ...

Recall from §3.6 that we let L_n denote the "latching category" whose objects are surjective Δ-morphisms $\sigma : [n] \to [m]$ with $m < n$ and whose morphisms are the obvious triangles of *surjective* Δ-morphisms.

Definition 7.6.4. A *degenerate simplicial ring* is a functor

$$\begin{aligned} A : (\Delta^{\mathrm{epi}})^{\mathrm{op}} &\to \mathbf{An} \\ [n] &\mapsto A_n. \end{aligned}$$

A simplicial ring determines an *underlying degenerate simplicial ring* by restriction of functors. A *degenerate module* M over a degenerate simplicial ring A consists of an A_n-module M_n (for each $n \in \mathbb{N}$) and an $A(\sigma)$-linear map $M(\sigma) : M_m \to M_n$ defined functorially for each surjective Δ-morphism $\sigma : [n] \to [m]$. Given a degenerate module M over a degenerate simplicial ring A and an $n \in \mathbb{N}$, we call the A_n-module

$$D_n := \varinjlim \{M_m \otimes_{A_m} A_n : (\sigma : [n] \to [m]) \in \mathsf{L}_n\}, \tag{7.6.1}$$

the n^{th} *degeneracy module* of M. The maps $M_m \otimes_{A_m} A_n \to M_n$ determined by the structure maps $M(\sigma)$ for M induce a natural A_n-module map

$$D_n \to M_n. \tag{7.6.2}$$

We say that M is *split* if the natural maps (7.6.2) are injective (in which case we refer to $D_n \subseteq M_n$ as the *degenerate submodule*) and, for each $n \in \mathbb{N}$, there is an A_n-submodule $E_n \subseteq M_n$, called a *non-degenerate complement*, inducing a direct sum decomposition $M_n = D_n \oplus E_n$.

Lemma 7.6.5. *Suppose M is a split degenerate module over a degenerate simplicial ring A with non-degenerate complements $(E_0, E_1, \dots)$. Then for each $n \in \mathbb{N}$, the natural maps*

$$M(\sigma)|E_m : E_m \otimes_{A_m} A_n \to D_n \tag{7.6.3}$$

for $\sigma \in \mathsf{L}_n$ induce an isomorphism of A_n-modules

$$D_n = \bigoplus_{(\sigma:[n]\to[m])\in\mathsf{L}_n} E_m \otimes_{A_m} A_n. \tag{7.6.4}$$

Since $M_n = D_n \oplus E_n$, we hence have

$$M_n = \bigoplus_{(\sigma:[n]\to[m])\in\mathsf{L}_n} E_m \otimes_{A_m} A_n. \tag{7.6.5}$$

Proof. Note that when $n = 0$ we have $D_0 = 0$ and all of the statements of the lemma hold trivially. We prove the lemma by induction on n by constructing an inverse f to the map

$$\bigoplus_{(\sigma:[n]\to[m])\in\mathsf{L}_n} E_m \otimes_{A_m} A_n \to D_n \tag{7.6.6}$$

induced by the natural maps (7.6.3). By the universal property of the direct limit D_n and the adjointness of extension and restriction of scalars, such an A_n-module map f is specified by giving $A(\rho)$-linear maps

$$f_\rho : M_l \to \bigoplus_{\sigma\in\mathsf{L}_n} E_m \otimes_{A_m} A_n \tag{7.6.7}$$

for each surjective Δ-morphism $\rho : [n] \to [l]$ with $l < n$ so that the diagrams

$$\begin{tikzcd} & \bigoplus_{\sigma\in T_n} E_m \otimes_{A_m} A_n & \\ M_l \arrow[ur, "f_\rho"] & & M_k \arrow[ul, "f_\tau"'] \arrow[ll, "M(\theta)"] \end{tikzcd} \tag{7.6.8}$$

commute for each L_n-morphism as below.

$$\begin{tikzcd} & {[n]} \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\ {[l]} \arrow[rr, "\theta"] & & {[k]} \end{tikzcd}$$

But by induction we already know that the natural maps (7.6.3) yield an isomorphism

$$M_l = \bigoplus_{\theta:[l]\to[k]} E_k \otimes_{A_k} A_m$$

(where $\theta : [l] \to [k]$ runs over all surjective Δ-morphisms) and we can define our map

$$f_\rho : \bigoplus_{\theta:[l]\to[k]} E_k \otimes_{A_k} A_l \to \bigoplus_{\sigma\in\mathsf{L}_n} E_m \otimes_{A_m} A_n$$

by taking the summand indexed by $\theta : [l] \to [k]$ into the summand indexed by $\theta\rho : [n] \to [k]$ by the obvious map

$$\begin{aligned} E_k \otimes_{A_k} A_l &\to E_k \otimes_{A_k} A_n \\ e \otimes a &\mapsto e \otimes A(\rho)(a). \end{aligned}$$

It is tautological to check that the diagrams (7.6.8) commute and that our f is an inverse to (7.6.6). $\square$

Definition 7.6.6. A morphism of simplicial rings $i : A \to P$ is called *semi-symmetric* iff there are A_n-submodules $M_n \subseteq P_n$ (called a *basis*) for each $n \in \mathbb{N}$ satisfying the following properties:

(1) For each $n \in \mathbb{N}$, the induced map $\mathrm{Sym}^*_{A_n} M_n \to P_n$ is an isomorphism.
(2) For each surjective Δ-morphism $\sigma : [n] \to [m]$, the corresponding degeneracy map $P(\sigma) : P_m \to P_n$ takes $M_m \subseteq P_m$ into $M_n \subseteq P_n$. (Recall that this degeneracy map is injective, as it has a retract.)

The second condition above says that the degeneracy maps for P make the M_n into a degenerate module M over the degenerate simplicial ring underlying A. A semi-symmetric morphism i is called *symmetric* iff it has a basis $(M_0, M_1, \dots)$ so that the corresponding degenerate module M is split (Definition 7.6.4).

Definition 7.6.7. A morphism of simplicial rings is called *projective* (resp. *free*) iff it is symmetric and has a basis $(M_0, M_1, \dots)$ and non-degenerate complements $(E_0, E_1, \dots)$ with each E_n a projective (resp. *free*) A_n-module. Notice that, in light of the formulas (7.6.4) and (7.6.5) in Lemma 7.6.5, this implies that each D_n and each M_n is also a projective (resp. *free*) A_n-module.[8]

One can equivalently define a free morphism to be a map $i : A \to P$ such that there are subsets $X_n \subseteq P_n$ (called a *basis*) for each $n \in \mathbb{N}$ satisfying the following properties:

(1) For each $n \in \mathbb{N}$, the induced map $A_n[X_n] \to P_n$ is an isomorphism (i.e. P_n "is" a polynomial ring over A_n).
(2) For each surjective Δ-morphism $\sigma : [n] \to [m]$, the corresponding degeneracy map $P(\sigma) : P_m \to P_n$ takes $X_m \subseteq P_m$ into $X_n \subseteq P_n$.
(3) Let D_n denote the direct limit of the X_m over $(\sigma : [n] \to [m]) \in \mathsf{L}_n$. [9] We require the natural map $D_n \to X_n$ to be injective for every n. We call $D_n \subseteq X_n$ the *degenerate subset* of X_n and we call its complement $E_n := X_n \setminus D_n$ the *non-degenerate subset* of X_n.

We say that a free morphism is of *finite type* (or *degree-wise finite type* if there is any chance of confusion) if we can take every set X_n to be finite. A *trivial free* (resp. *projective*) *morphism* is a morphism which is both free (resp. projective) and a weak equivalence.

Remark 7.6.8. The "one can equivalently define" in the above definition requires some justification. Suppose $i : A \to P$ is a free morphism with basis $(X_0, X_1, \dots)$. For $n \in \mathbb{N}$, we view D_n as a subset of X_n via the natural map which is injective by hypothesis. We call $D_n \subseteq X_n$ the *degenerate subset of* X_n. Let $E_n := X_n \setminus D_n$ be the *non-degenerate*

[8] Projective (resp. free) modules are stable under extension of scalars and direct sums.
[9] Note that D_n "is" just the n^{th} latching object $\mathsf{L}_n(X)$ of "X," though in §3.6 we discussed latching objects only for simplicial sets, though the definition makes perfect sense for "purely degenerate simplicial sets" such as the X here.

complement. If we now let M_n be the free A_n-module on the set X_n, then we have $A_n[X_n] = \operatorname{Sym}^*_{A_n} M_n$ so i is certainly semi-symmetric with basis $(M_0, M_1, \dots)$. Furthermore, the natural A_n-module map (7.6.2) is nothing but the map of free A_n-modules induced by the inclusion of sets $D_n \hookrightarrow X_n$, so it is clear that the free A_n-module on the subset E_n will serve as a (free) non-degenerate complement.

To go the other way, suppose $i : A \to P$ is a symmetric morphism with basis $(M_0, M_1, \dots)$ and free non-degenerate complements $(E_0, E_1, \dots)$. Choose a basis $Y_n \subseteq E_n$ for each $n \in \mathbb{N}$. In light of the formula (7.6.5) of Lemma 7.6.5, we obtain a basis

$$X_n := \coprod_{(\tau:[n]\to[l])\in \mathsf{L}_n} Y_l$$

of M_n so that each surjective Δ-morphism $\sigma : [n] \to [m]$ induces an inclusion $M(\sigma) : X_m \hookrightarrow X_n$. This inclusion is nothing but the obvious inclusion

$$\coprod_{\pi:[m]\to[k]} Y_k \to \coprod_{\tau:[n]\to[l]} Y_l$$

taking the subset indexed by $\pi : [m] \to [k]$ bijectively onto the subset indexed by $\pi\sigma : [n] \to [k]$. (Note that, if $\pi, \pi' : [m] \rightrightarrows [k]$ are distinct, then $\pi\sigma$ and $\pi'\sigma$ are distinct because σ is surjective.) The natural map

$$\varinjlim \{X_m : (\sigma : [n] \to [m]) \in \mathsf{L}_n\} \to X_n$$

is injective because the natural map

$$\varinjlim \{M_m \otimes_{A_m} A_n : (\sigma : [n] \to [m]) \in \mathsf{L}_n\} \to M_n$$

is injective.

Remark 7.6.9. Suppose A is a simplicial ring and X is a simplicial set. The structure map $A \to A[X]$ for the free A-algebra on X may not be a free morphism of simplicial rings, because the natural map $D_n \to X_n$ (which is just the natural map $\mathsf{L}_n(X) \to X_n$ for the simplicial set X) may not be injective (Example 3.6.1). Never-the-less, we have seen in Corollary 7.6.3 that $A \to A[X]$ is a cofibration.

Similarly, if P is a projective A-module, $A \to \operatorname{Sym}^*_A P$ need not be a projective morphism. Indeed, when $P = \mathsf{F}_n M$ for a projective A_n-module M, the natural maps $D_m \to X_m$ will *not* be injective for all m (unless $M = 0$). In Example 7.6.10 below, we will give a very simple example showing that the map $A \to \operatorname{Sym}^*_A M$, although semi-symmetric, need not be symmetric, because (7.6.2) need not be injective.

Also note that not every free morphism $A \to P$ is *of the form* $A \to A[X]$ for a simplicial set X—this is because in the definition of free morphism we do not require $P(\sigma) : P_m \to P_n$ to take X_m into X_n for *all* Δ-morphisms $\sigma : [n] \to [m]$ (as would be the case if $P = A[X]$ for a simplicial set X), but rather only for *surjective* Δ-morphisms. This is why we have "two flavors" of cofibrations. The cofibrations of the "first flavor" are good for factoring a map as a cofibration followed by a surjection. The extra "flexibility" in the cofibrations of the "second flavor" makes them suitable for factoring a map that is already surjective as a cofibration followed by a trivial fibration.

Example 7.6.10. Let $s : \mathbb{Z} \to \mathbb{Z}[x]$ be the unique ring map and let $p : \mathbb{Z}[x] \to \mathbb{Z}$ be the unique ring map with $p(x) = 0$, so we have a retract diagram of rings as below.

$$\mathbb{Z} \xrightarrow{s} \mathbb{Z}[x] \xrightarrow{p} \mathbb{Z}$$

We can thing of this as a 1-truncated simplicial ring A with $A_0 = \mathbb{Z}$, $A_1 = \mathbb{Z}[x]$, $s = s_0^0$ and $d_1^0 = d_1^1 = p$. We can similarly view the diagram

$$\mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z}$$

as a module M over this 1-truncated simplicial ring. By general theory of coskeleta (§3.5), there will exist a simplicial ring A and an A-module M which agree with our A and M in degrees ≤ 1. The map $M_0 \otimes_{A_0} A_1 \to M_1$ (the natural map (7.6.2) when $n = 1$) is just the map p—it is not injective, so the semi-symmetric map $A \to \operatorname{Sym}_A^* M$ is not symmetric.

When $i : A \to P$ is a symmetric morphism with basis $(M_0, M_1, \dots)$ and $p : P \to B$ is a map of simplicial rings, we usually write $p_n : M_n \to B_n$ instead of the more correct $p_n|M_n : M_n \to B_n$. Similarly for free morphisms.

Let us first observe that free morphisms of simplicial rings have some "stability" properties that are similar to the stability properties we would expect from cofibrations:

Lemma 7.6.11. *Free morphisms of simplicial rings are stable under pushouts and compositions. If*

$$A^0 \to A^1 \to A^2 \to \cdots$$

is a diagram of free morphisms of simplicial rings indexed by $\mathbb{N}$ *with direct limit* A*, then the structure maps* $A^n \to A$ *are free morphisms.*

Proof. Suppose $i : A \to P$ is a projective (resp. free) morphism with basis $(M_0, M_1, \dots)$ and projective (resp. free) non-degenerate complements $(E_0, E_1, \dots)$. Let $A \to B$ be an arbitrary map of simplicial rings, $B \to Q = P \otimes_A B$ the pushout. It is straightforward to see that $B \to Q$ is free with basis $(M_0 \otimes_{A_0} B_0, M_1 \otimes_{A_1} B_1, \dots)$ and non-degenerate complements $(E_0 \otimes_{A_0} B_0, E_1 \otimes_{A_1} B_1, \dots)$. Since each E_n is projective (resp. free), so is each $E_n \otimes_{A_n} B_n$. The degeneracy maps for the degenerate B-module $M \otimes_A B$ are injective because the degeneracy maps for M are inclusions of direct summands. The other statements are similar and boil down to the fact that a "polynomial ring over a polynomial ring is a polynomial ring" and "a sequential direct limit of polynomial rings is a polynomial ring on the union of the variables." ☐

The proof of the following technical lemma is elementary, though tedious, and is included merely for the sake of completeness—the reader is urged to skip over it.

Lemma 7.6.12. *Suppose* $i : A \to P$ *is a symmetric morphism of simplicial rings with basis* $(M_0, M_1, \dots)$ *and non-degenerate complements* $(E_0, E_1, \dots)$*,* $f : A \to B$ *is an arbitrary morphism of simplicial rings, and* $\overline{p} : \operatorname{tr}_n P \to \operatorname{tr}_n B$ *is a map of* n*-truncated simplicial rings with* $\operatorname{tr}_n f = \overline{p} \operatorname{tr}_n i$*. Then*

$$p \mapsto (p_{n+1} : M_{n+1} \to B_{n+1})$$

establishes a bijection between the set of liftings of $\overline{p}$ *to a map* $p : \operatorname{tr}_{n+1} P \to \operatorname{tr}_{n+1} B$ *of* $(n+1)$*-truncated simplicial rings satisfying* $\operatorname{tr}_{n+1} f = p \operatorname{tr}_{n+1} i$ *and the set of* A_{m+1}*-module* *homomorphisms $q : E_{n+1} \to B_{n+1}$ making the diagrams*

(7.6.9)
$$\begin{tikzcd} E_{n+1} \arrow[r, "q"] \arrow[d, "d^i_{n+1}"'] & B_{n+1} \arrow[d, "d^i_{n+1}"] \\ P_n \arrow[r, "p_n"'] & B_n \end{tikzcd}$$

commute for $i = 0, \dots, n+1$.

Proof. Set $p_m := \overline{p}_m$ for $m \leq n$. Such a lifting p is certainly determined by the A_{n+1}-algebra map $p_{n+1} : P_{n+1} \to B_{n+1}$. Such an algebra map p_{n+1} will yield a lifting of $\overline{p}$ with $\operatorname{tr}_{n+1} f = p \operatorname{tr}_{n+1} i$ iff the square $S(\sigma)$ below commutes for every Δ-morphism $\sigma : [m] \to [k]$ with $m, k \leq n+1$.

(7.6.10)
$$\begin{tikzcd} P_k \arrow[r, "p_k"] \arrow[d, "P(\sigma)"'] & B_k \arrow[d, "B(\sigma)"] \\ P_m \arrow[r, "p_m"'] & B_m \end{tikzcd}$$

Notice that if σ happens to factor as $\sigma = \rho\tau$, then the square $S(\sigma)$ is obtained by concatenating $S(\rho)$ and $S(\tau)$ so we can check the commutativity of $S(\sigma)$ by checking commutativity of $\S(\rho)$ and $S(\tau)$. Since $\overline{p}$ was a map of n-truncated simplicial rings, we know $S(\sigma)$ commutes whenever $m, k \leq n$. We thus see that the commutativity of all the $S(\sigma)$ is equivalent to the commutativity of the squares $S(\partial^i_{n+1})$ $(i = 0, \dots, n+1)$ below

(7.6.11)
$$\begin{tikzcd} P_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "P(\partial^i_{n+1})=d^i_{n+1}"'] & B_{n+1} \arrow[d, "d^i_{n+1}=B(\partial^i_{n+1})"] \\ P_n \arrow[r, "p_n"'] & B_n \end{tikzcd}$$

and the squares $S(\sigma)$ for $\sigma : [n+1] \to [m]$ for $\sigma \in T$, shown below.

(7.6.12)
$$\begin{tikzcd} P_m \arrow[r, "p_m"] \arrow[d, "P(\sigma)"'] & B_m \arrow[d, "B(\sigma)"] \\ P_{n+1} \arrow[r, "p_{n+1}"'] & B_{n+1} \end{tikzcd}$$

By the universal property of the symmetric algebra $P_{n+1} = \operatorname{Sym}^*_{A_{n+1}} M_{n+1}$, the A_{n+1}-algebra map p_{n+1} is equivalent to the data of an A_{n+1}-module map $p_{n+1} : M_{n+1} \to B_{n+1}$. Using the direct sum decomposition $M_{n+1} = D_{n+1} \oplus E_{n+1}$ into the degenerate and non-degenerate summands, the universal property of direct sums, and the direct limit description of D_{n+1}, we see that the A_{n+1}-module map p_{n+1} is determined by giving an A_{n+1}-module map $q : E_{n+1} \to B_{n+1}$ and A_{n+1}-module maps $p_\sigma : M_m \otimes_{A_m} A_n \to B_{n+1}$ for each $\sigma \in \mathsf{L}_{n+1}$ satisfying the compatibility condition necessary to define a map out of the direct limit D_{n+1} (we will formulate this momentarily). By the adjointness of $\otimes$ and restriction of scalars, we will view p_σ as a map of A_m-modules $p_\sigma : M_m \to B_{n+1}$, regarding B_{n+1} as an A_m-module by restriction of scalars along $f_{n+1}A(\sigma) = B(\sigma)f_m$. The condition for the p_σ to define a map $D_{n+1} \to M_{m+1}$ is that the diagram

(7.6.13)
$$\begin{tikzcd} & P_{n+1} & \\ M_m \arrow[ur, "p_\sigma"] & & M_k \arrow[ul, "p_\tau"'] \arrow[ll, "P(\theta)"] \end{tikzcd}$$

should commute for each L_{n+1}-morphism as below.

$$\begin{tikzcd} & {[n+1]} \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\ {[m]} \arrow[rr, "\theta"] & & {[k]} \end{tikzcd}$$

Using the universal property of $P_m = \operatorname{Sym}_{A_m} M_m$, we see that commutativity of (7.6.12) is equivalent to the formula

(7.6.14)
$$p_\sigma = B(\sigma)p_m.$$

Using our description of p_{n+1} in terms of the p_σ (assuming these actually "glue") and q, we see that the commutativity of (7.6.11) is equivalent to the commutativity of the diagrams (7.6.9) plus the commutativity of the diagrams

(7.6.15)
$$\begin{tikzcd} M_m \arrow[r, "p_\sigma"] \arrow[d, "P(\partial^i_{n+1})P(\sigma)"'] & B_{n+1} \arrow[d, "d^i_{n+1}=B(\partial^i_{n+1})"] \\ P_n \arrow[r, "p_n"'] & B_n \end{tikzcd}$$

for each $\sigma \in T$. The whole point of the lemma is that, as long as p_σ is *defined* by (7.6.14) (as it must be in any lifting p), the diagrams (7.6.13) and (7.6.15) will commute as desired. For the commutativity of (7.6.13), just compute

$$\begin{aligned} p_\sigma P(\theta) &= B(\sigma)p_m P(\theta) \\ &= B(\sigma)B(\theta)p_k \\ &= B(\tau)p_k \end{aligned}$$

(the second equality uses the fact that $\overline{p}$ is a well-defined map of n-truncated simplicial rings). For the commutativity of (7.6.15), let $\tau : [n] \to [m]$ be the composition of $\partial^i_{n+1} : [n] \to [n+1]$ and $\sigma : [n+1] \to [m]$, then compute

$$\begin{aligned} B(\partial^i_{n+1})p_\sigma &= B(\partial^i_{n+1})B(\sigma)p_m \\ &= B(\tau)p_m \\ &= p_n P(\tau) \\ &= p_n P(\partial^i_{n+1})P(\sigma) \end{aligned}$$

(using the fact that $\overline{p}$ is a well-defined map of n-truncated simplicial rings for the third equality). □

Theorem 7.6.13. *Projective morphisms of simplicial rings are cofibrations.*

Proof. Suppose $i : A \to P$ is a projective morphism of simplicial rings. We need to show that there exists a lift l as indicated in any solid commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "h"] \\ P \arrow[r, "g"] \arrow[ur, dotted, "l"] & C \end{tikzcd}$$

of simplicial rings where h is a trivial fibration.

Fix a basis $(M_0, M_1, \dots)$ for i and a choice of non-degenerate complement $E_n \subseteq M_n$ for each $n \in \mathbb{N}$. Recall from §**??** that we let $\mathsf{M}_n(B/C)$ denote the set (in fact, A_n-module) of

$$(b_0, \dots, b_n, c) \in B_{n-1}^{[n]} \times C_n$$

satisfying

$$\begin{aligned} d_{n-1}^i b_j &= d_{n-1}^{j-1} b_i, \quad 0 \le i < j \le n \\ d_n^i c &= h_{n-1} b_i, \quad i \in [n] \end{aligned}$$

and that "h is a trivial fibration" is equivalent to surjectivity of each map

$$\begin{aligned} B_n &\to \mathsf{M}_n(B/C) \\ b &\mapsto (d_n^0 b, \dots, d_n^n b, h_n b). \end{aligned}$$

We construct the lift l inductively: To get started, we need to find a lift l_0 in the diagram of rings:

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "i_0"'] & B_0 \arrow[d, "h_0"] \\ P_0 \arrow[r, "g_0"] \arrow[ur, dotted, "l_0"] & C_0 \end{tikzcd}$$

By the universal property of the symmetric algebra $P_0 = \mathrm{Sym}^*_{A_0} M_0$, this is the same thing as finding a lift in the diagram of A_0-modules below.

$$\begin{tikzcd} & B_0 \arrow[d, "h_0"] \\ M_0 \arrow[r, "g_0"] \arrow[ur, dotted, "l_0"] & C_0 \end{tikzcd}$$

By definition of "projective module," we can find such a lift since $B_0 \to C_0 = \mathsf{M}_0(B/C)$ is surjective because h is a trivial fibration. Now suppose that we have constructed the solid diagram

$$\begin{tikzcd} \mathrm{tr}_n A \arrow[r, "\mathrm{tr}_n f"] \arrow[d, "\mathrm{tr}_n i"'] & \mathrm{tr}_n B \arrow[d, "\mathrm{tr}_n h"] \\ \mathrm{tr}_n P \arrow[r, "\mathrm{tr}_n g"] \arrow[ur, "\mathrm{tr}_n l"] & \mathrm{tr}_n C \end{tikzcd}$$

of n-truncated simplicial rings and we want to lift it to a diagram

$$\begin{tikzcd}
\operatorname{tr}_{n+1} A \arrow[r, "\operatorname{tr}_{n+1} f"] \arrow[d, "\operatorname{tr}_{n+1} i"'] & \operatorname{tr}_{n+1} B \arrow[d, "\operatorname{tr}_{n+1} h"] \\
\operatorname{tr}_{n+1} P \arrow[r, "\operatorname{tr}_{n+1} g"] \arrow[ur, dotted, "\operatorname{tr}_{n+1} l"] & \operatorname{tr}_{n+1} C
\end{tikzcd}$$

of $(n+1)$-truncated simplicial rings. By Lemma 7.9.2, it suffices to find an A_{n+1}-module homomorphism $q : E_{n+1} \to B_{n+1}$ such that

$$g_{n+1} = h_{n+1} q$$

and such that the following diagram commutes:

$$\begin{tikzcd}
E_{n+1} \arrow[r, "q"] \arrow[d, "d^i_{n+1}"'] & B_{n+1} \arrow[d, "d^i_{n+1}"] \\
A_n \arrow[r, "f_n"'] & B_n
\end{tikzcd}$$

The fact that $\operatorname{tr}_n l$ is a map of truncated simplicial rings ensures that we have a map of A_{n+1}-modules

$$\begin{aligned}
E_{n+1} &\to \mathsf{M}_{n+1}(B/C) \\
x &\mapsto (l_n d^0_{n+1} x, \dots, l_n d^{n+1}_{n+1} x, g_{n+1} x).
\end{aligned}$$

We can then find our q by lifting in the diagram of A_{n+1}-modules below.

$$\begin{tikzcd}
 & B_{n+1} \arrow[d] \\
E_{n+1} \arrow[r] \arrow[ur, dotted, "q"] & \mathsf{M}_{n+1}(B/C)
\end{tikzcd}$$

(Such a lift exists because E_{n+1} is projective and $B_{n+1} \to \mathsf{M}_{n+1}(B/C)$ is surjective.) $\square$

7.7. **Path space.** If B is a simplicial ring, then the path space B^I of §4.7 carries the structure of a simplicial ring.

DISCUSS: simplicial homotopy (§3.7) is internal so **sAn** via path spaces

Replace "trivial free morphism" in Lemma 7.9.1 with "trivial cofibration" and elaborate on the importance of this result: emphasize that the deformation retract is in **sAn**.

7.8. **Factorizations.** In this section, we will construct various factorizations of morphisms of simplicial rings which, among other things, will be used in §7.5 to prove that simplicial rings form a model category. The factorization construction in Theorem 7.8.6 is a loose conglomeration of [Q1, II.4.6-7], [And, I.6], and [Beh, Proposition 1.9]. With the possible exception of the finiteness statement in Theorem 7.8.6, the results of this section can all be obtained from the general results of Quillen in [Q1, II.4].

Lemma 7.8.1. *Any morphism $f : A \to B$ of simplicial rings can be factored as $f = qj$ where $j : A \to Q$ is a homotopy equivalence and $q : Q \to B$ is a fibration.*

Proof. If B is a simplicial ring, then the path space B^I of §4.7 carries the structure of a simplicial ring. Indeed,

$$B_n^I = \mathrm{Hom}_{\mathbf{sSets}}(\Delta[1] \times \Delta[n], B)$$

carries a natural ring structure because B is a ring object in simplicial sets. Furthermore, the evaluation maps $e_0, e_1 : B^I \to B$ and the constant path map $i : B \to B^I$ are maps of simplicial rings, and the homotopy equivalence $B \cong B^I$ is a homotopy equivalence of simplicial rings. Since all simplicial rings are fibrant (Corollary 5.6.6), the factorization of f from Lemma 4.7.6 is as desired. $\square$

We need another straightforward, though tedious, lemma.

Lemma 7.8.2. *Suppose A is a simplicial ring and $i : \mathrm{tr}_n A \to \mathrm{tr}_n P$ is a symmetric morphism of n-truncated simplicial rings. To lift i to a symmetric morphism of $(n+1)$-truncated simplicial rings, it suffices to give an A_{n+1}-module E_{n+1} (which will be a non-degenerate complement for the lifted morphism) and $A(\partial^i_{n+1})$-linear maps*

$$d^i_{n+1} : E_{n+1} \to P_n$$

for $i \in [n+1]$, so that we have an equality

$$d^i_n d^j_{n+1} = d^{j-1}_n d^i_{n+1} : E_{n+1} \to P_{n-1} \tag{7.8.1}$$

of $A(\partial^j_{n+1}\partial^i_n) = A(\partial^i_{n+1}\partial^{j-1}_n)$-linear maps for $0 \le i < j \le n+1$.

Remark 7.8.3. The data of $A(\partial^i_{n+1})$-linear maps $d^i_{n+1} : E_{n+1} \to P_n$ satisfying (7.8.1) is the same as the datum of an A_{n+1}-module map $E_{n+1} \to \mathsf{M}_{n+1}(P)$, where the $(n+1)^{\text{st}}$-matching object $\mathsf{M}_{n+1}(P)$ (§3.6) is regarded as an A_{n+1}-module via the scalar multiplication

$$a \cdot (p_0, \dots, p_{n+1}) := ((\partial^0_{n+1} a)p_0, \dots, (\partial^{n+1}_{n+1} a)p_{n+1})$$

(there is an i_n suppressed in the notation).

Proof. Choose a basis $(M_0, \dots, M_n)$ for i, so that the M_m form a (split) n-truncated degenerate module $\mathrm{tr}_n M$ over the n-truncated degenerate simplicial ring underlying A. Set

$$D_{n+1} := \varinjlim \{M_m \otimes_{A_m} A_{n+1} : (\sigma : [n+1] \to [m]) \in T_{n+1}\}$$

and let $\overline{\sigma} : M_m \to D_{n+1}$ denote the $A(\sigma)$-linear structure map to the direct limit. Set $M_{n+1} := D_{n+1} \oplus E_{n+1}$, $P_{n+1} := \mathrm{Sym}^*_{A_{n+1}} M_{n+1}$. For a T_{n+1}-morphism $\sigma : [n+1] \to [m]$, set $M(\sigma) : M_m \to M_{n+1}$ equal to $\overline{\sigma}$ followed by the inclusion $D_{n+1} \hookrightarrow M_{n+1}$. By the universal property of the direct limit, the maps $M(\sigma)$ are compatible with the structure maps for the n-truncated degenerate module $\mathrm{tr}_n M$, so we have constructed a (split) $(n+1)$-truncated module M lifting $\mathrm{tr}_n M$, where, as the notation suggests, $D_{n+1} \subseteq M_{n+1}$ is the $(n+1)^{\text{st}}$ degenerate submodule. In particular, applying the symmetric algebra functor, we obtain degeneracy maps for our lifted P compatible with the degeneracy maps for $\mathrm{tr}_n P$ and those for A. In particular, our new degeneracy maps satisfy the simplicial relations

$$s^i_n s^j_{n-1} = s^{j+1}_n s^i_{n-1} \qquad (i \le j)$$

expressing their compatibility with the degeneracy maps for $\mathrm{tr}_n P$.

We now need to define boundary maps for our lifted P compatible with our degeneracies, with the boundary maps for $\mathrm{tr}_n P$, and with the boundary maps for A. For a Δ-morphism

$\tau : [k] \to [n+1]$, to say that $P(\tau) : P_{n+1} \to P_k$ is compatible with $A(\tau)$ is to say it is obtained (using the symmetric algebra adjointness) from an $A(\tau)$-linear map

$$P(\tau) : M_{n+1} \to P_k. \tag{7.8.2}$$

Since we want (7.8.2) to be compatible with our degeneracy maps $P(\sigma)$, there is no choice about how to define the restriction $P(\tau)_D$ of (7.8.2) to D_{n+1}: We have to have $P(\tau)M(\sigma) = P(\sigma\tau) : M_m \to P_k$. Thinking of $M(\sigma)$ as the structure map $\overline{\sigma} : M_m \to D_{n+1}$, it is straightforward (and rather tautological) to check that the universal property of the direct limit does indeed yield a unique map $P(\tau)_D : D_{n+1} \to P_k$ with $P(\tau)_D M(\sigma) = P(\sigma\tau)$ for all $\sigma \in T_{n+1}$. Now, no matter how we define (7.8.2) on the non-degenerate complement E_{n+1}, it is clear from our construction of $P(\tau)_D$ that our maps $P(\tau)$ will be compatible with the degeneracies for P—that is, the simplicial relations

$$d^i_{n+1}s^j_n = \begin{cases} s^{j-1}_{n-1}d^i_n, & i < j \\ \mathrm{Id}_n, & 0 < j \le i \le j+1 \\ s^j_{n-1}d^{i-1}_n, & j+1 < i \end{cases}$$

will hold. So, it remains only to define the boundary maps $d^i_{n+1} = P(\partial^i_{n+1}) : E_{n+1} \to P_n$ in such a way that they are compatible with the the boundary maps for $\mathrm{tr}_n P$. Taking a look at the simplicial relations in Lemma 3.2.2, we see that it is enough to arrange the equalities in the statement of the lemma. $\square$

To construct our factorizations of simplicial rings, we will make heavy use of the matching objects

$$\mathsf{M}^k_n(A) := \varprojlim \{A_m : (\sigma : [m] \to [n]) \in \mathsf{M}^k_n\}$$

discussed in §3.6. Recall (§3.6) that M^k_n is the category of injective Δ-morphisms $\sigma : [m] \to [n]$ with $m \le k$ whose morphisms are the obvious commutative triangles of injective Δ-morphisms. We have "restriction" ring maps

$$A_n = \mathsf{M}^n_n(A) \to \mathsf{M}^{n-1}_n(A) \to \cdots \to \mathsf{M}^1_n(A) \to \mathsf{M}^0_n(X) = A_0^{[n]}. \tag{7.8.3}$$

We regard each $\mathsf{M}^k_n(A)$ as an A_n-module via these restriction maps. If A_n is noetherian, then each $\mathsf{M}^k_n(A)$ is a finitely generated A_n-module because $\mathsf{M}^k_n(A)$ is a submodule of the (finite!) product

$$\prod \{A_m : (\sigma : [m] \to [n]) \in \mathsf{M}^k_n\}$$

(regarding A_m in the factor indexed by σ as an A_n-module via $A(\sigma)$), each of whose factors is a finitely generated A_n-module (each of these $A(\sigma)$ is surjective because each σ is an injective Δ-morphism). In particular, when A_n is noetherian, the ring homomorphisms (7.8.3) are all finite type (since they are module finite), so, in particular, each $\mathsf{M}^k_n(A)$ is noetherian.

If $f : A \to B$ is, degree-wise, a finite type map of noetherian rings, then every induced map $\mathsf{M}^k_n(A) \to \mathsf{M}^k_n(B)$ is a finite type map of noetherian rings because it sits in a commutative diagram

$$\begin{tikzcd} A_n \arrow[r] \arrow[d] & B_n \arrow[d] \\ \mathsf{M}^k_n(A) \arrow[r] & \mathsf{M}^k_n(B) \end{tikzcd}$$

where the three other arrows are finite type.

Lemma 7.8.4. *Any map of simplicial rings $f : A \to B$ can be factored functorially as $f = ip$ where $i : A \to P$ is a cofibration and $p : P \to B$ is surjective.*

Finite version: *If f is a degree-wise finite type map of noetherian rings and we fix $N \in \mathbb{N}$ is finite, then we can can find a factorization $f = ip$ (though not functorially) where i is a finite type cofibration and p_k is surjective for $k \leq N$.*

Proof. □

Lemma 7.8.5. *Fix $n \in \mathbb{N} \cup \{\infty\}$. Suppose $f : A \to B$ is a map of simplicial rings which is surjective in degrees $< n$. Then we can factor f as $i : A \to P$ followed by $p : P \to B$ where:*

(1) i is a free morphism.
(2) $\mathrm{H}_k(p)$ is an isomorphism for all $k < n - 1$.

Theorem 7.8.6. *Any morphism $f : A \to B$ of simplicial rings can be factored as $f = ip$ where $i : A \to P$ is a free morphism and $p : P \to B$ is a trivial fibration.*

Finite Version: *Fix some $N \in \$$ and suppose that each f is (degree-wise) a finite type map of noetherian rings. Then we can factor f as $f = ip$ such that:*

(1) i is a finite type free map
(2) $p_k : P_k \to B_k$ is surjective for $k \leq N$
(3) $\mathrm{H}_k(p)$ is an isomorphism for $k \leq n$. ("p is n-connected").

Proof. We will refer to the situation where each f_n is a finite type morphism of noetherian rings as the "noetherian situation." For clarity, if $A[X]$ is the free A-algebra on a set X, we will write $[x] \in A[X]$ for the element corresponding to $x \in X$.

We get started by choosing a subset $X_0 \subseteq B_0$ generating B_0 as an A_0 algebra. In the noetherian situation, we choose X_0 finite. We set $P_0 := A_0[X_0]$, let $i_0 : A_0 \to P_0$ be the natural map, and let $p_0 : P_0 \to B_0$ be the unique A_0 algebra map with $p_0[x] = x$ for all $x \in X_0$, so $f_0 = p_0 i_0$.

Now we proceed inductively. We assume that we have constructed an n-truncated simplicial ring $\mathrm{tr}_n P \in \mathbf{s}_n\mathbf{An}$, a free map $\mathrm{tr}_n i : \mathrm{tr}_n A \to \mathrm{tr}_n P$ (of finite type in the noetherian situation) with basis $(X_0, X_1, \dots, X_n)$, and a map $\mathrm{tr}_n p : \mathrm{tr}_n P \to \mathrm{tr}_n B$ such that:

(1) $\mathrm{tr}_n f = (\mathrm{tr}_n p)(\mathrm{tr}_n i)$
(2) The map $p_k : P_k \to B_k$ is surjective for $k \leq n$.
(3) The map $p_k : \mathsf{M}_k(P) \to \mathsf{M}_k(B)$ is surjective for $k \leq n$.
(4) The map $p_k : \mathrm{H}_k(P) \to \mathrm{H}_k(B)$ is an isomorphism for $k < n$.
(5) The map $p_k : \mathsf{Z}_n(P) \to \mathrm{H}_n(B)$ is surjective.

We will prove that it is possible to extend all of this to find the same data satisfying the same conditions with n replaced everywhere by $n+1$. The desired factorization will then be the union, over all n, of these finite approximations (and the abusive notation $\mathrm{tr}_n P$, $\mathrm{tr}_n i$, $\mathrm{tr}_n p$ will be justified). The conditions above will ensure that p is a trivial fibration: By definition, this can be checked on the underlying simplicial abelian groups, so we can use Corollary 5.6.5.

To construct the desired lift, we make several *choices*:

(1) Choose a subset $W \subseteq B_{n+1}$ generating B_{n+1} as an A_{n+1} algebra, taking W finite in the noetherian situation.
(2) For each $w \in W \subseteq B_{n+1}$, $(d^0_{n+1}w, \dots, d^{n+1}_{n+1}w) \in \mathsf{M}_n(B)$, so, since $p_n : \mathsf{M}_n(P) \to \mathsf{M}_n(B)$ is surjective, we can choose $(w_0, \dots, w_{n+1}) \in \mathsf{M}_n(P)$ so that

$$(p_n w_0, \dots, p_n w_{n+1}) = (d^0_{n+1}w, \dots, d^{n+1}_{n+1}w).$$

(3) Choose some subset $S \subseteq \mathsf{M}_{n+1}(B)$ generating $\mathsf{M}_{n+1}(B)$ as a B_{n+1}-module (c.f. Remark 7.8.3 for the module structure). Each map $d^i_{n+1} : B_{n+1} \to B_n$ is surjective and $\mathsf{M}_{n+1}(B)$ is a B_{n+1}-submodule of $B_n^{[n+1]}$ (regarding the latter as a B_{n+1}-module by using d^i_{n+1} to define scalar multiplication on the i^{th} factor), so we can and do take S finite in the noetherian situation.
(4) For each $s = (s_0, \dots, s_{n+2}) \in S \subseteq \mathsf{M}_{n+1}(B)$ and each pair (i,j) with $0 \le i < j \le n+2$ choose some $s_{ij} \in P_n$ with

$$\begin{aligned} p_n(s_{ij}) &= d^i_{n+1}s_j \\ &\quad (= d^{j-1}_{n+1}s_i). \end{aligned}$$

Such a choice is possible because $p_n : P_n \to B_n$ is surjective.
(5) Choose a subset

$$K \subseteq \operatorname{Ker}(\mathrm{H}_n(p) : \mathsf{Z}_n(P) \to \mathrm{H}_n(B))$$

generating the kernel of this surjection as a P_n module, taking K finite in the noetherian situation.
(6) For each $x \in K \subseteq P_n$, $p_n(x)$ is zero in $\mathrm{H}_n(B)$, so we can choose some $z_x \in \mathsf{N}_{n+1}(B)$ with $d^0_{n+1}z_x = p_n(x)$.
(7) Choose a subset $Y \subseteq \mathsf{Z}_{n+1}(B) \subseteq B_{n+1}$ whose image in $\mathrm{H}_{n+1}(B)$ generates the latter as a B_{n+1} module. We can and do take Y finite in the noetherian situation.

For clarity, we now partially repeat the proof of Lemma 7.8.2. Let D_{n+1} be the direct limit of the sets X_m over the latching category L_{n+1} (§3.6) of all surjective non-identity Δ-morphisms $\sigma : [n+1] \to [m]$. (Note that D_{n+1} is finite when all these X_m are finite because L_{n+1} is a finite category.) For such a σ, let $\overline{\sigma} : X_m \to D_{n+1}$ denote the structure map to the direct limit.

Let T be the set of all formal symbols s_i for $s = (s_0, \dots, s_{n+2}) \in S$ and $i \in [n+2]$. (We say "formal symbols" to emphasize that s_i and s'_j are thought of as distinct elements of T whenever $s \neq s'$ or $i \neq j$, even though the elements $s_i, s'_j \in B_{n+1}$ may very well coincide.) Note that T is finite when S is finite. Set

$$\begin{aligned} E_{n+1} &:= W \coprod T \coprod K \coprod Y \\ X_{n+1} &:= D_{n+1} \coprod E_{n+1}. \end{aligned}$$

We will often write $\overline{\sigma} : X_m \to X_{n+1}$ as abuse of notation for the composition of $\overline{\sigma} : X_m \to D_{n+1}$ and the inclusion $D_{n+1} \subseteq X_{n+1}$. Let $P_{n+1} := A_{n+1}[X_{n+1}]$ and let $i_{n+1} : A_{n+1} \to P_{n+1}$ be the natural map.

For a $(\sigma : [n+1] \to [m]) \in \mathsf{L}_{n+1}$, we define the degeneracy map $P(\sigma) : P_m \to P_{n+1}$ to be the unique ring homomorphism making

$$\begin{tikzcd} P_m \arrow[r, "P(\sigma)"] & P_{n+1} \\ A_m \arrow[u, "i_m"] \arrow[r, "A(\sigma)"] & A_{n+1} \arrow[u, "i_{n+1}"'] \end{tikzcd}$$

commute and taking $X_m \subseteq P_m$ into $X_{n+1} \subseteq P_{n+1}$ via $\overline{\sigma}$. These degeneracy maps are compatible with the "old" degeneracy maps for the n-truncated simplicial ring $\operatorname{tr}_n P$ and, as the notation suggests, $D_{n+1}, E_{n+1} \subseteq X_{n+1}$ are the non-degenerate and degenerate basis elements, respectively, in the sense of Definition 7.6.7.

For $i \in [n+1]$, the boundary map $d^i_{n+1} = P(\partial^i_{n+1}) : P_{n+1} \to P_n$ is defined as follows: First of all, we need

$$\begin{tikzcd} P_{n+1} \arrow[r, "d^i_{n+1}"] & P_n \\ A_{n+1} \arrow[u, "i_{n+1}"] \arrow[r, "d^i_{n+1}"] & A_n \arrow[u, "i_n"'] \end{tikzcd} \tag{7.8.4}$$

to commute. The universal property of the free A_{n+1}-algebra $P_{n+1} = A_{n+1}[X_{n+1}]$ says that to give an A_{n+1}-algebra map d^i_{n+1} making (7.8.4) commute is the same thing as giving a map of sets

$$d^i_{n+1} : X_{n+1} \to P_n. \tag{7.8.5}$$

By (the proof of) Lemma 7.8.2, there is only one way to define the boundary maps (7.8.5) on D_{n+1} compatibly with the degeneracy maps $P(\sigma)$ defined in the previous paragraph (we must have $d^i_{n+1}\overline{\sigma} = P(\sigma\partial^i_{n+1})$) and to define the maps (7.8.5) lifting $\operatorname{tr}_n i : \operatorname{tr}_n A \to \operatorname{tr}_n P$ to a free morphism of $(n+1)$-truncated simplicial rings $\operatorname{tr}_{n+1} i : \operatorname{tr}_{n+1} A \to \operatorname{tr}_{n+1} P$ it suffices to define maps of sets

$$d^i_{n+1} : E_{n+1} \to P_n \tag{7.8.6}$$

(which, as the notation suggests, will be the restrictions of the maps (7.8.5) to E_{n+1}) satisfying

$$d^i_n d^j_{n+1} = d^{j-1}_n d^i_{n+1} : E_{n+1} \to P_{n-1}, \quad 0 \le i < j \le n+1. \tag{7.8.7}$$

We define (7.8.6) on the subsets W, K, and Y by setting

$$\begin{aligned} d^i_{n+1}(w) &:= w_i, \quad w \in W \\ d^0_{n+1}(x) &:= x, \quad x \in K \\ d^i_{n+1}(x) &:= 0, \quad x \in K,\ i > 0 \\ d^i_{n+1}(y) &:= 0, \quad y \in Y. \end{aligned} \tag{7.8.8}$$

We define (7.8.6) on T by setting

$$d^i_{n+1}(s_j) := s_{ij} \tag{7.8.9}$$

for each $s = (s_0, \dots, s_{n+2}) \in S$ and each pair (i,j) with $0 \le i < j \le n+2$. It is important to notice that this makes sense—i.e. that this definition actually defines $d^i_{n+1} s_j$ exactly once for each $s \in S$ and each $(i,j) \in [n+1] \times [n+2]$ (because either $0 \leq i < j \leq n+2$ or $0 \leq j < i+1 \leq n+2$).

As in the proof of Lemma 7.8.2, it is clear that our boundary maps d^i_{n+1}, thus defined, are compatible with the degeneracies for P. It remains only to check that they are compatible with the boundary maps d^j_n for $\operatorname{tr}_n P$. According to Lemma 7.8.2, we need to check that

$$d^i_n d^j_{n+1} = d^{j-1}_n d^i_{n+1} : E_{n+1} \to P_{n-1}$$

for all $0 \leq i < j \leq n+1$. This completes the construction of the lift $\operatorname{tr}_{n+1} i : \operatorname{tr}_{n+1} A \to \operatorname{tr}_{n+1} P$.

To define $p_{n+1} : P_{n+1} \to B_{n+1}$, with $f_{n+1} = p_{n+1} i_{n+1}$ making $\operatorname{tr}_{n+1} p : \operatorname{tr}_{n+1} P \to \operatorname{tr}_{n+1} P$ an $\mathbf{s}_{n+1}\mathbf{An}$ morphism extending $\operatorname{tr}_n p$ satisfying

$$\operatorname{tr}_{n+1} f = (\operatorname{tr}_{n+1} p)(\operatorname{tr}_{n+1} i),$$

we need only define a function $p_{n+1} : E_{n+1} \to B_{n+1}$ so that the diagram

(7.8.10)
$$\begin{tikzcd} P_{n+1} \arrow[r, "d^i_{n+1}"] \arrow[d, "p_{n+1}"'] & P_n \arrow[d, "p_n"] \\ B_{n+1} \arrow[r, "d^i_{n+1}"] & B_n \end{tikzcd}$$

commutes (Lemma 7.9.2). I claim that we can arrange this by setting

$$\begin{aligned} p_{n+1}(w) &:= w, \quad w \in W \\ p_{n+1}(x) &:= z_x, \quad x \in K \\ p_{n+1}(y) &:= y, \quad y \in Y. \end{aligned}$$

Indeed, commutativity of (7.8.10) on $W \subseteq P_{n+1}$ is equivalent to the equality

$$d^i_{n+1}(w) = p_n(w_i)$$

which we arranged by our choice of w_i (and our definition $d^i_{n+1}(w) := w_i$ in (7.8.8)), commutativity on $x \in K \subseteq P_{n+1}$ holds when $i > 0$ since both ways around the diagram yield 0 and when $i = 0$, this commutativity is equivalent to the equality

$$d^0_{n+1}(z_x) = p_n(x)$$

we arranged by our choice of z_x; commutativity on $Y \subseteq P_{n+1}$ holds since both ways around the diagram yield 0.

Now we argue that our extension to $n+1$ has all the desired properties. The map $p_{n+1} : P_{n+1} \to B_{n+1}$ is surjective by our choice of W, since p_{n+1} is a map of A_{n+1}-algebras and each $w \in W \subseteq P_{n+1}$ is in the image of p_{n+1}. To see that the stage $n+1$ map $\mathrm{H}_n(P) \to \mathrm{H}_n(B)$ is an isomorphism, first note that it was surjective at stage n and $\mathsf{Z}_n(P)$ is the same both at stage n and at stage $n+1$. Next note that our definition of d^i_{n+1} in (7.8.8) for $x \in K \subseteq X_{n+1} \subseteq P_{n+1}$ ensures that K is in fact contained in $\mathsf{N}_{n+1}(P) \subseteq P_{n+1}$, and our definition of $d^0_{n+1}(x)$ for $x \in K$ ensures that $\mathsf{N}_{n+1}(P) \to \mathsf{Z}_n(P)$ surjects onto the kernel of the surjection $\mathsf{Z}_n(P) \to \mathrm{H}_n(B)$. To see that $\mathsf{Z}_{n+1}(P) \to \mathrm{H}_{n+1}(B)$ is surjective, note that our definition $d^i_{n+1}(y) := 0$ in (7.8.8) ensures that $Y \subseteq P_{n+1}$ is in fact contained in $\mathsf{Z}_{n+1}(P)$ and our choice of $Y \subseteq \mathsf{Z}_{n+1}(B)$ and definition $p_{n+1}(y) := y$ then ensure the desired surjectivity.

For the final statement of the theorem, first factor $f = jq$ as in Lemma 7.8.1, then factor $j = ip$ as in the first part of the theorem. The free morphism i is also a weak equivalence by 2-out-of-3 because j and p are weak equivalences. The resulting factorization $f = i(pq)$ is hence as desired because pq, being a composition of fibrations, is also a fibration. $\square$

Remark 7.8.7. (Functoriality) The factorization described in the proof of Lemma 7.8.1 is clearly functorial in f. The factorizations described in the proof of Theorem 7.8.6 can be made functorial at the cost of the finiteness statement by eliminating the *choices* made in the above proof, as follows. We get started by taking $X_0 = B_0$, $P_0 = A_0[B_0]$, so that our map $p_0 : P_0 \to B_0$ comes with a natural *set-theoretic* section $s_0 : B_0 \to P_0$ given by $s_0(b) := [b]$. We then proceed inductively as in the proof above, assuming that each $p_k : P_k \to B_k$ also comes with a set-theoretic section $s_k : B_k \to P_k$. Instead of making the *choices* we made in that proof, we proceed as follows:

(1) We set $W := B_{n+1}$.
(2) For each $w \in W$ and each $i \in [n+1]$, we set $w_i := s_n d^i_{n+1}(w) \in P_n$ so that $p_n(w_i) = p_n s_n d^i_{n+1}(w) = d^i_{n+1}(w)$ because s_n is a section of p_n.
(3) Define K as the fibered product

$$\begin{tikzcd} K \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathsf{Z}_n(P) \arrow[d] \\ \mathsf{Z}_{n+1}(B) \arrow[r, "d^0_{n+1}"] & \mathrm{H}_n(B) \end{tikzcd}$$

and set $z_x := \pi_1(x) \in \mathsf{Z}_{n+1}(B)$ for $x \in K$. The image of the map π_2 is clearly the kernel of $\mathsf{Z}_n(P) \to \mathrm{H}_n(B)$.
(4) Set $Y := \mathsf{Z}_{n+1}(B) \subseteq B_{n+1}$.

It is clear that the sets W, K, Z and the definitions of w_i and z_x above are functorial in the map of simplicial rings $A \to B$. We then define P_{n+1}, i_{n+1}, and the maps p_{n+1} exactly as in the above proof. The map p_{n+1} comes with a tautological section $s_{n+1} : B_{n+1} \to P_{n+1}$ given by $s_{n+1}(b) = [b]$ because of our choice of $W = B_{n+1}$. We also define the boundary maps $d^i_{n+1} : P_{n+1} \to P_n$ as in the above proof, except we set

$$d^i_{n+1}[x] := \begin{cases} \pi_2(x), & i = 0 \\ 0, & i > 0. \end{cases}$$

All of these constructions are also functorial in $A \to B$.

In light of this remark, we have proved:

Corollary 7.8.8. *Morphisms of simplicial rings admit two functorial factorizations:*

(1) $f = ip$, where i is free (hence a cofibration by Theorem 7.6.13) and p is a trivial fibration, and
(2) $f = ip$, where i is a trivial free morphism (hence a trivial cofibration) and p is a fibration.

Remark 7.8.9. If one isn't interested in the finiteness statement of Theorem 7.8.6, then the first factorization of Theorem 7.8.6 can also be obtained formally by using the fact that the free algebra functor is left adjoint to the forgetful functor from rings to sets. This is what Illusie does in [Ill].

We can use Theorem 7.8.6 to say something about the structure of cofibrations of simplicial rings:

Corollary 7.8.10. *Suppose $i : A \to P$ is a cofibration of simplicial rings. Then the ring map $i_0 : A_0 \to P_0$ has the LLP with respect to all surjective ring maps.*

Proof. Suppose $f : B \to C$ is a surjective ring map and

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, "i_0"'] & B \arrow[d, "f"] \\ P_0 \arrow[r] \arrow[ur, dotted] & C \end{tikzcd}$$

is a commutative diagram of ring maps. Regarding f as a map of constant simplicial rings and using the fact that $A \mapsto A_0$ is left adjoint to the "constant simplicial ring" functor, we obtain a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B \arrow[d, "f"] \\ P \arrow[r] \arrow[ur, dotted] & C \end{tikzcd}$$

of simplicial rings. By Theorem 7.8.6 we can factor the map f of constant simplicial rings as $B \to Q \to C$ where $B \to Q$ is a cofibration and $Q \to C$ is a trivial fibration. It is clear from the proof of that theorem that, since f is surjective, we can arrange that $B \to Q$ is an isomorphism in degree zero. Since i is a cofibration, we can lift in the square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Q \arrow[d] \\ P \arrow[r] \arrow[ur, dotted] & C \end{tikzcd}$$

of simplicial rings because $Q \to C$ is a trivial fibration; the degree zero part of this lift then provides a lift in our original ring diagram. □

7.9. **Proof of Theorem 7.5.2.** Recall from §7.5 that the only model category axioms (§1) for **sAn** not immediate from the definitions (Definition 7.5.1) are: The factorization axiom and the lifting axiom. We established the factorization axiom in Corollary 7.8.8, and "half" of the lifting axiom is immediate from the definition of "cofibration." It remains only to show that trivial cofibrations have the LLP with respect to fibrations, which we will do in Lemma 7.9.4 after we prove the final statement of Theorem 7.5.2 as Lemma 7.9.3.

Lemma 7.9.1. *Every trivial free morphism of simplicial rings admits a deformation retract.*

Proof. We recycle the idea used to prove Lemma 7.8.1. Suppose $i : A \to P$ is a trivial free morphism. Let P^I be the path space for P, $i_P : P \to P^I$ the constant path, $e_0, e_1 : P^I \rightrightarrows P$ the evaluations. Since all simplicial rings are fibrant (by our definition of fibration and

Corollary 5.6.6), Lemma 4.7.6 says that we have a solid commutative square

$$\begin{tikzcd} A \arrow[r, "(\mathrm{Id}, i_P i)"] \arrow[d, "i"'] & A \times_P^{e_0} P^I \arrow[d, "e_1 \pi_2"] \\ P \arrow[r, equal] \arrow[ur, dotted, "(r,h)"] & P \end{tikzcd}$$

where $(\mathrm{Id}, i_P i)$ is a homotopy equivalence (hence a weak equivalence) and $e_1\pi_2$ is a fibration. Since i is a *trivial* free morphism, $e_1\pi_2$ is in fact a *trivial* fibration by 2-out-of-3, hence we can find the lift (r, h) as indicated by Theorem 7.6.13. Unravelling the commutativity of this diagram with Proposition **??**, we see that $r : P \to A$ retracts i and $h : P \to P^I$ can be viewed as a homotopy rel A from ir to the identity of P. □

Lemma 7.9.2. *Trivial free morphisms of simplicial rings have the LLP with respect to fibrations.*

Proof. This follows formally from the previous lemma using [Q1, II.3.4]. I will spell out the details here because I found Quillen's proof a little confusing. Consider a diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & B \arrow[d, "p"] \\ P \arrow[r, "k"] \arrow[ur, dotted, "l"] & C \end{tikzcd}$$

where $i : A \to P$ is a trivial free morphism and p is a fibration; we must find a lift l as indicated. By the previous lemma, i has a deformation retract $r : P \to A$. Let $h : P \to P^I$ be the homotopy witnessing this, so we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "i"'] & A^I \arrow[d, "i^I"] \\ P \arrow[r, "h"] & P^I \end{tikzcd}$$

with $e_0 h = ir$, $e_1 h = \mathrm{Id}_P$ expressing the homotopy rel A from ir to Id_P. The square

$$\begin{tikzcd} P \arrow[r, "k^I h"] \arrow[d, "jr"'] & C^I \arrow[d, "e_0"] \\ B \arrow[r, "p"] & C \end{tikzcd}$$

commutes by the computation

$$e_0 k^I h = k e_0 h = kir = pjr,$$

so we have an induced map

$$(jr, k^I h) : P \to B \times_C^{e_0} C^I.$$

The solid square

$$\begin{tikzcd} A \arrow[r, "i_B j"] \arrow[d, "i"'] & B^I \arrow[d, "(e_0, p^I)"] \\ P \arrow[r, "(jr, k^I h)"'] \arrow[ur, dotted, "H"] & B \times^{e_0}_C C^I \end{tikzcd}$$

commutes by the computations

$$e_0 i_B j = j = jri$$

and

$$p^I i_B j = p^I j^I i_A = k^I i^I i_A = k^I hi.$$

We claim that the map (e_0, p^I) is a trivial fibration. Assuming the claim, we have the lift H as indicated by Theorem 7.6.13. We obtain the desired lift l in the original diagram by setting $l := e_1 H$.

The claim follows from general facts about fibrations in the category of simplicial sets (either [Hov, 3.3.1], or the discussion in [Q1, II.2.3-4]), or slightly more directly by using the characterizations of fibrations and trivial fibrations in §**??** as follows: To prove that (e_0, p^I) is a trivial fibration it suffices (by Proposition 4.4.6) to prove that there is a lift as indicated in any solid diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "t"'] & B^I \arrow[d, "(e_0, p^I)"] \\ L \arrow[r] \arrow[ur, dotted] & B \times^{e_0}_C C^I \end{tikzcd}$$

where t is an injective map of simplicial sets. But one translates this into finding a lift in

$$\begin{tikzcd} (K \times \Delta[1]) \cup (L \times \{0\}) \arrow[r] \arrow[d] & B \arrow[d, "p"] \\ L \times \Delta[1] \arrow[r] & C \end{tikzcd}$$

(see the proof of Lemma 4.7.5 for a similar "translation") which can be done because p is a fibration (c.f. Proposition 5.6.3(3)). □

Lemma 7.9.3. *A map of simplicial rings is a cofibration (resp. trivial cofibration) iff it is a retract of a free morphism (resp. trivial free morphism).*

Proof. We saw in Theorem 7.6.13 that free morphisms are cofibrations. Any retract of a cofibration (resp. trivial cofibration) is a cofibration (resp. trivial cofibration) because this follows formally from our definition of cofibration (Definition 7.5.1), as we already mentioned in §7.5. This proves ($\Longleftarrow$). For ($\Longrightarrow$), suppose $j : A \to B$ is a cofibration (resp. trivial cofibration). Factor $j = ip$ where $i : A \to P$ is a free morphism and p is a trivial fibration (Theorem 7.8.6). If j is trivial, then so is i by 2-out-of-3. Since j is a cofibration and p is a trivial fibration, the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & P \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dotted, "s"] & B \end{tikzcd}$$

admits a completion as indicated, hence the diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "j"] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "j"] \\ B \arrow[r, "s"] & P \arrow[r, "p"] & B \end{tikzcd}$$

displays j as a retract of i. ☐

Lemma 7.9.4. *Trivial cofibrations of simplicial rings have the LLP with respect to fibrations.*

Proof. We showed in Lemma 7.9.2 that trivial free morphisms have the LLP with respect to fibrations and we showed in Lemma 7.9.3 that every trivial cofibration is a retract of a trivial free morphism, so this follows from formal diagram arguments (Lemma 1.5.2). ☐

7.10. **Properness.** In this section we will prove, among other related things, that our model structure on simplicial rings is both left and right proper, in the sense of Definition 1.4.1. We require some preliminary lemmas:

Lemma 7.10.1. *A retract of a flat ring homomorphism is again flat.*

Proof. Consider a commutative diagram of ring homomorphisms

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & A' \arrow[r, "p"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "j"] & B' \arrow[r, "q"] & B \end{tikzcd}$$

where g is flat and $pi = \mathrm{Id}$, $qj = \mathrm{Id}$. We must show that f is flat. It suffices to show that for any injective map $k : M \to N$ of A modules, $k \otimes_A B$ is also injective. Regard M, N as A' modules and k as a morphism (still injective of course!) of A modules by restriction of scalars along p. Then we have a commutative diagram

$$\begin{tikzcd} M \otimes_A B \arrow[r, "\mathrm{Id} \otimes j"] \arrow[d, "k \otimes_A B"'] & M \otimes_{A'} B' \arrow[r, "\mathrm{Id} \otimes q"] \arrow[d, "k \otimes_{A'} B'"] & M \otimes_A B \arrow[d, "k \otimes_A B"] \\ N \otimes_A B \arrow[r, "\mathrm{Id} \otimes j"] & N \otimes_{A'} B' \arrow[r, "\mathrm{Id} \otimes q"] & N \otimes_A B \end{tikzcd}$$

(of abelian groups, say) where the compositions of the rows are identities (i.e. it is a retract diagram—since $qj = \mathrm{Id}$), hence $k \otimes_A B$ is injective because $k \otimes_{A'} B'$ is injective as g is flat. Note that we use $pi = \mathrm{Id}$ to check that $\mathrm{Id} \otimes j$ is well-defined (A bilinear) as follows: the point is that, for $a \in A$ and $m \in M$, the element $a \cdot m$ of M (now viewed as an A' module via p) coincides with the element $i(a) \cdot M$, because the latter is defined to be $p(i(a)) \cdot M = a \cdot M$—one then computes

$$\begin{aligned} (\mathrm{Id} \otimes j)((a \cdot m) \otimes b) &= (a \cdot m) \otimes j(b) \\ &= (i(a) \cdot m) \otimes j(b) \\ &= m \otimes ((gi)(a) j(b)) \\ &= m \otimes ((jf)(a) j(b)) \\ &= m \otimes j(f(a) b) \\ &= (\mathrm{Id} \otimes j)(m \otimes (f(a) b)). \end{aligned}$$

□

Lemma 7.10.2. *A cofibration $i : A \to P$ of simplicial rings is degree-wise flat.*

Proof. A free morphism of simplicial rings is clearly degree-wise flat and any cofibration is a retract of a free morphism (Lemma 7.9.3), and is hence degree-wise flat by Lemma 7.10.1. □

Theorem 7.10.3. *The model category of simplicial rings (§7.5) is both left and right proper in the sense of Definition 1.4.1.*

Proof. Right properness follows formally from right properness of the model category **sSets** (Proposition **??**) in light of the fact that the forgetful functor $\mathbf{sAn} \to \mathbf{sSets}$ preserves inverse limits (in particular pullbacks), takes fibrations to fibrations (by definition of a fibration in **sAn**), and is faithful in the sense that a map of simplicial rings is a weak equivalence iff the underlying map of simplicial sets is a weak equivalence.

For left properness, suppose $f : A \to B$ is a weak equivalence and $i : A \to P$ is a cofibration. We want to show that $P \to B \otimes_A P$ is a weak equivalence. But i is degree-wise flat by Lemma 7.10.2, so this follows from Lemma 7.3.2. □

There are several variants of the properness conditions satisfied by the model category of simplicial rings:

Theorem 7.10.4. *If $f_i : A \to B_i$ $(i = 1, 2)$ are cofibrations of simplicial rings, $A \to C$ is an arbitrary map of simplicial rings, and $g : B_1 \to B_2$ is a cofibration (resp. weak equivalence) of simplicial rings with $gf_1 = f_2$, then*

$$g \otimes_A C : B_1 \otimes_A C \to B_2 \otimes_A C$$

is a cofibration (resp. weak equivalence) of simplicial rings.

Proof. First of all, a pushout of a cofibration in any model category is again a cofibration [Hov, 1.1.11]—for simplicial rings one can see this explicitly using the characterization of cofibrations in Lemma 7.9.3 by first noting that a pushout of a free morphism is clearly free, then note that pushouts (like any functor) take retracts to retracts. Evidently then, the only issue is the statement about weak equivalences, which follows from Lemma 7.10.2 and Lemma 7.3.2(1). □

Corollary 7.10.5. *If $f : A \to B$ is a (trivial) cofibration of simplicial rings and $A \to C$ is an arbitrary morphism of simplicial rings, then $f \otimes_A C : C \to B \otimes_A C$ is a (trivial) cofibration of simplicial rings.*

Proof. Apply the previous theorem with $B_1 = A$, $g = f$. □

7.11. **Derived tensor product.** Corollary 7.10.5 says that for any morphism $A \to C$ of simplicial rings, the pushout (tensor product) functor

$$\_ \otimes_A C : A/\mathbf{sAn} \to C/\mathbf{sAn} \tag{7.11.1}$$

is a left Quillen functor (§1.7), so we can form its left derived functor

$$\_ \otimes^{\mathsf{L}}_A C : \mathrm{Ho}(A/\mathbf{sAn}) \to \mathrm{Ho}(C/\mathbf{sAn}),$$

the *left derived tensor product of simplicial rings.*

7.12. **Model structure on modules.** In this section, we describe the model category structure on the category $\mathbf{Mod}(A)$ of modules over a simplicial ring A (§7.2), fixed throughout. The general ideas and proofs are similar, though easier, to what we did with simplicial rings, so for the proofs we will generally just refer back to the corresponding proofs for simplicial rings, letting the reader make the necessary modifications.

Definition 7.12.1. Let $f : M \to N$ be a morphism of A-modules. We say that f is a *weak equivalence* (or *quasi-isomorphism*) iff $\mathrm{H}_*(f) : \mathrm{H}_*(A) \to \mathrm{H}_*(B)$ is an isomorphism. We say that f is a *fibration* iff it satisfies the following equivalent conditions:

(1) The underlying map of simplicial sets is a fibration.
(2) The map $A \to B \times_{\mathrm{H}_0(B)} \mathrm{H}_0(A)$ is surjective.

(Proposition 5.6.3). We say that f is a *cofibration* iff it has the LLP with respect to all trivial fibrations.

The following facts are easy to see directly from the definitions:

(1) Weak equivalences satisfy 2-out-of-3.
(2) Fibrations and cofibrations form subcategories closed under retracts.

Definition 7.12.2. A morphism $i : M \to P$ is called *semi-split* iff there are A_n-submodules $B_n \subseteq P_n$ (called a *basis*) for each $n \in \mathbb{N}$ satisfying the following properties:

(1) For each $n \in \mathbb{N}$, the map $i_n : M_n \to P_n$ is injective and we have a direct sum decomposition of A_n-modules $P_n = M_n \oplus B_n$.
(2) For each surjective Δ-morphism $\sigma : [n] \to [m]$, the corresponding degeneracy map $P(\sigma) : P_m \to P_n$ takes $B_m \subseteq P_m$ into $B_n \subseteq P_n$. (Recall that this degeneracy map is injective, as it has a retract.)

Such a morphism i is called *split* iff, for each $n \in \mathbb{N}$, the induced A_n-module homomorphism

$$\varinjlim \{B_m \otimes_{A_m} A_n : (\sigma : [n] \to [m]) \in T_n\} \to B_n \tag{7.12.1}$$

is injective (call its image $D_n \subseteq M_n$ the *degenerate submodule* of M_n) and there is an A_n-submodule $E_n \subseteq B_n$ (called a *non-degenerate complement*) inducing a direct sum decomposition $B_n = D_n \oplus E_n$. If, furthermore, B_n and E_n can be taken projective (resp. free), then i is called *projective* (resp. *free*).

Theorem 7.12.3. *The category* $\mathbf{Mod}(A)$ *forms a model category (§1) with the weak equivalences, fibrations, and cofibrations as in Definition 7.12.1. A map of A-modules is a cofibration iff it is a retract of a free morphism (Definition 7.12.2).*

This theorem is proved in the same manner as Theorem 7.5.2. One begins by proving that every projective morphism of A-modules (in particular every free morphism of A-modules) is a cofibration (adapt the proof of Theorem 7.6.13). Then one makes use of the following factorization results:

Theorem 7.12.4. *Any morphism $f : M \to N$ of A-modules can be factored (functorially) as $f = ip$ where $i : M \to P$ is a free morphism and $p : P \to N$ is a trivial fibration. If, for all $n \in \mathbb{N}$, A_n is noetherian and M_n and N_n are finitely generated, then we can find such a factorization where all the P_n are also finitely generated. (Though the functorial factorization won't have this finiteness property.) We can also factor f (functorially) as $f = ip$ where i is a trivial free morphism and p is a fibration.*

Proof. Adapt Theorem 7.8.6. □

7.13. **Cotangent complex.**

Proposition 7.13.1. *The functor* $\Omega : \mathbf{sMap\,An} \to \mathbf{sAnMod}$ *is a left Quillen adjunction for the weak model structures above.*

8. Differential graded rings

In §8.1 we will recall some basic definitions and facts concerning differential graded rings (DGAs) and modules over them (c.f. [C1]). We will then explain how to equip the total chain complex $\mathsf{C}(A)$ of any simplicial ring A with a (strictly) graded-commutative multiplication using the *shuffle product* (§8.3), so that the usual differential on $\mathsf{C}(A)$ makes $\mathsf{C}(A)$ into a DGA. We will see that the normalized chain complex $\mathsf{N}(A) \subseteq \mathsf{C}(A)$ is a sub DGA (quasi-isomorphic to $\mathsf{C}(A)$). For an A-module M, we will see how to view $\mathsf{C}(M)$ as a module over the DGA $\mathsf{C}(A)$, and $\mathsf{N}(M)$ as a module over $\mathsf{N}(A)$ (§8.4). In particular, we will see that the homology $\mathrm{H}_*(A)$ of a simplicial ring A (§7.3) carries the structure of a strictly graded-commutative ring and the homology of an A-module carries the structure of a module over $\mathrm{H}_*(A)$.

We will also see that the DGA $\mathsf{N}(A)$ associated to a simplicial ring A is endowed with a system of *divided powers* compatible with the differential (§8.5). We will also see that this structure descends to the homology ring. It is unclear to me to what extent these are "folk theorems." Most of the statements proved here are alluded to, but not proved, in [Ill, p. 59]. The results presented here on divided powers are due essentially to Birgit Richter [Ric].

The results of this section can be summarized as follows.

Theorem 8.0.1. *The normalized chain complex defines a functor*

$$\mathsf{N} : \mathbf{sAn} \to \mathbf{PDDGA}$$

from the category **sAn** *of simplicial rings to the category* **PDDGA** *of strictly graded-commutative differential graded rings equipped with divided powers passing to homology.*

8.1. **Definitions.** A *graded-commutative ring* is a graded ring $A = \oplus_{n=0}^{\infty} A_n$ whose multiplication satisfies

$$ab = (-1)^{mn} ba \tag{8.1.1}$$

for homogeneous elements $a \in A_m, b \in A_n$. We will write $(-1)^a$ as abusive notation for $(-1)^m$ when $a \in A_m$. In this notation, we would write (8.1.1) simply as $ab = (-1)^{ab} ba$. A *strictly graded-commutative ring* is a graded-commutative ring with the additional property that $a^2 = 0$ for every homogeneous element a of odd degree. Note that graded-commutativity implies that $2a^2 = 0$ for such an a, so any graded commutative ring where 2 is a regular element is automatically strictly graded-commutative. Graded-commutative rings form a category **GrAn** whose morphisms $A \to B$ are ring homomorphisms taking A_n into B_n for every $n \in \mathbb{N}$.

Any ring can be viewed as a graded-commutative ring supported in degree zero. The exterior algebra $\wedge_A^{\bullet} M$ on a module M over a ring A is a graded-commutative A algebra.

For a graded-commutative ring A, an A *module* is a graded abelian group $M = \oplus_{n=0}^{\infty} M_n$ equipped with an A module structure which respects the grading in the sense that scalar multiplication $A \times M \to M$ takes $A_m \times M_n$ into M_{m+n}. Modules over A form an abelian category $\mathbf{Mod}(A)$ whose morphisms are maps which are both A module maps in the usual sense and maps of graded abelian groups. If $f : A \to B$ is a **GrAn** morphism, then B becomes an A module with scalar multiplication $a \cdot b := f(a)b$. The *tensor product* $M \otimes N$

of two A modules M, N is the A module obtained by quotienting the graded abelian group

$$\bigoplus_{n=0}^{\infty} \left(\bigoplus_{p+q=n} M_p \otimes_{\mathbb{Z}} N_q \right)$$

by the homogeneous (for the "outer" $\mathbb{N}$ grading) relations

$$(am) \otimes n - (-1)^{am} m \otimes (an). \tag{8.1.2}$$

A *differential graded ring* (for short: *DGA*) is a graded-commutative ring A equipped with a differential $d : A_\bullet \to A_{\bullet-1}$ satisfying $d^2 = 0$ and the *graded Leibniz rule*

$$d(ab) = (da)b + (-1)^a adb. \tag{8.1.3}$$

When discussing multiple DGAs we will often write d_A for the differential of a DGA A. Differential graded rings form a category **DGA** where a morphism is a morphism of graded-commutative rings commuting with the differentials. There is an obvious forgetful functor **DGA** $\to$ **GrAn**.

A *module* M over a DGA A is a module M over the underlying graded-commutative ring which is equipped with a differential

$$d_M : M_\bullet \to M_{\bullet-1}$$

which interacts with the differential d_A for A according to the graded Leibniz rule

$$d_M(a \cdot b) = d_A a \cdot b + (-1)^a a \cdot d_M(b). \tag{8.1.4}$$

If there is no chance of confusion we will simply write d instead of d_M. If $f : A \to B$ is a **DGA** morphism, then B becomes an A module with the usual scalar multiplication $a \cdot b := f(a)b$. The Leibniz rule is obtained from the commutativity of d_A, d_B, and f as follows:

$$\begin{aligned} d_B(a \cdot b) &= d_B(f(a)b) \\ &= d_B(f(a))b + (-1)^a f(a) d_B(b) \\ &= f(d_A(a))b + (-1)^a a \cdot d_B(b) \\ &= d_A(a) \cdot b + (-1)^a a \cdot d_B(b). \end{aligned}$$

For modules M, N over a DGA A, the tensor product $M \otimes N \in \mathbf{Mod}(A)$ is constructed by equipping the tensor product of underlying modules over the underlying graded-commutative ring with the differential

$$\begin{aligned} d : (M \otimes N)_\bullet &\mapsto (M \otimes N)_{\bullet-1} \\ d(m \otimes n) &:= d_M m \otimes n + (-1)^m m \otimes d_N n. \end{aligned}$$

To see that this is well defined we must check that it kills the relations (8.1.2). To this end we compute

$$\begin{aligned} & d((am) \otimes n - (-1)^{am} m \otimes (an)) \\ = {} & d_M(am) \otimes n + (-1)^{am}(am) \otimes d_N n - (-1)^{am} d_M m \otimes (an) - (-1)^{am+m} m \otimes d_N(an) \\ = {} & (d_A(a)m) \otimes n + (-1)^a (a d_M m) \otimes n + (-1)^{am}(am) \otimes d_N(n) \\ & -(-1)^{am} d_M m \otimes (an) - (-1)^{am+m} m \otimes (d_A(a)n) - (-1)^{am+a+a} m \otimes (a d_N n). \end{aligned}$$

We see that these six terms cancel in pairs by using the following three instances of the relations (8.1.2):

$$(d_A(a)m) \otimes n - (-1)^{(a-1)m} m \otimes (d_A(a)n)$$
$$(am) \otimes d_N n - (-1)^{am} m \otimes (a d_N n)$$
$$(a d_M m) \otimes n - (-1)^{a(m-1)} d_M m \otimes (an).$$

Similar (easier) calculations show that $d^2 = 0$ and that this d satisfies the Leibnitz rule (8.1.4). For a map of dgas $f : A \to B$, multiplication defines a B module homomorphism

$$\begin{aligned} B \otimes_A B &\to B \\ b_1 \otimes b_2 &\mapsto b_1 b_2. \end{aligned}$$

For any DGA A and any A module M, the *homology* of M is defined, as a graded abelian group, by

$$\mathrm{H}_n(M) := \mathrm{Ker}(d : M_n \to M_{n-1}) / \mathrm{Im}(d : M_{n+1} \to M_n).$$

In particular case $A = M$, the homology groups $\mathrm{H}_n(A)$ can be assembled into a graded-commutative ring

$$\mathrm{H}(A) := \oplus_{n=0}^{\infty} \mathrm{H}_n(A)$$

whose multiplication is given by $[a][b] := [ab]$ (it is easy to check that this is well-defined, associative, graded-commutative, and so forth). Similarly, the homology $\mathrm{H}(M)$ of an A module M becomes an $\mathrm{H}(A)$ module using the scalar multiplication $[a] \cdot [m] := [a \cdot m]$. Formation of homology is clearly functorial in the A module M, so we have a functor

$$\mathrm{H} : \mathbf{Mod}(A) \to \mathbf{Mod}(\mathrm{H}(A)).$$

We say that a map of A modules $f : M \to N$ is a *quasi-isomorphism* iff $\mathrm{H}(f)$ is an isomorphism of $\mathrm{H}(A)$-modules.

A morphism $f : A \to B$ of dgas induces a **GrAn** morphism

$$\mathrm{H}(f) : \mathrm{H}(A) \to \mathrm{H}(B)$$

by setting $\mathrm{H}(f)[a] := [f(a)]$. Thus we define a functor

$$\mathrm{H} : \mathbf{DGA} \to \mathbf{GrAn}.$$

We say that f is a *quasi-isomorphism* iff $\mathrm{H}(f)$ is an isomorphism.

8.2. **Examples.** ADD: free DGA on a chain complex

In this section we will give several examples of DGAs.

First of all, we can view any graded-commutative ring A as a DGA by giving it the zero differential. This defines a functor

$$i : \mathbf{GrAn} \to \mathbf{DGA}.$$

The functor i admits a right adjoint

$$\mathrm{Ker}\, d : \mathbf{DGA} \to \mathbf{GrAn}$$

taking a DGA A to the graded-commutative ring

$$\mathrm{Ker}\, d_A := \bigoplus_{n=0}^{\infty} \mathrm{Ker}(d : A_n \to A_{n-1}).$$

The point is that there is an obvious inclusion map of DGAs $i \operatorname{Ker} d_A \hookrightarrow A$ which is initial among maps from a dga with zero differential to A.

In particular, we can view any ring as a DGA supported in degree zero. When A is a ring, we use "differential graded A-algebra" as a synomnym for "differential graded ring under $i(A)$". If there is no chance of confusion, we will omit the i and simply write $i(A)$ for A. This results in a similar conflict of notation for modules as we saw in §7.2 for modules over a constant simplicial ring. We have a natural isomorphism of categories

$$\mathbf{Mod}(i(A)) = \mathbf{Ch}_{\geq 0}\mathbf{Mod}(A). \tag{8.2.1}$$

This isomorphism takes the tensor product of $i(A)$ modules to the tensor product of complexes of A modules.

For example, the de Rham complex

$$\Gamma(X, \Omega_X^\bullet) := \bigoplus_{n=0}^{\infty} \Gamma(X, \Omega_X^n)$$

of, say, a smooth manifold is a differential graded algebra under exterior differential and wedge product of forms. Similarly, if $A \to B$ is a ring map, then the de Rham complex

$$\Omega_{B/A}^\bullet := \bigoplus_{n=0}^{\infty} \wedge^n \Omega_{B/A}$$

is a differential graded A algebra.

The other main example is the Koszul complex [EGA III.1.1]. Suppose A is a ring and $\mathbf{f} = (f_1, \dots, f_n) \in A^n$. Then there is a unique differential d on the graded-commutative ring

$$K_\bullet(\mathbf{f}, A) := \bigoplus_{i=0}^{n} \wedge^i (A^n)$$

satisfying the graded Leibnitz rule and the formulas $de_i = f_i \in A = \wedge^0 A$, where $e_1, \dots, e_n \in A^n$ are the standard basis vectors.

The other main example is the symmetric algebra $\operatorname{Sym}_A^* M$ on a module M over a dga A. The forgetful functor

$$\begin{aligned} A/\mathbf{DGA} &\to \mathbf{Mod}(A) \\ (f : A \to B) &\mapsto B \end{aligned}$$

admits a left adjoint

$$\operatorname{Sym}_A^* \_ : \mathbf{Mod}(A) \to A/\mathbf{DGA}.$$

Here $\operatorname{Sym}_A^* M$ is the quotient of the tensor algebra

$$T(M) := A \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots$$

(this $T(A)$ is a non-commutative DGA under A) by the two sided ideal generated by all expressions

$$m_1 \otimes m_2 - (-1)^{|m_1 m_2|} m_2 \otimes m_1$$

where m_1, m_2 are homogeneous elements of M.

In the case of the DGA $i(A)$ associated to a ring A, we can use the isomorphism (8.2.1) to view the symmetric algebra as a functor

$$\mathbf{Ch}_{\geq 0}\mathbf{Mod}(A) \to i(A)/\mathbf{DGA} = \{\,\text{differential graded } A \text{ algebras}\,\}.$$

Running through the definitions, we see that, for a complex $M \in \mathbf{Ch}_{\geq 0}\mathbf{Mod}(A)$, we have

$$\operatorname{Sym}^*_{i(A)} M = \bigoplus_{n=0}^{\infty} \left(\bigoplus_{0p_0+1p_1+2p_2+\cdots=n} \operatorname{Sym}^{p_0}_A M_0 \otimes \wedge^{p_1}_A M_1 \otimes \operatorname{Sym}^{p_2}_A M_2 \otimes \cdots \right).$$

The inner direct sum and tensor product here are finite. The differential

$$d : (\operatorname{Sym}^*_{i(A)} M)_\bullet \to (\operatorname{Sym}^*_{i(A)} M)_{\bullet-1}$$

is characterized by the commutativity of the diagrams

$$\begin{tikzcd} M_n \arrow[r] \arrow[d, "d"'] & (\operatorname{Sym}^*_{i(A)} M)_n \arrow[d, "d"] \\ M_{n-1} \arrow[r] & (\operatorname{Sym}^*_{i(A)} M)_{n-1} \end{tikzcd}$$

where the horizontal arrows are the natural inclusions.

8.3. **Shuffle product.** Let A be a simplicial ring. In the rest of this section, we will explain how to view $\mathsf{C}(A) := \oplus_{n=0}^{\infty} A_n$ as a DGA, with the same differential

$$d_n := \sum_{i=0}^{n} (-1)^i d_n^i : A_n \to A_{n-1} \tag{8.3.1}$$

used to view $\mathsf{C}(A)$ is a chain complex (§7.3) . We will also see that $\mathsf{N}(A) = \oplus_{n=0}^{\infty} \mathsf{N}_n(A)$ is a sub-DGA. The inclusion $\mathsf{N}(A) \hookrightarrow \mathsf{C}(A)$ is then a quasi-isomorphism by Lemma 5.1.2. In particular, the homology $\mathrm{H}_*(A)$ of A (§7.3) carries the structure of a graded-commutative ring. The construction will be functorial in A, so we will obtain functors

$$\mathsf{C} : \mathbf{sAn} \to \mathbf{DGA} \tag{8.3.2}$$

$$\mathsf{N} : \mathbf{sAn} \to \mathbf{DGA}. \tag{8.3.3}$$

The dga $\mathsf{C}(A)$ associated to a simplicial ring enjoys various addition properties. It is *strictly graded commutative* in the sense that $a \cdot a = 0$ for every $a \in A_i$ for every *odd* i.[10] We will also see in the next section that $\mathsf{C}(A)$ has a natural divided power structure.

The multiplication maps for $\mathsf{C}(A)$ are given by

$$\begin{aligned} A_m \otimes A_n &\to A_{m+n} \\ a \otimes b &\mapsto a \cdot b \\ a \cdot b &:= \sum_{(\sigma,\tau)} \operatorname{sign}(\sigma,\tau) A(\sigma)(a) A(\tau)(b), \end{aligned} \tag{8.3.4}$$

where the sum runs over all (m,n)-shuffles (§5.7), and the juxtaposition on the right hand side is the product in the ring A_{m+n}. For now, we will write $a \cdot b$ to emphasize that we are refering to the product for $\mathsf{C}(A)$, to avoid possible confusion (when $m = n$) with the

[10] Graded commutativity in the usual sense only implies $2a \cdot a = 0$, but of course there is no difference if 2 is a unit in one of the rings (equivalently all of the rings) A_n.

product for the ring A_m, which we will always denote simply by juxtaposition. Notice that these multiplication maps are clearly natural in A.

If $a, b \in A_0$, then the product $a \cdot b$ defined by (8.3.4) is just the usual product ab of a and b in the ring A_0. More generally, if $a \in A_0$ and $b \in A_n$, then

$$a \cdot b = A([n] \to [0])(a)b$$

is just the usual product of the degeneracy $A([n] \to [0])(a) \in A_n$ of a and b in the ring A_n. That is, the A_0 module structure on A_n defined using (8.3.4) coincides with the usual A_0 module structure defined by restriction of scalars along the degeneracy map $A([n] \to [0]) : A_0 \to A_n$.

The multiplication (8.3.4) is graded-commutative because, for $a \in A_m$, $b \in A_n$, we compute

$$\begin{aligned} a \cdot b &= \sum_{(\sigma,\tau)} \operatorname{sign}(\sigma, \tau) A(\sigma)(a) A(\tau)(b) \\ &= \sum_{(\sigma,\tau)} (-1)^{mn} \operatorname{sign}(\tau, \sigma) A(\tau)(b) A(\sigma)(a) \\ &= (-1)^{mn} \sum_{(\tau,\sigma)} \operatorname{sign}(\tau, \sigma) A(\tau)(b) A(\sigma)(a) \\ &= (-1)^{mn} b \cdot a. \end{aligned}$$

The first two sums are over (m, n)-shuffles and the third is over (n, m)-shuffles. We have used the bijective correspondence $(\sigma, \tau) \mapsto (\tau, \sigma)$ between the former and the latter (§5.7) and the commutativity of A_{m+n}. In the case where $m = n$ is odd and $a = b$, the same observation shows that, in the sum defining $a \cdot a$, the summand indexed by (σ, τ) cancels with the one indexed by (τ, σ), thus we establish the strict graded commutativity.

To see that this multiplication is associative, fix $a \in A_m$, $b \in A_n$, $c \in A_p$. We first compute

$$\begin{aligned} (a \cdot b) \cdot c &= \sum_{\rho,\theta} \operatorname{sign}(\rho, \theta) A(\rho) \left(\sum_{\sigma,\tau} \operatorname{sign}(\sigma, \tau) A(\sigma)(a) A(\tau)(b) \right) A(\theta)(c) \\ &= \sum_{\rho,\theta} \sum_{\sigma,\tau} \operatorname{sign}(\rho, \theta) \operatorname{sign}(\sigma, \tau) A(\rho\sigma)(a) A(\rho\tau)(b) A(\theta)(c) \\ &= \sum_{\alpha,\beta,\gamma} \operatorname{sign}(\alpha, \beta, \gamma) A(\alpha)(a) A(\beta)(b) A(\gamma)(c). \end{aligned} \tag{8.3.5}$$

Here, (ρ, θ) runs over $(m+n, p)$-shuffles, (σ, τ) runs over (m, n)-shuffles, and (α, β, γ) runs over (m, n, p)-shuffles. We have used the first bijection

$$((\rho, \theta), (\sigma, \tau)) \mapsto (\rho\sigma, \rho\tau, \theta)$$

and sign relationship in Lemma 5.7.1. Computing $a \cdot (b \cdot c)$ using the second bijection and sign relationship from Lemma 5.7.1 will yield the same result.

We will check that our multiplication and differential are compatible (satisfy the graded Leibnitz Rule) in Lemma 8.3.2.

Lemma 8.3.1. *If $a \in \mathsf{N}_m(A) \subseteq A_m$ and $b \in \mathsf{N}_n(A) \subseteq A_n$, then $a \cdot b \in A_{m+n}$ is in $\mathsf{N}_{m+n}(A)$.*

Proof. Fix an $i \in \{1, \ldots, m+n\}$ and let $\delta^i := \delta^i_{m+n} : [m+n-1] \to [m+n]$ be the unique injective Δ-morphism whose image does not contain i. Assume $a \in \mathsf{N}_m(A)$, $b \in \mathsf{N}_n(A)$. We want to show that

$$\begin{aligned} d^i_{m+n}(a \cdot b) &= A(\delta^i)(a \cdot b) \\ &= A(\delta^i)\left(\sum_{\sigma,\tau} \operatorname{sign}(\sigma,\tau) A(\sigma)(a) A(\tau)(b)\right) \\ &= \sum_{\sigma,\tau} \operatorname{sign}(\sigma,\tau) A(\sigma\delta^i)(a) A(\tau\delta^i)(b) \end{aligned}$$

vanishes. Since σ is surjective and $i > 0$, we see that if $\sigma\delta^i$ isn't surjective, then its image misses some $j > 0$, so we can write $\sigma\delta^i = \delta^j_m \sigma'$, where $\delta^j_m : [m-1] \hookrightarrow [m]$ is the unique injective Δ-morphism whose image misses j. We then see that

$$\begin{aligned} A(\sigma\delta^i)(a) &= A(\delta^j_m \sigma')(a) \\ &= A(\sigma')(A(\delta^j_m)(a)) \\ &= A(\sigma')(d^j_m a) \\ &= 0 \end{aligned}$$

because $a \in \mathsf{N}_m(A)$. Similarly, if $\tau\delta^i$ fails to be surjective, then $A(\tau\delta^i)(b)$ vanishes. Therefore, in the sum calculating $d^i_{m+n}(a \cdot b)$ above, we need only sum over those (m,n)-shuffles (σ,τ) where $\sigma\delta^i$ and $\tau\delta^i$ are surjective. If $i = m+n$ there are no such shuffles (Lemma 5.7.2), so we're done. Otherwise $i \in \{1, \ldots, m+n-1\}$, and we see from the first part of Lemma 5.7.2 that the summand indexed by (σ,τ) cancels with the summand indexed by the shuffle (σ',τ') obtained by swapping i and $i+1$ in the permutation

$$(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n).$$

This is because

$$\operatorname{sign}(\sigma,\tau) = -\sigma(\sigma',\tau')$$

while $\sigma'\delta^i = \sigma\delta^i$ and $\tau\delta^i = \tau'\delta^i$. ∎

The above lemma shows that $\mathsf{N}(A)$ is a sub-DGA of $\mathsf{C}(A)$. (It is easy to see, as mentioned in §5.1, that the differential for $\mathsf{C}(A)$ takes $\mathsf{N}(A)$ into $\mathsf{N}(A)$.)

Lemma 8.3.2. *For a simplicial ring A, the multiplication* (8.3.4) *and differential* (8.3.1) *satisfy the graded Leibnitz rule*

$$d_{m+n}(a \cdot b) = d_m(a) \cdot b + (-1)^m a \cdot d_n(b)$$

for all $m, n \in \mathbb{N}$, $a \in A_m$, $b \in B_n$.

Proof. From the definitions, we compute

$$d_{m+n}(a \cdot b) = \sum_{i=0}^{m+n} \sum_{\sigma,\tau} (-1)^i \operatorname{sign}(\sigma,\tau) A(\sigma\delta^i)(a) A(\tau\delta^i)(b),$$

where (σ,τ) runs over all (m,n)-shuffles and $\delta^i : [m+n-1] \hookrightarrow [m+n]$ is the unique injective Δ-morphism whose image does not contain i. As in the previous proof, it follows from the first part of Lemma 5.7.2 that the summands indexed by those (σ, τ) for which $\sigma\delta^i$ and $\tau\delta^i$ are surjective cancel in pairs, so we can rewrite our sum as

$$\sum_{i=0}^{m+n} \sum_{\sigma,\tau} (-1)^i \operatorname{sign}(\sigma,\tau) A(\sigma\delta^i)(a) A(\tau\delta^i)(b)$$
$$+ \sum_{i=0}^{m+n} \sum_{\alpha,\beta} (-1)^i \operatorname{sign}(\alpha,\beta) A(\alpha\delta^i)(a) A(\beta\delta^i)(b),$$

where (σ, τ) runs over those (m, n)-shuffles for which $\sigma\delta^i$ isn't surjective and (α, β) runs over those (m, n)-shuffles for which $\beta\delta^i$ isn't surjective—these two sets are disjoint by Lemma 5.7.2. Using the bijections and sign rules from Lemma 5.7.2 we can rewrite this as

$$\sum_{j=0}^{m} \sum_{\sigma',\tau'} (-1)^j \operatorname{sign}(\sigma',\tau') A(\delta^j_m \sigma')(a) A(\tau')(b)$$
$$+ \sum_{j=0}^{n} \sum_{\alpha',\beta'} (-1)^{m+j} \operatorname{sign}(\alpha',\beta') A(\alpha')(a) A(\delta^j_n \beta')(b),$$

where (σ', τ') runs over $(m-1, n)$-shuffles and (α', β') runs over $(m, n-1)$-shuffles. Reorganizing these sums in the form

$$\sum_{\sigma',\tau'} \operatorname{sign}(\sigma',\tau') A(\sigma') \left(\sum_{j=0}^{m} (-1)^j A(\delta^j_m)(a) \right) A(\tau')(b)$$
$$+ (-1)^m \sum_{\alpha',\beta'} \operatorname{sign}(\alpha',\beta') A(\alpha')(a) A(\beta') \left(\sum_{j=0}^{n} (-1)^j A(\delta^j_n)(b) \right),$$

we recognize the right hand side of the desired equality. □

8.4. **Modules.** Let A be a simplicial ring, M an A module (§7.2). The construction from the previous section can also be used to endow $\mathsf{C}(M) = \oplus_{n=0}^{\infty} M_n$ with the structure of a module over the DGA $\mathsf{C}(A)$ using the expected formula $\sum_i (-1)^i d^i$ for the differential. Scalar multiplication is given by

$$
\begin{aligned}
A_p \otimes M_q &\to M_{p+q} \\
a \otimes m &\mapsto a \cdot m \\
a \cdot m &:= \sum_{(\sigma,\tau)} \operatorname{sign}(\sigma,\tau) A(\sigma)(a) \cdot M(\tau)(m),
\end{aligned}
\tag{8.4.1}
$$

where the sum runs over all (p, q)-shuffles (§5.7). The proof that $a \cdot (b \cdot m) = (a \cdot b) \cdot m$ is identical to the associativity calculation (8.3.5). The proof of the Leibnitz rule (8.1.4) is the same as the proof of Lemma 8.3.2. We also see that the quasi-isomorphic subcomplex $\mathsf{N}(M) \subseteq \mathsf{C}(M)$ is a module over $\mathsf{N}(A) \subseteq \mathsf{C}(A)$ by the same proof we used for Lemma 8.3.1.

8.5. **Divided powers.** Let A be a simplicial ring. In this section we explain how to equip the DGAs $\mathsf{C}(A)$ and $\mathsf{N}(A)$ of the previous section with a system of *divided powers*.

Let $A = \oplus_{n=0}^{\infty} A_n$ be a graded-commutative ring. Write $A_{*>0}$ for the set of homogeneous elements of A of positive degree. A *divided power structure* (or *PD structure*) on A is a set of maps

$$\gamma_n : A_i \to A_{ni},$$

defined for each $n \in \mathbb{N}$ and each integer $i > 0$ satisfying the following properties:

(PD1) $\gamma_0(a) = 1$, $\gamma_1(a) = a$ for all $a \in A_{*>0}$.
(PD2) For all *odd* $i \in \mathbb{N}$, all $n > 1$, and all $a \in A_i$, we have $\gamma_n(a) = 0$.
(PD3) $\gamma_n(\lambda a) = \lambda^n \gamma_n(a)$ for all $\lambda \in A_0$ and all $a \in A_{*>0}$.
(PD4) For all $m, n \in \mathbb{N}$ and all $a \in A_{*>0}$ we have

$$\binom{m+n}{m} \gamma_{m+n}(a) = \gamma_m(a)\gamma_n(a).$$

(PD5) For all $a, b \in A_{*>0}$ and all $n \in \mathbb{N}$ we have

$$\gamma_n(a+b) = \sum_{j+k=n} \gamma_j(a)\gamma_k(b).$$

(PD6) For all $a, b \in A_{*>0}$ and all $n \in \mathbb{N}$ we have

$$\gamma_n(ab) = n!\gamma_n(a)\gamma_n(b) = a^n \gamma_n(b) = \gamma_n(a) b^n.$$

(PD7) For all $a \in A_{*>0}$ and all $m, n \in \mathbb{N}$ we have

$$\gamma_n(\gamma_m(a)) = \frac{(mn)!}{n!(m!)^n} \gamma_{mn}(a).$$

For a DGA A, a *divided power structure* (or *PD structure*) on A is a divided power structure on the underlying graded-commutative ring A satisfying the following compatibility with the differential:

(PD8) $d\gamma_n(a) = \gamma_{n-1}(a)da$ for all $a \in A_{*>0}$ and all $n > 0$.

A PD structure on a DGA A is said to *pass to homology* iff the following is satisfied:

(PD9) For any $i > 1$, any $a \in A_i$, and any $n \in \mathbb{N}$, $\gamma_n(d(a)) = db$ for some $b \in A_{ni+1}$.

Conditions (PD1) and (PD4) imply, by induction on n, that $n!\gamma_n(a) = a^n$ for every $a \in A_{*>0}$, so if $\mathbb{Q} \subseteq A$, then A has a unique PD structure given by setting $\gamma_n(a) := a^n/(n!)$. (It is straightforward to check that this satisfies all of the properties above.) Condition (PD9) implies, as the terminology suggests, that the homology $\mathrm{H}_*(A)$ inherits a PD structure by setting $\gamma_n[a] := [\gamma_n(a)]$ for $a \in \mathrm{H}_{*>0}(A)$, this being well-defined in light of (PD9).

Let **PDDGA** denote the category of strictly graded-commutative DGAs equipped with a PD structure satisfying (PD1)-(PD9) above. A morphism in **PDDGA** is a morphism of DGAs compatible with all the maps γ_n. The remainder of this section is devoted to proving:

Theorem 8.5.1. *For any simplicial ring A, the DGA $\mathsf{N}(A)$ (§8.3) is equipped with a PD structure passing to homology. This structure is functorial in A, so that N can be viewed as functor $\mathsf{N} : \mathbf{sAn} \to \mathbf{PDDGA}$.*

In fact we will construct a PD structure on $\mathsf{C}(A)$ satisfying all of the above axioms, with the possible exception of (PD9), then we will argue that the maps γ_n take $\mathsf{N}(A)$ into itself so that $\mathsf{N}(A)$ also had a PD structure satisfying (PD1)-(PD8), then finally we will check the axiom (PD9) for $\mathsf{N}(A)$.

Discussion of Axiom (PD9)**.** I do not know whether the axiom (PD9) holds for $\mathsf{C}(A)$ itself, though I suspect that it does. The axiom (PD9) is rather restrictive and not as well-behaved as one might hope. It should be viewed as only a shadow of the "right" divided power axioms for DGAs with divided powers as "richly structured" as the divided powers on the normalized chain complex of a simplicial ring—see Remark 8.5.5, for example.

Let us give an example of two "perfectly nice" "**PDDGA**s" (we put this in quotes to mean we do not assume (PD9)); despite their common construction, only one will satisfy (PD9).

Example 8.5.2. Fix $n \in \mathbb{N}$. As in Definition 5.5.4, we let D^{n+1} denote the chain complex of abelian groups given by $\mathbb{Z}$ in degrees $n+1$ and n and zero in other degrees, with degree $n+1$ differential $d : \mathbb{Z} \to \mathbb{Z}$ given by the identity. By "general nonsense" we can form the free "**PDDGA**" on a chain complex of abelian groups. Our aim is to explicitly describe the free "**PDDGA**" A_{n+1} on the disk complex D_{n+1}—this will depend heavily on the parity of n. The "**PDDGA**" A_{n+1} is generated by a generator y of degree $|y| = n+1$ and a generator x of degree $|x| = n$; we have $dy = x$ and $dx = 0$.

Let us first discuss the case where n is odd. Then

$$A_{n+1} = \mathbb{Z}[x, \gamma_\bullet(y)]/x^2,$$

where the notation $\gamma_\bullet(y)$ is shorthand for adjoining variables $\gamma_0(y), \gamma_1(y), \gamma_2(y), \dots$ (which, as the notation suggests, will serve as the divided powers of $y = \gamma_1(y)$) subject to the relations $\gamma_0(y) = 1$ and

$$\gamma_m(y)\gamma_n(y) = \binom{m+n}{m} \gamma_{m+n}(y).$$

The notation $dy = x$ means (c.f. (PD8)) that $d\gamma_n(y) = x\gamma_{n-1}(y)$ for each positive integer n. This A_{n+1} is freely generated as a graded abelian group by

$$1 = \gamma_0(y),\ x,\ y = \gamma_1(y),\ x\gamma_1(y),\ \gamma_2(y),\ x\gamma_2(y),\ \gamma_3(y),\ \dots.$$

The boundaries of these generators are given, respectively, by

$$0,\ 0,\ x,\ 0,\ x\gamma_1(y),\ 0,\ x\gamma_2(y), \dots.$$

Notice that (PD9) is satisfied by A_{n+1}. We see that $\mathrm{H}^n(\mathbf{A}_{n+1}) = 0$ for $n > 0$ and $\mathrm{H}^0(A_{n+1}) = \mathbb{Z}$.

Now consider the case where n is even. Then

$$A_{n+1} = \mathbb{Z}[\gamma_\bullet(x), y]/y^2$$

has $\mathbb{Z}$-module basis

$$1 = \gamma_0(x),\ x = \gamma_1(x),\ y,\ \gamma_2(x),\ xy = \gamma_1(x)y,\ \gamma_3(x),\ \dots$$

with respective boundaries

$$0,\ 0,\ x,\ 0,\ 2\gamma_2(x),\ 0,\ 3\gamma_3(x), \dots.$$

Notice that (PD9) is not satisfied: for example, $\gamma_2(x)$ is not a boundary even though x is a boundary. The homology algebra of A_{n+1} is given by

$$\mathrm{H}_*(A_{n+1}) = \bigoplus_{m=0}^{\infty}(\mathbb{Z}/m\mathbb{Z})\gamma_m(x)$$

(note that the degree of $\gamma_m(x)$ is mn) with the multiplication rule

$$\gamma_k(x)\gamma_m(x) = \binom{m+k}{m}\gamma_{m+k}(x),$$

where the binomial coefficient is read modulo $m+k$. This ring certainly does not have divided powers and is rather unusual.

We start by defining the maps γ_n. By a calculation similar to the associativity calculation (8.3.5) for $\mathsf{C}(A)$, we see that the n^{th} power map

$$\begin{aligned} A_i &\mapsto A_{ni} \\ a &\mapsto a^n = \underbrace{a \cdot a \cdots a}_{n} \end{aligned}$$

is given by

$$a^n = \sum_{\sigma^1,\ldots,\sigma^n} \operatorname{sign}(\sigma^1,\ldots,\sigma^n)A(\sigma^1)(a)\cdots A(\sigma^n)(a),$$

where the sum runs over all $(i,i,\ldots,i)$-shuffles $(\sigma^1,\ldots,\sigma^n)$. The symmetric group S_n acts on such shuffles by

$$g\cdot(\sigma^1,\ldots,\sigma^n) := (\sigma^{g(1)},\ldots,\sigma^{g(n)}).$$

As long as $i>0$ or $n=1$ this is a free action.

Each $g \in S_n$ acts by swapping blocks of size i in the permutation $(\sigma^1,\ldots,\sigma^n)$, so we have

$$\operatorname{sign}(g\cdot(\sigma^1,\ldots,\sigma^n)) = \operatorname{sign}(\sigma^1,\ldots,\sigma^n) \qquad (i \text{ even}). \tag{8.5.1}$$

For even $i>0$, we define

$$\gamma_n(a) := \sum_{[\sigma^1,\ldots,\sigma^n]} \operatorname{sign}(\sigma^1,\ldots,\sigma^n)A(\sigma^1)(a)\cdots A(\sigma^n)(a), \tag{8.5.2}$$

where the sum is over orbits $[\sigma^1,\ldots,\sigma^n]$ of the S_n action on $\mathbf{Shuff}(i,i,\ldots,i)$. The point is that the summand indexed by $[\sigma^1,\ldots,\sigma^n]$ is independent of the chosen representative $(\sigma^1,\ldots,\sigma^n)$ of the orbit in light of (8.5.1) and the commutativity of the ring A_{ni}. For $n=0$, we define $\gamma_n(a) := 1$ and for $n=1$, we define (consistently with (8.5.2)) $\gamma_1(a) = a$. For $n>1$ and odd $i>0$, we set $\gamma_n(a) := 0$.

The axioms (PD1)-(PD7) are established by straightforward combinatorial arguments (counting the number of shuffles) that will be left to the reader. The first tricky axiom to check is (PD8):

Lemma 8.5.3. *For all even m all $n>0$ and all $a \in A_m$, we have*

$$d_{mn}(\gamma_n(a)) = \gamma_{n-1}(a)\cdot d_m(a).$$

Proof. From the definitions, we compute

$$d_{mn}(\gamma_n(a)) = \sum_{i=0}^{mn}(-1)^i \sum_{[\overline{\sigma}]} \operatorname{sign}[\overline{\sigma}] \prod_{k=1}^{n} A(\sigma^k\delta^i)(a), \tag{8.5.3}$$

where $\delta^i = \delta^i_{mn} : [mn-1] \hookrightarrow [mn]$ is the monic map missing i and the sum is over

$$[\overline{\sigma}] = [\sigma^1, \ldots, \sigma^n] \in \mathbf{Shuff}(m, \ldots, m)/S_n.$$

For a fixed $i \in [mn]$, let us call $[\overline{\sigma}]$ *multisurjective for* i iff each $\delta^i\sigma^k$ is surjective (this is obviously well-defined on S_n orbits). We first argue, as in previous proofs, that for any fixed i, the terms in the second sum indexed by multisurjective $[\overline{\sigma}]$ cancel in pairs. As in the proof of Lemma 5.7.2, there are no multisurjective $[\overline{\sigma}]$ unless $i \in \{1, \ldots, mn-1\}$, and in that case $[\overline{\sigma}]$ is multisurjective iff i and $i+1$ appear in different parts of the partition $\overline{\sigma}$ of $\{1, \ldots, mn\}$. Furthmore, swapping i and $i+1$ in a multisurjective $[\overline{\sigma}]$ yields a new multisurjective $[\overline{\sigma}']$ with opposite sign (in particular they aren't in the same S_n orbit) and with the property $\delta^i\sigma^k = \delta^i(\sigma')^k$ for all k (after ordering the σ^k appropriately, since we work with orbits). Evidently then, the term indexed by $[\overline{\sigma}]$ cancels with the one indexed by $[\overline{\sigma}']$.

For each $i \in [mn]$, let T_i be the set of $[\overline{\sigma}]$ which are *not* multisurjective, so that the second sum in (8.5.3) need only run over T_i. Consider some $[\overline{\sigma}] \in T_i$ and *choose* some representative $(\sigma^1, \ldots, \sigma^n)$ of this S_n orbit. Since $[\overline{\sigma}]$ isn't multisurjective, it must be that $(\sigma^k)^{-1}(j) = \{i\}$ for a unique $j \in [m]$ (because σ^k itself *is* surjective and δ^i only misses i). That is, $\sigma^k_j = i$ and $\sigma^k_{j+1} = i+1$. In particular, exactly *one* of the $\delta^i\sigma^k$ fails to be surjective. (When $i = 0$ the unique σ^k that fails to be surjective is the part of the corresponding partition $\overline{\sigma}$ of $\{1, \ldots, mn\}$ containing 1 and when $i = mn$, the unique σ^k that fails to be surjective is the part of the partition containing mn.) We can then write our one nonsurjective map as $\delta^i\sigma^k = \delta^j_m\tau$ for some surjective $\tau : [mn-1] \to [m-1]$. Via the bijection in the beginning of §5.7, we can view this τ as an $m-1$ element subset of $\{1, \ldots, mn-1\}$, and we can then let σ be the complementary $mn-m$ element subset to obtain a shuffle $(\sigma, \tau) \in \mathbf{Shuff}(mn-m, m-1)$. Now forget about the σ^k for which $\delta^i\sigma^k$ failed to be surjective, so we have $n-1$ disjoint m-element subsets

$$\sigma^1, \ldots, \sigma^{k-1}, \sigma^{k+1}, \ldots, \sigma^n \subseteq \{1, \ldots, mn\}.$$

Shifting these to replace the m elements of $\{1, \ldots, mn\}$ that were in σ^k, and reindexing, we obtain a partition

$$\{1, \ldots, mn-m\} = \alpha^1 \coprod \cdots \coprod \alpha^{n-1}$$

into m element subsets: that is,

$$(\alpha^1, \ldots, \alpha^{n-1}) \in \mathbf{Shuff}(\underbrace{m, \ldots, m}_{n-1}).$$

It is then straightforward to check that the corresponding surjections $\alpha^l : [mn-m] \to [m]$ are related to the surjections $\sigma^l : [mn] \to [m]$ and the surjection $\sigma : [mn-1] \to [mn-m]$ above by

$$\sigma^1\delta^i = \alpha^1\sigma, \ldots, \sigma^{k-1}\delta^i = \alpha^{k-1}\sigma, \sigma^{k+1}\delta^i = \alpha^k\sigma, \ldots, \sigma^n\delta^i = \alpha^{n-1}\sigma \tag{8.5.4}$$

(everything in sight is surjective, so one need only look at minimal elements of preimages to determine equality). It is clear that the S_{n-1} orbit $[\overline{\alpha}] \in \mathbf{Shuff}(m, \ldots, m)/S_{n-1}$ depends only on $[\overline{\sigma}]$ and not on our choice of representative, and that nothing else above depends on this choice of representative, so we have described a map

$$\begin{aligned}(8.5.5) \qquad \coprod_{i=0}^{mn} T_i &\to \mathbf{Shuff}(mn-m, m-1) \times \mathbf{Shuff}(m, \dots, m)/S_{n-1} \times [m] \\ [\overline{\sigma}] &\mapsto ((\sigma, \tau), [\overline{\alpha}], j).\end{aligned}$$

It is not hard to see that this map is bijective and that

$$(8.5.6) \qquad (-1)^i \operatorname{sign}[\overline{\sigma}] = (-1)^j \operatorname{sign}(\sigma, \tau) \operatorname{sign}[\overline{\alpha}].$$

Next we take a look at $\gamma_{n-1}(a) \cdot d_m(a)$, which is given by

$$\begin{aligned}&\sum_{(\sigma,\tau)} \operatorname{sign}(\sigma,\tau) A(\sigma)(\gamma_{n-1}(a)) A(\tau) \left(\sum_{j=0}^{m} (-1)^j A(\delta_m^j)(a) \right) \\ &= \sum_{(\sigma,\tau)} \operatorname{sign}(\sigma,\tau) A(\sigma) \left(\sum_{[\overline{\alpha}]} \operatorname{sign}[\overline{\alpha}] \prod_{k=1}^{n-1} A(\alpha^k)(a) \right) A(\tau) \left(\sum_{j=0}^{m} (-1)^j A(\delta_m^j)(a) \right) \\ &= \sum_{(\sigma,\tau)} \sum_{[\overline{\alpha}]} \sum_{j=0}^{m} (-1)^j \operatorname{sign}(\sigma,\tau) \operatorname{sign}[\overline{\alpha}] A(\delta_m^j \tau)(a) \prod_{k=1}^{n-1} A(\alpha^k \sigma)(a),\end{aligned}$$

where $(\sigma, \tau) \in \mathbf{Shuff}(mn-m, m-1)$ and $[\overline{\alpha}] \in \mathbf{Shuff}(m, \dots, m)/S_{n-1}$. As in the paragraph above, the summands here are indexed by a set which is bijective correspondence with with the set indexing the (non-cancelling) summands in (8.5.3), and the corresponding summands are equal in light of (8.5.4) and the sign rule (8.5.6). □

The proof that the maps γ_n take $\mathsf{N}(A)$ into itself is almost identical to the proof of Lemma 8.3.1 and will be left to the reader. Finally we check (PD9):

Lemma 8.5.4. *If $b \in \mathsf{N}_i(A)$ is a boundary in $\mathsf{N}(A)$, then $\gamma_n(b) \in \mathsf{N}_{ni}(A)$ is also a boundary in $\mathsf{N}(A)$.*

Proof. This is trivial unless i is even, which we now assume. Say $b = d_{i+1}^0 a$ for some $a \in \mathsf{N}_{i+1}(A)$. By definition of $\gamma_n(b)$, we have

$$\begin{aligned}\gamma_n(b) &:= \sum_{\overline{\sigma}} \operatorname{sign}(\overline{\sigma}) \left(\prod_{k=1}^{n} A(\sigma^k)(b) \right) \\ &= \sum_{\overline{\sigma}} \operatorname{sign}(\overline{\sigma}) \left(\prod_{k=1}^{n} A(\sigma^k)(d_{i+1}^0 a) \right),\end{aligned}$$

where the sum runs over all $\overline{\sigma} = (\sigma^1, \dots, \sigma^n) \in \mathbf{Shuff}(i, \dots, i)/S_n$ and we have implicitly picked some representative of each S_n orbit.

We view $\overline{\sigma}$ as a partition of $\{1, \dots, ni\}$ into n subsets

$$\sigma^k = \{\sigma_1^k < \cdots < \sigma_i^k\} \qquad (k \in \{1, \dots, n\})$$

of size i. The corresponding surjective Δ-morphisms σ^k are given by

$$\begin{aligned}
\sigma^k : [ni] &\to [i] \\
0, \dots, \sigma_1^k - 1 &\mapsto 0 \\
\sigma_1^k, \dots, \sigma_2^k - 1 &\mapsto 1 \\
&\vdots \\
\sigma_i^k, \dots, ni &\mapsto i.
\end{aligned}$$

Define a surjective Δ-morphism $\tau(\sigma^k) : [ni+1] \to [i+1]$ by

$$\begin{aligned}
\tau(\sigma^k) : [ni+1] &\to [i+1] \\
0 &\mapsto 0 \\
1, \dots, \sigma_1^k &\mapsto 1 \\
\sigma_1^k + 1, \dots, \sigma_2^k &\mapsto 2 \\
&\vdots \\
\sigma_{i-1}^k + 1, \dots, \sigma_i^k &\mapsto i \\
\sigma_i^k + 1, \dots, ni+1 &\mapsto i+1.
\end{aligned} \tag{8.5.7}$$

Then the diagram

$$\begin{tikzcd}
{[ni]} \arrow[r, "\sigma^k"] \arrow[d, "\partial^0_{ni+1}"'] & {[i]} \arrow[d, "\partial^0_{i+1}"] \\
{[ni+1]} \arrow[r, "\tau(\sigma^k)"] & {[i+1]}
\end{tikzcd}$$

commutes, so we have

$$A(\sigma^k) d^0_{i+1} = d^0_{ni+1} A(\tau(\sigma^k)).$$

Then we have

$$\begin{aligned}
\gamma_n(b) &= \sum_{\overline{\sigma}} \operatorname{sign}(\overline{\sigma}) \left(\prod_{k=1}^n d^0_{ni+1} A(\tau(\sigma^k))(a) \right) \\
&= d^0_{ni+1} \left(\sum_{\overline{\sigma}} \operatorname{sign}(\overline{\sigma}) \prod_{k=1}^n A(\tau(\sigma^k))(a) \right).
\end{aligned}$$

It remains only to prove that the term x in the parentheses is in the normalized chain complex $\mathsf{N}_{ni+1}(A)$—i.e. that $d^j_{ni+1}(x) = 0$ for every $j \in \{1, \dots, ni+1\}$. We use our usual tricks. For such a j, let $T(j)$ denote the set of those $\overline{\sigma}$ for which the composition $\tau(\sigma^k)\partial^j_{ni+1} : [ni] \to [i+1]$ is surjective for every $k \in \{1, \dots, n\}$. Since $j > 0$, it is clear from the formula (8.5.7) for $\tau(\sigma^k)$ that the image of $\tau(\sigma^k)\partial^j_{ni+1}$ always contains $0 \in [i+1]$, so if this composition fails to be surjective, then it will factor through some $\partial^s_{i+1} : [i] \to [i+1]$ with $s > 0$. So, if $\overline{\sigma}$ is not in $T(j)$, then there will be some $k \in \{1, \dots, n\}$ for which $d^j_{ni+1} A(\tau(\sigma^k))(a) = 0$ because $d^s_{i+1}(a) = 0$ since $a \in \mathsf{N}_{i+1}(A)$. Thus only the summands with $\overline{\sigma} \in T(j)$ will contribute to $d^j_{ni+1}(x)$, so we can write

$$d^j_{ni+1}(x) = \sum_{\overline{\sigma} \in T(j)} \operatorname{sign}(\overline{\sigma}) \prod_{k=1}^n A(\tau(\sigma^k)\partial^j_{ni+1})(a).$$

Now we argue that the terms in this sum cancel in pairs. First of all, the set $T(ni+1)$ is empty: there must be some $l \in \{1, \dots, n\}$ so that $ni = \sigma_i^k$ (because the σ^k form a partition), and then $\tau(\sigma^l)\partial_{ni+1}^{ni+1}$ won't be surjective because $i+1 \in [i+1]$ is not in its image. For similar reasons we see that $T(1) = \emptyset$. So we can assume $j \in \{2, \dots, ni\}$. Given such a j and a fixed $\overline{\sigma} \in T(j)$, there is a unique $l \in \{1, \dots, n\}$ and a unique $s \in \{1, \dots, i\}$ for which $j = \sigma_s^l$. It cannot be that $j-1 = \sigma_{s-1}^l$, because then $\tau(\sigma^l)\partial_{ni+1}^j$ would not be surjective (its image would not contain s). So $j-1$ cannot be in the part σ^l of the partition $\overline{\sigma}$, so there is some $m \in \{1, \dots, n\}$ not equal to l and some $t \in \{1, \dots, i\}$ so that $j-1 = \sigma_t^m$. Now we argue as we have done in the previous proofs that we can obtain a new $\overline{\theta} \in T(j)$ by swapping the positions of j and $j-1$ in $\overline{\sigma}$ (put $j-1 \in \theta^l$ and $j \in \theta^m$). This $\overline{\theta}$ has $\operatorname{sign}(\overline{\theta}) = -\operatorname{sign}(\overline{\sigma})$ and $\tau(\sigma^k)\partial_{ni+1}^j = \tau(\theta^k)\partial_{ni+1}^j$ for every k, so that the summand indexed by $\overline{\sigma}$ cancels with the one indexed by $\overline{\theta}$. □

Remark 8.5.5. The proof actually shows much more than the statement of the lemma. It shows that a choice of a with $da = b$ gives a *canonical choice* of c with $dc = \gamma_n(b)$. It is not entirely clear that the formula for c in terms of a is given solely in terms of the *PDDGA* $\mathsf{N}(A)$, but it is certainly a universal formula natural in the *simplicial ring* A. Even a typical PDDGA satisfying (PD9) might not come with such a "canonical choice" of witnesses to (PD9). The upshot is that the divided power structure on the normalized chain complex of a simplicial ring is *much* richer than one might initially suspect; it will satisfy many axioms much more elaborate than (PD9).

8.6. **Application to Tor.** Suppose A is a ring and B_1, B_2 are A-algebras. Then we can *forget* that the B_i are A-algebras and just think of them as A-modules, then form the graded A-module

$$\operatorname{Tor}^A(B_1, B_2) := \bigoplus_n \operatorname{Tor}_n^A(B_1, B_2).$$

In fact this graded A-module carries lots of additional structure: It is in fact a strictly graded-commutative A-algebra equipped with divided powers—we will call this the *Tor algebra*. A formal definition of this additional structure is as follows: Calculate "the" *derived* tensor product $B_1 \otimes_A^{\mathsf{L}} B_2$ in the derived category of simplicial rings (§7.11), then take the homology of the simplicial ring $B_1 \otimes_A^{\mathsf{L}} B_2$, noting that the latter has various additional structures by Theorem 8.0.1.

In the present text, we haven't tried to construct a model category structure on the category of (strictly graded-commutative) DGAs with divided powers; it is not so clear that such a thing exists. If we had developed such additional structures, then we could presumably argue that we could compute the Tor algebra using DGA resolutions instead of simplicial resolutions. For the sake of the example discussed below, we are going to assume that this is the case. That is, we will assume the Tor algebra $\operatorname{Tor}^A(B_1, B_2)$ can be computed as follows: First, find a strictly graded-commutative differential graded A-algebra C (supported in nonnegative degrees, as always) with the following properties:

(1) $\mathrm{H}_0(C) = B_1$ and $\mathrm{H}_n(C) = 0$ for $n \neq 0$.
(2) C has divided powers compatible with the passage to homology (if c is a boundary then $\gamma_n(c)$ is a boundary).
(3) C is flat as an A-module (each C_n is a flat A-module).

Now form the differential graded A-algebra $C \otimes_A B_2$ and note that it is also equipped with divided powers compatible with passage to homology, so that $\mathrm{H}_\bullet(C \otimes_A B_2)$ is a strictly graded-commutative graded A-algebra with divided powers. We want to assume that this is the Tor algebra. Let us at least note that

$$\mathrm{H}_n(C \otimes_A B_2) = \mathrm{Tor}_n^A(B_1, B_2)$$

as A-modules because our assumptions on C certainly imply that C (viewed just as a complex of A-modules) is a flat resolution of B_1. The issue is to prove that the strictly graded-commutative graded A-algebra $\mathrm{H}_\bullet(C \otimes_A B_2)$ is independent of the choice of C satisfying the above properties. Our present machinery allows us to prove this only for those C which arise as normalized complexes of simplicial A-algebras. (We haven't developed a theory of derived tensor product of DGAs.) Anyway, let us ignore this issue for a moment and consider a simple example.

Example 8.6.1. Let $A = \mathbb{Z}[\epsilon]/\epsilon^2$ and let $B_1 = B_2 = \mathbb{Z}$, regarded as an A-algebra via $\epsilon \mapsto 0$. We can easily find a resolution of $\mathbb{Z}$ by free A-*modules*: Just take the chain complex which is A in every (non-negative) degree and where every boundary map is multiplication by ϵ. Tensoring this complex over A with $\mathbb{Z}$ yields the complex which is $\mathbb{Z}$ in every degree with all boundary maps zero, so

$$\mathrm{Tor}_n^A(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$$

for all n. This describes $\mathrm{Tor}^A(\mathbb{Z}, \mathbb{Z})$ as a graded A-*module*, but we want to understand it as a graded A-*algebra*. Consider the differential graded A-algebra

$$C := A[x, \gamma_\bullet(y)]/x^2$$

where $|x| = 1$, $|y| = 2$, $dx = \epsilon$, $dy = \epsilon x$ and $\gamma_\bullet(y)$ is shorthand for adjoining variables $\gamma_0(y), \gamma_1(y), \gamma_2(y), \ldots$ (which, as the notation suggests, will serve as the divided powers of $y = \gamma_1(y)$) subject to the relations $\gamma_0(y) = 1$ and

$$\gamma_m(y)\gamma_n(y) = \binom{m+n}{m} \gamma_{m+n}(y).$$

The notation $dy = \epsilon x$ is shorthand for $d\gamma_n(y) = \gamma_{n-1}(y)\epsilon x$ for each positive integer n. As a graded A-module, C is free with basis

$$\gamma_0(y) = 1,\ x\gamma_0(y) = x,\ \gamma_1(y) = y,\ x\gamma_1(y) = xy,\ \gamma_2(y),\ x\gamma_2(y),\ \ldots$$

This C is strictly graded commutative with divided powers compatible with passage to homology. For $n \in \mathbb{N}$, the kernel of $d_{2n+1} : C_{2n+1} \to C_{2n}$ is generated as an A-module by $\epsilon x \gamma_n(y)$. But (perhaps up to a sign) $\epsilon x \gamma_n(y)$ is equal to $d\gamma_{n+1}(y)$, so C has zero homology in odd degrees. In degree $2n$ for n positive, the kernel of d_{2n} is generated as an A-module by $\epsilon\gamma_n(y)$, which (possibly up to a sign) is the boundary of $x\gamma_n(y)$; so $\mathsf{N}_n(C) = 0$ for $n \neq 0$. In degree zero, $\mathrm{H}_0(C) = \mathbb{Z}$ because the kernel of d_0 is $A \cdot 1 = C_0$, but the image of $d_1 : C_1 \to C_0$ is $A \cdot \epsilon$. The differential graded A-algebra $C \otimes_A \mathbb{Z}$ is obtained from C by setting $\epsilon = 0$. The boundary maps are then zero, so this differential graded A-algebra (with divided powers, etc.) is equal to its own homology. We find that

$$\mathrm{Tor}^A(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}[x, \gamma_\bullet(y)]/x^2.$$

Notice that the degree m part of this graded A-algebra is indeed isomorphic to $\mathbb{Z}$ as an A-module for each m (generated by $\gamma_n(y)$ if $m = 2n$ is even and by $x\gamma_n(y)$ if $m = 2n+1$ is odd).

9. Derived algebraic geometry

The purpose of this section is to introduce the basic notions of derived algebraic geometry: derived schemes, properties of morphisms of derived schemes, factorizations of maps of derived schemes, derived fibered products, and the cotangent complex. We warn the reader at the outset that we will make no attempt at all to give the most general possible treatment of derived algebraic geometry. In order to keep everything self-contained (we will use only the basic notions from the theory of simplicial rings in §7) and as concrete as possible, we will ultimately limit ourselves roughly to the category of schemes quasi-projective over some fixed noetherian base ring.

9.1. **Affine morphisms.** We begin by establishing some facts about affine morphisms of schemes which we will need later. Although these results should be standard, they cannot be found, in the generality we need, in [EGA, II.1.6].

Recall [EGA, II.1.6] that a morphism of schemes $f : X \to Y$ is affine iff $f^{-1}(U)$ is affine for every affine open subscheme $U \subseteq Y$. Equivalently, f is affine iff $f_*\mathcal{O}_X$ is quasi-coherent and the natural map of Y-schemes

$$X \to \operatorname{Spec}_Y f_*\mathcal{O}_X \tag{9.1.1}$$

is an isomorphism. Affine morphisms are closed under composition (this is immediate from the definition) and base change (the best way to prove this is to note that the formation of $\operatorname{Spec}_Y \_$ is compatible with base change, or apply [EGA, II.1.6.2(ii)] with $S = Y$). The question of whether f is affine is local on Y:

Lemma 9.1.1. *Let $f : X \to Y$ be a morphism of schemes, $\{Y_i\}$ an open cover of Y. Then f is affine iff each induced map $f_i : X_i := f^{-1}(Y_i) \to Y_i$ is affine.*

Proof. Since affine morphisms are stable under base change, the issue is to prove that f is affine when each f_i is affine. Since pushforward commutes with base change along open embeddings and the question of quasi-coherence is local, $f_*\mathcal{O}_X$ is certainly quasi-coherent. Isomorphy for (9.1.1) can be checked locally on Y. Since the Spec_Y construction is compatible with base change, the base change of (9.1.1) along $Y_i \hookrightarrow Y$ is just the natural map

$$X_i \to \operatorname{Spec}_{Y_i} f_{i*}\mathcal{O}_{X_i},$$

which is an isomorphism since f_i is affine. □

Remark 9.1.2. Actually, one can form a locally ringed space $\operatorname{Spec}_Y A$ over Y for *any* $\mathcal{O}_Y$-algebra A—not necessarily quasi-coherent—and this more general relative spectrum has all the expected properties (compatibility with base change, in particular), so the first part of the argument in the above proof is not strictly necessary.

We will need the following generalization of [EGA, II.1.6.3]:

Lemma 9.1.3. *In any commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

where $X \to Z$ is affine and $Y \to Z$ is separated the map f is affine.

Proof. (Lurie) Factor f as the graph morphism $\Gamma_f = (\mathrm{Id}, f) : X \to X \times_Z Y$ followed by the projection $\pi_2 : X \times_Z Y \to Y$. The diagrams

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Gamma_f"'] & Y \arrow[d, "\Delta"] \\ X \times_Z Y \arrow[r, "(f,\mathrm{Id})"] & Y \times_Z Y \end{tikzcd}$$

are cartesian and Δ is a closed embedding (hence an affine morphism) since $Y \to Z$ is separated; affine morphisms are closed under composition and base change so $f = \pi_2 \Gamma_f$ is affine. □

We will also need the following generalization of [EGA, II.1.6.2(iv)]:

Lemma 9.1.4. *Given a commutative diagram of schemes*

$$\begin{tikzcd} & Y' \arrow[d] \arrow[dr] & \\ & & Y \arrow[dd] \\ X' \arrow[r] \arrow[dr] & Z' \arrow[dr] & \\ & X \arrow[r] & Z \end{tikzcd}$$

where the diagonal arrows are affine, the induced map $X' \times_{Z'} Y' \to X \times_Z Y$ is affine.

Proof. The question is local on $W := X \times_Z Y$ (Lemma 9.1.1), so we can work near a fixed point $w \in W$. Let x, y, z denote the images of w in X, Y, Z, respectively. Pick affine neighborhoods S, T, U of x, y, z in X, Y, Z so that $X \to Z$ takes S into U and $Y \to Z$ takes T into U. Then $S \times_U T$ is an affine open neighborhood of w in W and it will suffice to show that the preimage $(S \times_U T)'$ of $S \times_U T$ in $X' \times_{Z'} Y'$ is affine. Let S', T', U' be the preimages of S, T, U under the diagonal maps. These are affine since the diagonal maps are affine, so $S' \times_{U'} T'$ is affine, and so the proof is complete because it is easy to see that the two open subschemes $(S \times_U T)'$ and $(S' \times_{U'} T')$ of $X' \times_{Z'} Y'$ coincide. □

(The case $Z = Z'$ is [EGA, II.1.6.2(iv)].)

9.2. **Simplicially ringed spaces.** A *simplicially ringed space* is a pair $X = (X, B)$ consisting of a topological space X together with a simplicial sheaf of rings (called the *structure sheaf*) B on X. Note that, in particular, each B_n is a B_0-module. Since B a simplicial sheaf of rings, $B(U)$ is a simplicial ring for each open subspace $U \subseteq X$ and each stalk B_x is a simplicial ring. The pair $X_0 := (X, B_0)$ is a ringed space sometimes called the *underlying ringed space* of (X, B). If X_0 is a *locally* ringed space (i.e. each stalk $B_{0,x}$ is a local ring), we say that X is a *local simplicially ringed space.*

A *morphism* of simplicially ringed spaces from $X = (X, B)$ to $Y = (Y, A)$ is a pair $(f, f^\sharp)$ consisting of a map of topological spaces $f : X \to Y$ and a map $f^\sharp : f^{-1}A \to B$ of simplicial sheaves of rings on X. If X and Y are local, then in the definition of a morphism from X to Y we also demand that the map of ringed spaces

$$(f, f_0^\sharp) : X_0 \to Y_0$$

be a map of *locally ringed spaces* (stalks of $f_0^\sharp$ should be local maps of local rings). We denote the category of local simplicially ringed spaces by **sLRS**, though we emphasize that this is probably bad notation, for **sLRS** is certainly not the same as the category of simplicial locally ringed spaces.

We can (and will) regard any (locally) ringed space X as a (local) simplicially ringed space by taking the constant simplicial sheaf of rings for the structure sheaf.

If $X = (X, B)$ is a local simplicially ringed space, we can consider the ideal $I \subseteq B_0$ given by the image of

$$d_1^0 - d_1^1 : B_1 \to B_0$$

(recall that this image is an ideal because the degeneracy map $B_0 \to B_1$ is a section of both d_1^0 and d_1^1). We let $i : \pi_0(X) \hookrightarrow X_0$ denote the corresponding closed embedding of locally ringed spaces, with ideal I; we write $\pi_0(B)$ for the structure sheaf of $\pi_0(X)$, so that $i_*\pi_0(B) = B_0/I = \mathrm{H}_0(B)$. The homology sheaves $\mathrm{H}_n(B)$ (defined for simplicial sheaves of rings exactly as they were defined for simplicial rings in §7.2 and §7.4) are modules over $\pi_0(B)$ (in fact they form a strictly graded-commutative graded $\pi_0(B)$-algebra). Formation of $\pi_0(X)$ and the modules $\mathrm{H}_n(B)$ is functorial in the local simplicially ringed space X in the sense that an **sLRS** map

$$f : (X, B) \to (Y, A)$$

induces a map of locally ringed spaces

$$\pi_0(f) : \pi_0(X) \to \pi_0(Y) \tag{9.2.1}$$

and maps

$$\mathrm{H}_n(f) : \pi_0(f)^* \,\mathrm{H}_n(A) \to \mathrm{H}_n(B) \tag{9.2.2}$$

of $\pi_0(B)$-modules.

Definition 9.2.1. An **sLRS** map $f : (X, B) \to (Y, A)$ is a *quasi-isomorphism* iff the map (9.2.1) and all the maps (9.2.2) are isomorphisms.

It is immediate from the naturality of the maps (9.2.1) and (9.2.2) that quasi-isomorphisms satisfy "two-out-of-three".

9.3. **Spec of a simplicial ring.** Let A be a simplicial ring. Let $X := \operatorname{Spec} A_0$. Each A_n is an A_0 algebra via the structure map $A_0 \to A_n$ corresponding to the unique Δ-morphism $[n] \to [0]$; in particular, each A_n is an A_0 module. Then $A^\sim$ is a simplicial sheaf of rings on the topological space X and each $A_n^\sim$ is quasi-coherent as an $\mathcal{O}_X$-module. The pair $(\operatorname{Spec} A_0, A^\sim)$ is hence a simplicially ringed space (§9.2), which will be denoted $\operatorname{Spec} A$. The underlying ringed space $(\operatorname{Spec} A_0, A_0^\sim)$ is just the scheme X, so, in particular, $\operatorname{Spec} A$ is a local simplicially ringed space. The construction of $\operatorname{Spec} A$ is contravariantly functorial in the simplicial ring A and in fact defines a fully faithful functor

$$\operatorname{Spec} : (\mathbf{sAn})^{\mathrm{op}} \to \mathbf{sLRS}.$$

For the fullness: If

$$(f, f^\sharp) : \operatorname{Spec} B \to \operatorname{Spec} A$$

is a map of local simplicially ringed spaces, then certainly the induced map of underlying locally ringed spaces

$$(\operatorname{Spec} B_0, B_0^\sim) \to (\operatorname{Spec} A_0, A_0^\sim)$$

is $\operatorname{Spec}(A_0 \to B_0)$ for a ring map $A_0 \to B_0$. The map $f^\sharp : f^{-1}A^\sim \to B^\sim$ of simplicial sheaves of rings is the same as a map of simplicial sheaves of rings $f^\sharp : A^\sim \to f_* B^\sim$ on the scheme $\operatorname{Spec} A_0$. But $f_*(B^\sim)$ is just $B^\sim$, regarding the B_n as A_0-modules by

$$A_0 \to B_0 \to B_n$$

so the $f_n^\sharp : A_n^\sim \to B_n^\sim$ are maps of quasi-coherent $A_0^\sim$-algebras, hence they correspond to ring maps $A_n \to B_n$ and the compatibility of the $f_n^\sharp$ with the boundaries and degeneracies for $A^\sim$ and $B^\sim$ is equivalent to the compatibility of the maps $A_n \to B_n$ with the boundaries and degeneracies for A and B.

9.4. **Derived schemes.** An object (X, B) of **sLRS** isomorphic (in **sLRS**) to $\operatorname{Spec} A$ (for a simplicial ring A) is called an *affine derived scheme*. A *derived scheme* is an object (X, B) of **sLRS** which is locally an affine derived scheme—i.e. the topological space X admits an open cover $\{U_i\}$ such that each local simplicially ringed space $(U_i, B|U_i)$ is an affine derived scheme. Morphisms of derived schemes are, by definition, morphisms in **sLRS**, so that derived schemes form a full subcategory **DS** $\subseteq$ **sLRS**.

Example 9.4.1. For example, every scheme X can be viewed as a derived scheme (X, A), where A is the constant simplicial sheaf of rings associated to the sheaf of rings $\mathcal{O}_X$.

If $X = (X, B)$ is a derived scheme, then the underlying locally ringed space $X_0 = (X, B_0)$ is a scheme and each B_n is a quasi-coherent B_0-module (the question is local and it is clearly true for an affine derived scheme). In fact, derived schemes are characterized by these two properties: If (X, B) is a local simplicially ringed space such that $X_0 = (X, B_0)$ is a scheme and each B_n is quasi-coherent, then (X, B) is a derived scheme: Indeed, the question is local, so we can assume $X_0 \cong \operatorname{Spec} A_0$ for some ring A_0 (in fact $A_0 := B_0(X)$ is a canonical choice). Since the category of quasi-coherent sheaves on X_0 is equivalent to the category of A-modules, each we can write $B \cong A^\sim$ for some simplicial ring A (in fact $A := B(X)$ is a canonical choice), so $X \cong \operatorname{Spec} A$.

It is clear from this latter characterization of derived schemes that if $X = (X, B)$ is a derived scheme and $U \subseteq X$ is an open subspace, then $(U, B|U)$ is also a derived scheme, denoted simply by U if there is no chance of confusion. Furthermore, if the scheme $U_0 = (U, B_0|U)$ is affine, then U is an affine derived scheme. If $x \in U \subseteq X$, then we will say that U is a *neighboorhood* of x, or an *affine neighborhood of* x if U is an affine derived scheme. With this notion of affine neighborhoods, it is clear that affine neighborhoods of x in the derived scheme X are in bijective correspondence with affine neighborhoods of x in the usual scheme $X_0 = (X, B_0)$.

Suppose $f : X \to Y$ is a morphism of derived schemes and $x \in X$. We can find an affine neighborhood $V \cong \operatorname{Spec} A$ of $f(x)$ in Y and an affine neighborhood $U \cong \operatorname{Spec} B$ of x in $f^{-1}(V)$ (indeed, the data of such U and V is the same data as an affine neighborhood of x in the usual morphism of schemes $f_0 : X_0 \to Y_0$). The morphism of affine derived schemes $f|U : U \to V$ then corresponds to some map of simplicial rings $A \to B$ (§9.3). Evidently then, a morphism of derived schemes can be "covered by Spec of simplicial ring maps" in the same way that a morphism of (usual) schemes can be "covered by Spec of ring maps".

9.5. **Cosimplicial approach.** If (X, B) is a derived scheme, then $X_0 = (X, B_0)$ is a scheme and we can form the schemes $X_n := \operatorname{Spec}_{X_0} B_n$, noting that B_n is a quasi-coherent $\mathcal{O}_{X_0} = B_0$-algebra, so the X_n are schemes, affine over X_0. We denote the structure map by $p_n : X_n \to X_0$, or simply by p if there is no chance of confusion. We have $p_{n*}\mathcal{O}_{X_n} = B_n$. The simplicial structure on B endows these X_n with the structure of a cosimplicial scheme, which we denote

$$X_\bullet := \operatorname{Spec}_{X_0} B_\bullet.$$

For a Δ-morphism $\sigma : [m] \to [n]$, the structure map $X(\sigma) : X_m \to X_n$ is the morphism of affine X_0-schemes defined by

$$X(\sigma) := \operatorname{Spec}_{X_0}(B(\sigma) : B_n \to B_m).$$

We will often write σ instead of $X(\sigma)$ if there is no chance of confusion. By Lemma 9.1.3 the morphism $X(\sigma)$ is itself an affine morphism because we have a commutative diagram

$$\begin{tikzcd} X_m \arrow[rr, "X(\sigma)"] \arrow[dr, "p_m"'] & & X_n \arrow[dl, "p_n"] \\ & X_0 & \end{tikzcd}$$

where p_m and p_n are affine morphisms (note that affine morphisms are separated).

The derived scheme (X, B) can be recovered from the cosimplicial scheme $X_\bullet$ by the formula

$$B = p_*\mathcal{O}_{X_\bullet}.$$

Indeed, we could equivalently *define* a derived scheme to be a cosimplicial scheme $X_\bullet$ where every structure morphism $X(\sigma) : X_m \to X_n$ is an affine morphism.

The formation of the cosimplicial scheme $X_\bullet$ is functorial in the derived scheme X, so that a map of derived schemes $f : X \to Y$ induces a map of cosimplicial schemes $f_\bullet : X_\bullet \to Y_\bullet$.

Example 9.5.1. For example, if we view a (usual) scheme X as a "constant" derived scheme as in Example **??**, then $X_\bullet$ is the constant cosimplicial scheme associated to the scheme X.

9.6. **Properties of maps of derived schemes.** If $\mathbf{P}$ is a property of maps of schemes (e.g. "closed embeddings," "smooth maps," etc.) and $f : X \to Y$ is a map of derived schemes then we say that f has $\mathbf{P}$ *degree-wise* iff each map of schemes $f_n : X_n \to Y_n$ has property $\mathbf{P}$. For example, we can speak of *degree-wise smooth* and *degree-wise flat* maps of derived schemes. In fact, we will generally drop the "degree-wise" and simply speak of *smooth, flat, affine, finite type, ...* maps of derived schemes. We similarly define properties of derived schemes "degree-wise," so that, for example, a derived scheme X is *noetherian* iff each scheme X_n is noetherian.

If (X, B) is a derived scheme, the ideal $I \subseteq B_0$ defined in §9.2 is quasi-coherent (it is the image of a map of quasi-coherent sheaves), so $\pi_0(X)$ is a scheme. For similar reasons, the $H_n(B)$ are quasi-coherent sheaves on $\pi_0(X)$. (We will return to this point in §9.7.) We define *quasi-isomorphisms* for derived schemes exactly as we did for local simplicially ringed spaces in Definition 9.2.1.

We now clarify some issues related to the definitions of "degree-wise flat" and "degree-wise affine."

Lemma 9.6.1. *Consider a commutative diagram of schemes*

$$\begin{tikzcd} X' \arrow[r, "q"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "p"] & Y \end{tikzcd}$$

where p and q are affine morphisms. Then f' is flat iff the induced map

$$f^{-1}(p_*\mathcal{O}_{Y'}) \to q_*\mathcal{O}_{X'} \tag{9.6.1}$$

is a flat map of sheaves of rings on X.

Proof. Both questions of flatness are local on f, so we can assume that X and Y (hence also X' and Y') are affine, so our commutative diagram is Spec of a diagram of rings as below.

$$\begin{tikzcd} B' & B \arrow[l, "j"'] \\ A' \arrow[u, "f'"] & A \arrow[l, "i"'] \arrow[u, "f"'] \end{tikzcd} \tag{9.6.2}$$

We first prove that if f' is flat, then (9.6.1) is a flat map of sheaves of rings on $X = \operatorname{Spec} B$. This can be checked on stalks. The stalk of $f^{-1}(p_*\mathcal{O}_{Y'})$ at a point $x \in X$ (a prime ideal of B) with image $y = f(x)$ in Y (we view y as a prime ideal of A) is given by

$$\begin{aligned} (f^{-1}(p_*\mathcal{O}_{Y'}))_x &= (p_*\mathcal{O}_{Y'})_y \\ &= \varinjlim \mathcal{O}_{Y'}(p^{-1}U) \\ &= \varinjlim \mathcal{O}_{Y'}(p^{-1}U_a) \\ &= \varinjlim \mathcal{O}_{Y'}(U_{i(a)}) \\ &= \varinjlim A'_{i(a)} \\ &= i(A \setminus y)^{-1}A'. \end{aligned}$$

Some explanation is needed. The first direct limit runs over open neighborhoods U of y in $Y = \operatorname{Spec} A$; we then immediately replace this direct limit with the direct limit over the cofinal set of such neighborhoods given by the basic opens

$$\begin{aligned} U_a &:= \{w \in \operatorname{Spec} A : a \notin w\} \\ &= \operatorname{Spec}(A_a) \end{aligned}$$

with $a \in A \setminus y$, then we note that the preimage $p^{-1}U_a$ of such a basic open in $Y' = \operatorname{Spec} A'$ is just the basic open $U_{i(a)}$ in Y'. We write $S^{-1}A$ to denote the localization of a ring A at a multiplicative subset $S \subseteq A$. A similar calculation shows that

$$(q_*\mathcal{O}_{X'})_x = j(B \setminus x)^{-1}B'$$

so that the stalk of (9.6.1) at x is just the bottom arrow in the commutative square of ring maps below.

(9.6.3)
$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d] & B' \arrow[d] \\ i(A \setminus y)^{-1}A' \arrow[r] & j(B \setminus x)^{-1}B' \end{tikzcd}$$

The corresponding pushout square is given by:

(9.6.4)
$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d] & B' \arrow[d] \\ i(A \setminus y)^{-1}A' \arrow[r] & f'i(A \setminus y)^{-1}B' \end{tikzcd}$$

The bottom arrow in (9.6.4) is flat because f' is flat by hypothesis and flatness is stable under base change. By commutativity of (9.6.2) and the fact that $y = f^{-1}(x)$, we see that

$$f'i(A \setminus y) \subseteq j(B \setminus x),$$

so the induced map $f'i(A\setminus y)^{-1}B' \to j(B\setminus x)^{-1}B'$ is a localization, so it is flat. The bottom arrow in (9.6.3) is the composition of the bottom arrow in (9.6.4) and this localization, so it is a composition of flats hence is flat.

For the converse, it suffices to prove that for any prime ideal $z \in B'$ (point of X'), the composition

$$A' \to B' \to B'_z = (B' \setminus z)^{-1}z \tag{9.6.5}$$

is flat. Let $x := q(z) = j^{-1}(z)$. Then $j(B \setminus x) \subseteq B' \setminus z$, so (9.6.5) will factor as the composition

$$A' \to i(A \setminus y)^{-1}A' \to j(B \setminus x)^{-1}B' \to B'_z = (B' \setminus z)^{-1}z.$$

The first and final maps in this latter composition are localizations, so they are flat, and the middle map is the stalk of (9.6.1) at x, so it is flat by hypothesis, hence (9.6.5) is flat as desired. □

Lemma 9.6.2. *Let $f : (X, B) \to (Y, A)$ be a map of derived schemes. The induced map of schemes $f_n : X_n \to Y_n$ is flat iff $f_n^\sharp : f^{-1}A_n \to B_n$ is a flat map of sheaves of rings on X. In particular, f is degree-wise flat iff $f^\sharp : f^{-1}A \to B$ is a degree-wise flat map of simplicial sheaves of rings on X.*

Proof. Apply the previous lemma to the commutative diagram of schemes

$$\begin{tikzcd} X_n \arrow[r, "p"] \arrow[d, "f_n"'] & X_0 \arrow[d, "f_0"] \\ Y_n \arrow[r, "p"] & Y_0 \end{tikzcd}$$

noting that the horizontal maps are affine and $B_n = p_*\mathcal{O}_{X_n}$, $A_n = p_*\mathcal{O}_{Y_n}$. (Also note that, on the level of topological spaces, f_0 coincides with the map $f : X \to Y$.) □

The next lemma says that the whole theory of derived schemes is just the theory of schemes with "extra data in the structure sheaf" ("higher nilpotents")—all the "topology" of a scheme or map of schemes is in X_0 (of f_0).

Lemma 9.6.3. *A morphism $f : X \to Y$ of derived schemes is degree-wise affine iff $f_0 : X_0 \to Y_0$ is affine.*

Proof. Obviously f_0 has to be affine for f to be degree-wise affine; to see that this is sufficient, note that we have a commutative diagram

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "f_n"'] & X_0 \arrow[d, "f_0"] \\ Y_n \arrow[r] & Y_0 \end{tikzcd}$$

where the horizontal arrows and f_0 are affine, hence the composition $X_n \to Y_0$ is affine and we conclude that f_n is affine by Lemma 9.1.3. $\square$

Lemma 9.6.4. *Let $i : X \to V$ be a morphism of derived schemes. The following are equivalent:*

(1) The map i is a quasi-isomorphism and a (degree-wise) closed embedding.
(2) The map i is affine and for every affine open derived subscheme $U = \operatorname{Spec} P$ of V, the map $i : i^{-1}(U) \to U$ is given by $\operatorname{Spec}(P \to B)$ for a trivial fibration of simplicial rings $P \to B$.
(3) The map i is affine and each point v of V has an affine open neighborhood $U = \operatorname{Spec} P$ of V such that the map $i : i^{-1}(U) \to U$ is given by $\operatorname{Spec}(P \to B)$ for a trivial fibration of simplicial rings $P \to B$.

Proof. For (1) implies (2), first note that i is affine by Lemma 9.6.3 since i_0, being a closed embedding, is affine. So for every affine open $\operatorname{Spec} P = U \subseteq V$, we can write $i^{-1}(U) \to U$ as $\operatorname{Spec}(P \to B)$ for some map of simplicial rings $P \to B$. Since i is a degree-wise closed embedding, so is $i^{-1}(U) \to U$, hence $P \to B$ is degree-wise surjective because

$$(i^{-1}(U)_n \to U_n) = \operatorname{Spec}(P_n \to B_n).$$

Since i is a quasi-isomorphism, $\pi_0(P) \to \pi_0(B)$ is a ring isomorphism and since

$$\begin{aligned} \mathrm{H}_n(V)|U &= \mathrm{H}_n(U) \\ &= \mathrm{H}_n(P)^{\sim} \end{aligned}$$

(we will see this in §9.7 below) and similarly for $i^{-1}(U)$, we see that $P \to B$ is a quasi-isomorphism of simplicial rings; since we just saw that it is also degree-wise surjective, it is a trivial fibration by Corollary 5.6.5.

Clearly (2) implies (3). For (3) implies (1), note that the formation of π_0 and H_n is local in nature (§9.7), so the question of whether i is a quasi-isomorphism is local on V; similarly, the question of whether i is a degree-wise closed embedding is local on V. $\square$

Definition 9.6.5. A map of derived schemes $i : X \to V$ satisfying the equivalent conditions of the above lemma is called a *trivial cofibration*.

Every trivial cofibration is, in particular, a finite type affine morphism. Notice that for a map of simplicial rings $P \to B$, the corresponding map $\operatorname{Spec} B \to \operatorname{Spec} P$ of affine derived schemes is a trivial cofibration iff $P \to B$ is a trivial fibration of simplicial rings.

9.7. **Modules and homology.** Let (X, B) be a simplicially ringed space (§9.2). We define a *B-module* exactly as we defined modules over a simplicial ring in §7.2: a B-module M thus consists of B_n-modules M_n and, for each Δ-morphism $\sigma : [m] \to [n]$, a $B(\sigma)$-linear map $M(\sigma) : M_n \to M_m$ functorial in σ.

As in §7.2 and §7.4, the homology $\mathrm{H}_n(M)$ of a B-module M is defined by the following formulas:

$$\begin{aligned}
\mathsf{N}_n(M) &:= \operatorname{Ker}\left(\prod_{i=1}^{n} d_n^i : M_n \to \prod_{i=1}^{n} M_{n-1}\right) \\
\mathsf{Z}_n(M) &:= \operatorname{Ker}\left(\prod_{i=0}^{n} d_n^i : M_n \to \prod_{i=0}^{n} B_{n-1}\right) \\
&= \operatorname{Ker}(d_n^0 : \mathsf{N}_n(M) \to \mathsf{Z}_{n-1}(M)) \\
\mathrm{H}_n(M) &:= \mathsf{Z}_n(M)/d_{n+1}^0 \mathsf{N}_{n+1}(M) \\
&= \operatorname{Cok}(\mathsf{N}_{n+1}(M) \otimes_{B_{n+1}} B_n \to \mathsf{Z}_n(M)).
\end{aligned}$$

Evidently $\mathsf{N}_n(M)$, $\mathsf{Z}_n(M)$, and $\mathrm{H}_n(M)$ are B_n-modules. Since the formation of kernels and cokernels commutes with restriction to open subsets, we have

$$\begin{aligned}
\mathsf{N}_n(M|U) &= \mathsf{N}_n(M)|U \\
\mathsf{Z}_n(M|U) &= \mathsf{Z}_n(M)|U \\
\mathrm{H}_n(M|U) &= \mathrm{H}_n(M)|U.
\end{aligned} \tag{9.7.1}$$

Next suppose (X, B) is a derived scheme with associated cosimplicial scheme $X_\bullet = \operatorname{Spec}_{X_0} B_\bullet$. Since the maps $p_n : X_n \to X_0$ are affine morphisms, pushforward p_{n*} gives an equivalence of categories between quasi-coherent $\mathcal{O}_{X_n}$-modules and quasi-coherent B_n-modules, with inverse given by the "relative" version of $M \mapsto M^\sim$. A B-module M is called *quasi-coherent* iff each M_n is a quasi-coherent B_n-module (in the essential image of $p_{n*} : \mathbf{Qco}(X_n) \to \mathbf{Mod}(B_n)$). If X_n is locally noetherian, we define a *coherent* B_n-module by replacing "$\mathbf{Qco}(X_n)$" with "$\mathbf{Coh}(X_n)$". If each X_n is locally noetherian, then we have an evident notion of coherent B-modules. Equivalently, a B_n-module M is quasi-coherent (resp. coherent in the locally noetherian situation) iff, for each affine open derived subscheme $\operatorname{Spec} A = U \subseteq (X, B)$, we have $M|U \cong N^\sim$ for some A_n-module N (resp. some finitely generated A_n-module N). (Note that $N^\sim$ here is defined by regarding N as a module over A_0 via $A_0 \to A_n$; the resulting $B_0 = A_0^\sim$-module has a natural $B_n = A_n^\sim$-module structure.) A B-module M is then quasi-coherent (resp. coherent in the locally noetherian setting) iff, for each affine open derived subscheme $\operatorname{Spec} A = U \subseteq (X, B)$, we have $M \cong N^\sim$ for some A-module N (resp. some degree-wise finitely generated A-module N).

Now suppose $(X, B) = \operatorname{Spec} A$ for a simplicial ring A and $M = N^\sim$ for an A-module N (§7.2). Then $d_n^i : M_n \to M_{n-1}$ is just $(d_n^i : N_n \to N_{n-1})^\sim$ and we see that

$$\begin{aligned}
\mathsf{N}_n(M) &= \mathsf{N}_n(N)^\sim \\
\mathsf{Z}_n(M) &= \mathsf{Z}_n(N)^\sim \\
\mathrm{H}_n(M) &= \mathrm{H}_n(N)^\sim.
\end{aligned}$$

If each A_n is noetherian and each N_n is a finitely generated as an A_n-module, then each of the A_n-modules $\mathsf{N}_n(N)$, $\mathsf{N}_n(N)$, $\mathrm{H}_n(N)$ is finitely generated.

Lemma 9.7.1. *Suppose (X, B) is a derived scheme (resp. locally noetherian derived scheme) and M is a quasi-coherent (resp. coherent) B-module. Then $\mathsf{N}_n(M)$, $\mathsf{Z}_n(M)$, and $\mathrm{H}_n(M)$ are quasi-coherent (resp. coherent) B_n-modules for each $n \in \mathbb{N}$.*

Proof. The question is local and the formation of these modules is local (9.7.1), so we reduce to the discussion above. $\square$

It is—for the author at least—easier to think about quasi-coherent sheaves on the scheme X_n than to think about quasi-coherent modules on the ringed space (X, B_n). We will now give an explicit description of the homology modules $\mathrm{H}_n(B)$ from this "more geometric" point of view. This will be useful in §9.10.

For the Δ-morphism $\partial_n^i : [n-1] \hookrightarrow [n]$, we usually just write $\partial_n^i : X_{n-1} \to X_n$ instead of $X(\partial_n^i)$. The map of schemes $\partial_n^i : X_{n-1} \to X_n$ is a closed embedding because $B(\partial_n^i) = d_n^i : B_n \to B_{n-1}$ is a surjection of quasi-coherent sheaves of rings on the scheme X_0 (d_n^i has a section because the Δ-morphism ∂_n^i has a retract). We will write

$$d_n^i : \mathcal{O}_{X_n} \to \partial_{n*}^i \mathcal{O}_{X_{n-1}} \tag{9.7.2}$$

for the surjection of quasi-coherent $\mathcal{O}_{X_n}$-modules (coherent if X_n is locally noetherian) corresponding to the closed embedding $\partial_n^i : X_{n-1} \to X_n$. This notation is justified by the fact that

$$p_{n*}(d_n^i : \mathcal{O}_{X_n} \to \partial_{n*}^i \mathcal{O}_{X_{n-1}}) = d_n^i : B_n \to B_{n-1}. \tag{9.7.3}$$

We will also write d_n^i for the map

$$d_n^i : (\partial_n^i)^* \mathcal{O}_{X_n} \to \mathcal{O}_{X_{n-1}} \tag{9.7.4}$$

corresponding to (9.7.2) under the adjunction $((\partial_n^i)^*, \partial_{n*}^i)$.

Define

$$\begin{aligned}
\mathsf{N}_n(X_\bullet) &:= \operatorname{Ker}\left(\prod_{i=1}^n d_n^i : \mathcal{O}_{X_n} \to \prod_{i=1}^n \partial_{n*}^i \mathcal{O}_{X_{n-1}} \right) \\
\mathsf{Z}_n(X_\bullet) &:= \operatorname{Ker}\left(\prod_{i=0}^n d_n^i : \mathcal{O}_{X_n} \to \prod_{i=0}^n \partial_{n*}^i \mathcal{O}_{X_{n-1}} \right) \\
&= \operatorname{Ker}(d_n^0 : \mathsf{N}_n(X_\bullet) \to \partial_{n*}^0 \mathsf{Z}_{n-1}(X_\bullet)) \\
\mathrm{H}_n(X_\bullet) &:= \partial_{n+1\,*}^0 \mathsf{Z}_n(X_\bullet) / d_{n+1}^0(\mathsf{N}_{n+1}(X_\bullet)) \\
&= \operatorname{Cok}(d_{n+1}^0 : (\partial_{n+1}^0)^* \mathsf{N}_{n+1}(X_\bullet) \to \mathsf{Z}_n(X_\bullet)).
\end{aligned}$$

The last equality here is a sheafified version of the discussion in Remark 7.3.1. Notice that $\mathsf{N}_n(X_\bullet)$, $\mathsf{Z}_n(X_\bullet)$, and $\mathrm{H}_n(X_\bullet)$ are all quasi-coherent sheaves on X_n, coherent if X_n is locally noetherian. It is clear from (9.7.3) and the fact that p_{n*} is exact on quasi-coherent sheaves that

$$\begin{aligned}
p_{n*} \mathsf{N}_n(X_\bullet) &= \mathsf{N}_n(B) \\
p_{n*} \mathsf{Z}_n(X_\bullet) &= \mathsf{Z}_n(B) \\
p_{n*} \mathrm{H}_n(X_\bullet) &= \mathrm{H}_n(B).
\end{aligned}$$

9.8. **Fibered products.** The category **DS** of derived schemes, like the category **Sch** of schemes, has all finite inverse limits.

To form the fibered product $(W, D) = (X, A) \times_{(Z,C)} (Y, B)$, set $W_n := X_n \times_{Z_n} Y_n$. The cosimplicial structure of $X_\bullet$, $Y_\bullet$ and $Z_\bullet$ endows $W_\bullet$ with the structure of a cosimplicial scheme with the structure map for a Δ-morphism $\sigma : [m] \to [n]$ given by

$$W(\sigma) := X(\sigma) \times_{Z(\sigma)} Y(\sigma) : X_m \times_{Z_m} Y_m \to X_n \times_{Z_n} Y_n. \tag{9.8.1}$$

To show that this W is a derived scheme, we need to show that each map (9.8.1) is affine. This is because $X(\sigma)$, $Y(\sigma)$ and $Z(\sigma)$ are affine (Lemma 9.1.4). The structure sheaf D of the fibered product (W, D) is given, as usual, by

$$D_\bullet = p_* \mathcal{O}_{W_n}.$$

It is trivial to see that this W has the correct universal property.

Let $\pi_1 : (W, D_0) \to (X, A_0)$, $\pi_2 : (W, D_0) \to (Y, B_0)$, $\pi : (W, D_0) \to (Z, C_0)$ denote the projections on the level of underlying schemes. Notice that the structure sheaf D is not equal to $\pi_1^* A \otimes_{\pi^* C} \pi_2^* B$. But of course we don't expect this, because this is not even true in degree zero (i.e. it isn't true for fibered products of schemes).

Notice that if **P** is a property of morphisms of schemes which is stable under base change, then the corresponding property for maps of derived schemes (defined degree-wise) is also stable under base change.

9.9. **Resolution properties.** A noetherian scheme X is said to have the *resolution property* iff every coherent sheaf on X is a quotient of a bundle (locally free coherent sheaf). Unfortunately, having the "resolution property" is not a particularly "robust" property. We instead seek a stronger property which is more robust, yet is also enjoyed by a wide class of (noetherian) schemes.

Given an invertible sheaf L on a scheme X and a global section $s \in \Gamma(X, L)$, we let

$$X_s := \{x \in X : s(x) \neq 0\}$$

denote the non-vanishing locus of s. This X_s is an open subscheme of X and the inclusion $X_s \hookrightarrow X$ is an affine morphism (see [EGA, II.5.5.8] or [SGA6, II.2.2.3.1]).

Theorem 9.9.1. *Let X be a noetherian scheme. The following are equivalent:*

1. *The opens X_s, for $s \in \Gamma(X, L)$, L an invertible sheaf on X, form a basis for the topology of X.*
2. *The opens X_s as in the previous part which are affine cover X.*
3. *For every coherent sheaf $\mathcal{F}$ on X, there is a surjection $V \to \mathcal{F}$ of coherent $\mathcal{O}_X$-modules, where V is a finite direct sum of invertible sheaves on X.*
4. *Same as the previous condition, but require $\mathcal{F}$ to be a coherent ideal sheaf.*

Proof. Apply [SGA6, II.2.2.3] to the family of all invertible sheaves. □

A noetherian scheme X is called *divisorial* iff it satisfies the equivalent conditions of the above theorem. Evidently a divisorial noetherian scheme has the resolution property.

Proposition 9.9.2. *(1) Any noetherian affine scheme is divisorial.*

(2) Any separated regular noetherian scheme (e.g. any quasi-compact scheme smooth and separated over a field) is divisorial.

(3) An open subscheme of a divisorial noetherian scheme is a divisorial noetherian scheme.

(4) If $f : X \to Y$ is an affine morphism of noetherian with Y divisorial, then X is a divisorial noetherian scheme.

(5) A closed subscheme of a divisorial noetherian scheme is a divisorial noetherian scheme.

(6) If $f : X \to Y$ is a quasi-projective morphism of noetherian schemes and Y is divisorial, then X is divisorial.

Proof. For (1), note that the second condition of Theorem 9.9.1 holds trivially since $X = X_1$ is affine, for $1 \in \Gamma(X, \mathcal{O}_X)$. For (2), see [SGA6, II.2.2.7.1]. For (3), check the first or second criterion in Theorem 9.9.1. For (4), check the second condition of Theorem 9.9.1, noting that the formation of the nonvanishing locus Y_s of a global section commutes with pullback: $X_{f^*s} = f^{-1}(Y_s)$. A closed embedding is an affine morphism so (5) is a special case of (4). For (6), we can reduce, in light of (3) and (5), to proving that X is divisorial whenever $f : X \to Y$ is a projective morphism of noetherian schemes with Y divisorial. Pick an f-relatively ample invertible sheaf $\mathcal{O}_X(1)$. We will check the third condition of Theorem 9.9.1. Let $\mathscr{F}$ be a coherent sheaf on X. By the Fundamental Theorem of Projective Morphisms [EGA, III.2.2.1], we can find $n \in \mathbb{N}$ such that the natural map

$$f^*(f_*(\mathscr{F}(n))) \to \mathscr{F}(n)$$

is surjective. The same theorem also says that $f_*(\mathscr{F}(n))$ is a coherent sheaf on Y, so, since Y is divisorial, there is a surjection $V \to f_*(\mathscr{F}(n))$ with V a finite direct sum of invertible sheaves on Y. Since f^* preserves surjections, $f^*V \to \mathscr{F}(n)$ is surjective. Since tensoring with an invertible sheaf preserves surjections, $(f^*V)(-n) \to \mathscr{F}$ is surjective. Since $(f^*V)(n)$ is a finite direct sum of invertible sheaves on X, we're done. $\square$

Let X be a ringed space, B an $\mathcal{O}_X$-algebra, $f : M \to B$ a map of $\mathcal{O}_X$-modules. We say that f *generates B as an $\mathcal{O}_X$-algebra* iff the induced map $\operatorname{Sym}^*_X M \to B$ is surjective (has surjective stalks).

Theorem 9.9.3. *Let $f : X \to Y$ be an affine morphism of noetherian schemes.*

(1) If f is finite type, then there is a coherent sheaf $\mathscr{G}$ on Y and a map $\mathscr{G} \to f_*\mathcal{O}_X$ of quasi-coherent sheaves on Y generating $f_*\mathcal{O}_X$ as an $\mathcal{O}_Y$-algebra. In fact, if $\mathscr{F}' \to f_*\mathcal{O}_X$ is any surjection of quasi-coherent sheaves on Y, we can choose our $\mathscr{G} \to f_*\mathcal{O}_X$ to factor through $\mathscr{F}' \to f_*\mathcal{O}_X$. Note that such a map yields a factorization

$$X = \operatorname{Spec}_Y f_*\mathcal{O}_X \to \operatorname{Spec}_Y \operatorname{Sym}^*_Y \mathscr{G} \to Y$$

of f where the first arrow is a closed embedding.

(2) For any coherent sheaf $\mathscr{F}$ on X, there is a coherent sheaf $\mathscr{G}$ on Y and a surjection $f^*\mathscr{G} \to \mathscr{F}$ of coherent sheaves on X. In fact, if $\mathscr{F}' \to f_*\mathscr{F}$ is any surjection of quasi-coherent sheaves on Y, we can arrange that the corresponding map $\mathscr{G} \to f_*\mathscr{F}$ factors through $\mathscr{F}' \to f_*\mathscr{F}$.

(3) If Y has the resolution property we can take the $\mathscr{G}$ in the first two parts to be a bundle. In particular, when f is finite type, we can factor f as a closed embedding followed by the projection $E \to Y$ for a vector bundle E over Y.

Proof. We will just prove (2); the proof of (1) is similar, but easier. Cover Y by finitely many open affines $\{U_i = \operatorname{Spec} A_i\}$. Set

$$f_i := f|f^{-1}(U_i) : f^{-1}(U_i) \to U_i, \quad \mathscr{F}_i := \mathscr{F}|f^{-1}(U_i), \quad \mathscr{F}'_i := \mathscr{F}'|U_i.$$

Since f is affine, we can write $f_i = \operatorname{Spec}(A_i \to B_i)$ for some A_i-algebra B_i and $\mathscr{F}_i = M_i^\sim$ for some finitely generated B_i-module M_i. In the formula $\mathscr{F}_i = M_i^\sim$, $M_i^\sim$ is the one for B_i-modules; we also have $(f_i)_*\mathscr{F}_i = M_i^\sim$ where $M_i^\sim$ is the one for A_i-modules and M_i is regarded as an A_i-module by restriction of scalars along $A_i \to B_i$. We can also write $\mathscr{F}'_i = M'_i$ for some A_i-module M'_i (not necessarily finitely generated) so the surjection $\mathscr{F}' \to f_*\mathscr{F}$ corresponds to a surjection $M'_i \to M_i$ of A_i-modules. Since $M'_i \to M_i$ is surjective and M_i is finitely generated, we can find, for each i, finitely many elements $m_{i,j} \in M'_i$ whose images in M_i generate M_i as an B_i-module. (To prove (1), take $m_{i,j} \in B_i$ generating B_i as an A_i-algebra.) Let $S_i \subseteq M'_i$ denote the A_i-submodule of M'_i generated by the $m_{i,j}$, so that $S_i \otimes_{A_i} B_i \to M_i$ is a surjection of B_i-modules. Then $S_i^\sim$ is a coherent sheaf on U_i contained in the quasi-coherent sheaf $(M'_i)^\sim = \mathscr{F}'_i$ with the property that $f_i^* S_i^\sim \to \mathscr{F}_i$ is surjective. By [EGA, I.9.4.7], we can find coherent subsheaves $\mathscr{G}_i \subseteq \mathscr{F}'_i$ such that $\mathscr{G}_i|U_i = S_i^\sim$. Let $\mathscr{G} := \oplus_i \mathscr{G}_i$, $\mathscr{G} \to f_*\mathscr{F}$ the natural map. This $\mathscr{G}$ is a coherent sheaf on Y and the map $\mathscr{G} \to f_*\mathscr{F}$ is as desired since the surjectivity of $f^*\mathscr{G} \to \mathscr{F}$ can be checked locally on Y and it is clear on the U_i because the image of $\mathscr{G} \to f_*\mathscr{F}$ on U_i contains the image of $S_i^\sim \to f_{i*}\mathscr{F}_i$. If Y has the resolution property, then we can find a surjection $\mathscr{G}' \to \mathscr{G}$ with $\mathscr{G}'$ and bundle and replace $\mathscr{G}$ with $\mathscr{G}'$. $\square$

9.10. **Factoring maps of derived schemes.** We need a "global" (and "finite type") analog of the "projective" maps of simplicial rings defined in §**??** to use in our factorizations of maps of derived schemes.

Definition 9.10.1. Say (Y, A) is a derived scheme. An A_n-module M is called a *bundle* (or an A_n-*bundle* for emphasis) iff $M \cong p_{n*}M'$ for a bundle (locally free sheaf of locally finite rank) M' on $Y_n = \operatorname{Spec}_{Y_0} A_n$. Equivalently, M is a bundle iff, for any affine open derived subscheme $\operatorname{Spec} C = U \subseteq (Y, A)$, there is a finitely presented projective C_n-module P so that $M|U \cong P^\sim$ as $A_n|U$-modules. (To make sense of $P^\sim$ here, one views P as a C_0-module by restriction of scalars along $C_0 \to C_n$, noting that $\operatorname{Spec} C_0 = (U, A_0|U)$, and noting also that $P^\sim$, thus defined, becomes a module over $A_n|U = C_n^\sim$.)

Definition 9.10.2. A map of derived schemes $V \to Y = (Y, A)$ is called *affine projective* iff $V \cong \operatorname{Spec}_Y P$ (as a derived scheme over Y) for some map of sheaves of simplicial rings $A \to P$ which is symmetric (Definition 7.6.6) with a basis $(M_0, M_1, \dots)$ and non-degenerate complements $(E_0, E_1, \dots)$ so that E_n is an A_n-bundle for each $n \in \mathbb{N}$. (By Lemma 7.6.5, this implies that each M_n is also an A_n-bundle.)

Let me try to explain this in more geometric terms—that is, let me reformulate this definition entirely in terms of the corresponding map of cosimplicial schemes $V_\bullet \to Y_\bullet$. To say that this map is affine projective is equivalent to the following:

(1) For each $n \in \mathbb{N}$, the map $V_n \to Y_n$ is the projection map for a vector bundle: i.e. $V_n = \operatorname{Spec}_{Y_n} \operatorname{Sym}^*_{Y_n} M'_n$ for a locally free locally finite rank sheaf M'_n on the scheme Y_n (this M'_n is related to the M_n is the above definition by $M_n = p_{n*}M'_n$).
(2) For each surjective Δ-morphism $\sigma : [n] \to [m]$, the structure map $\sigma : V_n \to V_m$ for the cosimplicial scheme $V_\bullet$ is "induced by" (i.e. is $\operatorname{Spec}_{Y_n} \operatorname{Sym}^*_{Y_n}$ of) a map $\sigma^* M'_m \to M'_n$ of locally free sheaves on Y_n (this latter σ is the structure map σ :

$Y_n \to Y_m$ for $Y_\bullet$). (This condition corresponds to the condition that "degeneracies take bases into bases" (i.e. that the M_n have the structure of a degenerate module over the degenerate simplicial ring underlying A) in the definition of a symmetric morphism of simplicial rings.)

(3) For each $n \in \mathbb{N}$, there is a locally free locally finite rank subsheaf $E'_n \subseteq M'_n$ so that the maps $\sigma^* M'_m \to M'_n$ from the previous part induce a direct sum decomposition

$$M'_n = E'_n \oplus \varinjlim \sigma^* M'_m,$$

where the direct limit is over the category T_n (§**??**) of surjective, non-identity Δ-morphisms $\sigma : [n] \to [m]$. In geometric terms, this says that the vector bundle V_n over Y_n in fact splits as a sum of two bundles $V_n = V_n^D \oplus V_n^E$ (one might prefer to write $V_n = V_n^D \times_{Y_n} V_n^E$), where $V_n^D = \varprojlim \sigma^* V_m$. (This E'_n is related to the non-degenerate complement E_n in the definition by $E_n = p_{n*} E'_n$.)

Remark 9.10.3. It is clear from the definition that an affine projective morphism is (degree-wise) affine, finite type, smooth, etc. Indeed, if $\pi : V \to Y$ is affine projective, then for any affine open derived subscheme $U = \operatorname{Spec} A \subseteq Y$, the map $\pi^{-1}(U) \to U$ is given by $\operatorname{Spec}(A \to P)$ for a degree-wise finite type projective morphism of simplicial rings $A \to P$. (It might be worth recalling from Theorem 7.6.13 that projective morphisms of simplicial rings are cofibrations.)

Theorem 9.10.4. *Suppose $Y = (Y, A)$ is a noetherian derived scheme such that Y_0 is divisorial and $f : (X, B) \to (Y, A)$ is a finite-type affine map of derived schemes.*

(1) We can factor f as $j : X \to V$ followed by $q : V \to Y$, where j is a trivial cofibration (Definition 9.6.5) and q is affine projective.
(2) Any two such factorizations map to a third such factorization.

Proof. Notice that every morphism of schemes involved here is affine and every sheaf is quasi-coherent, so in principle nothing is lost by pushing everything forward to the scheme $Y_0 = (Y, A_0)$ and *carefully* factoring $A \to f_* B$ using much the same factorization procedure we used in Theorem 7.8.6. The issue is that one has to be *very careful* to make sure that it is possible to use *bundles* everywhere we used (often very large) free modules in that proof. I find it is easier to see that this can be done if we phrase the proof in rather more geometric terms; in particular we will make use of the description of the homology sheaves of a derived scheme in §9.7.

We first note that since Y_0 is divisorial and the Y_n are affine over Y_0, each of the schemes Y_n is divisorial (Proposition 9.9.2(4)), hence has the resolution property. The map $X_n \to Y_n$ is also affine, so each X_n is also divisorial, hence has the resolution property.

As in the proof of Theorem 7.8.6, we are going to construct the factorization inductively. Start by noting that $f_0 : X_0 \to Y_0$ is a finite type affine map of noetherian schemes and Y_0 has the resolution property, so by Theorem 9.9.3(1), we can find a bundle M_0 on Y_0 and a map $M_0 \to f_{0*} \mathcal{O}_{X_0}$ generating $f_* \mathcal{O}_{X_0}$ as an $\mathcal{O}_{Y_0}$-algebra—i.e. so that $\operatorname{Sym}^*_{Y_0} M_0 \to f_* \mathcal{O}_{X_0}$ is surjective. Then $\operatorname{Spec}_{Y_0}$ of this surjection yields a closed embedding

$$j_0 : X_0 \hookrightarrow V_0 := \operatorname{Spec}_{Y_0} \operatorname{Sym}^*_{Y_0} M_0$$

of X_0 into the bundle V_0 over Y_0 so that f_0 factors as the closed embedding followed by the projection $q_0 : V_0 \to Y_0$.

Assume we have constructed an n-truncated derived scheme $V = (V, P)$ and a factorization

$$\operatorname{tr}_n X \xrightarrow{j} V \xrightarrow{q} \operatorname{tr}_n Y \tag{9.10.1}$$

of $\operatorname{tr}_n f$ with the following properties:

(1) The map $q : V \to \operatorname{tr}_n Y$ is (truncated) affine projective with basis $(M_0, \ldots, M_n)$ and non-degenerate complements $(E_0, \ldots, E_n)$, each M_k and each E_k an A_k-bundle, say with $E_k = p_{k*}E'_k$, $M_k = p_{k*}M'_k$ for locally free coherent sheaves M'_k and E'_k on Y_n.
(2) The map j is a (degree-wise) closed embedding.
(3) The map $j_k : \mathrm{H}_k(V_\bullet) \to j_{k*}\,\mathrm{H}_k(X_\bullet)$ is an isomorphism for $k < n$.
(4) The map $j_n : \mathsf{Z}_n(V_\bullet) \to j_{n*}\,\mathrm{H}_n(X_\bullet)$ is surjective.

Our start above gives us all of this when $n = 0$. We need only prove that we can lift our factorization (9.10.1) to an analogous factorization of $\operatorname{tr}_{n+1} f$ satisfying all of the above properties with "n" replaced by "$n + 1$".

Let us remark that in light of (**??**), the surjectivity of the map in (**??**) is equivalent to the surjectivity of its adjoint map $j_n : j_n^*\mathsf{Z}_n(V_\bullet) \to \mathrm{H}_n(X_\bullet)$. We also note that since j_0 is a closed embedding, isomorphy for the map in (**??**) when $k = 0$ says exactly that $\pi_0(X) \to \pi_0(V)$ is an isomorphism.

For each $i \in [n + 1]$, we have a solid commutative diagram of schemes:

$$\begin{tikzcd}
X_n \arrow[r, "\partial^i_{n+1}"] \arrow[d, "j_n"] \arrow[dd, bend right=60, "f_n"'] & X_{n+1} \arrow[d, dotted, "j_{n+1}"] \arrow[dd, bend left=60, "f_{n+1}"] \\
V_n \arrow[r, dotted, "\partial^i_{n+1}"] \arrow[d, "q_n"] & V_{n+1} \arrow[d, dotted, "q_{n+1}"] \\
Y_n \arrow[r, "\partial^i_{n+1}"'] & Y_{n+1}
\end{tikzcd} \tag{9.10.2}$$

We need to construct V_{n+1}, etc. making the resulting diagram commute and satisfying various properties.

Define a quasi-coherent sheaf $\overline{W}$ on Y_{n+1} by the cartesian daigram of quasi-coherent sheaves:

$$\begin{tikzcd}
\overline{W} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \prod_{i=0}^{n+1}(\partial^i_{n+1})_* q_{n*}\mathcal{O}_{V_n} \arrow[d] \\
(f_{n+1})_*\mathcal{O}_{X_{n+1}} \arrow[r, "((f_{n+1})_* d^i_{n+1})"'] & \prod_{i=0}^{n+1}(f_{n+1})_*(\partial^i_{n+1})_*\mathcal{O}_{X_n}
\end{tikzcd} \tag{9.10.3}$$

The right vertical arrow is a product of pushforwards under affine morphisms of the surjection

$$j_n^\flat : \mathcal{O}_{V_n} \to j_{n*}\mathcal{O}_{X_n} \tag{9.10.4}$$

corresponding to the closed embedding j_n, hence it is surjective, and hence π_1 is also surjective. By Theorem 9.9.3(1) we can find a locally free coherent sheaf W' on Y_{n+1}

and a map $W' \to \overline{W}$ of quasi-coherent sheaves on Y_{n+1} so that the evident composition abusively denoted $\pi_1 : W' \to (f_{n+1})_* \mathcal{O}_{X_{n+1}}$ induces a surjection

$$\operatorname{Sym}_{Y_{n+1}} W' \to (f_{n+1})_* \mathcal{O}_{X_{n+1}}. \tag{9.10.5}$$

For $i \in [n+1]$, let

$$\pi_2^i : W' \to (\partial^i_{n+1})_* q_{n*} \mathcal{O}_{V_n} \tag{9.10.6}$$

denote the composition of $W' \to \overline{W}$, π_2, and the projection to the i^{th} factor of the product.

Next define a coherent sheaf $\overline{K}$ on X_{n+1} by the cartesian diagram

$$\begin{tikzcd} \overline{K} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & (\partial^0_{n+1})_* j_n^* \mathsf{Z}_n(V_\bullet) \arrow[d] \\ \mathsf{N}_{n+1}(X_\bullet) \arrow[r, "d^0_{n+1}"'] & (\partial^0_{n+1})_* \mathsf{Z}_n(X_\bullet) \end{tikzcd} \tag{9.10.7}$$

of coherent sheaves on X_{n+1}. The right vertical arrow is the pushforward under the closed embedding ∂^0_{n+1} of the map which is surjective by (**??**), hence it is surjective, and hence π_1 is surjective. By Theorem 9.9.3(2) we can find a bundle K' on Y_{n+1} and a surjection $f^*_{n+1} K' \to \overline{K}$. The composition of this surjection and the map π_1 in (9.10.7) yield a map $f^*_{n+1} K' \to \mathsf{N}_{n+1}(X_\bullet)$. The composition of the map $(f^*_{n+1}, (f_{n+1})_*)$-adjoint to the latter and $(f_{n+1})_*$ of the inclusion $\mathsf{N}_{n+1}(X_\bullet) \subseteq \mathcal{O}_{X_{n+1}}$ is a map

$$K' \to (f_{n+1})_* \mathcal{O}_{X_{n+1}}. \tag{9.10.8}$$

Using the commutative diagram (9.10.2), we obtain a map

$$K' \to (\partial^0_{n+1})_* q_{n*} \mathcal{O}_{V_n} \tag{9.10.9}$$

by composing the $(f^*_{n+1}, (f_{n+1})_*)$-adjoint of our chosen surjection, $(f_{n+1})_*$ of the map π_2 in (9.10.7), and the appropriate pushforward of the inclusion $\mathsf{Z}_n(V_\bullet) \subseteq \mathcal{O}_{V_n}$.

Since $\mathsf{Z}_{n+1}(X_\bullet)$ is a coherent sheaf on X_{n+1}, Theorem 9.9.3(2) says we can also find a bundle S' on Y_{n+1} and a surjection

$$f^*_{n+1} S' \to \mathsf{Z}_{n+1}(X). \tag{9.10.10}$$

(Actually we only need the composition $f^*_{n+1} S' \to \mathrm{H}_{n+1}(X)$ to be surjective.)

Let D'_{n+1} be the direct limit of the $\sigma^* M'_m$ as $\sigma : [n+1] \to [m]$ runs over the category T_{n+1} (§**??**). (We will often use σ as abuse of notation for $Y(\sigma)$.) Let $\overline{\sigma} : \sigma^* M'_m \to D'_{n+1}$ be the structure map to the direct limit. Since we can alternatively express D'_{n+1} as

$$D'_{n+1} = \bigoplus_{(\sigma:[n+1]\to[m]) \in T_{n+1}} \sigma^* E'_m$$

(c.f. Lemma 7.6.5) it is clear that D'_{n+1} is a locally free coherent sheaf on Y_{n+1}, as are

$$\begin{aligned} E'_{n+1} &:= W' \oplus K' \oplus S' \\ M'_{n+1} &:= D'_{n+1} \oplus E'_{n+1}. \end{aligned}$$

Let

$$\begin{aligned} D_{n+1} &:= (p_{n+1})_* D'_{n+1} \\ E_{n+1} &:= (p_{n+1})_* E'_{n+1} \\ M_{n+1} &:= (p_{n+1})_* M'_{n+1} \end{aligned}$$

be the corresponding A_{n+1}-bundles. Let $q_{n+1} : V_{n+1} \to Y_{n+1}$ be the vector bundle defined by

$$V_{n+1} := \operatorname{Spec}_{Y_{n+1}} \operatorname{Sym}^*_{Y_{n+1}} M'_{n+1}.$$

For the Δ-morphism $\partial^i_{n+1} : [n] \to [n+1]$, the corresponding map $\partial^i_{n+1}(V) : V_n \to V_{n+1}$ for our $(n+1)$-truncated derived scheme V is defined as follows. We want the bottom square of (9.10.2) to commute. This is equivalent to saying that our map $\partial_{n+1}(V)$ is obtained from a map of quasi-coherent sheaves

$$d^i_{n+1} : M'_{n+1} \to (\partial^i_{n+1})_* q_{n*}\mathcal{O}_{V_n} = (\partial^i_{n+1})_* \operatorname{Sym}_{Y_n} M'_n \tag{9.10.11}$$

on Y_{n+1}. Furthermore, for each $(\sigma : [n+1] \to [m]) \in T_{n+1}$, we need the diagrams

$$\begin{tikzcd} V_n \arrow[r, "\partial^i_{n+1}"] \arrow[d, "q_n"'] & V_{n+1} \arrow[r, "V(\sigma)"] \arrow[d, "q_{n+1}"] & V_m \arrow[d, "q_m"] \\ Y_n \arrow[r, "\partial^i_{n+1}"'] & Y_{n+1} \arrow[r, "Y(\sigma)"'] & Y_m \end{tikzcd} \tag{9.10.12}$$

to commute. This commutativity is equivalent to saying that the composition of

$$\overline{\sigma} : \sigma^* M'_m \to D'_{n+1} \subseteq M'_{n+1}$$

and the map (9.10.11) must coincide with the map

$$\sigma^* M'_m \to (\partial^i_{n+1})_* q_{n*}\mathcal{O}_{V_n}$$

which is $((\partial^i_{n+1})^*, (\partial^i_{n+1})_*)$-adjoint to the map

$$(\sigma\partial^i_{n+1})^* M'_m = (\partial^i_{n+1})^* \sigma^* M'_m \to q_{n*}V_n$$

corresponding to the "big square" in (9.10.12) (this big square is already defined for our n-truncated simplicial objects). It is then tautological to check that we can *define* (9.10.11) on the degenerate submodule $D'_{n+1} \subseteq M'_{n+1}$ using the universal property of the direct limit D'_{n+1} in a unique manner so that the diagrams (9.10.12) will commute. We can then define the map d^i_{n+1} in (9.10.11) on the non-degenerate complement $E'_{n+1} \subseteq M'_{n+1}$ in any way we see fit. On the summand $W' \subseteq E'_{n+1}$, we define d^i_{n+1} by setting $d^i_{n+1}|W' := \pi^i_2$ equal to the map π^i_2 in (9.10.6). On the summand $K' \subseteq E'_{n+1}$, we define $d^i_{n+1}|K'$ to be zero when $i \neq 0$ and we define $d^0_{n+1}|K'$ to be the map (9.10.9). We declare $d^i_{n+1}|S' := 0$ for all i.

We next need to define the map $j_{n+1} : X_{n+1} \to V_{n+1}$. We want the "right triangle" in (9.10.2) to commute. This is equivalent to saying that j_{n+1} corresponds to a map

$$j_{n+1} : M'_{n+1} \to (f_{n+1})_* \mathcal{O}_{X_{n+1}} \tag{9.10.13}$$

of quasi-coherent $\mathcal{O}_{Y_{n+1}}$-modules. We also want the top square in (9.10.2) to commute. This is equivalent to saying that the square

$$\begin{tikzcd} M'_{n+1} \arrow[r, "j_{n+1}"] \arrow[d, "d^i_{n+1}"'] & (f_{n+1})_* \mathcal{O}_{X_{n+1}} \arrow[d] \\ (\partial^i_{n+1})_* q_{n*}\mathcal{O}_{V_n} \arrow[r] & (\partial^i_{n+1})_* f_{n*}\mathcal{O}_{X_n} \end{tikzcd} \tag{9.10.14}$$

of quasi-coherent $\mathcal{O}_{Y_{n+1}}$-modules commutes. (The right vertical arrow in (9.10.14) is $(f_{n+1})_*$ of

$$d^i_{n+1} : \mathcal{O}_{X_{n+1}} \to (\partial^i_{n+1})_* \mathcal{O}_{X_n}$$

and the bottom horizontal arrow in (9.10.14) is the pushforward of (9.10.4).)

First of all, if we want (9.10.13) to be compatible with the maps $X(\sigma)$ and $Y(\sigma)$ for $\sigma \in T_{n+1}$, then there is no choice about how to define (9.10.13) on the degenerate submodule D'_{n+1}. It is tautological to check that, using the universal property of the direct limit D'_{n+1} (and the fact that we started with a map of n-truncated simplicial objects!), if we define the composition of $\overline{\sigma} : \sigma^* M'_m \to D'_{n+1}$ and (9.10.13) in the unique way compatible with $X(\sigma)$ and $Y(\sigma)$, then the resulting maps "glue" to a map

$$j_{n+1}|D'_{n+1} : D'_{n+1} \to (f_{n+1})_* \mathcal{O}_{X_{n+1}},$$

this map has the property that (9.10.14) commutes when restricted to D'_{n+1}, and, regardless of how we define $j_{n+1}|E'_{n+1}$, the map j_{n+1} will be compatible with the $X(\sigma)$ and $Y(\sigma)$ for $\sigma \in T_{n+1}$. We thus reduce to defining the map (9.10.13) on the non-degenerate submodule $E'_{n+1} \subseteq M'_{n+1}$ and checking that (9.10.14) commutes on E'_{n+1}.

We define (9.10.13) on the summand $W' \subseteq E'_{n+1}$ to be the map we abusively denoted π_1 when we constructed W'. We define (9.10.13) on the summand $K' \subseteq E'_{n+1}$ to be the map (9.10.8). We define (9.10.13) on the summand $S' \subseteq E'_{n+1}$ to be the composition of the map $S' \to (f_{n+1})_* \mathsf{Z}_{n+1}(X_\bullet)$ which is $(f^*_{n+1}, (f_{n+1})_*)$-adjoint to (9.10.10) and the map given by $(f_{n+1})_*$ of the inclusion $\mathsf{Z}_{n+1}(X_\bullet) \subseteq \mathcal{O}_{X_{n+1}}$. The commutativity of (9.10.14) on W' follows from the definitions and the commutativity of (9.10.3). The commutativity of (9.10.14) on K' is trivial when $i \neq 0$ (both ways around are zero) and follows from the commutativity of (9.10.7) when $i = 0$. The commutativity of (9.10.14) on S' is easy to see from the definitions of the maps involved (both ways around are zero for all i).

It remains only to check the properties (1)-(4). Certainly (1) holds by our construction of V_{n+1} and q_{n+1}. Property (2) is equivalent to surjectivity of the map

$$\operatorname{Sym}_{Y_{n+1}} M'_{n+1} \to (f_{n+1})_* \mathcal{O}_{X_{n+1}} \tag{9.10.15}$$

induced by (9.10.13). Our construction of (9.10.13) on W' ensures that the restriction of (9.10.15) to

$$\operatorname{Sym}_{Y_{n+1}} W' \subseteq \operatorname{Sym}_{Y_{n+1}} M'_{n+1}$$

is the surjection (9.10.5), so (9.10.15) is certainly surjective. For (3), we need to check that

$$j_n : \mathrm{H}_n(V_\bullet) \to j_{n*} \mathrm{H}_n(X_\bullet) \tag{9.10.16}$$

is an isomorphism. Since this map was surjective at stage n, it will remain surjective at stage $n+1$, so it is enough to check that the map

$$d^0_{n+1} : \mathsf{N}_{n+1}(V_\bullet) \to (\partial^0_{n+1})_* \mathsf{Z}_n(V_\bullet) \tag{9.10.17}$$

surjects onto the kernel of the map

$$(\partial^0_{n+1})_* j_n : (\partial^0_{n+1})_* \mathsf{Z}_n(V_\bullet) \to (\partial^0_{n+1})_* j^*_n \mathrm{H}_n(X_\bullet). \tag{9.10.18}$$

A moment's thought with the definition of homology shows that the kernel of (9.10.18) is exactly the image of the map π_1 in (9.10.7).

Consider the map

$$K' \to (q_{n+1})_*\mathcal{O}_{V_{n+1}} = \mathrm{Sym}_{Y_{n+1}} M'_{n+1}$$

including K' as a summand of the degree one part of the symmetric product. Since the maps $d^i_{n+1}|K'$ are zero for $i \neq 0$, the adjoint map $q^*_{n+1}K' \to \mathcal{O}_{V_{n+1}}$ factors through $\mathsf{N}_{n+1}(V_\bullet) \subseteq \mathcal{O}_{V_{n+1}}$. Pulling back along $j_{n+1} : X_{n+1} \hookrightarrow V_{n+1}$, we obtain a map

$$f^*_{n+1}K' = j^*_{n+1}q^*_{n+1}K \to j^*_{n+1}\mathsf{N}_{n+1}(V_\bullet). \tag{9.10.19}$$

Because of the way we defined $d^0_{n+1}|K'$, the composition of (9.10.19) and

$$j^*_{n+1}d^0_{n+1} : j^*_{n+1}\mathsf{N}_{n+1}(V_\bullet) \to j^*_{n+1}(\partial^0_{n+1})_*\mathsf{N}_n(V_\bullet) = (\partial^0_{n+1})_*j^*_n\mathsf{N}_n(V_\bullet)$$

is nothing but the composition of the surjection $f^*_{n+1}K' \to \overline{K}$ and the map π_1 in (9.10.7). □

Theorem 9.10.5. *Suppose Y is a degree-wise noetherian derived scheme and $f : X \to Y$ is a degree-wise finite-type map of derived schemes such that $f_0 : X_0 \to Y_0$ can be factored as a closed embedding $i : X_0 \to M_0$ into a divisorial noetherian scheme M_0 followed by a smooth finite type map $p_0 : M_0 \to Y_0$. (These hypotheses on f_0 are satisfied, for example, if Y_0 is divisorial and f_0 is quasi-projective.) Then any such factorization can be extended to a factorization $f = pi$ where i is a trivial cofibration and p is degree-wise smooth and degree-wise finite type. Any two such factorizations of f map to a third such factorization.*

9.11. Derived fibered products.

9.12. Cotangent complex.

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