Homotopy Theory of Spheres Seminar

I - Early Theory - Before Adams' Spectral Seq.

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Primary Structures

These are homotopy operations which were important in the study of the homotopy groups of spheres and which are now also of intrinsic interest.

Composition

Simple composition of representative maps gives a map

\[ \Pi_p(S^q) \times \Pi_q(S^r) \rightarrow \Pi_p(S^r) \]

\[ \beta \circ \alpha \rightarrow \alpha \circ \beta \]

Note that we do not have a \( \otimes \) sign, if

\[ \beta, \beta' \in \Pi_p(S^q) \]

\[ \alpha, \alpha' \in \Pi_q(S^r) \]

then

\[ (\alpha + \alpha') \circ \beta \neq \alpha \circ \beta + \alpha' \circ \beta \]

in general, although

\[ \alpha \circ (\beta + \beta') = \alpha \circ \beta + \alpha \circ \beta' \]

Whitehead Product

This is a pairing

\[ \Pi_p(X) \times \Pi_q(X) \rightarrow \Pi_{p+q-1}(X) \]

\[ \alpha \otimes \beta \rightarrow [\alpha, \beta] \]
defined as follows:
If
\[ [t] = \alpha \in \pi_p(X) \]
\[ [g] = \beta \in \pi_q(X) \]
then \([\alpha, \beta]\) is the class of the map
\[ S^{p+q-1} \xrightarrow{u_{p,q}} S^p \vee S^q \xrightarrow{f_{ \nu g}} X \wedge X \xrightarrow{\text{fold}} X \]

\(u_{p,q}\) may be described by

\[
\begin{array}{cc}
S^{p+q-1} & \\
\downarrow & \\
(S^{p-1} \times D^q) \cup (D^p \times S^{q-1}) & \\
\downarrow & \downarrow \\
D^p \vee D^q & D^p \vee D^q \\
\downarrow & \downarrow \\
S^p \vee S^q & S^p \vee S^q
\end{array}
\]

We use the bracket notation because the Whitehead product satisfies anticommutativity and 'Jacobi' identities similar to those of a Lie product in a Lie ring.

Join

This is a pairing
\[ \pi_p(S^m) \times \pi_q(S^n) \rightarrow \pi_{p+q+1}(S^{m+n+1}) \]

\[ \alpha \otimes \beta \rightarrow \alpha \ast \beta \]
defined as follows:
If
\[ \alpha \in \Pi^p_m(S^n) \]
\[ \beta \in \Pi^q_S(S^n) \]
then \( \alpha \ast \beta \) is the equivalence class of the suspension (see below) of
\[ S^{p+q} \simeq S^p \# S^q \xrightarrow{f \# g} S^m \# S^n \simeq S^{m+n} \]
\( (A \# B = A \times B / A \vee B) \)

**Suspension**

The suspension functors are well known. We prefer to use the reduced suspension:
\[ S^X = X \times I / X \times 1 \cup X_0 \times I \]

The points of \( S^X \) are pairs \((x, t) \in X \times I\) under proper identification.

Given \( \alpha : X \to Y \)
\[ S\alpha : S^X \to S^Y \]
sends \((x, t)\) into \((\alpha x, t)\)

We give a second 'adjoint' point of view.
Let \( \Delta X \) denote the Moore loop space of \( X \); the points of \( \Delta X \) are the maps
\([0,r] \to X\) such that \(0 \leq r < \infty\) and \(0, r\) go into \(x_0\).

Now let \(\lambda\) be a non-negative continuous function on \(SX\) positive except at the base point.

We define a map
\[
\iota : X \to \Lambda SX
\]
by
\[
\iota(x) : [0, \lambda(x)] \to SX
\]
by \(\iota(x)(t) = (x, t/\lambda(x))\).

We give also a third point of view of suspension due to Blakers and Massey:

The difficulty of homotopy groups is that they do not obey excision, i.e., given \(A, B \subset X\), the maps

\[
\pi_r(A, A \cap B) \to \pi_r(X, B)
\]

\[
\pi_r(B, B \cap A) \to \pi_r(X, A)
\]

induced by inclusion are not, in general, isomorphisms. Blakers and Massey defined homotopy groups of a triad which measure this failure to be an isomorphism; that is, there are long exact sequences.
Let $S^n = E^{n\uparrow} \cup E^{n\downarrow}$ be the usual decomposition of the sphere into hemispheres. In the diagram

\[
\begin{array}{cccc}
\pi_r(E^{n\uparrow}, S^n) & \to & \pi_r(S^{n+1}, E^{n\downarrow}) & \to & \pi_r(S^{n+1}, E^{n\downarrow}) \\
\downarrow S & & & & \uparrow S \\
\pi_{r-1}(S^n) & \cong & \pi_r(S^{n+1}) & \\
\end{array}
\]

$E$ is (up to sign) the Freudenthal suspension.

A crude form of the Freudenthal theorem asserts that

\[
\pi_r(S^n) \cong \pi_{r+n}(S^{n+1})
\]

is an isomorphism (onto) if $r \leq 2n-2$

and is onto if $r = 2n-1$.

This just says that $\pi_r(S^{n+1}, E^{n\uparrow}, E^{n\downarrow}) = 0$ if $r \leq 2n-2$. 

Range

The Freedman-Thurston Theorems lead naturally to the concept of range. The group $\Pi_r(S^h)$ is said to be in the stable range if $r \leq 2n-2$.

In this case $E^m : \Pi_r(S^h) \to \Pi_{r+m}(S^{h+m})$ will always be an isomorphism.

If $2n-1 \leq r \leq 3n-2$ then $\Pi_r(R)$ is said to be in the metastable range; if $3n-2 < r$, premetastable. The significance of metastability will emerge later.

The Hopf Construction

If $[A, B]$ denotes the set of homotopy classes of maps $f: A \to B$ then the Hopf construction is a map

$$[S^p \times S^q, X] \to \Pi_{p+q+1}(SX)$$

defined by composition of a representative of $S^p \times S^q$ with $S^{p+q+1}, S^{p+q+1} \nu S^{q+1} \nu S^p \xrightarrow{i} S(S^p \times S^q)$

where $i$ is a homotopy equivalence.
Toda Brackets

These are 'higher order' composition operations.
Suppose we have
$$SP \overset{f}{\to} S^q \overset{g}{\to} S^r \overset{h}{\to} S^t$$
such that
$$(*) \quad kg = 0 \quad \text{and} \quad gf = 0$$

Then we may construct a diagram

$$\begin{array}{ccc}
S^p & \xrightarrow{f} & S^q \\
\downarrow & & \downarrow \\
S^{p+1} & \xrightarrow{g} & S^r \\
\downarrow & & \downarrow \\
C^+S^p & \xrightarrow{G} & C^+S^t \\
\end{array}$$

Where $F, G$ are homotopies for $kg = 0, gf = 0$ respectively. It is trivial to verify that $F \circ f$ and $h \circ G$ agree on $SP$.

and so fit together to give a continuous map $S^{p+1} \to S^t$; this map has a homotopy class which is well defined only modulo a certain subgroup of $\pi_{p+1}(S^t)$; we get a map
(subset of $\pi_p(S^2) \times \pi_q(S^3) \times \pi_r(S^6) \rightarrow \pi_{p+q+r}(S^2)$)

The Hopf Invariant

Let $f: S^{2n-1} \rightarrow S^n$. If $y \in S^n$ then $f$ is homotopic to a map $g$ such that $g^{-1}(y)$ is a smooth submanifold of $S^{2n-1}$. For any two distinct points $y_1, y_2 \in S^n$ we may therefore assume $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are such subsets of $S^{2n-1}$ and define the Hopf invariant of $f$ to be their linking number.

More algebraically let $Y = S^n U e^{2n}$ be the union of the $n$-sphere and a $2n$-cell attached to it via the map $f$ (which is defined on the boundary $S^{2n-1}$ of $e^{2n}$). Then

$H^1(Y) \cong \mathbb{Z}$ generated by $u$

$H^{2n}(Y) \cong \mathbb{Z}$

One defines (up to sign) the Hopf invariant of $f$ to be the integer $H(f)$ given by

$u \cup u = H(f) \cdot v$. 
Defn. A map 
\[ f : S^p \times S^q \to S^n \]
is said to be of type \((\alpha, \beta)\)
\[ \alpha \in \pi_p(S^n), \beta \in \pi_q(S^n) \]
if
\[ [f \circ \iota_1] = \alpha \]
\[ [f \circ \iota_2] = \beta \]

where \( \iota_1 : S^p \to S^p \times S^q \)
\( \iota_2 : S^q \to S^p \times S^q \)
are the natural injections.

Suppose \( S^{n-1} \times S^{n-1} \xrightarrow{f} S^{n-1} \) is of type \((r, s)\) where \( r, s \) are integers and \( c \) is a generator of \( \pi_{n-1}(S^{n-1}) \). One proves that for the Hopf construction \( c(f) \) of \( f \),
\[ H(c(f)) = r \cdot s \] (up to sign).

Using this Hopf showed that \( \pi_3(S^2) \) is \( \simeq \mathbb{Z} \) and is generated by his famous fibering map \( S^3 \to S^5 \) (see e.g. Steenrod, Topology of Fiber Bundles). There are similar fiberings \( S^7 \to S^4 \) and \( S^15 \to S^8 \).

G.W. Whitehead generalized the Hopf invariant as follows: he viewed the Hopf invariant as a homomorphism \( H \).
H: $\tilde{\Pi}_{2n-1}(S^n) \rightarrow \mathbb{Z} = \tilde{\Pi}_{p-1}(S^{n-1})$
and sought to find similar homomorphisms of homotopy groups. In the exact homotopy sequence of the pair $(S^p \times S^q, S^p \vee S^q)$ one gets a splitting in every dimension via the maps induced by $\iota, \cdot p$, and $\iota_2 \cdot p_2$ in the diagram

$$
\begin{array}{c}
S^p \times S^q \\
\downarrow \iota_2 \cdot p_2
\end{array}
\begin{array}{c}
\iota \cdot p \\
\downarrow \iota \cdot p
\end{array}
\begin{array}{c}
S^p \vee S^q
\end{array}

This yields short exact sequences:

$$
0 \rightarrow \tilde{\Pi}_{r+1}(S^p \times S^q, S^p \vee S^q) \rightarrow \tilde{\Pi}_r(S^p \vee S^q) \rightarrow \tilde{\Pi}_r(S^p \times S^q) \rightarrow 0
$$

What we are interested in is the specific projection that this induces,

$$
\tilde{\Pi}_{r+1}(S^p \times S^q, S^p \vee S^q) \rightarrow \tilde{\Pi}_r(S^p \vee S^q).
$$

Now let $p = q = n$; one gets a commutative diagram as below if $E^3$ is an isomorphism that is, if $r = 2 \leq 2(2n-3) - 2$

$$
\begin{array}{c}
\tilde{\Pi}_{r+1}(S^h \times S^n, S^h \vee S^n) \\
\downarrow \text{smash}
\end{array}
\begin{array}{c}
\tilde{\Pi}_{r-1}(S^2, S^h) \\
\downarrow \text{pinch}
\end{array}
\begin{array}{c}
\pi^* E^3
\end{array}
\rightarrow \text{Generalized Hopf invariant}

\tilde{\Pi}_{r-2}(S^{2h-3}) \leftarrow \pi^n \tilde{\Pi}_r(S^h)$$
The point is to get a map from the \((r-n)\)-stem into, roughly, the \((r-2n)\)-stem. Whitehead proved

1) if \(f : S^r \times S^q \to S^h\) is of type \((\alpha, \beta)\) and \(c(f) \in T_{p+q+1}(S^{h+1})\), then a certain suspension of \(c(f)\) has homotopy class \(\pm \alpha \cdot \beta\).

2) the homomorphism (we call it \(H(f)\)) associated with such an \(f\) can be fit into an exact sequence starting with about 

\[ \pi_{3n-1}(S^h) \]

Replacing \(\Lambda S^X\) by \(X^\infty\)

For a space \(X\) with base point \(x_0\) and for any integer \(n\) we define the reduced product space \(X^n\) as an identification space of \(X^n\) (the \(n\)-fold cartesian product of \(X\)), identifying two \(n\)-tuples if the ordered \(n\)-tuples obtained by deleting all occurrences of \(x_0\) are identical. There is a natural inclusion \(X^n \subseteq X^{n+1}\) for each \(n\). Let \(X^\infty\) be the direct limit of the \(X^n\) with suitable topology.

We recall that we previously defined a map

\[ X \to \Lambda S^X \]

We will define a map

\[ X^\infty \to \Lambda S^X \]

by sending a point \((x_1, \ldots, x_n) \in X^n\) (representing \(X^n\))
into the Moore loop

\((X_1, X_2, \ldots, X_n)\) \quad [\text{using the usual multiplication}] \\
[\text{in Moore loop spaces}] \\
\text{Since } L(x_0) \text{ is the identity in } \Lambda S^n, \text{ we respect the identifications made in defining } X_n \text{ for each } n. \text{ From } W \text{ we get a map}

\[(X_\infty, X) \rightarrow (\Lambda S^n X, X)\]

\(X\) being identified on the one hand with \(X_1 \circ X_\infty\) and on the other hand with \(L(X) \subset \Lambda S^n X\). James proved that this map induces an isomorphism of long exact homotopy sequences. We replace some terms in the sequence of \((\Lambda S^n X, X)\) via the diagram

\[
\begin{array}{cccccc}
\cdots & \rightarrow & \pi_r(X) & \rightarrow & \pi_r(X_\infty) & \rightarrow & \pi_r(X_\infty, X) & \rightarrow & \pi_{r-1}(X) & \cdots \\
\downarrow \quad \Downarrow & \downarrow & \quad \downarrow \quad \Downarrow & \downarrow & \quad \downarrow \quad \Downarrow & \downarrow & \quad \downarrow \quad \Downarrow & \downarrow & \quad \downarrow \quad \Downarrow \\
\cdots & \rightarrow & \pi_r(\Lambda S^n X) & \rightarrow & \pi_r(\Lambda S^n X, X) & \rightarrow & \pi_{r-1}(X) & \cdots
\end{array}
\]

Now let \(X = S^n\). \(S^n_2\) is \(S^n \times S^n\) with \((x, x_0), (x_0, x)\) identified. It is easy to see that \(S^n_2 \sim S^n \vee \Sigma S^n\) (use the relationship \(S^n \times S^n = (S^n \vee \Sigma S^n) \cup (S^n - \{x_0\} \times S^n - \{x_0\})\)).

Let \(f: S^n_2 \rightarrow S^{2n}\) be the map which identifies \(S^n\) to a point: \((S^n_2, S^n) \rightarrow (S^{2n}, *)\). We extend this map to a map \((S^n_\infty, S^n) \rightarrow (S^{2n}_\infty, *)\) as follows: if \((x_1, \ldots, x_m) \in X^n\), let \(g\) map the corresponding point of \(S^n_{x_1} \subset S^n_\infty\) into the point of \(S^{2n}_\infty\) corresponding to the \((m^2)\)-tuple
Theorem. For $n$ odd, or $k < 3n-3$ (roughly),
\[ g_x : \Pi_{\nu} (S_{\infty}^n, S^n) \to \Pi_{\nu} (S_{\infty}^{2n}) \]
is an isomorphism. For $n$ even $g_x$ is an isomorphism on the 2-primary components of these groups.

The diagram below comes from the above one by replacing $x$ by $S^n$

\[
\begin{array}{cccc}
\Pi_{\nu + 1}(S^{n+1}) & \rightarrow & \Pi_{\nu} (S_{\infty}^{n+1}, C^+S^n, C^-S^n) & \rightarrow \\
\downarrow & & \downarrow & \\
\Pi_{\nu}(S^n) & \rightarrow & \Pi_{\nu} (\Delta S_{\infty}^n, S^n) & \rightarrow \\
\downarrow & & \downarrow & \\
\Pi_{\nu}(S_{\infty}^n) & \rightarrow & \Pi_{\nu} (S_{\infty}^{2n}, S^n) & \rightarrow \\
\downarrow & & \downarrow & \\
& & \Pi_{\nu}(S_{\infty}^{2n}) & \rightarrow
\end{array}
\]

From this we get a map $H$:

\[
\begin{array}{cccc}
\Pi_{\nu}(S_{\infty}^n, S^n) & \xrightarrow{g_x} & \Pi_{\nu}(S_{\infty}^{2n}) & \\
\Pi_{\nu + 1}(S_{\infty}^{n+1}, C^+S^n, C^-S^n) & \xrightarrow{h} & \Pi_{\nu + 1}(S_{\infty}^{2n+1}) & \\
\Pi_{\nu + 1}(S_{\infty}^{n+1}) & \xrightarrow{H} & \\
\end{array}
\]

This map fits into an exact sequence of James
Indicative of the nature of James' results is the following assertion:

\[ \text{Nil} (S^{n+1}) \] has no element of order \( 2^k \) where

\[ k = \begin{cases} 
  n+1 & \text{(n even)} \\
  2n+2 & \text{(n odd)}
\end{cases} \]

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**Figure Space Methods**

Serre made the following construction: Given a space \( X \), let \( X_0 = X \) and let \( T_1 \) be a universal cover for \( X \) (supposing it exists).

Let \( X_1 = \Omega T_1 \). Continue in this manner;

\( \text{Nil}^{n+1}(X) \cong \text{Nil}_1(X_n) \). Serre uses 2 spectral sequences and gets results like the following:

1. \( \text{Nil}_r(S^n) \) is finite except for \( r = n \)
2. \( r = 4m-1 \)

\[ p \text{Nil}_r(S^{2m}) \cong p \text{Nil}_{r-1}(S^{2m-1}) \oplus p \text{Nil}_r(S^{4m-1}) \]

where the isomorphism is induced by:

\[ (\alpha, \beta) \rightarrow E\alpha + [\ell_{2m}, \ell_{2m}] \cdot \beta \]

\[ S^i \beta \rightarrow S^{4m-1} [\ell_{2m}, \ell_{2m}] S^{2m} \text{ (Whitehead product)} \]

Hilton has also generalized the Hopf invariant using \( \text{Nil}_r(S^n \vee S^n) \).
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Beginning of Adams' Spectral Sequence

This is a real step forward in the use of algebraic and homological, as opposed to geometric methods. Its start was given by the introductions of cohomology operations. The operations we will be concerned with are the Steenrod squares, \( S^i : H^n(X; \mathbb{Z}_2) \to H^{n+i}(X; \mathbb{Z}_2) \) defined for all \( i \geq 0 \) and each space \( X \), with the following properties:

1. \( f : Y \to X \); then \( S^i f^* = f^* S^i \)
2. \( S^0 = \text{identity} \)
3. \( S^n = X \) is \( n \)-dimensional
4. Cartan formula \( S^n x y = \sum_{i=0}^{n} S^{n-i} x S^i y \)

Under composition \( \otimes \) for the \( S^i \) form an algebra over \( \mathbb{Z}_2 \). This is graded by grad \( (S^i)^k := (k + \cdots + i) \). And if we denote this algebra by \( A \), there is an action of \( A \) on \( H^*(X; \mathbb{Z}_2) \) for all space \( X \), such that \( H^*(X; \mathbb{Z}_2) \) is a graded \( A \)-module; \( A \cdot H^8 \subset H^{p+8} \).

The Cartan formula gives a map \( \psi : A \to A \otimes A \) given by \( \psi(S^i) = \sum_{k=0}^{i} S^{i-k} \otimes S^k \). This is an algebra homomorphism and makes \( A \) into a Hopf algebra.

We would like to use homological properties of \( A \) to get the homotopy groups of spheres in the stable range. First we consider the cohomology of \( A \).
$\mathbb{Z}_2$ is a graded $A$ module as follows. $S^0 \in A$ is identically $S^0 \cdot 1 = 1$, and everything else in $A$ acts like 0. The degree of $1 \in \mathbb{Z}_2$ will be 0. Then the cohomology of $A$ is $\text{Ext}^*_A(\mathbb{Z}_2, \mathbb{Z}_2)$. This is defined in the usual way, i.e., let $\{C_i\}$ be an $A$-free resolution of $\mathbb{Z}_2$. We have an exact sequence

$$\cdots \rightarrow C_3 \xrightarrow{d_3} \cdots \rightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \rightarrow 0$$

where the $C_i$'s are graded free $A$ modules, and the $d$'s and $\varepsilon$ are degree preserving maps. Let $\text{Hom}_A(C_0, \mathbb{Z}_2)$ denote the set of $A$-homomorphisms from $C_0 \rightarrow \mathbb{Z}_2$ while lowering degree by 0. Then we get a complex

$$\cdots \xrightarrow{\text{Hom}_A(C_3, \mathbb{Z}_2)} \text{Hom}_A(C_2, \mathbb{Z}_2)$$

and taking its homology we get $\text{Ext}_A^*(\mathbb{Z}_2, \mathbb{Z}_2)$. Moreover this has a ring structure induced by $\Psi$, i.e., $S[C \otimes C]$ is a resolution of $\mathbb{Z}_2$; then the complex

$$C \otimes C \otimes \cdots \otimes C$$

has an $A$ module structure given by $a(c \otimes c_2) = \Psi(a)(c \otimes c_2) = \otimes a(c) \otimes a(c_2)$ if $\Psi(a) = \otimes a(\otimes a)$.

Furthermore there is a map $m: C \rightarrow C \otimes C$. Composing $m$ with the obvious map $\Phi$ induced by the map of $A$:

$$\text{Hom}_A(C, \mathbb{Z}_2) \rightarrow \text{Hom}_A(C \otimes C, \mathbb{Z}_2)$$

gives the product. Now we are ready to state the main theorem.
There is a spectral sequence $\tilde{E}_r^{s,t}$ with differential $d_r: \tilde{E}_r^{s,t} \to \tilde{E}_{r+1}^{s+t+1, t-1}$ such that

1. $E_2^{s,t} = \text{Ext}_{\mathbb{F}_2}^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$

2. $E_r^{s,t} \Rightarrow \pi_\ast^S$, i.e. a sequence of groups $B^{i,j}_r \Rightarrow \pi^S_m = B^{0,m}_0 \supset B^{1,m+1}_1 \supset B^{2,m+2}_2 \supset \ldots$

where $B^{s,t}/B^{s+1,t+1} \cong E_\infty^{s,t}$

The $E_\infty^{s,t}$ which contribute to $\pi^S_m$ are those where $t-s = m$.

$\pi^S_m = 2$ primary component of the stable $m$ stem of homotopy groups of spheres, i.e. $\lim_k [S^{K+m} : S^K]$.

Look at this for small $s,t$. $\text{Ext}_{\mathbb{F}_2}^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$ has an element in each dimension $\leq 2^{i-1}$. One can easily calculate:

where product is that of Ext. Now the differentials $d_r$ go up $r$ and back $1$ \( r \). The product structure is such that $d_r$ is a derivation at $r$. This is very helpful in computing the $d_r$'s, which are in general very difficult.
But we use product structure. Clear that only possible non-zero d is $d_r h_1$ for some $r$. If $d_r h_1 \neq 0$ then, $d_r h_1 = h_0^{r+2}$.

So $0 = d_0 = d_r h_1 = h_0 d_r h_1 = h_0^{r+3} \neq 0$.

Also there is a product in $\Pi^S_3$, $\Pi^S_4 \otimes \Pi^S_2 \to \Pi^S_6$ induced by composition. This induces a product in $E^\infty$ which is the same as that induced from $E_2$. So we can use the products to calculate the group extensions. Mut by $h_2$ corresponds to mut by 2 in $2\Pi^S_3$ so we have $2\Pi^S_0 = \mathbb{Z}$; $2\Pi^S_1 = 2\Pi^S_2 = 2\Pi^S_6 = \mathbb{Z}_2$; $2\Pi^S_4 = 2\Pi^S_5 = 0$; $2\Pi^S_3 = \mathbb{Z}_8$ and $2\Pi^S_7 = \mathbb{Z}_{16}$. These are right answers.

Question Let $L$ be where this comes from.

Will set up an exact couple to give this spectral sequence. For convenience, let $H^*(X)$ denote reduced cohomology with coef. $\mathbb{Z}_2$. Let $X$ and $Y$ be s. d. CW complexes with $H^*(X); H^*(Y)$ finitely generated in each dimension. Let $\{C_s\}$ be an A-free resolution of $H^*(Y)$. Define a realization of $C \cdot \{C_s\}$ to be a collection $\{\{T_x(R; K_s) = \text{Hom}_{\mathbb{Z}_2}(C_s; H^*(R))\}$ where $T_x(R; K_s) = \lim_{x \to \infty} [S^{x+k} R; S^k K_s]$ ($K_s$ will be a stably a product $K(\pi, n)^n$ and a sequence of spaces $M_s$ and maps $f_s: M_{s-1} \to K_s$ such that $M_{s-1} = Y_s$.}
$M_5 \rightarrow K_5 \cup \text{CM}_{M_5-1}$ and

$$K_5 \xrightarrow{f_5} K_5 \cup M_{M_5-1} = M_5 \xrightarrow{f_5} K_{5+1}$$

and induces $d_5^*: H^*(K_{5+1}) \rightarrow H^*(K_5)$

$$C_5$$

and induces $e: C_0 \rightarrow H^*(Y)$.

One obtains such a realization by induction. First choose $f_0 \in \pi_0 [Y; K_0]$. Form $M_0 = K_0 \cup f_0 [Y]$. This gives an exact sequence

$$0 \rightarrow H^1(M_0) \xrightarrow{f_0*} H^1(K_0) \xrightarrow{f_0*} H^1(Y) \rightarrow 0$$

since $f_0*$ is epi. Now $H^*(M_0) = \ker e$. We have a map $d_0 : C_1 \rightarrow \ker e = H^1(M_0)$. Choose $k_1 \supset H^1(K_1) = C_1$ and map $f_1 : M_0 \rightarrow K_1$ such that $f_1* = d_0$. Continue in the same way.

Now look at the sequence

$$M_{M_5-1} \xrightarrow{f_5} K_5 \xrightarrow{f_5} M_5 \xrightarrow{f_5} \text{SM}_{M_5-1}$$

where $f$ is inclusion of $M_5$ into $\text{SM}_{M_5-1} = M_5 \cup \text{CM}_{K_5}$.

Apply functor $\pi_4(X; -)$ and get

$$\pi_4(X; M_{M_5-1}) \rightarrow \pi_4(X; K_5) \rightarrow \pi_4(X; M_5) \rightarrow \pi_4(X; \text{SM}_{M_5-1}) \rightarrow$$

$$\pi_3(X; M_{M_5-1})$$

Summing everything we get an exact couple.
and assume $g_{s*}$ is zero and $\pi_+(X; K_s) = 0$ if $s < 0$.

Let $\pi_+(X; K_s) = E^{s, s+1}_s$. This is $\text{Hom}_A(H^*(K_s); H^*(X))$ = $\text{Hom}_A(C_s; H^*(X))$ and the map $d_s$ of the exact couple is the induced by $d : C_{s+1} \to C_s$.

So $E^{s, s+1}_s = \text{Ext}^s_A(H^*(Y); H^*(X))$.

Claim this converges to $\pi_+(X; Y)$.

Setting $X = Y = S^0$, we obtain the desired spectral sequence.
Homotopy Theory of Spheres Seminar
The Lower Central Series of Group Complexes
7-7-65 Speaker: D.M. Kan

It has long been known that the homology groups of a space yield information about
its homotopy groups. For example:

**Poincaré** If \( \pi_0 X = 0 \) there is an epi
\[
\pi_i X \to H_i X = \text{abelianization} \pi_i X
\]

**Hurewicz** If \( \pi_i X = 0 \), \( 0 \leq i \leq n \), \( n \geq 2 \)
then
\[
\begin{align*}
\pi_n X & \xrightarrow{\cong} H_n X \\
\pi_{n+1} X & \xrightarrow{\text{epi}} H_{n+1} X
\end{align*}
\]

**J.H.C. Whitehead** If \( \pi_i X = 0 \), \( 0 \leq i \leq n \), \( n \geq 2 \)
then the Hurewicz homos \( \pi_i X \to H_i X \)
may be put in a certain exact sequence
\[
\cdots \to \pi_{i+1} X \to H_{i+1} X \to \Gamma_i X \to \pi_i X \to \Gamma_{i-1} X \to \cdots
\]
where \( \Gamma_i X = \text{im} ( \pi_i X^{l-1} \to \pi_i X^l ) \).

The Hurewicz theorem is equivalent to the fact that \( \Gamma_i X = 0 \) for \( i \leq n \).

Our purpose is to make sense out of these results and to show
that in fact \( \pi_* X \) is related to \( H_* X \)
in a much stronger way.
We shall need both semi-simplicial (ss.) complexes and ss group complexes. These correspond respectively to topological spaces and topological groups. On the category $G^{\Delta}$ of ss. group complexes, a group homotopy relation may be defined between maps and is analogous to that for topological groups. Indeed the singular functor 

$S: \text{topological groups} \to G^{\Delta}$

preserves this relation.

If $X$ is a connected ss. complex with base, we may define an ss. group complex $GX$ which serves as the loop space of $X$. The group homotopy type of $GX$ determines the homotopy type of $X$ and vice versa.

If $G$ is the category of groups, a functor $T: G \to G$ induces a functor $T: G^{\Delta} \to G^{\Delta}$. Such an induced functor preserves group homotopies.

**Example:** For $C \in G$ and an fixed prime $p$ let $T_C = \{ \sigma, \tau \}, \sigma^p \tau = \tau^p \sigma, \sigma, \tau \in C$.

The subgroup of $C$ generated by commutators, $p$th powers.
If $X$ is a connected s.s. complex with base, the filtration $\cdots \subset T^nGX \subset \cdots \subset T^2GX \subset TGX = GX$ gives rise to a homotopy exact couple. This spectral sequence $E(X)$ is a homotopy invariant of $X$. There is a sequence of spectral sequences induced by suspension $E(X) \to E(SX) \to E(S^2X) \to \cdots$.

The limit spectral sequence $\widehat{E}(X)$ is the Adams spectral sequence for the prime $p$.

Taking $p = 0$ so $TC = [C, C]$, a sort of integral Adams spectral sequence is obtained.

**The Hurewicz Homomorphism**

Let $X$ be a connected s.s. complex with base. The loop space $GX$ is an s.s. free group complex with $\pi_n GX = \pi_{n-1} X$. Let $AX =$ abelianization of $GX$. Then $AX$ is a free abelian group complex with $\pi_{n-1} AX = \widehat{H}_n X$. The Hurewicz map arises from the natural homomorphism $GX \to AX$.

\[
\begin{align*}
\pi_{n-1} AX & \xrightarrow{\sim} \widehat{H}_n X \\
\uparrow & \uparrow \\
\pi_{n-1} GX & \xrightarrow{\sim} \pi_n X
\end{align*}
\]
**A Certain Exact Sequence**

There is a fibration $\Gamma_2 GX \rightarrow GX \rightarrow AX$

If $X$ is simply connected, there is an isomorphism $\Gamma_{n-1} X \cong \Pi_{n-2} \Gamma_2 GX$ such that

$$
\begin{array}{ccccccccc}
\rightarrow & \Pi_n GX & \rightarrow & \Pi_n AX & \rightarrow & \Pi_{n-1} \Gamma_2 GX & \rightarrow & \Pi_{n-1} GX & \rightarrow & \cdots \\
SS & SS & SS & SS & SS & SS & SS & SS & SS & SS
\end{array}
$$

commutes.

If $X$ is simply connected, the Hurewicz theorem is thus equivalent to the following:

connectivity $\Gamma_2 GX \geq 1 + $ connectivity $GX$

**The Lower Central Series**

For $B \in G$ let $\Gamma_1 B = B$ and

$\Gamma_{r+1} B = [\Gamma_r B, B]$ for $r \geq 1$. This filtration of $B$ by normal subgroups is called the lower central series.

$$
\cdots \subset \Gamma_n B \subset \cdots \subset \Gamma_2 B \subset \Gamma_1 B = B
$$

If $B$ is a free group then there is a natural isomorphism $\frac{\Gamma_r B}{\Gamma_{r+1} B} \cong L_r(\frac{\Gamma_r B}{\Gamma_2 B})$ where

$L = \sum_{r=0}^{\infty} \frac{L_r}{r!}$ is the free Lie ring functor.
If now $B$ is an s.s. free group complex then $\prod_* \frac{F_\ast B}{F_{\ast+1} B}$ is determined by $\prod_* \frac{F_\ast B}{F_{\ast+1} B}$.

This follows from a theorem of Dold. Namely, let $A$ be the category of abelian groups, $T : A \to G$ a functor and $T : A \to G^\Delta$ the induced functor. If $B, B' \in A^\Delta$ are free abelian with $\prod_* B \cong \prod_* B'$ then $TB, TB'$ have the same group homotopy type, so $\prod_* TB \cong \prod_* TB'$.

**A Spectral Sequence**

If $X$ is a connected s.s. complex with base then the fibrations

$\xrightarrow{\sim} \xrightarrow{\sim} \xrightarrow{\sim} \xrightarrow{\sim}$

\[ \xrightarrow{\sim} F_3 G_X \xrightarrow{\sim} F_2 G_X \xrightarrow{\sim} G_X \]

\[ \xrightarrow{\sim} \xrightarrow{\sim} \xrightarrow{\sim} \xrightarrow{\sim} \]

\[ \xrightarrow{\sim} \frac{F_3 G_X}{F_4 G_X} \xrightarrow{\sim} \frac{F_2 G_X}{F_3 G_X} \xrightarrow{\sim} A_X \]

give rise to a homotopy exact couple. For this spectral sequence

\[ E'(X) = \sum_{r=1}^\infty \frac{\prod_* F_r G_X}{\prod_* F_{\ast+1} G_X} = \prod_* L(AX) = E'(H_* X) \]

For $X$ simply connected, $E(X)$ converges to $\prod_* G_X$. This follows from a theorem due
to Curtis. Namely, that

Connectivity $\Gamma_{\frac{1}{3}} G X \geq \{ i \log_2 r \} + \text{connectivity } GX$

where $\{ \cdot \}$ denotes the next integer $\geq \cdot$.

Furthermore, one obtains a new proof of the result that if $X$ simply connected, finite
then $\pi_1 X$ is computable.

The above may be generalized
as follows. Let $Y$ be an $ss$ complex,
and $X$ a connected $ss$ complex with base.

There are fibrations of function complexes:

$$
\begin{array}{c}
\vdots \\
(\Gamma_{\frac{1}{3}} G X)^Y \\ \\
(\Gamma_{\frac{1}{2}} G X)^Y \\ \\
(\Gamma_{\frac{1}{3}} G X)^Y \\
\end{array}
\xrightarrow{	ext{ fibre sequence }}
\begin{array}{c}
(\Gamma_{\frac{1}{2}} G X)^Y \\ \\
(AX)^Y \\
\end{array}
$$

The associated spectral sequence has

$$
E' = \pi_* \sum_n \left( L^n AX \right)^Y = \sum_n H^*_Y \left( Y, \pi_* L^n AX \right) = E' \left( H^*_Y, H^*_X \right)
$$

If $\pi_i X = 0$, $i \leq 1$ and $Y$ finite then

$E$ converges to $\sum_{k=2}^{\infty} [ S^k Y \to X ]$

If $X$ is also finite then $[ S^k Y \to X ]$ is computable for $k \geq a$. 

On the Groups $E'(S^{n+1})$

$E'(S^{n+1}) = \prod_* \Sigma L^r K(\mathbb{Z}, n) = \prod_* LK(\mathbb{Z}, n)$

The k-fold suspension homomorphism is

$\sigma_k: \prod_* LK(\mathbb{Z}, n) \to \prod_{*+k} LK(\mathbb{Z}, n+k)$

Assume now and henceforth that $n$ is even.

Then $\sigma_k$ is a monomorphism onto a direct summand.

By an argument of Dold-Puppe this stability implies:

$\prod_* L^r K(\mathbb{Z}, n) = 0$ unless $r = 1$ or $r = p^j$

for some $j > 0$, $p$ prime.

$p \prod_* L^{p^j} K(\mathbb{Z}, n) = 0$ for $j > 0$ and $p$ prime.

A simple consequence of this is that $\prod_* S^{n+1}$ is finite except for $i = n+1$.

Let $\alpha_{n+1} \in \prod_{*+n+2} L^2 K(\mathbb{Z}, n+1) = \mathbb{Z}$ be a generator.

If $\alpha_{n+1}: K(\mathbb{Z}, 2n+2) \to L^2 K(\mathbb{Z}, n+1)$ represents $\alpha_{n+1}$, then $L(\alpha_{n+1})$ induces the composition homomorphism

$\alpha_{n+1}: \prod_* LK(\mathbb{Z}, 2n+2) \to \prod_* LK(\mathbb{Z}, n+1)$

One obtains the following direct sum decomposition:

$\prod_{*+n+2} LK(\mathbb{Z}, n) \to \prod_{*+n+1} LK(\mathbb{Z}, n+1)$
Thus \( E'(S^{n+2}) = E'(S^{n+1}) + E'(S^{2n+3}) \)

Hence \( \Pi_q S^{n+2} \) is finite except for \( q = n+2, 2n+3 \)

**A Version of the Hopf Invariant 1 Problem**

There is a suspension map of spectral sequences \( E(S^{n+1}) \rightarrow E(S^{n+2}) \)

If there exists a map \( GS^{2n+3} \rightarrow \Gamma_2 GS^{n+1} \) such that

\[
\begin{align*}
\text{GS}^{2n+3} & \quad \rightarrow \quad \Gamma_2 \text{GS}^{n+1} \\
\text{AS}^{2n+3} = K(\mathbb{Z}, 2n+2) & \xrightarrow{\alpha^n+1} L^2 K(\mathbb{Z}, n+1) = L^2 \text{AS} \\
\end{align*}
\]

commutes, then it induces a map

\( E(S^{2n+3}) \rightarrow E(S^{n+2}) \) and

\( E(S^{n+2}) = E(S^{n+1}) + E(S^{2n+3}) \)

Consequently \( \Pi_q S^{n+2} = \Pi_{q-1} S^{n+1} + \Pi_q S^{2n+3} \)

and hence there is a map \( S^{2n+3} \rightarrow S^{n+2} \) of Hopf invariant 1.

Conversely if such a map \( S^{2n+3} \rightarrow S^{n+2} \) exists then there is a map \( GS^{2n+3} \rightarrow \Gamma_2 GS^{n+1} \) such that the above diagram commutes,
An Application

The diagram

\[ GS^{2n+3} \xrightarrow{[\cdot, \cdot]} \Gamma_2 GS^{n+2} \]
\[ \downarrow \]
\[ AS^{2n+3} = K(\mathbb{Z}, 2n+2) \xrightarrow{2\alpha_n} L^2 K(\mathbb{Z}, n+1) = L^2 AS^{n+2} \]

commutes.

If \( p \) is an odd prime then

\[ E'(S^{n+2}; \rho) \times E'(S^{n+1}; \rho) + E'(S^{2n+3}; \rho) \]

Using the suspension map \( E(S^{n+1}; \rho) \rightarrow E(S^{n+2}; \rho) \)
and the map \( E(S^{2n+3}; \rho) \rightarrow E(S^{n+2}; \rho) \)
induced by \([\cdot, \cdot] : GS^{2n+3} \rightarrow \Gamma_2 GS^{n+2}\)

it follows that

\[ E(S^{n+2}; \rho) = E(S^{n+1}; \rho) + E(S^{2n+3}; \rho) \]

Hence

\[ \pi_g(S^{n+2}; \rho) = \pi_{g-1}(S^{n+1}; \rho) + \pi_g(S^{2n+3}; \rho) \]
Let \( X \) be a simply connected semi-simplicial complex with base point. We wish to study \( \pi_\cdot \pi(X) \).

We can define a semi-simplicial group complex \( G \) which serves as the loop space of \( X \), and \( \pi_q \pi(X) \cong \pi_q(G) \). \( (G) \) is a free group, and the face and degeneracy operators are homomorphisms.

Now take \( \Gamma_2 G < G-1 \), the commutator subgroup \( (\Gamma_2 G) \), defined by \( (G \pi(G)) \), and define:

\[
\begin{align*}
\pi \Gamma_2 G & < G - 1 \\
\pi & \downarrow \\
\pi G & = A \pi, \text{ free abelian.}
\end{align*}
\]

e.g. if \( X = S^1 \), \( A \pi = K(\mathbb{Z}, n) \)

We have the commutative diagram:

\[
\begin{align*}
\pi_{q+1}(X) & \cong \pi_q(G) \\
\downarrow & \downarrow \\
H_{q+1}(X) & \cong \pi_q(A \pi)
\end{align*}
\]

where the left map is the usual homotopy homomorphism and the right map is induced by the natural projection.

We may filter \( G \pi \) thusly:

\[
\begin{align*}
\cdots & \pi_{r} G \pi < \pi_{r-1} G \pi < \cdots < \pi_{3} G \pi < \pi_{2} G \pi < \pi_{1} G \pi < \pi G \pi
\end{align*}
\]

\[
\begin{align*}
\pi_{r} G \pi & = [\pi_{r} G \pi, G \pi]
\end{align*}
\]
Now \( \pi_{G} G \cong L^r(AX) \), where \( L^r \) is a functor from abelian groups to abelian groups defined purely in terms of \( AX \). (If \( M \) is an abelian group, take
\[ N(M) = \text{the non-associative algebra generated by } M = M + M \otimes M + M \otimes (M \otimes M) + (M \otimes M) \otimes M + \cdots. \]
Let \( I \) be the ideal generated by \( \sum x_{ij} + \sum y_{ij} x_{ij} \) and \( \sum x_{ij} y_{ij} \), \( \sum z_{ij} \), \( \sum x_{ij} y_{ij} \) (where here \( \sum x_{ij} = x \otimes y \)).
Then \( L^r(M) = N(M) / I \). \( \sum_{r \geq 1} L^r(AX) = L_0(AX) \)

Now take the homotopy exact couple of the filtration, we are interested in \( E^1_{r,n}(X) = \pi_{-n}(L^r(AX)) \).

In particular, we wish to find \( E^1_{r,q}(\mathbb{Z},n) \) or that is \( \pi_{-q} L^r K(\mathbb{Z},n) \). We will denote \( K(\mathbb{Z},n) \) by just \( (\mathbb{Z},n) \) henceforth. The results that follow are due to Kan, Schlesinger, & Curtis.

\( \text{3. Now } L^1 = \text{identity functor, so } \pi_{-0} L^1(\mathbb{Z},n) = \mathbb{Z} \text{ in } \text{dim } n, 0 \text{ otherwise}. \)

\( \pi_{-n} L^2(\mathbb{Z},n) \) is generated by elements \( w_1, w_2, \ldots, w_n \), odd
where \( \text{dim } w_i = n + 1 \) and \( 2w_i = 0 \), except that \( w_n \) has no order if \( n \) is odd.

\( \pi_{-1} L^0(\mathbb{Z},2m) \), \( m \) an odd prime, has generators \( a_1, a_2, \ldots, a_m \) of order \( p \).
The stable dimension of $\alpha_k$ is $(2p-2)k - k$. There is nothing of infinite order.

Also, for $r \geq 2^k$, $L^r(\mathbb{Z}, n)$ is $n+1+k$ connected.

\[ \begin{array}{cccc}
| & | & | & | \\
| n+0 & n+1 & N_2 & N_2 \\
| \cdots & \cdots & \cdots & \cdots \\
| b & 0 & N_2 & N_2 \\
| 1 & 0 & \mathbb{Z} & \mathbb{Z} \\
\hline
| & | & | & | \\
| n+0 & n+1 & \mathbb{Z}_p & \mathbb{Z}_p \\
\end{array} \]

$2p-2$

C. Consider the k-fold suspension:

$\prod_4 L^r(\mathbb{Z}, n) \xrightarrow{\sigma^k} \prod_{q+k} L^r(\mathbb{Z}, n+k)$.

The suspension has these properties:

1. If $n$ is even, $\sigma^k$ is a monomorphism onto a direct summand.
2. For an element $u \in \text{im } \sigma$, $(i^r)u = 0$ for $r \geq 2$, $i = 1, 2, \ldots, r-1$. The order will be $\leq \text{g.c.d. } (r)$. If $r = p^i$, $\text{order } r u = p$

   $r \neq p^i$, $\text{order } r u = 1$.

Hence $\prod_4 L^r(\mathbb{Z}, \text{even})$ has exponent $p$ (if non-trivial) if $r = p^i$, and $= 0$ if $r \neq $ prime power.
3. If \( r \) is odd (only non-trivial possibility)
\[ \prod_q L^r(\mathbb{Z}, \text{even}) \xrightarrow{S} \prod_{q+1} L^r(\mathbb{Z}, \text{even}) \text{ is an isomorphism} \]

4. If \( r \) is even and \( n \) is even
\[ \prod_q L^r(\mathbb{Z}, n) \xrightarrow{S} \prod_{q+1} L^r(\mathbb{Z}, n+1) \]
\[ \oplus \]
\[ \prod_{q+1} L^{2r}(\mathbb{Z}, 2n+2) \]
\[ \text{by \( wn+1 \) \text{ being even, there is a \#1} \]

provides a direct sum decomposition, for we have
\[ (\mathbb{Z}, 2n+2) \xrightarrow{wn+1} L^2(\mathbb{Z}, n+1) \text{ and so get} \]
\[ L^{2r}(\mathbb{Z}, 2n+2) \xrightarrow{L^2(\mathbb{Z}, n+1)} L^{2r}(\mathbb{Z}, n+1) \rightarrow L^r(\mathbb{Z}, n+1) \]
where \( wn+1(\mathbb{Z}, 2n+2) = \frac{1}{2} \left[ L_{n+1}^0, L_{n+1}^1 \right] \) (Whitehead product)

\[
= \sum (-1)^* \left[ \begin{array}{c}
\text{degeneracy by} \ n+1 \\
\text{complementary degeneracy by} \ n+1
\end{array} \right]
\]

\[ * = \text{sign of degeneracy} \]
\[ \text{e.g.} \ 2^1 = \frac{1}{2} \left[ [si, si, J - [si, si, J] \right] \]

Hence:
\[
\text{Cor: All we need is } \prod_q L^r(\mathbb{Z}, \text{even}), \text{ all } r.
\]

\[
1. \ \text{Let } r=2^j. \ \prod_q L^{2^j}(\mathbb{Z}, n) = \ ? \ (n \text{ even})
\]

We have \( 0 \rightarrow L^{2^j}(\mathbb{Z}, n) \xrightarrow{2} L^{2^j}(\mathbb{Z}, n) \xrightarrow{\eta} L^{2^j}(\mathbb{Z}, n+1) \rightarrow 0 \)
where \((\mathbb{Z}, n)\) is a vector space over \( \mathbb{Z}_2 \).
Lemma: \( \pi_{q-2} \mathbb{L}_p^2 (\mathbb{Z}_p, 2n-2) \rightarrow \pi_q \mathbb{L}_p^2 (\mathbb{Z}_p, 2n) \)
\( \oplus \)
\( \pi_q \mathbb{L}_p^{2n-1} (\mathbb{Z}_p, 2n-1) \)
\( \oplus \)
\( \pi_q \mathbb{L}_p^{2n} (\mathbb{Z}_p, 2n) \)
Composition by unspecified elements gives a direct sum decomposition.
\( \pi_q \mathbb{L}_p^2 (\mathbb{Z}_p, 2n) \) has generators
\( a_1, \ldots, a_n, b_1, \ldots, b_n, \quad \text{dim } a_i = (2p-2)i-1 \)
\( \text{dim } b_i = (2p-2)i. \)

Thm: \( \mathbb{L}_p^2 (\mathbb{Z}_p, n) \) has for generators \( \{\gamma_I\} \), \( I = i_1, i_2 \)
\( w_{i_2-i_1} = w_{i_1} w_{i_2-i_1} \ldots w_{i_{k-1}} \) where
\( i_k \equiv 0 \pmod{(2p-2)}, \ i_k \leq (p-1)n \), and \( i_k \leq p^k \) for \( k = 2, \ldots, j \), and where if \( i = (2p-2)h-1 \) then \( w_i = a_h \)
and if \( i = (2p-2)h \) then \( w_i = b_h. \)

\( \Im \gamma_I \) is those \( w_I \) with \( i_j \equiv 1 \pmod{(2p-2)} \)

E. \( \mathbb{L}_p^2 (\mathbb{Z}, n) \) has \( \{\beta_I\} \) for generators, where \( I = (i_1, i_2) \)
and \( \eta_I (\beta_I) = w_I \) and \( i_1 \leq n, i_2 \leq 2i_1, \ldots, i_j \leq 2i_{j-1} \)
and \( i_j \) odd. Filtration of \( \beta_I \) (that is, lowest sphere on which it is defined) is \( i_1 \). Hence it occurs in \( \pi_k (\Omega S^{i_1+1}) \).
This induces
\[ \pi_q L^2(\mathbb{Z}, n) \xrightarrow{2^x} \pi_q L^2(\mathbb{Z}, n) \xrightarrow{\eta_*} \pi_q L^2(\mathbb{Z}, n) \]
\[ \oplus \]
\[ O \]
\[ \pi_x L^2(\mathbb{Z}, n) \text{ is generated by elements } \pi_{q-1} L^2(\mathbb{Z}, n) \]
\[ w_1, \ldots, w_n \text{ each of order } 2, \]
\[ w_n = \frac{1}{2} \left[ l_{w_1} l_{w_2} \right] \]

5. So in general if \( r \) is \( 2^j \), \( n \) arbitrary
\[ \pi_q L^2(\mathbb{Z}, n) \xrightarrow{\sigma} \pi_{q+1} L^2(\mathbb{Z}, n+2) \oplus \pi_{q+1} L^2(\mathbb{Z}, 2n+2) \]

Thm. a) \( \pi_x L^2(\mathbb{Z}, n) \) has for free generators over \( \mathbb{Z}_2 \) \( \{ w_I \} \) where \( I = (i_1, i_2, \ldots, i_j) \),
\[ i_1 \leq n, i_2 \leq 2n, \ldots, i_j \leq 2^j n-1 \]
and \( w_I \in \pi_q L^2(\mathbb{Z}, n) \), \( q = n + \sum i_k \)
\[ w_I = \sigma^{x_k} w_{i_1} \sigma^{x_k} w_{i_2} \ldots \sigma^{x_k} w_{i_j} \text{ where } \sigma^{x_k} \text{ is suitable suspension.} \]

b) The image of \( \eta_* \) is exactly those \( w_I \) for which \( i_j \) is odd in \( \pi_x L^2(\mathbb{Z}, n) \xrightarrow{\eta_*} \pi_x L^2(\mathbb{Z}, n) \).
Also \( w_{i_1} w_{i_2} = 0 \) if \( j > 2i \).
After $L^2$, only the indices are listed. The circled terms are some that persist in the spectral sequence to $\infty$. The crossed terms are some that do not. This can be seen by looking at the known homotopy groups of spheres.
Homotopy Groups of Spheres Seminar

The Adams Spectral Sequence - Revisited

7-21-65

Let $A(p) =$ Steenrod Algebra of "stable" cohomology operations $H^*(X, \mathbb{Z}_p) \rightarrow H^*(X, \mathbb{Z}_p)$

$A = A(2)$ is generated by $\xi \eta : H^*(X, \mathbb{Z}_2) \rightarrow H^*(X, \mathbb{Z}_2)$

A basis is given by $\xi^a \eta^b$, with $a \geq 2, b \geq 2$.

With the Adem relations $\xi^a \eta^b = \sum_{i=0}^{\min(a, b)} (b-i)^{a-i} \xi^{a+i} \eta^i$.

for $a < 2b$, determining the algebra structure.

Thm (Adams) There is a spectral sequence

$E^2_{p,q} \Rightarrow E^{p+q}_{\infty}$

$E^2_{p,q} = \text{Ext}^{s,t}_{A(p)}(\mathbb{Z}_p, \mathbb{Z}_p)$

2) $E^{s,s+t}_\infty = \text{Graded Group associated to a filtration of } \pi^s_{\text{even}}$

3) Products can be introduced in each $E_r$ such that $E_2 = \text{The Algebra } \text{Ext}^{s,t}_{A(p)}(\mathbb{Z}_p, \mathbb{Z}_p)$

The $d_r$ are derivations and the multiplication in $E_r$ induces the one in $E_{r+1}$.

In $E_\infty$, the product from the $E_r$ coincides (up to sign) with the composition product.

The computation of $\text{Ext}^{s,t}_{A}(\mathbb{Z}_2, \mathbb{Z}_2)$

Let $0 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow \mathbb{Z}_2 \rightarrow 0$ be an exact sequence of left $A$-modules, with the $C_0$ free, and $d$ of degree $0$. Form the chain complex
Then \( \text{Ext}^3_A(\mathbb{Z}_2, \mathbb{Z}_2) = H^5(\text{Hom}_A(\mathbb{Z}_2, \mathbb{Z}_2)) \)

where \( \text{Hom}_A(M,N) = \{ \text{maps } f : M \rightarrow N \text{ lowering degree by } 3 \} \).

\( \text{Hom}_A(\mathbb{Z}_2, \mathbb{Z}_2) \) has a very simple structure. As an \( A \)-module it is trivial because this is the case for \( \mathbb{Z}_2 \). It is just the graded \( \mathbb{Z}_2 \) vector space with one copy of \( \mathbb{Z}_2 \) for each free generator of \( C \). More precisely:

\[ \text{Hom}_A(\mathbb{Z}_2, \mathbb{Z}_2) \cong \text{Hom}(\frac{\mathbb{Z}_2}{I(A)C_0}, \mathbb{Z}_2) \text{ where } I(A) = \mathbb{Z}_2 \]

We have \( \text{Hom}_A(\mathbb{Z}_2, \mathbb{Z}_2) \rightarrow \text{Hom}_A(\mathbb{Z}_{s+1}, \mathbb{Z}_2) \)

\[ \text{Hom}(\frac{\mathbb{Z}_2}{I(A)C_0}, \mathbb{Z}_2) \rightarrow \text{Hom}(\frac{\mathbb{Z}_{s+1}}{I(A)C_0}, \mathbb{Z}_2) \]

\[ \Rightarrow \mathbb{Z} \rightarrow 0 \Rightarrow d_{s+1} : \mathbb{Z}_{s+1} \rightarrow I(A)C_0 \]

\[ \Rightarrow \text{Ker } d_s \subseteq I(A)C_0 \]

Def: \( \mathbb{E} \)-form an indecomposable basis for an \( A \)-module \( M \). \( \mathbb{E} \) is a \( \mathbb{Z}_2 \)-basis for \( M/IM \), where \( \mathbb{E} \) means projection \( M \rightarrow M/IM \rightarrow 0 \).

If \( C = \bigoplus A e_i \rightarrow M \) is given by \( d(e_i) = \mathbb{E}_i \) one shows by induction on the components \( M^2 \equiv M \) that \( d \) is onto. Also, since\n
\[ I(A)C_0 = \frac{M}{I(A)M} \]

it follows \( \text{Ker } d \subseteq I(A)C_0 \).

One now constructs \( C_5 \xrightarrow{d_5} \text{Ker } d_{s-1} \rightarrow 0 \) such that \( \text{Ker } d_5 \subseteq I(A)C_5 \) inductively, beginning of course with \( 1 \rightarrow \mathbb{Z}_2 \rightarrow 0 \). To obtain a resolution \( C_0 \xrightarrow{d_0} \mathbb{Z}_2 \rightarrow 0 \).
with the differential $s = 0$ in $\text{Hom}_A(C_0, Z_2)$. Such a resolution is called "minimal".

For example, $\ker d_0 = I(A)$ and it is well known
\[ \bigoplus_{z \geq 0} \mathbb{Z} \otimes \mathbb{Z}_2 \]
form an indecomposable basis of the $A$-module $I(A)$. Hence $C_1$ has generators in dimension $t = 2^z$ and
\[ \text{Ext}^1_A(Z_2, Z_2) = \bigoplus_{z \geq 0} \mathbb{Z}_2 t = 2^z \text{ with generator } h_z \]

It is necessary also to know the multiplicative structure of $\text{Ext}^1_A(Z_2, Z_2)$, when computing with a minimal resolution as above, the Yoneda product can be used. This coincides (to sign) with the product induced by $A \Delta \to A \otimes A$. It is defined as follows:

\[ C_n \to C_{n+1} \to \cdots \to C_1 \to C_0 \to Z_2 \to 0 \]

Let $\alpha, \beta$ be cocycles in $\text{Hom}_A(C_n, Z_2)$ representing $\alpha, \beta \in \text{Ext}^1_A(Z_2, Z_2)$, lift $\alpha$ by $\alpha_0 \cdots \alpha_n$

using $\alpha = d_{n+1} = 0$ and then $\alpha_C$ are free.

Define $\bar{\alpha}, \bar{\beta}$ to be the class of $B \otimes \alpha$ in $B \otimes C_n$. This is well-defined.
By other methods, Adams has computed
\[ \operatorname{Ext}^2_{A}(\mathbb{Z}_2, \mathbb{Z}_2) = \begin{cases} 
\mathbb{Z}_2 & \text{corresponding to } h_i h_j \neq i + 1 \\
0 & \text{otherwise}
\end{cases} \]

\[ \operatorname{Ext}^3_{A}(\mathbb{Z}_2, \mathbb{Z}_2) = \begin{cases} 
\mathbb{Z}_2 & \text{corresponding to } h_i h_j h_k \text{ subject to } h_i^3 - h_i = h_i^{-1} h_i h_i^{-1} \text{ and } h_i h_i^2 = 0 \\
\mathbb{Z}_2 \text{ in dim } 2, (2^{i^2 - 1} + 2) \cdot 2^{i - 1} \\
0 & \text{otherwise} \quad \text{(Conjecture)}
\end{cases} \]

A Crude Vanishing Thm

Let \( M \) be an \( A \)-module, \( S^1 / \mathbb{Z}_2 = 0 \) so \( S^1 \)
acts as a differential operator on \( M \). Let \( H^*(M) \)
be the cohomology.

Lemma 1: If \( 0 \to M' \to M \to M'' \to 0 \) is exact and \( H^* \)
of any two = 0 Then \( H^* \) of The Third = 0.

Proof: The long exact sequence

Lemma 2: \( H^*(A) = H^*(A, A) = 0 \). Hence \( H^*(C) = 0 \)
for any free \( A \)-module \( C \).

Proof: \( S^1 / \mathbb{Z}_2 = \begin{cases} 
\mathbb{Z}_2 & n \text{ even} \\
0 & n \text{ odd}
\end{cases} \). The Lemma follows by looking at the action of \( S^1 \)
on the basis \( \mathbb{Z}_2 \).

Thm: \( H^*(M) = 0 \Rightarrow \operatorname{Ext}^s_{A}(M, \mathbb{Z}_2) = 0 \) for \( t-s < s \)

Proof: By induction on \( s \). For \( s=0 \) there is nothing to prove. Suppose \( \cdots \to C_s \xrightarrow{d_s} C_{s-1} \xrightarrow{d_{s-1}} \cdots \to M \to 0 \) is a minimal resolution. By induction and Lemmas 1 and 2
$H^r(\ker d_r) = 0 \quad \forall r \geq 0$. Assume the generators of $C_s$ are in dimensions $\geq 25$. We get a diagram:

$$
\begin{array}{ccc}
0 & \rightarrow & C_s \\
\uparrow & & \uparrow \\
\ker(d_{s-1}) & \rightarrow & \ker(d_{s-1}) \\
\uparrow & & \uparrow \\
ker(d_{s-1}) & \rightarrow & 0 \\
\end{array}
$$

Since the resolution is minimal, there are no relations in $\ker(d_{s-1})^{(25+1)}$ and hence $(\ker d_s)^2 = 0$ for $t < 25+2$. The same then holds for $C_{s+1}$ QED.

To apply this to $\text{Ext}_A(Z_2, Z_2)$ we use Lemma 2 and

$$
0 \rightarrow \frac{I(A)}{A_{\mathbb{Z}_2}^s} \rightarrow \frac{A}{A_{\mathbb{Z}_2}^s} \rightarrow Z_2 \rightarrow 0
$$

inducing

$$
\text{Ext}^{s+1}_A(Z_2, Z_2) = \text{Ext}^{s+1}_A\left(\frac{A}{A_{\mathbb{Z}_2}^s}, Z_2\right) \oplus \text{Ext}^{s-1}_A\left(\frac{I(A)}{A_{\mathbb{Z}_2}^s}, Z_2\right)
$$

\[\begin{align*}
\text{if } & t = s, \\
& \text{if } 0 \leq t < s
\end{align*}\]

This result \Rightarrow the higher stable homotopy groups are finite. In fact if $x \in \pi_0^s F_{2,0}$ then $2^x = 0$.

It can also be used to prove convergence of the Adams Spectral Sequence in the special case $[S^0, S^0]$ under consideration.

Adams Vanishing and Periodicity Theorems

Let $A_t = \text{subalgebra of } A$ generated by $S_2^c, \quad 0 \leq r$

$A_\infty = A$
The injections \( A_p \to A_r \), \( p < r \) induce \( \text{Ext}^s_{A_r}(M,N) \to \text{Ext}^s_{A_p}(M,N) \)
where \( M \) and \( N \) are \( A_r \)-modules. Assume \( H^0(M) = 0 \), \( \mu^\infty = 0 \), \( \forall \ t < 2 \).

**Thm 1** \[ \text{Ext}^{s,t}_{A_r}(M,\mathbb{Z}_2) = 0 \] for \( t-s < g + T(s) \)

where \( T(s) = \left\{ \begin{array}{ll}
 2s & \text{if } s = 0 \\
 2s-1 & \text{if } 1 \equiv 2 \\
 2s-2 & \text{if } 2 \equiv 3  \\
 2s-2 & \text{if } 3 \\
 2s & \text{if } 4 \\
 \end{array} \right. \)

**Thm 2** \[ \text{Ext}^{s,t}_{A_r}(M,\mathbb{Z}_2) \to \text{Ext}^{s,t}_{A_p}(M,\mathbb{Z}_2), \ r > p \] is an isomorphism for \( t-s < g + T(s-1) + 2^{p+1} \), \( s \geq 1 \).

**Thm 3** There are elements \( p_r \in \text{Ext}^{2^r,3^r}_{A_r}(\mathbb{Z}_2,\mathbb{Z}_2), \ r \geq 1 \)
such that

\[ \forall x : x \mapsto x \cdot p_r : \text{Ext}^{s,t}_{A_r}(M,\mathbb{Z}_2) \to \text{Ext}^{s+2^r,3^r+t+3^r}_{A_r}(M,\mathbb{Z}_2) \]
is an isomorphism if \( t-s < 2 + U(s) \) for some function \( 3s \leq U(s) \leq 5s \)

\[ \forall x : x \cdot p_r = (p_{r-1})^2 \]

\[ \forall x : x \cdot p_r = (p_{r-1})^2 \]

In the range where \( \text{Ext}^{5^t}_{A_r}(M,\mathbb{Z}_2) \cong \text{Ext}^{3^t}_{A_r}(M,\mathbb{Z}_2) \)

\( p_r \) on the right corresponds to the Massey secondary operation \( \langle x, h^2_r, h_{r+1} \rangle \) on \( \text{Ext}^s \).

**Indications of proofs** \( A_0 = \) Exterior Algebra generated by \( S_2 \).
\( A_1 = \) \( S_2 \)-dim. algebra over \( \mathbb{Z}_2 \) generated by \( S_2, S_2 \).

It is easy to compute \( \text{Ext}^{5,t}_{A_1}(A_0,\mathbb{Z}_2) \) completely by the minimal resolution method and get Thm 1 in
This case. Then Thm 1 for \( r=1 \) follows for any \( M \) which can be expressed by extensions of \( A_0 \), using long exact sequences for \( \text{Ext} \).

Thm 2 supports Thm 1 for higher \( r \).

Thm 1 applies to Thm 2 as follows. Consider

\[
0 \to k \to A_r \otimes M \to M \to 0
giving
\[
\begin{align*}
\text{Ext}^{s,t}_{A_r} (M, \mathbb{Z}_2) \\
\text{Ext}^{s,t}_{A_r} (A_r \otimes M, \mathbb{Z}_2) \leftarrow \text{Ext}^{s,t}_{A_r} (M, \mathbb{Z}_2) \leftarrow \text{Ext}^{s,t}_{A_r} (k, \mathbb{Z}_2).
\end{align*}
\]

Now \( k^t = 0 \) for \( t < 9 + 2^{p+1} \) so \( \text{Ext}^{s,t}_{A_r} (k, \mathbb{Z}_2) = 0 \) if \( t - s < 9 + (s-1) + 2^{p+1} \),
and \( \iota^* \) becomes an isomorphism in this range.

One needs \( \iota^*(k) = 0 \) and the vertical "change of rings" isomorphism which depend on the structure of the algebras \( A_r \). Using these techniques, Thms 1 and 2 can be proved simultaneously by induction on \( r \).

For Thm 3, Thm 1 \( \Rightarrow \) \( h_0^r h_{r+1} = h_0^r x = 0 \) for \( x \) in a certain range, so \( \langle x, h_0^r h_{r+1} \rangle \) is defined. For \( i: A_r \to A_{r+1} \), \( \iota^*(h_{r+1}) = 0 \) so if \( S_c = h_0^r h_{r+1} \), then \( \iota^* \) is a cocycle representing \( Pr \in \text{Ext}^{s,t}_{A_r} (\mathbb{Z}_2, \mathbb{Z}_2) \). Explicitly, computation shows \( x \mapsto x \cdot (i^* p_2 \cdot \text{Ext}^{s,t}_{A_r} (A_0, \mathbb{Z}_2)) \). Explicitly, computation shows \( x \mapsto x \cdot (i^* p_2 \cdot \text{Ext}^{s,t}_{A_{r+1}} (A_0, \mathbb{Z}_2)) \).
is an isomorphism in a certain range. The proof continues by using $K$ and an inductive argument as above.

The function $T$ of Thm 1 does not quite give the edge of $\text{Ext}_A(Z_2, Z_2)$. The "best possible" edge may be read off the Table because of periodicity. This vanishing Theorem gives a bound on the order of elements in $2\pi^S$ of approximately $2^{b/2}$.

**Table and Differentials**

The Table gives $\text{Ext}_A^{5^S}(Z_2, Z_2)$ for $t-s \leq 27$, all non-zero differentials in this range, and the value of $2^{5^S}$.

$d_2(h_0) = h_0 h_3^2$ holds because the composition product in $\pi^S$ is anti-commutative $\Rightarrow \exists \gamma^2 = 0$ where $h_3 \leftrightarrow 0$.

$\Rightarrow h_0 h_2^2$ can't survive to $E_{20}$. Using Secondary Cohomology operations, Adams has shown $d_2(h_i) = h_0 h_2 i$.

$\forall i \geq 4 \Rightarrow \# \text{ elements of Kpt Invariant } 1$ except in $T_3(5^2), T_7(5^4), T_{15}(5^8)$.

The differentials $d_3(h_0 h_1 h_2)$ are calculated from Toda's calculations of $\pi^S_{\leq 19}$. In Adams Berkeley Notes, the listed differentials $d_3(h_0 h_2 h_4) = h_0 h_2 d_0$ are incorrect. The other non-zero differentials are correct. Their proof in the range $t-s \leq 27$ follows using the fact that $\partial_x = x \partial^x y = y \partial^x y$ if $\partial^x, \partial^y = 0$.
| 15 |  |  |  |  |  | p_{1h_1}^{1h_1} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 14 |  |  |  |  |  |  |  |  |  |  |  |
| 13 |  |  |  |  |  |  |  |  |  |  |  |
| 12 |  |  |  |  |  | p_{2h_1}^{2h_1} |  | p_{2h_0}^{2h_0} |  | p_{2h_0}^{2h_0} |  | p_{2h_0}^{2h_0} |
| 11 |  |  |  |  |  |  |  |  |  |  |  |
| 10 | p_{1h_2}^{1h_2} |  | p_{1h_1}^{1h_1} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |
| 9 | p_{1h_1}^{1h_1} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |
| 8 | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |
| 7 | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |  | p_{1h_0}^{1h_0} |
| 6 | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |
| 5 | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |
| 4 | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |
| 3 | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |
| 2 | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |
| 1 | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |  | h_{1h_0}^{1h_0} |
| 0 |  |  |  |  |  |  |  |  |  |  |  |

\[ \pi_{C_{g}}(S) \]

\[ \begin{align*}
S_{g}(c_{0}) &= h_{1h_0}^{1h_0} \\
S_{g}(r_{0}) &= h_{1h_0}^{1h_0} \\
\end{align*} \]

\[ \begin{align*}
S_{g}(c_{1}) &= h_{1h_0}^{1h_0} \\
S_{g}(r_{1}) &= h_{1h_0}^{1h_0} \\
\end{align*} \]

\[ \begin{align*}
S_{g}(A) &= h_{1h_0}^{1h_0} \\
S_{g}(A) &= h_{1h_0}^{1h_0} \\
\end{align*} \]

\[ \begin{align*}
(Z_{2})^{2} Z_{2} + Z_{2} Z_{2} + Z_{2} &
\begin{align*}
(Z_{2})^{2} &+ Z_{2} Z_{2} + Z_{2} \\
\end{align*} \]
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\[ A \cdot 3 \]

\[ H^* (A) \]

\[ p = 2 \]
Homotopy theory of Spheres Seminar 8/1/65

Toda brackets and Massey Products in the Adams Spectral Sequence

Speaker: Kahn

References:

3. Adams: Berkeley Notes
4. Barratt: Seattle Notes
5. M. Moss: Cambridge Thesis
6. Toda: "Composition Methods in the homology groups of spheres"
7. May: Princeton Thesis

The results discussed here are reported to occur in a more general form in (5). Here we restrict our attention to the mod 2 Adams spectral sequence for the 2-component of \( \mathbb{H}_* (5) \); \( A = \text{mod} \ 2 \text{ Steenrod Algebra} \).

Let \( (K,d) \) be a chain algebra which is associative. If \( a, b, c \) are cycles and \( \overline{a}, \overline{b}, \overline{c} \) their respective homology classes with \( \overline{a} \overline{b} = 0 \) and \( \overline{b} \overline{c} = 0 \), the Massey product \( \langle \overline{a}, \overline{b}, \overline{c} \rangle \) is defined as follows:

\[\overline{a} \overline{b} = 0 \Rightarrow \exists u \ni du = ab\]
\[\overline{b} \overline{c} = 0 \Rightarrow \exists v \ni dv = bc\]

Then, using the associativity of \( K \), \( uc = (-1)^{du} \overline{a} nu \) is a cycle. The set of homology classes of all such cycles is called the Massey product \( \langle \overline{a}, \overline{b}, \overline{c} \rangle \). As a result of the leeway in choosing \( u \) and \( v \), it turns out to be a coset of \( \overline{a} \ (-) + (--) \overline{c} \).
Examples: the terms \((E^p, d_r)\) of the Adams spectral sequence for \(\Pi_\ast(\mathbb{Z})\) are associative chain algebras and hence have Massey products defined in their homology \((E_\ast, d_\ast)\) is not invariant and need not be an associative chain algebra. However, the bar construction (see [1] sections 2.1, 2.2) may be realized (see [2] and [3]) yielding an associative chain algebra \((E_\ast, d_\ast)\) and hence yielding Massey products in \(E_2 = H(E_\ast, d_\ast)\). Many such Massey products are calculated in [1].

As in [2], we use the "smash" product to define composition in \(\Pi_\ast(\mathbb{Z})\) and relate it to the products in the Adams spectral sequence. Our purpose is to relate (as far as possible), the Toda brackets (defined in the first lecture of this series) and Massey products in the Adams spectral sequence.

Let \(\alpha \in \Pi_{p+h}(S^k)\), \(\beta \in \Pi_{q+k}(S^t)\), \(\gamma \in \Pi_{r+k}(S^r)\) be \(\alpha \wedge \beta = 0\), \(\beta \wedge \gamma = 0\). Let \(a, b, c\) be representatives of \(\alpha, \beta, \gamma\) respectively and let \(A_i\) and \(B_i\) be nullhomotopies of \(a \wedge b\) and \(b \wedge c\) respectively. Let \(H_n\) be the identity map \(S^n \to S^n\) for each \(n\).

Consider \(H: S^{s+1} \to S^{p+q+r}\), \(S = p+h+q+k+r+1\), given by

\[
H(d_s(x, t)) = \begin{cases} 
(\alpha \wedge (b \wedge (c + B_{2r})))((p+h \wedge B_{2r-1}))(x), & \frac{1}{2} \leq t \leq 1 \\
(A_{1-2t} \wedge B_r)(c + B_{2r-1})(x), & 0 \leq t < \frac{1}{2}
\end{cases}
\]

where \(d_s: S^s \times I \to S^{s+1}\) in the usual fashion. This differs from the usual stable Toda bracket \(\langle a, \beta, \gamma \rangle\) by a sign discussed in Chapt. III of [1] (see especially p. 26). Since we are using the mod 2 Adams spectral sequence, we may ignore this sign difference. Note that for \(\frac{1}{2} \leq t \leq 1\), \(H\) may be thought of as \(a \wedge B : (S^{p+q+k+r+1}) \to S^{p+q+r}\) and for \(0 \leq t < \frac{1}{2}\), \(H\) may be thought of as \(A \wedge c : (TS^{p+q+k+r+1}) \to S^{p+q+r}\) and \(H_{1/2}\) as \(a \wedge b \wedge c : S^s \to S^{p+q+r}\).

The multiplicative structure of the Adams spectral
Sequence will be treated as follows. Let $B_\ast$ be the unreduced bar construction on $A$, namely

$$B_n = B_n(A) = A \otimes A \otimes \cdots \otimes A \quad (\otimes \text{over } \mathbb{Z}_2)$$

where $A = \sum_{\sigma \in S_n} A_\sigma$. Then

$$\mathbb{Z}_2 \leftarrow B_0 \leftarrow B_0 \leftarrow \cdots$$

is known to be a free $A$-resolution of $\mathbb{Z}_2$ (see (1): 21, 22). $B_\ast \otimes B_\ast$ can be made into an $A$-module using the diagonal map $\psi: A \to A \otimes A$. $0 \leftarrow \mathbb{Z}_2 \leftarrow B_\ast \otimes B_\ast$ is an acyclic complex over $\mathbb{Z}_2 \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$. Hence there exists a mapping $\Delta: B_\ast \otimes B_\ast \to B_\ast \otimes B_\ast$ covering $\mathbb{Z}_2 \to \mathbb{Z}_2 \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

Let $x_0 = s^P \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots$

$y_0 = s^S \leftarrow y_1 \leftarrow y_2 \leftarrow \cdots$

$z_0 = s^R \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots$

be realizations of $B_\ast$. Let

$$s^P \leftarrow u_1 \leftarrow u_2 \leftarrow \cdots$$

$$s^S \leftarrow v_1 \leftarrow v_2 \leftarrow \cdots$$

$$s^R \leftarrow w_1 \leftarrow w_2 \leftarrow \cdots$$

be given by $u_m = U_{k+l=m} X_k \wedge Y_l$

$v_m = U_{k+l=m} Y_k \wedge Z_l$

$w_m = U_{k+l=m} X_k + Y_l + Z_l$

$q_m = U_{k+l=m} U_k \wedge Z_l$

$r_m = U_{k+l=m} X_k \wedge V_l$

Then $\{u_i\}$ is a realization of $B_\ast \otimes B_\ast$ as are $\{v_i\}, \{q_i\}, \{r_i\}$ and $\{w_i\}$ is a resolution of $B_\ast \otimes B_\ast \otimes B_\ast$. (see (2)). Also let
Let $F_i : U_i \rightarrow \overline{U}_i$
$J_i : V_i \rightarrow \overline{V}_i$
$h_i : Q_i \rightarrow \overline{W}_i$
$k_i : R_i \rightarrow \overline{W}_i$

be realizations of $\Delta$. Then \( h_i(f_i, 1) : W_i \rightarrow \overline{W}_i \) realizes $\Delta(\Delta f_i)$ and \( k_i(f_i, 1) : W_i \rightarrow \overline{W}_i \) realizes $\Delta(\Delta \Delta)$. In $\otimes$ it is proved that $h_i(f_i, 1)$ and $k_i(f_i, 1)$ are homotopic after composition with the map $\overline{W}_i \rightarrow \overline{W}_{i-1}$. It follows that the $E_r$ are differential rings for $r > 1$ with the ring structure in $E_2$ the usual one for $\text{Ext}_A(\mathbb{Z}_2, \mathbb{Z}_2)$.

Let $X_0 \leftarrow X_1 \rightarrow \cdots$ be a realization of a free $A$-resolution over $H^\bullet(X_0)$, $X_0$ a finite complex, and let $\{E_r^{s, t+s}\}$ be the Adams spectral sequence for $\prod_{\bullet}(X_0)$. For our use here, the most convenient description of $E_r^{s, t+s}$ is $\text{Im} \left[ \prod_{t'}(X_{s-r'}, X_s) \rightarrow \prod_{t'}(X_{s-r'}, X_{s+r}) \right]$.

**Definition.** In $E_r$, \( d^{s-r', t+1+s-r} r \rightarrow r \), $x = y \in E_r^{s, t+s}$ is called an honest relation if $d^{s-r', t+1+s-r-k} r+k+1 = 0$ for $k > 0$, $l > 1$.

**Proposition 1.** If $x \in \prod_{t'}(X_s)$ and $x \rightarrow 0$ in $\prod_{t'}(X_s)$ and $x \rightarrow a$ in $E_r^{s, t+s}$ and $d_r u = a$ is an honest relation, then $\exists \theta \in \prod_{t'+1}(X_{s-r'}, X_s) \cdot \theta$. $\text{Im} \theta$ in $E_r^{s-r', t'+1+s-r}$ is $u$ and $d_{t'+1} \theta = a$.

**Proof.** $u$ is represented by $\theta_0 \in \prod_{t'+1}(X_{s-r'}, X_s)$ such that
$$(\alpha - \delta_\theta \Theta) - 0$$

Now $j: \mathbb{C} \to \mathbb{C}$.

\[ \text{Now } \gamma (\alpha - \delta_\theta \Theta) = 0. \]

\[ \text{Case I: } i_{s-r+1} (\alpha - \delta_\theta \Theta) = 0 \Rightarrow \exists \Theta \in \Pi_{s+1} (X_{s-r+2}, X_s) \text{ s.t. } \]

$\delta_\theta \Theta = \alpha - \delta_\theta \Theta$. Then $\delta_\theta (\Theta + \Theta') = \alpha$ and $\text{Im} (\Theta + \Theta')$ in $E_r$ is $\alpha$, where $\Theta'$ is the image of $\Theta$ in $\Pi_{s+1} (X_{s-r+2}, X_s)$.

Then $\Theta = \Theta + \Theta'$ satisfies the conclusion of the proposition.

\[ \text{Case II: } i_{s-r+1} (\alpha - \delta_\theta \Theta) = 0. \]

Then $i_{s-r+1} i_{s-r+1} (\alpha - \delta_\theta \Theta) = 0$

$\Rightarrow \exists \alpha \in \Pi_r (X_{s+1}, X_s)$ s.t. $i_{s-r+1} (\alpha - \delta_\theta \Theta)$, and $\alpha$ is an image in $E_{s-r+1}$ the element $\alpha$. Since $(\alpha - \delta_\theta \Theta) = 0$

in $\Pi_r (X_0)$, $\alpha_s = 0$ in $\Pi_r (X_0)$. Hence $\alpha_s$ must be 0 in some $E_r$. By the hypothesis of homotopy, $\alpha_s = 0$ in $\Pi_r (X_{s-r+1}, X_{s+2})$.

We can now apply cases I and II to $\alpha_s$, and if necessary to $\alpha_{s+1}, \alpha_{s+2}, \ldots$. If Case II repeats indefinitely,

then by the convergence of the Adams spectral sequence,

$$(\alpha - \delta_\theta \Theta) = 0$$

and $\Theta = \Theta'$, satisfies the conclusion of the proposition.

Otherwise, Case I eventually occurs and we have $\Theta \in \Pi_{s+1} (X_{s-r+2}, X_s)$.

$\exists \delta_\theta \Theta = \alpha$ and $i_{s-r+1} (\alpha - \delta_\theta \Theta) = i_{s-r+1} (\alpha - \delta_\theta \Theta)$. Let $H_1$

$H_1 : X_{s-r+2}$ be a map $\Rightarrow [H_1] = (\alpha - \delta_\theta \Theta)$ and

$[H_1] = \alpha$. Let $H_1 : (T, 5, 0) \to (X_{s-r+1}, X_{s+1})$ represent $\Theta$

with $H_1 = H_1$. Define $H_2 : (T, 5, 5) \to (X_{s-r+1}, X_{s+1})$ by
\[ H''(x,t) = \begin{cases} H'(x, 2(t-\frac{1}{2})) & \frac{1}{2} \leq t \leq 1 \\ H(x, 2t) & 0 \leq t \leq \frac{1}{2} \end{cases} \]

Let \( \psi \) be the class of this map in \( \pi_{t+1}(X_{S'}; X_0) \). Then \( d^+ \psi \cdot \theta_0 = \alpha \) and the image of \( (\psi \cdot \theta_0) \) in \( E_{S' + t + 1} \) is \( u \). Q.E.D.

Before proceeding to the main theorem, let us record an easy technical lemma.

**Lemma 2** Suppose \( A \subset X \) and \( \pi_{t+1}(X,A) \xrightarrow{\sim} \pi_{t+1}(X/A) \). If \( f, g : (T^s, s) \rightarrow (X,A) \), \( f|st = g|st \), and \( h : S^t \times I \rightarrow A \), \( h|st = f|st \) \((= g|st)\) then \( F : S^{t+1} \rightarrow T^s \times I \rightarrow T \times X \) by ful hug agrees in \( \pi_{t+1}(X) \) with \( d(f,g) \) modulo the image of \( \pi_{t+1}(A) \rightarrow \pi_{t+1}(X) \).

**Remark 3** The hypothesis on \((X,A)\) of Lemma 2 is satisfied if "everything" is in the stable range (Using the Blakers-Massey triad theorem).

**Theorem 4** Let the 2-component of \( \pi_*(S) \) contain \( \alpha, \beta, \gamma \) such that \( \alpha \beta = 0, \beta \gamma = 0 \). Let \( a \in E_{t+1}, b \in E_{t+2}, c \in E_{t+s}, d \in E_{t'+1} \) converge to \( \alpha, \beta, \gamma \) respectively, \( r \geq 1 \). Given that \( d_r u = ab \) and \( d_r v = bc \) are honest relations, then the cycle \((mc + aw)\) of \((E_r, d_r)\) converges to \( \langle \alpha, \beta, \gamma \rangle \).

**Proof** We may choose representatives \( f = \alpha, g = \beta, h = \gamma \). \( f : S^{t+1} \rightarrow X_{S'}, g : S^{t+2} \rightarrow Y_{S'}, h : S^{t+3} \rightarrow Y_{S'} \). The hypothesis of honesty says that
\[ F_{t+1, t} \circ (f \circ g) : S^{t+1} \times S^{t+2} \rightarrow U_{t+1, t+2} \]
extends to \( F_{t+1, t} : T(S^{t+1} \times S^{t+2}) \rightarrow U_{t+1, t+2} \) so that \( F_{t+1, t} \) represents \( u \). Similarly,
\[ \text{Can be extended to} \quad G_{S^2+S_3} : T(S^3 \land S^3) \rightarrow \overline{V_{S^2+S_3}}. \]

So that \( G_{S^2+S_3} \) represents \( \eta \). Now \( H = h_{S^2+S_3} \circ (F_{S^2+S_3} \land h) \) represents \( \eta \) and \( K = k_{S^2+S_3} \circ (F_{S^2+S_3} \land h) \). We may assume that \( U \) is a compound mapping cylinder of a sequence of fibrations. Hence we may assume that

\[ h_{S^2+S_3} \circ (F_{S^2+S_3} \land h) \mid S^{t+1} \land S^{S^2+S_3} \land S^{S^3} \]

and

\[ k_{S^2+S_3} \circ (F_{S^2+S_3} \land h) \mid S^{t+1} \land S^{S^2+S_3} \land S^{S^3} \]

project to the same maps in \( S^0 \land S^8 \land S^r \). Call the projection \( \pi : \overline{W_{S^2+S_3}} \rightarrow S^0 \land S^8 \land S^r \). Then \( d(\pi_0 H, \pi_0 K) \) represents \( \langle x_1, y \rangle \). Now

\[ H' = H \mid S^{t+1} \land S^{S^2+S_3} \land S^{S^3} \land S^r \]

and

\[ K' = K \mid S^{t+1} \land S^{S^2+S_3} \land S^{S^3} \land S^r \]

are homotopic in \( \overline{W_{S^2+S_3}} \). Let \( D : S^t \times I \rightarrow \overline{W_{S^2+S_3}} \)

where \( E = t_{S^2+S_3} \). \( D_1 = H' \) and \( D_0 = K' \). Then \( (\text{HuDuK}) : T \times S^t \rightarrow \overline{W_{S^2+S_3}} \).

By Lemma 2, \( \pi_0 (\text{HuDuK}) \) represents \( \langle x_1, y \rangle \) modulo elements of filtration \( S^0 \land S^8 \land S^r \).

It is also clear that the image of \( \text{HuDuK} \) in \( E_r \)

\[ t \rightarrow S^1_0 + S^2_3 = \langle \eta + \mu \rangle \]

Example (Barratt) \( \eta \{q\} \in \langle u, v, X \rangle \) where \( X = \{d_0\} \) in May's notation (see \( \theta \)).

Proof \( d_{S^2} (\) is clearly honest and \( d_{S^2} \) is honest since \( h_2, h_4, h_2 \land h_4, h_2 \land h_4 \) are all permanent cycles.
Hence $h_2 f_0 \in \langle h_3 g_0, h_3 d_0 \rangle$ converges to an element in $\Omega, 2 \theta, K$. But, in $E_2$ (according to $\Theta$), we have $h_2 f_0 = h_3 g$.
Q.E.D.

Remarks In the case $r=1$, the Massey product in $E_2$ is defined in terms of the bar construction (see 1, 2.1+2.2). In order for the present approach to apply for $r=1$, we need that the homotopy $D : S^F \times I \rightarrow \overline{W}_{s_1+s_2+s_3}$ rather than $\rightarrow \overline{W}_{s_1+s_2+s_3-1}$. It is reported that Moss proves the case $r=1$ using a definition of the Massey product based on the Yoneda product in $\text{Ext}$. It seems likely that with this approach he uses the original definition of Toda brackets rather than the one used here.
Homotopy Theory of Spheres Seminar

VII: Adam's Spectral Sequence: Computation

8/5/65 Speaker J.P. May

The main purpose of this talk is to simplify the computation of $\text{Ext}_A(Z_2, Z_2)$, the $E_2$ term of the Adams spectral sequence for computing 2-primary components of $\pi_n$. We do have a 'classical' acyclic resolution of $Z_2$ over $A$ from the bar construction,

$$B(A) = T(SI(A)) /$$

'T' - tensor algebra

'A' - 'supension'

but in practice this is unwieldy.

Milnor-Moore thm on structure of Hopf alg.

filter: $F_p A = I(A)^{-p}$, $p<0$

$F_p A = A$ $p \geq 0$

$E_0 A$ is primitively generated

By Milnor-Moore Theorem 5.18

$E_0 A \simeq \bigvee P E_0 A$
"P" primitive elements
V universal enveloping algebra of restricted Lie algebra.

Filter $\overline{B}(A)$ by $\otimes$ filtration (essentially)
Look at $E^0 \overline{B}(A) = \overline{B}(E^0A) = E'$ of resulting spectral sequence.

$E^2 = H^* \overline{E}(E^0A)$. Is easy to compute differentials
Now $\vee P E^0 A \cong E^0 A$ calculate $H^*$ by dual spectral sequence. We can get much smaller spectral sequence resolution for $\vee P E^0 A$ then for $A$.

Milnor's results on structure of $A$ (see Annals of Math. Vol. 63)

Let $x_i = x_1 \cdot \ldots \cdot x_n$ admissible monomial $x_i$ have dual $x_i^*$, $A^* = \text{polynomial alg. on } \{x_i^*\}$
Dualize back and got new basis for $A$.

If $R^c = (x_1, \ldots, x_n)$ let $PC(R) = (x_1^*)^* \cdot \ldots \cdot (x_n^*)^*$
New basis is written: Consider all matrices of integers $> 0$
\[
\begin{array}{c|c|c}
* & x_{0,1} & x_{0,2} \\
\hline
x_{1,0} & x_{1,1} & \cdots \\
x_{2,0} & \cdots & \\
\end{array} = X
\]

Define
\[
R(x) = r_i = \sum_j 2^j x_{i,j}, \quad B(x) = \prod_{i,j} t_{i,j}!
\]
\[
S(x), \quad S_j = \sum_i x_{i,j}
\]
\[
T(x), \quad t_n = \sum_{i+j=n} x_{i,j}
\]

Then
\[
P(R), \quad P(S) = \sum_{X \in \mathbb{Z}} B(X) P(T(X))
\]

Consider those sequences \( R \) with \( r_i = 2^d, \quad r_k = 0 \) \( k \neq i \). Those \( e \) in \( R \) only project to primitive \( e \) of the associated grading. Call \( P(R) = P_i \) basis for the Lie algebra in associated grading.

Now \( [P_i, P_k] = S_i, k+1 P_{i+k} \) for \( i \geq k \)

\[
(p_i^\dagger)^{(2)} = 0
\]

much simpler product than that in \( A \).

\( P_i \) is \((X_i^*)^{2^d})^w \) projected to assoc. grading.

Hence (by results on Lie algebras, etc.) have reduced \( \text{Ext}_A (\mathbb{Z}, \mathbb{Z}) \) problem to computing

\[ H^* (V(L) \otimes \Gamma(SL)) \]

\( \Gamma = \text{divided polynomials} \)
Give this a differential \( d \gamma_r(x) = x \gamma_{r-1}(x) \)

\[
d(\gamma_r(x)) \cdots \gamma_n(x^n) = \\
\sum_{i<j} \gamma_i[x_i,x_j] \cdots \gamma_{r-1}(x_i) \gamma_{r+1}(x) + \\
\sum x_i \cdots \gamma_{r+1}(x_i) \cdots
\]

We can prove that this is a complex and a free resolution of \( \mathbb{Z} \).

Then form \( \overline{X}(V) = \mathbb{Z}_2 \otimes_{\mathbb{Z}_2} V(L) \otimes \Gamma(SL) \).

Consider \( \overline{X}(V)^* \). Give \( \Gamma \) its natural coalgebra structure, \( \delta_r(x) \rightarrow \sum_i \gamma_i(x) \otimes \delta_j(x) \)

\( \overline{X}(V)^* = \) polynomial algebra on \( SL^* \).

If \( R_j = (P_j)^* \), then \( S(R_j) = \sum_{k=1}^{j-1} R_k R_{j-k} \).

This is a polynomial alg \( P(R_j) \). Its coh. is coh. of \( E^2 \) of the spectral sequence which converges to coh. of \( A \).

Look at \( \overline{X}_n = P(R_j | j \leq n) \), diff. and is subalgebra

\( \overline{X}_n \otimes \mathbb{Z}_2 \), where \( \mathbb{Z}_2 = P \{ R_n \} \).
Filter $\bar{X}_n^*$ by homological degree on $\bar{X}_{n-1}^*$. (This is a decreasing filtration). We get a spectral sequence: $d_0 = 0$.

$$E_2 = H^*(\bar{X}_{n-1}^*) \otimes \mathbb{Z}$$

because of filtration, $E_3 = E_\infty = E^0 H^*(\bar{X}_n^*)$.

To calculate differentials, imbed $V(L) \otimes P(SL)$ in $\hat{B}(E^0A)$. $B$ has shuffle product

$$[x_1, \ldots, x_{m+n}] = \sum_{\pi} [x_{\pi(1)}, \ldots, x_{\pi(m+n)}]$$

$\pi$ is all $(m+n)$-shuffles.

Embedding given by $\delta_r(x) \to x^r$ where $x^r = [x_1^r, \ldots, x]$, $r$ factors.

Also $Sq^f_8: E_r \to E_{r+1}$

$$(R^i)^2$$

is a cocycle in $(\bar{X})^*$. Call $(R_2^2)^2 = b_2^i$. One has $S_2^n (b_2^0)^2 = h_0^2 h_1^n$ where $h_0 = R_0^i$

$h_i = R_i$ corresponding to $h_i$ in coh. of $A$. 

Near $3t = s$, if $x \in E_{2n}$, then $h_0, h_{2n+1} x = 0$ in $E_{2n}$.

Compute Massey products in $E_{2n-1}$. $(b_2)^{2n} x = p^n x$ (in $E_2$).

We know $S_a R_0^2 = R_0 R_1$ knowing that $S_a R_2^0 = (R_2^0)^2 = b_2 \Rightarrow S_2 S_a R_2^0 = S_2 b_2 = S_a (R_1 R_0) = (R_1^0)^2 R_2^0 + (R_1^0)^3$

**Charts**

$h_i e_0 = h_i f_i$

Now $S_2 h_i e_0 = h_i S_2 e_0 = h_i^3 d_0 = h_0^2 h_2 d_0$

$\therefore S_2 f_0 = h_0 h_2 d_0$

Have $j$ in $26 - stcm$

$p' h_i e_0 = h_0^2 j$

$h', p' e_0$

$e_0 p' h_i$

$S_2 (e_0 p' h_i) \neq 0$ hence $S(j) \neq 0$

$h_2 i = h_0 j$ hence $S(i) \neq 0$.

The $h_0, h_j$ are in the image of the $J$-homomorphism.

Also $p_i h, p_i h, p_i h_0, p_i h_0 h_2$

Also $p_i h_i h_j$
Suppose $\alpha_i$ in $H^\ast(A)$, $\alpha_{ij}$ rep. cocycles, and $\alpha_{ij}$ s.t. $\sum_{k=1}^{j-1} \alpha_{ik} \alpha_{k+j} = \sum_{r=1}^{h-1} \alpha_{r+j}$ for $j = 1 \leq h-1$.

Easy to verify $\sum_{r=1}^{h-1} \alpha_{r+j} \alpha_{r+1}$ is a cocycle.

$\xi$ coh. classes of such $\xi = \langle \alpha_1, \ldots, \alpha_n \rangle$.

Morse n-tuple product.

Define homology ops. whose images necessarily generate $H^\ast(A)$ (A any augmented algebra).

These generalize Morse products.

Consider $\overline{B}(A)$ differential coalgebra with coproduct $[a_1 \ldots a_n] \mapsto \sum [a_1] \otimes [a_{i+1} \ldots a_n]$.

Hence dual $\overline{C}(A)$ (close to co-bar construction) is a differential algebra. Consider $\overline{B}(\overline{C}(A))$.

Basis of $H_i^\ast(A)$ $\leftrightarrow$ gens. of $A$

Basis of $H^\ast(A)$ $\leftrightarrow$ all relations in $A$

Hence $H_i^\ast, H_{-i}^\ast$ "determine" $A$

Take $H_i^\ast, H_{i-1}^\ast$ $(H^\ast(A))$ get $H^\ast(A)$!

$\overline{B}(\overline{C}(A))$ has 2 differentials.
\[ d' \left[ a_1, \ldots, a_n \right] = \sum \left[ a_1, \ldots, (a_i, a_i+1, \ldots, a_n) \right] \]
\[ d'' \left[ a_1, \ldots, a_n \right] = \sum \left[ a_1, \ldots, (d''a_i, 1, \ldots, a_n) \right] \]

They commute; let \( d = d' + d'' \). Grade by total degree.

Filter in 2 ways

Then \( d \) comes from \( d' \) only. Hence

\[ E' = H_n \left( T \left( \text{SI}(A) \right) \right) \]

\[ = A^* \text{ essentially} \]

Using this see \( H_\ast \left( B \left( \text{C}(A) \right) \right) = A^* \)

Now filter by homological degree in \( B \). Then in \( E_0 \)

\( d \) comes from \( d'' \).

Hence

\[ E' = B \left( H_\ast \left( A \right) \right) \]
\[ E^2 = H_\ast \left( H^\ast \left( A \right) \right) \text{ which is what we want.} \]

We know \( E^\infty \) additively. Comultiplicatively it is

\[ (E^0A)^* = E^\infty \]

\[ \text{want these 2 cols. } H_i \left( H^\ast \left( A \right) \right) \]

\[ \text{dits.} \]

\[ \text{know } E_0 \text{ here} \]
Products of $[\{h_i\}]$ gen. all diagonal:

$[\{h_i, h_{i+1}, \ldots \}]$

Hence differentials are cohomology operations — defined on $\otimes$ products of $H^\ast(A)$.

But all the first two columns are killed except such products since $E_\infty$ is on diagonal.

There generalize Massey products. If $d_i x_i = 0$,

$x = [[x_1, \ldots, x_n]]$ will satisfy

$S_{n-1} x = [E_{n-1}, \ldots, 1 \otimes n]$. If $S_i x = 0$, $s < n - 1$.

These generalize useful & common ops — the complete solution may have to be expressed in terms of them.
Motivation + General Philosophy (see Adams notes)

Let \( x \rightarrow y \) be a cohomology theory, an induced map \( f^* \in \text{Hom}(\tilde{H}^\ast(y), \tilde{H}^\ast(x)) \). If \( f^* = 0 \), let \( z = y \cup x \); we have

\[
0 \rightarrow \tilde{H}^\ast(x) \rightarrow \tilde{H}^\ast(z) \rightarrow \tilde{H}^\ast(y) \rightarrow 0
\]

determines an element in \( \text{Ext}^1(\tilde{H}^\ast(y), \tilde{H}^\ast(x)) \)

Let \( A \) be an algebra acting as cohomology operations, then \( f^* \) determines above elements of \( \text{Hom}(\tilde{H}^\ast(y), \tilde{H}^\ast(x)) \) and \( \text{Ext}^1(\tilde{H}^\ast(y), \tilde{H}^\ast(x)) \)

As an example let \( X = S^{2n+1}, Y = S^m \) and \( H^\ast = KU^\ast \)

then \( \tilde{R}^0(Z) = J \oplus J \quad J \text{ integers} \)

so clearly \( 0 \rightarrow \tilde{R}^0(Z) \rightarrow \tilde{H}^\ast(X)(\tilde{H}^\ast(Y)) \rightarrow 0 \)

Let \( A \) be the algebra of the Adams operations. These comprise all of the additive operations in \( K^\ast \) theory.

They have the following properties:

1) \( A \in \mathbb{Z}^+ \) for \( \psi^k \cdot K^0 \rightarrow K^0 \) a natural ring homomorphism.
2) \( \psi^0 = \text{identity} \)
3) \( \psi^k \cdot \tilde{K}(S^m) \) is null by \( k^m \)
4) \( \psi^i \cdot \psi^j = \psi^j \cdot \psi^i \)
5) \( E \) a complex line bundle \( \psi^k(E) = E^k \)

The \( \psi^k \) are characterized by 1 and 5.

Choose generators \( a \) and \( b \) for \( \tilde{K}^\ast(Y) \) such that \( b \) restricts from \( K^{-1}(x) \) and \( a \) goes into a generator of \( \tilde{K}(y) \). Then we get:

\[
\psi^k(b) = k^m b \quad \psi^k(a) = k^m a + \lambda_k b
\]

\[
\psi^k \cdot \psi^l = \psi^k \cdot \psi^l = \begin{pmatrix} 0 & (1 - k^m) \\ \frac{k^m}{l^m} & (1 - k^m) \end{pmatrix}
\]

\[
\lambda_k = \frac{k^m}{l^m} \cdot (1 - l^m)
\]

If \( k \) is an integer then all \( \lambda_k \) are integers and extension is trivial since a can be varied by a multiple of \( b \), hence \( \lambda_k \) are determined mod an integer.
This number determines the extension and can be shown that 
\[ \text{Ext}^k \left( \overline{F}^k(\mathbb{F}_p), \overline{V}^k(\mathbb{F}_p) \right) \] 
are finite groups of order the same order as the image of the complex \( F \) homomorphism.

\[ \lambda \text{ is the same invariant as Adams } c(1) \]

We now proceed to sketch construction of spectral sequence

Let \( \text{lim}^n \) : inverse limit. This functor is

1) left exact
2) has rt. derived functors \( \text{lim}^n \)
3) if \( G_1 \to G_2 \to \ldots \to G_i \) are abeliangps. \( \text{lim}^n (G_i) = 0 \) \( n > 1 \)
4) if \( \{ G_i \} \) is an inverse system of top.

\[ \text{cont. maps then } \lim^n (G_i) = 0 \] \( n > 0 \).

General reference for \( \text{lim}^n \):
1) Nöbeling: Topology I 97-63
2) Grothendieck: Complements to Chapter 0 in Chapter III of the Elements.

**Def:** A top. gp. \( G \) will be separated if \( \exists \) a decreasing sequence \( \ldots > G_i > G_{i+1} \to \ldots \)

of subgps. which define the topology of \( G \) and \( \text{lim}^0 (G_i) = 0 \) \( n = 0 \).

\[ \text{lim}^0 = 0 \Rightarrow \text{ G is Hausdorff.} \]

We get the following general limit sequence.

\[ 0 \to \text{lim}^0 G_i \to G \to \text{lim}^0 (G_i / G_1) \to \text{lim}^1 (G_i) \to 0 \]

\[ \text{lim}^0 = 0 \Rightarrow G \text{ complete.} \]

Let \( C' \) : category of special top. gps. and closed cont. homomorphisms.

**Lem:** \( C' \) is abelian.

**Proof:** Note that image + coimage have same top.

It condition on category is changed + compact + condition of closed is unnecessary.

This category has countable inverse limits.

We define a filtration on \( H^n \) by \( F^n \overline{H}^n = \text{ker } \{ \overline{H}^n (X) \to \overline{H}^n (X^{(n)}) \} \)

\[ F^n \overline{H}^n (x) = \bigoplus F^n \overline{H}^n (x) \]

**Lem:** \( \text{If } H^n (pt) \neq 0 \) \( n > 0 \) then

1) \( \overline{H}^n (x) = F^n \overline{H}^n (x) \)
2) \( \exists K^n (x) \) is finitely generated if \( H^n (X, H^{n} (pt)) = 0 \) of finite type \( A. \)

We will assume \( H^{n} (pt) \) cofinite \( A. \)
The limit \( \lim_{\kappa} H^\kappa(X) \) is compact.

\[
\lim_{\kappa} H^\kappa(X) = \overline{H^\kappa(X)} \quad \text{by definition.}
\]

Each \( \varphi \cdot H^\kappa(X) \) is compact Hausdorff, and \( \varphi \cdot H^\kappa(X) = \lim_{\kappa} \varphi \cdot H^\kappa(X) \).

Ring of cohomology operations.

For definition of spectrum and associated concepts, see Whitehead-Generalized Homology Theory.

\[
\tilde{H}^\kappa(\eta) = \lim_{\kappa} H^{\kappa + \kappa}(\eta).
\]

If \( M \to \eta \) is a map of spectra, \( \eta \to \tilde{H}^\kappa(\eta) \) is also.

Let \( A^{\eta} = \tilde{H}^{\eta}(\eta) \), where \( H \) in this definition is the cohomology theory associated to \( \eta \).

\( A^{\eta} \) is a compact Hausdorff top ring.

We get a map \( A^{\eta} \otimes \tilde{H}^\kappa(X) \to \tilde{H}^{\kappa + \kappa}(X) \) where \( A^{\eta} \) acts as a ring of continuous maps on \( \tilde{H}^\kappa(X) \). Furthermore \( F^{\eta} \otimes H^{\eta}(A) \to F^{\eta} H^{\eta}(A) \).

Des: An \( A^{\eta} \) module \( M \) of finite type is a top. \( A^{\eta} \) module if:

1. \( A^{\eta} \otimes M^{\eta} \to M^{\eta} \) is continuous giving \( M^{\eta} \) the discrete topology
2. every homogeneous element acts continuously on \( M^{\eta} \)
3. the topology of \( M^{\eta} \) is given by a homogeneous filtration such that the two conditions (1) hold.

Prop: \( A^{\eta} \) modules of finite type and continuous degree preserving \( A^{\eta} \) module maps form an Abelian category.

Des: A free \( A^{\eta} \) module is one of the form \( \otimes \mathbb{Z} \cdot A^{\eta} \).

Prop: A countable increasing chain of finitely generated free \( A^{\eta} \) module is projective.

(To be continued)
Homotopy Theory of Spheres Seminar 8/25/65

Adams spectral sequence for generalized cohomology theories

Speaker: Prof. D.W. Anderson

Let $C^*$ be a cohomology theory. We assume $C^i(\text{one point}) = 0$ for $i \gg 0$, and is finite for all $i$. In fact, by simply shifting dimensions we may assume $C^i(\text{point}) = 0$ for $i > 0$. Let $A^{**}$ be the algebra of cohomology operations for $C^{**}$. It has a topology given by a countable decreasing sequence of groups such that the quotient of any two adjacent groups in the series is finite. On $C^{**}(X)$ we have a decreasing filtration $F^i C^{**}(X)$ satisfying $F^{i_0}(C^{**}(X)) = C^{**}(X)$ for some $i_0$, and $F^i C^{**}(X)/F^{i+1} C^{**}(X)$ is a finite group for all $i$. Again we assume $i_0 = 0$. We wish to find a resolution of $C^{**}(X)$ as an $A^{**}$-module. We have a diagram

$$
0 \leftarrow C^{**}(X) \leftarrow F_0 \leftarrow K_0 \leftarrow 0
$$

$$
\downarrow \quad \downarrow \quad \downarrow
$$

$$
\vdots \quad \vdots \quad \vdots
$$

$$
0 \leftarrow C^{**}(X)/F^2 C^{**}(X) \leftarrow F_2 \leftarrow K_2 \leftarrow 0
$$

$$
\downarrow \quad \downarrow \quad \downarrow
$$

$$
\vdots \quad \vdots \quad \vdots
$$

$$
0 \leftarrow C^{**}(X)/F^1 C^{**}(X) \leftarrow F_1 \leftarrow K_1 \leftarrow 0
$$

$$
\downarrow \quad \downarrow \quad \downarrow
$$

$$
\vdots \quad \vdots \quad \vdots
$$

$$
0 \leftarrow C^*(X)/F^0 C^{**}(X)
$$

where $F_i$ is a free $A^{**}$-module with one generator for each element of (the finite group) $C^{**}(X)/F^i C^{**}(X)$. Taking the lim of the continuous surjections $F_i \rightarrow \cdots$ of compact groups we get $F_0$ and a map onto $C^{**}(X)$. The construction is canonical and the resolutions $F_i$ map onto each other, given an $A^{**}$-map of a similar
The \( K_i \) are kernels (in the abelian category) and are also topologized nicely (= filtered by groups with all adjacent quotients finite).

Now construct for each \( K_i \) a diagram like (1), replacing the group \( C^{**}(X) \). Iterate this construction over the kernels of the new infinite sequence of diagrams so obtained, and continue the process. In this way we get a family of objects \( \mathcal{F}(\nu_1, \ldots, \nu_k) \) (where \( \nu_k \) an integer \( \geq 1 \)), with \( \mathcal{F} = \mathcal{F}(i) \). If \( \mu_k > \nu_k \) for \( k = 1, 2, \ldots, n \) we have a map \( \mathcal{F}(\mu_1, \ldots, \mu_n) \to \mathcal{F}(\nu_1, \ldots, \nu_n) \). This gives us a resolution

\[
0 \to C^{**}(X) \to \mathcal{F}(\infty) \to \mathcal{F}(\infty, \infty) \to \cdots
\]

where \( \mathcal{F}(\infty, \infty, \ldots, \infty) \) is an \( n \)-fold limit, and all are projective.

The \( \mathcal{F}(\nu_1, \ldots) \) can all be represented by \( CW \)-spectra. In the diagram corresponding to (1) of \( CW \)-spectra and maps the horizontal arrows (only) are turned around. Thus \( X \) maps into each \( X_i \) (for \( X_i \)) and hence into \( X_\infty = \lim X_i \). The \( K_i \) map into \( C^{**}(\cdot) \), the mapping cone of \( X \to [\text{Spectrum for } \mathcal{F}(i)] \).

Now apply the functor \( \{ Y, \cdot \}^* \). This gives a series of spectral sequences which are described in terms of the path taken from \( C^{**}(X) \) to the particular \( \mathcal{F}(\nu_1, \ldots, \nu_k) \):

\[
E_1^{\nu_k} (\text{path}) = \text{Hom}^*(\text{5th } \mathcal{F}(\cdot) \text{ in that path}, C^{**}(X))
\]

for any path out from \( C^{**}(X) \) in the diagram (1). This \( \text{Hom}^* \) will be compact since each \( \mathcal{F}(\cdot) \) is finitely generated and \( C^{**}(Y) \) is in our category. Hence we can take inverse limits over all the paths. It will be an exact functor and will commute.
with taking cohomology.

\[ E_2^{s,t}(\text{inverse limit}) = \text{Ext}^{s,t}(\tilde{\mathcal{C}}^\infty(x), \tilde{\mathcal{C}}^\infty(y)) \]

Look at \( K\mathcal{C}^* = \text{complex K-theory mod } p \). Groups of a point continue indefinitely in positive dimensions, periodic with period 2. Note \( \psi^k \) acts on \( K\mathcal{C}^* \) as an operation of degree 0 for \( k \) prime to \( p \).

\[(\psi^k - k^{n_1})(\psi^k - k^{n_2}) \cdots (\psi^k - k^{n_r}) = 0\]

for some collection \((n_1, \ldots, n_r)\). There are only finitely many different powers of \( k \mod p \), hence we can pick \( k \) a generator of \( \mathbb{Z}_p \) and divide up \( K\mathcal{C}^0(x) \) according to the eigenvalues of \( \psi^k \).

\[ K\mathcal{C}^0(x) = K\mathcal{C}^0(x; k) \oplus K\mathcal{C}^0(x; k^2) \oplus \cdots \oplus K\mathcal{C}^0(x, k^{n}) \]

Periodicity map carries \( K\mathcal{C}^0(x; k^i) \to K\mathcal{C}^0(x, k^{i+1}) \) etc. Now \( K\mathcal{C}^*(x, k) \) is a cohomology theory which is periodic; the period is \( 2(p-1) \).

Thus \( K\mathcal{C}^r(\text{point}; k) \) is given by

<table>
<thead>
<tr>
<th>r</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>\ldots</th>
<th>2(p-1)</th>
<th>(p-1)</th>
<th>2(p-1)+1</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>group</td>
<td>\mathbb{Z}_p</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
<td>0</td>
<td>\mathbb{Z}_p</td>
<td>0</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

Thus (renaming) \( K\mathcal{C}^*(x) \) is a cohomology theory periodic with period \( 2(p-1) \). Define a new theory \( K\mathcal{C}^* \) by

\[ K\mathcal{C}^r(\text{pt.}) = \begin{cases} 0 & i > 0 \\ K\mathcal{C}^i(\text{pt.}) & i \leq 0 \end{cases} \]

(Take a representing spectrum for \( K \), an \((i-1)\)-connected covering for \( i^{th} \) term). Call the spectrum \( BU \). Then

\[ BU^0 = BU \times \mathbb{Z}_p \]

\[ BU^{i+1} = BU \]

\[ BU^i = BSU \]

Adams showed that for \( H^* \) = ordinary cohomology, \( H^*(BU; \mathbb{Z}_p) = \alpha_* / \alpha_*(q') \) where \( \alpha_* \) is the Steenrod Algebra and

\[ [\beta_p, q'] = q' \] (see paper on Chern characters).

We want to compute \( K\mathcal{C}^* (bu) = \alpha^{**} \).
we now change terminology to avoid confusion, calling the
$\tilde{\mathbb{A}}^{\infty}$ $R^{\infty}$ instead

$$E_{\beta}^{qr} = H^{r}(\tilde{\mathbb{A}}^{q}; H^{p}(\tilde{\mathbb{A}^{r}}))$$

$$E_{\beta}^{p} = g_{p} k C^{\infty}(\tilde{\mathbb{A}}^{q})$$

The $KC^{q}$ theory looks as follows:

---

If $K$-theory has only 1-differential (true for $BU$), all the boundary
of $K$-theory will be non-zero in $K$-theory and will be lost.

Next we compute $KC_{0}$ not $2$ only. The cohomology operator
in integral $K$-theory are half those of mod-2 theory (only the
Bockstein operator is missing). So we may compute mod-2 theory in
part by proceeding with integral theory.

$$H^{0}(\tilde{\mathbb{A}}^{0}; \mathbb{Z}_{2}) = \mathbb{A}_{2}/\mathbb{A}_{2}(\mathbb{S}_{2}, \mathbb{S}_{2})$$

Compute the cohomology of $\mathbb{A}_{2}/\mathbb{A}_{2}(\mathbb{S}_{2}, \mathbb{S}_{2})$ with respect
to left multiplication by $\mathbb{S}_{2}$. This is $\mathbb{S}_{2}$ is the first differential of the exact
sequence. This is (additively) isomorphic to the homology of
$\mathbb{A}_{2}/(S_{2}^{0}, S_{2}^{0}) \cdot \mathbb{A}_{2}$ (applying the canonical automorphism $\times 2$ to
everything) with respect to right multiplication by $S_{2}$, above.
dual: \( a^* = \mathbb{Z}_2 \left[ \xi_1, \xi_2, \ldots \right] \)

\[ \left( a^*_x / (\xi_0^1, \xi_0^2) a^*_x \right)^* = \mathbb{Z}_2 \left[ \xi_1^2, \xi_2^2, \xi_3^2, \xi_4^2, \xi_5^2, \ldots \right] \]

(this takes some computation). Remember \( \varphi(\xi_k) \equiv \sum_{i=0}^k \xi_i \Theta_{k-1} \Theta_{k-2} \).

Fact: \( d_3^* \) is a derivation: \( d_3(\xi_k) = \xi_k \Theta_2 \). Therefore the homology of \( a^*_x / a^*_x(\xi_0^1, \xi_0^2) \) with respect to left multiplication by \( \xi_0^2 \) is (additively) \( E(\xi_1^2, \xi_2^2, \xi_3^2, \ldots) \) where \( E \) means "exterior algebra". Therefore \( H^*(\_\_) = 0 \) in odd dimensions, and so the spectral sequence stops at \( E_2 \). We have a periodicity map \( \Pi \) lowering dimension by \( 2(\nu-1) \); \( \Pi \in \mathcal{R}^{2(\nu-1)} \). The left annihilators of \( \Pi \) and the right annihilators of \( \Pi \) are identical and are equal to the \( K \)-theory boundaries of dimension \( \nu = 0 \) in \( K \)-theory.

**Theorem** \( \mathcal{R}^{\nu(*)} \) (annihilators of \( \Pi \)) is abelian.

Call this algebra \( \Psi^* \). We can show \( \Psi^* \) is generated by elements \( \psi_1 = \psi^2 - 1 \). We can show \( \psi^2 \) together with the Bockstein operation \( \psi \) is generated out \( \text{ann}(\xi) \). We can also show \( \psi_1 \) corresponds to \( \xi_0^2 \).
1) Let $\mathcal{A}$ be a category, $A, B$ objects. The product of $A$ and $B$, if it exists, is denoted by $A \times B$ with morphisms $A \leftarrow \mathcal{P} \rightarrow B$ and with the universal property that given $f: C \rightarrow A$ and $g: C \rightarrow B$ there exists $\mathcal{P}g: C \rightarrow A \times B$ such that the following diagram commutes.

$$
\begin{array}{ccc}
A & \mathcal{P} & \rightarrow & B \\
\downarrow{f} & \downarrow{g} & & \downarrow{g} \\
C & \rightarrow & B
\end{array}
$$

It is easily seen that the product is unique up to equivalence.

Similarly, the coproduct, if it exists, is $A \oplus B$ with morphisms $A \rightarrow A \oplus B \rightarrow B$ and the analogous universal property represented by the diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{i} & A \oplus B \\
\downarrow{f} & \downarrow{g} & \downarrow{g} \\
C & \rightarrow & B
\end{array}
$$

Again, $A \oplus B$ is unique up to equivalence.

A category $\mathcal{A}$ is pointed if there exists an object $*$ in $\mathcal{A}$ such that if $A$ is any object in $\mathcal{A}$ then there is a unique morphism $*:A \rightarrow *$ and a unique morphism $*:*: \rightarrow A$. $*$ is therefore unique up to equivalence.

One says that a morphism $f:A \rightarrow B$ is trivial if it factors so:

$$
\begin{array}{ccc}
A & \mathcal{P} & \rightarrow & B \\
\downarrow{f} & \downarrow{g} & & \\
\downarrow{g} & & \rightarrow & \ \\
B & \rightarrow & B
\end{array}
$$

For example, the category of abelian groups and homomorphisms is pointed, the point being the zero group.
On the other hand, the category of topological spaces and continuous maps is not pointed, for although there is a natural projection from any space onto a point, there is no natural inclusion of a point in an arbitrary space. Again, there is a natural inclusion of the empty set in any topological space, but no projection.

A category has an initial point if there is an object $P$ and a unique morphism $P \to A$ for any object $A$. A category has a terminal point if there is an object $Q$ and a unique morphism $A \to Q$ for every object $A$.

Thus the category of topological spaces and continuous maps has an initial point, the empty set, and a terminal point, the point.

If a category is pointed, there is a unique morphism $A \circ B \to A \times B$. This is because one has $A \circ B \to A$ and $A \times B \to \circ B$.

and similarly $A \circ B \to B$

and so $f_{A \circ B} : A \circ B \to A \times B$.

For example, in the category of abelian groups and homomorphisms this map is an isomorphism.

In the category of topological spaces with base points and continuous base point preserving maps (a pointed category) $\times$ equals the cross product and $\circ$ equals disjoint union with base points identified and the natural map $A \circ B \to A \times B$ amounts to the inclusion $A \times B \cup A \times B \to A \times B$ where $a$ is the base point of $A$ and $b$ is the base point of $B$.

Now we will define an additive category. Let $\mathcal{C}$ be a category with product and coproduct (that is, product and coproduct exist
for any two objects of the category. Let \( \mathcal{A} \) be pointed.

Let the map \( A \oplus B \to A \times B \) be an isomorphism (equivalence).

There is always a natural diagonal map \( \Delta : A \to A \times A \)
and similarly a map \( \phi : A \oplus A \to A \).

If \( f, g : A \to B \), define the sum of \( f \) and \( g \) by
\[
A \xrightarrow{\Delta} A \times A \xrightarrow{\phi} B \oplus B \xrightarrow{f \oplus g} B
\]

It can easily be shown that under this operation
\( \text{Hom}_\mathcal{A}(A, B) \) is a monoid with identity \( A \to B \).

\( \mathcal{A} \) is said to be an additive category if this monoid
is a group for all \( A, B \).

2)

Let \( K \) be the field \( \mathbb{R} \) (reals), \( \mathbb{C} \) (complexes), or \( \mathbb{H} \) (quaternions).
Let \( X \) be a topological space. Consider the category of vector
bundles over \( X \) with field \( K \) and vector bundle maps.

There is a special vector bundle over \( X \), the one where
the fibre is zero. Denote this by \( * \). Given any bundle \( E \)
there are natural maps \( * \to E \to * \), and in this way the category
is pointed.

The Whitney sum of \( P, Q \), bundles over \( X \), is those \( (p, q) \)
in \( P \times Q \) such that \( \pi_p(p) = \pi_q(q) \), this set with the subspace
topology, etc. The Whitney sum provides a coproduct for the
Category.

\( K^{\mathbb{R} \times X} \) with the obvious projection is the canonical bundle
over \( X \) for any integer \( n \geq 0 \), where \( K^n = K \times K \times \cdots \times K \) \( n \) times.
\( K^{\mathbb{R} \times X} \) is also called the \( n \)-dimensional trivial bundle.

A bundle \( E \) over \( X \) is said to be invertible if there exists
a bundle \( F \) over \( X \) such that \( E \otimes F = K^{\mathbb{R} \times X} \).
If $X$ is compact, every vector bundle is invertible.

Let $\mathcal{A}_K(X)$ be the category of invertible vector bundles with field $K$ over $X$. $\mathcal{A}_K(X)$ is pointed, has coproduct as before.

A map $f: X \to Y$ induces a map $f^*: \mathcal{A}_K(Y) \to \mathcal{A}_K(X)$. For any $P$ in $\mathcal{A}_K(Y)$ is contained in a trivial bundle $K^n \times Y$. Extend $f$ to $f': K^n \times X \to K^n \times Y$. $f'^{-1}(P)$ is an invertible bundle over $X$.

The association $X \mapsto \mathcal{A}_K(X)$ is therefore a contravariant functor from topological spaces and maps into pointed categories.

3)

Let $\mathcal{A}$ be any category with coproducts and $\mathcal{J}$ a distinguished collection of objects such that the isomorphism classes of $\mathcal{J}$ form a set, and if $A, B$ are objects in $\mathcal{J}$, $A \otimes B$ is in $\mathcal{J}$.

Let $F$ be the free abelian group generated by the set of isomorphisms of elements of $\mathcal{J}$. Let $\langle A \rangle$ be the isomorphism class of $A$. Let $R$ be the subgroup of $F$ generated by elements of the form $\langle A \otimes B \rangle = \langle A \rangle \cdot \langle B \rangle$.

Then the Grothendieck group $G^\mathcal{J}_R = F/R$ and $[A]$ represents the equivalence class of $b\langle A \rangle$ in this group. If the equivalence classes of $\mathcal{A}$ form a set, $G(\mathcal{A})$ denotes $G^\mathcal{A}_R$.

For instance, let $K$ be a field and consider the category of finite dimensional vector spaces over the field. Isomorphism classes of these are uniquely determined by a single integer $n \geq 0$, the dimension. Then $G(\mathcal{A}) = \mathbb{Z}$, the integers.

Now we had a functor from topological spaces to pointed categories given by $X \mapsto \mathcal{A}_K(X)$. We define the Grothendieck group of $X$, for $K$ equal to the reals, complexes, or quaternions, by $G(\mathcal{A}_K(X))$ and denote them respectively by $K_0(X)$, $K_G(X)$, $K_{Sp}(X)$,
or just $K(X)$ where the field is understood.

Now we consider just this sort of situation in a slightly different context. Let $R$ be a ring. Let $P$ be the category consisting of all finitely generated projective left $R$ modules and $R$ homomorphisms.

This category has a Grothendieck group $G(R)$ and there is a canonical map $Z \to G(R)$ given by $1 \to [R]$. Note that in the previous example where $K$ was a field and everything was projective this map was an isomorphism.

Now $Z \to G(R) \to P(R) \to 0$ defines $P(R)$, the cokernel, as the projective class group of $R$.

Returning to the vector bundles and topological spaces we have a natural map $Z \to K(X)$ given by $1 \to [\text{trivial bundle}]$. The cokernel, denoted by $\overline{K}(X)$, is called the projective class group.

Let $\mathcal{R}$ denote the category of rings and ring homomorphisms. If $f: R \to S$, $S$ and all $S$ modules may be considered $R$ modules by the action $r \# s = f(r)s$. If $P$ is any finitely generated projective left $R$ module, then $S \otimes_R P$ is a finitely generated left $S$ module. The assignment $P \to S \otimes_R P$ induces the map $G(f): G(R) \to G(S)$, and $G$ is easily seen to be a functor on $\mathcal{R}$, and so is $P$, the projective class operator.

In reality, the range categories of $G$ and $P$ are not the same.

Let $\mathcal{A}$ be the category of abelian groups and homomorphisms.

$P: \mathcal{R} \to \mathcal{A}$.
Let $\mathcal{A}_o$ be the category whose objects $A$ are abelian groups together with maps $z \rightarrow A$. (i.e. $A$ has a distinguished element, the image of 1.) And those morphisms are homomorphisms $f$ such that 

$$
\begin{array}{c}
\eta_A \\
\eta_B
\end{array}
\xrightarrow{f}
\begin{array}{c}
A \\
B
\end{array}
$$

commutes.

$\mathcal{A}_o$ too has a coproduct, but not quite the usual one (direct sum).

Define $A\oplus B = A\oplus B/\text{subgroup generated by } (\eta_A(1) - \eta_B(1))$.

Define $Z \rightarrow A\oplus B$ by $z \rightarrow [\eta_A(z) \oplus 0] = [0 \oplus \eta_B(z)]$. $[,] = \text{equivalence class}$.

$\mathcal{A}_o$ is not pointed. $\mathcal{A}_o$ has the ordinary product, and $Z \rightarrow A \times B$ is given by $z \rightarrow \eta_A(z) \times \eta_B(z)$.

Now $\mathcal{A}_o$ is the proper range for $G$. $G : \mathcal{A} \rightarrow \mathcal{A}_o$.

There is a natural functor $\mathcal{A}_o \rightarrow \mathcal{A}$, which takes the cokernel of $\eta_A : Z \rightarrow A$, and one of course has commutative.

It is natural to ask whether the contragredient functor $K$ and the covariant functor $G$ behave nicely with respect to coproducts and products.

$\mathcal{A}$ has a product. The product of rings $R$ and $S$ is simply $R \times S$, the cross product, with termwise addition and multiplication. $R \rightarrow R \times S \rightarrow S$ becomes on application of $G$ $G(R) \leftarrow G(R \times S) \rightarrow G(S)$ and one can easily check that $G(R \times S)$ is the product of $G(R)$ and $G(S)$ in $\mathcal{A}_o$. 

$\text{5.}$
Analogously, the contravariant functor $K$ takes the co-product of two topological spaces, which is their disjoint union, into a product in $\mathcal{A}_0$. This follows easily from the fact that the union is disjoint and so the bundles above the union are simply sums of bundles over the two original spaces.

6)

Now assume the rings are commutative. If $P, Q$ are finitely generated projective left modules over $R$, $[P], [Q] \rightarrow [P \oplus Q]$ gives a well-defined bilinear map $G(R) \times G(R) \rightarrow G(R)$ which makes $G(R)$ a ring.

If $K$ is an arbitrary commutative ring, let $R(K)$ denote $G(K)$ with ring structure. Thus $R$ is a functor from commutative rings to commutative rings. The map $Z \rightarrow R(K)$ is a ring homomorphism.

If $P$ is a finitely generated projective module over $K$, a commutative ring, let $E(P)$ denote the exterior algebra of $P$.

$$E(P)_0 = K, \quad E(P)_1 = P, \quad \text{and } E(P)_r = 0 \text{ for } r \text{ large.}$$

$$E(K^n)_k = K_{(n)}^\bigoplus_k, \quad E(R \otimes Q)_k = \bigoplus_{i+j=k} E(P)_i \otimes E(Q)_j.$$  

Let $R(K)[[t]]$ denote the ring of formal power series in one indeterminate $t$. $R(K)[[t]]$ is the formal power series with leading coefficients 1, and is a group under multiplication.

We define a map $\lambda: R(K) \rightarrow R(K)[[t]]$ by

$$\lambda([P]) = \sum_{j=0}^{\infty} \lambda^k[P] t^k, \quad \text{where } \lambda^k[P] = [E(P)_k].$$

Note that $\lambda^k[P \oplus Q] = \sum_{i+j=k} \lambda^i[P] \lambda^j[Q]$. With these definitions $\lambda^0[P] = 1$ and $\lambda^1[P] = [P]$.  

Let \( x_1, \ldots, x_n, \ldots \) be a countable set of things which can be added and multiplied. Then the Newton formulas are:

\[
P_1 = x_1 \\
P_2 = p_1 x_1 - 2x_2 \\
P_3 = p_2 x_1 - p_1 x_2 + 3x_3 \\
\vdots \\
P_{n+1} = \sum_{j=1}^{n} (-1)^{j+1} p_n x_1 - jx_j + (-1)^{n+1}(n+1)x_{n+1}
\]

The original significance of the Newton formulas was this: Consider \( x_1, \ldots, x_n \) as indeterminates for a polynomial ring. The symmetric polynomials form a subring. Any symmetric polynomial can be expressed in terms of the so-called elementary symmetric polynomials. In particular:

- \( x_1 + \ldots + x_n = x_1 \) is the first elementary symmetric polynomial.
- \( x_1^2 + \ldots + x_n^2 \) is not elementary, but is symmetric. \( p_2 \) is the formula for expressing it as the sum of the first and second elementary polynomials.

Similarly, \( p_3 \) is the formula for expressing \( x_1^3 + \ldots + x_n^3 \) as the sum of \( x_1^2 + \ldots + x_n^2 \) and the first three elementary symmetric polynomials, and so on.

In our situation, the Adams operations can be defined by:

\[
\psi^1 = \chi^1 \\
\psi^2 = \psi^1 \chi^1 - 2\chi^2 \\
\vdots \\
\psi^{n+1} = \sum_{j=1}^{n} (-1)^{j+1} \psi^{n+1-j} \chi^j + (-1)^{n+1}(n+1)\chi^{n+1}
\]