ABSTRACT MEAN VALUES

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1. Introduction. The entropic law for a binary operation $+$ on a set S, that is,

$$(x + y) + (z + w) = (x + z) + (y + w) \quad \text{for all} \quad x, y, z, w \quad \text{in} \quad S,$$

has been studied by many authors (see [13] and the references given there). In particular, this law or its n-ary generalization is a natural condition to assume in characterizing mean value functions on the real numbers, [1], [2]. This paper is a study of the n-ary generalization of the entropic law and has two main objects (i) to describe the structure of an n-ary operation on an abstract set satisfying the law, (ii) to give algebraic versions of some of the characterizations of mean value functions on the real numbers which have been discussed by various authors.

In [9] and [12], Kolmogorov and Nagumo have shown that if M_n, $n = 1, 2, 3, \cdots$ is an infinite sequence of strictly increasing continuous symmetric idempotent functions on the real numbers such that for all $k < n$, and all $x_1, x_2, \cdots, x_n$,

$$M_n(x_1, x_2, \cdots, x_n) = M_n(M_k, M_k, \cdots, M_k, x_{k+1}, \cdots, x_n),$$

(where M_k denotes $M_k(x_1, x_2, \cdots, x_k)$), then each M_n is a generalized arithmetic mean.

We prove that if the same algebraic conditions hold for an infinite sequence of operations $M_1, M_2, \cdots, M_n, \cdots$ on a set S with the continuity and order conditions replaced by the assumption that S contains an element g such that the mapping $x \to M_n(x, g, g, \cdots, g)$ is one-one onto S, for each n, then each M_n is an arithmetic mean on a certain commutative semigroup $(S, +)$.

We obtain the above result as a consequence of the algebraic analogue of the following result due to Aczél [1]. If M is a continuous, strictly increasing idempotent function of n variables on the reals such that, for any $n \times n$ matrix of real numbers, M satisfies the generalized entropic law

$$M\{M(\mathbf{r}_1), M(\mathbf{r}_2) \cdots, M(\mathbf{r}_n)\} = M\{M(\mathbf{c}_1), M(\mathbf{c}_2), \cdots, M(\mathbf{c}_n)\}$$

where $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n, \mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$ are the row and column vectors of the matrix, then M is a generalized weighted arithmetic mean.

We prove that if an n-ary operation M on a set S satisfies the algebraic conditions assumed by Aczél and if S contains an element g satisfying the regularity condition that at least two of the mappings

$$x \to M(x, g, g, \cdots, g), x \to M(g, x, g, \cdots, g), \cdots, x \to M(g, g, \cdots, g, x)$$

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are one-one onto S, then we can construct a commutative semigroup with zero, $(S, +)$ such that M is a weighted arithmetic mean on $(S, +)$ in the sense that

$$M(x_1, x_2, \cdots, x_n) = \tau_1 x_1 + \tau_2 x_2 + \cdots + \tau_n x_n$$

where $\tau_1, \tau_2, \cdots, \tau_n$ are fixed endomorphisms of the semigroup which commute in pairs and whose sum $\tau_1 + \tau_2 + \cdots + \tau_n$ is the identity mapping.

In the preceding result, if we do not assume that M is idempotent but that S contains one idempotent element, with the same regularity properties as the element g above, then we obtain a similar characterization for M but without the condition that $\tau_1 + \tau_2 + \cdots + \tau_n$ is the identity mapping. This result generalizes a theorem of Toyoda [14] who assumed a stronger solvability of equations condition on the operation M.

Using the above result, we derive a structure theorem for an n-ary operation M on a set S satisfying the generalized entropic law. We assume only that S contains at least one element with certain regularity properties. Then we may construct on S a commutative semigroup with zero such that

$$M(x_1, x_2, \cdots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + d$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are fixed pairwise commuting endomorphisms of the semigroup and d is a fixed element of S. This generalizes the structure theory for entropic quasigroups given by Murdoch [11] and Bruck [3], as well as a theorem obtained by Aczél [1] for the real continuous case.

2. **Notation and terminology.** The algebra consisting of a set S and an n-ary operation Π defined on S will be denoted by (S, Π). We will often use parentheses-free notation in denoting the value of an operation and write $\Pi x_1 x_2 \cdots x_n$ for the element which the n-ary operation Π assigns to the sequence $x_1, x_2, \cdots, x_n$ of elements of S. Mappings of S into itself will be written on the left. Thus α is an endomorphism for the operation Π if

$$\Pi(\alpha x_1)(\alpha x_2) \cdots (\alpha x_n) = \alpha \Pi x_1 x_2 \cdots x_n \quad \text{for all} \quad x_1, x_2, \cdots, x_n .$$

A modified vector notation will be useful. We will write $\mathbf{x}_1^n$ for the sequence $x_1, x_2, \cdots, x_n$, or write simply $\mathbf{x}$ if there is no ambiguity. If $x_1, x_2, \cdots, x_N$ is a sequence of N elements, we will write $\mathbf{x}_i^j$ for the subsequence $x_i, x_{i+1}, \cdots, x_j$. If $i = j$, $\mathbf{x}_i^j$ is the element x_i and if $i > j$, $\mathbf{x}_i^j$ is to be interpreted as the empty sequence. If α is a mapping of S into itself and x is a sequence $x_1, x_2, \cdots, x_n$ of elements of S, then, by $(\alpha \mathbf{x})$ we will mean the sequence $\alpha x_1, \alpha x_2, \cdots, \alpha x_n$. Thus, the condition that α is an endomorphism with respect to the operation Π may be written as

$$\Pi(\alpha \mathbf{x}) = \alpha \Pi \mathbf{x} \quad \text{for all} \quad \mathbf{x}.$$

Similarly, if $\alpha_1, \alpha_2, \cdots, \alpha_n$ is a sequence of mappings of S into S, we will write $\boldsymbol{\alpha}$ for this sequence and write $(\boldsymbol{\alpha} x)$ for the sequence $\alpha_1 x, \alpha_2 x, \cdots, \alpha_n x$ where x is an element of S.

Since exponents will not be needed, we will write x^i for a sequence of length i each term of which is x. In general, if E is any expression, E^i or $(E)^i$ will denote the sequence of i E's. E^0 will denote the empty sequence.

If g is an element of S, the mapping $\tau_i : S \to S$, $i = 1, 2, \cdots, n$ defined by

$$\tau_i : x \to \Pi g^{i-1} x g^{n-i} \quad \text{for all} \quad x$$

will be called the i-th translation of g. If such a mapping is one-one onto S, then g will be called i-regular. If each translation is one-one onto S, then g will be called regular. An element e of S will be called an i-unit if its i-th translation is the identity mapping and will be called a unit if this is so for all i.

An element g of S will be called an idempotent if $\Pi g^n = g$ and the operation Π will be said to be idempotent if $\Pi x^n = x$ for all x.

We say that Π is symmetric if, for each permutation ϕ on $\{1, 2, \cdots, n\}$, Π satisfies, for all $x_1, x_2, \cdots, x_n$,

$$\Pi x_1 x_2 \cdots x_n = \Pi x_{\phi 1} x_{\phi 2} \cdots x_{\phi n} .$$

We say that Π is associative if it satisfies the following generalization of the binary associative law,

$$\Pi^2 \mathbf{x}_1^{2n-1} = \Pi \mathbf{x}_1^{n-1} \Pi \mathbf{x}_n^{2n-1}, \quad \text{for all} \quad \mathbf{x}_1^{2n-1}.$$

We will need to consider $n \times n$ arrays of n^2 elements of S. We denote the element in the i-th row, j-th column by x_{ij} and the array by X or (x_{ij}), $i, j = 1, 2, \cdots, n$. The rows and columns of such an array X are sequences of n elements which we will denote by $\mathbf{r}_i$, $\mathbf{c}_j$ respectively and $\Pi\mathbf{r}_i$, $\Pi\mathbf{c}_j$ will denote the elements which the n-ary operation Π assigns to the sequences $\mathbf{r}_i$ and $\mathbf{c}_j$. By ΠX, we will mean the sequence

$$\Pi\mathbf{r}_1, \Pi\mathbf{r}_2, \cdots, \Pi\mathbf{r}_n$$

having $\Pi\mathbf{r}_i$ as its i-th element and $\Pi\Pi X$ or $\Pi^2 X$ will denote the element $\Pi(\Pi\mathbf{r}_1)(\Pi\mathbf{r}_2) \cdots (\Pi\mathbf{r}_n)$ of S.

If Π satisfies the following law

$$\Pi(\Pi\mathbf{r}_1)(\Pi\mathbf{r}_2) \cdots (\Pi\mathbf{r}_n) = \Pi(\Pi\mathbf{c}_1)(\Pi\mathbf{c}_2) \cdots (\Pi\mathbf{c}_n) \quad \text{for all} \quad n \times n \text{ arrays } X,$$

then Π will be called entropic and this law will be called the entropic law. This name is due to Etherington for a binary operation [4].

If we use the usual matrix notation, X^T to denote the transpose of X, that is, the $n \times n$ array having x_{ji} in its i-th row, j-th column, then the entropic law may be written as

$$\Pi^2 X = \Pi^2 X^T \quad \text{for all} \quad n \times n \text{ arrays } X.$$

An extension of this notation to the case where the entropic law is a relation between two operations and the array X is not longer square, is discussed in [7].

3. **Properties of entropic operations.** We begin with a study of some of the properties of an entropic operation Π on a set S. The main theorem we

obtain in this section is a characterization of an entropic operation on a set which contains a two-place regular idempotent.

LEMMA 3.1. *If* (S, Π) *is an entropic algebra with a unit element, then* Π *is symmetric and associative.*

Proof. Let e be the Π-unit in S. Let $x_1, x_2, \cdots, x_n$ be any elements of S and let ϕ be any permutation on $\{1, 2, \cdots, n\}$. The entropic law applied to the $n \times n$ array having, for $i = 1, 2, \cdots, n$, the element x_i in row i, column $\phi^{-1}i$, and e in every other place, gives the symmetric property of Π immediately.

Now let $x_1, x_2, \cdots, x_{2n-1}$ be any $2n - 1$ elements of S and consider the array X having $x_1, x_2, \cdots, x_n$ as its first row, $x_n, x_{n+1}, \cdots, x_{2n-1}$ as its n-th column, and e in every other place. The entropic law applied to X reduces to the associative law.

LEMMA 3.2. *If* (S, Π) *is an entropic algebra with a unit, then there is a comutative semigroup* $(S, +)$ *with a zero element, such that*

$$\Pi x_1 x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n$$

for all $x_1, x_2, \cdots, x_n$ *in* S.

Proof. If e is the unit, we define an operation $+$ on S by

$$x + y = \Pi x e^{n-2} y$$

for all x, y in S. By Lemma 3.1, the operation $+$ is commutative and associative. Since e is a Π-unit, $(S, +)$ has e as a zero.

If $\Pi \mathbf{x}_1^i e^{n-i} = x_1 + x_2 + \cdots + x_i$, then

$$\begin{aligned} \Pi \mathbf{x}_1^{i+1} e^{n-i-1} &= \Pi \mathbf{x}_1^i e^{n-i-1} x_{i+1} \\ &= \Pi \mathbf{x}_1^i e^{n-i-1} (\Pi e^{n-1} x_{i+1}) \\ &= \Pi (\Pi \mathbf{x}_1^i e^{n-i}) e^{n-2} x_{i+1} \\ &= (x_1 + x_2 + \cdots + x_i) + x_{i+1} . \end{aligned}$$

Hence, by induction, $\Pi \mathbf{x} = x_1 + x_2 + \cdots + x_n$.

Our first generalization of this consists of replacing the condition that the algebra has a unit with the requirement that it contains a regular idempotent.

LEMMA 3.3. *If* (S, Π) *is an entropic algebra, then the i-th translation by an idempotent* g *is an endomorphism (automorphism if* g *is i-regular). Furthermore, any two such translations by an idempotent commute.*

Proof. Let g be an idempotent in S and let $x_1, x_2, \cdots, x_n$ be any n elements of S. Let X be the $n \times n$ array having $x_1, x_2, \cdots, x_n$ as its i-th row and g in every other place. Then

$$\Pi \mathbf{c}_j = \tau_i x_j , \quad \text{for all} \quad j,$$

where τ_i is the i-th translation by g. Now, from $\Pi^2 X = \Pi^2 X^T$, we obtain

$$\Pi g^{i-1}(\Pi \mathbf{x}) g^{n-i} = \Pi(\tau_i \mathbf{x}).$$

That is,

$$\tau_i \Pi \mathbf{x} = \Pi(\tau_i \mathbf{x}).$$

Hence τ_i is an endomorphism. Clearly, if g is i-regular, τ_i is an automorphism.

To prove that τ_i, τ_j commute, we apply the entropic law to the array X having an element x in its i-th row, j-th column, and g in every other place.

The next two lemmas enable us to construct from an entropic algebra with a regular idempotent, an entropic algebra with a unit.

LEMMA 3.4. *Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be endomorphisms of the entropic algebra (S, Π) which commute in pairs. Then the operation Σ on S defined by*

$$\Sigma x_1 x_2 \cdots x_n = \Pi(\alpha_1 x_1)(\alpha_2 x_2) \cdots (\alpha_n x_n)$$

is entropic. Furthermore, the α_i are Σ-endomorphisms.

Proof. Let $X = (x_{ij})$ be an $n \times n$ array of elements of S. Then

$$\Sigma^2 X = \Sigma \Sigma \mathbf{r}_1 \Sigma \mathbf{r}_2 \cdots \Sigma \mathbf{r}_n$$
$$= \Pi(\alpha_1 \Pi \mathbf{r}_1')(\alpha_2 \Pi \mathbf{r}_2') \cdots (\alpha_n \Pi \mathbf{r}_n')$$

where r_i' is the sequence $\alpha_1 x_{i1}, \alpha_2 x_{i2}, \cdots, \alpha_n x_{in}$.
Hence $\Sigma^2 X = \Pi\Pi(\alpha_1 \mathbf{r}_1')\Pi(\alpha_2 \mathbf{r}_2') \cdots \Pi(\alpha_n \mathbf{r}_n')$ since each α_i is a Π-endomorphism. That is,

$$\Sigma^2 X = \Pi^2 Y$$

where Y is the $n \times n$ array having $\alpha_i \alpha_j x_{ij}$ in its i-th row, j-th column. Similarly,

$$\Sigma^2 X^T = \Sigma \Sigma \mathbf{c}_1 \Sigma \mathbf{c}_2 \cdots \Sigma \mathbf{c}_n$$
$$= \Pi\Pi(\alpha_1 \mathbf{c}_1')\Pi(\alpha_2 \mathbf{c}_2') \cdots (\alpha_n \mathbf{c}_n')$$

where c_j' is sequence $\alpha_1 x_{1j}, \alpha_2 x_{2j}, \cdots, \alpha_n x_{nj}$. That is,

$$\Sigma^2 X^T = \Pi^2 Z$$

where Z is the $n \times n$ array having $\alpha_i \alpha_j x_{ji}$ in its i-th row, j-th column.

Now $\alpha_i \alpha_j = \alpha_j \alpha_i$ and hence Z has $\alpha_j \alpha_i x_{ji}$ in its i-th row, j-th column. That is, Z is the transpose of Y. Hence, Σ is entropic since

$$\Sigma^2 X = \Pi^2 Y = \Pi^2 Y^T = \Pi^2 Z = \Sigma^2 X^T.$$

To prove that α_i is a Σ-endomorphism, we compute. For any elements $x_1, x_2, \cdots, x_n$ in S

$$\begin{aligned}\alpha_i\Sigma\mathbf{x} &= \alpha_i\Pi(\alpha_1 x_1)(\alpha_2 x_2)\cdots(\alpha_n x_n)\\ &= \Pi(\alpha_i\alpha_1 x_1)(\alpha_i\alpha_2 x_2)\cdots(\alpha_i\alpha_n x_n)\\ &= \Pi(\alpha_1\alpha_i x_1)(\alpha_2\alpha_i x_2)\cdots(\alpha_n\alpha_i x_n)\\ &= \Sigma(\alpha_i x_1)(\alpha_i x_2)\cdots(\alpha_i x_n)\\ &= \Sigma(\alpha_i\mathbf{x}).\end{aligned}$$

Hence, α_i is a Σ-endomorphism.

LEMMA 3.5. *Let g be a regular idempotent in the entropic algebra (S, Π). Then the operation Σ on S defined by*

$$\Sigma x_1 x_2 \cdots x_n = \Pi(\tau_1^{-1}x_1)(\tau_2^{-1}x_2)\cdots(\tau_n^{-1}x_n)$$

(where τ_i is the i-th translation by g) is entropic and has g as a Σ-unit.

Proof. By Lemma 3.3, the τ_i are Π-automorphisms which commute in pairs. Hence the τ_i^{-1} are also Π-automorphisms which commute in pairs, and so by Lemma 3.4, the operation Σ is entropic. Direct computation shows that g is a Σ-unit.

We are now in a position to state the first main result of this section.

THEOREM 3.1. *Let the entropic algebra (S, Π) contain a regular idempotent. Then there is a commutative semigroup with a zero $(S, +)$, and automorphisms $\tau_1, \tau_2, \cdots, \tau_n$ of $(S, +)$, which commute in pairs, such that*

$$\Pi x_1 x_2 \cdots x_n = \tau_1 x_1 + \tau_2 x_2 + \cdots + \tau_n x_n \quad \textit{for all} \quad x_1, x_2, \cdots, x_n \quad \textit{in} \quad S.$$

Proof. Define an operation Σ on S as in Lemma 3.5. Then, by Lemma 3.1, there is a commutative semigroup $(S, +)$ with a zero element, such that

$$\Sigma x_1 x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n .$$

Hence, $\Pi x_1 x_2 \cdots x_n = \tau_1 x_1 + \tau_2 x_2 + \cdots + \tau_n x_n$ where, by Lemma 3.4, the τ_i are Σ-automorphisms which commute in pairs.
Since

$$x + y = \Sigma x g^{n-2} y,$$

we have

$$\begin{aligned}\tau_i x + \tau_i y &= \Sigma(\tau_i x)g^{n-2}(\tau_i y)\\ &= \Sigma(\tau_i x)(\tau_i g)^{n-2}(\tau_i y)\\ &= \tau_i\Sigma x g^{n-2} y\\ &= \tau_i(x + y).\end{aligned}$$

Hence, the τ_i are automorphisms of the semigroup and the theorem is proved.

The final result of this section generalizes Theorem 3.1 by removing the restriction of regularity on the idempotent for all but two argument places. The result we obtain is a generalization of work of Toyoda [14]. First we prove a lemma which helps simplify the computation later.

LEMMA 3.6. *Let Π be an entropic n-ary operation on a set S and let ϕ be a permutation of $\{1, 2, \cdots, n\}$. Then the operation Σ on S defined by*

$$\Sigma x_1 x_2 \cdots x_n = \Pi x_{\phi 1} x_{\phi 2} \cdots x_{\phi n}$$

is entropic.

Proof. Let $X = (x_{ij})$ be an $n \times n$ array of elements of S and let X_ϕ denote the $n \times n$ array having $x_{\phi i, \phi j}$ in its i-th row, j-th column. Then

$$\Sigma^2 X = \Pi^2 X_\phi , \qquad \Sigma^2 X^T = \Pi^2 X_\phi^T .$$

Hence $\Sigma^2 X = \Sigma^2 X^T$ and so Σ is entropic.

THEOREM 3.2. *Let the entropic algebra (S, Π) contain an idempotent, regular in two places. Then there is a commutative semigroup with zero $(S, +)$ and pair-wise-commuting endomophisms $\tau_1, \tau_2, \cdots, \tau_n$ of $(S, +)$, two of which are automorphisms, such that*

$$\Pi x_1 x_2 \cdots x_n = \tau_1 x_1 + \tau_2 x_2 + \cdots + \tau_n x_n \quad \textit{for all} \quad x_1, x_2, \cdots, x_n \quad \textit{in} \quad S.$$

Proof. In view of Lemma 3.6, there is no loss of generality if we assume in this proof that Π is 1-regular and 2-regular. In this case, it will appear that τ_1, τ_2 are the automorphisms. In the general case, where Π is p and q-regular, τ_p and τ_q are the automorphisms.

Let g be the idempotent which is 1 and 2-regular. The mappings, $x \to \Pi x g^{n-1}$, $x \to \Pi g x g^{n-2}$ are one-one onto S and hence the binary operation $\oplus$ on S defined by

$$x \oplus y = \Pi x y g^{n-2}$$

has g as a regular idempotent. Furthermore, if we apply the entropic property of Π to the $n \times n$ array having as entries, x_{ij} for $i, j = 1, 2$ and g in every other place, we obtain

$$(x_{11} \oplus x_{12}) \oplus (x_{21} \oplus x_{22}) = (x_{11} \oplus x_{21}) \oplus (x_{12} \oplus x_{22}).$$

Hence the operation $\oplus$ is entropic.

We now apply Theorem 3.1 for the case $n = 2$. Thus there is a commutative semigroup with zero $(S, +)$ and commuting automorphisms τ_1, τ_2 of $(S, +)$ such that

$$x \oplus y = \tau_1 x + \tau_2 y.$$

That is, $\Pi x y g^{n-2} = \tau_1 x + \tau_2 y$. Note that τ_1, τ_2 are Π-translations by g. We define an n-ary operation Σ, which by Lemma 3.4 is entropic on S.

$$\Sigma x_1 x_2 \cdots x_n = \Pi(\tau_1^{-1} x_1)(\tau_2^{-1} x_2) x_3 x_4 \cdots x_n .$$

Thus, $\Sigma x_1 x_2 g^{n-2} = x_1 + x_2$.

Apply the entropic property of Σ to the $n \times n$ array having $x_1, g, g, \cdots, g$ as its first row, $x_2, g, g, \cdots, g$ as its second row, $g, x_3, g, g, \cdots g$ as its third row and g in every other place. We get

$$\begin{aligned}\Sigma x_1 x_2 x_3 g^{n-3} &= \Sigma(\Sigma x_1 x_2 g^{n-2})(\Sigma g^2 x_3 g^{n-3}) g^{n-2} \\ &= (x_1 + x_2) + \tau_3 x_3\end{aligned}$$

where τ_3 is the translation $x \to \Sigma g^2 x g^{n-3}$ and is thus by previous results an endomorphism for Σ and also for $(S, +)$.

Now, assume $\Sigma x_1 x_2 \cdots x_i g^{n-i} = x_1 + x_2 + \tau_3 x_3 + \cdots + \tau_i x_i$ where the mappings $\tau_3, \cdots, \tau_i$ are pairwise commuting endomorphisms of $(S, +)$. Apply the entropic property of Σ to the $n \times n$ array having $x_t, g, g, \cdots, g$ as row t, for $t = 1, 2, \cdots, i$, and $g, x_{i+1}, g, g, \cdots, g$ as row $i + 1$, with g in every other place. We get

$$\begin{aligned}\Sigma \mathbf{x}_1^{i+1} g^{n-i-1} &= \Sigma(\Sigma \mathbf{x}_1^i g^{n-i})(\Sigma g^i x_{i+1} g^{n-i-1}) g^{n-2} \\ &= (x_1 + x_2 + \tau_3 x_3 + \cdots + \tau_i x_i) + \tau_{i+1} x_{i+1}\end{aligned}$$

where τ_{i+1} is the mapping $x \to \Sigma g^i x_{i+1} g^{n-i-1}$ which is a translation of Σ, hence an endomorphism of Σ and of $(S, +)$. Furthermore, this mapping commutes with all other translations by g. A simple induction now shows that

$$\Sigma \mathbf{x} = x_1 + x_2 + \tau_3 x_3 + \cdots + \tau_n x_n$$

and since $\Pi \mathbf{x} = \Sigma(\tau_1 x_1)(\tau_2 x_2) x_3^n$, we have

$$\Pi \mathbf{x} = \tau_1 x_1 + \tau_2 x_2 + \tau_3 x_3 + \cdots + \tau_n x_n$$

where τ_1, τ_2 are automorphisms and $\tau_3, \cdots, \tau_n$ are endomorphisms of $(S, +)$. In fact, τ_i is the i-th translation by g with respect to the operation Π. Thus, the τ_i commute in pairs. This concludes the proof of the theorem.

4. **Abstract means.**

We will call an n-ary operation Π on a set S, an Aczél mean, if (i) Π is entropic (ii) Π is idempotent. These are the conditions other than continuity and order which Aczel [1], [2] assumes for the continuous real case.

THEOREM 4.1. *Let Π be an Aczél mean on a set S containing an element regular in at least two places. Then there is on S a commutative semigroup with zero $(S, +)$ such that*

$$\Pi x_1 x_2 \cdots x_n = \tau_1 x_1 + \tau_2 x_2 + \cdots + \tau_n x_n$$

where $\tau_1, \tau_2, \cdots, \tau_n$ are fixed pairwise commuting endomorphisms of the semigroup such that $\tau_1 + \tau_2 + \cdots + \tau_n = 1$.

Proof. We have already proved most of this in Theorem 3.2. Since $\Pi x^n = x$ for all x, $\tau_1 + \tau_2 + \cdots + \tau_n$ is the identity mapping.

The conclusions of Theorem 4.1 take an especially simple form if we assume Π to be symmetric.

COROLLARY. *If Π is a symmetric Aczél mean on a set S containing a regular element, then*

$$\Pi x_1 x_2 \cdots x_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

where $+$ is the operation in a commutative semigroup with zero $(S, +)$, the elements of this semigroup admitting unique division by n.

Proof. By Theorem 4.1

$$\Pi x_1 x_2 \cdots x_n = \tau_1 x_1 + \tau_2 x_2 + \cdots + \tau_n x_n$$

where $\tau_1 + \tau_2 + \cdots + \tau_n$ is the identity mapping. Since Π is symmetric and the semigroup has a zero, $\tau_1 = \tau_2 = \cdots = \tau_n$. Hence $n \cdot \tau_i x = x$ for all x in S. That is, the elements of the semigroup admit division by n. For each x, an element y such that $n \cdot y = x$ is unique since if $n \cdot y_1 = n \cdot y_2$, then $n \cdot \tau_i y_1 = n \cdot \tau_i y_2$ or $y_1 = y_2$.

We now turn to the generalization of the Kolmogorov-Nagumo results. We will call an infinite sequence of n-ary operations $\overset{n}{\Pi}$, $n = 1, 2, 3, \cdots$, on a set S, a Kolmogorov-Nagumo mean or, briefly, a K-N mean if (i) $\overset{n}{\Pi}$ is symmetric for each n, (ii) $\overset{n}{\Pi}$ is idempotent for each n, (iii) for each n and $k \leq n$,

$$\overset{n}{\Pi}\mathbf{x} = \overset{n}{\Pi}(\overset{k}{\Pi}\mathbf{x}_1^k)^k \mathbf{x}_{k+1}^n \quad \text{for all} \quad x_1, x_2, \cdots, x_n .$$

These are the conditions other than continuity and order which Kolmogorov [9] and Nagumo [12] assume for the continuous real case.

THEOREM 4.3. *Let $\overset{n}{\Pi}$, $n = 1, 2, 3, \cdots$, be a* K-N *mean on a set S and let S contain an element which is regular for each operation $\overset{n}{\Pi}$. Then there is a commutative semigroup with zero, $(S, +)$, admitting the rationals as operators, such that, for each n,*

$$\overset{n}{\Pi} x_1 x_2 \cdots x_n = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Proof. We first show that each $\overset{n}{\Pi}$ is entropic. Let $X = (x_{ij})$, $i, j = 1, 2, \cdots, n$, be an $n \times n$ array of elements of S and let $\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_n$ be the row sequences of X, $\mathbf{c}_1, \mathbf{c}_2, \cdots \mathbf{c}_n$ the column of sequences of X. The two sequences of n^2 elements x_{ij}

$$\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n, \qquad \mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$$

differ only in the order of the elements and so by the symmetry of $\overset{n^2}{\Pi}$

$$\overset{n^2}{\Pi}\mathbf{r}_1\mathbf{r}_2 \cdots \mathbf{r}_n = \overset{n^2}{\Pi}\mathbf{c}_1\mathbf{c}_2 \cdots \mathbf{c}_n .$$

By property (iii) of a K-N mean, we can write this as

$$\overset{n^2}{\Pi}(\overset{n}{\Pi}\mathbf{r}_1)^n\mathbf{r}_2 \cdots \mathbf{r}_n = \overset{n^2}{\Pi}(\overset{n}{\Pi}\mathbf{c}_1)^n\mathbf{c}_2 \cdots \mathbf{c}_n .$$

After n applications of this property and the symmetry of $\overset{n^2}{\Pi}$, we obtain

$$\overset{n^2}{\Pi}(\overset{n}{\Pi}\mathbf{r}_1)^n(\overset{n}{\Pi}\mathbf{r}_2)^n \cdots (\overset{n}{\Pi}\mathbf{r}_n)^n = \overset{n^2}{\Pi}(\overset{n}{\Pi}\mathbf{c}_1)^n(\overset{n}{\Pi}\mathbf{c}_2)^n \cdots (\overset{n}{\Pi}\mathbf{c}_n)^n.$$

Again, by the symmetry of $\overset{n^2}{\Pi}$, we may write this as

$$\overset{n^2}{\Pi}(\overset{n}{\Pi}\mathbf{r}_1\overset{n}{\Pi}\mathbf{r}_2 \cdots \overset{n}{\Pi}\mathbf{r}_n)^n = \overset{n^2}{\Pi}(\overset{n}{\Pi}\mathbf{c}_1\overset{n}{\Pi}\mathbf{c}_2 \cdots \overset{n}{\Pi}\mathbf{c}_n)^n.$$

An application of property (iii) of the K-N mean to each side enables us to write this as

$$\overset{n^2}{\Pi}(\overset{n}{\Pi}\overset{n}{\Pi}\mathbf{r}_1\overset{n}{\Pi}\mathbf{r}_2 \cdots \overset{n}{\Pi}\mathbf{r}_n)^n(\overset{n}{\Pi}\mathbf{r}_1\overset{n}{\Pi}\mathbf{r}_2 \cdots \overset{n}{\Pi}\mathbf{r}_n)^{n-1}$$
$$= \overset{n^2}{\Pi}(\overset{n}{\Pi}\overset{n}{\Pi}\,\mathbf{c}_1\,\overset{n}{\Pi}\,\mathbf{c}_2 \cdots \overset{n}{\Pi}\,\mathbf{c}_n)^n(\overset{n}{\Pi}\,\mathbf{c}_1\,\overset{n}{\Pi}\,\mathbf{c}_2 \cdots \overset{n}{\Pi}\,\mathbf{c}_n)^{n-1}.$$

Repeating this use of property (iii) of a K-N mean and the symmetry of $\overset{n^2}{\Pi}$, we obtain

$$\overset{n^2}{\Pi}(\overset{n}{\Pi}\overset{n}{\Pi}\mathbf{r}_1\overset{n}{\Pi}\mathbf{r}_2 \cdots \overset{n}{\Pi}\mathbf{r}_n)^{n^2} = \overset{n^2}{\Pi}(\overset{n}{\Pi}\overset{n}{\Pi}\mathbf{c}_1\overset{n}{\Pi}\mathbf{c}_2 \cdots \overset{n}{\Pi}\mathbf{c}_n)^{n^2}.$$

That is,

$$\overset{n^2}{\Pi}(\overset{n}{\Pi}\overset{n}{\Pi}X)^{n^2} = \overset{n^2}{\Pi}(\overset{n}{\Pi}\overset{n}{\Pi}X^T)^{n^2}.$$

Since $\overset{n^2}{\Pi}$ is idempotent, we can write this as

$$\overset{n}{\Pi}\overset{n}{\Pi}X = \overset{n}{\Pi}\overset{n}{\Pi}X^T.$$

Hence, $\overset{n}{\Pi}$ is entropic.

By Theorem 4.2, we now have, for $n = 2, 3, 4, \cdots$ a commutative semigroup with zero defined on S such that

$$\overset{n}{\Pi}x_1x_2 \cdots x_n = \frac{x_1 \oplus_n x_2 \oplus_n \cdots \oplus_n x_n}{n}$$

where we write $\oplus_n$ for the addition in the semigroup corresponding to $\overset{n}{\Pi}$. Since we used the same $\overset{n}{\Pi}$-regular element to construct $\oplus_n$, these semigroups have the same zero. We now show that, for each n, $x \oplus_2 y = x \oplus_n y$ for all x, y. Let g be the $\overset{n}{\Pi}$-regular element of S used in constructing the semigroups. Then, by Theorem 4.1

$$\begin{aligned}\frac{x \oplus_n y}{n} &= \overset{n}{\Pi}xyg^{n-2}\\ &= \overset{n}{\Pi}(\overset{2}{\Pi}xy)^2g^{n-2}\\ &= \frac{(x \oplus_2 y)/2 \oplus_n (x \oplus_2 y)/2}{n}\end{aligned}$$

where division by n refers to the $\oplus_n$ semigroup and division by 2 to the $\oplus_2$ semigroup. Hence

$$x \oplus_n y = \frac{x \oplus_2 y}{2} \oplus_n \frac{x \oplus_2 y}{2}.$$

Let $y = g$ in this. We get $x = x/2 \oplus_n x/2$. Hence halves of elements with respect to the $\oplus_2$ operation are also halves with respect to the $\oplus_n$ operation. Thus

$$x \oplus_n y = \frac{x \oplus_2 y}{2} \oplus_2 \frac{x \oplus_2 y}{2}$$
$$= x \oplus_2 y.$$

Hence,

$$\overset{n}{\Pi} x_1 x_2 \cdots x_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

where we write $+$ for $\oplus_2$. Since the elements in $(S, +)$ allow unique division by $n = 2, 3, 4, \cdots$, the semigroup admits the rational numbers as operators.

5. **Finite entropic algebras.** We have described in Theorem 3.2 the structure of entropic algebras containing an idempotent. The purpose of this section is to obtain sufficient properties of finite entropic algebras to enable us to weaken considerably, in this finite case, the assumptions we need for characterizing entropic algebras which do not contain an idempotent. Throughout this section, (S, Π) will be a finite entropic algebra. We use continually the following lemma, the proof of which we omit.

LEMMA 5.1. *If $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n$ are elements of S such that, for all x, y,*

$$\Pi \mathbf{a}_1^{i-1} x \mathbf{a}_{i+1}^n = \Pi \mathbf{a}_1^{i-1} y \mathbf{a}_{i+1}^n$$

implies $x = y$, then, for any b in S there is a unique element x in S such that $\Pi \mathbf{a}_1^{i-1}\, x\, \mathbf{a}_{i+1}^n = b$.

If J is a non-empty subset of $\{1, 2, \cdots, n\}$, we will say that g is J-regular in (S, Π) if g is j-regular for all j in J.
The fundamental result for this section is contained in the next lemma.

LEMMA 5.2. *If $\Pi \mathbf{a} = b$ in (S, Π) and b is J-regular, then, for each i in J, a_i is J-regular.*

Proof. Let $i, j \in J$ and let x, y be elements of S such that

$$\Pi(a_i)^{i-1} x (a_i)^{n-i} = \Pi(a_i)^{i-1} y (a_i)^{n-i}$$

where we write $(a_i)^k$ for a sequence of k a_i's.

Let X be the $n \times n$ array having $a_1, a_2, \cdots, a_n$ for each column except the j-th which has x in the i-th place, and b in every other place. Let Y be the $n \times n$ array obtained from X by replacing x by y in the i-th row, j-th column. Then $\Pi^2 X = \Pi^2 Y$. Since Π is entropic, $\Pi^2 X^T = \Pi^2 Y^T$. An application of the j-regularity and then of the i-regularity of b yields $x = y$. By Lemma 5.1, this concludes the proof.

LEMMA 5.3. *If g is i-regular in (S, Π), then so is Πg^n.*

Proof. Let x, y be elements of S such that

$$\Pi(\Pi g^n)^{i-1}x(\Pi g^n)^{n-i} = \Pi(\Pi g^n)^{i-1}y(\Pi g^n)^{n-i}.$$

Let t be the element of S satisfying $\Pi g^{i-1}tg^{n-i} = g$. By Lemma 5.2, t is i-regular. Let x_1, y_1 be such that

$$\Pi t^{i-1}x_1t^{n-i} = x, \qquad \Pi t^{i-1}y_1t^{n-i} = y.$$

Now let X be the $n \times n$ array having $t, \cdots, t, x_1, t, \cdots, t$ [$(i-1)$ t's preceding x and $(n-i)$ following] as its i-th row and g in every other place. Let Y be the $n \times n$ array obtained by replacing x_1 by y_1 in the i-th row, i-th column. Then $\Pi^2 X = \Pi^2 Y$ and so $\Pi^2 X^T = \Pi^2 Y^T$. From this and the fact that g is i-regular, $x_1 = y_1$. Hence $x = y$.

LEMMA 5.4. *Let $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n$ be J-regular elements in (S, Π) where $i \in J$ and J contains at least two elements. Then, for any b in S, there is a unique x in S such that*

$$\Pi \mathbf{a}_1^{i-1}x\mathbf{a}_{i+1}^n = b.$$

Proof. Let x, y be such that

$$\Pi \mathbf{a}_1^{i-1}x\mathbf{a}_{i+1}^n = \Pi \mathbf{a}_1^{i-1}y\mathbf{a}_{i+1}^n.$$

We will prove that $x = y$. Let X be the $n \times n$ array which has $a_1, \cdots, a_{i-1}$, $x, a_{i+1}, \cdots, a_n$ as its i-th row, $t_1, \cdots, t_{i-1}, a_1, t_{i+1}, \cdots, t_n$ as its j-th row where $j \in J$, $j \neq i$ and t_k satisfies $\Pi(a_k)^{i-1}t_k(a_k)^{n-i} = a_1$ for $k = 1, \cdots, i-1$, $i+1, \cdots, n$, and has $a_1, \cdots, a_{i-1}, a_1, a_{i+1}, \cdots, a_n$ for every other row. Let Y be the $n \times n$ array obtained from X by replacing x by y. Then $\Pi^2 X = \Pi^2 Y$ and so $\Pi^2 X^T = \Pi^2 Y^T$. Two applications of the i-regularity of a_i yield $x = y$.

We now have all the results we need for the finite case. Specifically, from the above lemmas, we known that if a finite entropic algebra contains an element g which is i and j-regular, then (i) Πg^n is also i and j-regular, (ii) there is a unique element t in S, also i and j-regular, such that (for $i < j$)

$$\Pi g^{i-1}(\Pi g^n)g^{j-i-1}tg^{n-j} = g$$

(iii) for any b in S, there is a unique element x in S such that

$$\Pi g^{i-1}xg^{j-i-1}tg^{n-j} = b$$

where t is the element described in (ii).

It is quite easy to prove that in a finite entropic algebra (S, Π), the set of all J-regular elements (where J contains at least two elements) is closed under Π. Thus, we can sum up the results of this section as: if a finite entropic algebra (S, Π) contains at least one J-regular element, then (S, Π) contains a J-regular subalgebra, where by J-regular subalgebra of an entropic algebra, finite or infinite, we mean a subalgebra of J-regular elements such that if $i \in J$ and $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots a_n$ belong to the subalgebra, then there is a unique element x, for each b, such that $\Pi \mathbf{a}_1^{i-1} \, x \, \mathbf{a}_{i+1}^n = b$. Furthermore, if b is in the subalgebra, so is x.

6. **The structure of entropic algebras.** We discuss in this section the structure of an entropic algebra (S, Π) which does not contain a regular idempotent. We construct a new operation Σ on S in terms of Π such that Σ is entropic and S contains a Σ-idempotent. If certain regularity conditions are assumed for (S, Π), then this Σ-idempotent is also regular in (S, Σ) and hence we are able to use the results of §3 to describe the structure of the operation Σ.

LEMMA 6.1. *Let $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n$ be elements in the entropic algebra (S, Π). Then the operation Σ on S defined by*

$$\Sigma \mathbf{x} = \Pi \mathbf{a}_1^{i-1} (\Pi \mathbf{x}) \mathbf{a}_{i+1}^n$$

is entropic.

Proof. In view of Lemma 3.6. it is sufficient to prove this for $i = 1$. Let X be an $n \times n$ array (x_{ij}) of elements of S. We compute $\Sigma^2 X$ and $\Sigma^2 X^T$.

$$\begin{aligned} \Sigma^2 X &= \Sigma(\Sigma \mathbf{r}_1)(\Sigma \mathbf{r}_2) \cdots (\Sigma \mathbf{r}_n) \\ &= \Pi\{\Pi(\Sigma \mathbf{r}_1)(\Sigma \mathbf{r}_2) \cdots (\Sigma \mathbf{r}_n)\} \mathbf{a}_2^n . \end{aligned}$$

Since $\Sigma \mathbf{r}_i = \Pi(\Pi \mathbf{r}_i) \mathbf{a}_2^n$, we obtain $\Sigma^2 X = \Pi(\Pi^2 Y) \mathbf{a}_2^n$ where Y is the $n \times n$ array having $\Pi \mathbf{r}_i, a_2, a_3, \cdots, a_n$ as its i-th row. Since Π is entropic,

$$\begin{aligned} \Pi^2 Y &= \Pi^2 Y^T \\ &= \Pi(\Pi^2 X) \Pi(a_2)^n \cdots \Pi(a_n)^n \\ &= \Pi(\Pi^2 X^T) \Pi(a_2)^n \cdots \Pi(a_n)^n \\ &= \Pi^2 Z^T \end{aligned}$$

where Z is the $n \times n$ array having $\Pi \mathbf{c}_i, a_2, a_3, \cdots, a_n$ as its i-th row. Hence, $\Pi^2 Y = \Pi^2 Z$ and

$$\begin{aligned} \Sigma^2 X &= \Pi(\Pi^2 Z) \mathbf{a}_2^n \\ &= \Pi\{\Pi(\Sigma \mathbf{c}_1)(\Sigma \mathbf{c}_2) \cdots (\Sigma \mathbf{c}_n)\} \mathbf{a}_2^n \\ &= \Sigma(\Sigma \mathbf{c}_1)(\Sigma \mathbf{c}_2) \cdots (\Sigma \mathbf{c}_n) \\ &= \Sigma^2 X^T . \end{aligned}$$

Hence, Σ is entropic.

LEMMA 6.2. *Let g, t be elements in an entropic algebra (S, Π) such that*

$$\Pi g^{i-1}(\Pi g^{n})g^{j-i-1}tg^{n-j} = g.$$

Then the operation Σ on S defined by

$$\Sigma \mathbf{x} = \Pi g^{i-1}(\Pi \mathbf{x})g^{j-i-1}tg^{n-j}$$

is entropic and has g as an idempotent.

Proof. This follows immediately by Lemma 6.1 and direct computation of Σg^{n}.

We are now in a position to state the first structure theorem for entropic algebras.

THEOREM 6.1. *Let (S, Π) be an entropic algebra such that S contains a two place regular subalgebra. Then, there is a commutative semigroup $(S, +)$, pairwise commuting endomorphisms $\alpha_1, \alpha_2, \cdots, \alpha_n$ of $(S, +)$ and a one-one mapping ϕ of S onto itself, such that*

$$\Pi x_1 x_2 \cdots x_n = \phi(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \quad \textit{for all} \quad x_1, x_2, \cdots, x_n \quad \textit{in} \quad S.$$

Proof. If g is i, j-regular in $(S, +)$, then by the results of the preceding section, there is an i-j regular element t such that $\Pi g^{i-1}(\Pi g^{n})g^{j-i-1}tg^{n-j} = g$. Furthermore, for k either i or j, and any b in S, the equation

$$\Pi g^{i-1}(\Pi g^{k-1}xg^{n-k})g^{j-i-1}tg^{n-j} = b$$

has a unique solution. Hence g is i, j-regular with respect to the operation Σ on S defined by

$$\Sigma \mathbf{x} = \Pi g^{i-1}(\Pi \mathbf{x})g^{j-i-1}tg^{n-j}$$

and by Lemma 6.2, Σ is entropic with g as an idempotent. By Theorem 3.2, there is a commutative semigroup with zero $(S, +)$, and pairwise commuting endomorphisms $\alpha_1, \alpha_2, \cdots, \alpha_n$ of $(S, +)$ (with α_i, α_j automorphisms) such that

$$\Sigma \mathbf{x} = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \quad \text{for all} \quad x_1, x_2, \cdots \quad x_n \quad \text{in} \quad S.$$

Again, by the results of the previous section, the mapping

$$\phi^{-1} : x \to \Pi g^{i-1}xg^{j-i-1}tg^{n-j}$$

is one-one onto S. Thus, $\Pi \mathbf{x} = \phi \Sigma \mathbf{x}$ and the theorem follows.

One problem remains—the description of the mapping ϕ in the above theorem in terms of the semigroup. We need for this the following lemma.

LEMMA 6.3. *Let $(S, +)$ be a commutative semigroup with zero and let ϕ_1, $\phi_2, \cdots, \phi_n$ be one-one mappings of S onto itself such that*

$$\begin{aligned}
&\phi_1(x_{11} + x_{12} + \cdots + x_{1n}) \\
&\qquad + \phi_2(x_{21} + x_{22} + \cdots + x_{2n}) + \cdots + \phi_n(x_{n1} + x_{n2} + \cdots + x_{nn}) \\
&\quad = \phi_1(x_{11} + x_{21} + \cdots + x_{n1}) + \phi_2(x_{12} + x_{22} + \cdots + x_{n2}) \\
&\qquad + \cdots + \phi_n(x_{1n} + x_{2n} + \cdots + x_{nn}) \quad \textit{for all} \quad x_{ij}.
\end{aligned}$$

Then there is an automorphism α *of* $(S, +)$ *and fixed elements* $a_1, a_2, \cdots, a_n$ *such that for each* i

$$\phi_i x = \alpha x + a_i \quad \textit{for all} \quad x.$$

Proof. In the above identity, for fixed i and all j except 1, i, let $x_{jj} = \phi_j^{-1}0$, and let all other x_{pq} be zero except x_{1i} and x_{i1}. Let $x_{1i} = \phi_1^{-1}0$. We get

$$\phi_i x_{i1} = \phi_1 x_{i1} + \phi_i \phi_1^{-1}0 \quad \text{for all} \quad x_{i1}.$$

Since ϕ_1, ϕ_i are permutations on $(S, +)$, we have

$$\phi_i x = \phi_1 x + k_i \quad \text{for all} \quad x$$

where k_i is a fixed regular element of S. Substituting for the ϕ_i in the original identity and cancelling the k_i, which we may do since they are regular elements, we get

$$\begin{aligned}\phi_1(x_{11} + x_{12} + \cdots + x_{1n}) + \phi_1(x_{21} + x_{22} + \cdots + x_{2n})& \\ + \cdots + \phi_1(x_{n1} + x_{n2} + \cdots + x_{nn})& \\ = \phi_1(x_{11} + x_{21} + \cdots + x_{n1}) + \phi_1(x_{12} + x_{22} + \cdots + x_{n2})& \\ + \cdots + \phi_1(x_{1n} + x_{2n} + \cdots + x_{nn}).&\end{aligned}$$

In this, let $x_{ii} = \phi_1^{-1}0$, $i \neq 1, 2$, and let all other x_{ij} be zero except x_{11}, x_{12}. We get

$$\phi_1(x_{11} + x_{12}) + \phi_1 0 = \phi_1 x_{11} + \phi_1 x_{12} \quad \text{for all} \quad x_{11}, x_{12}.$$

Let $x_{11} = x_{12} = \phi_1^{-1}0$ in this. It follows that $\phi_1 0$ has an additive inverse and hence is a regular element. We now define a one-one mapping α of S onto itself by

$$\phi_1 x = \alpha x + \phi_1 0 \quad \text{for all} \quad x.$$

It follows immediately that

$$\alpha(x + y) = \alpha x + \alpha y \quad \text{for all} \quad x, y.$$

Hence α is an automorphism of $(S, +)$. Now

$$\begin{aligned}\phi_i x &= \phi_1 x + k_i \\ &= \alpha x + \phi_1 0 + k_i \\ &= \alpha x + a_i\end{aligned}$$

where a_i, as the sum of two regular elements is a regular element. This concludes the proof of the lemma.

We are now in a position to prove the main structure theorem for entropic algebras.

THEOREM 6.2. *Let* (S, Π) *be an entropic algebra such that* S *contains a regular* *subalgebra. Then there is a commutative semigroup with zero* $(S, +)$ *such that*

$$\Pi x_1 x_2 \cdots x_n = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + d \quad \textit{for all} \quad x_1, x_2, \cdots, x_n$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ *are fixed pairwise commuting automorphisms of the semigroup and* d *is a fixed regular element in it.*

Proof. From Theorem 6.1, we know that there is a commutative semigroup with zero defined on S and a one-one mapping ϕ of S onto itself such that, for all $x_1, x_2, \cdots, x_n$,

$$\Pi x_1 x_2 \cdots x_n = \phi(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n)$$

where $\beta_1, \beta_2, \cdots, \beta_n$ are pairwise commuting automorphisms of the semigroup. We wish to determine ϕ in terms of the semigroup structure.

The entropic law $\Pi^2 X = \Pi^2 X^T$, in terms of ϕ, β_i and the semigroup operation is

$$\phi\{\beta_1\phi(\beta_1 x_{11} + \cdots + \beta_n x_{1n}) + \cdots + \beta_n\phi(\beta_1 x_{n1} + \cdots + \beta_n x_{nn})\}$$
$$= \phi\{\beta_1\phi(\beta_1 x_{11} + \cdots + \beta_n x_{n1}) + \cdots + \beta_n\phi(\beta_1 x_{1n} + \cdots + \beta_n x_{nn})\}.$$

In this, write $\sigma_i = \beta_i\phi\beta_i^{-1}$ and y_{ij} for $\beta_i\beta_j x_{ij}$. Cancelling by ϕ on the left we obtain the identity of the preceding lemma.

$$\sigma_1(y_{11} + \cdots + y_{1n}) + \cdots + \sigma_n(y_{n1} + \cdots + y_{nn})$$
$$= \sigma_1(y_{11} + \cdots + y_{n1}) + \cdots + \sigma_n(y_{1n} + \cdots + y_{nn}).$$

Hence, there is an automorphism α of the semigroup and elements $a_1, a_2, \cdots, a_n$ such that

$$\beta_i\phi\beta_i^{-1}x = \alpha x + a_i .$$

From this, it follows that $\phi x = \beta_i^{-1}\alpha\beta_i x + \beta_i^{-1}a_i$. This gives us some information about β_i and a_i, but all we need for our purposes is that $\phi x = \rho x + d$, where ρ is an automorphism of the semigroup and d is a fixed regular element in it. We now have

$$\Pi x_1 x_2 \cdots x_n = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + d$$

where $\alpha_i = \rho\beta_i$ and is an automorphism of the semigroup.

It remains to check that the α_i, α_j commute in pairs. This follows immediately by applying the entropic law $\Pi^2 X = \Pi^2 X^T$ with Π written in terms of the semigroup $(S, +)$ to the $n \times n$ array having x in its i-th row, j-th column and the zero of the semigroup in all other places.

COROLLARY 1. *If* (S, Π) *is a finite entropic groupoid containing a regular element, then the conclusion stated in the theorem holds.*

COROLLARY 2. *If* $(Q, \circ)$ *is an entropic quasigroup, then there is an abelian group* $(Q, +)$, *commuting automorphisms* α_1, α_2 *of* $(Q, +)$ *and an element* d *of* $(Q, +)$ *such that*

$$x \circ y = \alpha_1 x + \alpha_2 y + d \quad \textit{for all} \quad x, y \quad \textit{in} \quad Q.$$

The result in Corollary 2 has been obtained by Bruck [3].

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