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ASPECTS OF HARRISON'S HOMOLOGY THEORY

BY

PATRICK JOHN FLEURY
A.B., Brown University, 1964
A.M., University of Illinois, 1965

THESIS

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY PATRICK JOHN FLEURY

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BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Michael Barr Irving Reiner

In Charge of Thesis

Paul T. Bateman

Head of Department

Recommendation concurred in†

Leon McCulloh

John W. Gray

Avak Balakhanian

Joseph Rotman

Committee on Final Examination†

† Required for doctor's degree but not for master's.

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PREFACE

The study of Harrison's cohomology groups was initiated by Harrison himself in (8) and continued by Barr in (2). Both of these papers dealt only with algebras over fields although all definitions hold true over an arbitrary ring. Here we make an attempt (suggested by Professor Barr) to study Harrison's groups in general over any ring. Unfortunately, complete meaningful results seem to be obtainable only when the ring contains the rational numbers. However, with mild assumptions on the existence of units in arbitrary rings, interesting partial results are available. Further, using an idempotent arising out of the shuffles of Harrison's theory, one gets an interesting splitting of Hochschild's complex.

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THE SPLITTINGS

1. The Hochschild Complex

Let k be an arbitrary commutative ring and let A be an arbitrary commutative algebra over k. Unless expressly mentioned to the contrary, all rings are assumed to have a unit element. The Hochschild homology (and cohomology) modules of A will be defined using the following complex. In dimension n, $n \geq 0$, we let

$$S_nA = A \otimes A^{(n)} \otimes A$$

denote the A-A bimodule in which $A^{(n)}$ denotes the tensor product of A taken with itself n times. Unless expressly mentioned to the contrary, all tensor products will be taken over the base ring k.

We define an A-A linear map $\partial_n \colon S_nA \longrightarrow S_{n-1}A$ in the following way. Let $a_0 \otimes a_1 \otimes \ldots \otimes a_{n+1} \in S_nA$. Set

$$\partial_n(a_0 \otimes \ldots \otimes a_{n+1}) = a_0a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1} - a_0 \otimes a_1a_2 \otimes \ldots \otimes a_{n+1} + \ldots + (-1)^n a_0 \otimes a_1 \otimes \ldots \otimes a_na_{n+1}$$

Then, because of Cartan and Eilenberg (5), page 174, we see that $\partial_{n-1}\partial_n = 0$, and we have defined a differential of degree -1. We shall denote the entire complex thus defined by S_*A.

We define the Hochschild homology and cohomology modules of A with coefficients in the A-A bimodule M to be:

$$\mathrm{Hoch}_*(A,M) = H(S_*A \otimes_{A \otimes A} M)$$

and

$$\mathrm{Hoch}^*(A,M) = H(\mathrm{Hom}_{A\text{-}A}(S_*A,M))$$

We shall denote the n-th homology and cohomology modules by $\mathrm{Hoch}_n(A,M)$ and $\mathrm{Hoch}^n(A,M)$ respectively.

The foregoing definitions can be somewhat simplified if we restrict the bimodules in which we are allowed to take coefficients. In the present case, we shall be concerned only with symmetric bimodules. That is, we are interested in those A-A bimodules, M, for which $am = ma$ for all a in A and m in M.

1.1 Proposition: If M is a symmetric A-A bimodule, then M is isomorphic to $A\otimes_A M$.

Proof: The A-A bimodule action on $A\otimes_A M$ is given by $(a_1\otimes a_2)(a\otimes m) = a_1 a\otimes m a_2$. We define $f\colon M \longrightarrow A\otimes_A M$ by $f(m) = 1\otimes m$ and $g\colon A\otimes_A M \longrightarrow M$ by $g(a\otimes m) = am$. The f and g are clearly A-A linear and $fg(a\otimes m) = f(am) = 1\otimes am = a\otimes m$. Furthermore, $gf(m) = g(1\otimes m) = 1m = m$. Thus, we see that f and g are isomorphisms.

We shall use the above isomorphism in the following way. If M is a symmetric A-A bimodule, then

$$\begin{aligned} S_n A\otimes_{A\otimes A} M &\simeq S_n A\otimes_{A\otimes A} A\otimes_A M \\ &= A\otimes A^{(n)}\otimes A\otimes_{A\otimes A} A\otimes_A M \\ &\simeq A\otimes A^{(n)}\otimes_A M \end{aligned}$$

In a similar manner, using the adjointness of tensor and hom we establish that

$$\mathrm{Hom}_{A\otimes A}(S_n A, M) \simeq \mathrm{Hom}_A(A\otimes A^{(n)}, M)$$

Proposition 1.1 now tells us that the category of symmetric A-A bimodules is naturally equivalent to the category of left A modules.

Thus, if we now consider left A modules as symmetric A-A bimodules, and we use the remarks following proposition 1.1, we may simplify Hochschild's complex in the following way. We set $C_n A = A \otimes A^{(n)}$ and we define $\partial_n : C_n A \longrightarrow C_{n-1} A$ by

$$\begin{aligned}\partial_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = {} & a_0 a_1 \otimes a_2 \otimes \ldots \otimes a_n - a_0 \otimes a_1 a_2 \otimes \ldots \otimes a_n \\ & + \ldots + (-1)^{n-1} a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} a_n \\ & + (-1)^n a_0 a_n \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_{n-1}\end{aligned}$$

Then, as before, $\partial_{n-1}\partial_n = 0$ and then we will have, if we denote the entire complex by $C_* A$

$$\mathrm{Hoch}_*(A,M) = H(C_* A \otimes_A M)$$

and

$$\mathrm{Hoch}^*(A,M) = H(\mathrm{Hom}_A(C_* A, M))$$

From now on we shall denote the element $a_0 \otimes a_1 \otimes \ldots \otimes a_n$ by $a_0[a_1, \ldots, a_n]$ in order to conform to the notation of Barr (1).

2. Differential Graded Algebras

2.1 Definition: A differential graded algebra (U, ∂) over the commutative k-algebra A is a graded algebra U over A equipped with a graded A-module endomorphism of degree -1, $\partial : U \longrightarrow U$, such that $\partial\partial = 0$ and the Leibniz formula is satisfied. I. e.,

$$\partial(u_1 \cdot u_2) = \partial(u_1) \cdot u_2 + (-1)^{\deg u_1} \cdot u_1 \cdot \partial(u_2)$$

We shall frequently abbreviate differential graded to DG.

If we consider the complex $C_* A$, we at once note that it is merely the underlying module of A tensored over k with itself many times in each dimension. Thus, C_*A looks quite a bit like the tensor algebra of A over k. It is thus natural to ask whether or not C_*A can be made into a DG-algebra. The answer is yes, however the multiplication is much more complicated than mere tensor multiplication.

In order to describe the multiplication, we must make use, first, of the fact that C_*A is a simplicial A-module with faces and degeneracies given by:

$$d^i_n(a_0[a_1,\ldots,a_n]) = a_0[a_1,\ldots,a_ia_{i+1},\ldots,a_n] \qquad 0\le i\le n$$

$$s^i_n(a_0[a_1,\ldots,a_n]) = a_0[a_1,\ldots,a_i,1,a_{i+1},\ldots,a_n] \qquad 0\le i\le n$$

These are easily seen to satisfy the following simplicial identities (see MacLane (11)):

$$d^i_{n-1}d^j_n = d^{j-1}_{n-1}d^i_n \quad i<j \qquad\qquad s^i_{n+1}s^j_n = s^{j+1}_{n+1}s^i_n \quad i\le j$$

$$d^i_n\, s^j_{n-1} = \begin{cases} s^{j-1}_{n-2}d^i_{n-1} & i<j \\ 1 & i=j,\ j+1 \\ s^j_{n-2}d^{i-1}_{n-1} & i>j+1 \end{cases}$$

Secondly, we make note of the fact that there is a natural isomorphism between $C_nA\otimes_A C_mA$ and $C_{m+n}A$.

We define the product of a generator in dimension i and one in dimension n-i as follows. If i equals 0 or n, then

$$s_{n,0}([a_1,\ldots,a_n]) = s_{0,n}([a_1,\ldots,a_n]) = [a_1,\ldots,a_n]$$

If i is not 0 or n, then

$$s_{i,n-i}([a_1,\ldots,a_n]) = [a_1]\otimes s_{i-1,n-i}([a_2,\ldots,a_n])$$
$$+ (-1)^i[a_{i+1}]\otimes s_{i,n-i-1}([a_1,\ldots,\hat{a}_{i+1},\ldots,a_n])$$

where the sign ^ denotes an omitted factor. We, of course extend these functions A-linearly, and then this becomes the "shuffle" multiplication.

2.2 Proposition: $\partial_n s_{i,n-i}([a_1,\ldots,a_n]) =$

$s_{i-1,n-i}(\partial_i[a_1,\ldots,a_i]\otimes[a_{i+1},\ldots,a_n])$

$+ (-1)^i s_{i,n-i-1}([a_1,\ldots,a_i]\otimes\partial_{n-i}[a_{i+1},\ldots,a_n])$

Proof: There is an A-linear map $C_iA\otimes_A C_iA \longrightarrow C_iA$ which is given by $a_0[a_1,\ldots,a_i]\otimes a_0'[a_1',\ldots,a_i'] \longrightarrow a_0a_0'[a_1a_1',\ldots,a_ia_i']$. Then, if we apply this map to the shuffle map of Eilenberg and MacLane (6), we will have our shuffle multiplication. The proposition then follows from theorem 5.2 of the paper cited.

The complex C_*A, together with the above defined multiplication, is now an augmented DG-algebra. That is, there exists a map of DG-algebras from C_*A to A where A is considered as a DG-algebra with trivial grading and differential. We are very interested in the kernel of this augmentation. In order to decide what that is, we note that the map ∂_1: $C_1A \longrightarrow C_0A$ is given by $\partial_1(a_0[a_1]) = a_0a_1 - a_0a_1 = 0$. Thus ∂_1 is zero, and the kernel of the augmentation must then consist of that part of C_*A of dimension greater than or equal to one. This kernel forms a subcomplex which we shall call J_*A, or, if A is understood, sometimes merely J_*.

Now consider J_*^2A, which we define to be that subcomplex of C_*A which is formed by all non-trivial shuffles. We now set

$$Ch_*A = J_*A/J_*^2A$$

Then the differential and grading of C_*A induce a differential and grading on the quotient complex Ch_*A. We now define the n-th Harrison homology and cohomology groups of A with coefficients in the left A-module M to be:

$$\mathrm{Harr}_n(A,M) = H_n(Ch_* A \otimes_A M)$$

and
$$\mathrm{Harr}^n(A,M) = H_n(\mathrm{Hom}_A(Ch_* A,M))$$

we denote the total homology and cohomology by $\mathrm{Harr}_*(A,M)$ and $\mathrm{Harr}^*(A,M)$ respectively.

3. The Action of Σ_n

Let Σ_n denote the full permutation group on n letters and let π be an arbitrary permutation. Any such permutation will define an A-isomorphism of $A\otimes A^{(n)}$ by

$$\pi^{-1}[\bar{a}_1,\ldots,a_n] = [\bar{a}_{\pi(1)},\ldots,a_{\pi(n)}]$$

Thus, we may make C_nA, $n\geq 1$, into a $k\Sigma_n$ module where k is our ground ring. We may then consider the shuffles $s_{i,n-i}$ as elements of the group ring.

Of special importance to us will be the element E_n of $k\Sigma_n$ defined in the following manner. Let the alternating representation $\mathrm{sgn}\colon \Sigma_n \longrightarrow k$ be defined by $\mathrm{sgn}(\pi) = 1$ if π is an element of the alternating subgroup of Σ_n and -1 otherwise. Then we may linearly extend sgn to a ring homomorphism also called $\mathrm{sgn}\colon k\Sigma_n \longrightarrow k$. We now set

$$E_n = \sum_{\pi \in \Sigma_n} (\mathrm{sgn}(\pi))\cdot\pi$$

Now, if $u \in k\Sigma_n$, then $u\cdot E_n = (\mathrm{sgn}(u))\cdot E_n$.

3.1 Lemma (Barr (2)): Let $a_0[\bar{a}_1,\ldots,a_n] \in J_nA$. Then

$$\partial_n E_n(a_0[\bar{a}_1,\ldots,a_n]) = 0.$$

Furthermore, if $u \in k\Sigma_n$, and $\partial_n u(a_0[a_1,\ldots,a_n]) = 0$ for all $a_0[a_1,\ldots,a_n] \in J_n A$ and arbitrary A, then u is some multiple of E_n.

Proof: Let us compute $\partial_n E_n(a_0[a_1,\ldots,a_n])$. If π^{-1} is one of the permutations in Σ_n, then the term $a_0 a_{\pi(1)}[a_{\pi(2)},\ldots,a_{\pi(n)}]$ occurs with coefficient $\mathrm{sgn}(\pi)$ in the boundary. This term also appears as the last term in the boundary of $\pi^{-1}\sigma^{-1}(a_0[a_1,\ldots,a_n])$ where $\sigma = (1\ldots n)$. That is, it will as the last term of the boundary of

$$a_0[a_{\pi(2)},\ldots,a_{\pi(n)},a_{\pi(1)}].$$

However, we note that $\mathrm{sgn}(\sigma) = (-1)^{n-1}$, so the second time the term appears, it has coefficient $(-1)^{2n-1}(\mathrm{sgn}(\pi)) = -\mathrm{sgn}(\pi)$. Furthermore, the term $a_0[a_{\pi(1)},\ldots,a_{\pi(i)}a_{\pi(i+1)},\ldots,a_{\pi(n)}]$ appears in the boundary of $\pi^{-1}(a_0[a_1,\ldots,a_n])$ and with opposite sign in the boundary of $\pi^{-1}(i\ i+1)(a_0[a_1,\ldots,a_n])$. Thus, we see that $\partial_n E_n$ is zero.

Now let u be any element of $k\Sigma_n$, and suppose $\partial_n u(a_0[a_1,\ldots a_n]) = 0$ for all $a_0[a_1,\ldots,a_n]$ in $C_n A$ and arbitrary A. Consider any permutation π^{-1} which appears in u. The term $a_0[a_{\pi(1)},\ldots,a_{\pi(i)}a_{\pi(i+1)},\ldots,a_{\pi(n)}]$ now appears in the boundary of $u(a_0[a_1,\ldots,a_n])$. This can be cancelled in all cases only by itself with opposite sign. Such a term will only be afforded in all cases by using $\pi^{-1}(i\ i+1)$. Thus every term of the form $\pi^{-1}(i\ i+1)$ appears in u along with π^{-1}, and this is sufficient to guarantee that u will be a multiple of E_n.

Now let us suppose that our ground ring is the integers, that u is in $k\Sigma_n$, and that $\partial_n u = 0$. Then $u = mE_n$ for some integer m. But

$$\begin{aligned} u\cdot E_n &= mE_n^2 \\ &= m\cdot n!\cdot E_n \\ &= (\mathrm{sgn}(u))\cdot E_n. \end{aligned}$$

Thus we see that $\mathrm{sgn}(u) = m\cdot n!$. This observation will be very helpful to us when our ground ring is a field of characteristic zero. Unfortunately, when it is a field of characteristic p, and $n \geq p$, we must modify the case for the integers to get any information about u.

4. The Exact Sequence

Now let the ground ring k be a field. we have defined the complex Ch_*A using the exact sequence

$$(4.1) \qquad 0 \longrightarrow J_*^2A \longrightarrow J_*A \longrightarrow Ch_*A \longrightarrow 0$$

We now need to know that this sequence splits, albeit non-naturally. Now, $J_n^2A \subseteq J_nA$ is simply $A\otimes\, s_{1,n-1}A^{(n)} \dagger \dots \dagger s_{n-1,1}A^{(n)}$ and is contained in $J_nA = A\otimes A^{(n)}$. The exact sequence of k-vector spaces

$$0 \longrightarrow s_{1,n-1}A^{(n)} \dagger \dots \dagger s_{n-1,1}A^{(n)} \longrightarrow A^{(n)}$$
$$\longrightarrow A^{(n)}/\, s_{1,n-1}A^{(n)} \dagger \dots \dagger s_{n-1,1}A^{(n)} \longrightarrow 0$$

certainly has a k-splitting. Thus, since A is a k-vector space and is then projective, we find that

$$0 \longrightarrow A\otimes\, s_{1,n-1}A^{(n)} \dagger \dots \dagger s_{n-1,1}A^{(n)} \longrightarrow A\otimes A^{(n)}$$
$$\longrightarrow A\otimes\{A^{(n)}/\, s_{1,n-1}A^{(n)} \dagger \dots \dagger s_{n-1,1}A^{(n)}\} \longrightarrow 0$$

is exact and split as a sequence of A-modules. We thus see that the exact sequence (4.1) is an exact sequence of complexes for which the sequence in the n-th dimension is split. Because of the splitting, both the sequences

$$(4.2)\quad 0 \longrightarrow \mathrm{Hom}_A(Ch_n A, M) \longrightarrow \mathrm{Hom}_A(J_n A, M) \longrightarrow \mathrm{Hom}_A(J_n^2 A, M) \longrightarrow 0$$

$$(4.3)\quad 0 \longrightarrow J_n^2 A \otimes_A M \longrightarrow J_n A \otimes_A M \longrightarrow Ch_n A \otimes M \longrightarrow 0$$

are exact for any A-module M. Thus we have two short exact sequences of complexes. Now, it is clear from Cartan and Eilenberg (5), page 169, that

$$H_n(J_* A \otimes_A M) = \mathrm{Hoch}_n(A,M) = \mathrm{Tor}_n^{A \otimes A}(A,M)$$

and

$$H_n(\mathrm{Hom}_A(J_* A, M)) = \mathrm{Hoch}^n(A,M) = \mathrm{Ext}^n_{A \otimes A}(A,M)$$

Thus, if we take homology, we have two long exact sequences:

$$\cdots \xrightarrow{\partial} H_n(J_*^2 A \otimes_A M) \longrightarrow \mathrm{Tor}_n^{A \otimes A}(A,M) \longrightarrow \mathrm{Harr}_n(A,M) \xrightarrow{\partial} H_{n-1}(J_*^2 A \otimes_A M) \longrightarrow \cdots$$

$$\cdots \xrightarrow{\delta} \mathrm{Harr}^n(A,M) \longrightarrow \mathrm{Ext}^n_{A \otimes A}(A,M) \longrightarrow H_n(\mathrm{Hom}_A(J_*^2 A, M)) \longrightarrow \mathrm{Harr}^{n+1}(A,M) \xrightarrow{\delta} \cdots$$

where ∂ and δ are the connecting homomorphisms.

5. The Splittings

We are interested in those cases for which Harrison's theory is a direct summand of Hochschild's theory. In (2), Barr has shown that, if k is a field of characteristic zero, then this is the case in every dimension for all commutative algebras. We shall give a variation of his proof and then use our own techniques to find splittings in certain dimensions for fields of characteristic p.

5.1 <u>Theorem</u>: Let k be a field of characteristic p. There are natural transformations of functors

$$\Phi_i(A,M)\colon \mathrm{Hoch}_i(A,M) \longrightarrow \mathrm{Harr}_i(A,M)$$

$$\Phi^i(A,M)\colon \mathrm{Harr}^i(A,M) \longrightarrow \mathrm{Hoch}^i(A,M)$$

If $1\leq i\leq p-1$, then $\Phi_i(A,M)$ is a split epimorphism and $\Phi^i(A,M)$ is a split monomorphism. If k happens to be a field of characteristic zero, then these splittings exist in each dimension.

The fact that there exist natural transformations of functors is clear since Harrison's complex was defined as a quotient of Hochschild's complex in a functorial manner. For the rest, we shall show that there exist projections $e_i\colon J_iA \longrightarrow Ch_iA$ which are also natural transformations and which split for $1\leq i\leq p-1$. We shall make use of several lemmas.

5.2 <u>Lemma</u>: Let $1\leq i\leq n$. Let $a_0[a_1,\dots,a_{n+1}] \in J_{n+1}A$. Then

$$\partial_{n+1}(a_0[a_1,\dots,a_{n+1}]) = (\partial_{i+1}(a_0[a_1,\dots,a_{i+1}])\boxtimes([a_{i+2},\dots,a_{n+1}])$$
$$+ (-1)^i(a_0[a_1,\dots,a_i])\boxtimes(\partial_{n+1-i}[a_{i+2},\dots,a_{n+1}])$$

<u>Proof</u>: $\partial_{n+1}(a_0\ a_1,\dots,a_{n+1}\ ) = a_0a_1[a_2,\dots,a_{n+1}]-a_0[a_1a_2,\dots,a_{n+1}]$
$+ \dots + (-1)^{n+1}a_0a_{n+1}[a_1,\dots,a_n]$

$= a_0a_1[a_2,\dots,a_{n+1}] - a_0[a_1a_2,\dots,a_{n+1}] + \dots$
$+ (-1)^{i+1}(a_0a_{i+1}[a_1,\dots,\hat{a}_{i+1},\dots,a_{n+1}])$
$- (-1)^{i+1}(a_0a_{i+1}[a_1,\dots,\hat{a}_{i+1},\dots,a_{n+1}])$
$+ \dots +(-1)^{n+1}(a_0a_{n+1}[a_1,\dots,a_n])$

$$= (\partial_{i+1}(a_0[a_1,\dots,a_{i+1}]))\otimes([a_{i+2},\dots,a_{n+1}])$$

$$+ (-1)^i(a_0[a_1,\dots,a_i])\otimes(a_{i+1}[a_{i+2},\dots,a_{n+1}])$$

$$- (-1)^i(a_0[a_1,\dots,a_i])\otimes([a_{i+1}a_{i+2},\dots,a_{n+1}]) + \dots$$

$$+ (-1)^i(-1)^{n-i+1}(a_0[a_1,\dots,a_i])\otimes(a_{n+1}[a_{i+1},\dots,a_n])$$

$$= (\partial_{i+1}(a_0[a_1,\dots,a_{i+1}]))\otimes([a_{i+2},\dots,a_{n+1}])$$

$$+ (-1)^i(a_0[a_1,\dots,a_i])\otimes(\partial_{n+1-i}([a_{i+1},\dots,a_{n+1}]))$$

We have noted earlier that each $s_{i,n-i}$ may be considered as an element of $k\Sigma_n$ and thus as an A-endomorphism of the A-module J_nA. We now define another element, s_n, of $k\Sigma_n$ in the following way. First, we set s_1 equal to zero. Next, if $n\geq 2$, we set

$$s_n = \sum_{i=1}^{n-1} s_{i,n-i}.$$

We already know that $s_{i,n-i}$ need not be a chain map. We, can now show, however, that s_n is a chain map.

5.3 Lemma (Barr (2)): $\partial_n s_n = s_{n-1}\partial_n$

Proof: We recall that, by proposition 2.2, we have,

$$\partial_n s_{i,n-i}(a_0[a_1,\dots,a_n]) = s_{i-1,n-i}((\partial_i(a_0[a_1,\dots,a_i])\otimes([a_{i+1},\dots,a_n])$$

$$+ (-1)^i s_{i,n-i-1}(a_0[a_1,\dots,a_i]\otimes(\partial_{n-i}[a_{i+1},\dots,a_n])$$

Thus, $\partial_n s_n(a_0[a_1,\dots,a_n]) = \partial_n(\sum_{i=1}^{n-1} s_{i,n-i}(a_0[a_1,\dots,a_n]))$

$$\sum_{i=1}^{n-1} s_{i-1,n-i}((\partial_i(a_0[a_1,\dots,a_i]))\otimes([a_{i+1},\dots,a_n]))$$

$$+ (-1)^i s_{i,n-i-1}((a_0[a_1,\dots,a_i])\otimes(\partial_{n-i}([a_{i+1},\dots,a_n])))$$

$$= -s_{1,n-2}((a_0[a_1])\otimes(\partial_{n-1}[a_2,\dots,a_n])) + s_{1,n-2}(\partial_2(a_0[a_1,a_2])\otimes([a_3,\dots,a_n]))$$

$$- \dots + (-1)^{n-2}s_{n-2,1}((a_0[a_1,\dots,a_{n-2}])\otimes(\partial_2([a_{n-1},a_n]))$$

$$+ s_{n-2,1}(\partial_{n-1}(a_0[a_1,\dots,a_{n-1}])\otimes[a_n])$$

$= s_{1,n-2}(\partial_{n-1}(a_0\ a_1,\ldots,a_n\ )) + \ldots + s_{n-2,1}(\partial_{n-1}(a_0\ a_1,\ldots,a_n\ ))$

$= s_{n-1}\partial_n(a_0\ a_1,\ldots,a_n\ )$

Thus the theorem is proved.

5.4 Lemma: $\mathrm{sgn}(s_{i,n-i}) = c_{i,n-i}$ (We use the symbol $c_{i,n-i}$ to represent the binomial coefficient of n objects taken i at a time.)

Proof: We shall first consider our ground ring to be the integers and look at $Z\Sigma_n$. Now we proceed by induction on n. The lemma is obvious for $n = 2$ since $s_{1,1} = e - (1\ 2)$ where e is the identity element of Σ_n. Now we assume we have proved the lemma for n-1. Then $s_{i,n-i}(a_0\ a_1,\ldots,a_n\ )$ may be written in the following way:

$$s_{i,n-i}(a_0\ a_1,\ldots,a_n\ ) = s_{i,n-i-1}(a_0\ a_1,\ldots,a_{n-1}\ )\otimes a_n$$
$$+ (-1)^{n-i}s_{i-1,n-i}(a_0\ a_1,\ldots,\hat{a}_i,\ldots a_n\ )\otimes a_i$$

Thus $s_{i,n-i} = s_{i,n-i-1} + (-1)^{n-i}s_{i-1,n-i}(i\ i+1\ i+2\ \ldots\ n)$. If we apply $E_n \varepsilon Z\Sigma_n$ to both sides of the above equation we will have

$$\mathrm{sgn}(s_{i,n-i})\cdot E_n = \{\mathrm{sgn}(s_{i,n-i-1}) + (-1)^{n-i}(\mathrm{sgn}(s_{i-1,n-i})\ \mathrm{sgn}((i\ \ldots n)))\}\cdot E_n$$
$$= \{\mathrm{sgn}(s_{i,n-i-i}) + (-1)^{2(n-1)}\mathrm{sgn}(s_{i-1,n-i})\}\cdot E_n$$
$$= \{c_{i,n-i-1} + c_{i-1,n-i}\}\cdot E_n$$
$$= \{c_{i,n-i}\}\cdot E_n$$

Thus $\mathrm{sgn}(s_{i,n-i}) = c_{i,n-i}$.

Now if we replace the ring of integers by any arbitrary commutative ring, there will be a canonical map, $\phi: Z \longrightarrow k$, which is given simply by taking unit to unit. This extends to a map $\psi: Z\Sigma_n \longrightarrow k\Sigma_n$. This second map will take $s_{i,n-i} \varepsilon Z\Sigma_n$ to the same element in $k\Sigma_n$ and it will also commute with the maps sgn from

$Z\Sigma_n$ to Z and $k\Sigma_n$ to k. Thus in the ring $k\Sigma_n$, the signature of $s_{i,n-i}$ is also $c_{i,n-i}$.

5.5 Corollary: $\mathrm{sgn}(s_n) = 2^n-2$.

Proof:
$$\begin{aligned}\mathrm{sgn}(s_n) &= \mathrm{sgn}(s_{1,n-1} + \dots + s_{n-1,1})\\ &= c_{1,n-1} + \dots + c_{n-1,1}\\ &= 2^n-2.\end{aligned}$$

5.6 Proposition: $((2^n-2)-s_n)\dots(2-s_n)s_{i,n-i} = 0$ for all $1\le i\le n$.

Proof: Once again, we shall consider our base ring to be the integers first. Thus, we have s_n and $s_{i,n-i}$ in $Z\Sigma_n$. We proceed by induction on n with the case for $n = 1$ being trivial. If $n = 2$, then $s_2 = s_{1,1} = e - (1\ 2)$. Thus,

$$\begin{aligned}(2 - s_2)s_{1,1} &= (2e - e + (1\ 2))(e - (1\ 2))\\ &= (e + (1\ 2))(e - (1\ 2))\\ &= (e^2 - (1\ 2)^2)\\ &= (e - e)\\ &= 0\end{aligned}$$

Now assume that the proposition is true for $n-1$. Then

$$\partial_n(((2^{n-1}-2)-s_n)\dots(2-s_n)s_{i,n-i}) =$$
$$(((2^{n-1}-2)-s_{n-1})\dots(2-s_{n-1}))(s_{i-1,n-i}(\partial_i\otimes 1)+ (-1)^i s_{i,n-i-1}(1\otimes\partial_{n-i}))$$

because of propositions 2.2 and 5.3. Then, by induction,

$$(((2^{n-1}-2)-s_{n-1})\dots(2-s_n))(s_{i-1,n-i}(\partial_i\otimes 1)) = 0$$

and $$(((2^{n-1}-2)-s_{n-1})\dots(2-s_n))(s_{i,n-i-1}(1\otimes\partial_{n-i})) = 0.$$

Thus, by proposition 3.1, we see that $((2^{n-1}-2)-s_n)\dots(2-s_n)s_{i,n-i}$ must be some multiple, say $r_{i,n-i}$, of E_n for all pairs i and $n-i$. Thus

$$\begin{aligned}((2^n-2)-s_n)((2^{n-1}-2)-s_n)\ldots(2-s_n)s_{i,n-i} &= ((2^n-2)-s_n)\ r_{i,n-i}\ E_n \\ &= r_{i,n-i}(2^n-2-\mathrm{sgn}(s_n))\ E_n \\ &= 0.\end{aligned}$$

by corollary 5.5. By applying the same tricks we used in lemma 5.4, we see that the proposition is true for an arbitrary commutative ring.

Now suppose we consider

$$e_n' = ((2^n-2)-s_n)((2^{n-1}-2)-s_n)\ldots(2-s_n) \in k\Sigma_n$$

where k is a field of characteristic p. Then look at $(e_n')^2$. If we expand $(e_n')^2$ as a polynomial in s_n, we see that every term, excepting only the first, is a multiple of $e_n's_n$. But each term of this form is zero by proposition 5.6. Thus we have

$$(e_n')^2 = \{\prod_{i=2}^{n}(2^i-2)\}((2^n-2)-s_n)\ldots(2-s_n)$$

If we could multiply e_n' by the inverse of $\prod_{i=2}^{n}(2^i-2)$ in our field k, we could convert e_n' into an idempotent map and be on our way. Unhappily, this is not always possible since that product may be equal to zero in the field k. We must then decide when it is possible to divide.

Certainly, it is possible to divide in dimension n by $\prod_{i=2}^{n}(2^i-2)$ when we are working over a field of characteristic zero. Furthermore, if we are working with a field of characteristic p where 2 is a primitive root modulo p (i.e., the order of 2 in the group of units modulo p is p-1) then we may divide by the above product in dimensions up to but not including p. When 2 is not a primitive root modulo p, we may also divide up to dimension p, but in order to show this, we must have some more facts at our disposal.

Let us recall that lemma 3.1 holds true for any ring. In particular, we may consider the ring of integers modulo p^a where p is the characteristic of our field, a is some non-negative integer, and p^a is the largest power of p dividing $\prod_{i=2}^{n}(2^i-2)$. We denote this ring by k'.

5.7 Lemma: Let n be any integer less than the prime p. Let us consider the ring Z_n. If p^a divides $\prod_{i=2}^{n}(2^i-2)$ then,

$$((2^n-2)-s_n)\dots(2-s_n) = p^a \sum_{\pi \in \Sigma_n} \alpha_\pi \pi \qquad\qquad \alpha_\pi \in Z$$

Proof: We fix p and proceed by induction. If p is two and n is one, the lemma is obvious. Now suppose the prime p is odd. once again, if n is one or two, the lemma is obvious. Let us assume it is true for n-1. Consider $((2^n-2)-s_n)\dots(2-s_n)$. this is expressible in the form we want if and only if it is congruent to zero in $k'\Sigma_n$. Since lemma 3.1 holds true there, we have

$$\begin{aligned}\partial_n(((2^n-2)-s_n)\dots(2-s_n)) &= (2^n-2)(((2^{n-1}-2)-s_{n-1})\dots(2-s_{n-1}))\partial_n \\ &\quad -s_{n-1}(((2^{n-1}-2)-s_{n-1})\dots(2-s_{n-1}))\partial_n \\ &= (2^n-2)(((2^{n-1}-2)-s_{n-1})\dots(2-s_{n-1}))\partial_n\end{aligned}$$

since the second term is zero by proposition 5.6.

Now suppose p^b divides 2^n-2. Then p^{a-b} divides $(2^{n-1}-2)..(2^2-2)$. By the induction assumption, we see that p^a divides $(2^n-2)((2^{n-1}-2)-s_{n-1})\dots(2-s_{n-1})$ since p^{a-b} divides $((2^{n-1}-2)-s_{n-1})\dots(2-s_{n-1})$. Thus, $_n(((2^n-2)-s_n)\dots(2-s_n))$ is zero in $k'\Sigma_{n-1}$. This tells us that $((2^n-2)-s_n)\dots(2-s_n)$ is a multiple of E_n in $k'\Sigma_n$. If we multiply the above by E_n, we have

$$((2^n-2)-s_n)\dots(2-s_n)\cdot E_n = \lambda\cdot E_n\cdot E_n$$

$$= n!\cdot\lambda\cdot E_n$$
$$= ((2^n-2)-\mathrm{sgn}(s_n))\dots(2-\mathrm{sgn}(s_n))\ E_n$$
$$= 0$$

Thus $n!\cdot\lambda = 0$. Since $n<p$, $n!$ is a unit in k', so λ must be zero. Thus we have shown that $((2^n-2)-s_n)\dots(2-s_n)$ is zero in $k'\Sigma_n$.

The above lemma tells us that we are now able to divide by the product $\prod_{i=2}^{n}(2^i-2)$ in a field of characteristic p in all dimensions less than p. Thus we now set

$$e_n = e'_n / \{\prod_{i=2}^{n}(2^i-2)\} \quad 2\leq n\leq p-1$$

We will then have the following proposition.

5.8 <u>Proposition</u>: 1.) $e_n : J_nA \longrightarrow J_nA$ is a chain map.

2.) $e_n^2 = e_n$

3.) the kernel of e_n consists of just those shuffles of dimension n.

<u>Proof</u>: 1.) $\partial_n e_n = \partial_n\{((2^n-2)-s_n)\dots(2-s_n)/\prod_{i=2}^{n}(2^i-2)\}$

$$= \{(2^n-2)((2^{n-1}-2)-s_{n-1})\dots(2-s_{n-1})\partial_n/\prod_{i=2}^{n}(2^i-2)\}$$
$$-\{s_{n-1}((2^{n-1}-2)-s_{n-1})\dots(2-s_{n-1})\partial_n/\prod_{i=2}^{n}(2^i-2)\}$$
$$= \{((2^{n-1}-2)-s_{n-1})\dots(2-s_{n-1})\partial_n\}/\prod_{i=2}^{n}(2^i-2)$$
$$= e_{n-1}\partial_n$$

since the second term above is zero by lemma 5.6.

2.) $e_n^2 = e_n\cdot\{((2^n-2)-s_n)\dots(2-s_n)/\prod_{i=2}^{n}(2^i-2)\}$. If we expand the right hand side, we see that every term involves s_n excepting only the first. But $e_ns_n = 0$. Thus

$$e_n^2 = e_n\left(\prod_{i=2}^{n}(2^i-2)/\prod_{i=2}^{n}(2^i-2)\right)$$
$$= e_n$$

3.) Certainly, applying e_n to a shuffle will yield zero. Now *suppose* $e_n(z) = 0$ *for some* z *in* J_nA. The leading term in e_n is 1. Thus, if we expand e_n as a polynomial in s_n, we will see that $z = \sum_{i=1}^{n} \alpha_i s_n^i(z)$ for some $\alpha_i \in k$ and thus z is a shuffle.

Now, from the idempotence of e_n, we may conclude that

$$J_nA = e_nJ_nA + (1-e_n)J_nA.$$

But $(1-e_n)J_nA$ is the kernel of e_n, and thus consists of the shuffles. Thus, e_nJ_nA is chain isomorphic to Ch_nA, and so when we apply the functors $\cdot \otimes_A M$ and $\mathrm{Hom}_A(\cdot, M)$ to J_nA and e_nJ_nA and take homology, we see that theorem 5.1 is proved.

6. The Second Splitting

From the foregoing, it is clear that a splitting of J_nA exists in all dimensions over a field of characteristic zero. It is equally clear that the same splitting does not exist in all dimensions when the characteristic of the field is greater than zero, since 2^p-2 is congruent to zero modulo p. In this second case, however, we do have another very interesting splitting. In order to investigate this, we shall need some facts about idempotents in arbitrary rings.

6.1 Proposition (Herstein (9)): Let T be a (possibly non-commutative) ring. Let a be a non-nilpotent element of T such that a^2-a is nilpotent. Then there is a polynomial q(x) which has integral coefficients and aq(a) is a non-zero idempotent.

Proof: Suppose $(a^2-a)^m = 0$. Then, if we expand $(a^2-a)^m$ and transfer all terms except a^m to the right hand side, we have $a^m = a^{m+1}p(a)$ where $p(x)$ is some polynomial which has integral coefficients. Now let $e = a^m\{p(a)\}^m$. If a^m is not zero, then neither is e, since $a^m = a^{m+1}p(a) = a\cdot a^m p(a) = a\cdot a^{m+1}p(a)\cdot p(a) = \ldots = a^{2m}\{p(a)\}^m = a^m\cdot e$.

Now we claim that e is idempotent. We have $e^2 = a^{2m}\{p(a)\}^{2m} = a^{m-1}\cdot a^{m+1}p(a)\{p(a)\}^{2m-1} = a^{m-1}\cdot a^m\{p(a)\}^{2m-1} = a^{m-2}\cdot a^{m+1}p(a)\{p(a)\}^{2m-2} = \ldots = a^m\{p(a)\}^m = e$.

Let us consider the polynomial $p(x)$ which we constructed above. We note that it depends on the integer m for which $(a^2-a)^m$ is zero. Thus, we should actually write $p_m(x)$ instead of just $p(x)$. We now wish to know exactly what $p_m(x)$ looks like.

6.2 Proposition: $p_m(x) = 1 + (1-x) + (1-x)^2 + \ldots + (1-x)^{m-1}$.

Proof: Clearly, $p_1(x)$ is 1, since $a^2-a = 0$ implies $a = a^2\, p_1(x)$. Now suppose $(a^2-a)^m = 0$. If we expand this, we will see that

$$a^{2m} - c_{m-1,1}a^{2m-1} + c_{m-2,2}a^{2m-2} - \ldots + (-1)^m a^m = 0$$

The above equation implies:

$$\begin{aligned} a^m &= a^{m+1}\cdot((-1)^{m+1}(-1)^{m-1}c_{m-1,1} + (-1)^{m+1}(-1)^{m-2}c_{m-2,2}a \\ &\qquad + \ldots + (-1)^{m+1}a^{m-1}\ ) \\ &= a^{m+1}(m - \{m(m-1)/2\}a + \ldots + (-1)^{m\;1}a^{m-1}) \\ &= a^{m+1}((m-1) - c_{m-3,2}a + \ldots + (-1)^{m-2}a^{m-2} \\ &\qquad + 1 - c_{m-2,1}a + \ldots + (-1)^{m-2}c_{m-2,1}a^{m-2} + (-1)^{m-1}a^{m-1}) \end{aligned}$$

because of the well-known formula involving binomial coefficients, $c_{i-1,j} + c_{i,j-1} = c_{i,j}$. Now the upper part of this equation is simply

$1 + (1-a) + (1-a)^2 + \ldots + (1-a)^{m-2}$ while the lower part is the expansion of $(1-a)^{m-1}$. Thus, $a^m = a^{m+1}(1 + (1-a) + \ldots + (1-a)^{m-1})$ so $p_m(x) = 1 + (1-x) + \ldots + (1-x)^{m-1}$.

Let us now return to our consideration of J_*A where A is a commutative algebra over the field k which has characteristic p. For the sake of simplicity, we shall temporarily assume that 2 is a primitive root modulo p. Let us now set

$$w_\ell = 2((2^{p-1}-2)-s_\ell)\ldots(2-s_\ell))$$

and consider w_ℓ as an element of $k\Sigma_\ell$. The reason we choose the coefficient 2 is explained by the following lemma.

6.3 Lemma: Let 2 be a primitive root modulo p. Then, $\prod_{i=2}^{p-1}(2^i-2)$ is congruent to $\frac{1}{2}$ modulo p.

Proof: First, we know that the product of all non-zero elements of Z_p is -1 since they are all roots of the polynomial $x^{p-1}-1$. Next, if i is less than j and both are less than p, then 2^i-2 is not equal to 2^j-2. If it were otherwise, then 2^i would be equal to 2^j and so 2^{j-i} would be equal to 1. This would contradict our assumption that 2 is a primitive root. Thus the factors in the above product are all different. There are p-2 of them. Thus there is only one non-zero element of Z_p which is not contained in the factorization. This element is obviously -2. Otherwise, $2^i-2 = -2$ would imply that 2^i is zero. Thus we have

$$\prod_{i=2}^{p-1}(2^i-2) = (-\tfrac{1}{2})(-2)(\prod_{i=2}^{p-1}(2^i-2))$$

$$= (-\tfrac{1}{2})(\prod_{n \in Z_p^\#} n)$$

$$= (-\tfrac{1}{2})(-1)$$
$$= (\tfrac{1}{2})$$

where $Z_p^{\#}$ is the multiplicative group of units modulo p.

The above tells us that the constant term of w is one and lets us prove the following proposition.

6.4 <u>Proposition</u>: $w_\ell^2 - w_\ell$ is nilpotent.

<u>Proof</u>: Let us remember that in the ring $Z\Sigma_\ell$, we have the equation (*) $((2^\ell-2)-s_\ell)\ldots(2-s_\ell)s_\ell = 0$. Now, we note that $2^\ell-2$ is congruent to $2^{\ell-(p-1)}-2$ modulo p. Thus, if we consider the sequence of factors of (*), we will have s_ℓ, $(2-s_\ell),\ldots,((2^{p-1}-2)-s_\ell),((2^p-2)-s_\ell)$, $\ldots,((2\ -2)-s_\ell)$ and if we reduce this sequence modulo p, we see that it repeats itself after p terms.

Suppose $\ell = m(p-1) + i$. Then, when we reduce (*) modulo p, we will have

$$(-1)^r((2^{p-1}-2)-s_\ell)^m\ldots((2^{i+1}-2)-s_\ell)^m((2^i-2)-s_\ell)^{m+1}\ldots(2-s_\ell)^{m+1}s^{m+1} = 0$$

as an element of $k\Sigma_\ell$ and $r = m-1$ if i is zero. It is m if i is not zero. The coefficient $(-1)^r$ is unimportant, however. What is important is $w_\ell^m s_\ell^m = 0$ if i is zero and $w_\ell^{m+1}s_\ell^{m+1} = 0$ if i is not zero.

Furthermore, by the remark following lemma 6.3, we see that w -1 is a polynomial in s_ℓ which is lacking a constant term. Now then, $w_\ell^2-w_\ell = w_\ell(w_\ell-1)$. Thus, if i is not zero,

$$\begin{aligned}(w_\ell^2-w_\ell)^{m+1} &= w_\ell^{m+1}(w_\ell-1)^{m+1}\\ &= w_\ell^{m+1}s_\ell^{m+1}H(s_\ell)\\ &= 0\end{aligned}$$

in $k\Sigma_\ell$ for some polynomial H(x) in k[x]. If i is zero, we use the same reasoning to see that $(w_\ell^2-w_\ell)^m = 0$.

Now we set

$$e_\ell = w_\ell^{m+1}\{p_{m+1}(w_\ell)\}^{m+1}$$

if $\ell = m(p-1) + i$ and i is not zero, and we set

$$e_\ell = w_\ell^m\{p_m(w_\ell)\}^m$$

if $\ell = m(p-1)$. From the foregoing, it is obvious that e_ℓ is an idempotent. Unfortunately, we do not yet know that it is a non-zero idempotent. It will be non-zero if w_ℓ is not zero as is shown by the following propositions.

6.5 <u>Proposition</u>: w_ℓ is not nilpotent if $\ell \geq p-1$.

<u>Proof</u>: Suppose w_ℓ were nilpotent. Then, $w_\ell^r = 0$ for some integer r. If we expand w_ℓ^r as a polynomial in s_ℓ, we find that the polynomial has constant term one. Thus, we see that $1 = s_\ell H(s_\ell)$ where $H(x)$ is just some polynomial. Thus s_ℓ is invertible.

Now we claim that if $n>\ell$, then s_n is invertible. We shall proceed by induction. If $n = \ell + 1$, then consider $1-s_{\ell+1}H(s_{\ell+1})$. Then,

$$\partial_{\ell+1}(1-s_{\ell+1}H(s_{\ell+1})) = (1-s_\ell H(s_\ell))\partial_{\ell+1} = 0.$$

Thus, $1-s_{\ell+1}H(s_{\ell+1}) = \lambda E_{\ell+1}$ for some λ. Then, $s_{\ell+1}H(s_{\ell+1}) = 1-\lambda E_{\ell+1}$. Raise both sides of the equation to the power p. Then we will have that $s_{\ell+1}^p\{H(s_{\ell+1})\}^p = 1-\lambda^p E_{\ell+1}^p$ but since $\ell \geq p-1$, $E_{\ell+1}^p = 0$. Thus, $s_{\ell+1}$ is invertible. By induction, s_n is invertible for $n \geq \ell$.

Suppose now that $n = m(p-1) + 1$ and $n>\ell$. Then, s_n is invertible, and $s_n q(s_n) = 1$ for some polynomial $q(x)$. If we apply E_n to both sides, we see $(\mathrm{sgn}(s_n))(\mathrm{sgn}(q(s_n))E_n = E_n$. But, $\mathrm{sgn}(s_n)$ is

$2^{m(p-1)+1}-2$ and this is zero. We now have a contradiction.

6.6 <u>Proposition</u>: w_ℓ is not zero if $\ell<p-1$

<u>Proof</u>: We shall proceed by a sort of backward induction. We know that w_{p-1} is not zero, nor is it nilpotent. If w_{p-2} were zero, then, $\partial_{p-1}w_{p-1} = w_{p-2}\partial_{p-1} = 0$. But then, $w_{p-1} = \lambda\cdot E_{p-1}$. Thus, we would have, $w^2_{p-1} = w_{p-1}\cdot\lambda\cdot E_{p-1} = 0$ since $\operatorname{sgn}(w_{p-1})$ is zero. But then w_{p-1} would be nilpotent and this is not true. Thus, w_{p-2} is not zero and the same reasoning shows that it is not nilpotent. Now we can show that w_{p-3} is neither zero nor nilpotent. Continuing in this way, the proposition is proved.

The two previous propositions show that e_ℓ is a non-zero idempotent. The next theorem shows that e_ℓ is a chain map.

6.7 <u>Theorem</u>: $\partial_\ell e_\ell = e_{\ell-1}\partial_\ell$

<u>Proof</u>: We assume that $\ell = m(p-1) + i$. First, if $i>1$, we have

$$\partial_\ell e_\ell = \partial_\ell w_\ell^{m+1}\{p_{m+1}(w_\ell)\}^{m+1} = w_{\ell-1}^{m+1}\{p_{m+1}(w_{\ell-1})\}^{m+1} = e_{\ell-1}\partial_\ell$$

If $i = 0$, the same reasoning works except we must replace $m+1$ by m. The only problem occurs when $i = 1$. Then we will have e_ℓ equal to $w_\ell^{m+1}\{p_{m+1}(w_\ell)\}^{m+1}$ and $e_{\ell-1}$ equal to $w^m_{\ell-1}\{p_m(w_{\ell-1})\}^m$. Now we note that $p_{m+1}(w_\ell) = p_m(w_\ell) + (1-w_\ell)^m$. Thus

$$\begin{aligned}\partial_\ell e_\ell &= \partial_\ell w_\ell^{m+1}\{p_{m+1}(w_\ell)\}^{m+1}\\ &= w_{\ell-1}^{m+1}\{p_{m+1}(w_{\ell-1})\}^{m+1}\partial_\ell\\ &= (w_{\ell-1}^{m+1}\{p_m(w_{\ell-1})^{\ m+1} + m w_{\ell-1}^{m+1}\{p_m(w_{\ell-1})^m(1-w_{\ell-1})^m\}\\ &\quad + \dots + w_{\ell-1}^{m+1}(1-w_{\ell-1})^{m(m-1)})\partial_\ell\end{aligned}$$

Now every term of the form $\alpha w_{\ell-1}^{m+1}\{p_m(w_{\ell-1})\}^{m+1-j}(1-w_{\ell-1})^{jm}$ is zero since

$1-w_{\ell-1}$ does not have a constant term and so every term of the above form will have a factor of the form $w_{\ell-1}^{m} s_{\ell-1}^{m}$ and this last is zero. Thus the only possible non-zero term is the first. So we have

$$\begin{aligned}
\partial_\ell e_\ell &= (w_{\ell-1}^{m+1}\{p_m(w_{\ell-1})\}^{m+1})\partial_\ell \\
&= (w_{\ell-1}^{m+1}\{p_m(w_{\ell-1})\}\{p_m(w_{\ell-1})\}^{m})\partial_\ell \\
&= (w_{\ell-1}^{m}\cdot\{p_m(w_{\ell-1})\}^{m})\partial_\ell \\
&= e_{\ell-1}\partial_\ell
\end{aligned}$$

since $w_{\ell-1}^{m} = w_{\ell-1}^{m+1}\cdot p_m(w_{\ell-1})$ by proposition 6.1.

Using the e_ℓ's we have constructed, we see that we have a natural splitting of the complex J_*A which is given in the n-th dimension by $(J_*A)_n = e_n(J_*A)_n + (1-e_n)(J_*A)_n$. We would now like to find out what the kernel of the splitting e_* is. In order to do this we shall apply the following filtration to J_*A. We let F_1J_*A be J_*A. Next let F_0J_*A be J_*^2A, the complex of non-trivial shuffles. If i is a negative integer, we set F_iJ_*A equal to the subcomplex whose n-th dimensional component is $s_n^{-i}(J_*^2A)_n$. If i is a positive integer, we set F_iJ_*A equal to J_*A. It is clear that each F_iJ_*A is a complex and that F_iJ_*A contains $F_{i-1}J_*A$ and so is a filtration. Note that the quotient complex F_1J_*A/F_0J_*A is just the complex Ch_*A.

6.8 <u>Proposition</u>: Let $\ell = m(p-1) + i$. Then $e_\ell(F_{-m}J_*A) = 0$.

<u>Proof</u>: Let $x \in (F_{-m}J_*A)$. Then, $x = s^m(y)$ where $y \in (J_*^2A)$. Then we consider $w_\ell^{m+1}\{p_{m+1}(w_\ell)\}^{m+1}(x) = w_\ell^{m+1}s_\ell^{m}\{p_{m+1}(w_\ell)\}^{m+1}(y) = 0$ since $w_\ell^{m+1}s_\ell^{m+1}s_{i,j} = 0$ for all i and j whose sum is ℓ.

6.9 <u>Proposition</u>: Let $\ell = m(p-1) + i$. Then, if $e_\ell(x) = 0$, we will have $x \in (F_{-m}J_*A)_\ell$.

Proof: We know that $e_\ell = 1 + \sum_{i=1}^{t} \alpha_i s^i$ for some integer t. Therefore, if $e_\ell(x) = 0$, we see that

$$x = \sum_{i=1}^{t} -\alpha_i s_\ell^i(x) = s_\ell\left(\sum_{i=1}^{t} -\alpha_i s_\ell^{i-1}(x)\right) = s_\ell(x_1)$$

for some x_1. Thus $e_\ell(x) = e_\ell s_\ell(x_1) = 0$. By the same reasoning, we see that $s_\ell(x_1) = s_\ell^2(x_2)$. Thus, $x \in s_\ell(J_*^2 A)$. Continuing in this manner, we find that $x \in s_\ell^m(J_*^2 A)_\ell$ for every m and the proposition is proved.

6.10 Theorem: Let k be a field of characteristic p where 2 is a primitive root. Let A be a commutative algebra over k and M a left A-module. Construct the complex J_*A and filter it as before. Let $\ell = m(p-1) + i$ where $1 \leq i \leq p-1$. Then there exist natural transformations

$$\Phi_\ell(A,M)\colon \mathrm{Hoch}_\ell(A,M) \longrightarrow H_\ell((J_*A/F_{-m}J_*A)\otimes_A M)$$

$$\Phi_\ell(A,M)\colon H^\ell(\mathrm{Hom}_A(J_*A/F_{-m}J_*A,M)) \longrightarrow \mathrm{Hoch}_\ell(A,M)$$

such that $\Phi_\ell(A,M)$ is a split epimorphism and $\Phi^\ell(A,M)$ is a split monomorphism.

Proof: In order to prove this theorem, we merely note that the complex $J_*A/F_{-m}J_*A$ is isomorphic to the cokernel of e_ℓ and then the proof follows immediately from the splitting.

In all that has gone before, the only property of fields that was used was the property that 2^i-2 has an inverse if it is not zero. Thus, theorem 6.10 could have been stated equally well for rings containing a field of characteristic p.

We have also assumed in the foregoing that 2 is a primitive root modulo p. If this is not the case, we may obtain a version of theorem

6.10 in the following way. Let n be the order of 2 in the multiplicative group of units modulo p. Then set

$$w_\ell = c((2^n-2)-s_\ell)\dots(2-s_\ell))$$

where c is the inverse of $\prod_{i=2}^{n}(2^i-2)$. All our theorems hold with slight modifications of proofs except for proposition 6.6. In that case we have

6.6' Proposition: If 2 is not a primitive root modulo p, then w_ℓ is not nilpotent for $\ell \leq p-1$.

Proof: We know that w_{p-1} is not nilpotent. Suppose w_{p-2} were. Then $w^m_{p-2} = 0$ so $w_{p-1} = \lambda \cdot E_{p-1}$ for some λ. Thus, applying the idempotent e_{p-1} which we constructed in 5, we see that $e_{p-1}w^m_{p-1} = e_{p-1}$ since $e_{p-1}s_{p-1} = 0$ and w^m_{p-1} is a polynomial in s_{p-1} with constant term 1. Also, since e_{p-1} has signature zero, $e_{p-1}\cdot\lambda\cdot E_{p-1} = 0$. Thus, e_{p-1} is zero and this is a contradiction. Thus, w_{p-2} is not nilpotent. If we continue in this way, the theorem is proved.

We now have the following version of theorem 6.10.

6.11 Theorem: Let k be a ring containing a field of characteristic p. Let n be the order of 2 in the group of units modulo p. Let A be a commutative algebra over k and M a left A-module. Construct J_*A and filter it as before. Let $\ell = mn + i$ where $1 \leq i \leq n$. Then there exist natural transformations

$$\Phi_\ell(A,M): \mathrm{Hoch}_\ell(A,M) \longrightarrow H_\ell((J_*A/F_{-m}J_*A)\otimes_A M)$$

$$\Phi^\ell(A,M): H^\ell(\mathrm{Hom}_A(J_*A/F_{-m}J_*A,M)) \longrightarrow \mathrm{Hoch}^\ell(A,M)$$

such that $\Phi_\ell(A,M)$ is a split epimorphism and $\Phi^\ell(A,M)$ is a split monomorphism.

RINGS CONTAINING THE RATIONALS

1. Adjoint Functors, Cotriples and Symm

We assume that the reader is familiar with most of the definitions in this section, but we include it to fix some notation and for the sake of completeness. For a more detailed study of triples and cotriples, the reader should refer to Barr and Beck (3) or Beck (4).

Let $\underline{A}$ and $\underline{B}$ be categories and $U: \underline{A} \longrightarrow \underline{B}$, $F: \underline{B} \longrightarrow \underline{A}$ be functors. We say that F is left adjoint to U (or coadjoint to U) and U is right adjoint to F (or adjoint to F) if there is a natural isomorphism of sets

$$\alpha: \mathrm{Hom}_{\underline{A}}(FB,A) \longrightarrow \mathrm{Hom}_{\underline{B}}(B,UA)$$

for all objects, A, in $\underline{A}$ and, B, in $\underline{B}$. We write $\alpha: F \longrightarrow\!| \; U$. In particular, we find

$$\mathrm{Hom}_{\underline{A}}(FB,FB) \simeq \mathrm{Hom}_{\underline{B}}(B,UFB) \text{ and } \mathrm{Hom}_{\underline{A}}(FUA,A) \simeq \mathrm{Hom}_{\underline{B}}(UA,UA).$$

Thus, there exist natural transformations $\varepsilon: FU \longrightarrow \underline{A}$ and $\eta: \underline{B} \longrightarrow UF$ called the unit and counit respectively. Here we are identifying a category $\underline{A}$ with its identity functor $1_{\underline{A}}$.

The unit and counit satisfy the following diagrams (see Beck (4)).

$$\begin{tikzcd} F \arrow[r, "\eta F"] \arrow[dr, "F"'] & FUF \arrow[d, "F\varepsilon"] \\ & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r, "U\eta"] \arrow[dr, "U"'] & UFU \arrow[d, "\varepsilon U"] \\ & U \end{tikzcd}$$

Here we are using U and F to represent the identity natural transformation of a functor as well as the functor itself.

Now set $G = FU$ and $\delta = F\eta U$. Then $\mathbf{G} = (G,\varepsilon,\delta)$ is a cotriple on the category $\underline{A}$. That is, G: $\underline{A} \longrightarrow \underline{A}$ is a functor and ε: $G \longrightarrow \underline{A}$, δ:$G \longrightarrow G^2$ are natural transformations such that

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[dr, "G"'] & G^2 \arrow[d, "\varepsilon G"', "G\varepsilon"] & G \arrow[l, "\delta"'] \arrow[dl, "G"] \\ & G & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] & G^2 \arrow[d, "\delta G"] \\ G^2 \arrow[r, "G\delta"] & G^3 \end{tikzcd}$$

commute.

For a particular example of adjoint functors, let us consider the following case. Let k be a commutative ring, ${}_k\underline{M}$ be the category of left k-modules and ${}_k\underline{Alg}$ be the category of commutative k-algebras. Then, there is a functor U: ${}_k\underline{Alg} \longrightarrow {}_k\underline{M}$ which assigns to each k-algebra its underlying k-module and to each algebra map its associated module map. It is then well known (see, for example, Beck (4)) that this functor has a left adjoint, S, which is the symmetric algebra functor. Then the counit ε: SU $\longrightarrow {}_k\underline{Alg}$ is the map which "remembers" multiplication and the unit η: ${}_k\underline{M} \longrightarrow$ US is front adjunction. We shall abbreviate the cotriple arising from this adjoint pair simply by S.

For another example of adjoint functors, we need to consider the following variation of the symmetric algebra functor. First, we recall that the category $\underline{R\text{-}M}$ of commutative ring modules is the category with objects (R,M) where R is a commutative ring and M is an R-module. A morphism in $\underline{R\text{-}M}$ is a pair (ϕ,f): $(R_0,M_0) \longrightarrow (R_1,M_1)$ such that ϕ: $R_0 \longrightarrow R_1$ is a ring homomorphism, f: $M_0 \longrightarrow M_1$ is a map of abelian groups and the following abelian group diagram

$$\begin{tikzcd} R_0 \otimes M_0 \arrow[r, "\phi \otimes f"] \arrow[d, "\mu_1"'] & R_1 \otimes M_1 \arrow[d, "\mu_2"] \\ M_0 \arrow[r, "f"] & M_1 \end{tikzcd}$$

commutes. The maps μ_1 and μ_2 are module multiplication.

If we now consider the category GCR of graded, strictly commutative rings, we find an underlying module from GCR to R-M. It is described by $U\hat{R} = (\hat{R}_0, \hat{R}_1)$ where $\hat{R}_i$ is the i-th direct summand of $\hat{R}$. Further, there is another functor, S, from R-M to GCR which is constructed by making S(R,M) into the symmetric algebra of M over R.

1.1 Proposition: $S \longrightarrow\!| \; U$: R-M ——— GCR

Proof: We note that US R-M. Thus the natural transformation R-M $\longrightarrow$ US is simply the identity. Now suppose $\hat{R}$ is an object of GCR and we have $(\phi,f)\colon (R,M) \longrightarrow U\hat{R}$. We need only show there is a unique $h\colon S(R,M) \longrightarrow \hat{R}$ with $Uh = (\phi,f)$. We now set $h = S(\phi,f)$. Then, obviously, $Uh = (\phi,f)$. Finally, h is unique since every morphism with domain S(R,M) is determined by its values on the zero-th and first dimensions.

The importance of the above proposition will become apparent later when we are forced to use colimit arguments.

Now suppose A is a commutative k-algebra and let us consider the k-algebras A, SA, S^2A, ... We have a map $\varepsilon\colon SA \longrightarrow A$. This gives rise to two maps, $S\varepsilon A\colon S^2A \longrightarrow SA$ and $\varepsilon SA\colon S^2A \longrightarrow SA$. In general, we have $n+1$ maps from $S^{n+1}A$ to S^nA given by $S^{n-i}\varepsilon S^iA$ for i ranging between zero and n. We also have maps from $S^{n+1}A$ to $S^{n+2}A$ given by $S^{n-i}\delta S^iA$. Huber has shown in (10) that A, SA, S^2A, ... together with the maps defined above form an augmented simplicial object over the category of commutative k-algebras.

Suppose E is any functor from ${}_k\underline{\mathrm{Alg}}$ to some abelian category $\underline{A}$. Then

$$\cdots \; ES^2A \rightrightarrows ESA \longrightarrow EA$$

will be a simplicial object in $\underline{A}$. To this simplicial object, we may associate a chain complex

$$\cdots \xrightarrow{\partial} ES^2A \xrightarrow{\partial} ESA \longrightarrow 0$$

which has $ES^{n+1}A$ in the n-th dimension and in which

$$\partial_n = \sum_{i=0}^{n} (-1)^i ES^{n-i} \varepsilon S^i A$$

The homology of this complex is denoted by $H_n(A,E)_S$, $n \geq 0$, and these are known as the homology objects of A with coefficients in E relative to the cotriple S.

We shall now describe a particular functor which we shall use as our coefficient functor. Consider two commutative k-algebras A and A' and a k-algebra morphism between them, $\phi\colon A' \longrightarrow A$. We can make $A \otimes A'$ into an A-module by operating on the first factor via A. We define the A-module Diff A' to be $(A \otimes A')/N$ where N is that submodule of $A \otimes A'$ generated by all elements of the form $a \otimes a_1' a_2' - a\phi(a_1') \otimes a_2' - a\phi(a_2') \otimes a_1'$ where $a \in A$ and $a_1', a_2' \in A'$. Then, it is easily seen that $\mathrm{Hom}_A(\mathrm{Diff}\, A', M) \simeq \mathrm{Der}(A', M)$ where $\mathrm{Der}(A', M)$ is the set of all k-linear maps $f\colon A' \longrightarrow M$ where M is an A-module and $f(a_1' a_2') = \phi(a_1') f(a_2') + \phi(a_2') f(a_1')$.

We now return to our consideration of the cotriple S. There is a unique map from S^nA to A which is arrived at by simply taking any composite of the $S^{n-i}\varepsilon S^iA$'s. We now have a complex over the category of A-modules by taking

$$\cdots \longrightarrow \mathrm{Diff}\ S^nA \longrightarrow \mathrm{Diff}\ S^{n-1}A \longrightarrow \cdots \longrightarrow \mathrm{Diff}\ SA \longrightarrow 0$$

The boundary maps are defined to be

$$\partial_n = \sum_{i=0} (-1)^i \mathrm{Diff}\ S^{n-i}\varepsilon S^iA$$

We now define the symmetric homology of A with coefficients in the A-module M to be the homology of the complex

$$\cdots \longrightarrow \mathrm{Diff}\ S^nA\otimes_A M \longrightarrow \cdots \longrightarrow \mathrm{Diff}\ S^2A\otimes_A M \longrightarrow \mathrm{Diff}\ SA\otimes_A M \longrightarrow 0$$

with the n-th boundary given by $\partial_n \otimes 1_M$. We denote the n-th homology module by $\mathrm{Symm}_n(A,M)$. Similarly, we define the symmetric cohomology of A to be the homology of the complex

$$0 \longrightarrow \mathrm{Hom}_A(\mathrm{Diff}\ SA,M) \longrightarrow \mathrm{Hom}_A(\mathrm{Diff}\ S^2A,M) \longrightarrow \cdots$$

with the boundary given by $\mathrm{Hom}_A(\partial_n,M)$. We denote the n-th symmetric cohomology module by $\mathrm{Symm}^n(A,M)$. We note that this second complex can be written as

$$0 \longrightarrow \mathrm{Der}(SA,M) \longrightarrow \mathrm{Der}(S^2A,M) \longrightarrow \cdots \longrightarrow \mathrm{Der}(S^nA,M) \longrightarrow$$

We are now in a position to state the main theorem of this chapter.

1.2 Theorem: Let k be any ring containing the rational numbers. Let A be any commutative k-algebra and M any A module. Then

$$\mathrm{Symm}_n(A,M) \simeq \mathrm{Harr}_{n+1}(A,M)$$
$$\mathrm{Symm}^n(A,M) \simeq \mathrm{Harr}^{n+1}(A,M)$$

In order to facilitate proving this theorem, we shall spread the proof out over several sections.

2. The First Proposition

2.1 Proposition : Let k be any commutative ring containing the rational numbers. Let $R = k[X]$ be the algebra of polynomials over a set X. Then, for any R-module M, $\mathrm{Harr}_n(R,M) = 0 = \mathrm{Harr}^n(R,M)$ for any $n>1$. Further, $\mathrm{Harr}^1(R,M) = \mathrm{Der}(R,M) = M^X$ and $\mathrm{Harr}_1(R,M) = \mathrm{Diff}\ R\otimes_R M = X\cdot M$ where $X\cdot M$ denotes the coproduct of X copies of M.

Proof: We note that $\otimes$ commutes with direct limits as does the idempotent e_n which we constructed in the last chapter. Further, homology commutes with direct limits. Now we note that $k[X]$ is the direct limit of the subalgebras $k[X_\alpha]$ where X_α ranges over all finite subsets of X. Thus, it suffices to show that Harrison's homology is zero when R is a polynomial algebra in a finite number of indeterminates.

We shall first prove the proposition for projective R-modules. Let $X^\#$ be any set isomorphic to and disjoint from X. Let us set $R^\# = k[X^\#]$. Now $R\otimes R^\# \simeq R\otimes R$ as a k-algebra and so we may identify the the two. Set $\tilde{R} = k[\tilde{X}]$ where $\tilde{X} = \{1\otimes x^\# - x\otimes 1 \in R\otimes R^\# \mid x \in X\}$. Then, $R\otimes\tilde{R} \simeq R\otimes R^\#$. Now we note that $\tilde{R}$ operates trivially on k via the augmentation $\tilde{R} \longrightarrow k$. Thus, we see that $R \simeq R\otimes k$ as an $R\otimes\tilde{R}$ module.

We now note that $R = k[X] \simeq k\otimes_Q Q[X]$ and $\tilde{R} \simeq k\otimes_Q Q[\tilde{X}]$ where Q is the field of rational numbers. Set $\hat{R} = Q[\tilde{X}]$. By theorem XI.3.1 of Cartan and Eilenberg (5), we have the following isomorphisms.

$$\begin{aligned}\mathrm{Tor}_*^{R\otimes\tilde{R}}(R,R) &\simeq \mathrm{Tor}_*^{R\otimes\tilde{R}}(R\otimes k, R\otimes k)\\ &\simeq \mathrm{Tor}_*^{R\otimes k\otimes_Q \hat{R}}(R\otimes_Q Q, R\otimes_Q Q)\end{aligned}$$

$$\simeq \mathrm{Tor}_*^{R\otimes_Q \hat{R}}(R\otimes_Q Q, R\otimes_Q Q)$$

$$\simeq \mathrm{Tor}_*^{R}(R,R)\otimes_Q \mathrm{Tor}_*^{\hat{R}}(Q,Q)$$

$$\simeq R\otimes_Q \mathrm{Tor}_*^{\hat{R}}(Q,Q)$$

Furthermore, $\mathrm{Tor}_*^{R\otimes R}(R,R) \simeq \mathrm{Tor}_*^{R\otimes R}(R,R) \simeq \mathrm{Hoch}_*(R,R) \simeq H(C_* R)$.

Now $\mathrm{Harr}_*(R,R)$ is a natural direct summand of $H(C_* R)$ and if we can show it is zero, we can conclude $\mathrm{Harr}_i(R,M)$ is zero for i 1 and M projective. Now

$$H(C_*\hat{R}\otimes_{\hat{R}} Q) \simeq \mathrm{Tor}_*^{\hat{R}\otimes\hat{R}}(\hat{R},Q)$$

$$\simeq \mathrm{Tor}_*^{\hat{R}}(R,Q)\otimes \mathrm{Tor}^{\hat{R}}(Q,Q)$$

$$\simeq Q\otimes \mathrm{Tor}_*^{\hat{R}}(Q,Q)$$

$$\simeq \mathrm{Tor}_*^{\hat{R}}(Q,Q)$$

where the above tensor products are taken over the field Q and not the ring k. We shall call a cycle, γ, in $C_n\hat{R}\otimes_{\hat{R}} Q$ alternating if $\varepsilon_n\gamma = \gamma$ where $\varepsilon_n = (1/(n!))\Sigma(\mathrm{sgn}(\pi))\pi$, $\pi \in \Sigma_n$.

2.2 Lemma: Every cycle in $C_n\hat{R}\otimes_{\hat{R}} Q$ is homologous to an alternating cycle.

Proof: Let $X = \{x_1,\ldots,x_m\}$. Then, $H(C_*\hat{R}\otimes_{\hat{R}} Q) = \mathrm{Tor}_*^{\hat{R}}(Q,Q)$ from above. From MacLane (11) page 205, it is well known that the above is a Q-vector space of dimension $c_{n,m-n}$. We shall now show that the alternating cycles span this space. Let us consider sequences of integers $1 \leq i_1 \leq i_2 \leq \ldots \leq i_n \leq m$. There are exactly $c_{n,m-n}$ such sequences. Look at $\varepsilon_n[x_{i_1},\ldots,x_{i_n}]$. Then these are $c_{n,m-n}$ alternating cycles which are linearly independent in $C_n\hat{R}\otimes_{\hat{R}} Q$. If we can show they are linearly independent modulo boundaries, we will be done. However, if we consider any boundary of the form $\partial_{n+1}[m_1,\ldots,m_{n+1}]$, where each m_i is a monomial, we see that each term of the boundary has an entry of degree at least two unless some of the m_i's happen to be units. In that case, every term of the boundary will have a unit except two which will cancel each other. Every boundary is a sum of boundaries of the form we have just discussed. Thus, the cycles $\varepsilon_n[x_{i_1},\ldots,x_{i_n}]$ are linearly independent modulo boundaries.

Now $\mathrm{Harr}_i(R,R) = e_i(H(C_*R)) = e_i(R\otimes_Q \mathrm{Tor}_i^R(Q,Q)$. Pick any cycle, γ, in Ch_iR. Then $\gamma \in C_iR$ and $\gamma = \varepsilon_i\gamma' + \partial_{i+1}\gamma''$ for $\gamma' \in C_iR$ and $\gamma'' \in C_{i+1}R$. But $\gamma = e_i\gamma = e_i(\varepsilon_i\gamma' + \partial_{i+1}\gamma'') = e_i\varepsilon_i\gamma' + \partial_{i+1}e_{i+1}\gamma''$. Now e_i has signature zero, so $e_i\varepsilon_i = (1/(n!))(e_iE_i) = 0$. Thus, $\gamma = \partial_{i+1}e_{i+1}\gamma''$. Thus $\mathrm{Harr}_i(R,R) = 0$ for $i>1$. Since any free R module is a coproduct of copies of R, and any projective R module is a retract of a free R-module, we see that $\mathrm{Harr}_i(R,M) = 0$ for any projective R-module M.

Also, $\mathrm{Harr}_1(R,M) = R\otimes M/M'$ where M' is the submodule generated by all elements of the form $[m_1m_2]\otimes m - [m_1]\otimes m_2m - [m_2]\otimes m_1m$ and $1\otimes m$ where m_1 and m_2 are monomials. Thus, it is easily seen that $\mathrm{Harr}_1(R,M)$ is isomorphic to $X\cdot M$ as was claimed and we are done for projective R- modules.

Now let M be any R- module. There exists an R-projective resolution of M say

$$\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

Then we may form the tensor product of the complexes X_* and Ch_*R

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & Ch_1R\otimes_R X_n \arrow[l] \arrow[d] & Ch_2R\otimes_R X_n \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 & Ch_1R\otimes_R X_{n-1} \arrow[l] \arrow[d] & Ch_2R\otimes_R X_{n-1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & Ch_1R\otimes_R X_0 \arrow[l] \arrow[d] & Ch_2R\otimes_R X_0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & 0 & 0 &
\end{tikzcd}$$

If we take homology going down, we get only the complex

$$\cdots \longrightarrow Ch_nR\otimes_R M \longrightarrow \cdots \longrightarrow Ch_2R\otimes_R M \longrightarrow Ch_1R\otimes_R M \longrightarrow 0$$

since Ch_iR is a projective R-module. The homology of this complex is $Harr_*(R,M)$.

On the other hand, if we take homology going across, we will get the complex

$$\cdots \longrightarrow \text{Diff } R\otimes_R X_n \longrightarrow \cdots \longrightarrow \text{Diff } R\otimes_R X_1 \longrightarrow \text{Diff } R\otimes_R X_0 \longrightarrow 0$$

Since Diff R is a free R-module, The homology of this complex is simply Diff $R\otimes_R M \simeq X\cdot M$. Since the homologies taken both ways must be equal because of MacLane (11), page 341, we have proposition 2.1 for homology. If we use universal coefficient theorems, we will get the proposition for cohomology.

3. The Finitely Generated Case

Let us now specialize to the case of a noetherian ground ring containing the rational numbers. In order to prove our main theorem, we must know the following proposition.

3.1 Proposition: Let M be a finitely generated k-module and N a finitely generated SM-module. Then, $Harr^i(SM,N)$ and $Harr_i(SM,N)$ are finitely generated SM-modules.

We shall need the following lemma.

3.2 Lemma: If k is any noetherian ring and M is a finitely generated k-module, Then SM is noetherian.

Proof: Since M is finitely generated, say by $\{x_1,..,x_n\}$, there is a free, finitely generated k-module, F, with free generators $\{\hat{x}_1,..,\hat{x}_n\}$ and an epimorphism of k-modules, $F \longrightarrow M \longrightarrow 0$ obtained by sending $\hat{x}_i$ to x_i. Then there is a map $SF \longrightarrow SM$ which is obviously a ring epimorphism. By the Hilbert Basis Theorem (Zariski and Samuel (15), page 201), SF is a noetherian ring and since an epimorphic image of a noetherian ring is noetherian, proposition 3.2 is proved.

If we can show $Hoch_i(SM,N)$ and $Hoch^i(SM,N)$ are finitely generated, then, since $Harr_i(SM,N)$ and $Harr^i(SM,N)$ are retracts of the above, we will be done. We must now immerse ourselves in the depths of relative homological algebra. Since our interest is not in this subject as such, we refer the reader to MacLane (11), chapter 9, for an exposition of it.

We note that $\mathrm{Hoch}_i(SM,N) = \mathrm{Tor}_i^{(SM\otimes SM,k)}(SM,N)$ where the right hand side stands for $(SM\otimes SM,k)$-relative homology theory. Similarly, we see $\mathrm{Hoch}^i(SM,N) = \mathrm{Ext}^i_{(SM\otimes SM,k)}(SM,N)$.

We want an $(SM\otimes SM,k)$-free allowable resolution of SM which will allow us to calculate whether or not $\mathrm{Tor}_i^{(SM\otimes SM,k)}(SM,N)$ is finitely generated. Let $M^{\#}$ be a k-module isomorphic to and distinct from M. Let the isomorphism send $m \in M$ to $m^{\#} \in M^{\#}$. Then $SM^{\#}$ is isomorphic to SM. Thus, $\mathrm{Tor}_i^{(SM\otimes SM^{\#},k)}(SM,N) \simeq \mathrm{Tor}^{(SM\otimes SM,k)}(SM,N)$ for any symmetric SM-bimodule. Of course, we define the action of $SM^{\#}$ on SM via the isomorphism. Let M' be that submodule of $SM\otimes SM^{\#}$ generated by elements of the form $1\otimes m^{\#} - m\otimes 1$. Then, M' is isomorphic to M so $SM\otimes SM' \quad SM\otimes SM$. Then, as an $SM\otimes SM'$-module, N is isomorphic to $N\otimes k$ where SM' acts on the ground ring k via the usual augmentation map $SM' \longrightarrow k$. Thus we have $\mathrm{Tor}_i^{(SM\otimes SM,k)}(SM,N) \simeq \mathrm{Tor}_i^{(SM\otimes SM',k)}(SM\otimes k,N\otimes k)$.

We shall now describe an (SM',k)-relatively free allowable resolution of k, i. e., we will build a complex

$$\cdots \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \xrightarrow{\varepsilon} k \longrightarrow 0$$

of (SM',k)-relative free modules which possesses a k-contracting homotopy of square zero. Using this complex, we will be able to get a useful $(SM\otimes SM',k)$-relatively free allowable resolution of $SM\otimes k$.

Let X_n be $SM'\otimes\Lambda_n M'$ where $\Lambda_n M'$ stands for the iterated exterior product of M' with itself n-times. (We recall that $M\wedge M = M\otimes M/L$ where L is the k-submodule of $M\otimes M$ generated by elements of the form $m_1\otimes m_2 + m_2\otimes m_1$.) We define a boundary homomorphism $\partial_n\colon SM'\otimes\Lambda_n M' \longrightarrow SM'\otimes\Lambda_{n-1}M'$

by $\partial_n(m'_1 \ldots m'_r \otimes m'_{r+1} \wedge \cdots \wedge m'_{r+n}) =$

$$\sum_{i=1}^{n} (-1)^i (m'_1 \cdot \ldots \cdot m'_r \cdot m'_{r+i} \otimes m'_{r+1} \wedge \cdots \wedge \hat{m}'_{r+i} \wedge \cdots \wedge m'_{r+n})$$

where $\hat{m}'_{r+i}$ signifies omitting m'_{r+i}. It is easy to see that ∂_n is a well-defined SM'-module homomorphism and that $\partial_{n-1}\partial_n = 0$.

We must now show that this complex is k-split. To define a k-module homomorphism from X_n to X_{n+1}, we need to define it on each k-summand of $SM' \otimes \Lambda_n M'$. We set $t_{r,n}(m'_1 \cdot \ldots \cdot m'_r \otimes m'_{r+1} \wedge \cdots \wedge m'_{r+n}) = 0$ if r is equal to zero. If r is not zero, then

$t_{r,n}(m'_1 \cdot \ldots \cdot m'_r \otimes m'_{r+1} \wedge \cdots \wedge m'_{r+n}) =$

$$(1/r+n) \sum_{j=1}^{r} m'_1 \cdot \ldots \cdot \hat{m}'_j \cdot \ldots \cdot m'_r \otimes m'_j \wedge m'_{r+1} \wedge \cdots \wedge m'_{r+n}$$

We should show that $t_{r,n}$ is a well-defined morphism, but this is wholly obvious since $\bar{t}_{r,n}: M' \times \ldots \times M' \longrightarrow S_{r-1}M' \otimes \Lambda_{n+1}M'$ ($M' \times \ldots \times M'$ stands for the cartesian product of M' with itself r+n times as a set) which is defined by the above formula is well-defined, k-linear, symmetric in the first r variables and skew symmetric in the last n variables. Thus, $\bar{t}_{r,n}$ has a unique factorization through $S_r M' \otimes \Lambda_n M'$ and that factorization gives rise to $t_{r,n}$.

Let $t_n: SM' \otimes {}_n M' \longrightarrow SM' \otimes \Lambda_{n+1} M'$ be the map on the direct sum which has components $t_{r,n}$ on each summand. We define t_{-1} from k to SM' to be mere front adjunction. We must now show that the t_n's so defined give us a k-homotopy with square zero.

There are three things involved here. First, we must have

$\varepsilon t_{-1} = 1_k$. Next $\partial_1 t_0 + t_{-1}\varepsilon = 1_{X_1}$. Thirdly, we must show that $\partial_{n+1} t_n + t_{n-1}\partial_n = 1_{X_n}$. The first equality is, of course, obvious.

Since $X_0 = SM'\otimes\Lambda_0 M' \simeq SM'$, we have two cases for the second equality, corresponding to whether or not the element we are dealing with, say x, is in degree zero. If so, then $t_0(x) = 0$. But also, $t_{-1}\varepsilon(x) = x$ and we are done. If x is not in degree zero, we may suppose it is homogeneous of degree $p>0$. Then $x = m_1'\cdot\ldots\cdot m_p'\otimes 1$. Thus, $\varepsilon(x) = 0$,

$t_0(x) = (1/p)\sum_{i=1}^{p} m_1'\cdot\ldots\cdot\hat{m}_i'\cdot\ldots\cdot m_p'\otimes m_i'$ and

$$\partial_1 t_0(x) = (1/p)\sum_{i=1}^{p} m_1'\cdot\ldots\cdot m_i'\cdot\ldots\cdot m_p'\otimes 1 = (p/p)\,x = x$$

For the third equality, we again have two cases. First if $x \in S_0 M'\otimes\Lambda_n M'$, then $t_n(x) = 0$. But, if $x = 1\otimes m_1'\wedge\ldots\wedge m_p'$ we will have

$$\begin{aligned}
t_{n-1}\partial_n(x) &= t_{n-1}\Big(\sum_{j=1}^{n} (-1)^{j-1} m_j'\otimes m_1'\wedge\ldots\wedge\hat{m}_j'\wedge\ldots\wedge m_n'\Big)\\
&= (1/n)\sum_{j=1}^{n} (-1)^{j-1}(1\otimes m_j'\wedge m_1'\wedge\ldots\wedge\hat{m}_j'\wedge\ldots\wedge m_n')\\
&= (1/n)\sum_{j=1}^{n} (-1)^{j-1}(-1)^{j-1}(1\otimes m_1'\wedge\ldots\wedge m_n')\\
&= (n/n)x\\
&= x
\end{aligned}$$

If $x \in S_r M'\otimes\Lambda_n M'$, for $r>0$, then $x = m_1'\cdot\ldots\cdot m_r'\otimes m_{r+1}'\wedge\ldots\wedge m_{r+n}'$ say.

Now $\partial_{n+1} t_{r,n}(x) = \partial_{n+1}\Big((1/r+n)\sum_{i=1}^{r} m_1'\cdot\ldots\cdot\hat{m}_i'\cdot\ldots\cdot m_r'\otimes m_i'\wedge m_{r+1}'\wedge\ldots\wedge m_{r+n}'\Big)$

$$= (1/r+n)\Big(\sum_{i=1}^{r}\big((n) m_1'\cdot\ldots\cdot m_r'\otimes m_{r+1}'\wedge\ldots\wedge m_{r+n}'$$

$$\sum_{j=1}^{n}(-1)^j m_1'\cdot\ldots\cdot\hat{m}_i'\cdot\ldots\cdot m_r'\cdot m_{r+j}'\otimes m_i'\wedge m_{r+1}'\wedge\ldots\wedge\hat{m}_{r+j}'\wedge\ldots\wedge m_{r+n}'\big)\Big)$$

$(*) = (n/r+n)x +$

$$\sum_{i=1}^{r}\sum_{j=1}^{n}(-1)^{j} m_1'\cdot\ldots\cdot\hat{m}_i'\cdot\ldots\cdot m_r'\cdot m_{r+j}'\otimes m_i'\wedge m_{r+1}'\wedge\ldots\wedge\hat{m}_{r+j}'\wedge\ldots\wedge m_{r+n}'$$

Also, $t_{r+1,n-1\;n}(x) =$

$$t_{r+1,n-1}\left(\sum_{j=1}^{n}(-1)^{j-1} m_1'\cdot\ldots\cdot m_r'\cdot m_{r+j}'\otimes m_{r+1}'\wedge\ldots\wedge\hat{m}_{r+j}\wedge\ldots\wedge m_{r+n}\right)$$

$$= \sum_{j=1}^{n}(-1)^{j-1}\Big((1/r+n)\Big(\sum_{i=1}^{r} m_1'\cdot\ldots\cdot\hat{m}_i'\cdot\ldots\cdot m_r'\cdot m_{r+j}'\otimes m_i'\wedge m_{r+1}\wedge\ldots\wedge\hat{m}_{r+j}\wedge\ldots\wedge m_{r+n}$$
$$+ m_1'\cdot\ldots\cdot m_r'\otimes m_{r+j}\wedge m_{r+1}\wedge\ldots\wedge\hat{m}_{r+j}\wedge\ldots\wedge m_{r+n}\Big)\Big)$$

$(**) = (n/r+n)x +$

$$\sum_{j=1}^{n}\sum_{i=1}^{r}(-1)^{j-1} m_1'\cdot\ldots\cdot\hat{m}_i'\cdot\ldots\cdot m_r'\cdot m_{r+j}'\otimes m_i'\wedge m_{r+1}'\wedge\ldots\wedge\hat{m}_{r+j}'\wedge\ldots\wedge m_{r+n}'$$

Then, every term after the $+$-sign in $(*)$ occurs with opposite sign after the $+$-sign of $(**)$. Thus, $t_{n-1}\partial_n(x) + \partial_{n+1}t_n(x) = (n/r+n)x + (r/r+n)x = x$. Thus t is a k-homotopy.

We now should show that $t^2 = 0$. This can be done by calculation. However, even if t^2 were not zero, then $t\partial t$ would be a contracting homotopy and $(tdt)^2 = 0$. This is shown by the identities $\partial t\partial t + t\partial t\partial = (1-t\partial)\partial t + t\partial(1-\partial t) = \partial t + t\partial = 1$ and $(t\partial t)^2 = (t\partial t)(t\partial t) = t(1-t\partial)(1-\partial t)t = t(1-t\partial-\partial t)t = 0$.

Now the resolution

$$\ldots \longrightarrow SM\otimes SM'\otimes\Lambda_n M' \longrightarrow \ldots \longrightarrow SM\otimes SM' \longrightarrow SM\otimes k \longrightarrow 0$$

is an $(SM\otimes SM',k)$-relatively free allowable resolution of $SM\otimes k$. Thus, $\mathrm{Tor}_*^{(SM\otimes SM',k)}(SM\otimes k,N\otimes k)$ is the homology of the complex

$$\ldots\longrightarrow SM\otimes SM'\otimes\Lambda_n M'\otimes_{SM\otimes SM'}N\otimes k \longrightarrow\ldots\longrightarrow SM\otimes SM'\otimes_{SM\otimes SM'}N\otimes k \longrightarrow 0$$

which is

$$\ldots \longrightarrow \Lambda_n M' \otimes N \otimes k \longrightarrow \ldots \longrightarrow M' \otimes N \otimes k \longrightarrow N \otimes k \longrightarrow 0$$

where the boundary of this complex is just zero. Thus the n-th homology module of this complex is $\Lambda_n M' \otimes N$. Since N is a finitely generated SM-module and M' is a finitely generated k-module, we see that $\Lambda_n M' \otimes N$ is a finitely generated SM-module. Thus $\mathrm{Harr}_i(SM,N)$ is finitely generated since it will be a retract of $\Lambda_n M' \otimes N$. The dual proof works for cohomology, so $\mathrm{Harr}^i(SM,N)$ is finitely generated. This completes the proof of proposition 3.1. I am indebted to Professor Barr for pointing out the resolution of which we make so much use.

4. The Residue Field Case

For this section, we need not assume that the ground ring, k, is noetherian. However, for expository reasons, we shall still assume that M is a finitely generated k-module and SM is its associated symmetric algebra. Our aim is to calculate the Harrison homology and cohomology of SM with coefficients in K where K is a residue field of SM. In order to do this, we must know something about the structure of $SM \otimes K$.

First, if $\{x_1, \ldots, x_n\}$ are generators of M, then $B = \{x_1 \otimes 1, \ldots, x_n \otimes 1\}$ are generators of $M \otimes K$. (The unit is the unit of K.) Since $M \otimes K$ is a K-vector space, we may assume that the set B contains a basis of $M \otimes K$ as a K-space. We may assume that this basis is $\{x_1 \otimes 1, \ldots, x_r \otimes 1\}$. Thus, as K-modules, $M \otimes K \simeq F \otimes K$ where F is the free k-module on the basis $\{\ell_1, \ldots, \ell_r\}$.

4.1 <u>Proposition</u>: $M^{(n)}\otimes K \simeq F^{(n)}\otimes K$

<u>Proof</u>: We proceed by induction. We have already established the case for $n = 1$. Suppose the theorem is true for $n-1$. Then

$$
\begin{aligned}
M^{(n)}\otimes K &\simeq M\otimes M^{(n-1)}\otimes K \\
&\simeq M\otimes M^{(n-1)}\otimes K\otimes_K K \text{ since } K\otimes_K K \quad K \\
&\simeq (M\otimes K)\otimes_K (M^{(n-1)}\otimes K) \\
&\simeq (F\otimes K)\otimes_K (F^{(n-1)}\otimes K) \\
&\simeq F\otimes F^{(n-1)}\otimes K \\
&\simeq F^{(n)}\otimes K
\end{aligned}
$$

4.2 <u>Proposition</u>: If S is the symmetric algebra functor over k, then, $SM\otimes K \simeq SF\otimes K$ as k-modules.

<u>Proof</u>: Since the tensor product commutes with coproducts, we have $TM\otimes K \simeq TF\otimes K$ where TM and TF are the respective tensor algebras of F and M. Now it is well known (see Quillen (14) for example) that SM is a direct summand of TM where the projection $TM \longrightarrow SM$ is given in the n-th dimension by $j_n = (1/n!)\sum\sigma$, $\sigma \in \Sigma_n$. Thus, $j_*\otimes 1$: $TM\otimes K \longrightarrow SM\otimes K$ and $j_*\otimes 1$: $TF\otimes K \longrightarrow SF\otimes K$ are retracts. Since j_* is obviously a natural transformation of functors, we have $SM\otimes K = SF\otimes K$.

4.3 <u>Corollary</u>: $SM^{(n)}\otimes K \simeq SF^{(n)}\otimes K$

4.4 <u>Corollary</u>: $SM\otimes SM^{(n)}\otimes_{SM} K \simeq SF\otimes SF^{(n)}\otimes_{SF} K$

<u>Proof</u>: From corollary 4.3, we have $SM^{(n)}\otimes K \simeq SF^{(n)}\otimes K$. However, $SM^{(n)}\otimes K \simeq SM^{(n)}\otimes SM\otimes_{SM} K \simeq SM\otimes SM^{(n)}\otimes_{SM} K$ and $SF^{(n)}\otimes K \simeq SF^{(n)}\otimes SF\otimes_{SF} K \simeq SF\otimes SF^{(n)}\otimes K$.

Corollary 4.4 now tells us that $J_* SM\otimes_{SM} K \simeq J_* SF\otimes_{SF} K$. Thus we

have $Ch_*SM\otimes_{SM}K \approx Ch_*SF\otimes_{SF}K$. By proposition 5.2', page 28, of Cartan and Eilenberg (5), we also have,

$$\begin{aligned} Hom_{SM}(J_*SM,K) &\approx Hom_{SM}(J_*SM,Hom_K(K,K)) \\ &\approx Hom_K(J_*SM\otimes_{SM}K,K) \\ &\approx Hom_K(J_*SF\otimes_{SF}K,K) \\ &\approx Hom_{SF}(J_*SF,K) \end{aligned}$$

Now SF is simply the polynomial algebra over k of $\{\ell_1,\ldots,\ell_r\}$. Thus, we have $Hoch_*(SM,K) \approx Hoch_*(k[x_1,\ldots,x_r],K)$ and similarly we have $Hoch^*(SM,K) \approx Hoch^*(k[x_1,\ldots,x_r],K)$. Now $Harr_*(SM,K) \approx e_*(Hoch_*(SM,K)) \approx e_*(Hoch_*(k[x_1,\ldots,x_r],K)) \approx Harr_*(SF,K)$ and dually for cohomology. But from section 2, we know that Harrison's homology theory and cohomology theory are zero in dimensions greater than one for a polynomial algebra.

4.5 <u>Theorem</u>: If M is finitely generated over k, Q is in k, and K is a residue field of SM, then, $Harr_i(SM,K) = 0 = Harr^i(SM,K)$ for $i>1$. Furthermore, $Harr_1(SM,K) \approx \text{Diff } SM\otimes_{SM}K \approx M\otimes K$ and $Harr^1(SM,K) \approx Der(SM,K) \approx Hom_k(M,K)$

<u>Proof</u>: Because of the foregoing, we need only prove the part concerning Harrison's homology and cohomology in the first dimension. Now, $Harr_1(SM,K) = SM\otimes K/L$ where L is the k-submodule generated by all elements of the form $[a_1a_2]\otimes 1 - [a_2]\otimes a_1\cdot 1 - [a_1]\otimes a_2\cdot 1$. Then consider the SM-exact sequence

$$0 \longrightarrow N \longrightarrow SM\otimes SM \longrightarrow \text{Diff } SM \longrightarrow 0$$

where N is the SM-submodule needed to define Diff SM. Then we will find that we have an exact sequence

$$N\otimes_{SM}K \longrightarrow SM\otimes SM\otimes_{SM}K \longrightarrow \text{Diff } SM\otimes_{SM}K \longrightarrow 0$$

The last term of this sequence is Diff $SM\otimes_{SM}K \simeq SM\otimes K/N\otimes_{SM}K \simeq SM\otimes K/L$ since $N\otimes_{SM}K$ is obviously L. Further, it is easy to see that Diff SM $\simeq$ $SM\otimes M$ so we obtain the second isomorphism. By dualizing, we obtain the proof for cohomology.

5. The General Case

Before we proceed any further, we find it convenient to state two lemmas concerning finitely generated modules over commutative rings.

5.1 Lemma: If E finitely generated over R and $I \subsetneq R$ is an ideal such that $IE = E$, Then there is an $r \in I$ such that $re = e$ for all $e \in E$.

Proof: This is simply lemma 2, page 215 of Zariski and Samuel (15).

5.2 Lemma: Let E be finitely generated over R. If $ME = E$ for all maximal ideals $M \subsetneq R$, then $E = 0$.

Proof: From lemma 5.1, we see that each maximal ideal contains at least one element p with $pe = e$ for all $e \in E$. If $1-p$ is a unit in R for any such p in any maximal ideal, we are done. If not, consider the ideal, I, generated by all the $1-p$'s. We claim that the ideal I must now be the entire ring. If not, Then $I \subsetneq M'$ where M' is a maximal ideal. Then there is $q \in M'$ and $qe = e$ for all $e \in E$. But then, $1-q \in I$. Thus, $1 = q + (1-q) \in M'$ and this is impossible. Since $I = R$, we see $1 = \sum_{i=1}^{\ell} r_i(1-p_i)$. But then

$$e = 1\cdot e = (\sum_{i=1}^{\ell} r_i(1-p_i))e = \sum_{i=1}^{\ell} r_i(1-p_i)e = 0.$$

We shall for the time being be concerned with noetherian rings. for the outlines of the theory of modules over such rings, we refer the reader to Zariski and Samuel (15), chapter 4.

5.3 Theorem: Let k be a noetherian ring containing the rational numbers. Let M be a finitely generated k-module and N a finitely generated SM-module. Then $\mathrm{Harr}^n(SM,N) = 0 = \mathrm{Harr}_n(SM,N)$ for $n>1$. Further, $\mathrm{Harr}_1(SM,N) \simeq \mathrm{Diff}\ SM \otimes N \simeq M \otimes N$ and $\mathrm{Harr}^1(SM,N) \simeq \mathrm{Der}(SM,N) \simeq \mathrm{Hom}_k(M,N)$.

Proof: The second part of the theorem is obtained using the same reasoning used in theorem 4.5. The first part of the theorem is a bit harder. In order to show it, we adapt for our purposes the proof of a theorem of Harrison (Harrison (8), theorem 2.2).

Suppose there is a finitely generated N with $\mathrm{Harr}^n(SM,N) = 0$, $n>1$. Consider the set of all submodules $\{N_i\}_{i \in I}$ such that $\mathrm{Harr}^n(SM,N/N_i) \neq 0$. This set is not empty since it certainly contains the zero submodule. Since k is noetherian, we know that SM is and we apply the maximal principle to this set. Thus, N' is a maximal submodule such that $\mathrm{Harr}^n(SM,N/N') \neq 0$.

Let P be any maximal ideal of SM. We recall that $N':P = \{x \in N \mid Px \subseteq N'\}$. We have an exact sequence

$$\mathrm{Harr}^n(SM,N':P/N') \longrightarrow \mathrm{Harr}^n(SM,N/N') \longrightarrow \mathrm{Harr}^n(SM,N/N':P)$$

since

$$0 \longrightarrow N':P/N' \longrightarrow N/N' \longrightarrow N/N':P \longrightarrow 0$$

is exact.

Since SM acts on N':P via the field SM/P, N':P/N' is a finite dimensional vector space over SM/P and homology commutes with direct sum, we see that $\mathrm{Harr}^n(SM,N':P/N') = 0$. Thus the following sequence is exact.

$$0 \longrightarrow \mathrm{Harr}^n(SM,N/N') \longrightarrow \mathrm{Harr}^n(SM,N/N':P)$$

Now, $\mathrm{Harr}^n(SM,N/N') \neq 0$ and N' is a maximal submodule with this property. Thus, since $N':P \supseteq N'$, either $\mathrm{Harr}_n(SM,N/N':P) = 0$ or $N':P = N'$. The first case is a contradiction so we must consider the second. Since N' is a submodule of N, it is the intersection of a finite number of primary submodules and the radicals of these submodules are prime. By theorem 11, page 214 of Zariski and Samuel (15), P is not contained in any of these radicals. Thus, by a remark on page 215 of (15), P is not contained in the set-theoretic union of the radicals. Thus, there is an α in P which is in none of the primes associated to N. We will then have an exact sequence of SM-modules

$$0 \longrightarrow N/N' \xrightarrow{\alpha} N/N' \longrightarrow N/N' \dagger \alpha \cdot N \longrightarrow 0$$

Now $N' \dagger \alpha \cdot N \neq N'$ since $N':\alpha = N'$. We now have the exact sequence

$$\mathrm{Harr}^n(SM,N/N') \longrightarrow \mathrm{Harr}^n(SM,N/N') \longrightarrow 0$$

because of the maximality of N'.

Thus, $P(\mathrm{Harr}^n(SM,N/N')) = \mathrm{Harr}^n(SM,N/N')$ for all maximal ideals in SM. Now, by proposition 3.1, $\mathrm{Harr}^n(SM,N/N')$ is finitely generated, so, lemma 5.2 tells us that it is zero and this is a contradiction. The dual proof works for homology.

In the still more general case where k is not necessarily noetherian and M and N are not necessarily finitely generated, we must employ subterfuges depending on colimits. By proposition 1.1, $S(-,-)$ is a functor on the category of ring modules and is a left adjoint.

thus it commutes with colimits. Now any ring k containing the rationals is the direct limit of its finitely generated subrings over the rationals, and thus is the colimit of noetherian rings. If M is any k-module, it is a direct limit of finitely generated k-modules $\{M_\gamma\}$. Each M_γ, when considered as a k_α-module, with k_α finitely generated over Q, is the direct limit of finitely generated k_α-modules, $M_{\gamma\beta}$. Thus, $S(k,M)$ is the colimit of $S(k_\alpha,M_{\gamma\beta})$. Similarly, N is the direct limit of $N_{\delta\nu}$ which are finitely generated k_α-modules. Thus, since direct limits commute with tensor products and idempotents, we see that $Ch_n S(k,M)\otimes_{S(k,M)}N$ is the direct limit of $Ch_n S(k_\alpha,M_{\gamma\beta})\otimes_{S(k_\alpha,M_{\gamma\beta})}N_{\delta\nu}$. (See Godement (7), page 10.) It is well-known that homology commutes with direct limits, so we have the following theorem.

5.4 Theorem: Let k be any commutative ring containing Q. Let M be a k-module and N an SM-module. Then $\mathrm{Harr}_n(SM,N) = 0$ for $n>1$ and $\mathrm{Harr}_1(SM,N) = \mathrm{Diff}\ SM\otimes_{SM}N$.

For cohomology we must change our proof since dualizing will get us an inverse limit and homology does not commute with inverse limits. In the notation we introduced prior to theorem 5.4, let us set $S(k_\alpha,M_{\gamma\beta}) = A$ and consider the complex Ch_*A. This will have no homology or cohomology above the first dimension for any coefficient module. Let R_n be the n-cycles of Ch_nA. Let $D_n = Ch_nA/R_n$. Then, D_n is isomorphic to the A-module of (n-1)-boundaries. We have an exact sequence

$$0 \longrightarrow R_n \longrightarrow Ch_nA \longrightarrow D_n \longrightarrow 0$$

Also we have an isomorphism $\eta_{n+1}: D_{n+1} \longrightarrow R_n$.

Now $\partial_n: Ch_n A \longrightarrow Ch_{n-1}A$ may be factored as

$$Ch_n A \xrightarrow{j_n} D_n \xrightarrow{\eta_n} R_{n-1} \xrightarrow{i_{n-1}} Ch_{n-1}A$$

Thus, the diagram below commutes.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & \mathrm{Hom}(R_{n+1},R_n) \arrow[r, "\eta^*_{n+2}"] & \mathrm{Hom}(D_{n+2},R_n) \arrow[d, "j^*_{n+2}"] & \\
\cdots \arrow[r] & \mathrm{Hom}(Ch_n A,R_n) \arrow[r, "\delta_{n+1}"] \arrow[d, "i^*_n"] & \mathrm{Hom}(Ch_{n+1}A,R_n) \arrow[r, "\delta_{n+2}"] \arrow[u, "i^*_{n+1}"] & \mathrm{Hom}(Ch_{n+2},R_n) \arrow[r] & \cdots \\
 & \mathrm{Hom}(R_n,R_n) \arrow[r, "\eta^*_{n+1}"] & \mathrm{Hom}(D_{n+1},R_n) \arrow[u, "j^*_{n+1}"] & & \\
 & & 0 \arrow[u] & &
\end{tikzcd}$$

Now, $\mathrm{Harr}^{n+1}(A,R_n) = 0$. Thus, $\mathrm{Ker}(\delta_{n+2}) = \mathrm{Im}(\delta_{n+1})$. However, both j^*_{n+2} and η^*_{n+2} are monomorphisms. Thus $\mathrm{Ker}(\delta_{n+2}) \subseteq \mathrm{Im}(j^*_{n+1})$. However the opposite inclusion also holds so $\mathrm{Ker}(\delta_{n+2}) = \mathrm{Im}(j^*_{n+1}) = \mathrm{Im}(j^*_{n+1}\eta^*_{n+1})$. Now, $\mathrm{Im}(\delta_{n+1}) = \mathrm{Im}(j^*_{n+1}\eta^*_{n+1}i^*_n)$. Thus $\mathrm{Im}(j^*_{n+1}\eta^*_{n+1}i^*_n) = \mathrm{Im}(j^*_{n+1}\eta^*_{n+1})$. Thus, $\mathrm{Im}(i^*_n) = \mathrm{Hom}(R_n,R_n)$. This implies there exists a map from $Ch_n A$ to R_n which is the identity when restricted to R_n. Thus the complex $Ch_* A$ splits and since the boundary factors in the manner shown above, this splitting is natural. Since the complex $Ch_* S(k,M)$ is the direct limit of these split complexes and naturality implies coherence with the colimit diagram. Thus the cohomology of the complex is zero too and we have the following theorem.

5.5 <u>Theorem</u>: Let k be any commutative ring containing the rationals. Let M be a k-module and N be an SM-module. Then, $\mathrm{Harr}^n(SM,N) = 0$ for $n>1$ and $\mathrm{Harr}^1(SM,N) \simeq \mathrm{Der}(SM,N) \simeq \mathrm{Hom}_k(M,N)$.

6. The Double Complex

We now have all the tools we need to complete the proof of theorem 1.2. Let us consider a double complex $E_{i,j}$ where $E_{i,j}$ $e_{i+1}(S^{j+1}A)^{(i+1)} \otimes M$. There are two boundary maps; the first is $D^I_i \colon E_{i,j} \longrightarrow E_{i-1,j}$ which is the restriction of the Hochschild boundary and the second is $D^{II}_j \colon E_{i,j} \longrightarrow E_{i,j-1}$ which is the cotriple boundary map for the cotriple S.

There is a map of k-modules $\psi A \colon A \longrightarrow SA$ by front adjunction. This gives rise to a contracting homotopy in the complex

$$\cdots \longrightarrow S^2A \otimes M \longrightarrow SA \otimes M \longrightarrow A \otimes M \longrightarrow 0$$

Thus, the n-fold tensor product of the homotopy composed with e_n gives rise to a contracting homotopy in

$$\cdots \longrightarrow e_n((S^2A)^{(n)}) \otimes M \longrightarrow e_n((SA)^{(n)}) \otimes M \longrightarrow e_n(A^{(n)}) \otimes M \longrightarrow 0$$

Thus, $H_{II}(E)$ is simply the complex $\mathrm{Ch}_* A \otimes_A M$ so $H_I H_{II}(E)$ is $\mathrm{Harr}_*(A,M)$. On the other hand, we have shown $H^j_I(E)$ is $\mathrm{Diff}\ S^{j+1}A \otimes_A M$ concentrated in bidegree $(j,0)$. Thus, $H_{II}H_I(E)$ is $\mathrm{Symm}_*(A,M)$ and by theorem 6.1, page 342, of MacLane (11), we are done.

PARTIAL RESULTS UP TO DIMENSION 2p

1. The Case of a Field of Characteristic p

Let us now consider a perfect field of characteristic p. It would be very nice if we were able to report that $\mathrm{Harr}_i(R,M) = 0$ for $i>1$ for all polynomial algebras R over k and all R-modules M. Then we would be able to show that Harrison's theory coincides with the theory afforded by the symmetric algebra cotriple for the case of characteristic p as well as for the characteristic zero case. Unfortunately, this is not true as Andre has shown by example (see Barr (2)). However, we can show that $\mathrm{Harr}_i(R,M) = 0$ for $1<i<2p$.

First we must examine the skew-commutative graded algebra functor. This is the left adjoint to the underlying functor, $\tilde{U}$, which goes from the category of graded, skew-commutative k-algebras to the category of graded k-modules. It may be constructed explicitly in the following manner. Let M be a graded k-module. Let TM be the tensor algebra on M with the following grading. The degree of $a_1 \otimes \dots \otimes a_i$ is $\deg(a_1) + \deg(a_2) + \dots + \deg(a_i)$. Then, $\tilde{S}M$ is TM modulo the two sided ideal generated by elements of the form $a_1 \otimes \dots \otimes a_i - (-1)^{\dot{a}_j \dot{a}_{j+1}} a_1 \otimes \dots \otimes a_j \otimes a_{j+1} \otimes \dots \otimes a_i$ where $\dot{a}_j$ stands for the degree of a_j. It is now easy to see that $\tilde{S}$ is left adjoint to $\tilde{U}$. Also, it is clear that

$$\tilde{S}M \simeq \Lambda(M_1 \oplus M_3 \oplus \dots) \otimes S(M_0 \oplus M_2 \oplus \dots)$$

where Λ is the exterior algebra functor and S is the symmetric algebra functor and both are defined with respect to the field k. We shall use S in the following lemma.

1.1 Lemma: Let R be a polynomial algebra. Then $\mathrm{Harr}_i(R,k) = 0$ for $1<i<2p$.

Proof: Once again we shall assume that R is finitely generated and afterwards use colimit arguments. We must now consider the spectral sequence of the complex $C_*R\otimes_R k$ which is obtained through the use of the filtration $C_*R\otimes_R k \supseteq J_*R\otimes_R k \supseteq J_*^2R\otimes_R k \supseteq \ldots.$ We set $F_0 = C_*R\otimes_R k$, $F_1 = J_*R\otimes_R k$, and, in general, $F_i = J_*^iR\otimes_R k$. Then we see that $F_i \supseteq F_{i+1}$ and we set $E^2_{r,s} = H_{r+s}(F_s/F_{s+1})$. We note that $E^2_{r,1}$ $\mathrm{Harr}_{r+1}(R,k)$.

Since the sequence is bounded both above and below, the sequence will converge to the homology of $C_*R\otimes_R k$ which we already know to be $\mathrm{Tor}^R_*(k,k)$. Assmus has shown in (1) that $\mathrm{Tor}^R_*(k,k)$ is a Hopf algebra over the field k and since the multiplication is the shuffle product, we know it is skew-commutative. Therefore, by the structure theorem of Borel (Milnor and Moore (12)), $\mathrm{Tor}^R_*(k,k)$ is the tensor product of an exterior algebra where the generators are of odd degree and monogenic algebras $k[x]/(x^{p^a})$, $a\geq 1$ where the generators are of even degree.

Now let an element of $\mathrm{Tor}^R_*(k,k)$ be called decomposable if it can be represented by a shuffle and indecomposable otherwise. Let W_* represent the complementary subspace to the decomposable objects in $\mathrm{Tor}^R_*(k,k)$. The Borel theorem tells us that $\mathrm{Tor}^R_n(k,k) = (SW_*)_n$ for $n<2p$. We wish to show that there exists an epimorphism

$$W_n \longrightarrow \mathrm{Tor}^R_n(k,k) \longrightarrow \mathrm{Harr}_n(k,k)$$

for $n<2p$.

We shall proceed by induction using the spectral sequence $\{E^m_{r,s}, \partial_m\}$. The case for $r+s=1$ is trivial since $\mathrm{Tor}^R_1(k,k)$ and $\mathrm{Harr}_1(R,k)$ are both the free k-module on the set X where $R=k[X]$. Since there are no decomposable elements in the first dimension, we see $W_1 = \mathrm{Tor}^R_1(k,k)$ and so $W_1 \longrightarrow \mathrm{Harr}_1(R,k)$ is epic. Furthermore, we note that the differentials out of $E^2_{0,1}$ are zero.

Assume now that the assertion is true for $r+s=n-1$. By Quillen (14), $E^2_{*,*}$ is generated as an algebra by $E^2_{*,1}$ up to degree $2p-1$. Thus, since the differentials are derivations, all differentials issuing from $E^m_{r,s}$ are zero for $r+s=n$. Thus, $E^\infty_{r,s} = E^2_{r,s}/B^\infty_{r,s}$ for $r+s=n$. Thus the graded module $\mathrm{Tor}^R_n(k,k)$ may be written as

$$\mathrm{Tor}^R_n(k,k) = \sum_{i=0}^{n} E^2_{i,n-i} = \sum_{i=0}^{n-2} E^2_{i,n-i}/B^\infty_{i,n-i} + \mathrm{Harr}_n(R,k).$$

Since $W_i \longrightarrow \mathrm{Harr}_i(R,k)$ is epimorphic for $i \leq n-1$, we have $\dim(E^2_{r,s}) = \dim(\tilde{S}_s\mathrm{Harr}_*(R,k))_{r+s} \leq \dim(\tilde{S}_sW_*)_{r+s}$ for $r+s \leq n$. Then,

$$\begin{aligned}
\dim(\mathrm{Tor}^R_n(k,k)) &= \dim\Big(\sum_{i=0}^{n-2} E^2_{i,n-i}/B^\infty_{i,n-i} \oplus \mathrm{Harr}_n(R,k)\Big) \\
&= \sum_{i=0}^{n-2} \dim(E^2_{i,n-i}/B^\infty_{i,n-i}) + \dim(\mathrm{Harr}_n(R,k)) \\
&= \sum_{i=0}^{n-2} \dim(\tilde{S}_nW_*)_n + \dim(W_n) \\
&= \dim(\mathrm{Tor}^R_n(k,k)
\end{aligned}$$

for $n<2p$. Thus $B^\infty_{i,n-i}$ must be zero for $0 \leq i \leq n-2$. Thus all differentials emanating from $E^2_{r,s}$ where $r+s=n$ must be zero. Thus, we see that

$$W_n \longrightarrow \mathrm{Tor}^R_n(k,k) \longrightarrow \mathrm{Harr}_n(R,k)$$

is epic. This implies $E^{\infty}_{r,1} = E^{2}_{r,1}$ for $r+1<2p$. Now, because of the convergence of the spectral sequence, we have

$$E^{2}_{r,1} = F_r(H_{r+1}(C_*R\otimes_R k))/F_{r+1}(H_{r+1}(C_*R\otimes_R k)).$$

But the right hand side is zero for $r>0$, since $H_{r+1}(C_*R\otimes_R k)$ is generated by cycles of the form $E_{r+1}([x_{i_1},\ldots,x_{i_{r+1}}])$ where E_{r+1} is the element of the group ring $k\Sigma_{r+1}$ which we constructed in the first chapter. These generators are contained in both $F_r(H_{r+1}(C_*R\otimes_R k)) = \mathrm{Im}(H_{r+1}(F_r) \longrightarrow H_{r+1}(F_0))$ and $F_{r+1}(H_{r+1}(C_*R\otimes_R k)) = \mathrm{Im}(H_{r+1}(F_{r+1}) \longrightarrow H_{r+1}(F_0))$. Thus, $E^{2}_{r,1} = \mathrm{Harr}_{r+1}(R,k) = 0$ for $1\leq r\leq 2p-2$. Using the same techniques we used before, we find that the lemma works for any set of variables.

1.2 <u>Proposition</u>: Let (X) denote the ideal in $k[X]$ generated by the set X. There is an R-exact sequence of complexes

$$(*)\ 0 \longrightarrow Ch_*R\otimes_R(X) \longrightarrow Ch_*R\otimes_R R \approx Ch_*R \longrightarrow Ch_*R\otimes_R k \longrightarrow 0$$

<u>Proof</u>: We consider the R-exact sequence of R-modules

$$0 \longrightarrow (X) \longrightarrow R \longrightarrow k \longrightarrow 0$$

Then $Ch_nR = R\otimes(R^{(n)}/sh_n(R))$ where $sh_n(R)$ is the shuffle submodule of $R^{(n)}$. Clearly, $Ch_nR\otimes_R M = (R^{(n)}/sh_n(R))\otimes M$ for any R-module M. Since k is a field, the following sequence is exact.

$$0 \longrightarrow (R^{(n)}/sh_n(R))\otimes(X) \longrightarrow (R^{(n)}/sh_n(R))\otimes R \longrightarrow (R^{(n)}/sh_n(R))\otimes k \longrightarrow 0$$

Moreover, the boundary homomorphisms in Ch_*R obviously commute with the homomorphisms of the exact sequence so the sequence $(*)$ is exact as a sequence of complexes.

If we now take homology arising from the complexes above, and consider the long exact sequence arising from that homology, we find we get an epimorphism $H_n(Ch_*R\otimes_R(X)) \longrightarrow H_n(Ch_*R)$ for $1<n<2p$. Now consider the exact sequence of R-modules

$$0 \longrightarrow (X)^{i+1} \longrightarrow (X)^{i} \longrightarrow (X)^{i}/(X)^{i+1} \longrightarrow 0$$

Using the above techniques, we get an exact sequence of complexes

$$0 \longrightarrow Ch_{*}R\otimes_{R}(X)^{i+1} \longrightarrow Ch_{*}R\otimes_{R}(X)^{i} \longrightarrow Ch_{*}R\otimes_{R}(X)^{i}/(X)^{i+1} \longrightarrow 0$$

Now R acts on $(X)^{i}/(X)^{i+1}$ via the augmentation $R \longrightarrow k$. Thus, since $(X)^{i}/(X)^{i+1}$ may be considered as a vector space over k and so a direct sum of copies of k and since homology commutes with direct sums, we find $H_n(Ch_{*}R\otimes_{R}(X)^{i}/(X)^{i+1}) = 0$ for $1<n<2p$. Thus the long exact sequence tells us that there is an epimorphism from $H_n(Ch_{*}R\otimes_{R}(X)^{i+1})$ to $H_n(Ch_{*}R\otimes_{R}(X)^{i})$ for $1<n<2p$. Thus, by induction, we see that there is an epimorphism from $H_n(Ch_{*}R\otimes_{R}(X)^{i})$ to $H_n(Ch_{*}R)$ for all i and 1 n 2p.

Let us now return to the study of J_nR. As a k-vector space, J_nR has a basis consisting of elements of the form $m_0\ m_1,\ldots,m_n = v$, where the m_i are monomials in R. Let $\deg(v) = \deg(m_0) + \ldots + \deg(m_n)$. If we have any arbitrary element of J_nR, we set its degree equal to the degree of the highest basis element in its expansion as a unique linear combination of elements of the above basis.

Now suppose c is a cycle in Ch_nR, $1<n<2p$. Then c is the image of some $\hat{c}$ under the canonical quotient mapping $J_nR \longrightarrow Ch_nR$. Let $\deg(\hat{c}) = t$. Then, because of the epimorphisms we calculated before, there is a cycle, c', in $J_nR\otimes_{R}(X)^{t+1}$ and $c'-\hat{c} \in J_n^2R + \partial_{n+1}J_{n+1}R$. Since the degree induces a grading wherever it goes, on $J_n^2R + \partial_{n+1}J_{n+1}R$ as well as J_nR, this cannot happen unless both $\hat{c}$ and c' are in the above complex. Thus, we see that the homology class of c in $H_n(Ch_{*}R)$ must be zero. Thus $H_n(Ch_{*}R) = 0$ for $1<n<2p$.

Once again, since homology commutes with direct sums, we see that $H_n(Ch_*R\otimes_R F) = 0$ for free R-modules F and $1<n<2p$. This, in turn, implies $H_n(Ch_*R\otimes_R P) = 0$ for $1<n<2p$ for all projective R-modules, P.

For a general R-module M, we need to consider a R-projective resolution of M, say

$$\cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

Tensor the projective resolution with the complex Ch_*R to get the double complex:

$$(*)\quad \begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 & Ch_1R\otimes_R X_i \arrow[l] \arrow[d] & Ch_2R\otimes_R X_i \arrow[l] \arrow[d] & \cdots \arrow[l] \\ & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 & Ch_1R\otimes_R M \arrow[l] \arrow[d] & Ch_2R\otimes_R M \arrow[l] \arrow[d] & \cdots \arrow[l] \\ & 0 & 0 & \end{tikzcd}$$

If we take homology going down in (*) we will get zero since each Ch_iR is a projective R-module. Taking homology across will not change this. On the other hand, if we take homology across, we will get a set of complexes like

$$\cdots \longrightarrow Harr_i(R,X_1) \longrightarrow Harr_i(R,X_0) \longrightarrow Harr_i(R,M) \longrightarrow 0$$

which must be acyclic. Thus, since $Harr_i(R,X) = 0$ for $1<i<2p$ we see

$\mathrm{Harr}_i(R,M) = 0$ for $1<i<2p$. It is easily seen that $\mathrm{Harr}_1(R,M) = M^X$
In order to complete the proof for cohomology, we make use of the following lemma from MacLane (11), page 78.

1.3 Lemma: If K is a chain complex composed of vector spaces over a field k, and V is any vector space over that field, There is a natural isomorphism $H^n(K,V) \simeq \mathrm{Hom}(H_n(K),V)$.

We now note that

$$\mathrm{Hom}_k(Ch_n R \otimes_R k,k) \simeq \mathrm{Hom}_R(Ch_n R,\mathrm{Hom}_k(k,k)) \simeq \mathrm{Hom}_R(Ch_n R,k)$$

from the adjointness of tensor and Hom. Thus we see that $\mathrm{Harr}^n(R,k) \simeq \mathrm{Hom}(\mathrm{Harr}_n(R,k),k)$ and this is zero for $1<n<2p$. From the exact sequences

$$0 \longrightarrow (X) \longrightarrow R \longrightarrow k \longrightarrow 0$$

and

$$0 \longrightarrow (X)^{i+1} \longrightarrow (X)^i \longrightarrow (X)^i/(X)^{i+1} \longrightarrow 0$$

as before, we find that there is an epimorphism from $\mathrm{Harr}^n(R,(X)^i)$ to $\mathrm{Harr}^n(R,R)$ for all i and $1<n<2p$. This implies that every cocycle in $\mathrm{Hom}_R(Ch_n R,R)$ is the sum of a coboundary and a cocycle which has its image in $(X)^i$ for all i. But then that cocycle must be zero and so the original cocycle is a coboundary which implies that $\mathrm{Harr}^n(R,R) = 0$
Thus $\mathrm{Harr}^n(R,F) = 0$ for $1<n<2p$ and all free R-modules F. This implies the theorem for projective R-modules. For an arbitrary R-module, we take a projective resolution and use the double complex as before.

Heretofore, we have been working with a perfect field. In the more general case, when k is not perfect, we only need to make minor adjustments. As a matter of fact, we only need to note that $k[X] \simeq k \otimes_{Z_p} Z_p[X]$ where Z_p is the prime subfield of k. Then we see

$$J_n k[X] = k[X] \otimes k[X]^{(n)} = k \otimes_{Z_p} (Z_p[X] \otimes_{Z_p} Z_p[X]^{(n)}) = k \otimes_{Z_p} J_n Z_p[X]$$

Thus, $Ch_n k[X] \otimes_{k[X]} M \simeq Ch_n Z_p X \otimes_{Z_p[X]} M$ where M is any $k[X]$ module. Since Z_p is finite, it is perfect and the foregoing arguments hold for k. We may now state the main theorem of this chapter.

1.4 Theorem: Let k be a field of characteristic $p>0$. Let $R = k[X]$ be a polynomial algebra over k and let M be an R-module. Then $Harr_n(R,M) = 0 = Harr^n(R,M)$ for $1<n<2p$ and $Harr_1(R,M) \simeq M^X$ and $Harr^1(R,M) \simeq Der(R,M)$.

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VITA

Patrick John Fleury was born on March 17, 1943 in Providence, Rhode Island. He received a Bachelor of Arts degree from Brown University in that city in June, 1964. He enrolled at the University of Illinois in September, 1964 and received a Master of Arts degree in June, 1965. He has been employed at McGill University in Montreal, Canada, since September, 1968.