On the deformation of rings and algebras: II

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This paper presents a deformation theory for filtered rings. A complete filtered ring is shown to be a deformation of its associated graded ring; in particular, it is given by a knowledge of the latter and certain homological information. The significant advance over Gerstenhaber [1] is that the additive structure, in particular the characteristic, should there be one, may change. The signal difficulty lies in constructing the necessary absolute cohomology groups. (For examples of the problems involved in this direction, cf. Dixmier [1] and Shukla [1].)

The classical algebraic obstruction theories, including those of Eilenberg-MacLane [1] and Hochschild [2], [3], [4], are in effect given a uniform treatment by § 1 of this paper. Section 2 contains the successive approximations to a filtered ring, and § 3 ends with the deformation of one filtered ring to another. Important but easily obtained results are stated without proof.

1. Homological preparation

Let $\mathcal{R}$ denote the category of all rings, associative or not. By an *epimorphism* in $\mathcal{R}$, we shall mean a morphism which is an epimorphism in the sense of sets, regardless of whether or not it may be an epimorphism in the categorical sense of Grothendieck [1].

DEFINITION 1. A *category of interest* is a subcategory $\mathcal{C}$ of $\mathcal{R}$ with the properties:

(1) If $A \to B$ is a morphism in $\mathcal{C}$, then its kernel and cokernel (in $\mathcal{R}$) are again in $\mathcal{C}$. [The cokernel of $A \to B$, coker $(A \to B)$ is the quotient of B by the ideal generated by the set-theoretic image of A in B, together with the natural projection $B \to$ coker $(A \to B)$. Similarly, the kernel is the set-theoretic kernel together with the natural injection. Under the usual categorical definitions both are in fact defined only up to isomorphism, and we so view them here.]

(2) The fibered product (cf. Gabriel [1]) of $\mathcal{C}$-morphisms $\varphi: A \to C$ and $\psi: B \to C$ (i.e., the object $A \times_C B = \{(a, b) \mid \varphi a = \psi c\}$ with the obvious ring structure and projection maps) is again in $\mathcal{C}$.

Remark. Had we considered categories of interest inside *almost abelian*

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categories (J. C. Moore [1]), then what follows could also have included the cases of groups and schemes.

The categories of associative, Lie, Jordan, commutative associative, and associative nilpotent algebras are examples of categories of interest. For the rest of this section, some category of interest $\mathcal{C}$ will be assumed fixed. By a *singular object* in $\mathcal{C}$ we shall mean a ring A such that the addition map $A \times A \rightarrow A$ is a morphism in $\mathcal{C}$. It follows that A is a zero ring, i.e., $ab = 0$ for all a, b in A. The following (not needed in the sequel) intrinsically characterizes singular objects.

THEOREM 1. *Singular objects are precisely the "groups relative to the category $\mathcal{C}$", i.e., the objects A such that*

(1) *for all x in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, A)$ carries an additive group structure, and*

(2) *given $X \rightarrow X'$, the induced mapping $\mathrm{Hom}_{\mathcal{C}}(X', A) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, A)$ is a morphism of groups.*

By the *bicenter* of a ring A, we shall mean the set of all x such that $xa = ax = 0$ for all a in A, and a morphism will be called *singular* if its image is contained in the bicenter. If $\varphi: C \rightarrow A$ is a singular morphism in $\mathcal{C}$, then $-\varphi$, defined by $-\varphi(c) = -(\varphi c)$ is again a morphism in $\mathcal{C}$. If $\psi: C \rightarrow B$ is a second singular morphism, then $-\varphi \times \psi: C \rightarrow A \times B$ (defined by $(-\varphi \times \psi)c = (-\varphi c, \psi c)$) is again in $\mathcal{C}$.

DEFINITION 2. The cokernel of $-\varphi \times \psi$ will be denoted by $A +_C B$; it is the direct product with the images of C in A and B identified and is again a ring in $\mathcal{C}$.

There are obvious morphisms $A \rightarrow A +_C B$ and $B \rightarrow A +_C B$.

LEMMA 1. *The "sum" $A +_C B$ has the following universal property. If*

$$\begin{array}{ccc} C & \xrightarrow{\psi} & B \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\alpha} & X \end{array}$$

is a commutative diagram of rings and morphisms in $\mathcal{C}$ in which φ, ψ and α are singular, then there exists a unique morphism $A +_C B \rightarrow X$ in C such that $A \rightarrow X$ is the composite $A \rightarrow A +_C B \rightarrow X$, and similarly for $B \rightarrow X$.

The concept of exact sequence in $\mathcal{C}$ is clear; a morphism of sequences is the usual commutative diagram (cf. Yoneda [1] and [2]). Given A and C in $\mathcal{C}$ an *extension* of A by C is a short exact sequence $\mathbf{E}: 0 \rightarrow C \rightarrow B \rightarrow A \rightarrow 0$, the objects and morphisms tacitly being in $\mathcal{C}$. The extension is *singular* if C is a singular object; it is *split* if there is a morphism $A \to B$ such that the composite $A \to B \to A$ is the identity morphism 1_A.

DEFINITION 3. Let A and C be in $\mathcal{C}$. An *A-structure* on C is a split extension $\mathbf{E}: 0 \to C \to B \to A \to 0$ together with a definite choice of splitting map. Equivalently, there is defined an action of A on C such that the group direct product with multiplication given by

$$(a, c)(a', c') = (aa', ac' + ca' + cc')$$

is again in $\mathcal{C}$.

By abuse of language, we may sometimes speak of the A-structure C. If C is singular, then we have the usual concept of an *A-module* in the category $\mathcal{C}$, cf. Gerstenhaber [2]. A morphism of A-structures is simply a morphism of short exact sequences carrying the splitting map of the first into that of the second. Kernels and cokernels are defined in the category of A-structures. If C' carries an A'-structure and a morphism $A \to A'$ is given, then C' carries an A-structure in a natural way. If, in addition, an A-structure C and a morphism $C \to C'$ in C are given, then we shall say that $A \to A'$ and $C \to C'$ are *compatible* if $C \to C'$ is a morphism of A-structures. Any morphism $\varphi: A \to C$ in C induces a natural A-structure on C. (Split the short exact sequence $0 \to C \to A \times C \to A \to 0$ by the diagonal map $A \to A \times C$. It is trivial to verify from the axioms that all the objects and morphisms are in C.) In particular, A and all its quotients carry *canonical* A-structure. If C_1 carries an A_1-structure and C_2 an A_2-structure, then $C_1 \times C_2$ carries an $A_1 \times A_2$-structure in a natural way.

DEFINITION 4. Suppose given an A-structure on C. Then a morphism $\varphi: C \to A$ is said to *conform* if the C-structure on C induced by the morphism is the canonical one, i.e., if $cc' = (\varphi c)c' = c \cdot \varphi(c')$ for all c and c' in C.

The Baer extension theory is meaningful for categories of interest. That is, fixing an A-module structure on M, the set of equivalence classes of singular extensions of A by M forms a group in the usual way, here denoted by $\mathcal{E}^2_{\mathcal{C}}(A, M)$ or simply by $\mathcal{E}^2(M)$ if A and $\mathcal{C}$ are understood. It is important to note that $\mathcal{E}^2_{\mathcal{C}}(A, M)$ does *not* necessarily vanish when M is an injective in the (abelian) category of A-modules relative to $\mathcal{C}$. It vanishes, for example, when $\mathcal{C}$ is the category of associative or Lie algebras over some fixed coefficient field, but does not vanish when $\mathcal{C}$ is the category of commutative associative algebras over a field, cf. Gerstenhaber [1, Ch. I, Th. 3].

For the purposes of the deformation theory, no present definition of the higher cohomology groups, including that of Gerstenhaber [2], is satisfactory; the germane definition of the third group, here denoted $\mathcal{E}^3_{\mathcal{C}}(A, M)$, follows.

(All concepts are tacitly relative to $\mathcal{C}$.)

Given an A-module structure on M, an exact sequence

$$\mathbf{E}\colon 0 \longrightarrow M \longrightarrow N \longrightarrow B \longrightarrow A \longrightarrow 0$$

will be called *admissible* if N carries a B-structure, the morphism $N \to B$ conforms, and $M \to N$ is a morphism of B-structures (where we may consider M as a B-structure by virtue of the morphism $B \to A$). Note that $M \to N$ is necessarily singular. A *morphism of admissible sequences* is a commutative diagram

$$\begin{array}{lcccccccccc} \mathbf{E}: 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow \mathbf{f} & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \\ \mathbf{E}'\colon 0 & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & B' & \longrightarrow & A' & \longrightarrow & 0 \end{array}$$

in which f_3 and f_0 are compatible, f_2 and f_1 are compatible, and f_3 and f_1 are compatible (where M is considered as a B-structure, and M' as a B'-structure). Let $C^3(A, M)$ be the category whose objects are admissible sequences beginning with A and terminating with M, with morphisms, those morphisms for which f_0 and f_3 are identity maps. Given two sequences in $C^3(A, M)$, they are considered equivalent if there is a morphism in $C^3(A, M)$ from one to the other, and we define the equivalence relation in $C^3(A, M)$ to be the one which this propagates.

LEMMA 2. *Given* $\mathbf{E}'$ *and* $\mathbf{E}''$ *in* $C^3(A, M)$, *they are equivalent if and only if there exists an* $\mathbf{E}$ *and morphisms* $\mathbf{E}'' \leftarrow \mathbf{E} \rightarrow \mathbf{E}''$.

The lemma is known in the Yoneda theory (Yoneda, [1], [2]); the proof here follows similar lines.

DEFINITION 5. $\mathcal{E}^3(A, M)$ is the set of equivalence classes in $C^3(A, M)$.

The group structure in $\mathcal{E}^3(A, M)$ is introduced precisely as in the Yoneda theory. Specifically, the zero element is the class of the sequence $\mathbf{0}\colon 0 \to M = M \to A = A \to 0$, where $M \to A$ is the zero map, the negative of the class of $\mathbf{E}\colon 0 \to M \to N \to B \to A \to 0$ is the class of the sequence formed from $\mathbf{E}$ by replacing $M \to N$ by its negative, and the sum of the class of $\mathbf{E}$ with that of $\mathbf{E}'\colon 0 \to M \to N' \to B' \to A \to 0$ is the class of the sequence

$$0 \longrightarrow (M \times M)/K \longrightarrow (N \times N')/K \longrightarrow B \times_A B' \longrightarrow A \longrightarrow 0 ,$$

where K is the kernel of the addition map $M \times M \to M$, i.e., $K = \{(m, -m)\}$.

THEOREM 2. $\mathcal{E}^3(A, M)$ *is a commutative group, contravariant as a functor of* A *and covariant as a functor of the* A*-module* M.

PROOF. The contravariance in A follows from the existence of fibered products. As for the covariance in M, note that given an admissible sequence

$0 \to M \to N \to B \to A \to 0$ and an A-module morphism $M \to M'$, then $M' +_M N$ is well-defined in $\mathcal{C}$. The group structure follows classical lines.

If φ is a derivation of A into the A-module M, and if $A + M$ denotes the split singular extension, then $(a, m) \to (a, m + \varphi(a))$ is an automorphism in $\mathcal{R}$ of $A + M$; if it is an automorphism in $\mathcal{C}$ then we shall say that φ is a derivation of A into M relative to $\mathcal{C}$. Denoting by $\mathrm{Der}_{\mathcal{C}}(A, M)$ the additive group of all such, we have the usual result on the exactness of the cohomology sequence. (The sequence here terminates with the third group since no higher ones have been defined, but can be extended by the derived functors of $\mathcal{E}^3$, as well as in some less obvious ways.)

THEOREM 3. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules. Then letting $\mathcal{C}$ and A be understood, there is a natural exact sequence* $0 \to \mathrm{Der}\, M' \to \mathrm{Der}\, M \to \mathrm{Der}\, M'' \to \mathcal{E}^2 M' \to \mathcal{E}^2 M \to \mathcal{E}^2 M'' \to \mathcal{E}^3 M' \to \mathcal{E}^3 M \to \mathcal{E}^3 M''$.

(Note. The cohomology groups of groups, of associative algebras, and of Lie algebras can be defined as derived functors of Der; (cf. Rinehart-Barr [1]).)

The following elementary theorem is the most important one in this section. In it 0 will denote also the zero module.

THEOREM 4. *The necessary and sufficient condition that the class of an admissible sequence $0 \to M \to N \to B \to A \to 0$ be the zero element of $\mathcal{E}^3(A, M)$ is that there exist a morphism of admissible sequences of the form*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \overline{N} & \longrightarrow & \overline{B} & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0\,, \end{array}$$

and when that is the case, we may assume, that $\overline{N} = N$ and that the morphism $\overline{N} \to N$ is the identity. We then have a singular extension $0 \to M \to \overline{B} \to B \to 0$.

PROOF. An immediate consequence of Lemma 2.

Despite its simplicity and importance, the theorem seems to have been overlooked in the classical literature.

If $\mathbf{E}$: $0 \to M \to N \to B \to A \to 0$ is equivalent to zero in $C^3(A, M)$, then the sequence $\mathbf{E}_0$: $0 \to 0 \to N \to \overline{B} \to A \to 0$ and morphism $\mathbf{E}_0 \to \mathbf{E}$ which exist by Theorem 4 are not necessarily unique. Calling such a sequence and morphism a *solution* for $\mathbf{E}$, we have

THEOREM 5. *If* $\mathbf{E}$ *is equivalent to zero in $C^3(A, M)$, then the set of equivalence classes of solutions for* $\mathbf{E}$ *is in a natural way a principal homogeneous space over $\mathcal{E}^2(A, M)$.*

PROOF. If $\mathbf{E}$ is equivalent to zero, then we have a diagram

$$\begin{array}{llllllllll}
\mathbf{E}_0: & & 0 & \longrightarrow & N & \longrightarrow & \overline{B} & \longrightarrow & A & \longrightarrow 0 \\
\downarrow & & & & \| & & \downarrow \pi & & \| & \\
\mathbf{E}: 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & B & \longrightarrow & A & \longrightarrow 0\,;
\end{array}$$

$\mathbf{E}_0$ and $\mathbf{E}'_0$ are equivalent if there exists a morphism

$$\begin{array}{lllllllll}
\mathbf{E}_0: 0 & \longrightarrow & N & \longrightarrow & \overline{B} & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow & & \| & & \\
\mathbf{E}'_0: 0 & \longrightarrow & N & \longrightarrow & \overline{B}' & \longrightarrow & A & \longrightarrow & 0\,.
\end{array}$$

The morphism $\overline{B} \to \overline{B}'$ is then an isomorphism. To see that $\mathcal{E}^2(A, M)$ actually operates, suppose that $\mathbf{F}: 0 \to M \to F \to A \to 0$ represents an element of that group. Then there is a morphism

$$\begin{array}{llllllllll}
\mathbf{E}_0 \times_A \mathbf{F}: & & 0 & \longrightarrow & N \times M & \longrightarrow & \overline{B} \times_A F & \longrightarrow & A & \longrightarrow 0 \\
\downarrow & & & & \text{add} \downarrow & & \downarrow & & \| & \\
\mathbf{E} \quad : 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & B & \longrightarrow & A & \longrightarrow 0\,,
\end{array}$$

where add is the addition map defined by $\text{add}\,(n, m) = n + m$, and the morphism $\overline{B} \times_A F \to B$ is given by $(\bar{b}, f) \to \pi\bar{b}$. We may thus set

$$\begin{aligned}
\mathbf{E}_0 + \mathbf{F} = \text{add}_* (\mathbf{E}_0 \times_A \mathbf{F}): 0 &\longrightarrow (N \times M)/\text{ker add} \\
&\longrightarrow (\overline{B} \times_A F)/\text{ker add} \longrightarrow A \longrightarrow 0\,.
\end{aligned}$$

It remains to show that given $\mathbf{E}_0$ and $\mathbf{E}'_0$, there is an $\mathbf{F}$ such that $\mathbf{E}'_0$ is equivalent to $\mathbf{E}_0 + \mathbf{F}$. To this end, set $N/M = K$ and observe that there is a natural morphism

$$\begin{array}{lllllllll}
\mathbf{E}_0: 0 & \longrightarrow & N & \longrightarrow & \overline{B} & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
\mathbf{K}: 0 & \longrightarrow & K & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0\,.
\end{array}$$

The same being true for $\mathbf{E}'_0$, and the morphisms $N \to K$ and $\overline{B} \to B$ being epimorphisms, we can form

$$\mathbf{E}_0 \times \mathbf{E}'_0: 0 \longrightarrow N \times_K N \longrightarrow \overline{B} \times_B \overline{B}' \longrightarrow A \longrightarrow 0\,.$$

Let $\text{diag}: N \to N \times_K N$ be the diagonal map, let $\text{diag}\, N$ be identified with its image in $\overline{B} \times_B \overline{B}'$, and set $(\overline{B} \times_B \overline{B}')/\text{diag}\, N = F$. There is a natural epimorphism $F \to A$ induced by the morphisms $\overline{B} \to A$ and $\overline{B}' \to A$. Now every element of $N \times_K N$ is uniquely representable in the form $(n, n + m)$; mapping this to (n, m) shows that $N \times_K N$ is isomorphic to $N \times M$. Further, $\overline{B} \times_B \overline{B}'$ is isomorphic to $\overline{B} \times_A F$, the morphism being the product of the morphisms

$\overline{B} \times_B \overline{B}' \to \overline{B}$ (projection on the first factor) and $\overline{B} \times_B \overline{B} \to F$ (reduction modulo diag N). Therefore, letting F denote the sequence $0 \to M \to F \to A \to 0$, it is the case that $\mathbf{E}_0 \times_K \mathbf{E}_0'$ is isomorphic to

$$\mathbf{E}_0 \times_A \mathbf{F}\colon 0 \longrightarrow N \times M \longrightarrow \overline{B} \times_A F \longrightarrow A \longrightarrow 0 \,.$$

Projection of $(n, n+m)$ on the second factor is just the addition map, add: $N \times M \to N$. Projecting $\mathbf{E}_0 \times_K \mathbf{E}_0'$ on the second factor gives just $\mathbf{E}_0'$. Therefore $\mathbf{F}$ has the property that $\mathbf{E}_0 + \mathbf{F} = \mathbf{E}_0'$, as required. This ends the proof.

One may note that, if $\mathcal{C}$ is a category of algebras over a coefficient ring k, then all the foregoing definitions can be relativized by considering only k-morphisms and only exact sequences of k-morphisms which are k-split; we write correspondingly $\operatorname{Rel} \mathcal{E}^2_{\mathcal{C}}(A, M)$ and $\operatorname{Rel} \mathcal{E}^3_{\mathcal{C}}(A, M)$. If A is projective as a k-module, then the relative groups coincide with the absolute ones.

The relationship between $\mathcal{E}^3(A, M)$ and the classical third cohomology group in important special cases is given by

THEOREM 6. 1. *Let $\mathcal{C}$ be an arbitrary category of interest, M be an A-module relative to $\mathcal{C}$, and $R^1\mathcal{E}^2_{\mathcal{C}}(A, M)$ denote the first right derived functor of $\mathcal{E}^2_{\mathcal{C}}(A, M)$ considered as a functor of M. Then there is a natural monomorphism $R^1\mathcal{E}^2_{\mathcal{C}}(A, M) \to \mathcal{E}^3_{\mathcal{C}}(A, M)$.*

2. *Let k be a fixed coefficient ring, and $\mathcal{C}$ denote either the category of associative algebras, Lie algebras, or commutative associative algebras over k (or, had we given the details, the category of groups). Then there is a natural morphism*

$$\operatorname{Rel} \mathcal{E}^3_{\mathcal{C}}(A, M) \longrightarrow H^3_{\mathcal{C}}(A, M) \,,$$

where if $\mathcal{C}$ is the category of associative algebras, then $H^3_{\mathcal{C}}(A, M)$ is defined as in Hochschild [1]. *If $\mathcal{C}$ is the category of Lie algebras, it is defined as in Chevalley-Eilenberg* [1]; *and if $\mathcal{C}$ is the category of commutative associative algebras, then it is defined as in Harrison* [1]. (*In the group case, it is defined as in Eilenberg-MacLane* [1]; cf. *also MacLane* [1] *for all the foregoing.*)

3. *If A is projective as a k-module, and if $\mathcal{C}$ is the category of associative algebras or Lie algebras over k (or if $\mathcal{C}$ is the category of all groups), then the morphisms*

$$R^1\mathcal{E}^2_{\mathcal{C}}(A, M) \longrightarrow \mathcal{E}^3_{\mathcal{C}}(A, M) \longrightarrow H^3_{\mathcal{C}}(A, M)$$

are isomorphisms.

PROOF. As for 1, if in the definition of $\mathcal{E}^3_{\mathcal{C}}(A, M)$ as equivalence classes of sequences $0 \to M \to N \to B \to A \to 0$, we had imposed the additional condition that N be singular (hence an A-module) the resulting group would have been just $R^1\mathcal{E}^2_{\mathcal{C}}(A, M)$; the monomorphism follows. Assertion 2 is just the basic computation of the algebraic obstruction theories, and can in effect be found in Hochschild [2] for the associative case, in Hochschild [3] for the Lie algebra case, in Harrison [1] for the commutative associative algebra case (and in Eilenberg-MacLane [1] or MacLane [1] for the group case). Assertion 3 follows from the fact that in the cases mentioned $H^3_{\mathcal{C}}(A, M)$ is isomorphic to $R^1\mathcal{E}^2_{\mathcal{C}}(A, M)$, and the composite morphism given in 3. is the classical isomorphism (Gerstenhaber [2]). It follows that each factor is an isomorphism. This ends the proof.

For the categories of part 3 of the theorem, we have also that $\mathcal{E}^3_{\mathcal{C}}(A, M) = R^2 \operatorname{Der}_{\mathcal{C}}(A, M)$, following Rinehart-Barr [1].

Note, finally, that if all the rings and modules under consideration are graded, and if we consider only morphisms of degree zero, then all the foregoing discussion continues to hold verbatim, and the groups Der, $\mathcal{E}^2$, $\mathcal{E}^3$ all acquire natural gradations. This is still true if the gradation is by the integers Z, and the objects are complete in their natural topologies; this will be the case in the next section.

2. Successive approximations and the rigidity theorem

Throughout this section, an equationally defined category $\mathcal{C}$ of algebras over a coefficient ring k will be understood as a category of interest. Examples include associative, Lie, Jordan, and commutative associative algebras, but nilpotent algebras are excluded unless an index of nilpotence is fixed. (The category of groups is also excluded.) We shall use *ring* and *algebra* interchangeably, but all morphisms will be understood to be k-module morphisms. We give now for the purposes of this paper

DEFINITION 1. A filtered ring A is a ring together with a decreasing sequence of submodules indexed by the integers,

$$A \supset \cdots \supset F_{-1}A \supset F_0A \supset F_1A \supset \cdots ,$$

such that $F_iA \cdot F_jA \subset F_{i+j}A$. A *complete* filtered ring is one which is complete in the sense of Eilenberg-Moore [1], or equivalently, exhaustive, separated, and complete in the sense of Bourbaki [1]. (Note that it carries a natural topology.)

In an equationally defined category, every ring has a natural completion.

DEFINITION 2. The weak associated graded ring wgr A of a filtered ring A is the direct sum $\bigoplus_{i=-\infty}^{\infty} F_iA/F_{i+1}A$, with the obvious k-module and multiplicative structures. A *graded ring* is a filtered ring B together with sections $s_i\colon F_iB/F_{i+1}B \to F_iB$ such that $\bigoplus s_i\colon \operatorname{wgr} B \to B$ is a morphism of k-algebras. The *associated graded ring* gr A of a filtered ring A is the completion of wgr A.

Warning. The definitions of graded ring and associated graded ring adopted here do not coincide with the usual ones (cf. MacLane [1]). Ordinarily a graded ring is one such that $\bigoplus s_i$ is an isomorphism. Here a graded ring is a special case of a filtered ring and therefore also has a completion which is again obviously graded. For example, if R is a ring and t a variable, then both the polynomial ring $R[t]$ and the power series ring $R[[t]]$ over R, with the obvious sections, are graded rings, the latter being the completion of the former.

If A is complete, then A and gr A are homeomorphic as topological spaces by a homeomorphism which in general is not canonical (cf. Gerstenhaber [1, Ch. III]).

Henceforth we shall tacitly assume that all filtered (*and hence also all graded*) *rings and modules under consideration are complete.*

If B is a graded ring, then we shall denote the submodule $s_i(F_iB/F_{i+1}B)$ of B by B_i and call it the i^{th} homogeneous part.

DEFINITION 3. A bifiltered ring B is one carrying two filtrations, F and F', such that setting $F_mF'_iB = F_mB \cap F'_iB$ makes each F_mB and each F'_iB into a (complete) filtered module. A filtered graded ring is one in which the second filtration is a gradation; in this case we write F_mB_i for $F_mF'_iB$.

Given a ring A, we shall denote by $A((t))$ the ring of formal power series $\sum a_it^i$ in which finitely many negative powers of t may appear.

DEFINITION 4. Let A be a filtered ring. Then App A is the filtered graded subring of $A((t))$ consisting of those series $\sum a_it^i$ for which $a_i \in F_iA$ for all i, with gradation induced by that of $A((t))$ and filtration defined by

$$F_m \operatorname{App} A = \{\sum a_it^i \mid a_i \in F_{m+i}A\} .$$

Note that the filtration on App A is non-negative, i.e., $F_0 \operatorname{App} A = \operatorname{App} A$. It follows that the $F_n \operatorname{App} A$ are ideals for all n. In what follows, we may write N-filtered for non-negatively filtered.

DEFINITION 5. The n^{th} approximation to a filtered ring A is the ring $\operatorname{App}_n A = \operatorname{App} A/F_{n+1} \operatorname{App} A$, with filtration and gradation inherited from App A.

Note that $\operatorname{App}_0 A$ is identical with gr A. The ring App A has a natural k-module endomorphism σ defined by $\sigma(\sum a_it^i) = \sum a_it^{i-1}$; we shall call σ the *shift*. If x and y are in App A then we have $\sigma(xy) = (\sigma x)y = x(\sigma y)$, and further, $F_m \operatorname{App} A = \sigma^m \operatorname{App} A$. Each of the rings $\operatorname{App}_n A$ inherits from App A an endomorphism with these properties; we shall continue to call it the shift and to denote it by σ.

LEMMA 1. *Let A be a filtered ring. Then,*

1. *The shift is a continuous k-module endomorphism of* $\operatorname{App} A$ *and of* $\operatorname{App}_n A$ *for every* n. *It is a monomorphism on* $\operatorname{App} A$; *its kernel on* $\operatorname{App}_n A$ *is* $F_n \operatorname{App}_n A$.

2. *There is a natural sequence of projections*

$$\operatorname{App}_0 A \xleftarrow{\pi_1} \operatorname{App}_1 A \xleftarrow{\pi_2} \operatorname{App}_3 A \longleftarrow \cdots .$$

Denoting the shift on $\operatorname{App}_n A$ *by* σ_n, *we have* $\pi_n \sigma_n = \sigma_{n-1} \pi_n$. *The kernel of* π_n *is* $F_n \operatorname{App}_n A$ *and* $\operatorname{proj\,lim} \operatorname{App}_n A$ *is naturally isomorphic with* $\operatorname{App} A$. *Further*, $\operatorname{proj\,lim} \sigma_n = \sigma$, *the shift of* $\operatorname{App} A$.

3. *Let* $(\sigma - 1)$ *denote the ideal of* $\operatorname{App} A$ *consisting of all elements of the form* $\sigma x - x$, $x \in \operatorname{App} A$, *and let* $(\operatorname{App} A)/(\sigma - 1)$ *be filtered by letting* $F_i[(\operatorname{App} A)/(\sigma - 1)]$ *be the image of* $(\operatorname{App} A)_i$ (*the* i^{th} *grading submodule*) *under the natural projection. Then* A *is canonically isomorphic with* $(\operatorname{App} A)/(\sigma - 1)$.

PROOF. Only 3 needs proof; and, for this, it is sufficient to observe that the morphism $\operatorname{App} A \to A$ given by $\sum a_i t^i \to \sum a_i$ (the sum necessarily converging) has kernel precisely $(\sigma - 1)$.

DEFINITION 6. A *shift ring* (B, σ) is an N-filtered graded ring B with a continuous k-module *shift* endomorphism σ, such that $\sigma(F_m B_i) = F_{m+1} B_{i-1}$, and such that $\sigma a \cdot b = a \cdot \sigma b = \sigma(ab)$. If, in addition, for some non-negative integer n we have $\ker \sigma = F_n B$, then B will be called an (abstract) n^{th} approximation ring. A morphism of shift rings will be assumed to carry the shift of the first into that of the second.

THEOREM 1. 1. App *and* App_n *are covariant functors from the category of filtered rings to that of shift rings*; $\operatorname{App}_n A$ *is an abstract* n^{th} *approximation ring.*

2. *A necessary and sufficient condition that a shift ring* (B, σ) *be of the form* $\operatorname{App} A$ *for some filtered ring* A *is that the shift endomorphism* σ *be a monomorphism.*

3. *Suppose given for every* $n \geqq 0$ *an* n^{th} *approximation ring* $(B^{(n)}, \sigma^{(n)})$ *together with an epimorphism* $(B^{(n+1)}, \sigma^{(n+1)}) \to (B^{(n)}, \sigma^{(n)})$ *of shift rings whose kernel is* $F_{n+1} B^{(n+1)}$. *Then* $\operatorname{proj\,lim} (B^{(n)}, \sigma^{(n)}) = (B, \sigma)$ *is a shift ring whose shift is a monomorphism*; *hence it is of the form* $\operatorname{App} A$ *for some* A.

PROOF. Only 2 needs proof. Observe that the elements of B of the form $\sigma x - x$ form an ideal $(\sigma - 1)$; the desired A is then just $B/(\sigma - 1)$.

Lemma 1 and Theorem 1 together assert that knowledge of a filtered ring is equivalent to knowledge of a graded ring $B^{(0)}$ and a sequence

$$(B^{(0)}, \sigma^{(0)}) \longleftarrow (B^{(1)}, \sigma^{(1)}) \longleftarrow \cdots$$

of abstract n^{th} approximation rings; $\sigma^{(0)}$ is the zero map.

DEFINITION 7. Given a shift ring (B, σ), let $(\mathbf{sh}\, B, \mathbf{sh}\, \sigma)$ denote the shift ring defined by setting $F_m(\mathbf{sh}\, B)_i = F_{m-1}B_{i+1}$. We shall also denote by sh the mapping from B to $\mathbf{sh}\, B$ which is pointwise the identity. (Note that sh is a module isomorphism and a homeomorphism, but does not preserve the filtration and gradation.) Multiplication in $\mathbf{sh}\, B$ is defined by setting $\mathrm{sh}\, a \cdot \mathrm{sh}\, b = \mathrm{sh}\,[\sigma(ab)]$; we set $(\mathbf{sh}\, \sigma)a = \mathrm{sh}\,(\sigma a)$.

It is important to note that every graded ring A is also filtered by setting $F_0A = A$, $F_1A = 0$, and hence is also trivially a shift ring with $\sigma = 0$. The multiplication in $\mathbf{sh}\, A$ is then identically zero.

LEMMA 2. *Given an arbitrary shift ring* (B, σ), $\mathbf{sh}\, B$ *becomes in a natural way a filtered graded B-structure by setting* $a \cdot \mathrm{sh}\, b = \mathrm{sh}\,(ab) = (\mathrm{sh}\, a) \cdot b$.

The proof is left to the reader. Henceforth, by abuse of language, we may speak of a *shift ring* B, the shift being understood. Note that the shift functor $\mathbf{sh}$ is an exact covariant functor from the category of shift rings to itself. In particular, it can be iterated; $\mathbf{sh}^n\, B$ is well defined for any shift ring B.

Given an n^{th} approximation ring $(B^{(n)}, \sigma^{(n)})$, a fundamental question is whether or not it is of the form $\mathrm{App}_n\, A$ for some filtered ring A, the shift endomorphism of $\mathrm{App}_n\, A$ being, of course, that induced from $\mathrm{App}\, A$. In view of Theorem 1, the first step is to determine under what conditions there exists an $(n+1)^{\text{st}}$ approximation ring $(B^{(n+1)}, \sigma^{(n+1)})$ from which $(B^{(n)}, \sigma^{(n)})$ can be obtained by reduction modulo $F_{n+1}B^{(n+1)}$. To this end, note that if $B^{(n)}$ is a fixed n^{th} approximation ring, and if for every $m < n$, $B^{(m)}$ denotes the m^{th} approximation ring obtained by reducing $B^{(n)}$ modulo $F_{m+1}B^{(n)}$, then we have $F_mB^{(n)} \cong \mathbf{sh}^m\, B^{(n-m)}$, and the sequence

$$\mathbf{E}_{n,m}\colon 0 \longrightarrow \mathbf{sh}^m\, B^{(n-m)} \longrightarrow B^{(n)} \longrightarrow B^{(m-1)} \longrightarrow 0$$

is exact. In particular, for $m = n = 1$, we have

$$\mathbf{E}_{1,1}\colon 0 \longrightarrow \mathbf{sh}\, B^{(0)} \longrightarrow B^{(1)} \xrightarrow{\pi_1} B^{(0)} \longrightarrow 0\ ,$$

where $\mathbf{sh}\, B^{(0)}$ is a zero ring and a module over $B^{(0)}$ isomorphic to $B^{(0)}$ itself except for the gradation. Since the concepts of zero ring and singular ring coincide in $\mathcal{C}$, $\mathbf{E}_{1,1}$ is a singular extension of $B^{(0)}$ by $\mathbf{sh}\, B^{(0)}$, and thus represents an element of $\mathcal{E}^2(B^{(0)}, \mathbf{sh}\, B^{(0)})$. Conversely, given an element of $\mathcal{E}^2(B^{(0)}, \mathbf{sh}\, B^{(0)})$ represented by the sequence $\mathbf{E}_{1,1}$, there is a unique structure of shift ring on $B^{(1)}$ such that π_1 is a morphism of shift rings with kernel $F_1B^{(1)}$. If we define such pairs $(B^{(1)}, \pi_1\colon B^{(1)} \to B^{(0)})$ and $(\overline{B}^{(1)}, \overline{\pi}_1)$ to be equivalent if there is an isomorphism $\varphi\colon B^{(1)} \to \bar{B}^{(1)}$ such that $\bar{\pi}_1\varphi = \pi_1$, then the equivalence classes are in effect just the elements of $\mathcal{E}^2(B^{(0)}, \mathbf{sh}\, B^{(0)})$. Accordingly, given a graded ring $B^{(0)}$, we may call $\mathcal{E}^2(B^{(0)}, \mathbf{sh}\, B^{(0)})$ its group of *first order infinitesimal deformations* to a filtered ring, and more generally, call $\mathcal{E}^2(B^{(0)}, \mathbf{sh}^n B^{(0)})$ the group of n^{th} order infinitesimal deformations for all $n \geqq 1$.

Consider now the sequences $\mathbf{E}_{n,n-1}$ and $\mathbf{E}_{n,0}$. Applying $\mathbf{sh}$ to the former, we can compose them to get

$$(\mathbf{E}_{n,0})\circ(\mathbf{sh}\,\mathbf{E}_{n,n-1})\colon 0 \longrightarrow \mathbf{sh}^{n+1} B^{(0)} \longrightarrow \mathbf{sh}\, B^{(n)} \longrightarrow B^{(n)} \longrightarrow B^{(0)} \longrightarrow 0$$
$$\begin{array}{ccc} \mathbf{sh}\, B^{(n)} & & B^{(n)} \\ & \searrow \quad \nearrow & \\ & \mathbf{sh}\, B^{(n-1)} & \\ & \nearrow \quad \searrow & \\ 0 & & 0\,. \end{array}$$

Note that in effect $B^{(0)}$ is merely a graded ring, since $F_1B^{(0)} = 0$.

THEOREM 2. *Let a shift ring* $(B^{(n)}, \sigma^{(n)})$ *be given, and let* $(\mathbf{E}_{n,0})\circ(\mathbf{sh}\,\mathbf{E}_{n,n-1})$ *be considered as an exact sequence of graded rings (letting the filtrations of* $B^{(n)}$ *and* $\mathbf{sh}\, B^{(n)}$ *be forgotten). Then*

1. $(\mathbf{E}_{n,0})\circ(\mathbf{sh}\,\mathbf{E}_{n,n-1})$ *is an admissible sequence of graded rings in the sense of* § 1, *and therefore represents an element, henceforth denoted* $\operatorname{obs}(B^{(n)}, \sigma^{(n)})$, *of* $\mathcal{E}^3(B^{(0)}, \mathbf{sh}^{n+1} B^{(0)})$. *In particular, for* $n = 1$, *there is a map*

$$\operatorname{obs}\colon \mathcal{E}^2(B^{(0)}, \mathbf{sh}\, B^{(0)}) \longrightarrow \mathcal{E}^3(B^{(0)}, \mathbf{sh}^2 B^{(0)})\ .$$

(The map is quadratic and in general is not a morphism of k-modules.)

2. *The necessary and sufficient condition that there exist an* $n+1^{\text{st}}$ *approximation ring* $(B^{(n+1)}, \sigma^{(n+1)})$ *together with a morphism* $\pi_{n+1}\colon B^{(n+1)} \to B^{(n)}$ *of shift rings with kernel* $F_{n+1}B^{(n+1)}$ *is that* $\operatorname{obs}(B^{(n)}, \sigma^{(n)}) = 0$. *When that is so, we shall call the pair* $(B^{(n+1)}, \pi_{n+1})$ *a solution for* $B^{(n)}$.

3. *Assume that* $\operatorname{obs}(B^{(n)}, \sigma^{(n)}) = 0$, *and define solutions* $(B^{(n+1)}, \pi_{n+1})$ *and* $(\bar{B}^{(n+1)}, \bar{\pi}_{n+1})$ *to be equivalent if there exists an isomorphism of shift rings* $\varphi\colon B^{(n+1)} \to \bar{B}^{(n+1)}$ *such that* $\bar{\pi}\varphi = \pi$. *Then the equivalence classes of solutions form in a natural way a principal homogeneous space over* $\mathcal{E}^2(B^{(0)}, \mathbf{sh}^{n+1} B^{(0)})$.

PROOF. We leave 1 to the reader. As for 2 and 3, note that as far as the structure of *graded* ring on $B^{(n+1)}$ is concerned, the assertions follow from Theorem 5 of § 1. However, the filtration on $B^{(n+1)}$ is completely determined by that on $B^{(n)}$, and the condition that π be a morphism of filtered rings with kernel $F_{n+1}B^{(n+1)}$; this ends the proof.

Combining the preceding with Theorem 1, one concludes that given an abstract n^{th} approximation ring $B^{(n)}$, if the successive obstructions to constructing a sequence of approximation rings and projections $B^{(n)} \leftarrow B^{(n+1)} \leftarrow \cdots$ all vanish, then $B^{(m)} = \operatorname{App}_m A$ for some A and all m; in particular, $B^{(n)} =$

$\mathrm{App}_n A$. (For $m < n$ we set, as usual, $B^{(m)} = B^{(n)}/F_{m+1}B^{(n)}$.) In particular, we have

COROLLARY 1. *Let* $(B^{(n)}, \sigma^{(n)})$ *be an* n^{th} *approximation ring. If*

$$\mathcal{E}^3(B^{(0)}, \mathbf{sh}^i\, B^{(0)}) = 0$$

for all $i > n$, *then there exists a filtered ring* A *such that* $B^{(n)} = \mathrm{App}_n A$.

Let an n^{th} approximation ring $B^{(n)}$ be fixed. In the set of pairs (A, ω), where A is a filtered ring and $\omega\colon \mathrm{App}_n A \to B^{(n)}$ an isomorphism of shift rings, let an equivalence relation be defined by setting $(A, \omega) \sim (A', \omega')$ if there exists a morphism $\varphi\colon A \to A'$ of filtered rings such that $\omega = \omega' \mathrm{App}_n \varphi$. Let $\mathrm{Fil}(B^{(n)})$ (the space of filtered rings with associated n^{th} approximation ring $B^{(n)}$) be the set of equivalence classes of these pairs. In the special case $n = 0$ we state

DEFINITION 7. A graded ring $B^{(0)}$ is *rigid* if $\mathrm{Fil}(B^{(0)})$ is reduced to a single element.

Observe that, if $(B^{(n)}, \sigma^{(n)})$ is an n^{th} approximation ring, then forming the associated graded ring $\mathrm{gr}\, B^{(n)}$ to $B^{(n)}$ as a filtered ring, we have a natural isomorphism of shift rings $\mathrm{App}\, B^{(0)} \cong \mathrm{gr}\, B^{(n)}$. (Note that $\mathrm{gr}\, B^{(n)}$ is in fact a doubly graded ring with an obvious shift endomorphism.) If $B^{(n)} = \mathrm{gr}\, B^{(n)}$, then we shall say that $B^{(n)}$ is graded. For such an n^{th} approximation ring, we have $\mathrm{obs}_n(B^{(n)}, \sigma) = 0$, for $\mathrm{App}_{n+1} B^{(0)}$ is a solution, and indeed a canonical one. It follows that the equivalence classes of solutions are in canonical one-one correspondence with the elements of $\mathcal{E}^2(B^{(0)}, \mathbf{sh}^{n+1} B^{(0)})$.

COROLLARY 2. *Let* $B^{(n)}$ *be a graded* n^{th} *approximation ring. If*

$$\mathcal{E}^2(B^{(0)}, \mathbf{sh}^i\, B^{(0)}) = 0$$

for $i > n + 1$, *then the elements of* $\mathrm{Fil}(B^{(n)})$ *are in canonical one-one correspondence with the elements of* $\mathcal{E}^2(B^{(0)}, \mathbf{sh}^{n+1} B^{(0)})$; *should the latter vanish, then* $\mathrm{Fil}(B^{(n)})$ *is reduced to a single element. In particular, if*

$$\mathcal{E}^2(B^{(0)}, \mathbf{sh}^i\, B^{(0)}) = 0$$

for all $i > 0$, *then the graded ring* $B^{(0)}$ *is rigid.*

The group of automorphisms, $\mathrm{Aut}\, B^{(n)}$, of an n^{th} approximation ring $B^{(n)}$ operates on $\mathrm{Fil}(B^{(n)})$ as follows. If τ is an automorphism, A a filtered ring, and $\omega\colon \mathrm{App}_n A \to B^{(n)}$ an isomorphism, then set $\tau(A, \omega) = (A, \tau\omega)$. The operation preserves equivalence classes and so defines the desired action. The action on $\mathrm{Fil}(B^{(n)})$, however, need not be effective. Let $\mathrm{Triv}\, B^{(n)}$ denote the normal subgroup consisting of all elements of $\mathrm{Aut}\, B^{(n)}$ whose action is trivial. The *modular group*, $\mathrm{Mod}\, B^{(n)}$ of an n^{th} approximation ring $B^{(n)}$ may be defined to be $\mathrm{Aut}\, B^{(n)}/\mathrm{Triv}\, B^{(n)}$.

3. The ray determined by a filtered ring

Henceforth, considering App A as a collection of power series $\sum a_i t^i$, the shift endomorphism will be denoted by t^{-1}.

DEFINITION 1. Let A be a filtered ring, and λ be any element of the coefficient ring k. Then $\mathrm{Def}_\lambda A$ will denote the ring App $A/(t^{-1} - \lambda)$, filtered by letting F_i Def A be the image of $(\mathrm{App}\, A)_i$ under the canonical projection.

The definition is extended to shift rings (B, σ) first by observing that in App A there is a k-module endomorphism App σ defined by $\mathrm{App}\,\sigma(\sum b_i t^i) = \sum (\sigma b_i) t^{i+1}$, and then by observing that the ideal $(t^{-1} - \lambda)$ is stable under App σ, so that the latter induces a k-module endomorphism $\mathrm{Def}_\lambda \sigma$ in $\mathrm{Def}_\lambda B$. The pair $(\mathrm{Def}_\lambda B, \mathrm{Def}_\lambda \sigma)$ is then again a shift ring. Considering for every fixed n the class of n^{th} approximation rings, and similarly the class of rings of the form App A, as a category with morphisms the morphisms of shift rings (i.e., considering these classes as full subcategories of the category of shift rings), we have

PROPOSITION 1. *The functor* Def_λ *is a functor from each of the following categories to itself: Filtered rings, N-filtered rings, non-positively filtered rings* (i.e., *filtered rings* A *with* $F_1 A = 0$), *filtered graded rings, shift rings,* n^{th} *approximation rings, rings of the form* App A.

The list is not to be considered exhaustive. Briefly, the functor Def_λ inserts a ring of any of the categories which we have considered into a one-parameter family parameterized by the elements of the coefficient ring k. Considering this we have

PROPOSITION 2. *The functor* Def_1 *is the identity functor,* $\mathrm{Def}_0 = \mathrm{gr}$, *and* $\mathrm{Def}_\lambda \mathrm{gr} = \mathrm{gr}\, \mathrm{Def}_\lambda = \mathrm{gr}$ *for all* λ *in* k.

The proposition asserts, in particular, that given any graded ring $B^{(0)}$, every point of Fil $B^{(0)}$ can be joined by a ray to the *origin*, namely $B^{(0)}$ itself.

Given a filtered ring A, there is a canonical morphism $F_0 A \to \mathrm{App}\, A$ defined by $a \to a t^0$. If A is N-filtered, then the composite morphism

$$A = F_0 A \longrightarrow \mathrm{App}\, A \longrightarrow \mathrm{App}\, A/(t^{-1} - \lambda) = \mathrm{Def}_\lambda A$$

will be denoted by Δ_λ. If A is a ring with non-positive filtration, in which case App A is the set of finite sums $a_0 + a_{-1} t^{-1} + \cdots + a_{-n} t^{-n}$, $a_{-i} \in F_{-i} A$, then the kernel of the morphism App $A \to A$ given by $\sum a_{-i} t^{-i} \to \sum \lambda^i a_{-i}$ contains the ideal $(t^{-1} - \lambda)$. There is therefore a canonical morphism

$$\Delta^\lambda\colon \mathrm{Def}_\lambda A \longrightarrow A\,.$$

PROPOSITION 3. *On the categories of N-filtered rings, shift rings,* n^{th}

approximation rings, and rings of the form App A, *the pair* $(\mathrm{Def}_\lambda, \Delta_\lambda)$ *is a natural transformation of the identity functor for every* λ *in* k, *while on the category of rings with non-positive filtration, the identity is a natural transformation of* $(\mathrm{Def}_\lambda, \Delta^\lambda)$.

A ring A will be said to have a *short filtration* if $A = F_0A$ and $F_2A = 0$. We have then a singular extension

$$0 \longrightarrow F_1A \longrightarrow F_0A \longrightarrow F_0A/F_1A \longrightarrow 0 ,$$

and $\mathrm{Fil}(\mathrm{gr}\, A)$ can be identified with $\mathcal{E}^2(F_0A/F_1A, F_1A)$. We can therefore consider $\mathrm{Def}_\lambda\, \beta$ for an element β of the latter group. Note also that $\mathcal{E}^2(F_0A/F_1A, F_1A)$ is a k-module, so that $\lambda\beta$ is defined for all λ in k.

PROPOSITION 4. *Let A be a ring with a short filtration. Then, for all β in $\mathcal{E}^2(F_0A/F_1A, F_1A)$, we have* $\mathrm{Def}_\lambda\, \beta = \lambda\beta$.

The proposition holds also for shift rings with short filtration and for n^{th} approximation rings with $n = 1$.

PROPOSITION 5. 1. *For any filtered ring A, we have* $\mathrm{App}\, \mathrm{Def}_\lambda A = \mathrm{Def}_\lambda\, \mathrm{App}\, A$.

2. *If* $B^{(0)} \xleftarrow{\pi_1} B^{(1)} \xleftarrow{\pi_2} B^{(2)} \longleftarrow \cdots$ *is a sequence of n^{th} approximation rings with kernel* $\pi_n = F_nB^{(n)}$, *then* $\mathrm{Def}_\lambda\, \mathrm{proj\,lim}\, B^{(n)} = \mathrm{proj\,lim}\, \mathrm{Def}_\lambda\, B^{(n)}$.

PROOF. 1. Given an N-filtered ring A, let $\mathrm{App}_+ A$ denote the subring of App A consisting of those series of the form $\sum a_it^i$ with $a_i = 0$ for $i < 0$. Then it is the case that $\mathrm{App}\, A/(t^{-1} - \lambda)$ is identical with $\mathrm{App}_+ A/(t^{-1} - \lambda)$; that is, every element of App A is congruent modulo the ideal $(t^{-1} - \lambda)$ to an element of $\mathrm{App}_+ A$.

Consider now $\mathrm{Def}_\lambda\, \mathrm{App}\, A$. Let the variable in App A be denoted by t, and that introduced in the definition of Def_λ be denoted by u. Then $\mathrm{Def}_\lambda\, \mathrm{App}\, A$ is the quotient of the ring of series of the form $\alpha = \sum a_{ij}t^iu^j$, $a_{ij} \in F_{i+j}A$ by the ideal $(u^{-1} - \lambda)$. In each series there are only a finite number of terms with negative powers of t or u; we write

$$\alpha = \sum\nolimits_{i \geqq -m} \sum\nolimits_{j \geqq -n} a_{ij}t^iu^j ,$$

where m and n depend on α. Since the filtration of App A is non-negative, we need, by the remark of the preceding paragraph, consider only those series $\sum a_{ij}t^iu^j$ in which $j \geqq 0$. On the other hand, one may readily verify that $\mathrm{App}\, A\, \mathrm{Def}_\lambda\, A$ *is the quotient of the ring of series of the form* $\sum_{i \geqq -m} \sum_{j \geqq -n} b_{ij}t^iu^j$ *with* $b_{ij} \in F_iA$ *and* $b_{ij} = 0$ for $j > i$, by the ideal $(t^{-1} - \lambda)$. We may write

$$\mathrm{Def}_\lambda\, \mathrm{App}\, A = \{\sum\nolimits_{i \geqq -m} \sum\nolimits_{j \geqq 0} a_{ij}t^iu^j \mid a_{ij} \in F_{i+j}A\}/(u^{-1} - \lambda) ,$$

and

$$\text{App Def}_\lambda\, A = \{\textstyle\sum_{i \geqq -m} \sum_{\substack{j \geqq -n \\ j<i}} b_{ij} t^i u^j \mid b_{ij} \in F_i A\}/(t^{-1} - \lambda)\ .$$

The mapping $at^iu^j \rightarrow at^{i+j}u^i$, $a \in F_{i+j}A$, extended in the obvious way to the power series, induces an isomorphism from the first to the second. We leave 2 to the reader.

PROPOSITION 6. 1. *Let* (B, σ) *be a shift ring. Then for every* λ *in* k, *we have* $\Delta_\lambda \sigma = \lambda\, \text{Def}_\lambda(\sigma) \cdot \Delta_\lambda$ *where, on the right,* λ *denotes the* k*-module endomorphism* $x \rightarrow \lambda x$ *of* $\text{Def}_\lambda B$.

2. *If* $(B^{(n)}, \sigma^{(n)})$ *is an* n^{th} *approximation ring, then for every* λ *in* k, *we have*

$$\text{obs}\,(\text{Def}_\lambda B^{(n)}, \text{Def}_\lambda \sigma^{(n)}) = \lambda^{n+1}\, \text{obs}\,(B^{(n)}, \sigma^{(n)})\ .$$

PROOF. *Part* 1 is obvious. Using it, we have the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbf{sh}^{n+1} B^{(0)} & \longrightarrow & \mathbf{sh}\, B^{(n)} & \searrow \mathbf{sh}\, B^{(n-1)} \nearrow & B^{(n)} & \longrightarrow & B^{(0)} & \longrightarrow & 0 \\
 & & \downarrow \lambda_{n+1} & & \downarrow \lambda\, \mathbf{sh}\,(\Delta_\lambda) & \downarrow \lambda\, \mathbf{sh}\,(\Delta_\lambda) & \downarrow \Delta_\lambda & & \| & & \\
0 & \longrightarrow & \mathbf{sh}^{n+1} B^{(0)} & \longrightarrow & \mathbf{sh}\, \text{Def}_\lambda B^{(n)} & \longrightarrow & \text{Def}_\lambda B^{(n)} & \longrightarrow & B^{(0)} & \longrightarrow & 0 \\
 & & & & \searrow & \downarrow & \nearrow & & & & \\
 & & & & & \mathbf{sh}\, \text{Def}_\lambda B^{(n-1)} & & & & &
\end{array}$$

where λ also stands, as before, for the k-module endomorphism $x \rightarrow \lambda x$.

Part 2 follows immediately.

The properties of the functor Def_λ given in Propositions 1 to 6 are more than sufficient to characterize it axiomatically.

THEOREM 1. *Let* λ *be an arbitrary element of the coefficient ring* k *and* De_λ *be a functor on the category of filtered rings carrying the categories of* N*-filtered, shift,* n^{th} *approximation rings, and rings of the form* $\text{App}\, A$ *into themselves; and suppose that on the latter categories there is given for every* A *a morphism* del_λ: $A \rightarrow \text{De}_\lambda A$ *such that the pair* $(\text{De}_\lambda, \text{del}_\lambda)$ *is a natural transformation of the identity. Suppose further that*

(1) $\text{gr De}_\lambda = \text{De}_\lambda \text{gr} = \text{gr}$;

(2) $\text{App De}_\lambda = \text{De}_\lambda \text{App}$;

(3) *if* A *is a ring with short filtration and* $\beta \in \mathcal{E}^2(A/F_1A, F_1A)$, *then* $\text{De}_\lambda \beta = \lambda\beta$;

(4) *if* (B, σ) *is a shift ring, then* $\text{del}_\lambda \cdot \sigma = \lambda \cdot \text{De}_\lambda(\sigma) \cdot \text{del}_\lambda$; *and*

(5) *if* $B^{(0)} \leftarrow B^{(1)} \leftarrow \cdots$ *is a sequence of* n^{th} *approximation rings as in Proposition* 5, *then* $\text{De}_\lambda \text{proj lim}\, B^{(n)} = \text{proj lim De}_\lambda B^{(n)}$.

Then De_λ *coincides with* Def_λ, *and* del_λ *with* Δ_λ *(up to isomorphism of functors).*

PROOF. It is in effect sufficient to prove for every n that De_λ and Def_λ coincide on n^{th} approximation rings. For $n = 1$, this follows from (3). Supposing this true for $n = m$, let an $m + 1^{\text{st}}$ approximation ring $(B^{(m+1)}, \sigma^{(m+1)})$ be given, denote as usual $B^{(m+1)}/F_{m+1}B^{(m+1)}$ by $B^{(m)}$, and given an element b of $B^{(m+1)}$, let πb denote its image in $B^{(m)}$. By means of the composite morphism $B^{(m+1)} \xleftrightarrow{\pi} B^{(m)} \longleftrightarrow \mathrm{Def}_\lambda B^{(m)}$, the ring $\mathbf{sh}\,\mathrm{Def}_\lambda B^{(m)}$ (which carries a $\mathrm{Def}_\lambda B^{(m)}$ structure) acquires a structure over $B^{(m+1)}$. We may therefore form

$$C = B^{(m+1)} + \mathbf{sh}\,\mathrm{Def}_\lambda B^{(m)} ,$$

and denoting by i the injection $\mathbf{sh}\,\mathrm{Def}_\lambda B^{(m)} \to \mathrm{De}_\lambda B^{(m+1)}$, there is a morphism $\varphi\colon C \to \mathrm{De}_\lambda B^{(m+1)}$ given by $\varphi(a, x) = \mathrm{del}_\lambda a + i(\mathbf{sh}\,\pi x)$. It is evident that $\mathrm{gr}\,\varphi$ is an epimorphism, whence so is φ, and $\mathrm{De}_\lambda B^{(m+1)}$ is therefore the quotient of C by some ideal. The same is true of $\mathrm{Def}_\lambda B^{(m+1)}$ and condition (4) shows that in both cases the ideal is the set of all elements of C of the form

$$\big(i(\mathbf{sh}\,\bar{a}),\ -\lambda(\mathrm{Def}_\lambda \sigma)(\Delta_\lambda \bar{a})\big) ,$$

where $\bar{a}$ is an arbitrary element of $B^{(m)}$. The rest is trivial.

Note now that, since the functors Def_λ are all from the category of filtered rings to itself, the composition of two such is meaningful.

COROLLARY. $\mathrm{Def}_\lambda \mathrm{Def}_\mu = \mathrm{Def}_{\lambda\mu}$.

PROOF. It is sufficient to observe that $\mathrm{Def}_\lambda \mathrm{Def}_\mu$ and $\mathrm{Def}_{\lambda\mu}$ have the same axiomatic properties.

It follows from the corollary that, if λ is a unit in k, then $\mathrm{Def}_\lambda A$ is canonically isomorphic to A for all filtered rings A.

We consider finally the question of deforming general filtered rings.

DEFINITION 2. A *deformation ring* (D, S) is a pair consisting of a bifiltered ring D with filtrations F and F', and a module endomorphism $S\colon D \to D$ with the properties

(1) S is a monomorphism,

(2) $S(F_iF'_jD) = F_{i+1}F'_{j-1}D$, and

(3) $(Sx)y = S(xy) = x(Sy)$ for all x and y in D.

THEOREM 2. *Let (D, S) be a deformation ring, and let* $\mathrm{gr}'\,D$ *denote the associated graded ring relative to the F' filtration, and set* $\sigma = \mathrm{gr}'\,S$. *Then* $\mathrm{gr}'\,D$ *is a ring of the form* $\mathrm{App}\,A$ *for some A, with σ being the shift. Further, A is just D/F_1D. Denoting the ideal of all elements of D of the form $Sx - \lambda x$ by $(S - \lambda)$, we have canonical isomorphisms $A = D/F_1D = D/(S)$, the filtration of $D/(S)$ being induced by the F' filtration of D; likewise, the rings $D_\lambda = D/(S - \lambda)$ are all filtered, the i^{th} filtering part being the image of F'_iD modulo $(S - \lambda)$, and the associated graded rings of the D_λ are all canonically isomorphic.*

In view of Theorem 2, we state

DEFINITION 3. A filtered ring B is a deformation of a filtered ring A if there is a deformation ring (D, S) such that setting $D_\lambda = D/(S - \lambda)$, we have $D_0 = A$ and $D_1 = B$.

It follows that, if B is a deformation of A, then gr B = gr A.

Considering the approximation ring App A as a graded ring with certain additional elements of structure, we can, in a manner precisely analogous to what has gone before, consider infinitesimal deformations of App A, and define the obstructions to these. The objects so obtained, being associated with A, may be considered, respectively, the infinitesimal deformations, and obstructions to infinitesimal deformatioms of A. As before, the successive approximations to a filtered ring whose associated graded ring is App A can be considered, and if all obstructions can be passed, then a deformation ring (D, S) will have been constructed. The concept of rigidity for a filtered ring A, the space Fil A, and the modular group Mod A are defined as the corresponding objects for App A.

We have omitted from discussion various important topics including the obstruction theory, and the relationship of the present theory to the analytic deformation theory of Froelicher-Nijenhuis [1] elaborated by Kodaira-Spencer (for the most recent developments in which, see Spencer [1]). These and other matters (including work which has followed upon the publication of Gerstenhaber [1]—in particular Rim [1], Nijenhuis-Richardson [1], and Richardson [1]) will be discussed in a monograph being prepared.

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