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DIVIDED POWERS AND KÄHLER DIFFERENTIALS

ASEEL KMAIL, JULIA KOZAK, AND HAYNES MILLER

ABSTRACT. Divided power algebras form an important variety of non-binary universal algebras. We identify the universal enveloping algebra and Kähler differentials associated to a divided power algebra over a general commutative ring, simplifying and generalizing work of Roby and Dokas.

1. Introduction

Divided power algebras were introduced by Henri Cartan [3] to describe the homology of Eilenberg Mac Lane spaces. They have subsequently been intensively studied [10] and have played important roles in other parts of mathematics, such as algebraic geometry, where they form the basis of the construction of crystalline cohomology [2]. They constitute an important example of a variety of algebras that are "non-binary," in the sense that their structure is not entirely encoded in a binary product.

In [8], Dan Quillen described a uniform process for defining a cohomology theory in any one of a wide class of algebraic structures. An important role in that construction is played by the category of "Beck modules" (see [1], for example) for an algebra A in a specified variety $\mathbf{V}$. This is the category, recognized long ago by Sammy Eilenberg [7] as providing the appropriate meaning of a "representation" in a general context, consists of the abelian objects in the slice category $\mathbf{V}/A$. In linear cases it can be identified with the category of modules over a unital associative algebra $U(A)$, the "universal enveloping algebra" of A. Beck modules form the coefficients in the Quillen cohomology theory defined on $\mathbf{V}$, and in fact Quillen homology is an appropriately defined derived functor of the abelianization functor, evaluated on the terminal object $1_A : A \downarrow A$ in $\mathbf{V}/A$. If $\mathbf{V}$ is the variety of commutative rings, $\mathrm{Ab}_A(1_A)$ is the A-module of Kähler differentials, and this suggests defining $\Omega_A^{\mathbf{V}} = \mathrm{Ab}_A(1_A)$ in a general variety of algebras $\mathbf{V}$.

This construction has been considered in detail by Ionnnis Dokas [6, 5] in case one is working with divided power algebras over a field. The goal of the present paper is to show how this works out over a general ring. Dokas proves the interesting result that if A is a DP algebra over a field then the module of divided power Kähler differentials is simply a DP Beck module structure on the usual commutative algebra module of Kähler differentials.

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Arab-American University, Jenin

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RESEARCH ARTICLE

The André–Quillen cohomology of commutative monoids

Bhavya Agrawalla[1] · Nasief Khlaif[2] · Haynes Miller[1]

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Abstract
We observe that Beck modules for a commutative monoid are exactly modules over a graded commutative ring associated to the monoid. Under this identification, the Quillen cohomology of commutative monoids is a special case of the André–Quillen cohomology for graded commutative rings, generalizing a result of Kurdiani and Pirashvili. To verify this we develop the necessary grading formalism. The partial cochain complex developed by Pierre Grillet for computing Quillen cohomology appears as the start of a modification of the Harrison cochain complex suggested by Michael Barr.

Keywords Commutative monoid · Harrison homology · Quillen cohomology

In his book *Homotopical Algebra* [34], Daniel Quillen described a homotopy theory of simplicial objects in any of a wide class of universal algebras, and corresponding theories of homology and cohomology. Quillen homology is defined as derived functors of an abelianization functor and in many cases can be computed using a cotriple resolution [4]. Coefficients for these theories are "Beck modules," that is, abelian objects in a slice category. The case of commutative rings was studied at the same time by Michel André [1].

An example of such an algebraic theory, one of long standing and increasing importance, is provided by the category **ComMon** of commutative monoids. The prime

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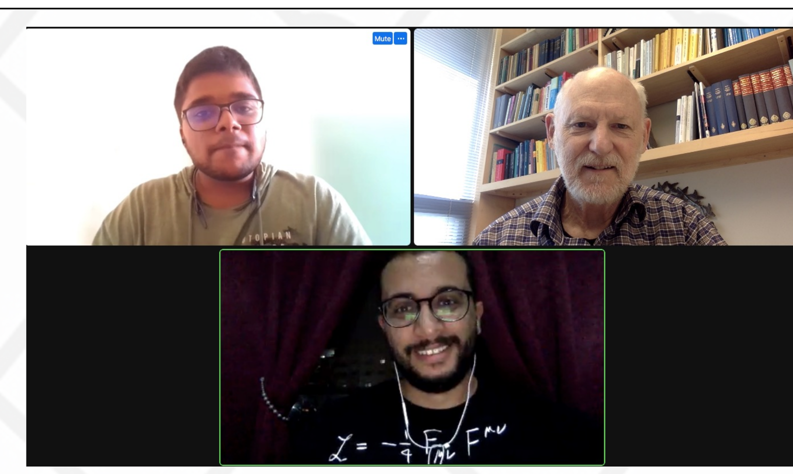
✉ Haynes Miller
hrm@math.mit.edu

Bhavya Agrawalla
bhavya@mit.edu

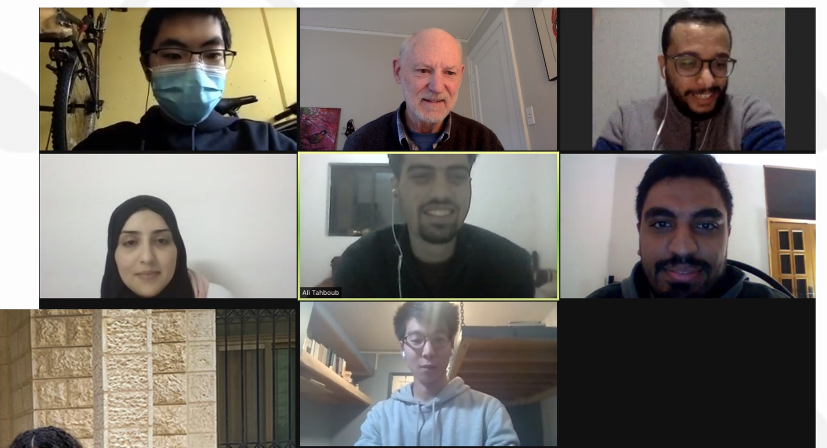
Nasief Khlaif
nkhlaif@birzeit.edu

1 Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

2 Department of Mathematics, Birzeit University, Ramallah, Palestine



Birzeit University, Ramallah



STEPS IN ANDERSON-BADAWI'S CONJECTURE ON N-ABSORBING AND STRONGLY N-ABSORBING IDEALS

MATIJA DELIC AND KHALID ADARBEH (*)

ABSTRACT. This article aims to solve positively Anderson-Badawi Conjecture of n-Absorbing and strongly n-absorbing ideals of commutative rings in the class of u-rings. The main result extends and recovers Anderson-Badawi's related result on Prufer domains [1, Corollary 6.9].

1. Introduction

Throughout this article, R denotes a commutative ring with $1 \neq 0$. In 2007, A. Badawi introduced the concept of 2-absorbing ideals of commutative rings as a generalization of prime ideals. He defined an ideal I of R to be 2-absorbing if whenever $a, b, c \in R$ and $abc \in I$, then ab or ac or bc is in I [2]. As in the case of prime ideals, 2-absorbing have a characterization in terms of ideals. Namely, I is 2-absorbing if whenever I_1, I_2, I_3 are ideals of R and $I_1I_2I_3 \subseteq I$, then I_1I_2 or I_1I_3 or I_2I_3 is contained in I [2, Theorem 2.13].

In 2011, D.F. Anderson, A. Badawi inspired from the definition of 2-absorbing ideals and defined the n-absobing ideals for any positive integer n. Where an ideal I is called n-absorbing ideal if whenever $x_1 \ldots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$ then there are n of the x_i's whose product is in I. Also they introduced the strongly-n-absorbing ideals as another generalization of prime ideals, where an ideal I of R is said to be a strongly n-absorbing ideal if whenever $I_1 \ldots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, , I_{n+1}$ of R, then the product of some n of the I_j's is contained in I. Obviously, a strongly n- absorbing ideal of R is also an n-absorbing ideal of R, and by the last fact in the previous paragraph, 2-absorbing and strongly 2 absorbing are the same. Moreover D.F. Anderson, A. Badawi were able to prove that n-absorbing and strongly n-absorbing are equivalent in the class of Prufer domains [1, Corollary 6.9], and they conjectured that these two concepts are equivalent in any commutative ring [1, Conjecture 1].

In 1975, Jr. P. Quartararo and H.S. Butts defined the u-rings to be those rings in which if an ideal I is contained in the union of ideals, then it must

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(*) Corresponding author.

An-Najah National University, Nablus

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Beck modules and alternative algebras

Nishant Dhankhar[a], Haynes Miller[a], Ali Tahboub[b], and Victor Yin[a]

[a]Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; [b]Department of Mathematics, Birzeit University, Ramallah, Palestine

ABSTRACT
We set out the general theory of "Beck modules" in a variety of binary algebras and describe them as modules over suitable "universal enveloping" unital associative algebras. We develop a theory of "noncommutative partial differentiation" to pass from the equations of the variety to relations in a universal enveloping algebra. We pay particular attention to the case of alternative algebras, defined by a restricted associative law, and determine the Poincaré polynomial of the universal enveloping algebra in the abelian case.

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1. Introduction

The notion of a "module" occupies an important place in the study of general algebraic systems. Most of these diverse notions are united under the theory of "Beck modules." Given an object A in any category $\mathbf{C}$, one may consider the "slice category" $\mathbf{C}/A$ of objects in $\mathbf{C}$ equipped with a map to A. A Beck module for A is then an abelian group object in $\mathbf{C}/A$. If $\mathbf{C}$ is the category of commutative rings, for example, a Beck module for A is simply an A-module, while if $\mathbf{C}$ is the category of associative algebras, a Beck module for A is an A-bimodule. Many other examples occur in the literature: Leibniz algebras [19], λ-rings [15], divided power rings [9], ….

This definition occurs in the thesis [5] of Jonathan Beck written under the direction of Samuel Eilenberg. Eilenberg himself had discussed such objects in [11], at least in the linear context, as the kernel of a "square zero extension." These kernels were understood to constitute "representations" of the algebra, and this structure was made explicit in various cases.

We review below the context of a "variety" $\mathbf{V}$ of algebras over a commutative ring K. In this case, for every $\mathbf{V}$-algebra A the category $\mathbf{Mod}_A$ of Beck A-modules is an abelian category with a single small projective generator. As a result, the category $\mathbf{Mod}_A$ is equivalent to the category of right modules over a canonical unital associative K-algebra $U_{\mathbf{V}}(A)$, the "universal enveloping algebra" for A.

This raises the question of identifying the structure of $U_{\mathbf{V}}(A)$ for various varieties $\mathbf{V}$ and $\mathbf{V}$-algebras A. Left and right multiplication determine a K-module map $A \oplus A \to U_{\mathbf{V}}(A)$, and hence a surjection of associative unital K-algebras $\mathrm{Tens}_K(A \oplus A) \to U_{\mathbf{V}}(A)$ (cf. [19]). Each defining equation determines a generator of the kernel of this map, by a process of "noncommutative differentiation" that we describe in detail.

We review some of the standard examples, and then focus on a somewhat less standard one, the variety of "alternative algebras" over K. This example has been considered before, but even over a field

CONTACT Haynes Miller hrm@math.mit.edu Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA.

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The Hermitian axiom on two-dimensional topological quantum field theories

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Honglin Zhu[a)]

AFFILIATIONS
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

[a)]Author to whom correspondence should be addressed: honglinz@mit.edu

ABSTRACT
We examine Atiyah's Hermitian axiom for two-dimensional complex topological quantum field theories. Building on the correspondence between 2D topological quantum field theories (TQFTs) and Frobenius algebras, we find the algebraic objects corresponding to Hermitian and unitary TQFTs, respectively, and prove structure theorems about them. We then clarify a few older results on unitary TQFTs using our structure theorems.

I. INTRODUCTION

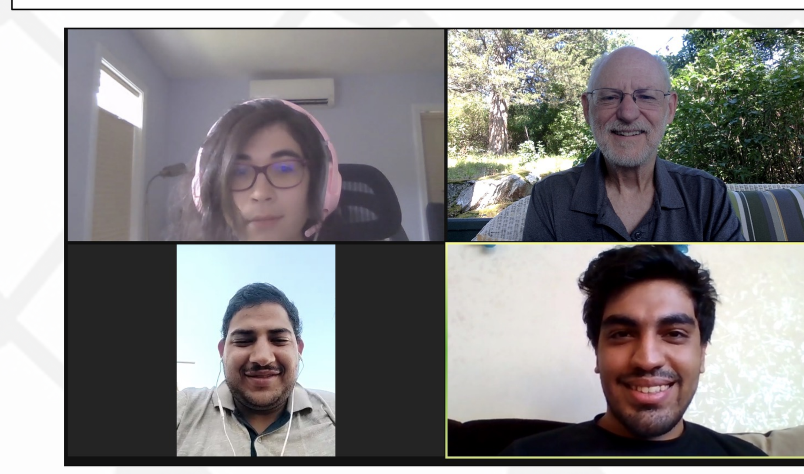
A topological quantum field theory (TQFT) in dimension d, by Atiyah's axiomatic formulation in Ref. 1, is a rule Z which assigns a vector space $Z(\Sigma)$ to each closed oriented $(d-1)$-dimensional manifold Σ (throughout this paper, all manifolds are understood to be compact, smooth, and orientable) and to each oriented smooth d-dimensional manifold M with boundary Σ an element $Z(M) \in Z(\Sigma)$ (see Sec. II).

Another model of a d-dimensional TQFT is to view it as a linear representation of $dCob$, the category of d-dimensional cobordisms. A cobordism from an oriented $(d-1)$-dimensional manifold Σ_1 to another oriented $(d-1)$-dimensional manifold Σ_2 is an oriented d-dimensional manifold M whose boundary is $\Sigma_1^* \cup \Sigma_2$, that is, the orientation of M matches that of the out-boundary Σ_2 and is the opposite of the orientation of the in-boundary Σ_1. We will write Σ^* for the $(d-1)$-dimensional manifold Σ with the orientation reversed. For a cobordism M from Σ_1 to Σ_2, we will write M^* for the manifold M with its orientation reversed. By the involutory axiom and the multiplicative axiom, the element $Z(M)$ can be regarded as a homomorphism from $Z(\Sigma_1)$ to $Z(\Sigma_2)$. Thus, a cobordism M from Σ_1 to Σ_2 can be viewed as a linear map from $Z(\Sigma_1)$ to $Z(\Sigma_2)$. For more discussions on the cobordism category $dCob$ and its connection to TQFTs, see Ref. 2.

In dimension two, there is a fascinating result that establishes an equivalence of categories between two-dimensional TQFTs over a field F and Frobenius algebras over F. Thus, for each 2D TQFT, there is a corresponding Frobenius algebra, and conversely, each Frobenius algebra gives rise to a 2D TQFT. This gives us a third way to understand TQFTs, and in particular, it allows us to use algebraic tools to study something geometric in nature. For more information, Kock offers a detailed examination of 2D TQFTs and Frobenius algebras in Ref. 2.

In the case of TQFTs over the complex numbers, when the vector spaces $Z(\Sigma)$ are equipped with nondegenerate Hermitian forms, Atiyah proposed an extra axiom [Eq. (1)], which relates a cobordism M to M^*, namely, that the linear map $Z(M)$ is the adjoint of $Z(M^*)$. We call a TQFT satisfying this axiom a Hermitian TQFT, and a Hermitian TQFT with positive-definite Hermitian forms a unitary TQFT (Theorem III.2).

The main focus of this paper is to relate the Hermitian and unitary conditions in each of the three models for TQFTs. Starting from Atiyah's axioms, we will derive the extra algebraic structures that the Hermitian and unitary conditions impose on the corresponding Frobenius algebras and clarify various arguments in the literature.



Palestine Polytechnic University, Hebron