

Research Article

Cohomology and Formal Deformations of Alternative Algebras*

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Abstract The purpose of this paper is to introduce an algebraic cohomology and formal deformation theory of alternative algebras. A short review of basics on alternative algebras and their connections to some other algebraic structures is also provided.

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1 Introduction

Deformation theory arose mainly from geometry and physics. In the latter field, the non-commutative associative multiplication of operators in quantum mechanics is thought of as a formal associative deformation of the pointwise multiplication of the algebra of symbols of these operators. In the sixties, Murray Gerstenhaber introduced algebraic formal deformations for associative algebras in a series of papers (see [11, 12, 13, 14, 15]). He used formal series and showed that the theory is intimately connected to the cohomology of the algebra. The same approach was extended to several algebraic structures (see [2, 3, 5, 6, 25]). Other approaches to study deformations exist; see [8, 9, 10, 17, 18, 19, 20, 24, 28]; also see [26] for a review.

In this paper, we introduce a cohomology and a formal deformation theory of alternative algebras. If A is left alternative algebra, then the algebra defined on the same vector space A with “opposite” multiplication $x \circ y := yx$ is a right alternative algebra and vice-versa. Hence, all the statements for left alternative algebras have their corresponding statements for right alternative algebras. Thus, we will only consider the left alternative algebra case in this paper. We also review the connections of alternative algebras to other algebraic structures. In Section 2, we review the basic definitions and properties related to alternative algebras. In Section 3, we discuss in particular all the links between alternative algebras and some other algebraic structures such as Moufang loops, Malcev algebras and Jordan algebras. In Section 4, we introduce a cohomology theory of left alternative algebras. We compute the second cohomology group of the 2 by 2 matrix algebra. It is known that, as an associative algebra, its second cohomology group is trivial, but we show that this is not the case as left alternative algebra. Finally, in Section 5, we develop a formal deformation theory for left alternative algebras and show that the cohomology theory introduced in Section 4 fits.

2 Preliminaries

Throughout this paper, \mathbb{K} is an algebraically closed field of characteristic 0.

2.1 Definitions

Definition 1 (see [29]). A left alternative \mathbb{K} -algebra (resp. right alternative \mathbb{K} -algebra) (\mathcal{A}, μ) is a vector space \mathcal{A} over \mathbb{K} and a bilinear multiplication μ satisfying the left alternative identity, that is, for all $x, y \in \mathcal{A}$,

$$\mu(x, \mu(x, y)) = \mu(\mu(x, x), y) \tag{2.1}$$

and respectively the right alternative identity, that is

$$\mu(\mu(x, y), y) = \mu(x, \mu(y, y)). \tag{2.2}$$

An alternative algebra is one which is both left and right alternative algebra.

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Lemma 2. Let \mathfrak{as} denote the associator, which is a trilinear map defined by

$$\mathfrak{as}(x, y, z) = \mu(\mu(x, y), z) - \mu(x, \mu(y, z)).$$

An algebra is alternative if and only if the associator $\mathfrak{as}(x, y, z)$ is an alternating function of its arguments, that is

$$\mathfrak{as}(x, y, z) = -\mathfrak{as}(y, x, z) = -\mathfrak{as}(x, z, y) = -\mathfrak{as}(z, y, x).$$

This lemma implies then that the following identities are satisfied:

$$\mathfrak{as}(x, x, y) = 0 \text{ (left alternativity)}, \quad \mathfrak{as}(y, x, x) = 0 \text{ (right alternativity)}, \quad \mathfrak{as}(x, y, x) = 0 \text{ (flexibility)}.$$

By linearization, we have the following characterization of left (resp. right) alternative algebras, which will be used in the sequel.

Lemma 3. A pair (\mathcal{A}, μ) is a left alternative \mathbb{K} -algebra (resp. right alternative \mathbb{K} -algebra) if and only if the identity

$$\mu(x, \mu(y, z)) - \mu(\mu(x, y), z) + \mu(y, \mu(x, z)) - \mu(\mu(y, x), z) = 0, \quad (2.3)$$

respectively,

$$\mu(x, \mu(y, z)) - \mu(\mu(x, y), z) + \mu(x, \mu(z, y)) - \mu(\mu(x, z), y) = 0, \quad (2.4)$$

holds.

Remark 4. When considering multiplication as a linear map $\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, the condition (2.3) (resp. (2.4)) may be written

$$\mu \circ (\mu \otimes \text{id} - \text{id} \otimes \mu) \circ (\text{id}^{\otimes 3} + \sigma_1) = 0, \quad (2.5)$$

respectively

$$\mu \circ (\mu \otimes \text{id} - \text{id} \otimes \mu) \circ (\text{id}^{\otimes 3} + \sigma_2) = 0, \quad (2.6)$$

where id stands for the identity map and σ_1 and σ_2 stand for transpositions generating the permutation group \mathcal{S}_3 which are extended to trilinear maps defined by

$$\sigma_1(x_1 \otimes x_2 \otimes x_3) = x_2 \otimes x_1 \otimes x_3, \quad \sigma_2(x_1 \otimes x_2 \otimes x_3) = x_1 \otimes x_3 \otimes x_2$$

for all $x_1, x_2, x_3 \in \mathcal{A}$.

In terms of associators, the identities (2.3) (resp. (2.4)) are equivalent to

$$\mathfrak{as} + \mathfrak{as} \circ \sigma_1 = 0 \quad (\text{resp. } \mathfrak{as} + \mathfrak{as} \circ \sigma_2 = 0). \quad (2.7)$$

Remark 5. The notions of subalgebra, ideal and quotient algebra are defined in the usual way. For general theory about alternative algebras see [29]. The alternative algebras generalize associative algebras. Recently, in [7], it was shown that their operad is not Koszul. The dual operad of right alternative (resp. left alternative) algebras is defined by associativity and the identity

$$\mu(\mu(x, y), z) + \mu(\mu(x, z), y) = 0, \quad (\text{resp. } \mu(\mu(x, y), z) + \mu(\mu(y, x), z) = 0).$$

The dual operad of alternative algebras is defined by the associativity and the identity

$$\mu(\mu(x, y), z) + \mu(\mu(y, x), z) + \mu(\mu(z, x), y) + \mu(\mu(x, z), y) + \mu(\mu(y, z), x) + \mu(\mu(z, y), x) = 0.$$

2.2 Structure theorems and examples

We have these following fundamental properties:

- **Artin's theorem.** In an alternative algebra, the subalgebra generated by any two elements is associative. Conversely, any algebra for which this is true is clearly alternative. It follows that expressions involving only two variables can be written without parenthesis unambiguously in an alternative algebra.

- **Generalization of Artin’s theorem.** Whenever three elements x, y, z in an alternative algebra associate (i.e. $as(x, y, z) = 0$), the subalgebra generated by those elements is associative.
- **Corollary of Artin’s theorem.** Alternative algebras are power-associative, that is, the subalgebra generated by a single element is associative. The converse need not hold: the sedenions are power-associative but not alternative.

Example 6 (4-dimensional alternative algebras). According to A. T. Gainov (see e.g. [16, p. 144]), there are exactly two alternative but not associative algebras of dimension 4 over any field. With respect to a basis $\{e_0, e_1, e_2, e_3\}$, one algebra is given by the following multiplication (the unspecified products are zeros):

$$e_0^2 = e_0, \quad e_0e_1 = e_1, \quad e_2e_0 = e_2, \quad e_2e_3 = e_1, \quad e_3e_0 = e_3, \quad e_3e_2 = -e_1.$$

The other algebra is given by

$$e_0^2 = e_0, \quad e_0e_2 = e_2, \quad e_0e_3 = e_3, \quad e_1e_0 = e_1, \quad e_2e_3 = e_1, \quad e_3e_2 = -e_1.$$

Example 7 (octonions). The octonions were discovered in 1843 by John T. Graves who called them Octaves and independently by Arthur Cayley in 1845. The octonions algebra which is also called Cayley Octaves or Cayley algebra is an 8-dimensional algebra defined with respect to a basis $\{u, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, where u is the identity for the multiplication, by the following multiplication table. The table describes multiplying the i th row elements by the j th column elements.

	u	e_1	e_2	e_3	e_4	e_5	e_6	e_7
u	u	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-u$	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	$-u$	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	$-u$	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	$-u$	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	$-u$	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	$-u$	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	$-u$

The octonions algebra is a typical example of alternative algebras. As stated early, the subalgebra generated by any two elements is associative. In fact, the subalgebra generated by any two elements of the octonions is isomorphic to the algebra of reals \mathbb{R} , the algebra of complex numbers \mathbb{C} or the algebra of quaternions \mathbb{H} , all of which are associative. See [4] for the role of the octonions in algebra, geometry and topology, and see also [1] where octions are viewed as quasialgebra.

3 Connections to other algebraic structures

We begin by recalling some basics of *Moufang* loops, Moufang algebras and Malcev algebras.

Definition 8. Let $(M, *)$ be a set with a binary operation. It is called a *Moufang loop* if it is a quasigroup with an identity e such that the binary operation satisfies one of the following equivalent identities:

$$x * (y * (x * z)) = ((x * y) * x) * z, \tag{3.1}$$

$$z * (x * (y * x)) = ((z * x) * y) * x, \tag{3.2}$$

$$(x * y) * (z * x) = (x * (y * z)) * x. \tag{3.3}$$

The typical examples include groups and the set of nonzero octonions which gives a nonassociative Moufang loop.

As in the case of Lie group, there exists a notion of analytic Moufang loop (see e.g. [27,30,31]). An analytic Moufang loop M is a real analytic manifold with the multiplication and the inverse, $g \mapsto g^{-1}$, being analytic mappings. The tangent space T_eM is equipped with a skew-symmetric bracket $[\cdot, \cdot] : T_eM \times T_eM \rightarrow T_eM$ satisfying Malcev’s identity, that is,

$$[J(x, y, z), x] = J(x, y, [x, z]) \tag{3.4}$$

for all $x, y, z \in T_eM$, and where J corresponds to Jacobi’s identity, that is,

$$J(x, y, z) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]].$$

Definition 9. A Malcev \mathbb{K} -algebra is a vector space over \mathbb{K} and a skew-symmetric bracket satisfying the identity (3.4).

The Malcev algebras are also called Moufang-Lie algebras. We have the following fundamental Kerdman theorem [23].

Theorem 10. For every real Malcev algebra there exists an analytic Moufang loop whose tangent algebra is the given Malcev algebra.

The connection to alternative algebras is given by the following proposition.

Proposition 11. The alternative algebras are Malcev-admissible algebras, that is, their commutators define a Malcev algebra.

Remark 12. Let \mathcal{A} be an alternative algebra with a unit. The set $U(\mathcal{A})$ of all invertible elements of \mathcal{A} forms a Moufang loop with respect to the multiplication. Conversely, not any Moufang loop can be imbedded into a loop of type $U(\mathcal{A})$ for a suitable unital alternative algebra \mathcal{A} . A counter-example was given in [34]. In [32], the author characterizes the Moufang loops which are imbeddable into a loop of type $U(\mathcal{A})$.

The Moufang algebras which are the corresponding algebras of a Moufang loop are defined as follows.

Definition 13. A left Moufang algebra (\mathcal{A}, μ) is one which is left alternative and satisfying the Moufang identity, that is,

$$\mu(\mu(x, y), \mu(z, x)) = \mu(\mu(x, \mu(y, z)), x).$$

The Moufang identities (3.1), (3.2), (3.3) are expressed in terms of associator by

$$\text{as}(x, y, z \cdot x) = x \cdot \text{as}(y, z, x), \quad \text{as}(x \cdot y, z, x) = \text{as}(x, y, z) \cdot x, \quad \text{as}(y, x^2, z) = x \cdot \text{as}(y, x, z) + \text{as}(y, x, z) \cdot x.$$

It turns out that in a characteristic different from 2, all left alternative algebras are left Moufang algebras. Also, a left Moufang algebra is alternative if and only if it is flexible, that is, $\text{as}(x, y, x) = 0$ for all $x, y \in \mathcal{A}$.

The alternative algebras are connected to Jordan algebras as follows. Given an alternative algebra (\mathcal{A}, μ) , then (\mathcal{A}, μ^+) , where $\mu^+(x, y) = \mu(x, y) + \mu(y, x)$, is a Jordan algebra, that is, the commutative multiplication μ^+ satisfies the identity $\text{as}_{\mu^+}(x^2, y, x) = 0$. For more nonassociative algebras theory, we refer to [21, 22, 33, 36, 37].

4 Cohomology of left alternative algebras

In this section, we introduce a cohomology theory for left alternative algebras fitting with deformation theory and compute the second cohomology group of 2×2 -matrix algebra viewed as an alternative algebra.

Let \mathcal{A} be a left alternative \mathbb{K} -algebra defined by a multiplication μ . A left alternative p -cochain is a linear map from $\mathcal{A}^{\otimes p}$ to \mathcal{A} . We denote by $\mathcal{C}^p(\mathcal{A}, \mathcal{A})$ the group of all p -cochains.

4.1 First differential map

Let id denotes the identity map on \mathcal{A} . For $f \in \mathcal{C}^1(\mathcal{A}, \mathcal{A})$, we define the first differential $\delta^1 f \in \mathcal{C}^2(\mathcal{A}, \mathcal{A})$ by

$$\delta^1 f = \mu \circ (f \otimes \text{id}) + \mu \circ (\text{id} \otimes f) - f \circ \mu.$$

We remark that the first differential of a left alternative algebra is similar to the first differential map of Hochschild cohomology of an associative algebra (1-cocycles are derivations).

4.2 Second differential map

Let $\varphi \in \mathcal{C}^2(\mathcal{A}, \mathcal{A})$, we define the second differential $\delta^2 \varphi \in \mathcal{C}^3(\mathcal{A}, \mathcal{A})$ by

$$\delta^2 \varphi = [\mu \circ (\varphi \otimes \text{id} - \text{id} \otimes \varphi) + \varphi \circ (\mu \otimes \text{id} - \text{id} \otimes \mu)] \circ (\text{id}^{\otimes 3} + \sigma_1), \quad (4.1)$$

where σ_1 is defined on $\mathcal{A}^{\otimes 3}$ by $\sigma_1(x \otimes y \otimes z) = y \otimes x \otimes z$.

Remark 14. The left alternative algebra 2-differential defined in (4.1) may be written using the Hochschild differential δ_H^2 as

$$\delta^2 \varphi = \delta_H^2 \varphi \circ (\text{id}^{\otimes 3} + \sigma_1).$$

Proposition 15. *The composite $\delta^2 \circ \delta^1$ is zero.*

Proof. Let $x, y, z \in \mathcal{A}$ and $f \in \mathcal{C}^1(\mathcal{A}, \mathcal{A})$. Then

$$\delta^1 f(x \otimes y) = \mu(f(x) \otimes y) + \mu(x \otimes f(y)) - f(\mu(x \otimes y)).$$

In order to simplify the notation, the multiplication is denoted by concatenation of terms and the tensor product is removed. Then, we have

$$\begin{aligned} & \delta^2(\delta^1 f)(x \otimes y \otimes z) \\ &= (xy)f(z) - f((xy)z) + [f(xy)]z + [xf(y)]z - [f(xy)]z + [(f(x))y]z \\ & \quad + (yx)f(z) - f((yx)z) + [f(yx)]z + [yf(x)]z - [f(yx)]z + [(f(y))x]z \\ & \quad - \{xf(yz) - f(x[yz]) + [f(x)](yz) + x(yf(z)) - xf(yz) + x[(f(y))z] \\ & \quad \quad + yf(xz) - f(y[xz]) + [f(y)](xz) + y(xf(z)) - yf(xz) + y[(f(x))z]\} \\ &= [(xy)f(z) + (yx)f(z) - x(yf(z)) - y(xf(z))] - [f((xy)z) + f((yx)z) - f(x(yz)) - f(y(xz))] \\ & \quad + [(xf(y))z + (f(y)x)z - (f(y))(xz) - x(f(y)z)] + [(f(x)y)z + (y(f(x))z - (f(x))(yz) - y(f(x)z))] \\ &= 0. \end{aligned}$$

After simplifying the terms which cancel in pairs, we group the remaining ones into brackets, so each bracket cancels using the left alternative algebra axiom (see (2.3)). \square

Example 16. Let $\mathcal{A} = \mathcal{M}_2(\mathbb{K})$ denote the associative algebra of 2 by 2 matrices over the field \mathbb{K} , considered as left alternative algebra of dimension 4. Let e_1, e_2, e_3 and e_4 be a basis of \mathcal{A} . The second cohomology group $H^2(\mathcal{A}, \mathcal{A})$ is three-dimensional and generated with respect to the canonical basis by $[f_1], [f_2]$ and $[f_3]$, where

$$\begin{aligned} f_1(e_2 \otimes e_4) &= e_1, & f_1(e_3 \otimes e_2) &= -e_3, & f_1(e_4 \otimes e_1) &= e_3, & f_1(e_4 \otimes e_2) &= e_4, \\ f_2(e_2 \otimes e_3) &= e_2, & f_2(e_3 \otimes e_1) &= -e_4, & f_2(e_3 \otimes e_3) &= e_3, & f_2(e_3 \otimes e_4) &= e_4, \\ f_3(e_2 \otimes e_3) &= e_1, & f_3(e_3 \otimes e_2) &= e_4. \end{aligned}$$

The non-specified terms of these generators are zeros. These generators were obtained independently using the softwares Maple and Mathematica.

Remark 17. It was implied from [35] that the second cohomology group of $\mathcal{A} = \mathcal{M}_2(\mathbb{K})$ is non-trivial. But the exact structure of this group was not known. We completely determine the structure by giving the dimension and generators.

4.3 Third differential map and beyond

Let $\psi \in \mathcal{C}^3(\mathcal{A}, \mathcal{A})$, we define the third differential $\delta^3 \psi \in \mathcal{C}^4(\mathcal{A}, \mathcal{A})$ as

$$\begin{aligned} \delta^3 \psi &= \mu \circ (\psi \otimes \text{id}) \circ (\text{id}^{\otimes 3} - \sigma_1) + \mu \circ (\text{id} \otimes \psi) \circ (\text{id}^{\otimes 3} - \sigma_2) - \psi \circ (\mu \otimes \text{id}^{\otimes 2}) \circ (\text{id}^{\otimes 3} + \sigma_2 \circ \sigma_1) \\ & \quad + \psi \circ (\text{id} \otimes \mu \otimes \text{id}) \circ (\text{id}^{\otimes 3} + \sigma_1 \circ \sigma_2) - \psi \circ (\text{id}^{\otimes 2} \otimes \mu) \circ (\text{id}^{\otimes 3} - \sigma_1); \end{aligned}$$

that is, for all $\psi \in \mathcal{C}^3(\mathcal{A}, \mathcal{A})$ and $x_1, \dots, x_4 \in \mathcal{A}$ we have

$$\begin{aligned} \delta^3 \psi(x_1, x_2, x_3, x_4) &= \mu(x_1 \otimes \psi(x_2 \otimes x_3 \otimes x_4)) - \mu(x_1 \otimes \psi(x_3 \otimes x_2 \otimes x_4)) + \mu(\psi(x_1 \otimes x_2 \otimes x_3) \otimes x_4) \\ & \quad - \mu(\psi(x_2 \otimes x_1 \otimes x_3) \otimes x_4) - \psi(\mu(x_1 \otimes x_2) \otimes x_3 \otimes x_4) - \psi(\mu(x_2 \otimes x_3) \otimes x_1 \otimes x_4) \\ & \quad + \psi(x_1 \otimes \mu(x_2 \otimes x_3) \otimes x_4) + \psi(x_3 \otimes \mu(x_1 \otimes x_2) \otimes x_4) - \psi(x_1 \otimes x_2 \otimes \mu(x_3 \otimes x_4)) \\ & \quad + \psi(x_2 \otimes x_1 \otimes \mu(x_3 \otimes x_4)). \end{aligned}$$

Remark 18. The third differential $\delta^3 \psi \in \mathcal{C}^4(\mathcal{A}, \mathcal{A})$ of a left alternative algebra \mathcal{A} may be written using the third Hochschild cohomology differential δ_H^3 as

$$\begin{aligned} \delta^3 \psi &= \delta_H^3 \psi - \mu \circ (\psi \otimes \text{id}) \circ \sigma_1 - \mu \circ (\text{id} \otimes \psi) \circ \sigma_2 - \psi \circ (\mu \otimes \text{id}^{\otimes 2}) \circ \sigma_2 \circ \sigma_1 \\ & \quad + \psi \circ (\text{id} \otimes \mu \otimes \text{id}) \circ \sigma_1 \circ \sigma_2 + \psi \circ (\text{id}^{\otimes 2} \otimes \mu) \circ \sigma_1. \end{aligned}$$

Proposition 19. *The composite $\delta^3 \circ \delta^2$ is zero.*

Proof. Let $x_1, \dots, x_4 \in \mathcal{A}$ and $f \in \mathcal{C}^2(\mathcal{A}, \mathcal{A})$. Then, by substituting ψ with $\delta^2 f$ in the previous formula and rearranging the terms we get

$$\begin{aligned} & \delta^3(\delta^2 f)(x_1 \otimes x_2 \otimes x_3 \otimes x_4) \\ &= x_1 [\delta^2 f(x_2 \otimes x_3 \otimes x_4) - \delta^2 f(x_3 \otimes x_2 \otimes x_4)] - [\delta^2 f(x_1 x_2 \otimes x_3 \otimes x_4) - \delta^2 f(x_3 \otimes x_1 x_2 \otimes x_4)] \\ & \quad + [\delta^2 f(x_1 \otimes x_2 x_3 \otimes x_4) - \delta^2 f(x_2 x_3 \otimes x_1 \otimes x_4)] - [\delta^2 f(x_1 \otimes x_2 \otimes x_3 x_4) - \delta^2 f(x_2 \otimes x_1 \otimes x_3 x_4)] \\ & \quad + [\delta^2 f(x_1 \otimes x_2 \otimes x_3) - \delta^2 f(x_2 \otimes x_1 \otimes x_3)] x_4 \\ &= 0, \end{aligned}$$

since $\delta^2 f(x \otimes y \otimes z) = \delta^2 f(y \otimes x \otimes z)$, for all $x, y, z \in \mathcal{A}$. \square

Let $\psi \in \mathcal{C}^4(\mathcal{A}, \mathcal{A})$ and δ_H^4 be the fourth Hochschild cohomology differential. We define the fourth differential $\delta^4 \psi \in \mathcal{C}^5(\mathcal{A}, \mathcal{A})$ for a left alternative algebra \mathcal{A} as

$$\delta^4 \psi = \sum_{\sigma \in \mathcal{S}_5} \delta_H^4 \psi \circ \sigma,$$

where σ is the extended map, which we still denote by σ , for a permutation $\sigma \in \mathcal{S}_5$ on $\mathcal{A}^{\otimes 5}$ defined by

$$\sigma(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)} \otimes x_{\sigma(4)} \otimes x_{\sigma(5)}.$$

By direct calculation, we can prove the following proposition.

Proposition 20. *The composite $\delta^4 \circ \delta^3$ is zero.*

One may complete the complex by considering $\delta^p = 0$ for $p > 4$. It is shown in [7] that the operad of alternative algebras is not Koszul, thus we think that there exist nontrivial p th differential maps for $p > 4$ by constructing a minimal model.

5 Formal deformations of left alternative algebras

In this section, we develop a deformation theory for alternative algebras and show that the cohomology introduced in the previous section fits with formal deformations of left alternative algebras.

Let (\mathcal{A}, μ_0) be a left alternative algebra. Let $\mathbb{K}[[t]]$ be the power series ring in one variable t and coefficients in \mathbb{K} and let $\mathcal{A}[[t]]$ be the set of formal power series whose coefficients are elements of \mathcal{A} (note that $\mathcal{A}[[t]]$ is obtained by extending the coefficients domain of \mathcal{A} from \mathbb{K} to $\mathbb{K}[[t]]$). Then, $\mathcal{A}[[t]]$ is a $\mathbb{K}[[t]]$ -module. When \mathcal{A} is finite-dimensional, we have $\mathcal{A}[[t]] = \mathcal{A} \otimes_{\mathbb{K}} \mathbb{K}[[t]]$. One notes that \mathcal{A} is a submodule of $\mathcal{A}[[t]]$. Given a \mathbb{K} -bilinear map $f : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, it admits naturally an extension to a $\mathbb{K}[[t]]$ -bilinear map $f : \mathcal{A}[[t]] \otimes \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]]$, that is, if $x = \sum_{i \geq 0} a_i t^i$ and $y = \sum_{j \geq 0} b_j t^j$ then

$$f(x \otimes y) = \sum_{i, j \geq 0} t^{i+j} f(a_i \otimes b_j).$$

Definition 21. Let (\mathcal{A}, μ_0) be a left alternative algebra. A *formal left alternative deformation* of \mathcal{A} is given by the $\mathbb{K}[[t]]$ -bilinear map $\mu_t : \mathcal{A}[[t]] \otimes \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]]$ of the form $\mu_t = \sum_{i \geq 0} \mu_i t^i$, where each μ_i is a \mathbb{K} -bilinear map $\mu_i : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ (extended to be $\mathbb{K}[[t]]$ -bilinear), such that for $x, y, z \in \mathcal{A}$, the following formal left alternativity condition holds:

$$\mu_t(x \otimes \mu_t(y \otimes z)) - \mu_t(\mu_t(x \otimes y) \otimes z) + \mu_t(y \otimes \mu_t(x \otimes z)) - \mu_t(\mu_t(y \otimes x) \otimes z) = 0. \quad (5.1)$$

5.1 Deformation equation and obstructions

The first problem is to give conditions about μ_i such that the deformation μ_t is alternative. Expanding the left-hand side of (5.1) and collecting the coefficients of t^k yield

$$\sum_{i+j=k; i, j \geq 0} \mu_i(x \otimes \mu_j(y \otimes z)) - \mu_i(\mu_j(x \otimes y) \otimes z) + \mu_i(y \otimes \mu_j(x \otimes z)) - \mu_i(\mu_j(y \otimes x) \otimes z) = 0, \quad k = 0, 1, 2, \dots$$

This infinite system, called the *deformation equation*, gives the necessary and sufficient conditions for the left alternativity of μ_t . It may be written as

$$\sum_{i=0}^k \mu_i(x \otimes \mu_{k-i}(y \otimes z)) - \mu_i(\mu_{k-i}(x, y) \otimes z) + \mu_i(y \otimes \mu_{k-i}(x \otimes z)) - \mu_i(\mu_{k-i}(y \otimes x) \otimes z) = 0, \quad k = 0, 1, 2, \dots \quad (5.2)$$

The first equation ($k = 0$) is the left alternativity condition for μ_0 . The second shows that μ_1 must be a 2-cocycle for the alternative algebra cohomology defined above ($\mu_1 \in Z^2(\mathcal{A}, \mathcal{A})$). More generally, suppose that μ_p is the first non-zero coefficient after μ_0 in the deformation μ_t . This μ_p is called the *infinitesimal* of μ_t .

Theorem 22. *The map μ_p is a 2-cocycle of the left alternative algebras cohomology of \mathcal{A} with coefficient in itself.*

Proof. In (5.2) make the following substitutions: $k = p$ and $\mu_1 = \dots = \mu_{p-1} = 0$. □

Definition 23. The 2-cocycle μ_p is said integrable if it is the first non-zero term, after μ_0 , of a left alternative deformation.

The integrability of μ_p implies an infinite sequence of relations which may be interpreted as the vanishing of the obstruction to the integration of μ_p .

For an arbitrary k , with $k > 1$, the k th equation of the system (5.2) may be written as

$$\delta^2 \mu_k(x \otimes y \otimes z) = \sum_{i=1}^{k-1} \mu_i(\mu_{k-i}(x \otimes y) \otimes z) - \mu_i(x \otimes \mu_{k-i}(y \otimes z)) + \mu_i(\mu_{k-i}(y \otimes x) \otimes z) - \mu_i(y \otimes \mu_{k-i}(x \otimes z)).$$

Suppose that the truncated deformation

$$\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \dots + t^{m-1}\mu_{m-1}$$

satisfies the deformation equation. The truncated deformation is extended to a deformation of order m , that is,

$$\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \dots + t^{m-1}\mu_{m-1} + t^m\mu_m,$$

satisfying the deformation equation if

$$\delta^2 \mu_m(x \otimes y \otimes z) = \sum_{i=1}^{m-1} \mu_i(\mu_{m-i}(x \otimes y) \otimes z) - \mu_i(x \otimes \mu_{m-i}(y \otimes z)) + \mu_i(\mu_{m-i}(y \otimes x) \otimes z) - \mu_i(y \otimes \mu_{m-i}(x \otimes z)).$$

The right-hand side of this equation is called the *obstruction* to finding μ_m extending the deformation.

We define a square operation on 2-cochains by

$$\mu_i \square \mu_j(x \otimes y \otimes z) = \mu_i(\mu_j(x \otimes y) \otimes z) - \mu_i(x \otimes \mu_j(y \otimes z)) + \mu_i(\mu_j(y \otimes x) \otimes z) - \mu_i(y \otimes \mu_j(x \otimes z)).$$

Then the obstruction may be written as

$$\sum_{i=1}^{m-1} \mu_i \square \mu_{m-i} \quad \text{or} \quad \sum_{i+j=m, i, j \neq m} \mu_i \square \mu_j.$$

A straightforward computation gives the following.

Theorem 24. *The obstructions are left alternative 3-cocycles.*

Remark 25. (1) The cohomology class of the element $\sum_{i+j=m, i,j \neq m} \mu_i \square \mu_j$ is the first obstruction to the integrability of μ_m . We consider now how to extend an infinitesimal deformation to a deformation of order 2. Suppose $m = 2$ and $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2$. The deformation equation of the truncated deformation of order 2 is equivalent to the finite system:

$$\begin{cases} \mu_0 \square \mu_0 = 0 & (\mu_0 \text{ is left alternative}), \\ \delta\mu_1 = 0 & (\mu_1 \in Z^2(\mathcal{A}, \mathcal{A})), \\ \mu_1 \square \mu_1 = \delta\mu_2. \end{cases}$$

Then $\mu_1 \square \mu_1$ is the first obstruction to integrate μ_1 and $\mu_1 \square \mu_1 \in Z^3(\mathcal{A}, \mathcal{A})$. The elements $\mu_1 \square \mu_1$ which are coboundaries permit to extend the deformation of order one to a deformation of order 2. But the elements of $H^3(\mathcal{A}, \mathcal{A})$ give the obstruction to the integrations of μ_1 .

(2) If μ_m is integrable, then the cohomological class of $\sum_{i+j=m, i,j \neq m} \mu_i \square \mu_j$ must be 0. In the previous example, μ_1 is integrable, implies that $\mu_1 \square \mu_1 = \delta^2\mu_2$, which means that the cohomology class of $\mu_1 \square \mu_1$ vanishes.

Corollary 26. *If $H^3(\mathcal{A}, \mathcal{A}) = 0$, then all obstructions vanish and every $\mu_m \in Z^2(\mathcal{A}, \mathcal{A})$ is integrable.*

5.2 Equivalent and trivial deformations

In this section, we characterize equivalent as well as trivial deformations of left alternative algebras.

Definition 27. Let (\mathcal{A}, μ_0) be a left alternative algebra and let (\mathcal{A}_t, μ_t) and (\mathcal{A}'_t, μ'_t) be two left alternative deformations of \mathcal{A} , where $\mu_t = \sum_{i \geq 0} t^i \mu_i$ and $\mu'_t = \sum_{i \geq 0} t^i \mu'_i$, with $\mu_0 = \mu'_0$. We say that the two deformations are *equivalent* if there exists a formal isomorphism $\Phi_t : \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]]$, that is a $\mathbb{K}[[t]]$ -linear map that may be written in the form

$$\Phi_t = \sum_{i \geq 0} t^i \Phi_i = \text{id} + t\Phi_1 + t^2\Phi_2 + \dots,$$

where $\Phi_i \in \text{End}_{\mathbb{K}}(\mathcal{A})$ and $\Phi_0 = \text{id}$ are such that the following relations hold:

$$\Phi_t \circ \mu_t = \mu'_t \circ (\Phi_t \otimes \Phi_t). \quad (5.3)$$

A deformation \mathcal{A}_t of \mathcal{A}_0 is said to be *trivial* if and only if \mathcal{A}_t is equivalent to \mathcal{A}_0 (viewed as a left alternative algebra on $\mathcal{A}[[t]]$).

In the following, we discuss the equivalence of two deformations. Condition (5.3) may be written as

$$\Phi_t(\mu_t(x \otimes y)) = \mu'_t(\Phi_t(x) \otimes \Phi_t(y)), \quad \forall x, y \in \mathcal{A}. \quad (5.4)$$

Equation (5.4) is equivalent to

$$\sum_{i \geq 0} \Phi_i \left(\sum_{j \geq 0} \mu_j(x \otimes y) t^j \right) t^i = \sum_{i \geq 0} \mu'_i \left(\sum_{j \geq 0} \Phi_j(x) t^j \otimes \sum_{k \geq 0} \Phi_k(y) t^k \right) t^i$$

or

$$\sum_{i,j \geq 0} \Phi_i(\mu_j(x \otimes y)) t^{i+j} = \sum_{i,j,k \geq 0} \mu'_i(\Phi_j(x) \otimes \Phi_k(y)) t^{i+j+k}.$$

Identifying the coefficients, we obtain that the constant coefficients are identical, that is,

$$\mu_0 = \mu'_0 \quad \text{because } \Phi_0 = \text{id}.$$

For the coefficients of t one finds

$$\Phi_0(\mu_1(x \otimes y)) + \Phi_1(\mu_0(x \otimes y)) = \mu'_1(\Phi_0(x) \otimes \Phi_0(y)) + \mu'_0(\Phi_1(x) \otimes \Phi_0(y)) + \mu'_0(\Phi_0(x) \otimes \Phi_1(y)).$$

Since $\Phi_0 = \text{id}$, it follows that

$$\mu_1(x, y) + \Phi_1(\mu_0(x \otimes y)) = \mu'_1(x \otimes y) + \mu_0(\Phi_1(x) \otimes y) + \mu_0(x \otimes \Phi_1(y)).$$

Consequently,

$$\mu'_1(x \otimes y) = \mu_1(x \otimes y) + \Phi_1(\mu_0(x \otimes y)) - \mu_0(\Phi_1(x) \otimes y) - \mu_0(x \otimes \Phi_1(y)). \quad (5.5)$$

The second-order conditions of the equivalence between two deformations of a left alternative algebra are given by (5.5), which may be written as

$$\mu'_1(x \otimes y) = \mu_1(x \otimes y) - \delta^1 \Phi_1(x \otimes y). \quad (5.6)$$

In general, if the deformations μ_t and μ'_t of μ_0 are equivalent, then $\mu'_1 = \mu_1 + \delta^1 f_1$. Therefore, we have the following proposition.

Proposition 28. *The integrability of μ_1 depends only on its cohomology class.*

Recall that two elements are cohomologous if their difference is a coboundary. The equation $\delta^2 \mu_1 = 0$ implies that

$$\delta^2 \mu'_1 = \delta^2(\mu_1 + \delta^1 f_1) = \delta^1 \mu_1 + \delta^2(\delta^1 f_1) = 0.$$

If $\mu_1 = \delta^1 g$, then

$$\mu'_1 = \delta^1 g - \delta^1 f_1 = \delta^1(g - f_1).$$

Thus, if two integrable 2-cocycles are cohomologous, then the corresponding deformations are equivalent.

Remark 29. Elements of $H^2(\mathcal{A}, \mathcal{A})$ give the infinitesimal deformations ($\mu_t = \mu_0 + t\mu_1$).

Proposition 30. *Let (\mathcal{A}, μ_0) be a left alternative algebra. There is, over $\mathbb{K}[[t]]/t^2$, a one-to-one correspondence between the elements of $H^2(\mathcal{A}, \mathcal{A})$ and the infinitesimal deformation of \mathcal{A} defined by*

$$\mu_t(x \otimes y) = \mu_0(x \otimes y) + t\mu_1(x \otimes y), \quad \forall x, y \in \mathcal{A}.$$

Proof. The deformation equation is equivalent to $\delta^2 \mu_1 = 0$, that is $\mu_1 \in Z^2(\mathcal{A}, \mathcal{A})$. □

Theorem 31. *Let (\mathcal{A}, μ_0) be a left alternative algebra and let μ_t be a one parameter family of deformation of μ_0 . Then μ_t is equivalent to*

$$\mu_t = \mu_0 + t^p \mu'_p + t^{p+1} \mu'_{p+1} + \dots,$$

where $\mu'_p \in Z^2(\mathcal{A}, \mathcal{A})$ and $\mu'_p \notin B^2(\mathcal{A}, \mathcal{A})$.

Proof. Suppose now that $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \dots$ is a one-parameter family of deformation of μ_0 for which $\mu_1 = \dots = \mu_{m-1} = 0$. The deformation equation implies $\delta\mu_m = 0$ ($\mu_m \in Z^2(\mathcal{A}, \mathcal{A})$). If further $\mu_m \in B^2(\mathcal{A}, \mathcal{A})$ (i.e. $\mu_m = \delta g$), then setting the morphism $f_t = \text{id} + t f_m$, we have, for all $x, y \in \mathcal{A}$,

$$\mu'_t(x \otimes y) = f_t^{-1} \circ \mu_t \circ (f_t(x) \otimes f_t(y)) = \mu_0(x \otimes y) + t^{m+1} \mu_{m+1}(x \otimes y) + \dots.$$

And again $\mu_{m+1} \in Z^2(\mathcal{A}, \mathcal{A})$. □

Corollary 32. *If $H^2(\mathcal{A}, \mathcal{A}) = 0$, then all deformations of \mathcal{A} are equivalent to a trivial deformation.*

In fact, assume that there exists a non trivial deformation of μ_0 . Following the previous theorem, this deformation is equivalent to

$$\mu_t = \mu_0 + t^p \mu'_p + t^{p+1} \mu'_{p+1} + \dots,$$

where $\mu'_p \in Z^2(\mathcal{A}, \mathcal{A})$ and $\mu'_p \notin B^2(\mathcal{A}, \mathcal{A})$. But this is impossible because $H^2(\mathcal{A}, \mathcal{A}) = 0$.

Remark 33. A left alternative algebra for which every formal deformation is equivalent to a trivial deformation is called rigid. The previous corollary provides a sufficient condition for a left alternative algebra to be rigid. In general, this condition is not necessary.

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