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Hopf Algebras with Divided Powers

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An algebra A, *over a field* K, with divided powers is a graded algebra

$$A = A_0 + A_1 + A_2 + \cdots$$

which is unitary, associative, commutative and has another structure specified by the functions γ^k described below. For each $k \geqslant 0$ and for each $n \geqslant 1$, there is given a set-theoretical map

$$\gamma^k : A_{2n} \to A_{2kn}$$

satisfying the following conditions

1) $\gamma^0(x) = 1, \qquad \gamma^1(x) = x,$
2) $\gamma^h(x) \cdot \gamma^k(x) = [(h + k)!/h!k!]\, \gamma^{h+k}(x),$
3) $\gamma^k(x + y) = \Sigma_{r+s=k}\, \gamma^r(x) \cdot \gamma^s(y),$
4) $\gamma^k(x \cdot y) = 0$ if x and y are homogeneous elements of odd degrees $(k \geqslant 2)$,
5) $\gamma^k(x \cdot y) = x^k \cdot y^k(y)$ if x and y are homogeneous elements of even degrees (positive for y).

Now let us consider two algebras with divided powers A' and A'', over the same field K. Then the tensor product $A = A' \otimes_K A''$ is itself a graded, unitary, associative, commutative algebra. It has divided powers in a canonical way: To give the definition, use the sum formula 3) and the product formulas 4) and 5), and notice the equality

$$x \otimes y = (x \otimes 1) \cdot (1 \otimes y).$$

In this way the algebra with divided powers $A' \otimes A''$ is well defined.

A *Hopf algebra with divided powers* A is an algebra with divided powers A plus two homomorphisms of algebras with divided powers

$$\Delta : A \to A \otimes A \qquad \epsilon : A \to K$$

such that Δ is coassociative and ϵ is a counit for Δ. A Hopf algebra with divided powers is a commutative Hopf algebra in the usual sense. A Hopf algebra A is *connected if* A_0 is isomorphic to K.

The aim of this paper is the proof of the following theorem, which will surprise nobody.

THEOREM. Let A be a connected Hopf algebra of finite type, with divided powers, over a field K. Then its dual Hopf algebra A^* is isomorphic to the universal enveloping algebra of a graded Lie algebra over the field K.

See Theorem 17 (p. 38). The property that the comultiplication preserves divided powers is essential in the proof.

Notice that the Lie algebra is *not restricted.* If we do not assume the existence of divided powers for the commutative Hopf algebra A, then, under a weaker condition, the dual Hopf algebra A^* is isomorphic to the universal enveloping algebra of a graded restricted Lie algebra (see Chap. 6 of Milnor–Moore [4]). Actually I prove a little more than the theorem above: As a Hopf algebra *with divided powers,* A is isomorphic to the "universal enveloping coalgebra" of a "graded Lie coalgebra". The condition of finite type is unnecessary.

I assume that the reader is more or less familiar with Milnor–Moore's paper [4].

Remark. In characteristic 2 we consider Hopf algebras having divided powers in any positive degree, *even and odd.* It is equivalent to consider Hopf algebras, with divided powers, having no homogeneous element of odd degree.

EXAMPLES. For a first example let us consider an Eilenberg–MacLane space $K(\pi, n)$ and a field K. Then the singular homology vector space

$$H(\pi, n, K) = \Sigma\, H_i(K(\pi, n), K)$$

has a natural structure as a Hopf algebra with divided powers over the field K, see [2]. For a second example let us consider a local ring R and its residue field K. Then the Tor vector space

$$B(R) = \Sigma\, \mathrm{Tor}_i^R(K, K)$$

has a natural structure as a Hopf algebra with divided powers over the field K, see [1]. In both examples, if the field K has characteristic 2, it is not true in full generality.

1. NOTATIONS

The ground field K is fixed. All vector spaces, algebras, coalgebras are *graded*

$$W = W_0 + W_1 + W_2 + \cdots.$$

The vector space W is *connected* if W_0 is isomorphic to K and it is *reduced* if W_0 is equal to 0. To a vector space W there corresponds a canonical reduced vector space $W^{\cdot}$ contained in W

$$W^{\cdot} = 0 + W_1 + W_2 + \cdots$$

and to a reduced vector space W there corresponds a canonical connected vector space ${}^{\cdot}W$ containing W

$${}^{\cdot}W = K + W_1 + W_2 + \cdots.$$

All homomorphisms have degree 0. Only tensor products over K are used

$$W \otimes W' = W \otimes_K W'.$$

Then τ denotes the *twisting homomorphism*

$$\tau : W \otimes W' \to W' \otimes W$$

mapping $x \otimes x'$ onto $(-1)^{pp'}x' \otimes x$ if x belongs to W_p and x' to $W'_{p'}$. It is useful to consider the *decomposition* $W = W_+ + W_-$ with

$$W_+ = W_0 + W_2 + W_4 + \cdots, \qquad W_- = W_1 + W_3 + W_5 + \cdots$$

if the characteristic is different from 2 and with

$$W_+ = W, \qquad W_- = 0$$

if the characteristic is equal to 2.

Let us consider a graded algebra A, called simply an *algebra*, with structural homomorphism

$$\Phi : A \otimes A \to A.$$

If this algebra is associative and has a unit $\eta : K \to A$, we shall consider the following homomorphisms

$$\Phi^k : A \otimes \cdots \otimes A \to A \qquad (k \text{ copies of } A)$$

with

$$\Phi^0 = \eta, \qquad \Phi^1 = Id, \qquad \Phi^{i+1} = \Phi^i \circ (\Phi \otimes Id).$$

Let us consider a graded coalgebra A, called simply a *coalgebra*, with structural homomorphism

$$\Delta : A \to A \otimes A.$$

If this coalgebra is coassociative and has a counit $\epsilon : A \to K$, we shall consider the following homomorphisms

$$\Delta^k : A \to A \otimes \cdots \otimes A \qquad (k \text{ copies of } A)$$

with

$$\Delta^0 = \epsilon, \qquad \Delta^1 = Id, \qquad \Delta^{i+1} = (\Delta \otimes Id) \circ \Delta^i.$$

A *Γ-algebra* is an algebra with divided powers as defined in the introduction. More precisely it is an algebra A with structural homomorphism Φ (the product $x \cdot y = \Phi(x \otimes y)$ is supposed to be unitary, associative, commutative) and with a set-theoretical map γ^k for each $k \geqslant 0$

$$\gamma^k : A_+^{\cdot} \to A_+^{\cdot}$$

fulfilling the following conditions

0) $\gamma^k(A_n) \subset A_{kn}$,
1) $\gamma^0(a) = 1, \qquad \gamma^1(a) = a,$
2) $\gamma^h(a) \cdot \gamma^k(a) = [(h+k)!/h!k!]\, \gamma^{h+k}(a),$
3) $\gamma^k(a' + a'') = \Sigma_{k'+k''=k}\, \gamma^{k'}(a') \cdot \gamma^{k''}(a'')),$
4) $\gamma^k(a' \cdot a'') = 0 \quad \text{if} \quad a' \in A_-, \quad a'' \in A_-, \quad k \geqslant 2,$
5) $\gamma^k(a' \cdot a'') = a'^k \cdot \gamma^k(a''^k) \quad \text{if} \quad a' \in A_+, \quad a'' \in A_+^{\cdot}.$

A homomorphism of Γ-algebras $f : A \to A'$ is a homomorphism of algebras compatible with the divided powers

$$f \circ \gamma^k(a) = \gamma^k \circ f(a) \qquad \text{for} \quad a \in A_+^{\cdot}.$$

The tensor product $A \otimes B$ of two Γ-algebras has a natural structure as a Γ-algebra. Then the two maps

$$\alpha : A \to A \otimes B, \qquad (\alpha(a) = a \otimes 1)$$
$$\beta : B \to A \otimes B, \qquad (\beta(b) = 1 \otimes b)$$

are homomorphisms of Γ-algebras. This property determines completely the Γ-algebra structure of the tensor product: for more details see [2, p. 7–04].

A *Hopf Γ-algebra* is a Hopf algebra with divided powers as defined in the introduction. More precisely it is a Γ-algebra A plus two homomorphisms of Γ-algebras

$$\Delta : A \to A \otimes A, \qquad \epsilon : A \to K,$$

such that Δ is coassociative and ϵ is a counit for Δ. A homomorphism of

Hopf Γ-algebras is a map which is both a homomorphism of Γ-algebras and a homomorphism of coalgebras. For a (connected) Hopf Γ-algebra A, the following reduced spaces are important. The reduced vector space $P = P(A)$ consists of the elements a of $\mathbf{A}^{\cdot}$ satisfying the following equality

$$\Delta(a) = a \otimes 1 + 1 \otimes a.$$

The reduced vector space $J = J(A)$ is equal to the following sum in $A^{\cdot}$

$$A^{\cdot} \cdot A^{\cdot} + \sum_{k \geqslant 2} K\gamma^k(A_+^{\cdot}).$$

The reduced vector space $Q = Q(A)$ is equal to the quotient $A^{\cdot}/J(A)$. The quotient $A/J(A)$ is denoted by $R = R(A)$.

For a vector space W, we can consider the vector space

$$T_n(W) = W \otimes \cdots \otimes W \qquad (n \text{ copies of } W).$$

If a homomorphism $\epsilon : W \to K$ is given, then a homomorphism

$$\epsilon_n{}^i : T_n(W) \to T_{n-1}(W), \qquad 1 \leqslant i \leqslant n,$$

is defined by the following equality

$$\epsilon_n{}^i(w_1 \otimes \cdots \otimes w_i \otimes \cdots \otimes w_n) = \epsilon(w_i)\, w_1 \otimes \cdots \otimes \hat{w}_i \otimes \cdots \otimes w_n .$$

For a vector space W, we can consider the vector space

$$T(W) = T_0(W) + T_1(W) + T_2(W) + \cdots$$

obtained by direct sum. Actually $T(W)$ has a double graduation. For

$$T_n(W)_k = \sum_{k_1 + \cdots + k_n = k} W_{k_1} \otimes \cdots \otimes W_{k_n}$$

the integer k is the *primary degree*, due to the graduation of W and the integer n is the *secondary degree.*

Let S_k denote the kth symmetric group. An element σ of S_k can be described as a reordering of an ordered set with k elements

$$\sigma(x_1, \ldots, x_k) = (x_{\sigma_1}, \ldots, x_{\sigma_k}).$$

Now let us consider the following set of integers

$$k_1 \geqslant 0, \ldots, k_m \geqslant 0, \qquad k_1 + \cdots + k_m = k.$$

We denote by X_i the following subset of the ordered set $(x_1, ..., x_k)$, with k_i elements,

$$(x_{k_1+\cdots+k_{i-1}+1}, ..., x_{k_1+\cdots+k_{i-1}+k_i}).$$

We define

$$S(k_1, ..., k_m) \subset S_k$$

as being the set of reorderings preserving the relative order of the elements of X_i for any i: for more details about these shuffles see [3].

The special case $k_1 = \cdots = k_m = i$ is specially interesting. We use the following notation

$$S(i \mid m) = S(i, ..., i) \subset S_{im}.$$

Let us consider the following equality with $k = im$

$$(x_1, ..., x_k) = (X_1, ..., X_m).$$

The elements of S_{im} permute the different x_i's and the elements of S_m permute the different X_j's. Thus we get a natural imbedding of S_m into S_{im}: write simply $S_m \subset S_{im}$. The subset $S(i \mid m)$ of the group S_{im} is invariant under the right action of the subgroup S_m. We consider the quotient set

$$S[i \mid m] = S(i \mid m)/S_m.$$

In a non-unique way we have a set-theoretical isomorphism

$$S(i \mid m) \cong S[i \mid m] \times S_m.$$

The surjective map

$$s(i \mid m) : S(i \mid m) \to S[i \mid m]$$

has much to do with divided powers, as we shall see later.

Notice that a product is denoted sometimes by a letter (Φ, $\bar{\Phi}$,...), sometimes by a point.

2. Tensor Hopf Γ-Algebras

Let us consider a vector space W and its associated vector space $T(W)$ in its primary graduation. We shall see that $T(W)$ has a natural structure as a Hopf Γ-algebra. For the definition of the structural homomorphisms we shall use the secondary graduation and let the group S_k act on the vector space $T_k(W)$ as follows. For the situation

$$\sigma \in S_k\,, \qquad w_i \in W_{n_i}\,, \qquad (i = 1,\dots, k)$$

we define

$$\sigma(w_1 \otimes \cdots \otimes w_k) = (-1)^{\Sigma n_i n_j}(w_{\sigma_1} \otimes \cdots \otimes w_{\sigma_k})$$

where the summation is over all pairs (i, j) with

$$1 \leqslant i < j \leqslant k, \qquad \sigma_i > \sigma_j\,.$$

Tensor products will appear in two different ways: in the definition of the vector space $T(W)$ and in the tensor product $T(W) \otimes T(W)$. We shall introduce brackets for the first type of tensor products and we shall use the notation $1 = (\ )$ in $K = T_0(W)$.

The structural homomorphism $\bar{\Delta} : T(W) \to T(W) \otimes T(W)$ is defined by the following equality

$$\bar{\Delta}(w_1 \otimes \cdots \otimes w_k) = \sum_{0 \leqslant i \leqslant k} (w_1 \otimes \cdots \otimes w_i) \otimes (w_{i+1} \otimes \cdots \otimes w_k).$$

The structural homomorphism $\bar{\Phi} : T(W) \otimes T(W) \to T(W)$ (shuffle product) is defined by the following equality

$$\bar{\Phi}[(w_1 \otimes \cdots \otimes w_i) \otimes (w_{i+1} \otimes \cdots \otimes w_{i+j})] = \sum \sigma(w_1 \otimes \cdots \otimes w_{i+j})$$

with summation over all elements σ of $S(i, j)$. The structural homomorphisms $\bar{\eta} : K \to T(W)$ and $\bar{\epsilon} : T(W) \to K$ are the canonical injection and projection due to the equality $K = T_0(W)$.

Now we have to define the map $\bar{\gamma}^k : T(W)_+^{\cdot} \to T(W)_+^{\cdot}$ for each $k \geqslant 0$. At first let us consider the following equality for $w_1 \otimes \cdots \otimes w_i$ in $T(W)_+^{\cdot}$

$$\begin{aligned}\bar{\Phi}^k[(w_1 \otimes \cdots \otimes w_i) \otimes \cdots \otimes (w_1 \otimes \cdots \otimes w_i)]\\ = \sum \sigma(w_1 \otimes \cdots \otimes w_i \otimes \cdots \otimes w_1 \otimes \cdots \otimes w_i)\end{aligned}$$

with summation over all elements σ of $S(i \mid k)$. The element

$$\sigma(w_1 \otimes \cdots \otimes w_i \otimes \cdots \otimes w_1 \otimes \cdots \otimes w_i)$$

depends only on the image of σ in $S[i \mid k]$. We get the following equality

$$\begin{aligned}\bar{\Phi}^k[(w_1 \otimes \cdots \otimes w_i) \otimes \cdots \otimes (w_1 \otimes \cdots \otimes w_i)]\\ = k! \sum \sigma(w_1 \otimes \cdots \otimes w_i \otimes \cdots \otimes w_1 \otimes \cdots \otimes w_i)\end{aligned}$$

with summation over all elements σ of $S[i \mid k]$.

Lemma 1. *For a vector space W, there is one and only one set of maps $\bar{\gamma}^k : T(W)_+^{\cdot} \to T(W)_+^{\cdot}$ satisfying the two following conditions*

1) $\bar{\gamma}^k(x + y) = \sum_{i+j=k} \bar{\Phi}[\bar{\gamma}^i(x) \otimes \bar{\gamma}^j(y)]$;

2) $\bar{\gamma}^k(w_1 \otimes \cdots \otimes w_i) = \sum_{\sigma \in S[i|k]} \sigma(w_1 \otimes \cdots \otimes w_i \otimes \cdots \otimes w_1 \otimes \cdots \otimes w_i)$.

Proof. Lemma 1 and Proposition 2 are proved together.

Proposition 2. *Let W be a vector space. Then $T(W)$ with its primary graduation and with the structural homomorphisms $\bar{\Phi}$, $\bar{\Delta}$, $\bar{\eta}$, $\bar{\epsilon}$ and maps $\bar{\gamma}^k$ described above is a Hopf Γ-algebra.*

Proof. Since the results of Lemma 1 and Proposition 2 are well known, the proofs are sketched only, for further details see [5, p. 101]. First, direct computations prove that $T(W)$ with its primary graduation and with the structural homomorphisms $\bar{\Phi}$, $\bar{\Delta}$, $\bar{\eta}$, $\bar{\epsilon}$ is a commutative Hopf algebra. Then it remains to prove Lemma 1, to prove that $T(W)$ with $\bar{\Phi}$ and $\bar{\gamma}^k$ is a Γ-algebra, not only an algebra, and to prove that $\bar{\Delta}$ is a homomorphism of Γ-algebras, not only a homomorphism of algebras.

Let us begin with the case of characteristic 0. It is clear that the second equality of Lemma 1 can be written

$$\bar{\gamma}^k(w_1 \otimes \cdots \otimes w_i) = (w_i \otimes \cdots \otimes w_i)^k/k!$$

Consequently, Lemma 1 is proved by setting $\bar{\gamma}^k(x) = x^k/k!$ for any element x of $T(W)_+^{\cdot}$. Then it is clear that $T(W)$ is a Γ-algebra and that $\bar{\Delta}$ is a homomorphism of Γ-algebras. Thus Lemma 1 and Proposition 2 are proved in the case of characteristic 0.

The proof in the case of positive characteristic can be deduced from the proof in the case of characteristic 0. All the definitions $T(W)$, $\bar{\Phi}$, $\bar{\Delta}$, $\bar{\eta}$, $\bar{\epsilon}$, $\bar{\gamma}^k$ are quite natural and are valid not only for a vector space W over a field K, but also for a graded module W over any commutative ring R.

Let us denote by Ω_i ($i = 1, 2, 3, 4$) Lemma 1 and Proposition 2 in the following cases:

1) The ring R is the field of all rational numbers and the module W is any vector space over this field.

2) The ring R is the ring of all rational integers and the module W is any abelian group which is free in all degrees.

3) The ring R is the ring of all rational integers and the module W is any graded abelian group.

4) The ring R is any field K and the module W is any vector space over this field.

We know that Ω_1 holds. We prove Ω_4 by showing that Ω_i implies Ω_{i+1}. Each time we use an auxiliary module $\tilde{W}$. For the first step, we consider an abelian group W which is free in all degrees, we choose a vector space $\tilde{W}$ over the rationals containing the abelian group W, we use the injection $T(W) \to T(\tilde{W})$. For the second step, we consider a graded abelian group W, we choose an abelian group $\tilde{W}$ which is free in all degrees and which has a quotient equal to W, we use the surjection $T(\tilde{W}) \to T(W)$. For the third step, we consider a vector space W over K, we choose a graded abelian group $\tilde{W}$ equal to W, we use the surjection $T(\tilde{W}) \to T(W)$. Now Lemma 1 and Proposition 2 are proved.

We shall use later the following result.

Proposition. 3. Let W be a reduced vector space. Let H be a subspace of $T(W)$, for both graduations, having the following properties

1) for any $n \geqslant 0$, the homogeneous elements of H of secondary degree equal to n are S_n-invariant;

2) the vector space H is a subalgebra of $T(W)$;

3) for any x of W_-, the element x of $T_1(W)$ belongs to H and for any x of W_+ and any $k \geqslant 0$, the element $x \otimes \cdots \otimes x$ of $T_k(W)$ belongs to H.

Then the vector space H is unique. For any $n \geqslant 0$, the homogeneous elements of H of secondary degree equal to n are exactly the S_n-invariant elements of $T_n(W)$.

Proof. Let us choose a well ordered basis $(w_i, i \in I)$ of the vector space W. Then for any $n \geqslant 0$, the vector space of the S_n-invariant elements of $T_n(W)$ has a basis consisting of the following elements

$$(w_{i_1} \otimes \cdots \otimes w_{i_1}) \cdot (w_{i_2} \otimes \cdots \otimes w_{i_2}) \cdots (w_{i_m} \otimes \cdots \otimes w_{i_m})$$

k_1 times w_{i_1}, k_2 times w_{i_2}, ..., k_m times w_{i_m} with

$$m \geqslant 0, \quad k_1 > 0, k_2 > 0, \ldots, k_m > 0, \quad i_1 < i_2 < \cdots < i_m,$$
$$n = k_1 + k_2 + \cdots + k_m \quad \text{and} \quad k_j = 1 \quad \text{if} \quad w_{i_j} \in W_-.$$

Then the rest of the proof is obvious.

3. Lie Coalgebras

A *Lie coalgebra* is a coalgebra L with structural homomorphism λ such that there exists a coassociative, counitary coalgebra A with structural homomorphisms Δ and ϵ and such that there exists an epimorphism of vector spaces $p : A \to L$, both giving the following commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\Delta} & A \otimes A & \xrightarrow{Id-\tau} & A \otimes A \\ \downarrow{\scriptstyle p} & & & & \downarrow{\scriptstyle p\otimes p} \\ L & & \xrightarrow{\qquad\lambda\qquad} & & L \otimes L \end{array}$$

All Lie coalgebras that we shall be considering are reduced. The coalgebra A, more precisely the coalgebra A with the epimorphism p, is called a *cover* of the Lie coalgebra L. Now we shall define and study a special cover, the universal enveloping coalgebra.

Let us consider a reduced Lie coalgebra L. Of course L has the underlying structure of a vector space. Consequently, the Hopf Γ-algebra $T(L)$ is well defined. For the moment we consider only the coassociative, counitary coalgebra $T(L)$. Since the vector space L is reduced, a homogeneous element of $T(L)$ of primary degree n has the following decomposition

$$\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_n\,, \qquad \alpha_j \in T_j(L).$$

By definition this element α satisfies the *Lie condition* if the following equality holds in $L \otimes L$

$$\lambda(\alpha_1) = (Id - \tau)(\alpha_2).$$

PROPOSITION. 4. *Let L be a reduced Lie coalgebra. In the set of all subcoalgebras of T(L) consisting of elements satisfying the Lie condition, there is a unique maximal one, denoted by U(L). This coalgebra is a cover of the Lie coalgebra.*

Proof. Let E be the specified set of subcoalgebras of $T(L)$. This set E is not empty, for the coalgebra (0) belongs to it. Further the sum (in the vector space sense) of subcoalgebras belonging to E is a subcoalgebra belonging to E, since the Lie condition is linear. Consequently, the sum of all subcoalgebras belonging to E is this unique maximal subcoalgebra $U(L)$. The coassociative, counitary coalgebra $U(L)$ is thus defined.

Let us denote by $\bar{p}$ the homomorphism of vector spaces mapping $U(L)$ into L in the following way $\bar{p}(\alpha_0 + \alpha_1 + \cdots + \alpha_n) = \alpha_1$. Actually this homomorphism is an epimorphism: use the equality $\bar{p} \circ \int p = p$ of the next proposition and notice that p is an epimorphism. The coalgebra $U(L)$ with the structural homomorphism $\bar{\Delta}$ and the epimorphism $\bar{p}$ is a cover of the Lie coalgebra L. The following equality (see the diagram above appearing in the definition of a cover)

$$\lambda \circ \bar{p} = (\bar{p} \otimes \bar{p}) \circ (Id - \tau) \circ \bar{\Delta}$$

is another way of writing the Lie condition, so the proposition is proved.

PROPOSITION 5. *Let L be a reduced Lie coalgebra. Then the coalgebra $U(L)$ with the homomorphism $\bar{p} : U(L) \to L$ has the following universal property. Let the coalgebra A with the epimorphism p be a cover of the Lie coalgebra L, then there exists one and only one homomorphism $\int p : A \to U(L)$ of coalgebras satisfying the equality $\bar{p} \circ \int p = p$.*

Because of this property, $U(L)$ is called the *universal enveloping coalgebra of L*.

Proof. Let us consider the following homomorphisms of vector spaces

$$p^k = T_k(p) \circ \Delta^k : A \to T_k(L).$$

The homomorphism p^k maps A_n onto 0 if $k > n$, since the vector space L is reduced. Consequently the homomorphism of vector spaces

$$\int p = \sum_{k \geqslant 0} p^k : A \to T(L)$$

is well defined. We have the equality

$$\bar{p} \circ \int p = p^1 = p$$

and we shall see that $\int p$ is a homomorphism of coalgebras. Let us use the isomorphism $T_{i+j}(W) \cong T_i(W) \otimes T_j(W)$ appearing in the explicit definition of the comultiplication $\bar{\Delta}$ of $T(W)$. Since the comultiplication Δ is coassociative, the following diagram is commutative

$$\begin{array}{ccccc} A & \xrightarrow{\Delta^{i+j}} & T_{i+j}(A) & \xrightarrow{T_{i+j}(p)} & T_{i+j}(L) \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ A \otimes A & \xrightarrow{\Delta^i \otimes \Delta^j} & T_i(A) \otimes T_j(A) & \xrightarrow{T_i(p) \otimes T_j(p)} & T_i(L) \otimes T_j(L). \end{array}$$

Thus we get the following commutative square

$$\begin{array}{ccc} A & \xrightarrow{p^{i+j}} & T_{i+j}(L) \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \cong} \\ A \otimes A & \xrightarrow{p^i \otimes p^j} & T_i(L) \otimes T_j(L); \end{array}$$

in other words

$$\left(\int p \otimes \int p\right) \circ \Delta = \bar{\Delta} \circ \int p$$

and $\int p$ is a homomorphism of coalgebras.

The uniqueness of the homomorphism $\int p$ of coalgebras such that $\bar{p} \circ \int p = p$ is a consequence of the following equalities

$$(\bar{p} \otimes \cdots \otimes \bar{p}) \circ \bar{\Delta}^k \circ \int p = (\bar{p} \otimes \cdots \otimes \bar{p}) \circ \left(\int p \otimes \cdots \otimes \int p\right) \circ \Delta^k$$
$$= (p \otimes \cdots \otimes p) \circ \Delta^k = p^k.$$

The proposition is proved.

PROPOSITION 6. *Let L be a reduced Lie coalgebra. Then the universal enveloping coalgebra U(L) has a canonical structure of a Hopf Γ-algebra, more precisely it is a Hopf Γ-subalgebra of the Hopf Γ-algebra T(L).*

Proof. Actually we have to prove that $U(L)$ is a Γ-subalgebra of $T(L)$. Let $V(L)$ be the Γ-subalgebra of $T(L)$ generated by $U(L)$. We shall prove that $V(L)$ is a subcoalgebra of $T(L)$ and that its elements satisfy the Lie condition. Then according to the maximal character of $U(L)$ (see Proposition 4), $V(L)$ and $U(L)$ are equal; in other words, $U(L)$ is a Γ-subalgebra of $T(L)$.

The homomorphism $\bar{\Delta} : T(L) \to T(L) \otimes T(L)$ of Γ-algebras maps $U(L)$ into $U(L) \otimes U(L)$ and consequently $V(L)$ into the Γ-subalgebra of $T(L) \otimes T(L)$ generated by $U(L) \otimes U(L)$. But this Γ-subalgebra is contained in the Γ-subalgebra $V(L) \otimes V(L)$. Therefore $V(L)$ is a subcoalgebra of $T(L)$.

The product $\alpha = \alpha'\alpha''$ of two elements of $T(L)$ satisfying the Lie condition satisfies the Lie condition. Indeed we have

$$\alpha' = \alpha_0' + \alpha_1' + \alpha_2' + \cdots, \qquad \alpha'' = \alpha_0'' + \alpha_1'' + \alpha_2'' + \cdots,$$
$$\alpha = \alpha_0 + \alpha_1 + \alpha_2 + \cdots,$$

with

$$\alpha_1 = \alpha_0'\alpha_1'' + \alpha_0''\alpha_1', \qquad \alpha_2 = \alpha_0'\alpha_2'' + \alpha_0''\alpha_2' + (Id + \tau)(\alpha_1' \otimes \alpha_1'')$$

(the graduation used is the secondary one) and we conclude by the equalities

$$\lambda(\alpha_1) = \alpha_0'\lambda(\alpha_1'') + \alpha_0''\lambda(\alpha_1'),$$

$$(Id - \tau)(\alpha_2) = \alpha_0'(Id - \tau)(\alpha_2'') + \alpha_0''(Id - \tau)(\alpha_2').$$

The i-th divided power $\beta = \bar{\gamma}^i(\alpha)$ of an element α of $T(L)_+^{\cdot}$ satisfying the Lie condition satisfies the Lie condition. Indeed we have

$$\alpha = \alpha_1 + \alpha_2 + \cdots, \qquad \beta = \beta_1 + \beta_2 + \cdots,$$

with

$$\beta_1 = 0, \qquad \beta_2 = 0 \quad \text{if} \quad i > 2, \qquad \beta_2 = \alpha_1 \otimes \alpha_1 \quad \text{if} \quad i = 2,$$

and the Lie condition is satisfied since α_1 belongs to L_+. Thus the elements of $V(L)$ satisfy the Lie condition since the elements of $U(L)$ satisfy the Lie condition. Furthermore $V(L)$ is a subcoalgebra of $T(L)$ containing $U(L)$. Consequently $U(L)$ and $V(L)$ are equal and the proposition is proved.

4. Supports of Hopf Γ-Algebras

Let us consider a connected Hopf Γ-algebra A with structural homomorphisms and maps Φ, Δ, η, ϵ, γ^k. Further, let us consider a connected coalgebra M with structural homomorphism μ. Finally, let us consider an epimorphism of vector spaces $\pi : A \to M$. We say that M, more precisely M with π, is a *support* of the Hopf Γ-algebra A if the following properties are satisfied (see Section 1 for the definition of P and J)

1) on $P(A)$ the homomorphism π is a monomorphism
2) on $J(A)$ the homomorphism π is the zero homomorphism
3) the following diagram is commutative

$$\begin{array}{ccccc} A & \xrightarrow{\Delta} & A \otimes A & \xrightarrow{Id-\tau} & A \otimes A \\ \downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle \pi \otimes \pi} \\ M & & \xrightarrow{\mu} & & M \otimes M \end{array}$$

Let us notice the following facts useful for the future. Since A has a counit ϵ, the image of $(Id - \tau) \circ \Delta$ is contained in $A^{\cdot} \otimes A^{\cdot}$. Consequently, the homomorphism μ maps $M^{\cdot}$ into $M^{\cdot} \otimes M^{\cdot}$. To the homomorphism π: $A \to M$, there corresponds a homomorphism $p : A \to M^{\cdot}$. Then $M^{\cdot}$ is a reduced Lie coalgebra and A is one of its covers. We consider M instead of $M^{\cdot}$ for technical reasons.

Lemma 7. *Let A be a connected Hopf Γ-algebra. Then the homomorphism $(Id - \tau) \circ \Delta$ maps $J(A)$ into $J(A) \otimes A + A \otimes J(A)$.*

Proof. By definition we have the equality

$$J(A) = A^{\cdot} \cdot A^{\cdot} + \sum_{k \geqslant 2} K \cdot \gamma^k(A_+^{\cdot}).$$

Since A has a counit we have

$$\Delta(A^{\cdot}) \subset (Id + \tau)(A \otimes A) + A^{\cdot} \otimes A^{\cdot}.$$

Consequently we have

$$\Delta(A^{\cdot}) \cdot \Delta(A^{\cdot}) \subset (Id + \tau)(A \otimes A) + (A^{\cdot} \cdot A^{\cdot}) \otimes A + A \otimes (A^{\cdot} \cdot A^{\cdot}).$$

But Δ is a homomorphism of algebras and we have

$$(Id - \tau) \circ \Delta(A^{\cdot} \cdot A^{\cdot}) \subset (A^{\cdot} \cdot A^{\cdot}) \otimes A + A \otimes (A^{\cdot} \cdot A^{\cdot}).$$

Now let us use the third property and the equality $\gamma^i \circ \tau = \tau \circ \gamma^i$ of the divided powers of $A \otimes A$. For $k \geqslant 2$ we get the following inclusions

$$\gamma^k(A^{\cdot} \otimes A^{\cdot})_+ \subset (A^{\cdot} \cdot A^{\cdot}) \otimes A + A \otimes (A^{\cdot} \cdot A^{\cdot})$$
$$\gamma^k \circ (Id + \tau)(A \otimes A)_+^{\cdot} \subset (Id + \tau)(A \otimes A)_+^{\cdot} + B$$

where B is the subspace of $A \otimes A$ generated by the elements $\omega \cdot \tau\omega$ with $\omega \in (A \otimes A)_+^{\cdot}$. But Δ is a homomorphism of Γ-algebras and we have

$$\Delta \circ \gamma^k(A_+^{\cdot}) = \gamma^k \circ \Delta(A_+^{\cdot})$$
$$\Delta(A_+^{\cdot}) \subset (Id + \tau)(A \otimes A)_+^{\cdot} + (A^{\cdot} \otimes A^{\cdot})_+ .$$

We use once more the third property of the divided powers of $A \otimes A$ and we get the following inclusion

$$\Delta \circ \gamma^k(A_+^{\cdot}) \subset (A^{\cdot} \cdot A^{\cdot}) \otimes A + A \otimes (A^{\cdot} \cdot A^{\cdot}) + (Id + \tau)(A \otimes A) + B$$

and consequently the following inclusion

$$(Id - \tau) \circ \Delta \circ \gamma^k(A_+^{\cdot}) \subset (A^{\cdot} \cdot A^{\cdot}) \otimes A + A \otimes (A^{\cdot} \cdot A^{\cdot}).$$

In summary $(Id - \tau) \circ \Delta$ maps $A^{\cdot} \cdot A^{\cdot}$ and $\gamma^k(A_+^{\cdot})$ for $k \geqslant 2$ into

$$(A^{\cdot} \cdot A^{\cdot}) \otimes A + A \otimes (A^{\cdot} \cdot A^{\cdot}).$$

Thus the lemma is proved.

Proposition 8. *Let A be a connected Hopf Γ-algebra. Then*

$$R(A) = A \Big/ \Big[A^{\cdot} \cdot A^{\cdot} + \sum_{k \geqslant 2} K\gamma^k(A_+^{\cdot})\Big]$$

is a support of the Hopf Γ-algebra A.

Proof. We consider the vector space $R(A) = A/J(A)$. Let $\tilde{\pi}$ be the canonical epimorphism of A onto $R(A)$. Lemma 7 proves that there is one and only one homomorphism

$$\tilde{\mu} : R(A) \to R(A) \otimes R(A) \qquad \text{with} \qquad (\tilde{\pi} \otimes \tilde{\pi}) \circ (Id - \tau) \circ \Delta = \tilde{\mu} \circ \tilde{\pi}.$$

The coalgebra $R(A)$ is a support provided on $P(A)$ the homomorphism $\tilde{\pi}$ is a monomorphism.

A Hopf Γ-algebra is said to be *finitely generated* if it has a finite number of generators as a Γ-algebra. For an element x of degree $n > 0$ of a Hopf Γ-algebra A

$$\Delta(x) - x \otimes 1 - 1 \otimes x \in \sum_{i,j<n} A^i \otimes A^j.$$

Consequently, a Hopf Γ-algebra is the union of its finitely generated Hopf Γ-subalgebras. For this assertion the fact that Δ is a homomorphism of Γ-algebras is essential. We argue in the following way. If B is a finitely generated Hopf Γ-subalgebra of A generated by $x_1, ..., x_m$ and if $y_1, ..., y_n$ are elements of A such that

$$\Delta(y_i) - y_i \otimes 1 - 1 \otimes y_i \in B \otimes B$$

then the Γ-subalgebra of A generated by the elements

$$x_1, ..., x_m, y_1, ..., y_n$$

is a finitely generated Hopf Γ-subalgebra of A. Then it is possible to prove the following assertion by induction on k: each finite set $(z_1, ..., z_n)$ of elements of A of degree smaller than k is contained in a finitely generated Hopf Γ-subalgebra of A. The different elements

$$\Delta(z_i) - z_i \otimes 1 - 1 \otimes z_i$$

can be written with a finite number of elements of A of degree smaller than $k - 1$. By hypothesis of the induction, those elements belong to a finitely generated Hopf Γ-subalgebra of A. The conclusion for $(z_1, ..., z_n)$ is a consequence of the assertion above with $(x_1, ..., x_m)$ a set of generators of the preceding Hopf Γ-subalgebra and with y_j equal to z_j. Thus a Hopf Γ-algebra is the union of its finitely generated Hopf Γ-subalgebras. Therefore it remains to prove that the homomorphism

$$\tilde{\pi} \mid P(A) : P(A) \to Q(A) = R^{\cdot}(A)$$

is a monomorphism for a connected and finitely generated Hopf Γ-algebra A. The proof goes by induction on the dimension of the vector space $Q(A)$, in other words on the minimal number of generators of A.

When the connected Hopf Γ-algebra A has only one generator x of degree n, there is an isomorphism $P(A) \cong Q(A)$. This is clear if we notice the following. If x belongs to A_-, the vector spaces A_i are equal to 0 except A_0 generated by 1 and A_n generated by x. If x belongs to A_+, the vector spaces

A_i are equal to 0 except A_{kn}, for $k = 0, 1, 2,...,$ generated by $\gamma^k(x) \neq 0$. Once more the fact that Δ is a homomorphism of Γ-algebras is essential.

Now let us go from the case $\dim Q(A) < k$ to the case $\dim Q(A) = k$. Let us choose k homogeneous elements $x_1 ,..., x_k$ generating the Hopf Γ-algebra A with $\deg x_k \geqslant \deg x_i$ for any i. Let A' be the Γ-subalgebra of A generated by $x_1 ,..., x_{k-1}$. According to the inequality $\deg x_k \geqslant \deg x_i$, it is a Hopf Γ-subalgebra of A. Let A'' be the quotient $A/A \cdot A'^{\cdot}$. It is a Hopf Γ-algebra too. Thus we have the following situation

$$A' \to A \to A'', \qquad \dim Q(A') = k - 1, \qquad \dim Q(A'') = 1.$$

Now let us consider the following commutative diagram

$$\begin{array}{ccccc} P(A') & \longrightarrow & P(A) & \longrightarrow & P(A'') \\ \downarrow{\scriptstyle \tilde{\pi}'} & & \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \tilde{\pi}''} \\ Q(A') & \xrightarrow{\beta} & Q(A) & \longrightarrow & Q(A''). \end{array}$$

Using the elements $x_1 ,..., x_k$, we see that β is a monomorphism. By the induction hypothesis, $\tilde{\pi}'$ and $\tilde{\pi}''$ are monomorphisms. Consequently $\tilde{\pi}$ is a monomorphism if the sequence

$$P(A') \to P(A) \to P(A'')$$

is exact. But it is a well-known result of the theory of Hopf algebras: see [4, Proposition 4.10]. Thus the proposition is proved.

PROPOSITION 9. *Let L be a reduced Lie coalgebra. Then the connected coalgebra $M = {}^{\cdot}L$ is a support of the connected Hopf Γ-algebra $U(L)$.*

Proof. More precisely, we consider the vector space $M = K + L$ and the following comultiplication $\mu : M \to M \otimes M$: on K, the homomorphism μ is equal to 0 and on L, the homomorphism μ is equal to the comultiplication $\lambda : L \to L \otimes L$. Further, we consider the canonical homomorphism $\bar{p} : U(L) \to L$ and the corresponding homomorphism $\tilde{\pi} : U(L) \to M$. We prove that the coalgebra M with this homomorphism $\tilde{\pi}$ is a support. We know that it is an epimorphism (see Proposition 4) and we have to verify the three axioms of the definition of a support.

The vector space $P(U(L))$ is contained in the vector space $P(T(L))$ which is equal to $T_1(L)$ according to the explicit definition of the comultiplication $\bar{\Delta}$ of $T(L)$. Consequently the restriction of $\tilde{\pi}$ to $P(U(L))$ is a monomorphism. The vector space $J(U(L))$ is contained in the vector space $J(T(L))$. But the vector space $J(T(L))$ is contained in the sum $\sum_{n \geqslant 2} T_n(L)$ since the multiplication $\bar{\Phi}$ maps $T_i(L) \otimes T_j(L)$ into $T_{i+j}(L)$ and since the map $\bar{\gamma}^k$ maps $T_i(L)_+$ into $T_{ik}(L)$. Consequently the restriction of $\bar{\pi}$ to $J(U(L))$ is equal to 0. Finally

$$(\bar{\pi} \otimes \bar{\pi}) \circ (Id - \tau) \circ \bar{\Delta} = \mu \circ \bar{\pi}$$

or equivalently

$$(\bar{p} \otimes \bar{p}) \circ (Id - \tau) \circ \bar{\Delta} = \lambda \circ \bar{p}$$

since $U(L)$ is a cover of the Lie coalgebra L (see Proposition 4). Thus the proposition is proved.

5. Poincaré–Birkhoff–Witt Theorem

Let us consider a connected Hopf Γ-algebra A with a support M. Out of the epimorphism $\pi : A \to M$ we can get new homomorphisms of vector spaces

$$\pi^k = T_k(\pi) \circ \Delta^k : A \to T_k(M).$$

Now let us introduce a filtration $F^n(A)$ on the Hopf Γ-algebra A, the so-called *Lie filtration* (see Lemma 10). By definition we have

$$F^n = F^n(A) = \operatorname{Ker} \pi^{n-1} : A \to T_{n-1}(M)$$

and we prove the following result.

Lemma 10. *Let A be a connected Hopf Γ-algebra with a support M. Then $F^n(A)$ contains $F^{n+1}(A)$ and π^n maps $F^n(A)$ into $T_n(M^\cdot)$.*

Proof. We shall use the homomorphisms

$$\epsilon : A \to K \quad \text{(counit)}, \qquad \epsilon : M \to K \quad (M_0 \simeq K),$$

and the associated homomorphisms (see Section 1)

$$\epsilon_k{}^i : T_k(A) \to T_{k-1}(A), \qquad \epsilon_k{}^i : T_k(M) \to T_{k-1}(M).$$

The proof of the lemma is an immediate consequence of the existence of the following commutative diagram with $1 \leqslant i \leqslant n$

$$\begin{array}{ccccc} A & \xrightarrow{\Delta^n} & T_n(A) & \xrightarrow{T_n(\pi)} & T_n(M) \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \epsilon_n{}^i} & & \downarrow{\scriptstyle \epsilon_n{}^i} \\ A & \xrightarrow{\Delta^{n-1}} & T_{n-1}(A) & \xrightarrow{T_{n-1}(\pi)} & T_{n-1}(M) \end{array} \qquad \begin{array}{c} \pi^n = T_n(\pi) \circ \Delta^n \\ \\ \pi^{n-1} = T_{n-1}(\pi) \circ \Delta^{n-1}. \end{array}$$

Thus π^n maps F^n into $T_n(M^\cdot)$ and the kernel is equal to F^{n+1}. In other words, we have a monomorphism of vector spaces

$$F^n/F^{n+1} \to T_n(M^\cdot).$$

By definition the *n-trace* of the Hopf Γ-algebra A in the support M is the image of this monomorphism. The *trace* of the Hopf Γ-algebra A in the support M is the direct sum of the n-traces. It is a vector subspace of $T(M^\cdot)$ for both graduations.

Here is a result we can consider as a *Poincaré–Birkhoff–Witt Theorem with divided powers*: by means of Proposition 6 and of Proposition 9, it is a result for Lie coalgebras.

THEOREM 11. *Let A be a connected Hopf Γ-algebra with a support M. Then the trace of A in M depends on the vector space M only. The n-trace is equal to the vector space of the S_n-invariant elements of $T_n(M^\cdot)$.*

Proof. Since F^1 is equal to $A^\cdot$ and since π^1 is equal to π, the 1-trace is equal to $T_1(M^\cdot)$. For the general proof we use Proposition 3, where W is the reduced vector space $M^\cdot$ and where H is the trace of A in M. We have to verify the three conditions appearing in Proposition 3. That is done by the three following propositions whose proofs appear in Section 8.

PROPOSITION 12. *Let A be a connected Hopf Γ-algebra with a support M. Then for any $n \geqslant 0$, the elements of the n-trace of A in M are S_n-invariant.*

PROPOSITION 13. *Let A be a connected Hopf Γ-algebra with a support M. Then the trace of A in M is a subalgebra of $T(M^\cdot)$.*

PROPOSITION 14. *Let A be a connected Hopf Γ-algebra with a support M. Then for any $k \geqslant 0$ and for any element x of $M_+^\cdot$, the element $\bar{\gamma}^k(x) =$* $x \otimes \cdots \otimes x$ of $T_k(M^\cdot)$ belongs to the k-trace of A in M.

The last three propositions are proved in Section 8. We shall use the following lemma for getting a corollary of the Poincaré-Birkhoff-Witt theorem.

LEMMA 15. *Let A be a connected Hopf Γ-algebra with a support M. Then the Lie filtration has the following property. The vector space F_k^n of the homogeneous elements of F^n of degree k is equal to 0 if n is large enough with respect to k.*

Proof. It suffices to prove, by induction on k, that the homomorphism π^n is a monomorphism for the homogeneous elements of degree k if n is large enough. If it is proved for all degrees smaller than k, we choose r and s (equal or not) such that π^r and π^s are monomorphism for all degrees smaller than k. Then we consider $n = r + s$ and we write

$$\pi^n = (\pi^r \otimes \pi^s) \circ \Delta$$

identifying $T_n(M)$ and $T_r(M) \otimes T_s(M)$. Now let us use the equality

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \alpha, \qquad \alpha \in A^{\cdot} \otimes A^{\cdot},$$

for a homogeneous element x of degree k. If $\pi^n(x)$ is equal to 0, then $(\pi^r \otimes \pi^s)(\alpha)$ is equal to 0. But on $(A^{\cdot} \otimes A^{\cdot})_k$, the homomorphism $\pi^r \otimes \pi^s$ is a monomorphism. Consequently α is equal to 0 and $\Delta(x) = x \otimes 1 + 1 \otimes x$. The preceding equality implies the following equality

$$\pi^n(x) = \pi(x) \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \pi(x).$$

Since $\pi^n(x)$ is equal to 0, the element $\pi(x)$ is equal to 0. In summary x is an element of $P(A)$ and an element of the kernel of π. According to the first property of a support, the element x is equal to 0. In other words π^n is a monomorphism for the homogeneous elements of degree k. The lemma is proved.

PROPOSITION 16. *Let A be a connected Hopf Γ-algebra with a support $\pi : A \to M$ and let A' be a connected Hopf Γ-algebra with a support $\pi' : A' \to M'$. Let $\rho : A \to A'$ be a homomorphism of coalgebras and let $\omega : M \to M'$ be a homomorphism of vector spaces such that the homomorphisms $\pi' \circ \rho$ and $\omega \circ \pi$ are equal. Then ρ is an isomorphism if and only if ω is an isomorphism.*

Proof. The definition of the Lie filtration and of the trace of a Hopf Γ-algebra with a support involves only the coalgebra structure of the Hopf Γ-algebra and the vector space structure of the support. Consequently the homomorphism ρ maps $F^n(A)$ into $F^n(A')$ and the homomorphism $T_k(\omega)$ maps the k-trace of A in M into the k-trace of A' in M'. After this remark it is clear that the proposition is a consequence of the following assertions.

1) The vector spaces $F_k{}^n(A)$ and $F_k{}^n(A')$ are equal to 0 for n large enough with respect to k; the vector space $F_k{}^1(A)$ is equal to $A_k^{\cdot}$ and the vector space $F_k{}^1(A')$ is equal to $A_k'^{\cdot}$. See Lemma 15.

2) There is a commutative diagram with exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_k^{n+1}(A) & \longrightarrow & F_k{}^n(A) & \longrightarrow & F_k{}^n(A)/F_k^{n+1}(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_k^{n+1}(A') & \longrightarrow & F_k{}^n(A') & \longrightarrow & F_k{}^n(A')/F_k^{n+1}(A') & \longrightarrow & 0 \end{array}$$

That is obvious.

3) The homomorphism ρ gives an isomorphism

$$F_k^1(A)/F_k^2(A) \cong F_k^1(A')/F_k^2(A')$$

if and only if the homomorphism ω gives an isomorphism $M_k \cong M_k'$. See Theorem 11.

4) For $n \geqslant 2$, the homomorphism ρ gives an isomorphism

$$F_k^n(A)/F_k^{n+1}(A) \cong F_k^n(A')/F_k^{n+1}(A')$$

if the homomorphism ω gives an isomorphism $M_i \cong M_i'$ for any $i < k$. See Theorem 11.

6. Results

Now we have a structure theorem.

THEOREM 17. *Let A be a connected Hopf Γ-algebra. Then*

$$Q(A) = A^{\cdot} \Big/ \Big[A^{\cdot} \cdot A^{\cdot} + \sum_{k \geqslant 2} K\gamma^k(A_+^{\cdot})\Big]$$

is a reduced Lie coalgebra. Its universal enveloping coalgebra $U(Q(A))$ is a Hopf Γ-algebra. The Hopf Γ-algebras A and $U(Q(A))$ are isomorphic.

Proof. By Proposition 8, the vector space $R(A) = {}^{\cdot}Q(A)$ is actually a support of the Hopf Γ-algebra A. Consequently it suffices to prove the following result.

PROPOSITION 18. *Let A be a connected Hopf Γ-algebra with a support M. Then $L = M^{\cdot}$ is a reduced Lie coalgebra. Its universal enveloping coalgebra $U(L)$ is a Hopf Γ-algebra. The Hopf Γ-algebras A and $U(L)$ are isomorphic.*

Proof. Let us consider the homomorphism $\pi : A \to M$ and the corresponding homomorphism $p : A \to L$ for $L = M^{\cdot}$. According to the beginning of Section 4, the coalgebra L is a Lie coalgebra and the coalgebra A is one of its covers. By Proposition 4, the universal enveloping coalgebra $U(L)$ of the Lie coalgebra L is one of its covers. Let us consider the homomorphism $\bar{p} : U(L) \to L$ and the corresponding homomorphism $\bar{\pi} : U(L) \to M$ for $M = {}^{\cdot}L$.

By Proposition 5, there is a homomorphism $\int p : A \to U(L)$ of coalgebras with $\bar{p} \circ \int p = p$ or equivalently with $\bar{\pi} \circ \int p = \pi$. But $\pi : A \to M$ is the canonical homomorphism of the support M of A and $\bar{\pi} : U(L) \to M$ is the canonical homomorphism of the support M of $U(L)$, by Proposition 9. We can apply Proposition 16. The homomorphism $\int p$ of coalgebras is an isomorphism. Thus the proposition is a consequence of the following result.

PROPOSITION 19. *Let A be a connected Hopf Γ-algebra with a support M. Then for the reduced Lie coalgebra $L = M^{\cdot}$, the canonical homomorphism* $\int p : A \to U(L)$ of Proposition 5 is a homomorphism of Hopf Γ-algebras.

Proof. See Section 8.

Here is a corollary of Theorem 17.

THEOREM 20. *The category of all connected Hopf Γ-algebras over the field K is equivalent to the category of all reduced Lie coalgebras over the field K.*

Proof. To a reduced Lie coalgebra L, there corresponds a unique connected Hopf Γ-algebra $U(L)$ by Proposition 6. To a connected Hopf Γ-algebra A, there corresponds a unique reduced Lie coalgebra $Q(A)$ by Proposition 8. The Hopf Γ-algebras A and $U(Q(A))$ are isomorphic by Proposition 18. The Lie coalgebras L and $Q(U(L))$ are isomorphic by Propositions 9 and 21. Thus the theorem is a consequence of the following result.

PROPOSITION 21. *Up to an isomorphism, a support of a connected Hopf Γ-algebra is determined by this Hopf Γ-algebra.*

Proof. Let M be a support of the Hopf Γ-algebra A, with canonical homomorphism π. By Proposition 8, there is another support $R(A)$ of the Hopf Γ-algebra A, with canonical homomorphism $\tilde{\pi}$. Since π is equal to 0 on $J(A)$ and since $R(A)$ is equal to $A/J(A)$, there is a homomorphism $\omega : R(A) \to M$ with $\omega \circ \tilde{\pi} = \pi$. By Proposition 16, it is an isomorphism and the proposition is proved.

7. PROOFS OF SOME LEMMAS

For the proofs of Propositions 12, 13, 14, 19 we need some auxiliary lemmas. Let us introduce a little more notation. Let us consider a vector space W and the corresponding vector space $T_n(W)$. Actually this vector space $T_n(W)$ is the direct sum of the vector spaces $T_{n,i}(W)$ defined as follows

$$T_{n,i}(W) = \sum W_{m_1} \otimes \cdots \otimes W_{m_n},$$

with summation over the set of the n-tuples $(m_1, ..., m_n)$ where i of the m_j's are positive and $n - i$ of the m_j's are equal to 0. As before, the multiplication of

$T(W)$ is denoted by $\bar{\Phi}$, the comultiplication by $\bar{\Delta}$ and the divided powers by $\bar{\gamma}^k$.

Now let us consider a connected Hopf Γ-algebra A. The multiplication of A is denoted by Φ, the comultiplication by Δ and the divided powers by γ^k. For any $n \geqslant 0$, the tensor product $T_n(A)$ is itself a Hopf Γ-algebra. The multiplication of $T_n(A)$ is denoted by ${}_n\Phi$, the comultiplication by ${}_n\Delta$ and the divided powers by ${}_n\gamma^k$. Further, we shall use the element 1_n equal to $1 \otimes \cdots \otimes 1$ in $T_n(A)$.

Now let us consider a connected Γ-algebra M. The multiplication of M is denoted by Φ and the divided powers by γ^k. For any $n \geqslant 0$, the tensor product $T_n(M)$ is itself a Γ-algebra. The multiplication of $T_n(M)$ is denoted by ${}_n\Phi$ and the divided powers by ${}_n\gamma^k$. Further, we shall use the element 1_n equal to $1 \otimes \cdots \otimes 1$ in $T_n(M)$. The connected Γ-algebra M is said to be *trivial* if

$$\Phi(M^{\cdot} \otimes M^{\cdot}) = 0, \qquad \gamma^k(M_+^{\cdot}) = 0, \qquad k \geqslant 2.$$

LEMMA 22. *Let A be a connected Hopf Γ-algebra with a support $\pi : A \to M$. Then M has one and only one Γ-algebra structure such that π is a homomorphism of Γ-algebras. The connected Γ-algebra M is trivial.*

Proof. We have uniqueness since π is an epimorphism and we have existence since π is the zero homomorphism on $J(A)$.

LEMMA 23. *Let A be a Hopf algebra and let a be an element of $A^{\cdot}$. Then the element $\Delta^n(a)$ of $T_n(A)$ and the product $\bar{\Phi}(\Delta^i(a) \otimes 1_{n-i})$ of the elements $\Delta^i(a)$ and 1_{n-i} of the algebra $T(A)$ have the same component in $T_{n,i}(A)$.*

Proof. We shall use the following decomposition

$$\Delta(x) = \delta(x) + d(x)$$

where $d(x)$ is the component of $\Delta(x)$ in $T_{2,2}(A)$. Then for any element ω of $T_{r,r}(A)$ there is an equality

$$\begin{aligned}&\bar{\Phi}[(d \otimes Id)(\omega) \otimes 1_s] \\ &\quad = (\delta \otimes Id) \circ \bar{\Phi}[(d \otimes Id)(\omega) \otimes 1_{s-1}] + (d \otimes Id) \circ \bar{\Phi}[\omega \otimes 1_s].\end{aligned}$$

The proof uses the explicit definition of $\bar{\Phi}$. Let us denote by $d^n(a)$ the component of $\Delta^n(a)$ in $T_{n,n}(A)$. We shall use the following equality

$$d^n(a) = (d \otimes Id) \circ d^{n-1}(a).$$

Now we apply the last but one equality with $r = i - 1$, $s = n - i$ and $\omega = d^{i-1}(a)$ and we get the following equality

$$\begin{aligned}&\bar{\Phi}[d^i(a) \otimes 1_{n-i}] \\ &\quad = (\delta \otimes Id) \circ \bar{\Phi}[d^i(a) \otimes 1_{n-i-1}] + (d \otimes Id) \circ \bar{\Phi}[d^{i-1}(a) \otimes 1_{n-i}].\end{aligned}$$

This equality proves the lemma by induction on n. If the lemma holds for $n - 1$, then the elements

$$\bar{\Phi}[d^i(a) \otimes 1_{n-i-1}], \qquad \bar{\Phi}[\Delta^i(a) \otimes 1_{n-i-1}], \qquad \Delta^{n-1}(a)$$

have the same component in $T_{n-1,i}(A)$ and the elements

$$\bar{\Phi}[d^{i-1}(a) \otimes 1_{n-i}], \qquad \bar{\Phi}[\Delta^{i-1}(a) \otimes 1_{n-i}], \qquad \Delta^{n-1}(a)$$

have the same component in $T_{n-1,i-1}(A)$. Further, the elements

$$\bar{\Phi}[d^i(a) \otimes 1_{n-i}], \qquad \bar{\Phi}[\Delta^i(a) \otimes 1_{n-i}]$$

have the same component in $T_{n,i}(A)$. Consequently by the last equality, the elements

$$\bar{\Phi}[\Delta^i(a) \otimes 1_{n-i}], \qquad \Delta^n(a) = (\delta \otimes Id) \circ \Delta^{n-1}(a) + (d \otimes Id) \circ \Delta^{n-1}(a)$$

have the same component in $T_{n,i}(A)$. The lemma is proved for n.

LEMMA 24. *Let M be a trivial Γ-algebra. Let us consider k elements of $T_n(M)$*

$$x_1 \in T_{n,i_1}(M), \ldots, x_k \in T_{n,i_k}(M).$$

Then the product ${}_n\Phi^k(x_1 \otimes \cdots \otimes x_k)$ of the elements x_j of the algebra $T_n(M)$ belongs to $T_{n,m}(M)$ with $m = i_1 + \cdots + i_k$

Proof. It suffices to prove the case $k = 2$. It is an obvious consequence of the equality $M^\cdot \cdot M^\cdot = 0$.

LEMMA 25. *Let M be a trivial Γ-algebra. Let us consider k elements of $T_n(M)$*

$$x_j = \bar{\Phi}(y_j \otimes 1_{n-i_j}), \qquad y_j \in T_{i_j}(M^\cdot).$$

Then the product ${}_n\Phi^k(x_1 \otimes \cdots \otimes x_k)$ of the elements x_j of the algebra $T_n(M)$ is equal to the product $\bar{\Phi}^{k+1}(y_1 \otimes \cdots \otimes y_k \otimes 1_{n-m})$ of the elements y_j and 1_{n-m} of the algebra $T(M)$ with $m = i_1 + \cdots + i_k$.

Proof. By multilinearity it suffices to prove the lemma with

$$y_j = a_{j,1} \otimes \cdots \otimes a_{j,i_j}, \qquad a_{j,h} \in M^{\cdot}.$$

Then we write

$$y_j \otimes 1_{n-i_j} = a_{j,1} \otimes \cdots \otimes a_{j,n}, \qquad a_{j,h} = 1 \quad \text{if} \quad h > i_j .$$

Now let us use the following equalities

$$x_j = \bar{\Phi}(y_j \otimes 1_{n-i_j}) = \sum_{\sigma^j \in S(i_j, n-i_j)} a_{j,\sigma_1^j} \otimes \cdots \otimes a_{j,\sigma_n^j} .$$

They give the following equality

$${}_n\Phi^k(x_1 \otimes \cdots \otimes x_k) = \sum_{\sigma^1 \in S(i_1, n-i_1)} \cdots \sum_{\sigma^k \in S(i_k, n-i_k)} \Omega(\sigma^1, \ldots, \sigma^k)$$

with $\Omega(\sigma^1, \ldots, \sigma^k)$ equal to

$$\begin{aligned} &{}_n\Phi^k[(a_{1,\sigma_1^1} \otimes \cdots \otimes a_{1,\sigma_n^1}) \otimes \cdots \otimes (a_{k,\sigma_1^k} \otimes \cdots \otimes a_{k,\sigma_n^k})] \\ &\quad = \pm\Phi^k(a_{1,\sigma_1^1} \otimes \cdots \otimes a_{k,\sigma_1^k}) \otimes \cdots \otimes \Phi^k(a_{1,\sigma_n^1} \otimes \cdots \otimes a_{k,\sigma_n^k}). \end{aligned}$$

But $\Phi^k(a_{1,\sigma_r^1} \otimes \cdots \otimes a_{k,\sigma_r^k})$ is equal to 1 if $\sigma_r^j > i_j$ for all j's, to $a_{j',\sigma_r^{j'}}$ if $\sigma_r^j > i_j$ for all j's except j' and to 0 otherwise. Therefore $\Omega(\sigma^1, \ldots, \sigma^j)$ is equal to 0 except when the set of elements

$$\sigma^1 \in S(i_1, n - i_1), \ldots, \sigma^k \in S(i_k, n - i_k)$$

has the following property Λ. We use the ordered set

$$(a_1, \ldots, a_n) = (a_{1,1}, \ldots, a_{1,i_1}, \ldots, a_{k,1}, \ldots, a_{k,i_k}, 1, \ldots, 1).$$

The property Λ holds if there exists one (and only one if the elements $a_{j,h}$ are general enough) permutation σ of $S(i_1, \ldots, i_k, n - m)$ such that for any r, the sets

$$(a_{1,\sigma_r^1}, \ldots, a_{k,\sigma_r^k}) \qquad \text{and} \qquad (a_{\sigma_r}, 1, \ldots, 1)$$

are equal up to the order. If the property Λ holds with the permutation σ, then

$$\Omega(\sigma^1, \ldots, \sigma^k) = \sigma(a_1 \otimes \cdots \otimes a_n) = \sigma(y_1 \otimes \cdots \otimes y_k \otimes 1_{n-m}).$$

Consequently we get the equalities

$$\begin{aligned} {}_n\Phi^k(x_1 \otimes \cdots \otimes x_k) &= \sum_{\sigma \in S(i_1, \ldots, i_k, n-m)} \sigma(y_1 \otimes \cdots \otimes y_k \otimes 1_{n-m}) \\ &= \bar{\Phi}^{k+1}(y_1 \otimes \cdots \otimes y_k \otimes 1_{n-m}). \end{aligned}$$

The lemma is proved.

LEMMA 26. *Let M be a trivial Γ-algebra. Let us consider an element x of $T_{n,i}(M)_+^{\cdot}$. Then the divided power ${}_n\gamma^h(x)$ of the element x of the Γ-algebra $T_n(M)$ belongs to $T_{n,m}(M)$ with $m = hi$.*

Proof. Let us consider $x = x' + x''$ and the equality

$$ {}_n\gamma^h(x) = \sum_{h'+h''=h} {}_n\Phi[{}_n\gamma^{h'}(x') \otimes {}_n\gamma^{h''}(x'')]. $$

If the lemma is proved for x' and for x'', then it is proved for x according to Lemma 24. Consequently it suffices to prove the lemma for

$$ x = a_1 \otimes \cdots \otimes a_n $$

i elements a_j belonging to $M^{\cdot}$ and $n - i$ elements a_j being equal to 1. Since the Γ-algebra M is trivial, the element ${}_n\gamma^h(x)$ is equal to 0 if h is greater than 1. The lemma is proved.

LEMMA 27. *Let M be a trivial Γ-algebra. Let us consider an element of $T_n(M)_+^{\cdot}$*

$$ x = \bar{\Phi}(y \otimes 1_{n-i}), \qquad y \in T_i(M^{\cdot}). $$

Then the divided power ${}_n\gamma^h(x)$ of the element x of the Γ-algebra $T_n(M)$ is equal to the product $\bar{\Phi}(\bar{\gamma}^h(y) \otimes 1_{n-m})$ of the elements $\bar{\gamma}^h(y)$ and 1_{n-m} of the Γ-algebra $T(M)$ with $m = hi$.

Proof. Let us consider $x = x' + x''$ and $y = y' + y''$ with

$$ x' = \bar{\Phi}(y' \otimes 1_{n-i}), \qquad x'' = \bar{\Phi}(y'' \otimes 1_{n-i}). $$

Let us suppose that the lemma holds for y' and for y'' and let us prove it for y. We use Lemma 25 and we get the following equalities

$$ \begin{aligned} {}_n\gamma^h(x) &= \sum_{h'+h''=h} {}_n\Phi[{}_n\gamma^{h'}(x') \otimes {}_n\gamma^{h''}(x'')] \\ &= \sum_{h'+h''=h} {}_n\Phi[\bar{\Phi}(\bar{\gamma}^{h'}(y') \otimes 1_{n-h'i}) \otimes \bar{\Phi}(\bar{\gamma}^{h''}(y'') \otimes 1_{n-h''i})] \\ &= \sum_{h'+h''=h} \bar{\Phi}^3[\bar{\gamma}^{h'}(y') \otimes \bar{\gamma}^{h''}(y'') \otimes 1_{n-hi}] \\ &= \bar{\Phi}\Big(\sum_{h'+h''=h} \bar{\Phi}[\bar{\gamma}^{h'}(y') \otimes \bar{\gamma}^{h''}(y'')] \otimes 1_{n-hi}\Big) = \bar{\Phi}(\bar{\gamma}^h(y) \otimes 1_{n-hi}). \end{aligned} $$

Thus the lemma holds for y. Consequently it suffices to prove the lemma for

$$ y = a_1 \otimes \cdots \otimes a_i \,. $$

Let us order the set $S(i, n-i)$: $\sigma^1,\dots,\sigma^k$. Then we have to consider

$$x = \sigma^1(a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes 1) + \cdots + \sigma^k(a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes 1)$$

We get the following equality

$$_n\gamma^h(x) = \sum_{h_1+\cdots+h_k=h} {}_n\Phi^k[{}_n\gamma^{h_1} \circ \sigma^1(a_1 \otimes \cdots \otimes 1) \otimes \cdots \otimes {}_n\gamma^{h_k} \circ \sigma^k(a_1 \otimes \cdots \otimes 1)].$$

But the Γ-algebra M is trivial. Then we have the following equalities

$$_n\gamma^h(x) = \sum_{\substack{h_1+\cdots+h_k=h \\ h_1\leqslant 1,\dots,h_k\leqslant 1}} {}_n\Phi^k[{}_n\gamma^{h_1} \circ \sigma^1(a_1 \otimes \cdots \otimes 1) \otimes \cdots \otimes {}_n\gamma^{h_k} \circ \sigma^k(a_1 \otimes \cdots \otimes 1)].$$

$$= \sum_{1\leqslant i_1<\cdots<i_h\leqslant k} {}_n\Phi^h[\sigma^{i_1}(a_1 \otimes \cdots \otimes 1) \otimes \cdots \otimes \sigma^{i_h}(a_1 \otimes \cdots \otimes 1)].$$

Now we can use the end of the proof of Lemma 25 in a special case: $i_j \equiv i$ and $a_{j,r} \equiv a_r$. Since we are no longer working with general elements $a_{j,r}$, the permutation σ of Property Λ is no longer unique. We use the following notation

$$S(i \mid h \mid n-m) = S(i,\dots,i, n-m) \subset S_n \qquad \text{with} \quad m = hi.$$

An element σ of S_n is described by the following equality

$$\sigma(x_1,\dots,x_n) = (x_{\sigma_1},\dots,x_{\sigma_n}).$$

We denote by X_t the following subset of the ordered set $(x_1,\dots,x_n)$, with i elements,

$$(x_{i(t-1)+1},\dots,x_{it}) = X_t, \qquad 1 \leqslant t \leqslant h.$$

Let us consider the following equality

$$(x_1,\dots,x_n) = (X_1,\dots,X_h, x_{hi+1},\dots,x_n).$$

The elements of S_n permute the different x_j's and the elements of S_h permute the different X_t's. Thus we get a natural imbedding of S_h into S_n: write simply $S_h \subset S_n$. The subset $S(i \mid h \mid n-m)$ of the group S_n is invariant under the right action of the subgroup S_h. We consider the quotient set

$$S[i \mid h \mid n-m] = S(i \mid h \mid n-m)/S_h.$$

Now let us come back to Property Λ in the special case: $i_j \equiv i$ and $a_{j,r} \equiv a_r$. We can replace the permutation σ of $S(i \mid h \mid n-m)$ by an element σ of

$S[i \mid h \mid n-m]$. In other words the summations over the set $S(i \mid h \mid n-m)$ can be replaced by summations over the set $S[i \mid h \mid n-m]$. Actually the summations of the first type must be replaced by summations of the second type since the condition $i_1 < \cdots < i_h$ appears in the last summation above describing ${}_n\gamma^h(x)$. Thus we get the following equalities

$$\begin{aligned}
{}_n\gamma^h(x) &= \sum_{1 \leqslant i_1 < \cdots < i_h \leqslant k} {}_n\Phi^h[\sigma^{i_1}(a_1 \otimes \cdots \otimes 1) \otimes \cdots \otimes \sigma^{i_h}(a_1 \otimes \cdots \otimes 1)] \\
&= \sum_{\sigma \in S[i \mid h \mid n-m]} \sigma(y \otimes \cdots \otimes y \otimes 1_{n-m}) \\
&= \sum_{\sigma' \in S(m, n-m)} \sum_{\sigma'' \in S[i \mid h]} \sigma'[\sigma''(y \otimes \cdots \otimes y) \otimes 1_{n-m}] \\
&= \bar{\Phi}[\bar{\gamma}^h(y) \otimes 1_{n-m}].
\end{aligned}$$

The lemma is proved.

8. Proofs of Some Propositions

Here are the proofs of four propositions appearing in the proofs of Theorems 11, 17, 20.

Proposition 12. *Let A be a connected Hopf Γ-algebra with a support M. Then for any $n \leqslant 0$, the elements of the n-trace of A in M are S_n-invariant.*

Proof. Let π be the canonical homomorphism of A onto M. Let μ be the structural homomorphism of the coalgebra M. Let i be an integer with $1 \leqslant i \leqslant n$. Let us consider the element σ of S_n described by the equality

$$\sigma(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n)$$

acting on $T_n(A)$ and on $T_n(M)$. To prove the lemma it suffices to verify the following equality

$$(Id - \sigma) \circ \pi^n(F^n(A)) = 0.$$

For that purpose, we use the identity homomorphisms I', I'', J', J'' of

$T_{i-1}(A)$, $T_{n-i-1}(A)$, $T_{i-1}(M)$, $T_{n-i-1}(M)$ respectively. Since π^{n-1} maps $F^n(A)$ onto 0, the equality above is a consequence of the following equalities

$$\begin{aligned}(Id - \sigma) \circ \pi^n &= (Id - \sigma) \circ T_n(\pi) \circ \Delta^n = T_n(\pi) \circ (Id - \sigma) \circ \Delta^n \\ &= T_n(\pi) \circ (Id - \sigma) \circ (I' \otimes \Delta \otimes I'') \circ \Delta^{n-1} \\ &= T_n(\pi) \circ (I' \otimes [(Id - \tau) \circ \Delta] \otimes I'') \circ \Delta^{n-1} \\ &= (T_{i-1}(\pi) \otimes [(\pi \otimes \pi) \circ (Id - \tau) \circ \Delta] \otimes T_{n-i-1}(\pi)) \circ \Delta^{n-1} \\ &= (T_{i-1}(\pi) \otimes (\mu \circ \pi) \otimes T_{n-i-1}(\pi)) \circ \Delta^{n-1} \\ &= (J' \otimes \mu \otimes J'') \circ T_{n-1}(\pi) \circ \Delta^{n-1} = (J' \otimes \mu \otimes J'') \circ \pi^{n-1}.\end{aligned}$$

The proposition is proved.

PROPOSITION 13. *Let A be a connected Hopf Γ-algebra with a support M. Then the trace of A in M is a subalgebra of $T(M^\cdot)$.*

Proof. Let us consider two elements a_1 and a_2 of $A^\cdot$. Further let us consider the following element

$$\begin{aligned}\pi^n \circ \Phi(a_1 \otimes a_2) &= T_n(\pi) \circ \Delta^n \circ \Phi(a_1 \otimes a_2) \\ &= T_n(\pi) \circ {}_n\Phi[\Delta^n(a_1) \otimes \Delta^n(a_2)] \\ &= {}_n\Phi[T_n(\pi) \circ \Delta^n(a_1) \otimes T_n(\pi) \circ \Delta^n(a_2)].\end{aligned}$$

By Lemma 23, the elements

$$T_n(\pi) \circ \Delta^n(a_i) \qquad \text{and} \qquad T_n(\pi) \circ \bar{\Phi}(\Delta^{p_i}(a_i) \otimes 1_{n-p_i})$$

have the same components in $T_{n,p_i}(M)$. Now we can apply Lemma 24. The element

$${}_n\Phi[T_n(\pi) \circ \Delta^n(a_1) \otimes T_n(\pi) \circ \Delta^n(a_2)]$$

and the element

$$\sum_{p_1+p_2=m} {}_n\Phi[T_n(\pi) \circ \bar{\Phi}(\Delta^{p_1}(a_1) \otimes 1_{n-p_1}) \otimes T_n(\pi) \circ \bar{\Phi}(\Delta^{p_2}(a_2) \otimes 1_{n-p_2})]$$

have the same components in $T_{n,m}(M)$. But we have the following equalities

$$\begin{aligned}T_n(\pi) \circ \bar{\Phi}(\Delta^{p_i}(a_i) \otimes 1_{n-p_i}) &= \bar{\Phi}(T_{p_i}(\pi) \circ \Delta^{p_i}(a_i) \otimes 1_{n-p_i}) \\ &= \bar{\Phi}(\pi^{p_i}(a_i) \otimes 1_{n-p_i}).\end{aligned}$$

In summary the elements

$$\pi^n \circ \Phi(a_1 \otimes a_2) \quad \text{and} \quad \sum_{p_1+p_2=m} {}_n\Phi[\bar{\Phi}(\pi^{p_1}(a_1) \otimes 1_{n-p_1}) \otimes \bar{\Phi}(\pi^{p_2}(a_2) \otimes 1_{n-p_2})]$$

have the same components in $T_{n,m}(M)$. Now we can apply Lemmas 24 and 25. The component in $T_{n,m}(M)$ of the element

$$_n\Phi[\overline{\Phi}(\pi^{p_1}(a_1) \otimes 1_{n-p_1}) \otimes \overline{\Phi}(\pi^{p_2}(a_2) \otimes 1_{n-p_2})]$$

is equal to the component in $T_{n,m}(M)$ of the element

$$\Phi^3[\pi^{p_1}(a_1) \otimes \pi^{p_2}(a_2) \otimes 1_{n-m}].$$

Consequently the elements

$$\pi^n \circ \Phi(a_1 \otimes a_2) \qquad \text{and} \qquad \sum_{p_1+p_2=m} \overline{\Phi}^3[\pi^{p_1}(a_1) \otimes \pi^{p_2}(a_2) \otimes 1_{n-m}]$$

have the same components in $T_{n,m}(M)$.

Now let us suppose that a_i is an element of $F^{r_i}(A)$. The preceding result for $n = r_1 + r_2 - 1$ shows that $\Phi(a_1 \otimes a_2)$ is an element of $F^{r_1+r_2}(A)$. The preceding result for $n = r_1 + r_2$ proves the equality

$$\pi^{r_1+r_2} \circ \Phi(a_1 \otimes a_2) = \overline{\Phi}[\pi^{r_1}(a_1) \otimes \pi^{r_2}(a_2)].$$

In other words the product of the element $\pi^{r_1}(a_1)$ of the r_1-trace and of the element $\pi^{r_2}(a_2)$ of the r_2-trace is equal to the element $\pi^{r_1+r_2} \circ \Phi(a_1 \otimes a_2)$ of the $(r_1 + r_2)$-trace. Then the trace of A in M is a subalgebra of $T(M^{\cdot})$.

PROPOSITION 14. *Let A be a connected Hopf Γ-algebra with a support M. Then for any $k \geqslant 0$ and for any element x of $M_+^{\cdot}$, the element $\tilde{\gamma}^k(x) = x \otimes \cdots \otimes x$ of $T_k(M^{\cdot})$ belongs to the k-trace of A in M.*

Proof. Let us consider an element a of $A_+^{\cdot}$ and let us use the following decomposition

$$\pi^n(a) = x_1 + \cdots + x_n\,, \qquad x_j \in T_{n,j}(M).$$

Further let us consider the following element

$$\begin{aligned}\pi^n \circ \gamma^k(a) &= T_n(\pi) \circ \Delta^n \circ \gamma^k(a) = T_n(\pi) \circ {}_n\gamma^k \circ \Delta^n(a) = {}_n\gamma^k \circ T_n(\pi) \circ \Delta^n(a)\\ &= {}_n\gamma^k \circ \pi^n(a) = {}_n\gamma^k(x_1 + \cdots + x_n)\\ &= \sum_{k_1+\cdots+k_n=k} {}_n\Phi^n[{}_n\gamma^{k_1}(x_1) \otimes \cdots \otimes {}_n\gamma^{k_n}(x_n)].\end{aligned}$$

Now we use Lemmas 24 and 26 and we get the following equality

$$\pi^n \circ \gamma^k(a) = \sum_{\substack{k_1+\cdots+k_n=k\\ 1\cdot k_1+\cdots+n\cdot k_n \leqslant n}} {}_n\Phi^n[{}_n\gamma^{k_1}(x_1) \otimes \cdots \otimes {}_n\gamma^{k_n}(x_n)].$$

Now let us suppose that $n = k - 1$. Then the equation and the inequality

$$k_1 + \cdots + k_n = k \qquad \text{and} \qquad 1 \cdot k_1 + \cdots + n \cdot k_n \leqslant k - 1$$

have no common solution and consequently

$$\pi^{k-1} \circ \gamma^k(a) = 0.$$

Now let us suppose that $n = k$. Then the equation and the inequality

$$k_1 + \cdots + k_n = k \qquad \text{and} \qquad 1 \cdot k_1 + \cdots + n \cdot k_n \leqslant k$$

have a unique common solution

$$k_1 = k \qquad \text{and} \qquad k_2 = \cdots = k_n = 0$$

and consequently

$$\pi^k \circ \gamma^k(a) = {}_k\gamma^k(x_1).$$

Now we use Lemmas 23 and 27 and we get the following equalities

$${}_k\gamma^k(x_1) = {}_k\gamma^k \circ \bar{\Phi}(\pi(a) \otimes 1_{k-1}) = \bar{\gamma}^k \circ \pi(a).$$

In summary

$$\pi^{k-1} \circ \gamma^k(a) = 0 \qquad \text{and} \qquad \pi^k \circ \gamma^k(a) = \bar{\gamma}^k \circ \pi(a).$$

In other words, the element $\bar{\gamma}^k \circ \pi(a)$ belongs to the k-trace. The proposition is proved with $x = \pi(a)$.

PROPOSITION 19. *Let A be a connected Hopf Γ-algebra with a support M. Then for the reduced Lie coalgebra $L = M^{\cdot}$, the canonical homomorphism $\int p : A \to U(L)$ of Proposition 5 is a homomorphism of Hopf Γ-algebras.*

Proof. By Proposition 5, we already know that $\int p$ is a homomorphism of coalgebras. There remains to prove that it is a homomorphism of Γ-algebras. The explicit definition of $\int p$ appears in the proof of Proposition 5:

$$\int p = \sum p^k \qquad \text{and} \qquad p^k = T_k(p) \circ \Delta^k.$$

Now let us use the following result appearing in the proof of Proposition 13. Proposition 13. For two elements a_1 and a_2 of $A\cdot$, the elements

$$\pi^n \circ \Phi(a_1 \otimes a_2) \qquad \text{and} \qquad \sum_{p_1+p_2=m} \bar{\Phi}^3[\pi^{p_1}(a_1) \otimes \pi^{p_2}(a_2) \otimes 1_{n-m}]$$

have the same components in $T_{n,m}(M)$. But $T_{n,n}(M)$ is equal to $T_n(L)$ and for $m = n$, the preceding result implies the following equality

$$p^n \circ \Phi(a_1 \otimes a_2) = \sum_{p_1+p_2=n} \bar{\Phi}[p^{p_1}(a_1) \otimes p^{p_2}(a_2)].$$

In other words

$$\int p \circ \Phi = \bar{\Phi} \circ \left[\int p \otimes \int p\right].$$

The homomorphism $\int p$ of coalgebras is a homomorphism of algebras.

Now let us use the following result appearing in the proof of Proposition 14. For an element a of $A_+^{\cdot}$ with

$$\pi^n(a) = x_1 + \cdots + x_n\,, \qquad x_j \in T_{n,j}(M),$$

the elements

$$\pi^n \circ \gamma^k(a) \qquad \text{and} \qquad \sum_{k_1+\cdots+k_n=k} {}_n\Phi^n[{}_n\gamma^{k_1}(x_1) \otimes \cdots \otimes {}_n\gamma^{k_n}(x_n)]$$

are equal. Let us use the decomposition

$$\Delta^n(a) = \Delta_1{}^n(a) + \cdots + \Delta_n{}^n(a), \qquad \Delta_j{}^n(a) \in T_{n,j}(A),$$

and Lemmas 23 and 27. We get the following equalities

$$\begin{aligned}
{}_n\gamma^h(x_j) &= {}_n\gamma^h \circ T_n(\pi) \circ \Delta_j{}^n(a) \\
&= {}_n\gamma^h \circ T_n(\pi) \circ \bar{\Phi}(\Delta_j{}^j(a) \otimes 1_{n-j}) = {}_n\gamma^h \circ \bar{\Phi}(T_j(\pi) \circ \Delta_j{}^j(a) \otimes 1_{n-j}) \\
&= {}_n\gamma^h \circ \bar{\Phi}(p^j(a) \otimes 1_{n-j}) = \bar{\Phi}(\bar{\gamma}^h \circ p^j(a) \otimes 1_{n-hj}).
\end{aligned}$$

Consequently we have the following equalities

$$\begin{aligned}
&\pi^n \circ \gamma^k(a) \\
&= \sum_{k_1+\cdots+k_n=k} {}_n\Phi^n[\bar{\Phi}(\bar{\gamma}^{k_1} \circ p^1(a) \otimes 1_{n-1\cdot k_1}) \otimes \cdots \otimes \bar{\Phi}(\bar{\gamma}^{k_n} \circ p^n(a) \otimes 1_{n-n\cdot k_n})] \\
&= \sum_{k_1+\cdots+k_n=k} \bar{\Phi}^{n+1}[\bar{\gamma}^{k_1} \circ p^1(a) \otimes \cdots \otimes \bar{\gamma}^{k_n} \circ p^n(a) \otimes 1_{n-1\cdot k_1-\cdots-n\cdot k_n}]
\end{aligned}$$

the last one by means of Lemma 25. The preceding equality for the component $T_{n,n}(M)$ equal to $T_n(L)$ can be rewritten in the following way

$$p^n \circ \gamma^k(a) = \sum_{\substack{k_1+\cdots+k_n=k \\ 1\cdot k_1+\cdots+n\cdot k_n=n}} \Phi^n[\bar{\gamma}^{k_1} \circ p^1(a) \otimes \cdots \otimes \bar{\gamma}^{k_n} \circ p^n(a)].$$

In other words

$$\int p \circ \gamma^k(a) = \bar{\gamma}^k \circ \int p(a).$$

The homomorphism $\int p$ of Hopf algebras is a homomorphism of Γ-algebras. The proposition is proved.

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