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Classification and construction of unitary topological field theories in two dimensions¹

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Abstract. We prove that unitary two-dimensional topological field theories are uniquely characterized by n positive real numbers $\lambda_1, \dots, \lambda_n$ which can be regarded as the eigenvalues of a hermitean handle creation operator. The number n is the dimension of the Hilbert space associated with the circle and the partition functions for closed surfaces have the form

$$Z_g = \sum_{i=1}^n \lambda_i^{g-1}$$

where g is the genus. The eigenvalues can be arbitrary positive numbers. We show how such a theory can be constructed on triangulated surfaces.

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1 Introduction

Topological quantum field theory (TQFT) [1, 4] has given considerable insight into a number of unsolved problems in theoretical physics [3, 5] and also provided an important research tool in pure mathematics [2, 6, 7]. Most work in this field has been devoted to studying particular examples of such theories and uncovering their physical content as well as their relation to low-dimensional topology.

The problem of classifying topological quantum field theories is a hard one in more than two dimensions since it is intimately tied up with the classification of topological manifolds in these dimensions. In two dimensions the situation is radically different since a compact orientable two-dimensional manifold is characterized topologically by its genus and the number of boundary components. In [5] it was shown that the two-dimensional theories have indeed a very simple structure.

The classification problem for TQFT in two dimensions was addressed in [8], see also [9], using triangulated surfaces and fields associated with vertices. In these papers topological theories were constructed with the partition function for a closed surface of genus g given by

$$Z_g = \lambda^{g-1} \tag{1}$$

where λ is an arbitrary positive number.

In the present paper we prove that the most general unitary TQFT in two dimensions is in fact a direct sum of theories of this type. The proof is very simple. It uses the fact that any two-dimensional surface can be constructed by gluing together spheres with three or fewer boundary components and therefore any unitary TQFT in two dimensions is determined by the partition functions for a sphere with three or fewer holes. Unitarity implies that an associated handle creation operator is hermitean and it follows that the theory is characterized up to equivalence by the values of partition functions for closed surfaces.

The classification of unitary TQFT on triangulated surfaces with the fields defined on links and taking values in a finite set has been discussed in a number of recent papers [10, 11, 12]. It was first shown in [10] that the weight factors for triangles can in this case be regarded as structure constants of a semisimple associative algebra and the physical Hilbert space can be identified with the center of the algebra. It follows that in these theories the number λ in (1) is always the inverse square of an integer. This is also the case in two-dimensional topological gauge theories [13, 14]. Theories with fields defined on links can easily be realized as vertex theories.

In the next section we describe the Atiyah axioms for unitary TQFT and point out simplifications that occur in two dimensions. We then proceed to show that the

theory is determined by the three loop function on the sphere and use the gluing axiom to show that the partition function for a closed surface is necessarily a sum of exponentials as in (1). In section 4 we construct a lattice TQFT of the most general form. In the final section we comment on the classification problem in higher dimensions.

2 The Atiyah axioms

In this section we recall the axioms for a unitary TQFT [4] as they apply in two dimensions. We assume that all manifolds are smooth, oriented and compact unless otherwise is stated. If M is an oriented manifold, we denote by M^* the same manifold with the orientation reversed.

A unitary TQFT in two dimensions comprises the following assignments: To each closed 1-dimensional manifold Σ there is assigned a finite dimensional Hilbert space \mathcal{H}_Σ with inner product \langle, \rangle_Σ . To each surface S (not necessarily closed) there is assigned an element $Z(S) \in \mathcal{H}_\Sigma$, called the partition function for S , where $\Sigma = \partial S$ is the boundary of S . The Hilbert space associated with the empty 1-dimensional manifold \emptyset is $\mathcal{H}_\emptyset = \mathbf{C}$ so $Z(S) \in \mathbf{C}$ if S is closed. These objects satisfy the following conditions:

1. If $f : \Sigma_1 \mapsto \Sigma_2$ is an orientation preserving diffeomorphism between 1-manifolds then there is an associated unitary mapping

$$U_f : \mathcal{H}_{\Sigma_1} \mapsto \mathcal{H}_{\Sigma_2} \tag{2}$$

such that $U_{g \circ f} = U_g U_f$ if g is an orientation preserving diffeomorphism from Σ_2 to another 1-manifold Σ_3 .

If f extends to an orientation preserving diffeomorphism between surfaces, $S_1 \mapsto S_2$, where $\partial S_i = \Sigma_i$, $i = 1, 2$, then $U_f(Z(S_1)) = Z(S_2)$.

2. For any 1-manifold Σ ,

$$\mathcal{H}_{\Sigma^*} = \mathcal{H}_\Sigma^* \tag{3}$$

i.e. we have an identification between these spaces, which is equivalent to giving a non-degenerate bilinear form

$$(\cdot, \cdot)_\Sigma : \mathcal{H}_\Sigma \times \mathcal{H}_{\Sigma^*} \mapsto \mathbf{C} \tag{4}$$

such that

$$(x, y)_\Sigma = (y, x)_{\Sigma^*}. \tag{5}$$

This form is preserved by diffeomorphisms, i.e. for $f : \Sigma_1 \mapsto \Sigma_2$ as in axiom 1,

$$(x, y)_{\Sigma_1} = (U_f x, U_{f^*} y)_{\Sigma_2} \quad (6)$$

for all $x \in \mathcal{H}_{\Sigma_1}$, $y \in \mathcal{H}_{\Sigma_2^*}$, where $f^* : \Sigma_1^* \mapsto \Sigma_2^*$ denotes f regarded as an orientation preserving map between Σ_1^* and Σ_2^* .

Defining the conjugate linear isomorphism $y \mapsto y^*$ from \mathcal{H}_{Σ} to \mathcal{H}_{Σ^*} by

$$(x, y^*)_{\Sigma} = \langle x, y \rangle_{\Sigma} \quad (7)$$

for $x, y \in \mathcal{H}_{\Sigma}$, we furthermore assume that $x^{**} = x$ for all $x \in \mathcal{H}_{\Sigma}$ and

$$Z(S^*) = Z(S)^* \quad (8)$$

for any surface S with boundary Σ .

3. If $\Sigma = \Sigma_1 \cup \Sigma_2$ is a disjoint union of 1-manifolds then $\mathcal{H}_{\Sigma} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$ and we have a corresponding factorization of the bilinear forms and unitary maps, i.e. $(\cdot, \cdot)_{\Sigma} = (\cdot, \cdot)_{\Sigma_1} \otimes (\cdot, \cdot)_{\Sigma_2}$, and if $f_1 : \Sigma_1 \mapsto \Sigma'_1$ and $f_2 : \Sigma_2 \mapsto \Sigma'_2$ are orientation preserving diffeomorphisms, then $U_f = U_{f_1} \otimes U_{f_2}$ where $f : \Sigma \mapsto \Sigma'_1 \cup \Sigma'_2$ denotes the diffeomorphism that equals f_1 on Σ_1 and f_2 on Σ_2 .

Let S_1 and S_2 be two surfaces such that $\partial S_1 = \Sigma_1 \cup \Sigma_3$ and $\partial S_2 = \Sigma_2 \cup \Sigma_3^*$. Let S be the surface obtained by gluing S_1 and S_2 together along their Σ_3 boundary component,

$$S = S_1 \cup_{\Sigma_3} S_2. \quad (9)$$

Then $Z(S) = (Z(S_1), Z(S_2))_{\Sigma_3}$ where, by abuse of notation, $(\cdot, \cdot)_{\Sigma_3}$ denotes the pairing

$$\mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_3} \otimes \mathcal{H}_{\Sigma_2} \otimes \mathcal{H}_{\Sigma_3}^* \mapsto \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \quad (10)$$

induced by (4).

Concretely, if $\{x_i\}$, $\{y_j\}$, $\{z_k\}$ are bases for \mathcal{H}_{Σ_1} , \mathcal{H}_{Σ_2} , \mathcal{H}_{Σ_3} , respectively, and $\{z_k^*\}$ is the dual basis for $\mathcal{H}_{\Sigma_3}^*$, then we can write

$$Z(S_1) = \sum_{i,k} c_{ik} x_i \otimes z_k \quad (11)$$

$$Z(S_2) = \sum_{j,l} d_{jl} y_j \otimes z_l^* \quad (12)$$

for suitable constants c_{ik} , d_{jl} , and

$$Z(S) = \sum_{i,j,k} c_{ik} d_{jk} x_i \otimes y_j. \quad (13)$$

4. Let Σ be an oriented 1-manifold and orient the cylinder $\Sigma \times [0, 1]$ such that $x \mapsto (x, 0)$ is an orientation reversing map from Σ onto $\Sigma \times \{0\}$ whereas $x \mapsto (x, 1)$ is an orientation preserving mapping from Σ onto $\Sigma \times \{1\}$. Using the canonical identification $\mathcal{H}_1^* \otimes \mathcal{H}_2 \approx \text{Hom}(\mathcal{H}_1, \mathcal{H}_2)$, we may according to axioms 2 and 3 regard $Z(\Sigma \times [0, 1])$ as a linear map from $\mathcal{H}_{\Sigma \times \{0\}}$ to $\mathcal{H}_{\Sigma \times \{1\}}$ and we assume that $Z(\Sigma \times [0, 1]) = U_f$ where $f : \Sigma \times \{0\} \mapsto \Sigma \times \{1\}$ is defined by $f((x, 0)) = (x, 1)$. Using the mapping U_f to identify the spaces $\mathcal{H}_{\Sigma \times \{0\}}$ and $\mathcal{H}_{\Sigma \times \{1\}}$ we can write

$$Z(\Sigma \times [0, 1]) = I \tag{14}$$

where I is the identity mapping.

We refer to [4] for a detailed discussion of the axioms. Here we make a few comments that apply especially in the two-dimensional case.

It is a standard consequence of the axioms that the unitary mapping U_f with f as in axiom 1 only depends on the homotopy class of the diffeomorphism f . If Σ_1 and Σ_2 are connected, i.e. circles, there is only one homotopy class of orientation preserving diffeomorphisms and we can write $U_f = U(\Sigma_1, \Sigma_2)$. By axiom 1 it follows that the mappings $U(\Sigma_1, \Sigma_2)$ yield a canonical identification of all the Hilbert spaces \mathcal{H}_Σ (where Σ is connected) with a single Hilbert space which we denote by \mathcal{H} with inner product \langle, \rangle . With these identifications all the mappings $U(\Sigma_1, \Sigma_2)$ are of course the identity and by (5) and (6) a unique symmetric bilinear form $(,)$ is defined on \mathcal{H} . Similarly, by (6) and (7) the $*$ -maps define a unique conjugate linear involution $x \mapsto x^*$ on \mathcal{H} given by

$$(x, y^*) = \langle x, y \rangle . \tag{15}$$

It follows from the assumed factorization properties of the inner products, bilinear forms and unitary mappings that the vectorspace associated to a 1-manifold with n boundary components $\Sigma_1, \dots, \Sigma_n$ can be identified with $\mathcal{H}^{\otimes n}$. If f is a diffeomorphism of $\Sigma_1 \cup \dots \cup \Sigma_n$ onto itself which permutes the boundary components, then the mapping induced by U_f acts on $\mathcal{H}^{\otimes n}$ by the corresponding permutation of factors in the tensor product. This implies that if S is a *connected* surface with $\partial S = \Sigma_1 \cup \dots \cup \Sigma_n$ then $Z(S) \in \mathcal{H}^{\otimes n}$ is a symmetric tensor since there exist orientation preserving diffeomorphisms that permute the boundary components of S in any prescribed way. Moreover, since any surface S possesses an orientation reversing diffeomorphism, we conclude that $Z(S^*) = Z(S)$ and hence, by (8), that

$$Z(S) = Z(S)^* \tag{16}$$

for any surface S .

Letting \mathcal{H}_R denote the real subspace of \mathcal{H} defined by

$$\mathcal{H}_R = \{x \in \mathcal{H} : x = x^*\} \quad (17)$$

we have a direct sum decomposition over \mathbf{R} ,

$$\mathcal{H} = \mathcal{H}_R \oplus i\mathcal{H}_R. \quad (18)$$

According to (15) and the symmetry of $(,)$ it follows that the restriction of the inner product to \mathcal{H}_R is a real inner product on \mathcal{H}_R which equals the restriction of the bilinear form to \mathcal{H}_R . We thus conclude from (16) that any two-dimensional unitary TQFT is effectively real, i.e. we might have started from the outset with real Hilbert spaces and partition functions satisfying the analogues of axioms 1-4 without (7) and (8).

3 Classification

Let us assume that we are given a unitary TQFT satisfying the axioms of the previous section. Let us denote the partition function for a connected surface of genus g with n boundary components by $Z_{g,n}$ and write $Z_{g,0} = Z_g$. We choose an orthonormal basis $\{x_i\}$ for \mathcal{H}_R which, according to (18), also constitutes an orthonormal basis for \mathcal{H} with $x_i^* = x_i$. The one loop function on the sphere can then be expressed as

$$Z_{0,1} = \sum_i d_i x_i. \quad (19)$$

The two and three loop functions on the sphere can similarly be written as

$$Z_{0,2} = \sum_{ij} q_{ij} x_i \otimes x_j \quad (20)$$

and

$$Z_{0,3} = \sum_{ijk} C_{ijk} x_i \otimes x_j \otimes x_k, \quad (21)$$

where q_{ij} and C_{ijk} are real and symmetric under interchange of the indices. By axioms 3 and 4 we have

$$\sum_i q_{ij} x_i = x_j^* \quad (22)$$

so $q_{ij} = \delta_{ij}$ with respect to the chosen basis.

We define a handle operator $H \in \text{End}(\mathcal{H})$ by gluing together two three loop functions:

$$H = \sum_{il} H_{il} x_i \otimes x_l^* \quad (23)$$

where

$$H_{il} = \sum_{jk} C_{ijk} C_{ljk}. \quad (24)$$

This is the same as regarding $Z_{1,2}$ as an operator on \mathcal{H} . Applying H to the vector $Z_{g,1}$ clearly gives $Z_{g+1,1}$. The operator H is hermitean by (8) i.e. symmetric on \mathcal{H}_R .

We now choose the basis $\{x_i\}$ to consist of the eigenvectors of H and let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of H (not necessarily distinct), $n = \dim \mathcal{H}$. Then by axiom 3

$$Z_g = \langle Z_{0,1}, H^g Z_{0,1} \rangle \quad (25)$$

$$= \sum_i \lambda_i^g |d_i|^2 \quad (26)$$

for any $g \geq 0$. Furthermore,

$$Z_{g+1} = \text{Tr} H^g = \sum_i \lambda_i^g \quad (27)$$

for all $g \geq 0$. It follows that

$$\sum_i \lambda_i^g |d_i|^2 = \sum_i \lambda_i^{g-1} \quad (28)$$

for all $g \geq 1$.

Eq. (28) implies that all the eigenvalues are nonnegative. In order to prove this we order the eigenvalues according to their absolute value so that $|\lambda_i| \geq |\lambda_j|$ if $i \leq j$ and assume $\lambda_1 \neq 0$. Clearly (28) cannot hold for all g unless λ_1 is positive and we conclude also that

$$\sum_{i=1}^k \lambda_i^g |d_i|^2 = k \lambda_1^{g-1} \quad (29)$$

where k is the multiplicity of λ_1 . Subtracting (29) from (28) and repeating the above argument we conclude that all the eigenvalues are nonnegative. Below we show that a zero eigenvalue cannot occur and we can choose a basis such that $d_i = \lambda_i^{-\frac{1}{2}}$.

Consider the operators C_i on \mathcal{H}_R defined by

$$C_i = \sum_{jk} C_{ijk} x_j \otimes x_k^*. \quad (30)$$

By the symmetry of C_{ijk} and the four loop function these operators are mutually commuting and symmetric so they can be simultaneously diagonalized and we can choose a new self-dual basis such that $C_{ijk} = \delta_{ij} \delta_{ik} C_{iii}$. Using the definition of the handle operator we now find that $C_{iii}^2 = \lambda_i$ and H is diagonal in this basis. If $\lambda_i = 0$, then $C_{iii} = 0$. This contradicts axiom 4 which states that

$$\sum_{ijk} C_{ijk} d_k x_i \otimes x_j^* = I \quad (31)$$

and implies that all $C_{iii} \neq 0$ and $C_{iii}d_i = 1$.

It is not hard to convince oneself that any positive operator can arise as a handle operator. In fact we give an explicit construction in the next section.

We conclude this section by showing that all the information in a two-dimensional unitary TQFT is contained in the spectrum of H . We begin by making precise what we mean by the equivalence of two unitary TQFT in two dimensions. Let T be a theory satisfying the axioms of section 2 and let T' be another one whose objects are distinguished from those of T by a prime. We say that T and T' are equivalent if for any 1-manifold Σ there exists a unitary mapping

$$V_\Sigma : \mathcal{H}_\Sigma \mapsto \mathcal{H}'_\Sigma \quad (32)$$

such that the following conditions hold:

1. For any orientation preserving diffeomorphism $f : \Sigma_1 \mapsto \Sigma_2$ between 1-manifolds

$$U'_f = V_{\Sigma_2} U_f V_{\Sigma_1}^*. \quad (33)$$

2. For any oriented surface S

$$Z'(S) = V_{\partial S}(Z(S)) \quad (34)$$

with the understanding that $Z'(S) = Z(S)$ if S is closed.

3. For any 1-manifold Σ we have

$$(V_\Sigma x, V_{\Sigma^*} y)'_\Sigma = (x, y)_\Sigma. \quad (35)$$

Let T, T' be a pair of unitary TQFT's whose handle operators have the spectrum $\lambda_1, \dots, \lambda_n$. Then, by previous arguments, they are both equivalent to theories where all Hilbert spaces for connected boundary components coincide and equal \mathcal{H} and \mathcal{H}' , respectively. We have seen that by a suitable choice of bases the three loop functions then take the form

$$Z_{3,0} = \sum_i \sqrt{\lambda_i} x_i \otimes x_i \otimes x_i. \quad (36)$$

Let us call such a basis a canonical basis. Furthermore, $q_{ij} = \delta_{ij}$, $d_i = \lambda_i^{-\frac{1}{2}}$ with respect to a canonical basis. An equivalence between T and T' is now obtained by mapping a canonical basis for \mathcal{H} to a canonical basis for \mathcal{H}' .

We finally remark that two unitary TQFT's whose partition functions take the same value for all closed surfaces have handle operators with identical spectra, in view of (27), and are therefore equivalent.

4 Construction

In [8] a field theoretical construction was given of a TQFT on triangulated surfaces, using fields defined on vertices, such that $Z_g = N^\chi$, where N could take any positive value and χ is the Euler characteristic. In order to obtain a theory whose partition functions for closed surfaces are sums of exponentials we can take a direct sum of theories of this type.

Let us explain what we mean by the direct sum of two TQFT's. Suppose we have two theories with loop functions $Z_{g,n}$ and $Z'_{g,n}$ and Hilbert spaces \mathcal{H} and \mathcal{H}' for each connected boundary component. The direct sum is a TQFT with the Hilbert space $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}'$ for each boundary component and loop functions $\tilde{Z}_{g,n} = Z_{g,n} + Z'_{g,n}$ where we regard $Z_{g,n} + Z'_{g,n}$ in a natural way as an element of $\tilde{\mathcal{H}}^{\otimes n}$, $n \neq 0$. Unitary maps and bilinear forms are defined in the obvious way. One can then check that the direct sum satisfies the axioms if the original theories do so.

Here we give an alternative and more direct construction of a TQFT with an arbitrary handle operator using triangulations and local weights. Let S be a triangulated surface of genus g and N_1, \dots, N_n a sequence of n not necessarily distinct positive numbers. Let $J = \{1, \dots, n\}$. A colouring of S is a mapping from the vertices of S , $\mathcal{V}(S)$, into J . The image of a vertex under a colouring is called its colour. Given a colouring ϕ of S we define the weight of the vertex v to be N_i where $i = \phi(v)$. The weight w_Δ of a triangle Δ in S with corners whose colours are i, j, k is defined to be $N_i^{-\frac{1}{2}}$ if $i = j = k$ but zero otherwise. The partition function for a closed surface S is now defined to be

$$Z(S) = \sum_{\phi} \prod_{v \in \mathcal{V}(S)} N_{\phi(v)} \prod_{\Delta \in \mathcal{T}(S)} w_\Delta, \quad (37)$$

where the sum is over all colourings of S and $\mathcal{T}(S)$ denotes the set of all triangles in S . It is easy to see that if S is connected the only colourings which contribute are those that assign the same colour to all vertices and

$$Z(S) = \sum_{i=1}^n N_i^\chi. \quad (38)$$

Now let S be a connected triangulated surface with b boundary components. Let $|\partial S|$ be the number of vertices in ∂S and define

$$\zeta_i(S) = N_i^{-\frac{1}{2}|\partial S|} \sum_{\phi} \prod_{v \in \mathcal{V}(S)} N_{\phi(v)} \prod_{\Delta \in \mathcal{T}(S)} w_\Delta, \quad (39)$$

where ϕ now runs over all colourings which are i on the boundary. Clearly $\zeta_i(S) = N_i^{\chi(S)}$. We define an n -dimensional Hilbert space \mathcal{H} by assigning a vector x_i to each

colour and let $\{x_i\}$ be an orthonormal basis for \mathcal{H} and we set $x_i^* = x_i$. The partition function for S is now defined by regarding $\zeta_i(S)$ as a coordinate of $Z(S)$ with respect to the chosen basis, i.e.

$$Z(S) = \sum_i \zeta_i(S) x_i^{\otimes b}. \quad (40)$$

It is not hard to check that the theory so defined satisfies all the axioms and $N_i^{-\frac{1}{2}}$ are the eigenvalues of the handle operator.

5 Discussion

In this paper we have classified all two-dimensional unitary TQFT's and shown how they can be obtained using locally defined weights on triangulated surfaces. Obviously one would like to generalize some of these results to higher dimensions.

The notions of equivalence and direct sums extend in a straightforward fashion to dimensions higher than 2. As we have seen any two-dimensional unitary TQFT can be written as a direct sum of theories for which the space associated with the circle is 1-dimensional. It is appropriate to call such theories irreducible. It seems to be of importance to introduce a notion of irreducibility and to establish associated decomposition properties for TQFT's in general. A large class of 3-dimensional TQFT's has been constructed [15, 16, 17, 18] generalizing the theory of Turaev and Viro [7] and these theories serve as candidates for irreducible theories since the dimension of the Hilbert space of the sphere is always 1 and thus they cannot be decomposed.

A natural question to ask is whether it is true in higher dimensions that the partition functions for closed manifolds determine a theory up to equivalence as is the case in two dimensions. If the partition functions for manifolds M with $\partial M = \Sigma$ span the Hilbert space associated to Σ for all boundaries Σ , then it is not hard to show that this is the case. However, this condition is in general not fulfilled. In the two-dimensional case it holds only if the spectrum of the handle operator is non-degenerate.

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