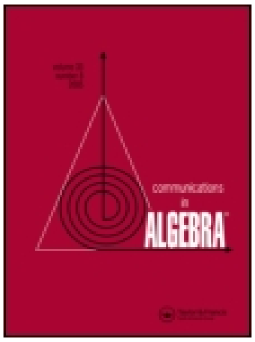

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ADEM-CARTAN OPERADS

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In this article, we introduce Adem-Cartan operads and prove that the cohomology of any algebra over such an operad is an unstable level algebra over the extended Steenrod algebra. Moreover, we prove that this cohomology is endowed with secondary cohomology operations.

Key Words: Cartan-Adem relations; Cohomology operations; E_∞ structures; Level algebras; Operads.

MSC (2000): 55-xx; 55S10; 18D50.

INTRODUCTION

The Steenrod Algebra $\mathcal{A}_p$ is one of the main computational tools of homotopy theory. Steenrod's operations were first introduced by Steenrod (1947) for $p = 2$ and for an odd prime in 1952 (Steenrod, 1952). The relations between these cohomology operations were determined by Adem (1952) and Cartan (1955). Cartan's proof relies on the computation of the singular cohomology of the Eilenberg-Mac Lane spaces at the prime p. Adem's proof is based on the computation of the homology of the symmetric group Σ_{p^2} acting on p^2 elements at the prime p. In the 1960s, Adams introduced secondary cohomology operations (Adams, 1960), which are an efficient tool to deal with realizability problems of unstable modules over $\mathcal{A}_2$. In this article, we extend these results to a more algebraically framework, at the prime 2.

The purpose of this article is to give an operadic description of algebras over the Steenrod algebra. More precisely we define Adem-Cartan operads and prove that the cohomology of an Adem-Cartan algebra is an unstable level algebra over the extended Steenrod algebra $\mathcal{B}_2$ (see Corollary 3.3.2). By a *level algebra*, we mean a commutative algebra (not necessarily associative) satisfying the following 4-term relation:

$$(a * b) * (c * d) = (a * c) * (b * d).$$

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These are algebras over the operad $\mathscr{L}$ev. An unstable algebra over $\mathscr{B}_2$ is a $\mathscr{B}_2$-module satisfying the usual conditions, namely the Cartan formula and Adem relations.

An Adem-Cartan operad is an operad $\mathscr{O}$ containing 2-cells $e_i \in \mathscr{O}(2)^{-i}$ and 4-cells $(G_n^m)_{0<m\leq n} \in \mathscr{O}(4)^{-n}$, whose differential satisfies relation (2.8). For any $\mathscr{O}$-algebra the cells e_i, usually the $\cup_i$-products, are responsible for the existence of Steenrod squares, whereas the cells G_n^m are responsible for the relations between them (Adem and Cartan relations). We prove that there exists a cofibrant Adem-Cartan operad $\mathscr{L}\mathrm{ev}^{AC}$, which is obtained by the process of attaching cells from the standard bar resolution of $\mathbb{F}_2$, and such that $\mathscr{L}\mathrm{ev}^{AC} \to \mathscr{L}\mathrm{ev}$ is a fibration. The cohomology of a $\mathscr{L}\mathrm{ev}^{AC}$-algebra A is a $\mathscr{L}\mathrm{ev}^{AC}$-algebra itself (this structure is however not natural). Hence the 4-cells G_{n+1}^m yield secondary cohomology operations $\theta^{m,n} : H^q(A) \to H^{4q-n-1}(A)$. We prove that given an Adem relation $\sum_i \mathrm{Sq}^{m_i}\mathrm{Sq}^{n_i}(a) = 0$, there exist maps $\psi^{m,n}$ from $\bigcap_i \mathrm{Sq}^{n_i} \subset H^*(A)$ to $H^*(A)/\sum_i \mathrm{Im}\,\mathrm{Sq}^{m_i}$ defined at the cochain level. We prove that these two maps coincide, that is $\psi^{m,n}(a) = [\theta^{m,n}(a)]$, for $a \in \bigcap_i \mathrm{Sq}^{n_i}$ (see Theorem 4.1.2).

Note that we recover some classical results on topological spaces. Since E_∞-operads are Adem-Cartan operads (see 2.3.5), any algebra over an E_∞-operad is an unstable algebra over the extended Steenrod algebra (see Kriz and May, 1995; May, 1970). Furthermore, since the cochain complex of a topological space $C^*(X; \mathbb{F}_2)$ is an algebra over an E_∞-operad (see Hinich and Schechtmann, 1987), then it has secondary cohomology operations. Thanks to the work of Kristensen (1963), we prove that these operations coincide with Adams operations. Moreover, we can extend these operations in a non-natural way to secondary cohomogy operations on the whole cohomology (see Theorems 4.2.2 and 4.2.3).

The article is presented as follows. Section 1 contains the background needed. In Section 2, Adem-Cartan operads are defined, the existence of $\mathscr{L}\mathrm{ev}^{AC}$ is proven, and we prove that E_∞-operads are Adem-Cartan operads. Section 3 is devoted to the main theorem. Section 4 is concerned with secondary cohomology operations, and Section 5 is devoted to proofs of technical lemmas stated in the different sections.

1. RECOLLECTIONS

The ground field is $\mathbb{F}_2$. In this article, a *vector space* means a differential $\mathbb{Z}$-graded vector space over $\mathbb{F}_2$, where the differential is of degree 1.

The symbol Σ_n denotes the symmetric group acting on n elements. Any $\sigma \in \Sigma_n$ is written $(\sigma(1)\cdots\sigma(n))$.

1.1. Operads

(Getzler and Jones, 1994; Ginzburg and Kapranov, 1994; Kriz and May, 1995; Loday, 1996). A (right) Σ_n*-module* is a (right) $\mathbb{F}_2[\Sigma_n]$-differential graded module. A Σ*-module* $\mathscr{M} = \{\mathscr{M}(n)\}_{n>0}$ is a collection of Σ_n-modules. Any Σ_n-module M gives rise to a Σ-module $\mathscr{M}$ by setting $\mathscr{M}(q) = 0$ if $q \neq n$ and $\mathscr{M}(n) = M$.

An *operad* is a right Σ-module $\{\mathscr{O}(n)\}_{n>0}$ such that $\mathscr{O}(1) = \mathbb{F}_2$, together with composition products:

$$\mathscr{O}(n) \otimes \mathscr{O}(i_1) \otimes \cdots \otimes \mathscr{O}(i_n) \longrightarrow \mathscr{O}(i_1 + \cdots + i_n)$$
$$o \otimes o_1 \otimes \cdots \otimes o_n \;\mapsto\; o(o_1, \ldots, o_n).$$

These compositions are subject to associativity conditions, unitary conditions, and equivariance conditions with respect to the action of the symmetric group. The equivariance conditions write

$$o(a_1 \cdot \tau_1, \dots, a_n \cdot \tau_n) = o(a_1, \dots, a_n) \cdot (\tau_1 \times \cdots \times \tau_n)$$

$$(o \cdot \sigma)(a_1, \dots, a_n) = o(a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n)}) \cdot \sigma(i_1, \dots, i_n),$$

where $\tau_1 \times \cdots \times \tau_n$ is the permutation of $\Sigma_{i_i+\cdots+i_n}$ such that τ_1 operates on the first i_1 terms, τ_2 on the next i_2 terms and so on; the permutation $\sigma(i_1, \dots, i_n)$ in $\Sigma_{i_i+\cdots+i_n}$ operates as σ on the n-blocks of size i_k.

For instance, for any $\sigma, \mu, \nu \in \mathcal{O}(2)$, one has

$$(\sigma \cdot (21))(\mu, \nu) = \sigma(\nu, \mu) \cdot (3412). \tag{1.1}$$

There is another definition of operads via $\circ_i$ operations

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \longrightarrow \mathcal{O}(n+m-1),$$

where $p \circ_i q$ is p composed with $n-1$ copies of the unit $1 \in \mathcal{O}(1)$ and with q at the ith position.

The forgetful functor from the category of operads to the category of Σ-modules has a left adjoint : the *free operad functor*, denoted by $\mathcal{F}$ree.

An *algebra over an operad* $\mathcal{O}$ or an $\mathcal{O}$*-algebra* A is a vector space together with evaluation maps

$$\mathcal{O}(n) \otimes A^{\otimes n} \longrightarrow A$$

$$o \otimes a_1 \otimes \cdots \otimes a_n \mapsto o(a_1, \dots, a_n).$$

These evaluation maps are subject to associativity conditions and equivariance conditions. These equivariance conditions write

$$(o \cdot \sigma)(a_1, \dots, a_n) = o(a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n)}).$$

A *graded algebra over an operad* $\mathcal{O}$ is an $\mathcal{O}$-algebra whose differential is zero.

1.2. Lemma. *Let A be a graded algebra ($d_A = 0$) over an operad $\mathcal{O}$. Assume that $o \in \mathcal{O}(n)$ is a boundary. Then $o(a_1, \dots, a_n) = 0$, for all $a_1, \dots, a_n$ in A.*

Proof. There exists $\omega \in \mathcal{O}(n)$ such that $d\omega = o$. The Leibniz rule implies that

$$d_A(\omega(a_1, \dots, a_n)) = o(a_1, \dots, a_n) + \sum_i \omega(a_1, \dots, d_A a_i, \dots, a_n)$$

and the result follows because $d_A = 0$. □

1.3. Operadic Cellular Attachment

(Berger and Moerdijk, 2003; Hinich, 1997). The category of operads is a closed model category. Weak equivalences are quasi-isomorphisms (i.e., isomorphisms in cohomology of the underlying vector spaces) and fibrations are epimorphisms. Cofibrations can be defined by the left lifting property with respect to the acyclic fibrations. For background material on closed model categories we refer to Dwyer and Spalinski (1995), Hovey (1999), and Quillen (1967).

Any morphism of operads

$$\mathcal{P} \longrightarrow \mathcal{Q}$$

can be factorized by a cofibration ($\rightarrowtail$) followed by an acyclic fibration ($\overset{\sim}{\twoheadrightarrow}$). This factorization can be realized using the inductive process of attaching cells (the category of operads is cofibrantly generated Hinich (1997)). An operad is *cofibrant* if the morphism from the initial object $\mathcal{F}\mathrm{ree}(0)$ to the operad is a cofibration. In order to produce a *cofibrant replacement* to an operad $\mathcal{O}$, one applies the inductive process of attaching cells to the canonical morphism $\mathcal{F}\mathrm{ree}(0) \longrightarrow \mathcal{O}$.

Let S^n_p be the free Σ_p-module generated by δt in degree n considered as a Σ-module. Let D^{n-1}_p be the free Σ_p-module generated by t in degree $n-1$ and dt in degree n, the differential sending t to dt. We have a canonical inclusion $i_n : S^n_p \longrightarrow D^{n-1}_p$ of Σ-modules (sending δt to dt). Let $f : S^n_p \longrightarrow \mathcal{O}$ be a morphism of Σ-modules. The cell D^{n-1}_p is attached to $\mathcal{O}$ along the morphism f via the following push-out:

$$\begin{array}{ccc} \mathcal{F}\mathrm{ree}(S^n_p) & \xrightarrow{i_n} & \mathcal{F}\mathrm{ree}(D^{n-1}_p) \\ {\scriptstyle \mathcal{F}\mathrm{ree}(f)}\downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow{i} & \mathcal{O} \coprod_\tau \mathcal{F}\mathrm{ree}(S^{n-1}_p). \end{array}$$

The main point of this process is that $f(\delta t)$, which was a cycle in $\mathcal{O}$, becomes a boundary in $\mathcal{O} \coprod_\tau \mathcal{F}\mathrm{ree}(S^{n-1}_p)$.

By iterating this process of cellular attachment, one gets a *quasi-free extension*: $\mathcal{O} \xrightarrow{i} \mathcal{O} \coprod_\tau \mathcal{F}\mathrm{ree}(V)$, that is if we forget the differential on V then $\mathcal{O} \coprod_\tau \mathcal{F}\mathrm{ree}(V)$ is the coproduct of $\mathcal{O}$ by a free operad over a free graded Σ-module V. Note that any cofibration is a retract of a quasi-free extension. A *quasi-free operad* is an operad which is free over a free Σ-module if we forget the differential.

The following proposition will be fundamental for our applications.

1.3.1. Proposition. *Let V be a free graded Σ_p-module together with*

$$d_V : V \longrightarrow \mathcal{O}(p) \oplus V$$

such that $d_V + d_{\mathcal{O}}$ is of square zero. Then if V is bounded above, the morphism $\mathcal{O} \rightarrow \mathcal{O} \coprod_\tau \mathcal{F}\mathrm{ree}(V)$ is a cofibration.

Proof. Let $V = \{V^i\}_{i \leq k}$ be a free graded Σ_p-module which is zero in degree more than k. We build the cofibration

$$\mathcal{O} \rightarrowtail \mathcal{O} \coprod_\tau \mathcal{F}\mathrm{ree}(V)$$

by induction on i. We attach operadic cells indexed by a basis B of V^k to the operad $\mathscr{O}$ along the map $\mathscr{F}\text{ree}\,(d_{V|_{V^k}})$:

$$\begin{array}{ccc} \mathscr{F}\text{ree}(\bigoplus_{b\in B} S_p^{k+1}) & \xrightarrow{i_{k+1}} & \mathscr{F}\text{ree}(\bigoplus_{b\in B} D_p^k) \\ {\scriptstyle \mathscr{F}\text{ree}(d_{V|_{V^k}})}\downarrow & & \downarrow \\ \mathscr{O} & \xrightarrow{i} & \mathscr{O}\coprod_\tau \mathscr{F}\text{ree}(\bigoplus_{b\in B} S_p^{n-1}). \end{array}$$

Suppose that we have built a cofibration $\mathscr{O} \to \mathscr{O}\coprod_\tau \mathscr{F}\text{ree}(\bigoplus_{j=i}^k V^j)$. The restriction of the map d_V to V^{i-1} extends by universality of the free operad functor to a well-defined map $f : \mathscr{F}\text{ree}(V^{i-1}) \to \mathscr{O}\coprod_\tau \mathscr{F}\text{ree}(\oplus_{j=i}^k V^j)$ as $d_V + d_{\mathscr{O}}$ is of square-zero we can make an operadic cell attachment of cells indexed by a basis B' of V^{i-1} along f:

$$\begin{array}{ccc} \mathscr{F}\text{ree}(\bigoplus_{b'\in B'} S_p^{i}) & \xrightarrow{i_i} & \mathscr{F}\text{ree}(\bigoplus_{b'\in B'} D_p^{i-1}) \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathscr{O}\coprod_\tau \mathscr{F}\text{ree}\left(\bigoplus_{j=i}^k V^j\right) & \xrightarrow{i} & \mathscr{O}\coprod_\tau \mathscr{F}\text{ree}(\bigoplus_{j=i-1}^k V^j). \end{array}$$

We then get a cofibration $\mathscr{O} \to \mathscr{O}\coprod_\tau \mathscr{F}\text{ree}(\bigoplus_{j=i-1}^k V^j)$. □

1.4. Homotopy Invariance Principle

Let $\mathscr{O}$ be a cofibrant operad. The category of $\mathscr{O}$-algebras is also a closed model category where weak equivalences are quasi-isomorphisms and fibrations are epimorphisms (Berger and Moerdijk, 2003 and Hinich, 1997).

Recall that the category of vector spaces is a closed model category, where weak equivalences are quasi-isomorphisms, and fibrations are epimorphisms. In this category, all objects are fibrant and cofibrant.

The following theorem is stated in Chataur (2001); its proof relies on a general result of Berger and Moerdijk (2002) about transfer of algebraic structure in closed model categories. Such a result was proven in caracteristic zero by Markl (2004) and for topological spaces by Boardman and Vogt (1973).

1.4.1. Theorem. Homotopy invariance principle (Chataur, 2001). *Let $\mathscr{O}$ be a cofibrant operad and assume that the morphism of vector spaces*

$$f : X \longrightarrow Y$$

is a weak equivalence between vector spaces. Assume that X is an $\mathscr{O}$-algebra. Then Y is also provided with an $\mathscr{O}$-algebra structure. For any cofibrant replacement $\widetilde{X}$ of X, there exists a sequence of quasi-isomorphisms of $\mathscr{O}$-algebras

$$X \longleftarrow \widetilde{X} \longrightarrow Y$$

such that the following diagram commute in cohomology

$$\begin{array}{ccc} & H^*(\widetilde{X}) & \\ \swarrow & & \searrow \\ H^*(X) & \xrightarrow{H^*(f)} & H^*(Y). \end{array}$$

1.4.2. Corollary. *Let $\mathcal{O}$ be a cofibrant operad. Let C be an $\mathcal{O}$-algebra and let H be its cohomology. Then H is a $\mathcal{O}$-algebra and there is a sequence of quasi-isomorphisms of $\mathcal{O}$-algebras*

$$\begin{array}{ccc} & \widetilde{C} & \\ {}^{\psi}\swarrow & & \searrow^{\Phi} \\ C & & H^*(C) \end{array}$$

where $\widetilde{C}$ is a cofibrant replacement of C and such that $H^*(\Phi) = H^*(\psi)$.

Proof. Set $X = C$, $Y = H^*(C)$. Choose a decomposition of vector spaces of $X = Z \oplus Q$, where Z is the kernel of the differential and let $f : X \to Y$ be the composite of the projections onto Z and onto H. Then f is a weak equivalence of vector spaces, since $H^*(f) = \mathrm{Id}$, and we can apply the previous theorem. □

2. ADEM-CARTAN OPERADS

In this section, we define Adem-Cartan operads and prove that a cofibrant Adem-Cartan operad exists, which is denoted by $\mathcal{L}\mathrm{ev}^{AC}$. In fact the operad $\mathcal{L}\mathrm{ev}^{AC}$ is the first step towards a resolution of the operad $\mathcal{L}\mathrm{ev}$. The latter governs level algebras, which are commutative algebras satisfying the 4-term relation (2.1). The idea of introducing level algebras instead of commutative algebras in order to deal with Cartan and Adem relations is inspired by the fact that these relations are not conditioned by the associativity of the product. In this section, we prove also that E_∞-operads are Adem-Cartan operads.

2.1. The Operad of Level Algebras

A *level algebra* A is a vector space together with a commutative product $*$ (not necessarily associative) satisfying the relation

$$(a * b) * (c * d) = (a * c) * (b * d), \qquad \forall\ a, b, c, d \in A. \tag{2.1}$$

2.1.1. Definition. Let $\mathbb{F}_2$ be the trivial representation of Σ_2 (generated by the operation μ) and $R_{\mathcal{L}\mathrm{ev}}$ be the sub-Σ_4-module of $\mathcal{F}\mathrm{ree}(\mathbb{F}_2)(4)$ generated by the elements $\mu(\mu, \mu) \cdot (\mathrm{Id} + \sigma)$ for all $\sigma \in \Sigma_4$. Then the operad $\mathcal{L}\mathrm{ev}$ is the operad

$$\mathcal{L}\mathrm{ev} = \mathcal{F}\mathrm{ree}(\mathbb{F}_2)/\langle R_{\mathcal{L}\mathrm{ev}} \rangle.$$

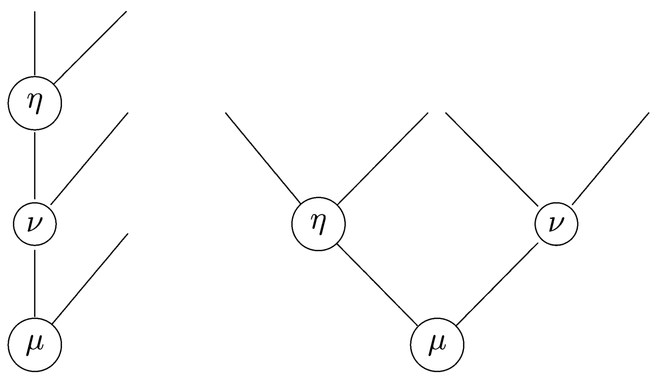
Algebras over this operad are level algebras.

2.1.2. Remark. Since a commutative and associative algebra is trivially a level algebra, there is a morphism of operads

$$\mathscr{L}\text{ev} \longrightarrow \mathscr{C}\text{om},$$

where $\mathscr{C}$om denotes the operad defining commutative and associative algebras.

2.1.3. Definition. For any Σ_2-module M, the vector space $\mathscr{F}\text{ree}(M)(4)$ is a direct sum of two Σ_4-modules: the one indexed by trees of *shape 1* denoted by $\mathbb{T}_1$ (that is the Σ_4-module generated by all the compositions $\mu \circ_1 \gamma$ for $\mu \in M$ and $\gamma \in \mathscr{F}\text{ree}(M)(3)$) and the one indexed by trees of *shape 2* denoted by $\mathbb{T}_2$ (that is the Σ_4-module generated by all the compositions $\mu(\nu, \eta)$ for $\mu, \nu, \eta \in M$).



$\mathbb{T}_1$ $\mathbb{T}_2$

As an example, since there is only one generator $\mu \in \mathscr{F}\text{ree}(\mathbb{F}_2)(2)$, the dimension of $\mathbb{T}_1(\mathscr{F}\text{ree}(\mathbb{F}_2))$ is 12 and the dimension of $\mathbb{T}_2(\mathscr{F}\text{ree}(\mathbb{F}_2))$ is 3, whereas the dimension of $\mathbb{T}_1(\mathscr{L}\text{ev})$ is 12 and the dimension of $\mathbb{T}_2(\mathscr{L}\text{ev})$ is 1.

2.2. Operads With a Σ_2-Projective Resolution

In this section, $\mathscr{O}$ is an operad such that $\mathscr{O}(2)$ is a Σ_2-projective resolution of $\mathbb{F}_2$. We define some elements in $\mathscr{O}(2)$ and $\mathscr{O}(4)$, which will play a role to define Adem-Cartan operads.

2.2.1. The Standard Bar Resolution. Let (21) be the non-trivial permutation of Σ_2. The standard bar resolution of Σ_2 over $\mathbb{F}_2$ is given by

$$\mathscr{W}^{-i} = \begin{cases} \langle e_i, e_i \cdot (21) \rangle, & \text{if } i \geq 0 \\ 0, & \text{if } i < 0 \end{cases}$$

$$d(e_i) = e_{i-1} + e_{i-1} \cdot (21), \quad \text{with } e_{-1} = 0.$$

Since $\mathscr{L}\text{ev} = \mathscr{F}\text{ree}(\mathbb{F}_2)/\langle R_{\mathscr{L}\text{ev}} \rangle$, there is a fibration of operads

$$p : \mathscr{F}\text{ree}(\mathscr{W}) \overset{\sim}{\twoheadrightarrow} \mathscr{F}\text{ree}(\mathbb{F}_2) \twoheadrightarrow \mathscr{L}\text{ev}.$$

The map $\mathcal{F}\text{ree}(\mathcal{W})(n) \to \mathcal{F}\text{ree}(\mathbb{F}_2)(n)$ is a quasi-ismorphism by Fresse (Proposition 3.1.2 of Fresse, 2004) and the map $\mathcal{F}\text{ree}(\mathbb{F}_2)(n) \to \mathcal{L}\text{ev}(n)$ is an isomorphism for $n < 4$ hence $p(n)$ is a quasi-isomorphism for $n < 4$. Furthermore $p(4)$ is a quasi-isomorphism on pieces of shape 1 (see 2.1.3).

Since $\mathcal{O}(2)$ is a Σ_2-projective resolution of $\mathbb{F}_2$, there exists a morphism of operads $m : \mathcal{F}\text{ree}(\mathcal{W}) \to \mathcal{O}$ such that $m(2)$ is an homotopy equivalence. Then the image $m(e_i) \in \mathcal{O}(2)$ is non zero. In the sequel, this image will be denoted also by e_i.

2.2.2. Notation. We use May's convention: for any integers i and j, the symbol (i, j) denotes $\frac{(i+j)!}{i!j!} \in \mathbb{F}_2$, if $i \geq 0$ and $j \geq 0$ and $(i, j) = 0$ otherwise. If the 2-adic expansion of i and j is $i = \sum a_k 2^k$ and $j = \sum b_k 2^k$, then $(i, j) = 0 \in \mathbb{F}_2$ if and only if there exists k such that $a_k = b_k = 1$.

For any $\sigma, \tau, \nu \in \mathcal{O}(2)$ we denote $\sigma \cdot (21)(\tau, \nu) \in \mathcal{O}(4)$ by $\sigma(\tau, \nu) \cdot Tw$, which is equal to the $\sigma(\nu, \tau) \cdot (3412)$ (see (1.1)).

2.2.3. Definition. We define some elements in $\mathcal{O}(4)$.

a) For $m > 0$, $[u_n^m]_x \in \mathcal{O}(4)^{-n-x}$ is a sum of elements of shape 2. More precisely, for any m such that $2^k \leq m \leq 2^{k+1} - 1$, one defines

$$[u_0^m]_x = e_x \cdot (21)^{m-1}(e_0, e_0), \qquad \text{and for} \quad n > 0,$$

$$\begin{aligned} [u_n^m]_x = \sum_{i=0}^{2^{k+1}-1} \sum_{0 \leq 2^{k+1}\delta - i \leq n} & (n - m + i, m - 1) \\ & \times [(i, m) e_x \cdot (21)^{m-1}(e_{2^{k+1}\delta-i}, e_{n+i-2^{k+1}\delta}) + (i-1, m) e_x \cdot (21)^{m-1} \\ & \times (e_{2^{k+1}\delta-i}, e_{n+i-2^{k+1}\delta} \cdot (21))]. \end{aligned} \tag{2.2}$$

b) For $p \in \mathbb{Z}$, the elements $\alpha_{n,p} \in \mathcal{O}(4)^{-n}$ are defined by

$$\alpha_{n,p} = \sum_{s=0}^{n-p} [u_{s+p}^{s+1}]_{n-s-p} + \sum_{s=0}^{n-p-1} [u_{s+p+1}^{s+1}]_{n-s-p-1} \cdot (3412). \tag{2.3}$$

2.2.4. Proposition. *The $[u_n^m]_x$'s satisfy the following properties*:

$$[u_n^m]_x = 0, \qquad \text{for } 0 < n < m, \tag{2.4}$$

$$[u_m^m]_x = e_x \cdot (21)^{m-1}(e_0, e_m) + e_x \cdot (21)^{m-1}(e_m, e_0 \cdot (21)), \tag{2.5}$$

$$\begin{aligned} \mathrm{d}[u_{n+1}^{m+1}]_x = {} & [u_n^{m+1}]_x(\mathrm{Id} + (2143)) + [u_n^m]_x(Tw + (4321)) \\ & + [u_{n+1}^{m+1}]_{x-1}(\mathrm{Id} + Tw). \end{aligned} \tag{2.6}$$

The $\alpha_{n,p}$'s satisfy the following property:

$$\begin{aligned} \alpha_{n,p} &= 0, \qquad \text{if } p < 0, \\ \mathrm{d}\alpha_{n,p} &= \alpha_{n-1,p-1} \cdot (\mathrm{Id} + (2143)) + \alpha_{n-1,p} \cdot (\mathrm{Id} + (4321)). \end{aligned} \tag{2.7}$$

2.2.5. Lemma. *Let $\mathcal{O}$ be an operad such that $\mathcal{O}(2)$ is a Σ_2-projective resolution of $\mathbb{F}_2$. For every graded $\mathcal{O}$-algebra A ($d_A = 0$), and for every $a \in A$, we define $D_n(a) := e_n(a, a)$. The following equality holds*:

$$[u_n^m]_x(a, a, a, a) = (n - 2m, 2m - 1)D_x D_{\frac{n}{2}}(a).$$

The proofs of the propositition and the lemma are postponed to the last section.

2.2.6. Example. By definition, we have for all $n \geq 0$

$$\begin{aligned}[u_n^1]_x &= \sum_k e_x(e_{2k}, e_{n-2k}) + \sum_l e_x(e_{2l+1}, e_{n-2l-1} \cdot (21)) \\ &= e_x(\psi(e_n)),\end{aligned}$$

where ψ is the coproduct in the standard bar resolution $\mathcal{W}$ of $\mathbb{F}_2$ (see Berger and Fresse, 2004; May, 1970).

$$[u_n^2]_x = \begin{cases} \sum_{0 \leq 2\delta \leq n} e_x \cdot (21)(e_{2\delta}, e_{n-2\delta} \cdot (21)^{\delta}) & \text{if } n \text{ even} \\ \sum_{0 \leq 4\delta - 1 \leq n} e_x \cdot (21)(e_{4\delta-1}, e_{n+1-4\delta}((12) + (21)) & \text{if } n \text{ odd.} \end{cases}$$

2.3. Adem-Cartan Operads

2.3.1. Definition. An *Adem-Cartan operad* is an operad $\mathcal{O}$ such that $\mathcal{O}(2)$ is a Σ_2-projective resolution of $\mathbb{F}_2$, and such that there is a distinguished family $(G_n^m)_{0<m\leq n}$ of elements in $\mathcal{O}(4)^{-n}$ subject to the following relations:

$$\begin{aligned} \mathrm{d}G_m^m &= G_{m-1}^{m-1} \cdot (3214)(\mathrm{Id} + (2143)) + G_{m-1}^{m-2} \cdot (\mathrm{Id} + (4321)) \\ &\quad + \alpha_{m-1,m-1-p} + \alpha_{m-1,p} \cdot (3214) \quad \textit{and} \quad \textit{for } n > m \qquad (2.8) \\ \mathrm{d}G_n^m &= G_{n-1}^m \cdot (\mathrm{Id} + (2143)) + G_{n-1}^{m-2} \cdot (\mathrm{Id} + (4321)) \\ &\quad + \alpha_{n-1,n-1-p} + \alpha_{n-1,p} \cdot (3214), \end{aligned}$$

where p is the integer part of $\frac{m-1}{2}$.

An *Adem-Cartan* algebra is an algebra over an Adem-Cartan operad.

Remark. We need to prove that the definition is consistent, i.e., that $d^2 = 0$. The relation (2.7) implies the following computations.

Assume first that $n > m + 1$, then $n - 1 > m$ and $n - 1 > m - 2$

$$\begin{aligned} (\mathrm{d})^2(G_n^m) &= \mathrm{d}(G_{n-1}^m \cdot (\mathrm{Id} + (2143)) + G_{n-1}^{m-2} \cdot (\mathrm{Id} + (4321))) \\ &\quad + \mathrm{d}(\alpha_{n-1,n-1-p} + \alpha_{n-1,p} \cdot (3214)) \end{aligned}$$

$$= G^m_{n-2} \cdot (\mathrm{Id} + (2143))^2 + G^{m-2}_{n-2} \cdot (\mathrm{Id} + (4321))(\mathrm{Id} + (2143))$$
$$+ G^{m-2}_{n-2} \cdot (\mathrm{Id} + (2143))(\mathrm{Id} + (4321)) + G^{m-4}_{n-2} \cdot (\mathrm{Id} + (4321))^2$$
$$+ \alpha_{n-2,n-2-p} \cdot (\mathrm{Id} + (2143)) + \alpha_{n-2,p} \cdot (3214)(\mathrm{Id} + (2143))$$
$$+ \alpha_{n-2,n-2-(p-1)} \cdot (\mathrm{Id} + (4321)) + \alpha_{n-2,p-1} \cdot (3214)(\mathrm{Id} + (4321))$$
$$+ \mathrm{d}(\alpha_{n-1,n-1-p} + \alpha_{n-1,p} \cdot (3214)).$$

Since (2143) and (4321) commute and are permutations of order 2, the 2 first lines vanish. Furthermore,

$$\mathrm{d}\alpha_{n-1,n-1-p} = \alpha_{n-2,n-2-p} \cdot (\mathrm{Id} + (2143)) + \alpha_{n-2,n-1-p} \cdot (\mathrm{Id} + (4321)),$$

and $(3214)(\mathrm{Id} + (4321)) = (\mathrm{Id} + (2143))(3214)$ imply.

$$\mathrm{d}\alpha_{n-1,p} \cdot (3214) = \alpha_{n-2,p-1} \cdot (3214)(\mathrm{Id} + (4321)) + \alpha_{n-2,p} \cdot (3214)(\mathrm{Id} + (2143)),$$

which yields $(\mathrm{d})^2(G^m_n) = 0$ for $n > m + 1$.

If $n = m + 1$, then $n - 1 = m$ and $n - 1 > m - 2$; thus terms in $\alpha_{u,v}$ will vanish as before and

$$(\mathrm{d})^2(G^m_{m+1}) = G^{m-1}_{m-1} \cdot (3214)(\mathrm{Id} + (2143))^2 + G^{m-2}_{n-2} \cdot (\mathrm{Id} + (4321))(\mathrm{Id} + (2143))$$
$$+ G^{m-2}_{n-2} \cdot (\mathrm{Id} + (2143))(\mathrm{Id} + (4321)) + G^{m-4}_{n-2} \cdot (\mathrm{Id} + (4321))^2$$
$$= 0.$$

If $m = n$, then

$$(\mathrm{d})^2(G^m_m) = \mathrm{d}(G^{m-1}_{m-1} \cdot (3214)(\mathrm{Id} + (2143)) + G^{m-2}_{m-1} \cdot (\mathrm{Id} + (4321)))$$
$$+ \mathrm{d}(\alpha_{m-1,m-1-p} + \alpha_{m-1,p} \cdot (3214))$$
$$= G^{m-2}_{m-2} \cdot [(3214)(\mathrm{Id} + (2143))]^2 + G^{m-3}_{m-2} \cdot (\mathrm{Id} + (4321))(3214)(\mathrm{Id} + (2143))$$
$$+ G^{m-2}_{m-2} \cdot (\mathrm{Id} + (2143))(\mathrm{Id} + (4321)) + G^{m-4}_{m-2} \cdot (\mathrm{Id} + (4321))^2$$
$$+ \alpha_{m-2,m-2-p'} \cdot (3214)(\mathrm{Id} + (2143)) + \alpha_{m-2,p'} \cdot (3214)^2(\mathrm{Id} + (2143))$$
$$+ \alpha_{m-2,m-2-(p-1)} \cdot (\mathrm{Id} + (4321)) + \alpha_{m-2,p-1} \cdot (3214)(\mathrm{Id} + (4321))$$
$$+ \mathrm{d}(\alpha_{m-1,m-1-p} + \alpha_{m-1,p} \cdot (3214)),$$

where p' is the integer part of $\frac{m-2}{2}$.

It's easy to check that the first two lines vanish. Furthermore, if $m = 2k + 1$ is odd, then $p = k$, $p' = k - 1$ and the last lines write

$$+ \alpha_{2k-1,k}(\mathrm{Id} + (4321)) \cdot (3214) + \alpha_{2k-1,k-1}(\mathrm{Id} + (2143))$$
$$+ \alpha_{2k-1,k} \cdot (\mathrm{Id} + (4321)) + \alpha_{2k-1,k-1} \cdot (\mathrm{Id} + (2143))(3214)$$
$$+ \mathrm{d}(\alpha_{2k,k} + \alpha_{2k,k} \cdot (3214)) = 0,$$

and if $m = 2k + 2$ is even, then $p = k$, $p' = k$ and the last lines write

$$+\alpha_{2k,k}(\mathrm{Id}+(4321))\cdot(3214)+\alpha_{2k,k}(\mathrm{Id}+(2143))$$
$$+\alpha_{2k,k+1}\cdot(\mathrm{Id}+(4321))+\alpha_{2k,k-1}\cdot(\mathrm{Id}+(2143))(3214)$$
$$+\mathrm{d}(\alpha_{2k+1,k+1}+\alpha_{2k+1,k}\cdot(3214))=0.$$

2.3.2. Theorem. *There exists a cofibrant Adem-Cartan operad, denoted* $\mathscr{L}\mathrm{ev}^{AC}$, *satisfying the following properties*:

a) $\mathscr{L}\mathrm{ev}^{AC}(2) = \mathscr{W}$;
b) *there is a fibration* $f : \mathscr{L}\mathrm{ev}^{AC} \to \mathscr{L}\mathrm{ev}$ *such that* $f(n)$ *is a quasi-isomorphism for* $n < 4$;
c) f *induces an isomorphism* $H^0(\mathscr{L}\mathrm{ev}^{AC}(n)) \simeq \mathscr{L}\mathrm{ev}(n)$.

Proof. The proof consists in building a sequence of cofibrant operads $\mathscr{L}\mathrm{ev}_m^{AC}$, $m \geq 0$, satisfying a) and b) where $\mathscr{L}\mathrm{ev}_m^{AC} \to \mathscr{L}\mathrm{ev}_{m+1}^{AC}$ is a cofibration obtained by operadic cellular attachment. More precisely, $\mathscr{L}\mathrm{ev}_0^{AC} = \mathscr{F}\mathrm{ree}(\mathscr{W})$ satisfies a) and b), according to 2.2.1. Assume $\mathscr{L}\mathrm{ev}_m^{AC}$ is built, satisfies a) and b), and contains a family of elements $G_n^k \in (\mathscr{L}\mathrm{ev}_m^{AC})^{-n}$ for $0 < k \leq m$ and $n \geq k$, satisfying the relation (2.8). Let V^{m+1} be the free graded Σ_4-module generated by elements G_n^{m+1} of degree $-n$ for $n \geq m+1$ with $d : V^{m+1} \to \mathscr{L}\mathrm{ev}_m^{AC}(4) \oplus V^{m+1}$ defined by the relation (2.8). Since $d^2 = 0$, according to Proposition 1.3.1,

$$\mathscr{L}\mathrm{ev}_m^{AC} \to \mathscr{L}\mathrm{ev}_m^{AC} \coprod_{\tau} \mathscr{F}\mathrm{ree}(V^{m+1}) =: \mathscr{L}\mathrm{ev}_{m+1}^{AC}$$

is a cofibration, obtained by a sequence of pushouts. It is clear that $\mathscr{L}\mathrm{ev}_{m+1}^{AC}$ satisfies a). By induction hypothesis, there exists a fibration $f_m : \mathscr{L}\mathrm{ev}_m^{AC} \to \mathscr{L}\mathrm{ev}$ such that $f_m(n)$ is a quasi-isomorphism for $n < 4$. Using the universal property of pushouts and sending G_n^{m+1} to zero in $\mathscr{L}\mathrm{ev}$, we obtain that there exists a fibration $f_{m+1} : \mathscr{L}\mathrm{ev}_{m+1}^{AC} \to \mathscr{L}\mathrm{ev}$ such that $f_{m+1}(n)$ is a quasi-isomorphism for $n < 4$. Let $\mathscr{L}\mathrm{ev}^{AC}$ be the limit over m of $\mathscr{L}\mathrm{ev}_m^{AC}$. It is clear that $\mathscr{L}\mathrm{ev}^{AC}$ satisfies a) and b) ($f : \mathscr{L}\mathrm{ev}^{AC} \to \mathscr{L}\mathrm{ev}$ is the limit of the f_m's).

To prove that $f : \mathscr{L}\mathrm{ev}^{AC} \to \mathscr{L}\mathrm{ev}$ induces an isomorphism $H^0(f)$, it suffices to prove that $H^0(f_1(4))$ is an isomorphism; indeed, the relations defining $\mathscr{L}\mathrm{ev}$ are generated by a Σ_4-module hence $H^0(\mathscr{L}\mathrm{ev}_1^{AC}(n)) \to \mathscr{L}\mathrm{ev}(n)$ will be an isomorphism, for all n; if $m > 1$, we do not introduce cells in degree -1, hence $H^0(f_m(n))$ will be an isomorphism for all n, hence $H^0(f(n))$ also.

$$H^0(\mathscr{L}\mathrm{ev}_1^{AC}(4)) = \mathbb{T}_1(\mathscr{F}\mathrm{ree}(\mathscr{W})^0(4))/\mathrm{d}(\mathbb{T}_1(\mathscr{F}\mathrm{ree}(\mathscr{W})^1(4))$$
$$\oplus\, \mathbb{T}_2(\mathscr{F}\mathrm{ree}(\mathscr{W})^0(4))/\mathrm{d}[\mathbb{T}_2(\mathscr{F}\mathrm{ree}(\mathscr{W})^1(4)) \oplus G_1^1 \cdot \mathbb{F}_2(\Sigma_4)]$$

the first summand being isomorphic to $\mathbb{T}_1(\mathscr{L}\mathrm{ev}(4))$ (see 2.2.1). To prove that the second summand is isomorphic to $\mathbb{T}_2(\mathscr{L}\mathrm{ev}(4))$, it suffices to prove that it is 1-dimensional (see 2.1.3). Let $X = e_0(e_0, e_0) \in \mathscr{F}\mathrm{ree}(\mathscr{W})^0(4)$ and $\overline{X}$ its class in the second summand.

i) For all σ in the dihedral group D_1 (σ satisfies $\{\sigma(1), \sigma(2)\} \subset \{1, 2\} \cup \{3, 4\}$), there exists $u = e_i(e_j, e_k)$, $i + j + k = 1$, such that $\mathrm{d}u = X + X \cdot \sigma$. Hence for all $\sigma \in D_1$ we have $\overline{X \cdot \sigma} = \overline{X}$.
ii) Since $\mathrm{d}G_1^1 = X + X \cdot (3214)$, for all $\sigma \in D_1$ we have $\overline{X \cdot (3214)\sigma} = \overline{X}$, i.e., for all $\tau \in D_2 = \{\tau | \{\tau(1), \tau(2)\} \subset \{1, 4\} \cup \{2, 3\}\}$ we have $\overline{X \cdot \tau} = \overline{X}$.
iii) Finally, equality $(3124) = (2134)(3214)$ implies $\overline{X \cdot (3124)\sigma} = \overline{X}$ for all $\sigma \in D_1$, or for all $\tau \in D_3 = \{\tau | \{\tau(1), \tau(2)\} \subset \{1, 3\} \cup \{2, 4\}\}$, $\overline{X \cdot \tau} = \overline{X}$.

Since $\Sigma_4 = D_1 \cup D_2 \cup D_3$ we get the result. □

2.3.3. Remark. For any Adem-Cartan operad $\mathcal{O}$, there is a morphism of operads $m : \mathcal{L}\mathrm{ev}^{AC} \to \mathcal{O}$ such that $m(2)$ is a quasi-isomorphism. The converse is true when $\mathcal{O}(2)$ is a Σ_2-projective resolution of $\mathbb{F}_2$. Hence, any Adem-Cartan algebra is an algebra over $\mathcal{L}\mathrm{ev}^{AC}$.

2.3.4. Corollary. *Any graded Adem-Cartan algebra is a level algebra.*

Proof. For any operad $\mathcal{O}$ and any $\mathcal{O}$-algebra A, $H^*(A)$ is a $H^*(\mathcal{O})$-algebra hence a $H^0(\mathcal{O})$-algebra. But A is a graded algebra over $\mathcal{L}\mathrm{ev}^{AC}$ (Remark 2.3.3) and $H^0(\mathcal{L}\mathrm{ev}^{AC}) = \mathcal{L}\mathrm{ev}$, $H^*(A) = A$, which yields the result. □

2.3.5. E_∞-operads. According to Remark 2.1.2, there is a morphism $\mathcal{L}\mathrm{ev} \longrightarrow \mathcal{C}\mathrm{om}$. An E_∞-operad E is a Σ_*-projective resolution of the operad $\mathcal{C}\mathrm{om}$: for every r, $E(r)$ is Σ_r-projective and there exists an acyclic fibration $E \to \mathcal{C}\mathrm{om}$. The structure of closed model category on operads implies that for any cofibrant operad $\mathcal{O} \to \mathcal{C}\mathrm{om}$ there exists a morphism $\mathcal{O} \to E$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}\mathrm{ree}(0) & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \sim \\ \mathcal{O} & \longrightarrow & \mathcal{C}\mathrm{om}. \end{array}$$

For instance, there exists a morphism $\mathcal{L}\mathrm{ev}^{AC} \to E$ such that the previous diagram commutes. Hence, any E_∞-operad is an Adem-Cartan operad. In particular, the algebraic Barratt-Eccles operad $\mathcal{BE}$ (studied in Berger and Fresse, 2004) is an Adem-Cartan operad.

3. ADEM-CARTAN ALGEBRAS AND THE EXTENDED STEENROD ALGEBRA

The aim of this section is to prove that the cohomology of an algebra over an Adem-Cartan operad carries an action of the extended Steenrod algebra.

3.1. The Extended Steenrod Algebra $\mathcal{B}_2$

3.1.1. Generalized Steenrod Powers. (Mandell, 2001, May, 1970). The *extended Steenrod algebra*, denoted by $\mathcal{B}_2$, is a graded associative algebra over $\mathbb{F}_2$

generated by the generalized Steenrod squares Sq^i of degree $i \in \mathbb{Z}$. These generators satisfy the Adem relation, if $t < 2s$

$$\mathrm{Sq}^t\,\mathrm{Sq}^s = \sum_i \binom{s-i-1}{t-2i} \mathrm{Sq}^{s+t-i}\,\mathrm{Sq}^i .$$

Note that negative Steenrod squares are allowed and that $\mathrm{Sq}^0 = \mathrm{Id}$ is not assumed. If $\mathcal{A}_2$ denotes the classical Steenrod algebra, then

$$\mathcal{A}_2 \cong \frac{\mathcal{B}_2}{\langle \mathrm{Sq}^0 + \mathrm{Id} \rangle}.$$

3.1.2. Definition. As in the classical case an *unstable module over* $\mathcal{B}_2$, is a graded $\mathcal{B}_2$-module together with the unstability condition

$$\mathrm{Sq}^n(x) = 0 \qquad \text{if } |x| < n.$$

An unstable level algebra over $\mathcal{B}_2$ is a graded level algebra $(A, *)$ which is an unstable module over $\mathcal{B}_2$, such that

$$\begin{cases} \mathrm{Sq}^{|x|}(x) = x * x, \\ \mathrm{Sq}^s(x * y) = \sum \mathrm{Sq}^t(x) * \mathrm{Sq}^{s-t}(y) \quad \text{(Cartan relation)}. \end{cases}$$

3.1.3. Remark. The category of unstable algebras over $\mathcal{A}_2$ is a full subcategory of the category of unstable level algebras over $\mathcal{B}_2$.

3.2. Cup-i Products

Let A be an algebra over an operad $\mathcal{O}$ such that $\mathcal{O}(2)$ is a Σ_2-projective resolution of $\mathbb{F}_2$. The evaluation map

$$\mathcal{O}(2) \otimes A^{\otimes 2} \longrightarrow A,$$

defines cup-i products $a \cup_i b = e_i(a, b)$ (e_i's were defined in 2.2.1). Steenrod squares are defined by $\mathrm{Sq}^r(a) = a \cup_{|a|-r} a = e_{|a|-r}(a, a)$. Following May (1970), define $D_n(a) = e_n(a, a) = \mathrm{Sq}^{|a|-n}(a)$. In that terminology, Adem relations read

$$\sum_k (k, v-2k) D_{w-v+2k} D_{v-k}(a) = \sum_l (l, w-2l) D_{v-w+2l} D_{w-l}(a), \tag{3.1}$$

and Cartan relations read

$$D_n(x * y) = \sum_{k=0}^{n} D_k(x) * D_{n-k}(y). \tag{3.2}$$

Note that in general, an $\mathcal{O}$-algebra does not satisfy these two relations.

3.3. Main Theorem

3.3.1. Theorem. *Any graded Adem-Cartan algebra is an unstable level algebra over the extended Steenrod algebra.*

Proof. We have already proven in Corollary 2.3.4 that a graded Adem-Cartan algebra A is a level algebra. Assume first that A is a module over $\mathscr{B}_2$ (Adem relation (3.1)). Hence the unstability condition reads

$$\mathrm{Sq}^n(x) = x \cup_{|x|-n} x = 0, \qquad \text{if } |x| - n < 0.$$

The next equality is also immediate:

$$\mathrm{Sq}^{|x|}(x) = x \cup_0 x = e_0(x, x) = x * x.$$

The Cartan relation is given by $\mathrm{d}G_n^1$; according to Lemma 1.2, $\mathrm{d}G_{n+1}^1(x, x, y, y) = 0$. Using relations (2.8), (2.3), and (2.5), one has

$$\begin{aligned} 0 &= G_n^1 \cdot (\mathrm{Id} + (2143))(x, x, y, y) + \alpha_{n,n}(x, x, y, y) + \alpha_{n,0} \cdot (3214)(x, x, y, y) \\ &= 2G_n^1(x, x, y, y) + [u_n^1]_0(x, x, y, y) + \sum_s [u_s^{s+1}]_{n-s}(y, x, x, y) \\ &\quad + [u_{s+1}^{s+1}]_{n-s-1} \cdot (3412)(y, x, x, y) \\ &= \sum_{l=0}^{n} e_0(e_l, e_{n-l} \cdot (21)^l)(x, x, y, y) + [u_0^1]_n(y, x, x, y) \\ &\quad + \sum_{s=0}^{n-1} e_{n-s-1} \cdot (21)^s(x * y, y \cup_{s+1} x) + e_{n-s-1} \cdot (21)^s(x \cup_{s+1} y, x * y). \end{aligned}$$

Using the commutativity of $*$ and $\cup_i$ for all i, one gets

$$\mathrm{d}G_{n+1}^1(x, x, y, y) = \sum_{l=0}^{n} D_l(x) * D_{n-l}(y) + D_n(x * y),$$

which gives the Cartan relation (3.2).

The proof of Adem relation (3.1) relies on Lemma 2.2.5, and on the relation $\mathrm{d}G_{n+1}^m(a, a, a, a) = 0$. Combined with the relation (2.8), one gets $\alpha_{n,p}(a, a, a, a) = \alpha_{n,n-p}(a, a, a, a)$, that is

$$\begin{aligned} &\sum_s [u_{p+s}^s(a, a, a, a) + u_{p+s}^{s+1}(a, a, a, a)]_{n-p-s} \\ &\quad = \sum_t [u_{n-p+t}^t(a, a, a, a) + u_{n-p+t}^{t+1}(a, a, a, a)]_{p-t} \\ &\quad \Rightarrow \sum_s [(p-s, 2s-1) + (p-s-2, 2s+1)] D_{n-p-s} D_{\frac{p+s}{2}}(a) \\ &\quad = \sum_t [(n-p-t, 2t-1) + (n-p-t-2, 2t+1)] D_{p-t} D_{\frac{n-p+t}{2}}(a). \end{aligned}$$

But $(x, y-2)+(x-2, y)=(x, y)$, hence

$$\sum_s (p-s, 2s+1) D_{n-p-s} D_{\frac{p+s}{2}}(a)=\sum_t (n-p-t, 2t+1) D_{p-t} D_{\frac{n-p+t}{2}}(a).$$

Since the first term is zero, as soon as $p-s$ is odd, we can set $s=p-2l$, and also $t=n-p-2k$. As a consequence,

$$\sum_l (2l, 2p-4l+1) D_{n-2p+2l} D_{p-l}(a)=\sum_k (2k, 2n-2p-4k+1) D_{2p-n+2k} D_{n-p-k}(a).$$

Using the 2-adic expansion, one gets

$$(2l, 2p-4l+1)=(l, p-2l); \tag{3.3}$$

by setting $w:=p$ and $v:=n-p$, one obtains

$$\sum_l (l, w-2l) D_{v-w+2l} D_{w-l}(a)=\sum_k (k, v-2k) D_{w-v+2k} D_{v-k}(a),$$

which is the Adem relation (3.1). □

3.3.2. Corollary. *Let $\mathscr{O}$ be an Adem-Cartan operad. The cohomology of any $\mathscr{O}$-algebra is an unstable level algebra over $\mathscr{B}_2$.*

Proof. By Remark 2.3.3, there is a morphism of operads $\mathscr{L}\mathrm{ev}^{AC} \to \mathscr{O}$, then it suffices to prove the theorem for $\mathscr{O}=\mathscr{L}\mathrm{ev}^{AC}$. Let A be a $\mathscr{L}\mathrm{ev}^{AC}$-algebra, then $H^*(A)$ is a level algebra. In order to prove the Adem-Cartan relations, we compute the boundaries of $G_n^1(a, a, b, b)$ and $G_n^m(a, a, a, a)$ for cocycles a and b that represents classes $[a]$ and $[b]$ in the cohomology. These boundaries give the Adem-Cartan relations between $e_o(a, b)$, $e_i(a, a)$, $e_j(b, b)$, which represent $[a]*[b]$, $D_i([a])$ and $D_j([b])$ respectively. □

In particular for algebras over an E_∞-operad, we recover the results of Kriz and May (1995) and May (1970).

3.3.3. Example. Given a $\mathscr{F}\mathrm{ree}(\mathscr{W})$-algebra, a natural question is to know whether it is possible to extend this structure into a structure of Adem-Cartan algebra. The following example shows that it cannot be done by imposing the triviality of all G_n^m's.

Let us consider the torus $\mathbb{T}^2=S^1\times S^1$. The algebraic model of the normalized singular cochains of $\mathbb{T}^2$ that we use is

$$A_{\mathbb{T}^2}=C^*(S^1)\otimes C^*(S^1).$$

The vector space $A_{\mathbb{T}_2}$ is generated by $1:=1\otimes 1$ in degree zero, $\alpha:=a\otimes 1$ and $\beta:=1\otimes b$ in degree 1, and $\alpha\beta=a\otimes b$ in degree 2. The differential is trivial on $A_{\mathbb{T}^2}$.

The $\mathscr{F}\mathrm{ree}(\mathscr{W})$-structure on $C^*(S^1)$ is given by $e_0(1,1)=1$, $e_1(a,a)=a$ where a is the generator in degree 1 and all the others are zero (see Berger and Fresse, 2004).

Hence the $\mathcal{F}\text{ree}(\mathcal{W})$-structure on $A_{\mathbb{T}^2}$ is given by the coproduct ψ on $\mathcal{F}\text{ree}(\mathcal{W})$. Since the differential is zero, one has $\mathrm{d}(G_3^1(\alpha, \beta, \alpha, \beta)) = 0$. But $\mathrm{d}G_3^1 = G_2^1(\mathrm{Id} + (2143)) + \alpha_{2,2} + \alpha_{2,0} \cdot (3214)$. Then, using the definition of the $\alpha_{n,p}$'s and the commutativity of the e_i's, one has

$$\begin{aligned}\mathrm{d}(G_3^1(\alpha, \beta, \alpha, \beta)) &= G_2^1(\alpha, \beta, \alpha, \beta) + G_2^1(\beta, \alpha, \beta, \alpha)\\ &\quad + e_0(e_1, e_1 \cdot (21))(\alpha, \beta, \alpha, \beta) + e_2(e_0, e_0)(\alpha, \beta, \alpha, \beta)\\ &= G_2^1(\alpha, \beta, \alpha, \beta) + G_2^1(\beta, \alpha, \beta, \alpha)\\ &\quad + e_1(\alpha, \beta)^2 + e_2(\alpha\beta, \alpha\beta).\end{aligned}$$

But

$$\begin{aligned}e_1(\alpha, \beta) &= \psi(e_1)(a \otimes 1, 1 \otimes b)\\ &= (e_0 \otimes e_1 + e_1 \otimes e_0 \cdot (21))((a, 1) \otimes (1, b))\\ &= e_0(a, 1) \otimes e_1(1, b) + e_1(a, 1) \otimes e_0(b, 1) = 0\end{aligned}$$

and

$$\begin{aligned}e_2(\alpha\beta, \alpha\beta) &= (e_0 \otimes e_2 + e_1 \otimes e_1 \cdot (21) + e_2 \otimes e_0)((a, a) \otimes (b, b))\\ &= a \otimes b = \alpha\beta\end{aligned}$$

thus

$$G_2^1(\alpha, \beta, \alpha, \beta) + G_2^1(\beta, \alpha, \beta, \alpha) = \alpha\beta,$$

which proves that the action of G_2^1 is nonzero.

4. OPERADIC SECONDARY COHOMOLOGY OPERATIONS

In this section, we prove that there exist secondary cohomology operations on the cohomology of an Adem-Cartan algebra A, and that these operations coincide with the Adams secondary operations in case $A = C^*(X, \mathbb{F}_2)$ is the singular cochain complex of a space X.

4.1. Secondary Cohomology Operations on Adem-Cartan Algebras

Let $\mathcal{O}$ be an Adem-Cartan operad and A be an $\mathcal{O}$-algebra. Then A is a $\mathcal{L}\text{ev}^{AC}$-algebra, hence $H^*(A)$ is endowed with a (non-natural) structure of $\mathcal{L}\text{ev}^{AC}$-algebra (see Corollary 1.4.2). Hence, for any (m, p), there are morphisms

$$\theta^{m,p} : H^n(A) \to H^{4n-p-1}(A)$$

given by $\theta^{m,p}(x) = G_{p+1}^m(x, x, x, x)$.

Besides from Corollary 3.3.2, the cohomology $H^*(A)$ is an unstable level algebra over $\mathscr{B}_2$. Let $x \in H^n(A)$ and

$$R_{Ad}(x) = \sum_i Sq^{m_i} Sq^{n_i}(x) = 0$$

be an Adem relation with $x \in \cap_i \operatorname{Ker}(\mathrm{Sq}^{n_i})$. Let $c \in A^n$, $\mathrm{d}c = 0$ representing x. Then there exists (m, p) such that $(\mathrm{d}G^m_{p+1})(c, c, c, c) = R_{Ad}(c)$: $p = 3n - m_i - n_i$ and m is a function of n and $m_i + n_i$. Since $\mathrm{Sq}^{n_i}(x) = 0$, there exists $b_i \in A^{n+n_i-1}$ such that $db_i = e_{n-n_i}(c, c)$. The element

$$b = \sum_i e_{n-m_i+n_i}(1, e_{n-n_i})(b_i, c, c) + e_{n-m_i+n_i-1}(b_i, b_i)$$

satisfies $d(G^m_{p+1}(c, c, c, c) + b) = 0$.

4.1.1. Proposition. *The class of $G^m_{p+1}(c, c, c, c) + b$ in $H^{m_i+n_i+n-1}(A)/\sum_i Im(\mathrm{Sq}^{m_i})$ does not depend on the choices of the b_i's and c.*

Proof. First, it does not depend on the choices of the b_i's. Let $b_i' \in A^{n+n_i-1}$ such that $\mathrm{d}b_i' = \mathrm{d}b_i = e_{n-n_i}(c, c)$, and

$$b' = \sum_i e_{n-m_i+n_i}(1, e_{n-n_i})(b_i', c, c) + e_{n-m_i+n_i-1}(b_i', b_i').$$

Then $\mathrm{d}(b + b') = 0$ and the following relation implies the result:

$$b + b' = \sum_i \mathrm{Sq}^{m_i}(b_i + b_i') + \mathrm{d}e_{n-m_i+n_i}(b_i + b_i', b_i).$$

Secondly, it does not depend on the choice of a representant c of x. Using the homotopy invariance principle as in 1.4.2, there is a zig-zag of acyclic fibrations of Adem-Cartan algebras

$$\begin{array}{ccc} & \widetilde{A} & \\ {}^{f}\swarrow & & \searrow^{g} \\ A & & H^*(A) \end{array}$$

with $H^*(f) = H^*(g)$. Let c and c' be two cocycles that represent x, and let $u, u' \in \widetilde{A}$ such that $\mathrm{d}u = 0$, $f(u) = c$, $g(u) = x$ and the same for u', c'. Since $\mathrm{Sq}^{n_i}(x) = 0$, there exists v_i, v_i' such that $\mathrm{d}v_i = \mathrm{Sq}^{n_i}(u)$ and $\mathrm{d}v_i' = \mathrm{Sq}^{n_i}(u')$. Let v, v' defined as b. Let us prove that

$$[G^m_{p+1}(c, c, c, c) + f(v)] = [G^m_{p+1}(c', c', c', c') + f(v')]$$

in $H^{n+m_i+n_i-1}(A)/\sum_i Im(\mathrm{Sq}^{m_i})$. This is equivalent to prove that $g(v) + g(v')$ is in the sum of all $Im(\mathrm{Sq}^{m_i})$. But

$$v = \sum_i e_{n-m_i+n_i}(v_i, \mathrm{d}v_i) + \mathrm{Sq}^{m_i}(v_i)$$

and $g(dv_i) = 0$ imply that $g(v) = \sum_i \mathrm{Sq}^{m_i}(g(v_i)) \in \sum Im(\mathrm{Sq}^{m_i})$, which proves the result. □

As a consequence, we have defined a map

$$\psi^{m,p} : \bigcap_i \mathrm{Ker}(\mathrm{Sq}^{n_i}) \subset H^n(A) \longrightarrow H^{n+m_i+n_i-1}(A) \Big/ \sum_i Im(\mathrm{Sq}^{m_i})$$

$$x \mapsto [G^m_{p+1}(c,c,c,c) + b].$$

4.1.2. Proposition. *Let* $\iota : \bigcap_i \mathrm{Ker}(\mathrm{Sq}^{n_i}) \to H^n(A)$ *be the canonical inclusion and* $\pi : H^{n+m_i+n_i-1}(A) \to H^{n+m_i+n_i-1}(A)/\sum_i Im(\mathrm{Sq}^{m_i})$ *be the canonical projection. Then*

$$\pi\theta^{m,p}\iota = \psi^{m,p}.$$

Proof. Using the proof of the previous proposition, it only remains to show that $[G^m_{p+1}(c,c,c,c) + b] = G^m_{p+1}(x,x,x,x)$ in $H^{m_i+n_i+n-1}(A)/\sum_i Im(\mathrm{Sq}^{m_i})$ for $x \in \bigcap_i \mathrm{Ker}(\mathrm{Sq}^{n_i})$. This is equivalent to prove that $g(v)$ is in $\sum_i Im(\mathrm{Sq}^{m_i})$ for v with $f(v) = b$, which has been already proven. □

4.2. Adams' Secondary Operations

Let $C^*(X;\mathbb{F}_2)$ denotes the singular cochains complex of a topological space X and $H^*(X,\mathbb{F}_2)$ its cohomology. We recall from Hinich and Schechtmann (1987) that $C^*(X;\mathbb{F}_2)$ is an algebra over a E_∞-operad, hence an Adem-Cartan algebra.

Adams (1960) defined (in an axiomatic way) stable secondary cohomology operations. His approach is topological, and uses the theory of so-called "universal examples". These operations correspond to Adem relations

$$R_{Ad} = \sum_i \mathrm{Sq}^{m_i}\,\mathrm{Sq}^{n_i},$$

and are denoted by Φ. Let us recall Adams' axioms:

Axiom 1. For any $u \in H^n(X;\mathbb{F}_2)$, $\Phi(u)$ is defined if and only if $\mathrm{Sq}^{n_i}(u) = 0$ for all n_i.

Axiom 2. If $\Phi(u)$ is defined then

$$\Phi(u) \in H^{m_i+n_i+n-1}(X;\mathbb{F}_2) \Big/ \sum_i Im(\mathrm{Sq}^{m_i}).$$

Axiom 3. The operation Φ is natural.

Axiom 4. Let (X, A) be a pair of topological spaces, we have the long exact sequence

$$\cdots H^{n-1}(A;\mathbb{F}_2) \xrightarrow{\delta^*} H^n(X,A;\mathbb{F}_2) \xrightarrow{j^*} H^n(X;\mathbb{F}_2) \xrightarrow{i^*} H^n(A;\mathbb{F}_2) \xrightarrow{\delta^*} \cdots.$$

Let $v \in H^n(X, A; \mathbb{F}_2)$ be a class such that ϕ is defined on $j^*(v) \in H^n(X; \mathbb{F}_2)$. Let $w_i \in H^*(A; \mathbb{F}_2)$ such that $\delta^*(w_i) = \mathrm{Sq}^{n_i}(v)$. Then, we have

$$i^*\Phi(j^*(v)) = \sum_i \mathrm{Sq}^{m_i}(w_i) \in H^*(A; \mathbb{F}_2) \Big/ i^*\Big(\sum_i Im(\mathrm{Sq}^{m_i})\Big).$$

Axiom 5. The operation Φ commutes with suspension.

Later on, Kristensen proved that these operations can be defined at the cochain level, using the existence of a coboundary which creates the stable secondary cohomology operation defined by Adams (Kristensen, 1963, Chapter 6). More precisely, for an Adem relation R_{Ad} and a class $x \in \bigcap_i \mathrm{Sq}^{n_i}$, Kristensen defines cochain operations θ such that the differential of $\theta(c)$ (c is a representant of x) gives a cocycle representing an Adem relation $R_{Ad}(x)$. If one chooses b_i such that $db_i = \mathrm{Sq}^{n_i}(c)$, then one gets a cocycle, and a cohomology class

$$Qu^r(c) = [\theta(c) + \sum_i (e_{n-m_i+n_i}(1, e_{n-n_i})(b_i, c, c) + e_{n-m_i+n_i-1}(b_i, b_i))].$$

Then,

4.2.1. Theorem (Kristensen, 1963, Theorem 6.1). *Any operation $x \mapsto Qu^r(c)$ satisfies axiom 1–5 of Adams.*

4.2.2. Corollary. *The maps $\psi^{m,p}$ coincide with the stable secondary cohomology operations of Adams.*

Proof. The proof relies on Theorem 4.2.1 with $\theta(c) = G^m_{p+1}(c, c, c, c)$. □

4.2.3. Theorem. *The stable secondary cohomology operations ϕ of Adams extend to maps $\theta^{m,p} : H^n(X, \mathbb{F}_2) \to H^{n+m_i+n_i-1}(X, \mathbb{F}_2)$. More precisely, if we denote by $\iota : \bigcap_i \mathrm{Ker}(\mathrm{Sq}^{n_i}) \to H^n(X, \mathbb{F}_2)$ and $\pi : H^{n+m_i+n_i-1}(X, \mathbb{F}_2) \to H^{m_i+n_i+n-1}(X, \mathbb{F}_2)/\sum_i Im(\mathrm{Sq}^{m_i})$, then*

$$\phi = \pi\theta^{m,p}\ \iota.$$

Proof. It is the translation of Theorem 4.1.2 for $A = C^*(X, \mathbb{F}_2)$. □

5. PROOF OF TECHNICAL LEMMAS

In this section, Proposition 2.2.4 and Lemma 2.2.5 are proven.

5.1. Lemma. *For any $i, j \leq 2^p - 1$, one has*

$$(i, j) = 0, \quad \text{if } i + j \geq 2^p, \quad \text{and}$$

$$(i, j) = (2^p - i - j - 1, j).$$

Proof. Let $\sum_{l=0}^{p-1} a_l 2^l$ and $\sum_{l=0}^{p-1} b_l 2^l$ be the 2-adic expansion of i and j respectively. Recall that $(i, j) = 1$ if and only if the 2-adic expansion of $i + j$ is $\sum(a_l + b_l)2^l$. If $i + j \geq 2^p$ this is not the case, thus $(i, j) = 0$.

If $(i, j) = 1$ then the 2-adic expansion of $2^p - 1 - i - j$ is $\sum_{l=0}^{p-1}(1 - a_l - b_l)2^l$, thus the 2-adic expansion of $(2^p - 1 - i - j) + j$ has for coefficients $(1 - a_l - b_l) + (b_l)$. Consequently $(2^p - 1 - i - j, j) = 1$. The converse is true by symmetry.

Note that the first assertion is a consequence of the second one, because if $(i + j) \geq 2^p$ then $(2^p - 1 - i - j) < 0$, and $(\alpha, \beta) = 0$ if $\alpha < 0$ or $\beta < 0$. □

5.2. Proof of Lemma 2.2.5

Using the commutativity of $\cup_x$, one has

$$\begin{aligned}
[u_n^m]_x(a, a, a, a) &= \sum_{i=0}^{2^{k+1}-1} \sum_{0 \leq 2^{k+1}\delta - i \leq n} (n - m + i, m - 1)(i, m)D_{2^{k+1}\delta - i}(a) \cup_x D_{n+i-2^{k+1}\delta}(a) \\
&\quad + (n - m + i, m - 1)(i - 1, m)D_{2^{k+1}\delta - i}(a) \cup_x D_{n+i-2^{k+1}\delta}(a) \\
&= \sum_{i,\delta}(n - m + i, m - 1)(i, m - 1)D_{2^{k+1}\delta - i}(a) \cup_x D_{n+i-2^{k+1}\delta}(a).
\end{aligned}$$

Let $0 \leq j \leq 2^{k+1} - 1$ such that $n + i \equiv -j\ [2^{k+1}]$, then there exists δ' such that $n + i - 2^{k+1}\delta = 2^{k+1}\delta' - j$, and Lemma 5.1 implies

$$\begin{aligned}
(n - m + i, m - 1) &= (2^{k+1}(\delta + \delta') - m - j, m - 1) \\
&= (j, m - 1) \qquad (5.1) \\
(n - m + j, m - 1) &= (i, m - 1).
\end{aligned}$$

As a consequence, if $i \neq j$ or $i = j$ and $\delta \neq \delta'$, the 2 following terms in $[u_n^m]_x(a, a, a, a)$

$$\begin{aligned}
&(n - m + i, m - 1)(i, m - 1)D_{2^{k+1}\delta - i}(a) \cup_x D_{2^{k+1}\delta' - j}(a) \\
&\quad + (n - m + j, m - 1)(j, m - 1)D_{2^{k+1}\delta' - j}(a) \cup_x D_{2^{k+1}\delta - i}(a)
\end{aligned}$$

vanish. Hence, if there exists (i, δ) such that $2^{k+1}\delta - i = n - (2^{k+1} - i)$, then

$$[u_n^m]_x(a, a, a, a) = (n - m + i, m - 1)^2 D_x D_{\frac{n}{2}}(a).$$

Relation (3.3) implies $(n - m + i, m - 1) = (2n - 2m + 2i, 2m - 1) = (n - 2m, 2m - 1)$. Furthermore, if $(n - 2m, 2m - 1) = 1$, then n is even, and we can pick $0 \leq i \leq 2^{k+1} - 1$ such that $\exists\delta$, $\frac{n}{2} = 2^{k=1}\delta - i$. □

5.3. Proof of Proposition 2.2.4

We have to prove relations (2.4), (2.5), (2.6), and (2.7). Relation (2.7) is straightforward using relation (2.6) and definition (2.3).

Proof of relations (2.4) *and* (2.5). Assume $n \le m$. The condition $0 \le 2^{k+1}\delta - i \le n \le m \le 2^{k+1} - 1$ implies δ equals 0 or 1. If $\delta = 0$, then $i = 0$. If $\delta = 1$, then $i + m \ge 2^{k+1}$ and $(i, m) = 0$ by Lemma 5.1. So $[u_n^m]_x$ writes

$$[u_n^m]_x = (n-m, m-1)e_x \cdot (21)^{m-1}(e_0, e_n) + \sum_{i=0}^{2^{k+1}-1} (n-m+i, m-1)(i-1, m)e_x \cdot (21)^{m-1}(e_{2^{k+1}-i}, e_{n+i-2^{k+1}} \cdot (21)).$$

If $n < m$, then $(n-m, m-1) = 0$ and $i - 1 + m \ge 2^{k+1}$ implies $(i-1, m) = 0$, which proves relation (2.4).

If $n = m$, $(i-1, m) = 0$ for all $i \ne 2^{k+1} - m$ and $(2^{k+1} - m, m-1)(2^{k+1} - 1 - m, m) = (0, m-1)(0, m)$ by virtue of Lemma 5.1. This proves relation (2.5).

Proof of relation (2.6). For the convenience of the reader, let

$$B^{\delta,i}_{x,[m],n} = e_x \cdot (21)^{m-1}(e_{2^{k+1}\delta-i}, e_{n+i-2^{k+1}\delta}),$$

where $[m]$ means m mod 2, then

$$[u_n^m]_x = \sum_{i,\delta}(n-m+i)[(i, m)B^{\delta,i}_{x,[m],n} + (i-1, m)B^{\delta,i}_{x,[m],n} \cdot (1243)].$$

Remarks.

a) Let $P = \{\mathrm{Id}, (2134), (2143), (1243)\} \in \Sigma_4$. Then the set $\mathcal{F} = \{B^{\delta,i}_{x,[m],n} \cdot \sigma, B^{\delta',j}_{x',[m+1],n'} \cdot \tau, \forall x, n, x', n', i, i', \delta, \delta', \forall \sigma, \tau \in P\}$ is a free system in $\mathcal{F}\mathrm{ree}(\mathcal{W})(4)$.
b) For any $\sigma \in P$, $B^{\delta,i}_{x,[m],n} \cdot \sigma T w = B^{\delta,i}_{x,[m],n} \cdot T w \sigma = B^{\delta,i}_{x,[m+1],n} \cdot \sigma$.

There are two cases to consider: if $2^k \le m \le 2^{k+1} - 2$ (then $m + 1 \le 2^{k+1} - 1$) or if $m = 2^{k+1} - 1$. Since computations are long but not difficult, we'll present only the first case:

$$\begin{aligned} \mathrm{d}[u_{n+1}^{m+1}]_{x+1} = \sum_{i,\delta} &\underbrace{(n-m+i, m)(i, m+1)}_{a_i}[B^{\delta,i+1}_{x+1,[m+1],n} \cdot (\mathrm{Id} + (2134)) \\ &+ B^{\delta,i}_{x+1,[m+1],n} \cdot (\mathrm{Id} + (1243)) + B^{\delta,i}_{x,[m+1],n+1} \cdot (\mathrm{Id} + Tw)] \\ &+ \underbrace{(n-m+i, m)(i-1, m+1)}_{b_i}[B^{\delta,i+1}_{x+1,[m+1],n} \cdot ((1243) + (2143)) \\ &+ B^{\delta,i}_{x+1,[m+1],n} \cdot (\mathrm{Id} + (1243)) + B^{\delta,i}_{x,[m+1],n+1} \cdot (\mathrm{Id} + Tw)(1243)], \end{aligned}$$

$$\begin{aligned} [u_n^{m+1}]_{x+1}(\mathrm{Id} + ((2143)) = \sum_{i,\delta} &\underbrace{(n-m+i-1, m)(i, m+1)}_{c_i} B^{\delta,i}_{x+1,[m+1],n}(\mathrm{Id} + (2143)) \\ &+ \underbrace{(n-m+i-1, m)(i-1, m+1)}_{d_i} B^{\delta,i}_{x+1,[m+1],n}((1243) \\ &+ (2134)), \end{aligned}$$

$$[u_n^m]_{x+1}(Tw+(4321)) = \sum_{i,\delta} \underbrace{(n-m+i, m-1)(i, m)}_{e_i} B^{\delta,i}_{x+1,[m],n}(Tw+(4321)) + \underbrace{(n-m+i, m-1)(i-1, m)}_{f_i} B^{\delta,i}_{x+1,[m],n}(1243) \times (Tw+(4321)).$$

Note that

$$B^{\delta,i}_{x+1,[m],n+1} \cdot (4321) = e_{x+1} \cdot (21)^m (e_{n+i-2^{k+1}\delta}, e_{2^{k+1}\delta-i}) \cdot (3412)(4321).$$

Hence by using relation (5.1), we get

$$\sum_{i,\delta}(n-m+i, m-1)(i, m) B^{\delta,i}_{x+1,[m],n} \cdot (4321) = \sum_{j,\delta'} \underbrace{(n-m+j-1, m)(j, m-1)}_{m_j} B^{\delta',j}_{x+1,[m+1],n}(2143)$$

and

$$\sum_{i,\delta}(n-m+i, m-1)(i-1, m) B^{\delta,i}_{x+1,[m],n}(1243)(4321) = \sum_{j,\delta'} \underbrace{(n-m+j, m)(j, m-1)}_{l_j} B^{\delta',j}_{x+1,[m+1],n}(1243),$$

$$[u_{n+1}^{m+1}]_x(1+Tw) = \sum_{i,\delta} \underbrace{(n-m+i, m)(i, m+1)}_{g_i} B^{\delta,i}_{x,[m+1],n+1}(\mathrm{Id}+Tw) + \underbrace{(n-m+i, m)(i-1, m+1)}_{h_i} B^{\delta,i}_{x,[m+1],n+1} \times (1243)(\mathrm{Id}+Tw).$$

Thus, to prove relation (2.6), it suffices to prove that the sum of all the coefficients of elements of $\mathcal{F}$ vanishes. For instance, the coefficient of $B^{\delta,i}_{x+1,[m+1],n} \cdot \mathrm{Id}$ is $a_{i-1} + a_i + b_i + c_i + e_i$, that is

$$\begin{aligned}
&(n-m+i-1, m)(i-1, m+1) + (n-m+i, m)(i, m+1) \\
&\quad + (n-m+i, m)(i-1, m+1) + (n-m+i-1, m)(i, m+1) \\
&\quad + (n-m+i, m-1)(i, m) \\
&= (n-m+i, m-1)(i-1, m+1) + (n-m+i, m-1)(i, m+1) \\
&\quad + (n-m+i, m-1)(i, m) = 0.
\end{aligned}$$

The coefficient of $B^{\delta,i}_{x+1,[m+1],n} \cdot (2143)$ is $b_{i-1} + c_i + m_i$, that is

$$(n-m+i-1, m)(i-2, m+1) + (n-m+i-1, m)(i, m+1) \\ + (n-m+i-1, m)(i, m-1) = 0.$$

All the other computations follow the same pattern. □

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