CHAPTER XIII

Lie Algebras

Introduction. In this chapter, Lie algebras are considered from a purely algebraical point of view, without reference to Lie groups and differential geometry. The "Jacobi identity" may be justified by the properties of the "bracket" operation $[x,y] = xy - yx$ in an associative algebra.

To each Lie algebra $\mathfrak{g}$ (over a commutative ring K) there corresponds a K-algebra $\mathfrak{g}^e$ (called the "enveloping algebra" of $\mathfrak{g}$), in such a way that the "representations" of $\mathfrak{g}$ in a K-module C are in a 1–1–correspondence with the $\mathfrak{g}^e$-module structures of C. Since $\mathfrak{g}^e$ has a natural augmentation $\varepsilon\colon \mathfrak{g}^e \to K$, it is a supplemented K-algebra. This at once leads to the homology and cohomology groups of $\mathfrak{g}$. To prove that these coincide with the ones hitherto considered (Chevalley-Eilenberg, *Trans. Am. Math. Soc.* 63 (1948), 85–124) we must assume that $\mathfrak{g}$ is K-free and apply the theorem of Poincaré-Witt (§ 3) which is an essential tool in the theory.

While the first two sections contain only definitions and results which are essentially trivial, because they do not use Jacobi's identity, this identity is essential for the theorem of Poincaré-Witt (§ 3). Once this theorem is established, the theory develops in a manner analogous to that for groups.

We do not touch upon the more advanced aspects of the homology theory of Lie algebras (Whitehead lemmas, Levi's theorem, semi-simple Lie algebras, etc.).

1. LIE ALGEBRAS AND THEIR ENVELOPING ALGEBRAS

We recall that a Lie algebra over a commutative ring K is a K-module $\mathfrak{g}$ together with a K-homomorphism $x \otimes y \to [x,y]$ of $\mathfrak{g} \otimes_K \mathfrak{g}$ into $\mathfrak{g}$ such that for $x, y, z \in \mathfrak{g}$:

$$[x,x] = 0 \tag{1}$$

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \qquad \text{(Jacobi's identity)}. \tag{2}$$

Condition (1) implies the condition

$$[x,y] + [y,x] = 0 \tag{1'}$$

and is equivalent with (1′) if in the ring K there is an element k with $2k = 1$.

A (left) $\mathfrak{g}$-representation of $\mathfrak{g}$ is a K-module A together with a K-homomorphism $x \otimes a \to xa$ of $\mathfrak{g} \otimes A$ into A such that

$$x(ya) - y(xa) = [x,y]a.$$

We now construct an associative K-algebra $\mathfrak{g}^e$ with the property that each (left) $\mathfrak{g}$-representation may be regarded as a (left) $\mathfrak{g}^e$-module and vice-versa. We shall call $\mathfrak{g}^e$ the *enveloping algebra* of $\mathfrak{g}$.

Let $T(\mathfrak{g})$ be the tensor algebra of the K-module $\mathfrak{g}$: this is the graded (associative) K-algebra such that $T_0(\mathfrak{g}) = K$ and $T_n(\mathfrak{g})$ is the n-fold tensor product (over K) of $\mathfrak{g}$ with itself. The product of elements $x_1 \otimes \cdots \otimes x_p$ and $y_1 \otimes \cdots \otimes y_q$ is $x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots y_q$. It is clear that a K-linear map $\mathfrak{g} \otimes_K A \to A$ admits a unique extension $T(\mathfrak{g}) \otimes_K A \to A$ satisfying $(x_1 \otimes \cdots \otimes x_n) \otimes a \to (x_1 \cdots (x_n a) \cdots)$. This converts A into a left $T(\mathfrak{g})$-module. Conversely any $T(\mathfrak{g})$-module A is obtained this way from a unique map $\mathfrak{g} \otimes A \to A$. In order that this map $\mathfrak{g} \otimes A \to A$ be a $\mathfrak{g}$-representation it is necessary and sufficient that the elements of $T(\mathfrak{g})$ of the form

$$x \otimes y - y \otimes x - [x,y] \qquad x,y \in \mathfrak{g} \tag{3}$$

annihilate A. Consequently, we are led to introduce the two-sided ideal $U(\mathfrak{g})$ of $T(\mathfrak{g})$ generated by the elements (3) and define the enveloping algebra of $\mathfrak{g}$ as $\mathfrak{g}^e = T(\mathfrak{g})/U(\mathfrak{g})$. Clearly left $\mathfrak{g}$-representations and left $\mathfrak{g}^e$-modules may be identified; we shall use the term *left* $\mathfrak{g}$-*module* to indicate either of the above.

We arrived at the enveloping algebra $\mathfrak{g}^e$ by the consideration of *left* representations $\mathfrak{g} \otimes A \to A$. A *right* representation $A \otimes \mathfrak{g} \to A$ with

$$(ax)y - (ay)x = a[x,y]$$

could equally well be used. Indeed, any K-homomorphism $A \otimes \mathfrak{g} \to A$ extends uniquely to a K-homomorphism $A \otimes T(\mathfrak{g}) \to A$ satisfying $a \otimes (x_1 \otimes \cdots \otimes x_n) = (\cdots(ax_1)\cdots x_n)$. This converts A into a right $T(\mathfrak{g})$-module. In order that $A \otimes \mathfrak{g} \to A$ be a right representation of A it is necessary and sufficient that the elements of the form (3) in $T(\mathfrak{g})$ annihilate A. We are thus led to the same enveloping algebra $\mathfrak{g}^e = T(\mathfrak{g})/U(\mathfrak{g})$. Thus right $\mathfrak{g}$-representations and right $\mathfrak{g}^e$-modules may be identified; we shall use the term *right* $\mathfrak{g}$-*module* to indicate either of the two.

The relation between $\mathfrak{g}$-representations and $\mathfrak{g}^e$-modules can be made more explicit by the use of the K-homomorphism

$$i\colon\ \mathfrak{g} \to \mathfrak{g}^e$$

defined by the fact that $\mathfrak{g} = T_1(\mathfrak{g})$. We then have

PROPOSITION 1.1. *Let f: $\mathfrak{g} \otimes A \to A$ be the map which defines A as a left $\mathfrak{g}$-representation. Then f admits a unique factorization $f = h(i \otimes A)$ where h: $\mathfrak{g}^e \otimes A \to A$ is a map defining A as a left $\mathfrak{g}^e$-module. Similarly for right representations and right modules.*

Since $T(\mathfrak{g})$ is a graded ring we have a natural augmentation ε: $T(\mathfrak{g}) \to T_0(\mathfrak{g}) = K$. Since ε is zero on $T_n(\mathfrak{g})$ for $n > 0$ it follows that the ideal $U(\mathfrak{g})$ is in the kernel of ε. Thus by passing to quotients we obtain the augmentation

$$\varepsilon\colon\ \mathfrak{g}^e \to K$$

which converts $\mathfrak{g}^e$ into a supplemented K-algebra. The augmentation ideal $I(\mathfrak{g})$ is generated by the image of i: $\mathfrak{g} \to \mathfrak{g}^e$.

As an example, consider the case of an *abelian* Lie algebra $\mathfrak{g}$ (i.e. $[x,y] = 0$ for $x,y \in \mathfrak{g}$). The enveloping algebra $\mathfrak{g}^e$ is then the quotient of $T(\mathfrak{g})$ by the two-sided ideal $U(\mathfrak{g})$ generated by the elements $x \otimes y - y \otimes x$; thus $\mathfrak{g}^e$ is the "symmetric algebra" of the K-module $\mathfrak{g}$. If $\mathfrak{g}$ is K-free with K-basis $\{x_\alpha\}$, then $\mathfrak{g}^e$ may be identified with the algebra $K[x_\alpha]$ of polynomials in the letters x_α.

A homomorphism f: $\mathfrak{g} \to \mathfrak{g}'$ of a Lie algebra $\mathfrak{g}$ into a Lie algebra $\mathfrak{g}'$ over the same ring K is a K-homomorphism satisfying $f([x,y]) = [fx,fy]$. Clearly f induces a map f^e: $\mathfrak{g}^e \to \mathfrak{g}'^e$ of supplemented algebras such that the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{f} & \mathfrak{g}' \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ \mathfrak{g}^e & \xrightarrow[f^e]{} & \mathfrak{g}'^e \end{array}$$

is commutative.

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two Lie algebras over the same ring K. The direct sum $\mathfrak{g} + \mathfrak{g}'$ (also called "direct product") is defined as a Lie algebra by setting

$$[(x,x'),(y,y')] = ([x,y],[x',y']).$$

If we identify x with $(x,0)$ and x' with $(0,x')$ then $\mathfrak{g}$ and $\mathfrak{g}'$ become subalgebras of $\mathfrak{g} + \mathfrak{g}'$, and $[x,x'] = 0$ for $x \in \mathfrak{g}$, $x' \in \mathfrak{g}'$. The inclusion maps $\mathfrak{g} \to \mathfrak{g} + \mathfrak{g}'$, $\mathfrak{g}' \to \mathfrak{g} + \mathfrak{g}'$ induce homomorphisms

$$\mathfrak{g}^e \to (\mathfrak{g} + \mathfrak{g}')^e, \qquad \mathfrak{g}'^e \to (\mathfrak{g} + \mathfrak{g}')^e$$

which in turn define a homomorphism

$$\varphi\colon\ \mathfrak{g}^e \otimes \mathfrak{g}'^e \to (\mathfrak{g} + \mathfrak{g}')^e.$$

PROPOSITION 1.2. *The homomorphism φ is an isomorphism of supplemented algebras.*

PROOF. The map $(x,x') \to x \otimes 1 + 1 \otimes x'$ of $\mathfrak{g} + \mathfrak{g}'$ into the tensor product of algebras $T(\mathfrak{g}) \otimes T(\mathfrak{g}')$ induces a homomorphism of K-algebras

$$\bar{\psi}\colon\ T(\mathfrak{g} + \mathfrak{g}') \to T(\mathfrak{g}) \otimes T(\mathfrak{g}').$$

After composing $\bar{\psi}$ with the natural map $T(\mathfrak{g}) \otimes T(\mathfrak{g}') \to \mathfrak{g}^e \otimes \mathfrak{g}'^e$ we find that $U(\mathfrak{g} + \mathfrak{g}')$ is mapped into zero. Thus we obtain a homomorphism

$$\psi\colon\ (\mathfrak{g} + \mathfrak{g}')^e \to \mathfrak{g}^e \otimes \mathfrak{g}'^e,$$

and it is trivial to verify that $\psi\varphi$ and $\varphi\psi$ are identity maps. Thus φ is an isomorphism.

The definition of a Lie subalgebra $\mathfrak{h}$ of a Lie algebra is obvious. We say that $\mathfrak{h}$ is an ideal if $[x,y] \in \mathfrak{h}$ for $x \in \mathfrak{g}, y \in \mathfrak{h}$. In view of the anti-commutativity of the bracket operation, there is no need to distinguish between left and right ideals. If $\mathfrak{h}$ is an ideal, then $\mathfrak{g}/\mathfrak{h}$ is again a Lie algebra with the bracket operation induced by that of $\mathfrak{g}$. Consider the composite map

$$\mathfrak{h} \xrightarrow{f} \mathfrak{g} \xrightarrow{i} \mathfrak{g}^e \tag{4}$$

where f is the inclusion, and let L denote the right ideal in $\mathfrak{g}^e$ generated by the image of if. Then L coincides with the left ideal generated by the image if, since in $\mathfrak{g}^e$ we have

$$if(x')i(x) = i(x)if(x') + if([x',x]) \qquad x' \in \mathfrak{h}, x \in \mathfrak{g}$$

PROPOSITION 1.3. *Let $\mathfrak{h}$ be an ideal in $\mathfrak{g}$ and $\varphi\colon\ \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ the natural homomorphism. Then $\varphi^e\colon\ \mathfrak{g}^e \to (\mathfrak{g}/\mathfrak{h})^e$ is an epimorphism and its kernel is the ideal L generated by the image of the composed map* (4).

PROOF. The fact that φ^e is an epimorphism is obvious. Clearly the image of if is in the kernel of φ^e. Thus φ^e induces a homomorphism $\bar{\varphi}\colon\ \mathfrak{g}^e/L \to (\mathfrak{g}/\mathfrak{h})^e$. We choose a function $u\colon\ \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ (not a homomorphism) which followed by φ is the identity. It is easily seen that the composite map

$$\mathfrak{g}/\mathfrak{h} \xrightarrow{u} \mathfrak{g} \xrightarrow{i} \mathfrak{g}^e \longrightarrow \mathfrak{g}^e/L$$

is independent of the choice of u and is a K-homomorphism. There results a K-algebra homomorphism $T(\mathfrak{g}/\mathfrak{h}) \to \mathfrak{g}^e/L$ which maps $U(\mathfrak{g}/\mathfrak{h})$ into zero. We thus obtain a map $\psi\colon\ (\mathfrak{g}/\mathfrak{h})^e \to \mathfrak{g}^e/L$ for which both compositions $\bar{\varphi}\psi$ and $\psi\bar{\varphi}$ are identity maps. Thus $\bar{\varphi}$ is an isomorphism.

As in the case of groups we have an antipodism

$$\omega\colon\ \mathfrak{g}^e \approx (\mathfrak{g}^e)^*$$

defined by the map $x_1 \otimes \cdots \otimes x_p \to (-1)^p x_p^* \otimes \cdots \otimes x_1^*$ of $T(\mathfrak{g})$ into $T(\mathfrak{g})^*$. As in the case of groups this allows us to convert a right $\mathfrak{g}$-module A into a left one, by setting

$$xa = -ax.$$

2. HOMOLOGY AND COHOMOLOGY OF LIE ALGEBRAS

For each Lie algebra $\mathfrak{g}$ over K, the (associative) K-algebra $\mathfrak{g}^e$ is a supplemented K-algebra, and therefore, following x,1, we have homology and cohomology groups of $\mathfrak{g}^e$. We shall write

$$H_n(\mathfrak{g},A) = \operatorname{Tor}_n^{\mathfrak{g}^e}(A,K), \qquad H^n(\mathfrak{g},C) = \operatorname{Ext}_{\mathfrak{g}^e}^n(K,C)$$

for any right $\mathfrak{g}$-module A and any left $\mathfrak{g}$-module C. Thus the homology and cohomology groups of $\mathfrak{g}$ are defined as those of the supplemented algebra $\mathfrak{g}^e$.

If $f\colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, we have the induced homomorphism $f^e\colon \mathfrak{g}^e \to \mathfrak{h}^e$ which in turn induces homomorphisms

$$F^f\colon H_n(\mathfrak{g},A) \to H_n(\mathfrak{h},A), \qquad A_{\mathfrak{h}},$$

$$F_f\colon H^n(\mathfrak{h},C) \to H^n(\mathfrak{g},C), \qquad {}_{\mathfrak{h}}C.$$

The homology group $H_0(\mathfrak{g},A)$ is the K-module $A \otimes_{\mathfrak{g}^e} K \approx A/AI$ where $I = I(\mathfrak{g})$ is the augmentation ideal in $\mathfrak{g}^e$. Clearly $AI = A\mathfrak{g}$ and therefore

(1) $$H_0(\mathfrak{g},A) = A/A\mathfrak{g}.$$

This K-module will also be denoted by $A_{\mathfrak{g}}$.

The cohomology group $H^0(\mathfrak{g},C)$ is the group $\operatorname{Hom}_{\mathfrak{g}^e}(K,C)$ which may be identified with the K-module of all *invariant* elements of C, i.e. all elements c such that $xc = 0$ for any $x \in \mathfrak{g}$. Denoting this module by $C^{\mathfrak{g}}$, we have

(1a) $$H^0(\mathfrak{g},C) = C^{\mathfrak{g}}.$$

The group $H^1(\mathfrak{g},C)$ has been described in x,1 as the group of all crossed homomorphisms $f\colon \mathfrak{g}^e \to C$ modulo the subgroup of principal crossed homomorphisms. Composing f with the map $i\colon \mathfrak{g} \to \mathfrak{g}^e$ we obtain a K-homomorphism $g\colon \mathfrak{g} \to C$ such that

$$x(gy) - y(gx) = g([x,y]) \qquad x,y \in \mathfrak{g}$$

which we call a *crossed homomorphism* of $\mathfrak{g}$ into C. Clearly the crossed homomorphisms of $\mathfrak{g}$ and those of $\mathfrak{g}^e$ are in a 1–1-correspondence given by the relation $g = fi$. The principal crossed homomorphisms $\mathfrak{g} \to A$

are those of the form $gx = xc$ for some fixed $c \in C$. We thus obtain again that $H^1(\mathfrak{g},C)$ may be identified with the group of crossed homomorphisms $\mathfrak{g} \to C$ reduced modulo principal homomorphisms.

If A has trivial $\mathfrak{g}$-operators (i.e. $xa = 0$ for all $a \in A, x \in \mathfrak{g}$), then we find

$$H_0(\mathfrak{g},A) = A = H^0(\mathfrak{g},A) \tag{2}$$

$$H^1(\mathfrak{g},A) = \operatorname{Hom}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}],A) \tag{3}$$

where $[\mathfrak{g},\mathfrak{g}]$ is the image of $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ under the map $x \otimes y \to [x,y]$.

We shall also interpret the group $H_1(\mathfrak{g},A)$ for A with trivial $\mathfrak{g}$-operators. We know from x,1,(4) that $H_1(\mathfrak{g},A) \approx A \otimes_K I/I^2$ where $I = I(\mathfrak{g})$ is the augmentation ideal. Since i maps $\mathfrak{g}$ into I and $[\mathfrak{g},\mathfrak{g}]$ into I^2 it defines a map $\varphi\colon \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to I/I^2$. On the other hand the map $T(\mathfrak{g}) \to \mathfrak{g}$ which is the identity on $T_1(\mathfrak{g}) = \mathfrak{g}$ and is zero on $T_n(\mathfrak{g})$ for $n \neq 1$, maps $U(\mathfrak{g})$ into $[\mathfrak{g},\mathfrak{g}]$ thus defining a map $I \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Since this map is zero on I^2 we obtain a map $\psi\colon I/I^2 \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Both compositions $\varphi\psi$ and $\psi\varphi$ are identities and we obtain an isomorphism

$$I/I^2 \approx \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]. \tag{4}$$

We thus have

$$H_1(\mathfrak{g},A) \approx A \otimes_K \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \tag{5}$$

if $\mathfrak{g}$ operates trivially on A.

3. THE POINCARÉ-WITT THEOREM

Throughout this section it will be assumed that the Lie algebra $\mathfrak{g}$ over K is K-free. A fixed K-base $\{x_\alpha\}$ will be chosen and it will be assumed that this K-base (or rather the set of indices) has been simply ordered.

We shall use the following notation: y_α will stand for the image of x_α under the map $i\colon \mathfrak{g} \to \mathfrak{g}^e$; if I is a finite sequence of indices $\alpha_1, \dots, \alpha_p$ we shall write $y_I = y_{\alpha_1} \cdots y_{\alpha_p}$; we say that I is *increasing* if $\alpha_1 \leqq \cdots \leqq \alpha_p$; we define $y_I = 1$ if I is empty, and we regard the empty set as increasing; the integer p will be called the *length* of I.

THEOREM 3.1. *The elements y_I, corresponding to finite increasing sequences I, form a K-base of the enveloping algebra $\mathfrak{g}^e$.*

COROLLARY 3.2. *$\mathfrak{g}^e$ is K-free.*

Since by 3.1, the elements y_α are linearly independent in $\mathfrak{g}^e$ we obtain

COROLLARY 3.3. *The map $i\colon \mathfrak{g} \to \mathfrak{g}^e$ is a K-monomorphism.*

PROOF of 3.1. We first show that the elements y_I corresponding to finite increasing sequences generate $\mathfrak{g}^e$. We denote by $F_p(\mathfrak{g}^e)$ the image of the submodule $\sum_{i \leq p} T_i(\mathfrak{g})$ of $T(\mathfrak{g})$ under the natural mapping $T(\mathfrak{g}) \to \mathfrak{g}^e$. It suffices to show that the elements y_I corresponding to increasing sequences I of length $\leqq p$ generate $F_p(\mathfrak{g}^e)$. Clearly the elements y_I corresponding to all sequences I of length $\leqq p$ generate $F_p(\mathfrak{g}^e)$. The conclusion thus follows by recursion from the following lemma (in which the fact that $\mathfrak{g}$ is K-free is not needed):

LEMMA 3.4. *For each sequence $a_1, \dots, a_p \in \mathfrak{g}$ and each permutation π of $(1, \dots, p)$ we have*

$$i(a_1) \cdots i(a_p) - i(a_{\pi(1)}) \cdots i(a_{\pi(p)}) \in F_{p-1}(\mathfrak{g}^e).$$

As usual $i\colon \mathfrak{g} \to \mathfrak{g}^e$ is the natural map. It clearly suffices to consider the case when π interchanges two consecutive indices $j, j+1$. In this case the conclusion is evident from the relation

$$i(a_j)i(a_{j+1}) - i(a_{j+1})i(a_j) = i([a_j,a_{j+1}]).$$

We now come to the more difficult part of the proof which consists in showing that the elements y_I of 3.1 are K-linearly independent. We shall denote by P the polynomial algebra $K[z_\alpha]$ on letters $\{z_\alpha\}$ in a 1–1-correspondence with the base $\{x_\alpha\}$. For each finite sequence I of indices $\alpha_1, \dots, \alpha_p$ we shall denote by z_I the element $z_{\alpha_1} \cdots z_{\alpha_p}$ of P.

LEMMA 3.5. *There exists a left representation of $\mathfrak{g}$ in P such that*

$$x_\alpha z_I = z_\alpha z_I \tag{1}$$

whenever $\alpha \leqq I$ (i.e. whenever $\alpha \leqq \beta$ for all $\beta \in I$).

Postponing the proof of the lemma, we can complete the proof of the theorem. The representation of $\mathfrak{g}$ in P induces a left $\mathfrak{g}^e$-module structure on P. If I is an increasing sequence of indices of length n it follows from (1) by recursion on n that $y_I \,.\, 1 = z_I$. Since the elements z_I are K-linearly independent in P, the same follows for the elements y_I of $\mathfrak{g}^e$.

PROOF of 3.5. In the graded algebra P, we denote as usual by P_p the K-module of homogeneous polynomials of degree p and set $Q_p = \sum_{i \leqq p} P_i$. Lemma 3.5 is an immediate consequence of the following inductive proposition:

(A_p). *For each integer p there is a unique homomorphism*

$$f\colon \mathfrak{g} \otimes Q_p \to P$$

such that

$$f(x_\alpha \otimes z_I) = z_\alpha z_I \qquad \alpha \leqq I,\ z_I \in Q_p \tag{1'}$$

$$f(x_\alpha \otimes z_I) \in Q_{q+1} \qquad z_I \in Q_q,\ q < p \tag{2}$$

$$f(x_\alpha \otimes f(x_\beta \otimes z_J)) = f(x_\beta \otimes f(x_\alpha \otimes z_J)) + f([x_\alpha,x_\beta] \otimes z_J),\ z_J \in Q_{p-1} \tag{3}$$

$$f(x_\alpha \otimes z_I) - z_\alpha z_I \in Q_q \qquad z_I \in Q_q,\ q \leqq p. \tag{4}$$

It is immediate that (2) is a consequence of (4); however we wrote (2) out explicitly in order to make it clear that the terms in (3) are well defined.

For $p = 0$, the definition $f(x_\alpha \otimes 1) = z_\alpha$ is forced by (1′) and trivially satisfies also (2)–(4).

Assume now that (A_{p-1}) is established for some $p > 0$. We shall show that the map f satisfying (A_{p-1}) admits a unique extension (also denoted by f) satisfying (A_p). We must define $f(x_\alpha \otimes z_I)$ for I of length p. If $\alpha \leqq I$, the definition is forced by (1′). If $\alpha \leqq I$ is false then I may be uniquely written as $I = (\beta, J)$ where $\alpha > \beta \leqq J$. Then $z_I = z_\beta z_J = f(x_\beta \otimes z_J)$ so that the left side of (3) is $f(x_\alpha \otimes z_I)$. In order to be able to use (3) as a definition we must verify that the right hand side of (3) is already defined. To this end we use (4) to write

$$f(x_\alpha \otimes z_J) = z_\alpha z_J + w, \qquad w \in Q_{p-1}.$$

Then the right hand side of (3) becomes

$$z_\beta z_\alpha z_J + f(x_\beta \otimes w) + f([x_\alpha, x_\beta] \otimes z_J).$$

This defines f in all cases, and (1′), (2) and (4) are clearly satisfied. As for (3) we only know that it holds if $\alpha > \beta \leqq J$. Because of the anti-symmetry of $[x_\alpha, x_\beta]$ it follows that (3) also holds if $\beta > \alpha \leqq J$. Since (3) trivially holds if $\alpha = \beta$, it follows that (3) is verified if either $\alpha \leqq J$ or $\beta \leqq J$. We shall show that this together with (1′) and (4) and together with the inductive assumption (A_{p-1}) implies (3) in all cases.

Indeed suppose that neither $\alpha \leqq J$ nor $\beta \leqq J$. Then J has positive length and $J = (\gamma, L)$ where $\gamma \leqq L$, $\gamma < \alpha$, $\gamma < \beta$. Using the abridged notation $f(x_\alpha \otimes z_I) = x_\alpha z_I$ we then have by the inductive assumption

$$\begin{aligned} x_\beta(z_J) = x_\beta(x_\gamma z_L) &= x_\gamma(x_\beta z_L) + [x_\beta, x_\gamma] z_L \\ &= x_\gamma(z_\beta z_L) + x_\gamma w + [x_\beta, x_\gamma] z_L \end{aligned}$$

where $w = x_\beta z_L - z_\beta z_L \in Q_{p-2}$. Applying x_α to both sides we have

$$x_\alpha(x_\beta z_J) = x_\alpha(x_\gamma(z_\beta z_L)) + x_\alpha(x_\gamma w) + x_\alpha([x_\beta, x_\gamma] z_L).$$

Since $\gamma \leqq (\beta, L)$, (3) may be applied to the term $x_\alpha(x_\gamma(z_\beta z_L))$; to the remaining two terms on the right we may apply (3) by the inductive assumption. Upon computation we obtain

$$(5)\quad \begin{aligned} x_\alpha(x_\beta z_J) = x_\gamma(x_\alpha(x_\beta z_L)) + [x_\alpha, x_\gamma](x_\beta z_L) + [x_\beta, x_\gamma](x_\alpha z_L) \\ + [x_\alpha, [x_\beta, x_\gamma]] z_L. \end{aligned}$$

Our assumptions on α and β were symmetric, so that (5) holds with α and β interchanged. Subtracting from (5) this yields

$$(6)\quad \begin{aligned} x_\alpha(x_\beta z_J) - x_\beta(x_\alpha z_J) = x_\gamma\{x_\alpha(x_\beta z_L) - x_\beta(x_\alpha z_L)\} + [x_\alpha, [x_\beta, x_\gamma]] z_L \\ - [x_\beta, [x_\alpha, x_\gamma]] z_L. \end{aligned}$$

Applying (3) we have

$$\begin{aligned} x\{x_{\gamma\alpha}(x_\beta z_L) - x_\beta(x_\alpha z_L)\} &= x_\gamma([x_\alpha,x_\beta]z_L) \\ &= [x_\alpha,x_\beta](x_\gamma z_L) + [x_\gamma,[x_\alpha,x_\beta]]z_L \\ &= [x_\alpha,x_\beta]z_J + [x_\gamma,[x_\alpha,x_\beta]]z_L. \end{aligned}$$

Substituting this in (6), we find that the three terms involving double brackets cancel by virtue of Jacobi's identity, and the final result is

$$x_\alpha(x_\beta z_J) - x_\beta(x_\alpha z_J) = [x_\alpha,x_\beta]z_J$$

as desired.

Theorem 3.1 was first proved by Poincaré (*Cambridge Philosophical Transactions* 18 (1899), 220–225, § III); a complete proof, based on the same principles, was given later by E. Witt (*Journ. für r.u.a. Math.* (*Crelle*) 177 (1937), 152–166; Hilfsatz, p. 153). The proof given here is modeled after Iwasawa.

4. SUBALGEBRAS AND IDEALS

If $\mathfrak{h}$ is a Lie subalgebra of a Lie algebra $\mathfrak{g}$ over K, then the inclusion map $\mathfrak{h} \to \mathfrak{g}$ induces a K-algebra homomorphism

$$(1) \qquad \varphi\colon \mathfrak{h}^e \to \mathfrak{g}^e$$

so that $\mathfrak{g}^e$ may be regarded either as a left or as a right $\mathfrak{h}^e$-module.

PROPOSITION 4.1. *If the K-modules $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are both K-free, then φ is a monomorphism and $\mathfrak{g}^e$ regarded as a left or right $\mathfrak{h}^e$-module is $\mathfrak{h}^e$-free.*

PROOF. In the exact sequence $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ of K-modules, the modules $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are K-free. Therefore the sequence splits and $\mathfrak{g}$ also is K-free. Furthermore, we can find a K-base of $\mathfrak{g}$ composed of two disjoint sets $\{x_\alpha\}_{\alpha \in A}$, $\{y_\beta\}_{\beta \in B}$ such that $\{x_\alpha\}$ is a K-base for $\mathfrak{h}$. We simply order the union $A \cup B$ of the disjoint sets A and B so that each $\alpha \in A$ precedes each $\beta \in B$. If we identify each element of $\mathfrak{g}$ with its image in $\mathfrak{g}^e$ under the monomorphism $i\colon\ \mathfrak{g} \to \mathfrak{g}^e$, then it follows from 3.1 that the elements of the form

$$x_{\alpha_1} \cdots x_{\alpha_p} y_{\beta_1} \cdots y_{\beta_q} \qquad \alpha_1 \leqq \cdots \leqq \alpha_p \in A, \qquad \beta_1 \leqq \cdots \leqq \beta_q \in B$$

of $\mathfrak{g}^e$ form a K-base for $\mathfrak{g}^e$, while the elements $x_{\alpha_1} \cdots x_{\alpha_p}$ form a K-base for $\mathfrak{h}^e$. This implies that (1) is a monomorphism and that the elements $y_{\beta_1} \cdots y_{\beta_q}$ form a left $\mathfrak{h}^e$-base for $\mathfrak{g}^e$. The proof that these elements also form a right $\mathfrak{h}^e$-base is similar.

We may now apply X,7.2 and X,7.3. We obtain

PROPOSITION 4.2. *Under the hypotheses of* 4.1 *we have*

$$H_n(\mathfrak{h},A) \approx H_n(\mathfrak{g},A \otimes_{\mathfrak{h}^e} \mathfrak{g}^e) \tag{2}$$

$$H^n(\mathfrak{h},C) \approx H^n(\mathfrak{g}, \mathrm{Hom}_{\mathfrak{h}^e}(\mathfrak{g}^e,C)), \tag{2a}$$

for each right $\mathfrak{h}$*-module* A *and each left* $\mathfrak{h}$*-module* C.

PROPOSITION 4.3. *Under the hypotheses of* 4.1 *we have*

$$H_n(\mathfrak{h},A) \approx \mathrm{Tor}_n^{\mathfrak{g}^e}(A,\mathfrak{g}^e \otimes_{\mathfrak{h}^e} K), \tag{3}$$

$$H^n(\mathfrak{h},C) \approx \mathrm{Ext}_{\mathfrak{g}^e}^n(\mathfrak{g}^e \otimes_{\mathfrak{h}^e} K,C), \tag{3a}$$

for each right $\mathfrak{g}$*-module* A *and each left* $\mathfrak{g}$*-module* C.

The module $\mathfrak{g}^e \otimes_{\mathfrak{h}^e} K$ appearing in (3) and (3a) may also be written as $H_0(\mathfrak{h},\mathfrak{g}^e)$ which has been computed in § 2 to be $\mathfrak{g}^e/\mathfrak{g}^e\mathfrak{h}$. If $\mathfrak{h}$ is an *ideal* in $\mathfrak{g}$ then $\mathfrak{g}^e\mathfrak{h}$ coincides with the ideal L of 1.3. Thus if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ we have the isomorphism

$$\mathfrak{g}^e \otimes_{\mathfrak{h}^e} K \approx (\mathfrak{g}/\mathfrak{h})^e.$$

COROLLARY 4.4. *If* $\mathfrak{h}$ *is an ideal in* $\mathfrak{g}$ *and the hypotheses of* 4.1 *are satisfied then*

$$H_n(\mathfrak{h},A) \approx \mathrm{Tor}_n^{\mathfrak{g}^e}(A,(\mathfrak{g}/\mathfrak{h})^e), \tag{4}$$

$$H^n(\mathfrak{h},C) \approx \mathrm{Ext}_{\mathfrak{g}^e}^n((\mathfrak{g}/\mathfrak{h})^e,C), \tag{4a}$$

for each right $\mathfrak{g}$*-module* A *and each left* $\mathfrak{g}$*-module* C. *These isomorphisms may be used to define a right* $\mathfrak{g}/\mathfrak{h}$*-module structure on* $H_n(\mathfrak{h},A)$ *and a left* $\mathfrak{g}/\mathfrak{h}$*-module structure on* $H^n(\mathfrak{h},C)$.

In XVI,6 we shall establish closer relations between the homology (and cohomology) groups of $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$.

5. THE DIAGONAL MAP AND ITS APPLICATIONS

For each Lie algebra $\mathfrak{g}$ over K, the diagonal map

$$D\colon\ \mathfrak{g}^e \to \mathfrak{g}^e \otimes \mathfrak{g}^e$$

is defined by the requirement

$$Dx = x \otimes 1 + 1 \otimes x, \qquad x \in \mathfrak{g}.$$

If we identify $\mathfrak{g}^e \otimes \mathfrak{g}^e$ with $(\mathfrak{g} + \mathfrak{g})^e$ as in 1.2, and consider the map $l\colon\ \mathfrak{g} \to \mathfrak{g} + \mathfrak{g}$ given by $lx = (x,x) = (x,0) + (0,x)$, then $D = l^e$. This diagonal map D is compatible with the augmentation (in the sense explained in XI,8) and is commutative and associative (in the sense defined in XI,4).

The diagonal map D may be combined with the antipodism $\omega\colon \mathfrak{g}^e \approx \mathfrak{g}^{e*}$ defined in § 1, to obtain a map

$$E\colon\ \mathfrak{g}^e \to \mathfrak{g}^e \otimes \mathfrak{g}^{e*} = (\mathfrak{g}^e)^e$$

as the composition

$$\mathfrak{g}^e \xrightarrow{D} \mathfrak{g}^e \otimes \mathfrak{g}^e \xrightarrow{\mathfrak{g}^e \otimes \omega} \mathfrak{g}^e \otimes \mathfrak{g}^{e*}.$$

It follows that the map E satisfies

$$Ex = x \otimes 1 - 1 \otimes x^*, \qquad x \in \mathfrak{g},$$

and that this condition determines E uniquely.

We first verify that the map E satisfies condition $(E.1)$ of x,6. To this end we denote by I and J the kernels of the respective augmentation maps

$$\varepsilon\colon\ \mathfrak{g}^e \to K, \qquad \rho\colon\ \mathfrak{g}^e \otimes \mathfrak{g}^{e*} \to \mathfrak{g}^e.$$

By IX,3.1, J is the left ideal generated by the elements $u \otimes 1 - 1 \otimes u^*$ for $u \in \mathfrak{g}^e$. In view of the relation

$$(uv) \otimes 1-1 \otimes (uv)^* = (u \otimes 1)(v \otimes 1-1 \otimes v^*)+(1 \otimes v^*)(u \otimes 1-1 \otimes u^*)$$

valid for $u,v \in \mathfrak{g}^e$, we find that J is the left ideal in $\mathfrak{g}^e \otimes \mathfrak{g}^{e*}$ generated by the elements

$$x \otimes 1 - 1 \otimes x^* = Ex \qquad x \in \mathfrak{g}.$$

Since the elements $x \in \mathfrak{g}$ generate the ideal I of $\mathfrak{g}^e$ it follows that J is the left ideal generated by the image of EI in $\mathfrak{g}^e \otimes \mathfrak{g}^{e*}$. This is precisely condition $(E.1)$ of x,6.

We now introduce the assumption that the Lie algebra $\mathfrak{g}$ is K-free. Then, by 3.2, $\mathfrak{g}^e$ also is K-free. Consequently, the diagonal map D may be used to define $\cup$- and $\cap$-products as in XI,7. Further we find that the conditions (i)–(vi) of XI,8 are satisfied by the maps D and ω. Consequently the considerations of XI,8 and XI,9 (reduction theorems) are applicable to the homology and cohomology groups of a K-free Lie algebra $\mathfrak{g}$.

Next (still under the assumption that $\mathfrak{g}$ is K-free) we shall show that condition $(E.2)$ of x,6 is satisfied, i.e. that $\mathfrak{g}^e \otimes \mathfrak{g}^{e*}$ regarded as a right $\mathfrak{g}^e$-module by means of the map E is $\mathfrak{g}^e$-projective. Since the map $\mathfrak{g}^e \otimes \omega$ is an isomorphism, it clearly suffices to show that $\mathfrak{g}^e \otimes \mathfrak{g}^e$ regarded as a right $\mathfrak{g}^e$-module using the map D, is $\mathfrak{g}^e$-free. To this end we identify $\mathfrak{g}^e \otimes \mathfrak{g}^e$ with $(\mathfrak{g} + \mathfrak{g})^e$ and notice that $D = l^e$, where $l\colon\ \mathfrak{g} \to \mathfrak{g} + \mathfrak{g}$ is the map of Lie algebras given by $lx = (x,x)$. Since l is a monomorphism and since Coker l is a K-module isomorphic with $\mathfrak{g}$ which is K-free, it follows from 4.1 that $(\mathfrak{g} + \mathfrak{g})^e$ is $\mathfrak{g}^e$-free.

Now that condition (E.1) and (E.2) of the "inverse process" have been verified, we may apply x,6.1. We obtain

THEOREM 5.1. *Let* $\mathfrak{g}$ *be a Lie algebra over K which is K-free, and let A be a two-sided* $\mathfrak{g}^e$*-module. Let* A_E *be the right* $\mathfrak{g}$*-module obtained from A by setting*

$$(a,x) \to ax - xa \qquad a \in A, x \in \mathfrak{g}$$

and let ${}_EA$ *be the left* $\mathfrak{g}$*-module obtained by*

$$(x,a) \to xa - ax \qquad a \in A, x \in \mathfrak{g}.$$

We then have isomorphisms

$$F^E\colon\ H_n(\mathfrak{g}^e,A) \approx H_n(\mathfrak{g},A_E)$$

$$F_E\colon\ H^n(\mathfrak{g},{}_EA) \approx H^n(\mathfrak{g}^e,A).$$

Furthermore if $\Lambda = \mathfrak{g}^e$ *and if X is a* Λ*-projective resolution of K (as a left* Λ*-module) then* $\Lambda^e \otimes_\Lambda X$ *is a* Λ^e*-projective resolution of* Λ *as a left* Λ^e*-module.*

In particular, let $\mathfrak{g}$ be the abelian Lie algebra with the letters $x_1, \ldots, x_n$ as a K-base. Then $\mathfrak{g}^e = K[x_1, \ldots, x_n] = \Lambda$, and we know from VIII,4 that $\Lambda \otimes E(x_1, \ldots, x_n)$ with a suitable differentiation operator is a $\mathfrak{g}^e$-projective resolution of K. It follows that $\Lambda^e \otimes E(x_1, \ldots, x_n)$ with a suitable differentiation operator is a Λ^e-projective resolution of Λ.

An application of x,6.2 gives

THEOREM 5.2. *If* $\mathfrak{g}$ *is a Lie algebra over K which is K-free then*

$$\dim \mathfrak{g}^e = \dim_{\mathfrak{g}^e} K.$$

If further the commutative ring K is semi-simple, then

$$\dim \mathfrak{g}^e = \text{gl.dim}\ \mathfrak{g}^e.$$

In view of the antipodism ω, there is no need to distinguish between $\text{l.dim}_{\mathfrak{g}^e} K$ and $\text{r.dim}_{\mathfrak{g}^e} K$ and between $\text{l.gl.dim}\ \mathfrak{g}^e$ and $\text{r.gl.dim}\ \mathfrak{g}^e$.

6. A RELATION IN THE STANDARD COMPLEX

For the purpose of the next section we shall establish here a relation valid in the normalized standard complex $N(\Lambda)$ of an arbitrary (associative) K-algebra Λ.

The notation $[x_1, \ldots, x_n]$ in the complex $N(\Lambda)$ introduced in IX,6 will be replaced here by $\{x_1, \ldots, x_n\}$ in order to avoid confusion with the brackets in the Lie algebras.

For each $y \in \Lambda$ we consider the Λ^e-endomorphisms $\sigma(y)$ and $\vartheta(y)$ of $N(\Lambda)$ defined by

$$\sigma(y)\{x_1, \ldots, x_n\} = \sum_{0 \leqq i \leqq n} (-1)^i \{x_1, \ldots, x_i, y, x_{i+1}, \ldots, x_n\} \tag{1}$$

$$\begin{aligned} \vartheta(y)\{x_1, \ldots, x_n\} = y\{x_1, \ldots, x_n\} - \{x_1, \ldots, x_n\}y \\ - \sum_{1 \leqq i \leqq n} \{x_1, \ldots, x_{i-1}, [y,x_i], x_{i+1}, \ldots, x_n\} \end{aligned} \tag{2}$$

where $[y,x] = yx - xy$.

PROPOSITION 6.1. *For each $y \in \Lambda$ we have the identity*

$$d\sigma(y) + \sigma(y)d - \vartheta(y) = 0, \tag{3}$$

where d is the differentiation operator of $N(\Lambda)$.

PROOF. Let $A(y)$ denote the left hand side of (3). We must show that for all $n \geqq 0$,

$$A(y)\{x_1, \ldots, x_n\} = 0. \tag{4}$$

This is immediate if $n = 0$. We now assume, by induction, that (4) holds for $n - 1$ ($n > 0$). In the complex $N(\Lambda)$ we have the contracting homotopy s defined in IX,6 and satisfying the identity

$$ds + sd = \text{identity}$$

when applied to any element of degree > 0. Thus for $n > 0$, relation (4) is equivalent to the pair of relations

$$sA(y)\{x_1, \ldots, x_n\} = 0, \tag{5}$$

$$sdA(y)\{x_1, \ldots, x_n\} = 0. \tag{6}$$

We recall that in the normalized complex we have $s(y\{x_1, \ldots, x_n\}y') = \{y, x_1, \ldots, x_n\}y'$ and that the right hand side is zero if $y = 1$. This rule easily implies

$$sd\sigma(y)\{x_1, \ldots, x_n\} = s(y\{x_1, \ldots, x_n\} - x_1\sigma(y)\{x_2, \ldots, x_n\})$$

$$s\sigma(y)d\{x_1, \ldots, x_n\} = s(x_1\sigma(y)\{x_2, \ldots, x_n\})$$

$$-s\vartheta(y)\{x_1, \ldots, x_n\} = -s(y\{x_1, \ldots, x_n\}).$$

Adding these relations yields (5).

To prove (6) we first compute the element

$$z = dA(y)\{x_1, \ldots, x_n\} = d\sigma(y)d\{x_1, \ldots, x_n\} - d\vartheta(y)\{x_1, \ldots, x_n\}.$$

An application of the inductive assumption yields

$$z = \vartheta(y)d\{x_1, \ldots, x_n\} - d\vartheta(y)\{x_1, \ldots, x_n\}.$$

We must show that $z \equiv 0$ mod the kernel of s. Calculating modulo this kernel we find that $d\vartheta(y)\{x_1, \ldots, x_n\}$ gives

$$yd\{x_1, \ldots, x_n\} - x_1\{x_2, \ldots, x_n\}y - [y,x_1]\{x_2, \ldots, x_n\} - \sum_{2 \leqq i \leqq n} x_1\{x_2, \ldots, x_{i-1},[y,x_i],x_{i+1}, \ldots, x_n\}$$

while $\vartheta(y)d\{x_1, \ldots, x_n\}$ gives

$$x_1y\{x_2, \ldots, x_n\} + y(d\{x_1, \ldots, x_n\} - x_1\{x_2, \ldots, x_n\})$$
$$-x_1\{x_2, \ldots, x_n\}y - \sum_{2 \leqq i \leqq n} x_1\{x_2, \ldots, x_{i-1},[y,x_i],x_{i+1}, \ldots, x_n\}.$$

The two results coincide and this concludes the proof.

Suppose now that Λ is a supplemented K-algebra with augmentation $\varepsilon: \Lambda \to K$. In the normalized standard complex $N(\Lambda,\varepsilon) = N(\Lambda) \otimes_\Lambda K$ we have endomorphisms induced by $\sigma(y)$ and $\vartheta(y)$. These will still be denoted by $\sigma(y)$ and $\vartheta(y)$. These operators are left Λ-endomorphisms of $N(\Lambda,\varepsilon)$ and we still have the relation (3). The explicit definition of $\sigma(y)$ is still given by formula (1), while the definition of $\vartheta(y)$ gets replaced by

$$(2') \qquad \vartheta(y)\{x_1, \ldots, x_n\} = y\{x_1, \ldots, x_n\} - \{x_1, \ldots, x_n\}(\varepsilon y) - \sum_{1 \leqq i \leqq n} \{x_1, \ldots, x_{i-1},[y,x_i],x_{i+1}, \ldots, x_n\}.$$

7. THE COMPLEX $V(\mathfrak{g})$

Throughout this section $\mathfrak{g}$ will denote a Lie algebra over K which is K-free.

We denote by $E(\mathfrak{g})$ the exterior algebra of the K-module $\mathfrak{g}$. The tensor product (over K)

$$V(\mathfrak{g}) = \mathfrak{g}^e \otimes E(\mathfrak{g})$$

is a left $\mathfrak{g}^e$-module and is $\mathfrak{g}^e$-free since $E(\mathfrak{g})$ is K-free. Using the grading of $E(\mathfrak{g})$ we define a grading in $V(\mathfrak{g})$ as

$$V_n(\mathfrak{g}) = \mathfrak{g}^e \otimes E_n(\mathfrak{g}).$$

Since $E_0(\mathfrak{g}) = K$ it follows that $V_0(\mathfrak{g}) = \mathfrak{g}^e$, and the augmentation $\varepsilon: \mathfrak{g}^e \to K$ defines an augmentation $\varepsilon: V(\mathfrak{g}) \to K$ which is zero on $V_n(\mathfrak{g})$, $n > 0$.

For $u \in \mathfrak{g}^e$, $x_1, \ldots, x_n \in \mathfrak{g}$, the element $u \otimes (x_1 \cdots x_n) \in \mathfrak{g}^e \otimes E(\mathfrak{g}) = V(\mathfrak{g})$ will be written as $u\langle x_1, \ldots, x_n\rangle$. If $u = 1$ we shall simply write $\langle x_1, \ldots, x_n\rangle$. Consequently the symbol $\langle\ \rangle$ will denote the element $1 \otimes 1$ of $\mathfrak{g}^e \otimes E(\mathfrak{g})$.

We now consider the normalized standard complex $N(\mathfrak{g}^e,\varepsilon)$ of the supplemented algebra $\mathfrak{g}^e$. Let

$$f\colon\ V(\mathfrak{g}) \to N(\mathfrak{g}^e,\varepsilon)$$

be the $\mathfrak{g}^e$-homomorphism defined by the requirement

$$f\langle x_1,\dots,x_n\rangle = \sum_{\pi}(-1)^{\tau(\pi)}\{x_{\pi(1)},\dots,x_{\pi(n)}\},$$

where the summation extends over all permutations π of $(1,\dots,n)$ and $\tau(\pi)$ is the signature of π. To verify that f is well defined we only need to observe that $f\langle x_1,\dots,x_n\rangle = 0$ if $x_i = x_j$ for some $0 \leqq i < j \leqq n$. In particular, the definition yields $f\langle\ \rangle = \{\ \}$.

If we choose a simply ordered K-base for $\mathfrak{g}$, we obtain in the usual fashion a K-base for $E(\mathfrak{g})$ which in turn induces a $\mathfrak{g}^e$-base for $V(\mathfrak{g})$. It follows then by inspection that f maps this $\mathfrak{g}^e$-base of $V(\mathfrak{g})$ into elements of $N(\mathfrak{g}^e,\varepsilon)$ which are $\mathfrak{g}^e$-linearly independent. Consequently f is a monomorphism. In the sequel we shall identify $V(\mathfrak{g})$ with a $\mathfrak{g}^e$-submodule of $N(\mathfrak{g}^e,\varepsilon)$ and regard f as an inclusion map.

THEOREM 7.1. *The submodule $V(\mathfrak{g})$ of $N(\mathfrak{g}^e,\varepsilon)$ is a subcomplex. The differentiation in $V(\mathfrak{g})$ is given by the formula*

$$d\langle x_1,\dots,x_n\rangle = \sum_{1\leqq i\leqq n}(-1)^{i+1}x_i\langle x_1,\dots,\hat{x}_i,\dots,x_n\rangle + \sum_{1\leqq i<j\leqq n}(-1)^{i+j}\langle [x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_n\rangle. \tag{1}$$

With the augmentation $\varepsilon\colon\ V(\mathfrak{g}) \to K$, the complex $V(\mathfrak{g})$ is a $\mathfrak{g}^e$-free resolution of K as a left $\mathfrak{g}^e$-module.

PROOF. Once formula (1) is proved, it will follow that $V(\mathfrak{g})$ is a subcomplex of $N(\mathfrak{g}^e,\varepsilon)$. For $n = 0$ formula (1) needs $d\langle\ \rangle = 0$ which is obviously correct. We now proceed by induction and assume that (1) holds for n.

We shall use the endomorphisms $\sigma(x)$ and $\vartheta(x)$ of the complex $N(\mathfrak{g}^e,\varepsilon)$ as defined by formulas (1) and (2′) of § 6. For $y,x_1,\dots,x_n \in \mathfrak{g}$, we obtain

$$\sigma(y)\langle x_1,\dots,x_n\rangle = \langle y,x_1,\dots,x_n\rangle \tag{2}$$

$$\vartheta(y)\langle x_1,\dots,x_n\rangle = y\langle x_1,\dots,x_n\rangle - \sum_{1\leqq i\leqq n}\langle x_1,\dots,[y,x_i],\dots,x_n\rangle. \tag{3}$$

The formula $d\sigma(y) + \sigma(y)d = \vartheta(y)$ (established in 6.1) together with (2) yields

$$d\langle y,x_1,\dots,x_n\rangle = d\sigma(y)\langle x_1,\dots,x_n\rangle = \vartheta(y)\langle x_1,\dots,x_n\rangle - \sigma(y)d\langle x_1,\dots,x_n\rangle.$$

Using (2) and (3) and the inductive assumption, this implies

$$\begin{aligned} d\langle y,x_1,\dots,x_n\rangle = y\langle x_1,\dots,x_n\rangle &+ \sum_{1\leq i\leq n}(-1)^i\langle [y,x_i],x_1,\dots,\hat{x}_i,\dots,x_n\rangle \\ &+ \sum_{1\leq i\leq n}(-1)^i x_i\langle y,x_1,\dots,\hat{x}_i,\dots,x_n\rangle \\ &- \sum_{1\leq i<j\leq n}(-1)^{i+j}\langle y,[x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_n\rangle. \end{aligned}$$

This is precisely the desired formula for $d\langle y,x_1,\dots,x_n\rangle$. Thus (1) is proved.

We have already exhibited a $\mathfrak{g}^e$-base for $V(\mathfrak{g})$, which is thus $\mathfrak{g}^e$-free.

The kernel of the augmentation $V_0(\mathfrak{g})\to K$ is the K-module generated by the elements of the form $x_1\cdots x_p\langle\ \rangle$, with $x_i\in\mathfrak{g}$, $p>0$. Since $x_1\cdots x_p\langle\ \rangle = d(x_1\cdots x_{p-1}\langle x_p\rangle)$, it follows that $V_1(\mathfrak{g})\to V_0(\mathfrak{g})\to K\to 0$ is exact. Thus to conclude the proof of the theorem it suffices to show that $H_q(V(\mathfrak{g}))=0$ for $q>0$. The following proof is due to J. L. Koszul.

We choose a simply ordered K-base $\{x_\alpha\}$ for $\mathfrak{g}$. The elements $\langle x_{\alpha_1},\dots,x_{\alpha_n}\rangle$ with $\alpha_1<\cdots<\alpha_n$, $n\geqq 0$ form a K-base for $E(\mathfrak{g})$. The elements $x_{\beta_1}\cdots x_{\beta_m}$ with $\beta_1\leqq\cdots\leqq\beta_m$, $m\geqq 0$, form by 3.1, a K-base for $\mathfrak{g}^e$. Consequently we obtain a K-base of $V(\mathfrak{g})=\mathfrak{g}^e\otimes E(\mathfrak{g})$:

$$(4)\quad x_{\beta_1}\cdots x_{\beta_m}\langle x_{\alpha_1},\dots,x_{\alpha_n}\rangle,\qquad \begin{array}{ll}\alpha_1<\cdots<\alpha_n, & n\geqq 0\\ \beta_1\leqq\cdots\leqq\beta_m, & m\geqq 0.\end{array}$$

We introduce the submodule $F_pV(\mathfrak{g})$ generated by the elements (4) with $m+n\leqq p$. In the quotient module $W_p=F_pV(\mathfrak{g})/F_{p-1}V(\mathfrak{g})$ we then have the K-base represented by the elements (4) with $m+n=p$. Furthermore, it follows from 3.4 that the class represented in W_p by an element (4) is independent of the order in which the elements $x_{\beta_1},\dots,x_{\beta_m}$ are written. The formula (1) for the differentiation d in $V(\mathfrak{g})$ implies

$$(5)\quad \begin{aligned} &d(x_{\beta_1}\cdots x_{\beta_m}\langle x_{\alpha_1},\dots,x_{\alpha_n}\rangle) \\ &\quad = \sum_{1\leq i\leq n}(-1)^{i+1}x_{\beta_1}\cdots x_{\beta_m}x_{\alpha_i}\langle x_{\alpha_1},\dots,\hat{x}_{\alpha_i},\dots,x_{\alpha_m}\rangle \end{aligned}$$

modulo $F_{m+n-1}V(\mathfrak{g})$. This implies that the modules $F_pV(\mathfrak{g})$ are subcomplexes and that the differentiation induced in W_p is given by the formula (5).

It is now clear that the complex $W=\sum_p W_p$ is the complex

$$K[x_\alpha]\otimes E(x_\alpha)$$

with the differentiation given by (5). This complex is isomorphic to the projective resolution of K as a left $K[x_\alpha]$-module constructed in VIII,4. It follows that $H_q(W)=0$ for $q>0$, and therefore that $H_q(W_p)=0$ for $q>0$.

Now consider the exact sequence

$$H_q(F_{p-1}V(\mathfrak{g})) \to H_q(F_pV(\mathfrak{g})) \to H_q(W_p), \qquad q > 0.$$

This implies that $H_q(F_{p-1}V(\mathfrak{g})) \to H_q(F_pV(\mathfrak{g}))$ is an epimorphism. Since $F_{-1}V(\mathfrak{g}) = 0$ we obtain $H_q(F_pV(\mathfrak{g})) = 0$ for $q > 0$ and all p. Since $V(\mathfrak{g}) = \cup_p F_pV(\mathfrak{g})$ it follows that $H_q(V(\mathfrak{g})) = 0$ for $q > 0$. This concludes the proof of the theorem.

8. APPLICATIONS OF THE COMPLEX $V(\mathfrak{g})$

We first show how the homology and cohomology groups of $\mathfrak{g}$ may be computed using the complex $V(\mathfrak{g})$.

If A is a right $\mathfrak{g}$-module, then the homology groups $H_q(\mathfrak{g},A)$ are the homology groups of the complex

$$A \otimes_{\mathfrak{g}^e} V(\mathfrak{g}) = A \otimes_{\mathfrak{g}^e} \mathfrak{g}^e \otimes E(\mathfrak{g}) = A \otimes E(\mathfrak{g}).$$

The differentiation operator in this complex is

$$d(a \otimes \langle x_1, \dots, x_n\rangle) = \sum_{1\leq i\leq n} (-1)^{i+1}(ax_i) \otimes \langle x_1, \dots, \hat{x}_i, \dots, x_n\rangle$$
$$+ \sum_{1\leq i<j\leq n} (-1)^{i+j}a \otimes \langle [x_i,x_j],x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n\rangle.$$

If C is a left $\mathfrak{g}$-module, the cohomology groups $H^q(\mathfrak{g},C)$ are the homology groups of the complex

$$\operatorname{Hom}_{\mathfrak{g}^e}(V(\mathfrak{g}),C) = \operatorname{Hom}_{\mathfrak{g}^e}(\mathfrak{g}^e \otimes E(\mathfrak{g}),C) = \operatorname{Hom}(E(\mathfrak{g}),C).$$

In this last complex, a q-cochain f: $E_q(\mathfrak{g}) \to C$ is simply a K-linear alternating function $f(x_1, \dots, x_q)$ of q variables in $\mathfrak{g}$, with values in C. The coboundary δf of such a cochain is the $q + 1$-cochain given by the formula

$$(\delta f)(x_1, \dots, x_{q+1}) = \sum_{1\leq i\leq q+1} (-1)^{i+1}x_if(x_1, \dots, \hat{x}_i, \dots, x_{q+1})$$
$$+ \sum_{1\leq i<j\leq q+1} (-1)^{i+j}f([x_i,x_j],x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{q+1}).$$

This description of the cohomology groups $H^q(\mathfrak{g},C)$ shows directly that these coincide with the cohomology groups of $\mathfrak{g}$ considered hitherto (C. Chevalley and S. Eilenberg, *Trans. Am. Math. Soc.* 63 (1948), 85–124).

We recall that the complex $V(\mathfrak{g})$ is a subcomplex of the normalized standard complex $N(\mathfrak{g}^e,\varepsilon)$. In this connection the following proposition will be useful.

PROPOSITION 8.1. *Every cochain $f \in \operatorname{Hom}_{\mathfrak{g}^e}(V(\mathfrak{g}),C)$ admits an extension $f' \in \operatorname{Hom}_{\mathfrak{g}^e}(N(\mathfrak{g}^e,\varepsilon),C)$. If f is a cocycle then f' may be chosen to be a cocycle.*

PROOF. The first fact follows from the observation that $V(\mathfrak{g})$ as a $\mathfrak{g}^e$-module is a direct summand of $N(\mathfrak{g}^e,\varepsilon)$. This is clear from the bases exhibited in § 7. Now assume that $\delta f = 0$. Since the cohomology groups obtained using $V(\mathfrak{g})$ and $N(\mathfrak{g}^e,\varepsilon)$ are isomorphic under the inclusion map, there exists a cocycle $g' \in \mathrm{Hom}_{\mathfrak{g}^e}(N(\mathfrak{g}^e,\varepsilon),C)$ whose restriction g to $V(\mathfrak{g})$ is cohomologous to f; then $f - g = \delta h$. Let h' be an extension of the cochain h. It follows that $f' = g' + \delta h'$ is an extension of f and $\delta f' = 0$ as desired.

The next application of the complex $V(\mathfrak{g})$ has to do with dimension.

THEOREM 8.2. *If* $\mathfrak{g}$ *has a K-base composed of n elements, then*

$$\dim \mathfrak{g}^e = \dim_{\mathfrak{g}^e} K = n.$$

If further the commutative ring K is semi-simple then

$$\mathrm{gl.dim}\, \mathfrak{g}^e = n.$$

PROOF. In view of 5.2, we only need to prove $\dim_{\mathfrak{g}^e} K = n$. Since $E_q(\mathfrak{g}) = 0$ for $q > n$, it follows that the complex $V(\mathfrak{g})$ is n-dimensional and thus $\dim_{\mathfrak{g}^e} K \leqq n$. Now consider the group $E_n(\mathfrak{g})$, with $\mathfrak{g}$ operating on the left by

$$y.\langle x_1, \ldots, x_n\rangle = \sum_{1 \leqq i \leqq n} \langle x_1, \ldots, [y,x_i], \ldots, x_n\rangle.$$

Let f be a $(n - 1)$-cochain with values in $E_n(\mathfrak{g})$; an easy computation (cf. Exer. 12) shows that $\delta f = 0$; thus $H^n(\mathfrak{g},E_n(\mathfrak{g}))$ is isomorphic to the K-module of n-cochains $E_n(\mathfrak{g}) \to E_n(\mathfrak{g})$, which is obviously isomorphic to K. Hence $\dim_{\mathfrak{g}^e} K \geqq n$.

Next we pass to the question of computing the products using the complexes $V(\mathfrak{g})$. We begin with the external products for two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ over K, both of which are K-free. As agreed upon in § 1, we shall systematically identify $(\mathfrak{g} + \mathfrak{h})^e$ with $\mathfrak{g}^e \otimes \mathfrak{h}^e$. As we have seen in XI,5, to compute the products $\perp$ and $\top$ we need a map

$$f\colon\ V(\mathfrak{g}) \otimes V(\mathfrak{h}) \to V(\mathfrak{g} + \mathfrak{h})$$

while for the products $\vee$ and $\wedge$ we need a map

$$g\colon\ V(\mathfrak{g} + \mathfrak{h}) \to V(\mathfrak{g}) \otimes V(\mathfrak{h}).$$

The answer to both of these problems is quite trivial here since the identification $(\mathfrak{g} + \mathfrak{h})^e = \mathfrak{g}^e \otimes \mathfrak{h}^e$ and the natural isomorphism $E(\mathfrak{g} + \mathfrak{h}) \approx E(\mathfrak{g}) \otimes E(\mathfrak{h})$ imply a natural isomorphism

$$V(\mathfrak{g} + \mathfrak{h}) \approx V(\mathfrak{g}) \otimes V(\mathfrak{h}) \tag{1}$$

compatible with the $(\mathfrak{g} + \mathfrak{h})^e$-operators and the differentiations.

For the internal products $\smile$ and $\frown$ we assume that $\mathfrak{g}$ is an abelian Lie algebra. Then $\mathfrak{g}^e$ is a commutative algebra. As we have seen in XI,5, to compute the products $\smile$ and $\frown$ we need a map

$$V(\mathfrak{g}) \otimes_{\mathfrak{g}^e} V(\mathfrak{g}) \rightarrow V(\mathfrak{g}).$$

To obtain such a map it suffices to regard $V(\mathfrak{g}) = \mathfrak{g}^e \otimes E(\mathfrak{g})$ as a $\mathfrak{g}^e$-algebra, and verify that this map is compatible with the differentiation (cf. Exer. 15).

We finally consider the products $\cup$ and $\cap$ defined using the diagonal map D: $\mathfrak{g}^e \rightarrow \mathfrak{g}^e \otimes \mathfrak{g}^e = (\mathfrak{g} + \mathfrak{g})^e$. According to XI,5, we need a map

$$V(\mathfrak{g}) \rightarrow V(\mathfrak{g}) \otimes V(\mathfrak{g}).$$

This is given by the maps $\mathfrak{g}^e \rightarrow \mathfrak{g}^e \otimes \mathfrak{g}^e$ and $E(\mathfrak{g}) \rightarrow E(\mathfrak{g}) \otimes E(\mathfrak{g})$ both defined by $x \rightarrow x \otimes 1 + 1 \otimes x$, $x \in \mathfrak{g}$. If we carry out the explicit computation and apply this map to find the cup product of cochains we obtain the classical formula for the multiplication of alternating multilinear forms. Explicitly, consider cochains $f \in \operatorname{Hom}(E_p(\mathfrak{g}),C)$, $f' \in \operatorname{Hom}(E_q(\mathfrak{g}),C')$, where C and C' are left $\mathfrak{g}$-modules. If $C \otimes C'$ is regarded as a left $\mathfrak{g}$-module by means of the map D, we find that the cochain $f \cup f' \in \operatorname{Hom}(E_{p+q}(\mathfrak{g}),C \otimes C')$ is given by

$$(f \cup f')(x_1, \ldots, x_{p+q}) = \Sigma \pm f(x_{i_1}, \ldots, x_{i_p}) \otimes f'(x_{j_1}, \ldots, x_{j_q}),$$

the sum being extended over all partitions of the sequence $(1, \ldots, p+q)$ into two increasing sequences $(i_1, \ldots, i_p)$ and $(j_1, \ldots, j_q)$. The sign is the signature of the permutation $(i_1, \ldots, i_p, j_1, \ldots, j_q)$.

EXERCISES

1. Given an associative K-algebra Λ, define

$$[x,y] = xy - yx \qquad\qquad x,y \in \Lambda$$

and prove that this assigns to Λ the structure of a Lie algebra, denoted by $\mathfrak{l}(\Lambda)$. Show that for any Lie algebra $\mathfrak{g}$ (over K), the map i is a Lie algebra homomorphism

$$i\colon\ \mathfrak{g} \rightarrow \mathfrak{l}(\mathfrak{g}^e).$$

2. Given a Lie algebra $\mathfrak{g}$ and an associative algebra Λ (both over the same ring K), show that any Lie algebra homomorphism f: $\mathfrak{g} \rightarrow \mathfrak{l}(\Lambda)$ admits a unique factorization

$$\mathfrak{g} \xrightarrow{i} \mathfrak{g}^e \xrightarrow{h} \Lambda$$

where h is a K-algebra homomorphism. Show further that this property of the pair $(\mathfrak{g}^e,i)$ characterizes this pair uniquely up to an isomorphism.

Show that in order that there exists an associative K-algebra Λ and a Lie algebra monomorphism $f\colon \mathfrak{g} \to \mathfrak{l}(\Lambda)$, it is necessary and sufficient that $i\colon \mathfrak{g} \to \mathfrak{g}^e$ be a monomorphism.

3. For a given Lie algebra $\mathfrak{g}$, let $\bar{\mathfrak{g}}$ denote the image of the homomorphism $i\colon \mathfrak{g} \to \mathfrak{g}^e$; we regard $\bar{\mathfrak{g}}$ as a Lie algebra. Show that the inclusion map $\bar{\mathfrak{g}} \to \mathfrak{g}^e$ satisfies the criterion of Exer. 2 and thus we may identify $\mathfrak{g}^e$ with $(\bar{\mathfrak{g}})^e$.

4. Given a Lie algebra $\mathfrak{g}$ over K, consider the associative K-algebra $\Lambda = \operatorname{Hom}_K(\mathfrak{g},\mathfrak{g})$ and the map

$$\rho\colon \mathfrak{g} \to \mathfrak{l}(\Lambda)$$

given by

$$(\rho x)y = [x,y].$$

Show that ρ is a homomorphism of Lie algebras and that $\rho = 0$ if and only if $\mathfrak{g}$ is a commutative (i.e. $[\mathfrak{g},\mathfrak{g}] = 0$). As an application show that if $\mathfrak{g} \neq 0$ then the natural map $i\colon \mathfrak{g} \to \mathfrak{g}^e$ is not zero.

5. Let M be a K-module. Consider the graded K-module $A(M) = \sum_{k\geq 1} A^k(M)$, where

$$A^1(M) = M, \qquad A^k(M) = \sum_{0<i<k} A^i(M) \otimes_K A^{k-i}(M) \qquad \text{for } k > 1.$$

Define the mapping $A(M) \otimes_K A(M) \to A(M)$ by the inclusion maps $A^k(M) \otimes_K A^h(M) \to A^{k+h}(M)$. We call $A(M)$ the *free non-associative K-algebra* (without unit element) *over M*. In $A(M)$ consider the two-sided ideal $J(M)$ generated by the elements

$$xx \text{ and } x(yz) + y(zx) + z(xy), \qquad x,y,z \in A(M).$$

Show that the quotient $L(M) = A(M)/J(M)$ is a (graded) Lie algebra; we call $L(M)$ the *free Lie algebra over M*. Show that the map $j\colon M \to L(M)$, defined by composition $M = A^1(M) \to A(M) \to L(M)$, is a monomorphism. Show that every K-homomorphism $f\colon M \to \mathfrak{g}$ into a Lie algebra $\mathfrak{g}$ admits a unique factorization $M \xrightarrow{j} L(M) \longrightarrow \mathfrak{g}$, where φ is a homomorphism of Lie algebras over K.

6. Let M be a K-module, and k a K-homomorphism of M into a Lie K-algebra $\mathfrak{l}$. Suppose that each K-homomorphism $f\colon M \to \mathfrak{g}$ into a Lie K-algebra $\mathfrak{g}$ admits a unique factorization

$$M \xrightarrow{k} \mathfrak{l} \xrightarrow{\psi} \mathfrak{g},$$

where ψ is a homomorphism of Lie K-algebras. Prove that there exists a unique isomorphism $\alpha\colon \mathfrak{l} \approx L(M)$ such that $\alpha k = j$. This gives an axiomatic description of the pair $(L(M),j)$.

7. Consider the tensor algebra $T(M)$ of the K-module M. Show that the natural injection $M \to T(M)$ admits a unique factorization $M \xrightarrow{j} L(M) \xrightarrow{i} T(M)$, where i is a homomorphism of the Lie algebra $L(M)$ into the Lie algebra $\mathfrak{l}(T(M))$. This mapping i is compatible with with the gradings in $L(M)$ and $T(M)$. Show that $T(M)$ may be identified with the enveloping algebra $L(M)^e$ of $L(M)$. If $\bar{L}(M)$ denotes the image of i, show that $\bar{L}(M)$ is the Lie subalgebra of $\mathfrak{l}(T(M))$ generated by the elements of degree 1 in $T(M)$, i.e. by M.

8. Prove the following theorem: if M is a K-free module, then $L(M)$ is K-free and $i: L(M) \to T(M)$ is a monomorphism; thus the Lie subalgebra $\bar{L}(M)$ of $\mathfrak{l}(T(M))$, generated by M, is K-free and isomorphic to $L(M)$.

[Hint: if $L(M)$ is K-free, then, by 3.3 and Exer. 7, i is a monomorphism. Hence the theorem is proved when K is a field. For any commutative ring K, and any K-free module M, there exists a free abelian group A such that $M = A \otimes K$; show that $L(M) = L(A) \otimes K$. This reduces the proof to showing that $L(A)$ is Z-free when A is Z-free; it will be sufficient to prove that $i: L(A) \to T(A)$ is a monomorphism. Let A_I be the subgroup of A generated by any finite subset I of the base of A; then $T(A_I) \to T(A)$ is a monomorphism, which reduces the proof to the case of a finitely generated free abelian group. Let now A be an abelian group with a finite base; for proving that $i: L(A) \to T(A)$ is a monomorphism, observe that, for each prime p, $L(A) \otimes Z_p \to T(A) \otimes Z_p$ is a monomorphism of degree zero, since the theorem is proved for a field; then apply VII, Exer. 12 to each graded component $L_k(A) \otimes Z_p \to T_k(A) \otimes Z_p$.]

9. Show that any representation satisfying 3.5 automatically satisfies condition (4) and therefore is unique.

10. Show that if $\mathfrak{g}$ is K-free and $\mathfrak{g}^e$ is commutative then $\mathfrak{g}$ is an abelian Lie algebra.

11. Given a map $K \to L$ (of commutative rings) examine the effects of this change of ground ring upon the homology and cohomology groups of a Lie algebra.

12. Let $\mathfrak{g}$ be a Lie algebra with a K-base $x_1, \dots, x_n$. Define the *constants of structure* c_{ijk} by the relations

$$[x_i, x_j] = \sum_k c_{ijk} x_k.$$

Express the axioms of the Lie algebra in terms of c_{ijk}. Prove that in the complex $K \otimes_{\mathfrak{g}^e} V(\mathfrak{g})$ we have

$$d\langle x_1, \dots, x_n\rangle = \sum_{\substack{1 \leqq i \leqq n \\ 1 \leqq j \leqq n}} (-1)^i c_{ijj} \langle x_1, \dots, \hat{x}_i, \dots, x_n\rangle.$$

13. Under the conditions of Exer. 12, $\mathfrak{g}$ is said to be *unimodular* if for any $y \in \mathfrak{g}$, the relation

$$\sum_{1 \leq j \leq n} x_1 \cdots x_{j-1}[y,x_j]x_{j+1} \cdots x_n = 0$$

holds in $E(\mathfrak{g})$. Show that this is equivalent with

$$d\langle x_1, \ldots, x_n\rangle = 0$$

in the complex $K \otimes_{\mathfrak{g}^e} V(\mathfrak{g})$.

14. (Alternative description of the complex $V(\mathfrak{g})$). Let $\Lambda = (K,d)$ be the ring of dual numbers over K and consider the K-module $\Lambda \otimes_K \mathfrak{g}$ with endomorphism d. Let $T(\Lambda \otimes \mathfrak{g})$ be the tensor algebra over K of the K-module $\Lambda \otimes \mathfrak{g}$. The map i: $x \to 1 \otimes x$ will be used to identify $\mathfrak{g}$ with a submodule of $\Lambda \otimes \mathfrak{g}$ and thus also of $T(\Lambda \otimes \mathfrak{g})$. In $T(\Lambda \otimes \mathfrak{g})$ introduce a grading written with lower indices in which the elements $x \in \mathfrak{g}$ have degree 1 and the elements dx ($x \in \mathfrak{g}$) have degree 0. The endomorphism d of $\Lambda \otimes \mathfrak{g}$ may now be extended uniquely to an antiderivation d of $T(\Lambda \otimes \mathfrak{g})$, i.e. a K-endomorphism satisfying

$$d(uv) = (du)v + (-1)^p u(dv)$$

for u of degree p in $T(\Lambda \otimes \mathfrak{g})$. This operator d satisfies $dd = 0$ and is of degree -1 (with respect to the lower indices).

Let L be the two-sided ideal in $T(\Lambda \otimes \mathfrak{g})$ generated by the elements

(1) $$xx$$

(2) $$(dx)y - y(dx) - [x,y]$$

(3) $$(dx)(dy) - (dy)(dx) - d[x,y]$$

for $x,y \in \mathfrak{g}$.

Prove that L is a homogeneous ideal and is stable under d. Consider the K-algebra

$$W(\mathfrak{g}) = T(\Lambda \otimes \mathfrak{g})/L$$

which is a left $\mathfrak{g}^e$-complex over K.

Use the maps

$$i\colon\ \mathfrak{g} \to \Lambda \otimes \mathfrak{g}, \qquad j = di\colon\ \mathfrak{g} \to \Lambda \otimes \mathfrak{g}$$

to obtain maps

$$i'\colon\ T(\mathfrak{g}) \to T(\Lambda \otimes \mathfrak{g}), \qquad j'\colon\ T(\mathfrak{g}) \to T(\Lambda \otimes \mathfrak{g})$$

$$i^*\colon\ E(\mathfrak{g}) \to W(\mathfrak{g}), \qquad j^*\colon\ \mathfrak{g}^e \to W(\mathfrak{g})$$

$$\varphi = j^* \otimes i^*\colon\ \mathfrak{g}^e \otimes E(\mathfrak{g}) \to W(\mathfrak{g}).$$

Prove that φ is an isomorphism of graded K-modules and is an isomorphism of the complexes $V(\mathfrak{g})$ and $W(\mathfrak{g})$.

[Hint to the last part: denote by M the ideal of $T(\Lambda \otimes \mathfrak{g})$ generated by the elements (2). Prove that $j' \otimes i'\colon\ T(\mathfrak{g}) \otimes T(\mathfrak{g}) \to T(\Lambda \otimes \mathfrak{g})$ induces an isomorphism $T(\mathfrak{g}) \otimes T(\mathfrak{g}) \approx T(\Lambda \otimes \mathfrak{g})/M$.]

15. Let $\mathfrak{g}$ be a Lie algebra with a K-base; then $W(\mathfrak{g})$ (Exer. 14) is a graded differential algebra and $\mathfrak{g}^e$ a subalgebra of degree 0, thus $W(\mathfrak{g})$ is a two-sided $\mathfrak{g}^e$-module. The multiplication of $W(\mathfrak{g})$ defines a map

$$W(\mathfrak{g}) \otimes_{\mathfrak{g}^e} W(\mathfrak{g}) \to W(\mathfrak{g}) \tag{1}$$

which is compatible with the structures of two-sided $\mathfrak{g}^e$-modules. Let A be a left $\mathfrak{g}^e$-module; (1) defines

$$W(\mathfrak{g}) \otimes_{\mathfrak{g}^e} A \to \operatorname{Hom}_{\mathfrak{g}^e}(W(\mathfrak{g}), W(\mathfrak{g}) \otimes_{\mathfrak{g}^e} A), \tag{2}$$

where $\operatorname{Hom}_{\mathfrak{g}^e}$ is related to the left $\mathfrak{g}^e$-module structures. Let n be the number of the elements of the K-base of $\mathfrak{g}$; (2) induces

$$W_{n-k}(\mathfrak{g}) \otimes_{\mathfrak{g}^e} A \to \operatorname{Hom}_{\mathfrak{g}^e}(W_k(\mathfrak{g}), W_n(\mathfrak{g}) \otimes_{\mathfrak{g}^e} A) \tag{3}$$

for any integer k; this is a map φ_k of the module of $(n - k)$-chains (with coefficients in A) into the module of k-cochains (with coefficients in $W_n(\mathfrak{g}) \otimes_{\mathfrak{g}^e} A \approx E_n(\mathfrak{g}) \otimes_K A$). Show that the collection of maps φ_k commute (up to the sign) with the boundary and coboundary operators, and that each φ_k is an isomorphism. Compute explicitly the left operations of $\mathfrak{g}$ on $E_n(\mathfrak{g}) \otimes_K A$, and establish the natural isomorphisms

$$H_{n-k}(\mathfrak{g},A) \approx H^k(\mathfrak{g},E_n(\mathfrak{g}) \otimes_K A).$$

For $k = n$ and $A = K$ (with trivial operators) we find again $H^n(\mathfrak{g},E_n(\mathfrak{g})) = K$ (cf. 8.2).