Homotopy Theory of Spheres Seminar

I — Early Theory – Before Adams Spectral Seq.)

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Primary Structures

These are homotopy operations which were important in the study of the homotopy groups of spheres and which are now also of intrinsic interest

Composition

Simple composition of representative maps gives a map

$$\pi_p(S^q) \times \pi_q(S^r) \longrightarrow \pi_p(S^r)$$
$$\beta \qquad \alpha \longrightarrow \alpha \circ \beta$$

Note that we do not have a $\otimes$ sign, if

$$\beta, \beta' \in \pi_p(S^q)$$
$$\alpha, \alpha' \in \pi_q(S^r)$$

then

$$(\alpha+\alpha')\circ\beta \neq \alpha\circ\beta + \alpha'\circ\beta$$

in general, although

$$\alpha\circ(\beta+\beta') = \alpha\circ\beta + \alpha\circ\beta'$$

Whitehead Product

This is a pairing

$$\pi_p(X) \otimes \pi_q(X) \longrightarrow \pi_{p+q-1}(X)$$
$$\alpha \otimes \beta \longrightarrow [\alpha, \beta]$$

defined as follows:
If

$$[f] = \alpha \in \pi_p(X)$$
$$[g] = \beta \in \pi_q(X)$$

then $[\alpha, \beta]$ is the class of the map

$$S^{p+q-1} \xrightarrow{u_{p,q}} S^p \vee S^q \xrightarrow{f \vee g} X \vee X \xrightarrow{\text{'fold'}} X \ ;$$

$u_{p,q}$ may be described by

$$\begin{array}{c}
S^{p+q-1} \\
\downarrow \\
(S^{p-1} \times D^q) \cup (D^p \times S^{q-1}) \\
\begin{array}{cc} \downarrow & \downarrow \\ D^q & D^p \\ \downarrow & \downarrow \\ S^q & \vee \ S^p \end{array}
\end{array}$$

We use the bracket notation because the Whitehead product satisfies anticommutativity and 'Jacobi' identities similar to those of a Lie product in a Lie ring.

<u>Join</u>

This is a pairing

$$\pi_p(S^m) \otimes \pi_q(S^n) \longrightarrow \pi_{p+q+1}(S^{m+n+1})$$
$$\alpha \otimes \beta \longrightarrow \alpha * \beta$$

defined as follows:

If

$$[f] = \alpha \in \pi_p(S^m)$$
$$[g] = \beta \in \pi_q(S^n)$$

then $\alpha * \beta$ is the equivalence class of the suspension (see below) of

$$S^{p+q} \simeq S^p \# S^q \xrightarrow{f \# g} S^m \# S^n \simeq S^{m+n}$$

$(A \# B = A \times B / A \vee B)$

Suspension

The suspension functors are well known. We prefer to use the reduced suspension:

$$SX = X \times I / X \times \dot{I} \cup X_0 \times I$$

The points of SX are pairs (x,t) $x \in X$, $t \in I$ under proper identification.

Given $f : X \to Y$,

$$Sf : SX \to SY$$

sends (x,t) into (fx,t)

We give a second 'adjoint' point of view.

Let ΛX denote the Moore loop space of X; the points of ΛX are the maps

$[0,r] \to X$ such that $0 \le r < \infty$ and $0, r$ go into x_0.

Now let λ be a non-negative continuous function on SX positive except at the base point.

We define a map

$$i : X \to \Lambda SX$$

by

$$i(x) : [0, \lambda(x)] \longrightarrow SX$$

by

$$i(x)(t) = (x, t/\lambda(x)).$$

We give also a third point of view of suspension due to Blakers and Massey:

The difficulty of homotopy groups is that they do not obey excision, i.e., given $A, B \subset X$, the maps

$$\pi_r(A, A \cap B) \longrightarrow \pi_r(X, B)$$
$$\pi_r(B, A \cap B) \longrightarrow \pi_r(X, A)$$

induced by inclusion are <u>not</u>, in general, isomorphisms. Blakers and Massey defined homotopy groups of a triad which measure this failure to be an isomorphism; that is, there are long exact sequences

$$\to \pi_r(A, A\cap B) \to \pi_r(X, B) \to \pi_r(X; A, B) \to \pi_{r-1}(A, A\cap B) \to$$

$$\to \pi_r(B, A\cap B) \to \pi_r(X, A) \to \pi_r(X; A, B) \to \pi_{r-1}(B, A\cap B) \to$$

Let $S^n = E^{n+} \cup E^{n-}$ be the usual decomposition of the sphere into hemispheres. In the diagram

$$\begin{tikzcd}
\pi_r(E^{n+1+}, S^n) \arrow[r] \arrow[d, "\wr"] & \pi_r(S^{n+1}, E^{n+1-}) \arrow[r] & \pi_r(S^{n+1}; E^{n+1+}, E^{n+1-}) \\
\pi_{r-1}(S^n) \arrow[r, "E"] & \pi_r(S^{n+1}) \arrow[u, "\wr"] &
\end{tikzcd}$$

E is (up to sign) the Freudenthal suspension.

A crude form of the Freudenthal theorem asserts that

$$\pi_r(S^n) \xrightarrow{E} \pi_{r+1}(S^{n+1})$$

is an isomorphism (onto) if $r \leq 2n-2$
and is onto if $r = 2n-1$.
This just says that $\pi_r(S^{n+1}; E^{n+1+}, E^{n+1-}) = 0$
if $r \leq 2n-2$

Range

The Freudenthal Theorems lead naturally to the concept of range. The group

$$\pi_r(S^n)$$

is said to be in the stable rang if

$$r \le 2n-2$$

In this case $E^m: \pi_r(S^n) \to \pi_{r+m}(S^{n+m})$ will always be an isomorphism.
If $2n-1 \le r \le 3n-2$ then $\pi_r(R)$ is said to be in the metastable range; if $3n-2 < r$, premetastable. The significance of metastability will emerge later.

The Hopf Construction

If $[A, B]$ denotes the set of homotopy classes of maps $f: A \to B$, then the Hopf construction is a map

$$[S^p \times S^q, X] \longrightarrow \pi_{p+q+1}(SX)$$

$$\alpha \longrightarrow C\alpha$$

defined by composition of a representative of $S\alpha$ with $S^{p+q+1} \hookrightarrow S^{p+q+1} \vee S^{q+1} \vee S^{p+1} \xrightarrow[\simeq]{i} S(S^p \times S^q)$ where i is a homotopy equivalence.

Toda Brackets

These are 'higher order' composition operations.

Suppose we have

$$S^p \xrightarrow{f} S^q \xrightarrow{g} S^r \xrightarrow{h} S^t$$

such that

$$(*) \quad \begin{array}{l} gf \sim 0 \\ hg \sim 0 \end{array}$$

Then we may construct a diagram

$$\begin{tikzcd}
 & C^+S^p \arrow[r, "C^+f"] & C^+S^q \arrow[rrd, "F"] & & \\
S^{p+1} \arrow[ru] \arrow[rd] & S^p \arrow[u, phantom, "\cup"] \arrow[d, phantom, "\cap"] \arrow[r, "f"] & S^q \arrow[r, "g"] & S^r \arrow[r, "h"] & S^t \\
 & C^-S^p \arrow[rru, "G"] & & &
\end{tikzcd}$$

Where F, G are homotopies for $hg \sim 0$, $gf \sim 0$ respectively. It is trivial to verify that $F\,C^+f$ and hG agree on S^p and so fit together to give a continuous map $S^{p+1} \to S^t$; this map has a homotopy class which is well defined only modulo a certain subgroup of $\pi_{p+1}(S^t)$; we get a map

$\left(\begin{array}{l}\text{subset of}\\ \text{satisfying} *\end{array}\right. \pi_p(S^\ell)\times\pi_q(S^r)\times\pi_r(S^t)\Big) \longmapsto \pi_{p+1}(S^t)\big/\text{subgroup}$

The Hopf Invariant

Let $f: S^{2n-1}\to S^n$. If $y\in S^n$ then f is homotopic to a map g such that $g^{-1}(y)$ is a smooth submanifold of S^{2n-1}. For any two distinct points $y_1, y_2 \in S^n$ we may therefore assume $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are such subsets of S^{2n-1} and define the Hopf invariant of f to be their linking number.

More algebraically let $Y = S^n \cup_f e^{2n}$ be the union of the n-sphere and a $2n$-cell attached to it via the map f (which is defined on the boundary S^{2n-1} of e^{2n}). Then

$H^n(Y)\approx \mathbb{Z}$ generated by say u
$H^{2n}(Y)\approx \mathbb{Z}$ " " " v

One defines (up to sign) the Hopf invariant of f to be the integer $H(f)$ given by

$$u\cup u = H(f)\cdot v .$$

Defn. A map
$$f: S^p \times S^q \to S^n$$
is said to be of type (α, β)
$\alpha \in \pi_p(S^n)$, $\beta \in \pi_q(S^n)$ if
$$[f \circ \iota_1] = \alpha$$
$$[f \circ \iota_2] = \beta$$
where
$$\iota_1 : S^p \to S^p \times S^q$$
$$\iota_2 : S^q \to S^p \times S^q$$
are ~~th~~ the natural injections.

Suppose $S^{n-1} \times S^{n-1} \xrightarrow{f} S^{n-1}$ is of type $(r\iota, s\iota)$ where r, s are integers and ι is a generator of $\pi_{n-1}(S^{n-1})$. One proves that for the Hopf construction $c(f)$ of f,
$$H(c(f)) = r \cdot s \qquad \text{(up to sign)}.$$

Using this Hopf showed that $\pi_3(S^2)$ is $\approx \mathbb{Z}$ and is generated by his famous fibering map $S^3 \to S^2$ (see e.g. Steenrod, Topology of Fibre Bundles) There are similar fiberings
$$S^7 \to S^4 \quad \text{and} \quad S^{15} \to S^8$$

G.W. Whitehead generalized the Hopf invariant as follows: he viewed the Hopf invariant as a homomorphism H,

$$H : \pi_{2n-1}(S^n) \longrightarrow \mathbb{Z} = \pi_{n-1}(S^{n-1})$$

and sought to find similar homomorphisms of homotopy groups. In the exact homotopy sequence of the pair $(S^p \times S^q; S^p \vee S^q)$ one gets a splitting in every dimension via the maps induced by $\iota_1 \circ p_1$ and $\iota_2 \circ p_2$ in the diagram

$$\begin{tikzcd} & S^p \arrow[dr, "\iota_1"] & \\ S^p \times S^q \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & S^p \vee S^q \\ & S^q \arrow[ur, "\iota_2"'] & \end{tikzcd}$$

This yields short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \pi_{r+1}(S^p \times S^q, S^p \vee S^q) \arrow[r] & \pi_r(S^p \vee S^q) \arrow[r] & \pi_r(S^p \times S^q) \arrow[r] \arrow[l, bend left, "\text{splitting}"] & 0 \end{tikzcd}$$

what we are interested in is the specific projection that this induces,

$$\pi_{r+1}(S^p \times S^q, S^p \vee S^q) \longleftarrow \pi_r(S^p \vee S^q).$$

Now let $p = q = n$; one gets a commutative diagram as below if E^3 is an isomorphism, that is, if $r-2 \leq 2(2n-3)-2$

$$\begin{tikzcd} \pi_{r+1}(S^n \times S^n, S^n \vee S^n) \arrow[d, "\text{smash}"] & \pi_r(S^n \vee S^n) \arrow[l] & \pi_r(S^n) \arrow[l, "\text{pinch}"'] \arrow[lldd, "\text{generalized Hopf invariant}"] \\ \pi_{r+1}(S^{2n}, *) & & \\ \pi_{r-2}(S^{2n-3}) \arrow[u, "E^3"] & & \end{tikzcd}$$

The point is to get a map from the $(r-n)$-stem into, roughly, the $(r-2n)$-stem. Whithead proved

i) if $f: S^p \times S^q \to S^n$ is of type (α, β) and $c(f) \in \pi_{p+q+1}(S^{n+1})$, then a certain suspension of $c(f)$ has homotopy class $\pm \alpha * \beta$.

ii) the homomorphism (we call it $H(f)$) associated with such an f can be fit into an exact sequence starting with about

$\pi_{3n-1}(S^n) \xrightarrow{E} \cdots$

Replacing ΛSX by X_∞

For a space X with base point x_0 and for any integer n we define the reduced product space X_n as an identification space of X^n (the n-fold Cartesian product of X) indentifying two n-tuples if the ordered p-tuples obtained by deleting all occurences of x_0 are identical. There is a natural inclusion $X_n \subset X_{n+1}$ for each n. Let X_∞ be the direct limit of the X_n with suitable topology.

We recall that we previously defined a map

$$X \xrightarrow{i} \Lambda SX.$$

We will define a map

$$X_\infty \xrightarrow{w} \Lambda SX$$

by sending a point $(x_1, \ldots x_n) \in X^n$ (representing a point of X_∞)

2. into the Moore loop

$$\iota(x_1)\cdot\iota(x_2)\cdot\;\cdots\;\cdot\iota(x_n) \qquad \left[\begin{array}{l}\text{using the usual multiplication}\\ \text{in Moore loop spaces}\end{array}\right]$$

Since $\iota(x_0)$ is the identity in ΛSX, w respects the identifications made in defining X_n for each n. From w we get a map

$$(X_\infty, X) \rightarrow (\Lambda SX, X)$$

X being indentified on the one hand with $X_1 \subset X_\infty$ and on the other hand with $\iota(X) \subset \Lambda SX$. James proved that this map induces an isomorphism of long exact homotopy sequences. We replace some terms in the sequence of $(\Lambda SX, X)$ via the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_r(X) \arrow[r] \arrow[d, "\cong"] & \pi_r(X_\infty) \arrow[r] \arrow[d, "w_*", "\cong"'] & \pi_r(X_\infty, X) \arrow[r] \arrow[d, "w_*", "\cong"'] & \pi_{r-1}(X) \arrow[d, "\cong"] \\
\cdots \arrow[r] & \pi_r(X) \arrow[r] \arrow[dr] & \pi_r(\Lambda SX) \arrow[r] \arrow[d, phantom, "\cong"] & \pi_r(\Lambda SX, X) \arrow[r] \arrow[d, phantom, "\cong"] & \pi_{r-1}(X) \\
& & \pi_{r+1}(SX) \arrow[r] & \pi_{r+1}(SX; C^+X, C^-X) \arrow[ur] &
\end{tikzcd}$$

Now let $X = S^n$. S^n_2 is $S^n \times S^n$ with $(x, x_0), (x_0, x)$ identified. It is easy to see that $S^n_2 \approx S^n \cup e^{2n}$ (Use the relationship $S^n \times S^n = (S^n \vee S^n) \cup (S^n - \{x_0\} \times S^n - \{x_0\})$)
Let $f: S^n_2 \rightarrow S^{2n}$ be the map which identifies S^n to a point: $(S^n_2, S^n) \xrightarrow{f} (S^{2n}, *)$. We extend this map to a map $(S^n_\infty, S^n) \xrightarrow{g} (S^{2n}_\infty, *)$ as follows: if $(x_1, \ldots, x_m) \in X^m$ let g map the corresponding point of $S^n_m \subset S^n_\infty$ into the point of S^{2n}_∞ corresponding to the $\binom{m}{2}$-tuple

$$\left(f(x_1,x_2), f(x_1,x_3), \ldots\ldots, f(x_{m-1},x_m)\right) \in \left(S^{2n}_{\binom{m}{2}}\right)$$

Theorem. For n odd, or $r < 3n-3$ (roughly),

$$g_* : \pi_r(S^n_\infty, S^n) \to \pi_r(S^{2n}_\infty)$$

is an isomorphism. For n even g_* is an isomorphism on the 2-primary components of these groups.

The diagram below comes from the above one by replacing X by S^n

$$\begin{tikzcd}
 & \pi_{r+1}(S^{n+1}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \pi_r(S^{n+1}; C^+S^n, C^-S^n) \arrow[d, phantom, "\cong" rotate=90] & \\
\pi_r(S^n) \arrow[r] & \pi_r(\Omega S S^n) \arrow[r] \arrow[d, "w_*"] & \pi_r(\Omega S S^n, S^n) \arrow[r] \arrow[d, "w_*"] & {} \\
 & \pi_r(S^n_\infty) & \pi_r(S^n_\infty, S^n) \arrow[d, "g_*"] & \\
 & & \pi_r(S^{2n}_\infty) &
\end{tikzcd}$$

From this we get a map H:

$$\begin{tikzcd}
\pi_r(S^n_\infty, S^n) \arrow[r, "g_*"] \arrow[d, phantom, "\cong" rotate=90] & \pi_r(S^{2n}_\infty) \arrow[d, phantom, "\cong" rotate=90] \\
\pi_{r+1}(S^{n+1}; C^+, C^-) \arrow[r, "h"] & \pi_{r+1}(S^{2n+1}) \\
\pi_{r+1}(S^{n+1}) \arrow[u] \arrow[ur, "H"'] &
\end{tikzcd}$$

This map fits into an exact sequence of James

Indicative of the nature of James' results is the following assertion:

$\pi_r(S^{n+1})$ has no element of order 2^k where

$k = n+1$ (n even)

or

$k = 2n+2$ (n odd)

Fibre Space Methods

Serre made the following construction: Given a space X, let $X_0 = X$ and let T_1 be a universal cover for X (supposing it exists). Let $X_1 = \Omega T_1$. Continue in this manner; $\pi_{n+1}(X) \approx \pi_1(X_n)$. Serre uses 2 spectral sequences and gets results like the following:

1) $\pi_r(S^n)$ is finite except for a) $r=n$ b) $n=2m$, $r=4m-1$

2) $_p\pi_i(S^{2m}) \approx {}_p\pi_{i-1}(S^{2m-1}) \oplus {}_p\pi_i(S^{4m-1})$

where the isomorphism is induced by

$$(\alpha, \beta) \longrightarrow E\alpha + [\iota_{2m}, \iota_{2m}] \circ \beta$$

$$S^i \xrightarrow{\beta} S^{4m-1} \xrightarrow{[i_{2m}, i_{2m}]} S^{2m}$$ (Whitehead product)

Hilton has also generalized the Hopf invariant using $\pi_r(S^n \vee S^n)$.

Homotopy of Spheres Seminar
6/30/65

Begining of Adams' Spectral Sequence

This is a real step forward in the use of algebraic and homological, as opposed to geometric methods. Its start was given by the introductions of cohomology operations. The operations we will be concerned with are the Steenrod squares, $Sq^i: H^n(X;Z_2) \to H^{n+i}(X;Z_2)$ defined for all $i \geq 0$ and each space X, with the following properties

(1) $f: Y \to X$; then $Sq^i f^* = f^* Sq^i$

(2) $Sq^0 =$ identity

(3) $Sq^n x = x^2$ is x is n dimensional

(4) Cartan formula $Sq^n xy = \sum_{i=0}^{n} Sq^{n-i} x \, Sq^i y$

Under composition ~~these for~~ the Sq^i; form an algebra over Z_2. This is graded by grad $(Sq^{i_k} \cdots Sq^{i_1}) = i_k + \cdots + i_1$. And if we denote this algebra by A, there is an action of A on $H^*(X;Z_2)$ for all space X, such that $H^*(X;Z_2)$ is a graded A module; $a_p \cdot H^q \subset H^{p+q}$.

The cartan formula gives a map $\psi: A \to A \otimes A$ given by $\psi(Sq^i) = \sum_{k=0}^{i} Sq^{i-k} \otimes Sq^k$. This is an algebra Homomorphism and makes A into a Hopf algebra.

We would like to use homological properties of A to get the homotopy groups of spheres in the stable range. First we consider the cohomology of A.

Z_2 is a graded A module as follows $Sq^0 \in A$ is identity $Sq^0 \cdot 1 = 1$, and everything else in A acts like 0. The degree of $1 \in Z_2$ will be 0. Then the cohomology of A is $Ext_A^{s,t}(Z_2, Z_2)$. This is defined in the usual way., ie Let $\{C_s\}$ be an A-free resolution of Z_2., we have an exact sequence

$$\cdots \to C_s \xrightarrow{d_{s-1}} \cdots \to C_1 \xrightarrow{d_0} C_0 \xrightarrow{\epsilon} Z_2 \to 0$$

where the C_i's are graded free A modules, and the d's and ϵ are degree preserving maps. Let $Hom_A^t(C_s; Z_2)$ denote the set of A-homomorphisms from $C_s \to Z_2$ which lower degree by t. Then we get a complex

$$\cdots \leftarrow Hom_A^t(C_s; Z_2) \xleftarrow{d_{s-1}^*} Hom_A^t(C_{s-1}; Z_2) \leftarrow \cdots$$

and taking its Homology we get $Ext_A^{s,t}(Z_2, Z_2)$.

Moreover this has a ring structure induced by ψ. ie, If $\{C_s\} = C$ is an A free resolution of Z_2; then the complex $C \otimes C$ $\{(C \otimes C)_m = \sum_{i+j=m} C_i \otimes C_j\}$ has an A module structure given by $a(c_1 \otimes c_2) = \psi(a)(c_1 \otimes c_2) = \sum a_i c_1 \otimes a_i' c_2$ if $\psi(a) = \sum a_i \otimes a_i'$

Furthermore there is a map $m: C \to C \otimes C$. Composing m_* with the obvious map of induced by the map of $Hom_A(C; Z_2) \otimes Hom_A(C; Z_2) \to Hom_A(C \otimes C'; Z_2)$ gives the product.

Now we are ready to state the main theorem:

There is a spectral sequence $\{E_r^{s,t}\}$ with differential $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,\,t+r-1}$ such that

(1) $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(Z_2; Z_2)$

(2) $E_r^{s,t} \Rightarrow {}_2\pi^S$, ie $\exists$ a sequence of groups $B^{i,j}$ $\ni$ ${}_2\pi^S_m = B^{0,m} \supset B^{1,m+1} \supset B^{2,m+2} \supset \cdots$

where $B^{s,t}/B^{s+1,t+1} \simeq E_\infty^{s,t}$

[ie the $E_\infty^{s,t}$'s which contribute to ${}_2\pi^S_m$ are those where $t-s=m$.]

${}_2\pi^S_m$ = 2 primary component of the stable m stem of homotopy groups of spheres, ie ${}_2\lim_K [S^{K+m}; S^K]$

Look at this for small s,t. $\mathrm{Ext}_A^{1,*}(Z_2, Z_2)$ has an element in each dimension of form $2^i - 1$. One can easily calculate:

E^2:

s \ $t-s$	0	1	2	3	4	5	6	7
4	h_0^4							$h_0^3 h_3$
3	h_0^3			$h_1^3 = h_0^2 h_2$				$h_0^2 h_3$
2	h_0^2		h_1^2	$h_0 h_2$			h_2^2	$h_0 h_3$
1	h_0	h_1		h_2				h_3
0	1							

where product is that of Ext. Now the differentials d_r go up r and back 1. The product structure is such that d_r is a derivation $\forall r$. This is very helpful in computing the d's, which are in general very difficult

But we use product structure. Clear that only possible non-zero d is $d_r h_1$ for some r. If

$d_r h_1 \neq 0$ then, $d_r h_1 = h_0^{r+2}$

so $0 = d_r 0 = d_r h_0 h_1 = h_0 d_r h_1 = h_0^{r+3} \neq 0$.

Also there is a product in ${}_2\pi^S$; $\pi_k^S \otimes \pi_l^S \to \pi_{k+l}^S$ induced by composition. This induces a product in E^∞ which is the same as that induced from E_2. So we can use the products to calculate the group extensions. Mult by h_0 corresponds to mult by 2 in ${}_2\pi^S$ so we have ${}_2\pi_0^S = Z$; ${}_2\pi_1^S = {}_2\pi_2^S = {}_2\pi_6^S = Z_2$ $\pi_4^S = \pi_5^S = 0$; ${}_2\pi_3^S = Z_8$ and ${}_2\pi_7^S = Z_{16}$. These are right answers.

Question Left is where this comes from. Will set up an exact couple to give this spectral sequence. For convenience, Let $H^*(X)$ denote reduced cohomology with coef. Z_2. Let X and Y be f. d. C.W. complexes with $H^*(X)$; $H^*(Y)$ finitely generated in each dimension. Let $\{C_s\}$ be an A free resolution of $H^*(Y)$. Define a realization of $C = \{C_s\}$ to be a collection of: a sequence of spaces K_s such that $H^*(K_s) \cong_A C_s$ and $\pi_t(R; K_s) = \mathrm{Hom}^t(C_s; H^*(R))$ {where $\pi_t(R; K_s)$ $= \lim\limits_k [S^{t+k} R; S^k K_s]$} ($K_s$ will be stably a product of $K(\pi, n)$'s and a sequence of spaces M_s and maps $f_s: M_{s-1} \to K_s$ such that $M_{-1} = Y$;

$M_s = K_s \cup_{f_s} CM_{s-1}$ and

$$K_s \xrightarrow{j_s} K_s \cup_{f_s} M_{s-1} = M_s \xrightarrow{f_s} K_{s+1}$$

induces $d_s : \underset{\substack{\shortparallel \\ C_{s+1}}}{H^*(K_{s+1})} \to \underset{\substack{\shortparallel \\ C_s}}{H^*(K_s)}$ and $f_{-1} : Y \to K_0$ induces $\epsilon : C_0 \to H^*(Y)$

One obtains such a realization by induction: First choose $f_0 \in \pi_0[Y; K_0] \ni f_{0*} = \epsilon : C_0 \to H^*(Y)$. Form $M_0 = K_0 \cup_{f_0} CY$. This gives an exact sequence

$$0 \to H^t(M_0) \xrightarrow{j} H^t(K_0) \xrightarrow{f_{0*}} H^t(Y) \to 0$$

since f_{0*} is epi. Now $H^*(M_0) \cong \operatorname{Ker} \epsilon$ ∴ we have a map $d_0 : C_1 \to \operatorname{Ker} \epsilon = H^*(M_0)$. Choose $K_1 \ni H^*(K_1) = C_1$ and map $f_1 : M_0 \to K_1$ such that $f_1^* = d_0$. continue in same way.

Now Look at sequence

$$\cdots \to M_{s-1} \xrightarrow{f_s} K_s \xrightarrow{j_s} M_s \xrightarrow{g} SM_{s-1}$$

where g is inclusion of M_s into $SM_{s-1} = M_s \cup_{j_s} CK_s$.
apply functor $\pi_t(X; \ )$ and get

$$\cdots \to \pi_t(X; M_{s-1}) \to \pi_t(X; K_s) \to \pi_t(X; M_s) \to \underset{\substack{\wr\shortparallel \\ \pi_{t-1}(X; M_{s-1})}}{\pi_t(X; SM_{s-1})} \to$$

summing everything we get an exact couple

$$\begin{tikzcd}
\sum_{s,t} \pi_t(X; M_s) \arrow[rr, "q_s^*"] & & \sum_{s,t} \pi_t(X; M_{s-1}) \arrow[dl, "f_{s*}"] \\
& = \sum_{s,t} \pi_t(X; K_s) \arrow[ul, "j_{s*}"] &
\end{tikzcd}$$

and assume q_{s*} is iso and $\pi_t(X; K_s) = 0$ if $s < 0$.

Let $\pi_t(X; K_s) = E_1^{s,t}$ This is $\mathrm{Hom}_A^t(H^*(K_s); H^*(X))$ $= \mathrm{Hom}_A^t(C_s; H^*(X))$ and the map d_1 of the exact couple $= f_* j_*$ is that induced by $d: C_{s+1} \to C_s$.

So $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(Y); H^*(X))$.

Claim this converges to ${}_2\pi_{t-s}(X; Y)$

Setting $X = Y = S^0$, we obtain the desired spectral sequence.

Homotopy Theory of Spheres Seminar

The Lower Central Series of Group Complexes

7-7-65 Speaker: D. M. Kan

It has long been known that the homology groups of a space yield information about its homotopy groups. For example:

<u>Poincaré</u> If $\pi_0 X = 0$ there is an epi

$$\pi_1 X \twoheadrightarrow H_1 X = \text{abelianization}\, \pi_1 X$$

<u>Hurewicz</u> If $\pi_i X = 0$, $0 \le i < n$, $n \ge 2$

then $\begin{cases} \pi_n X \xrightarrow{\approx} H_n X & \text{iso} \\ \pi_{n+1} X \twoheadrightarrow H_{n+1} X & \text{epi} \end{cases}$

<u>J.H.C. Whitehead</u> If $\pi_i X = 0$, $0 \le i < n$, $n \ge 2$ then the Hurewicz homos $\pi_i X \to H_i X$ may be put in a certain exact sequence

$$\cdots \to \pi_{i+1} X \to H_{i+1} X \to \Gamma_i X \to \pi_i X \to \Gamma_{i-1} X \to \cdots$$

where $\Gamma_i X = \operatorname{im}(\pi_i X^{i-1} \to \pi_i X^i)$.

The Hurewicz theorem is equivalent to the fact that $\Gamma_i X = 0$ for $i \le n$.

Our purpose is to make sense out of these results and to show that infact $\pi_* X$ is related to $H_* X$ in a much stronger way.

We shall need both semi-simplicial (s.s.) complexes and ss group complexes. These correspond respectively to topological spaces and topological groups. On the category $\mathcal{G}^\Delta$ of s.s. group complexes, a group homotopy relation may be defined between maps and is analygous to that for topological groups. Infact the singular functor

Sin: topological groups $\longrightarrow \mathcal{G}^\Delta$

preserves this relation.

If X is a connected s.s. complex with base, we may define an s.s. group complex GX which serves as the loop space of X. The group homotopy type of GX determines the homotopy type of X and vice versa.

If $\mathcal{G}$ is the category of groups, a functor $T: \mathcal{G} \longrightarrow \mathcal{G}$ induces a functor $T: \mathcal{G}^\Delta \longrightarrow \mathcal{G}^\Delta$. Such an induced functor preserves group homotopies.

<u>Example</u>: For $C \varepsilon \mathcal{G}$ and a fixed prime p let

$$TC = \{[\sigma, \tau], \rho^p \mid \sigma, \tau, \rho \varepsilon C\},$$ i.e.

subgroup of C generated by commutators, p^{th} powers, If X is a connected s.s. complex with base, the filtration $\cdots \subset T^n GX \subset \cdots \subset T^2 GX \subset TGX \subset GX$ gives rise to a homotopy exact couple. This spectral sequence $E(X)$ is a homotopy invariant of X. There is a sequence of spectral sequences induced by suspension

$$E(X) \longrightarrow E(SX) \longrightarrow E(S^2X) \longrightarrow \cdots$$

The limit spectral sequence $\bar{E}(X)$ is the Adams spectral sequence for the prime p.

Taking $p = 0$ so $TC = [C, C]$, a sort of integral Adams spectral sequence is obtained.

The Hurewicz Homomorphism

Let X be a connected s.s. complex with base. The loop space GX is an s.s. free group complex with $\pi_{n-1} GX = \pi_n X$. Let AX = abelianization of GX. Then AX is a free abelian group complex with $\pi_{n-1} AX = \tilde{H}_n X$. The Hurewicz map arises from the natural homomorphism $GX \longrightarrow AX$.

$$\begin{array}{ccc} \pi_{n-1} AX & \approx & \tilde{H}_n X \\ \uparrow & & \uparrow \\ \pi_{n-1} GX & \approx & \pi_n X \end{array}$$

A Certain Exact Sequence

There is a fibration

$$\begin{array}{ccc} \Gamma_2 GX & \longrightarrow & GX \\ & & \downarrow \\ & & AX \end{array}$$

If X is simply connected there is an isomorphism $\Gamma_{n-1}X \approx \pi_{n-2}\Gamma_2 GX$ such that

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_n GX & \longrightarrow & \pi_n AX & \longrightarrow & \pi_{n-1}\Gamma_2 GX & \longrightarrow & \pi_{n-1} GX & \longrightarrow \cdots \\ & \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & \\ \cdots \longrightarrow & \pi_{n+1} X & \longrightarrow & \tilde{H}_{n+1} X & \longrightarrow & \Gamma_n X & \longrightarrow & \pi_n X & \longrightarrow \cdots \end{array}$$

commutes.

If X is simply connected, the Hurewicz theorem is thus equivalent to the following:

connectivity $\Gamma_2 GX \geqq 1 +$ connectivity GX

The Lower Central Series

For $B \varepsilon \mathcal{G}$ let $\Gamma_1 B = B$ and $\Gamma_{r+1} B = [\Gamma_r B, B]$ for $r \geqq 1$. This filtration of B by normal subgroups is called the lower central series.

$$\cdots \subset \Gamma_n B \subset \cdots \subset \Gamma_2 B \subset \Gamma_1 B = B$$

If B is a free group then there is a natural isomorphism $\dfrac{\Gamma_r B}{\Gamma_{r+1} B} \approx L^r\left(\dfrac{\Gamma_1 B}{\Gamma_2 B}\right)$ where $L = \sum_{r=0}^{\infty} L^r$ is the free Lie ring functor.

If now B is an s.s. free group complex then $\pi_* \dfrac{\Gamma_r B}{\Gamma_{r+1} B}$ is determined by $\pi_* \dfrac{\Gamma_1 B}{\Gamma_2 B}$

This follows from a theorem of Dold. Namely, let $\mathcal{A}$ be the category of abelian groups, $T: \mathcal{A} \longrightarrow \mathcal{G}$ a functor and $T: \mathcal{A}^\Delta \longrightarrow \mathcal{G}^\Delta$ the induced functor. If $B, B' \varepsilon \mathcal{A}^\Delta$ are free abelian with $\pi_* B \approx \pi_* B'$ then TB, TB' have the same group homotopy type, so $\pi_* TB \approx \pi_* TB'$.

A Spectral Sequence

If X is a connected s.s. complex with base then the fibrations

$$\begin{array}{ccccccc} \cdots \longrightarrow & \Gamma_3 GX & \longrightarrow & \Gamma_2 GX & \longrightarrow & GX \\ & \downarrow & & \downarrow & & \downarrow \\ & \dfrac{\Gamma_3 GX}{\Gamma_4 GX} & & \dfrac{\Gamma_2 GX}{\Gamma_3 GX} & & AX \end{array}$$

give rise to a homotopy exact couple. For this spectral sequence

$$E'(X) = \sum_{r=1}^{\infty} \pi_* \frac{\Gamma_r GX}{\Gamma_{r+1} GX} = \pi_* L(AX) = E'(H_* X)$$

For X simply connected, $E(X)$ converges to $\pi_* GX$. This follows from a theorem due to Curtis. Namely, that

connectivity $\Gamma_r GX \geqq \{\log_2 r\}$ + connectivity GX

where $\{i\}$ denotes the next integer $\geqq i$.

Furthermore one obtains a new proof of the result that if X simply connected, finite then $\pi_* X$ is computable.

The above may be generalized as follows. Let Y be an ss complex, and X a connected ss complex with base. There are fibrations of function complexes.

$$\begin{tikzcd}
\cdots \arrow[r] & (\Gamma_3 GX)^Y \arrow[r] \arrow[d] & (\Gamma_2 GX)^Y \arrow[r] \arrow[d] & (GX)^Y \arrow[d] \\
& \left(\frac{\Gamma_3 GX}{\Gamma_2 GX}\right)^Y & \left(\frac{\Gamma_2 GX}{\Gamma_3 GX}\right)^Y & (AX)^Y
\end{tikzcd}$$

The associated spectral sequence has

$$E^1 = \pi_* \sum_n (L^n AX)^Y = \sum_n H^*(Y; \pi_* L^n AX) = E^1(H_* Y; H_* X)$$

If $\pi_i X = 0$, $i \leq 1$ and Y finite then

E converges to $\sum_{k=2}^{\infty} [S^k Y \to X]$

If X is also finite then $[S^k Y \to X]$ is computable for $k \geqq 2$.

On the Groups $E'(S^{n+1})$

$$E'(S^{n+1}) = \pi_* \sum L^r K(Z,n) = \pi_* LK(Z,n)$$

The k-fold suspension homomorphism is

$$\sigma^k : \pi_q LK(Z,n) \longrightarrow \pi_{q+k} LK(Z,n+k)$$

assume now and henceforth that n is even. Then σ^k is a monomorphism onto a direct summand.

By an argument of Dold-Puppe this stability implies:

$\pi_* L^r K(Z,n) = 0$ unless $r=1$ or $r=p^j$ for some $j>0$, p prime.

$p\, \pi_* L^{p^j} K(Z,n) = 0$ for $j>0$ and p prime

A simple consequence of this is that $\pi_i S^{n+1}$ is finite except for $i = n+1$

Let $\alpha_{n+1} \in \pi_{2n+2} L^2 K(Z,n+1) = Z$ be a generator. If $\alpha_{n+1} : K(Z,2n+2) \longrightarrow L^2 K(Z,n+1)$ represents α_{n+1}, then $L(\alpha_{n+1})$ induces the composition homomorphism

$$\alpha_{n+1} : \pi_* LK(Z,2n+2) \longrightarrow \pi_* LK(Z,n+1)$$

One obtains the following direct sum decomposition:

$$\begin{matrix} \pi_q LK(Z,2n+2) \xrightarrow{\alpha_{n+1}} \\ \qquad\qquad \pi_q LK(Z,n+1) \\ \pi_{q-1} LK(Z,n) \xrightarrow{\sigma} \end{matrix}$$

Thus $E'(S^{n+2}) = E'(S^{n+1}) + E'(S^{2n+3})$

Hence $\pi_q S^{n+2}$ is finite except for $q = n+2, 2n+3$

A Version of the Hopf Invariant 1 Problem

There is a suspension map of spectral sequences $E(S^{n+1}) \xrightarrow{\sigma} E(S^{n+2})$
If there exists a map $GS^{2n+3} \longrightarrow \Gamma_2 GS^{n+1}$ such that

$$\begin{tikzcd}
GS^{2n+3} \arrow[rr] \arrow[d] & & \Gamma_2 GS^{n+1} \arrow[d] \\
AS^{2n+3} = K(Z, 2n+2) \arrow[rr, "\alpha_{n+1}"] & & L^2 K(Z, n+1) = L^2 AS
\end{tikzcd}$$

commutes, then it induces a map $E(S^{2n+3}) \longrightarrow E(S^{n+2})$ and

$$E(S^{n+2}) = E(S^{n+1}) + E(S^{2n+3})$$

Consequently $\pi_q S^{n+2} = \pi_{q-1} S^{n+1} + \pi_q S^{2n+3}$ and hence there is a map $S^{2n+3} \longrightarrow S^{n+2}$ of Hopf invariant 1.

Conversely if such a map $S^{2n+3} \longrightarrow S^{n+2}$ exists then there is a map $GS^{2n+3} \longrightarrow \Gamma_2 GS^{n+1}$ such that the above diagram commutes.

An Application

The diagram

$$\begin{tikzcd}
GS^{2n+3} \arrow[r, "{[\ ,\ ]}"] \arrow[d] & \Gamma_2 GS^{n+2} \arrow[d] \\
AS^{2n+3} = K(Z, 2n+2) \arrow[r, "2\alpha_n"] & L^2 K(Z, n+1) = L^2 A S^{n+2}
\end{tikzcd}$$

commutes.

If p is an odd prime then

$$E'(S^{n+2};p) \approx E'(S^{n+1};p) + E'(S^{2n+3};p)$$

Using the suspension map $E(S^{n+1};p) \to E(S^{n+2};p)$ and the map $E(S^{2n+3};p) \longrightarrow E(S^{n+2};p)$ induced by $[\ ,\ ] : GS^{2n+3} \longrightarrow \Gamma_2 GS^{n+2}$ it follows that

$$E(S^{n+2};p) \approx E(S^{n+1};p) + E(S^{2n+3};p)$$

Hence $\pi_q(S^{n+2};p) = \pi_{q-1}(S^{n+1};p) + \pi_q(S^{2n+3};p)$

Homotopy Theory of Spheres Seminar

7/14/65 Speaker: E.B. Curtis

A. Let X be a simply connected semi-simplicial complex with base point. We wish to study $\pi_*(X)$.

We can define a semi-simplicial group complex GX which serves as the loop space of X, and $\pi_{q+1}(X) \cong \pi_q(GX)$. $(GX)_n$ is a free group, and the face and degeneracy operators are homomorphisms.

Now take $\Gamma_2 GX \subset GX$, the commutator subgroup, $(\Gamma_2 GX)_n = [(GX)_n, (GX)_n]$, and define:

$$\begin{array}{l} \Gamma_2 GX \subset GX \\ \qquad\qquad \downarrow \text{ fibration} \\ \qquad\qquad GX/\Gamma_2 GX = AX \text{, free abelian.} \end{array}$$

e.g. if $X = S^{n+1}$, $AX = K(Z, n)$

We have the commutative diagram:

$$\begin{array}{ccc} \pi_{q+1}(X) & \cong & \pi_q(GX) \\ \downarrow & & \downarrow \\ H_{q+1}(X) & \cong & \pi_q(AX) \end{array}$$

where the left map is the usual Hurewicz homomorphism and the right map is induced by the natural projection.

We may filter GX thusly:

$$\cdots \Gamma_{r+1} GX \subset \Gamma_r GX \subset \cdots \subset \Gamma_3 GX \subset \Gamma_2 GX \subset GX$$
$$\downarrow$$
$$\Gamma_r GX/\Gamma_{r+1} GX \qquad \text{where } \Gamma_{r+1} GX = [\Gamma_r GX, GX]$$

Now $\Gamma_r GX / \Gamma_{r+1} GX \cong L^r(AX)$, where L^r is a functor from abelian groups to abelian groups defined purely in terms of AX. (If M is an abelian group, take $N(M)$ = the non-associative algebra generated by M $= M + M\otimes M + M\otimes(M\otimes M) + (M\otimes M)\otimes M + \cdots$. Let I be the ideal generated by $[x,y]+[y,x]$, and $[[x,y],z]+[[z,x],y]+[[y,z],x]$ (where here $[x,y] = x\otimes y$) Then $L^r(M) = N^r(M)/I$). $\sum_{r\geq 1} L^r(AX) = L(AX)$ = the Lie ring generated by AX.

Now take the homotopy exact couple of the filtration. We are interested in $E^1_{r,n}(X) = \pi_n(L^r(AX))$

In particular, we wish to find $E^1_{r,q}(S^{n+1})$ or that is $\pi_q L^r K(Z,n)$. We will denote $K(Z,n)$ by just (Z,n) henceforth. The results that follow are due to Kan, Schlesinger, & Curtis.

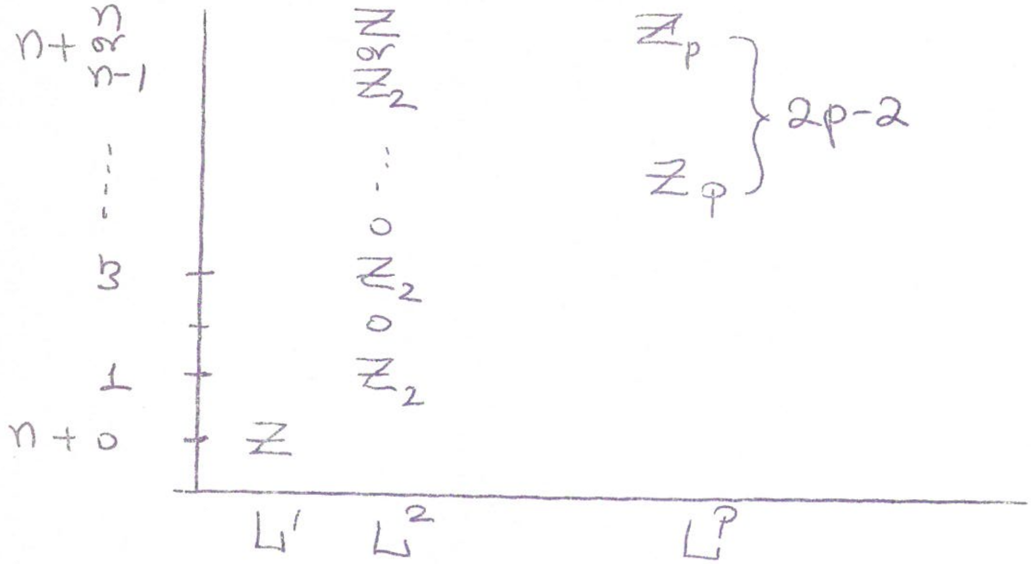
B. Now L^1 = identity functor, so $\pi_* L^1(Z,n)$ = Z in dim n, 0 otherwise.

$\pi_* L^2(Z,n)$ is generated by elements $w_1, w_3, w_5 \cdots w_n$ n odd or w_{n-1} n even where $\dim w_i = n+i$ and $2w_i = 0$, except that w_n has ∞ order if n is odd.

$\pi_* L^p(Z,2m)$, p an odd prime, has generators $a_1, a_2, \cdots a_m$ of order p.

The stable dimension of a_k is $(2p-2)\cdot k-1$. There is nothing of infinite order.

Also, for $r \geq 2^k$, $L^r(\mathbb{Z},n)$ is $n-1+k$ connected.



C. Consider the k-fold suspension:

$$\pi_q L^r(\mathbb{Z},n) \xrightarrow{\sigma^k} \pi_{q+k} L^r(\mathbb{Z},n+k).$$

The suspension has these properties:

1. If n is even, σ^k is a monomorphism onto a direct summand.

2. For an element $u \in \operatorname{im}\sigma$, $\binom{r}{i}u = 0$ for $r \geq 2$, $i = 1,2,\ldots r-1$. The order will be $\leq \gcd\limits_{1\leq i\leq r-1} \binom{r}{i}$.

If $r = p^j$, $\text{order}_r\, u = p$

$r \neq p^j$, $\text{order}_r\, u = 1$.

Hence $\pi_* L^r(\mathbb{Z}, \text{even})$ has exponent p (if non-trivial) if $r = p^j$, and $= 0$ if $r \neq$ prime power.

3. If r is odd (p^j only non-trivial possibility)

$\pi_q L^r(Z, \text{even}) \xrightarrow{\sigma'} \pi_{q+1} L^r(Z, \text{even}+1)$ is an isomorphism

4. If r is even and n is even

$$\begin{array}{l} \pi_q L^r(Z,n) \xrightarrow{\ \sigma\ } \pi_{q+1} L^r(Z,n+1) \\ \oplus \\ \pi_{q+1} L^{r/2}(Z,2n+2) \nearrow_{w_{n+1}\circ} \end{array}$$

(composition)

(n being even, there is a w_{n+1})

provides a direct sum decomposition, for we have

$(Z, 2n+2) \xrightarrow{w_{n+1}} L^2(Z, n+1)$ and so get

$L^{r/2}(Z, 2n+2) \xrightarrow{L^{r/2}(w_{n+1})} L^{\frac{r}{2}} L^2(Z, n+1) \longrightarrow L^r(Z, n+1)$,

where $w_{n+1}(i_{2n+2}) = \frac{1}{2}[[i_{n+1}, i_{n+1}]]$ (Whitehead product)

generator

$= \sum (-1)^* \left[\begin{array}{ll}\text{degeneracy } i_{n+1} & \text{complementary degeneracy } i_{n+1} \\ \text{raising dimension by } n+1, & \text{raising dimension by } n+1\end{array}\right]$

$*$ = sign of degeneracy.

e.g. $w_1 = \frac{1}{2}\left[[s_0 i_1, s_1 i_1] - [s_1 i_1, s_0 i_1]\right]$.

Hence:

Cor: All we need is $\pi_* L^r(Z, \text{even})$, all r.

I. Let $r = 2^j$. $\pi_* L^{2^j}(Z, n) = ?$ (n even)

We have $0 \to L^{2^j}(Z, n) \xrightarrow{2} L^{2^j}(Z, n) \xrightarrow{\eta} L^{2^j}(Z_2, n) \to 0$

where (Z_2, n) is a vector space over Z_2.

Lemma:

$$\begin{tikzcd}
\pi_{q-2} L^{p^j}(Z_p, 2n-2) \arrow[rr] & & \pi_q L^{p^j}(Z_p, 2n) \\
\oplus & & \\
\pi_q L^{p^{j-1}}(Z_p, 2np-1) \arrow[uurr, "(\ )\circ"] & & \\
\oplus & & \\
\pi_q L^{p^{j-1}}(Z_p, 2np) \arrow[uuuurr, "(\ )\circ"'] & &
\end{tikzcd}$$

composition by unspecified elements

gives a direct sum decomposition.

$\pi_* L^p(Z_p, 2n)$ has generators $a_1, \dots, a_n$, $b_1, \dots, b_n$. $\dim a_i = (2p-2)i-1$, $\dim b_i = (2p-2)\cdot i$.

Thm: $L^{p^j}(Z_p, n)$ has for generators $\{w_I\}$, $I = i_1 \cdots i_j$,

$w_{i_1 i_2 \cdots i_j} = w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_j}$ where

$i_k \equiv 0, -1 \mod (2p-2)$, $i_1 \le (p-1)n$, and $i_k \le p \cdot i_{k-1}$ for $k = 2, \dots, j$, and where if $i = (2p-2)h - 1$, $w_i = a_h$ and if $i = (2p-2)\cdot h$, $w_i = b_h$.

Im η_* is those w_I with $i_j \equiv -1 \mod (2p-2)$

E. $L^{2^j}(Z, n)$ has $\{\beta_I\}$ for generators, where $I = (i_1, \dots, i_j)$ and $\eta_*(\beta_I) = w_I$ and $i_1 \le n$, $i_2 \le 2i_1, \dots, i_j \le 2i_{j-1}$ and i_j odd. Filtration of β_I (that is, lowest sphere on which it is defined) is i_1. Hence it occurs in $\pi_*(\Omega S^{i_1+1})$.

This induces

$$\begin{tikzcd} & & 0 \arrow[dl] & \\ \pi_q L^{2^j}(Z,n) \arrow[r, "\partial_*"] & \pi_q L^{2^i}(Z,n) \arrow[r, "\eta_*"] & \pi_q L^{2^i}(Z_2,n) \arrow[d, "\partial"] \\ & & \pi_{q-1} L^{2^j}(Z,n) \arrow[d] \\ & & 0 \end{tikzcd}$$

$\pi_* L^2(Z_2,n)$ is generated by elements $w_1, \cdots, w_n$ each of order 2.

$w_n = \frac{1}{2} [\![i_n, i_n]\!]$

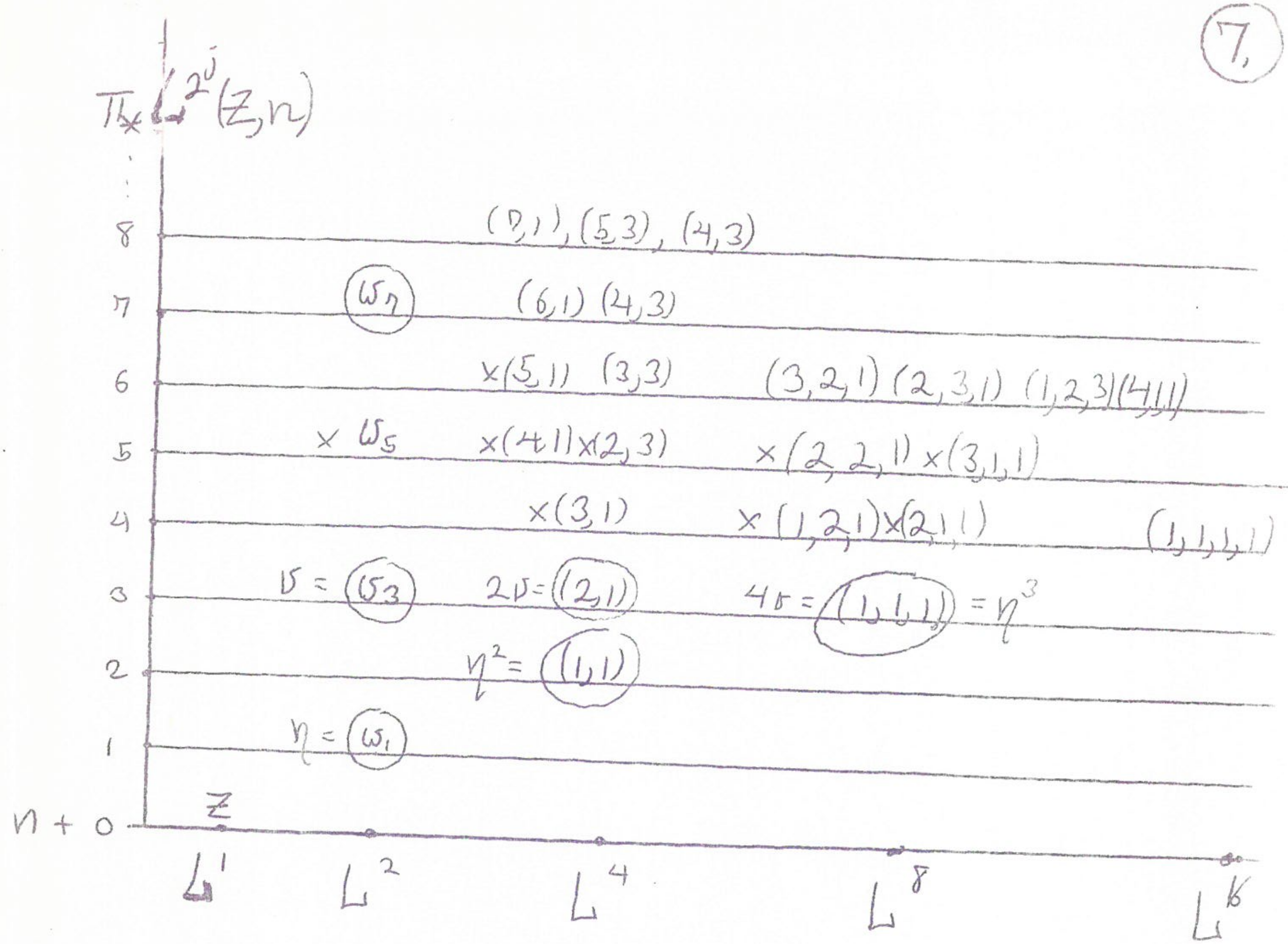
5. So in general if r is 2^j, n arbitrary

$$\begin{tikzcd} \pi_q L^r(Z_2,n) \arrow[r, "\sigma"] & \pi_{q+1} L^r(Z_2,n+1) \\ \oplus & \\ \pi_{q+1} L^{r/2}(Z_2,2n+2) \arrow[uur, "w_{n+1}\circ"'] & \end{tikzcd}$$

Thm: a.) $\pi_* L^{2^j}(Z_2,n)$ has for free generators over Z_2 $\{w_I\}$ where $I = (i_1, i_2, \cdots i_j)$
$i_1 \leq n$, $i_2 \leq 2i_1, \cdots, i_j \leq 2i_{j-1}$
and $w_I \in \pi_q L^{2^j}(Z_2,n)$, $q = n + \sum i_k$
$w_I = \sigma^* w_{i_1} \circ \cdots \circ \sigma^* w_{i_j}$ where σ^* = suitable suspension.

b.) The image of η_* is exactly those w_I for which i_j is odd in $\pi_* L^{2^j}(Z,n) \xrightarrow{\eta_*} \pi_* L^{2^j}(Z_2,n)$

Also $w_i \circ w_j = 0$ if $j > 2i$.



After L^2 only the indices are listed.
The circled terms are some that persist in the spectral sequence to ∞.
The crossed terms are some that do not.
This can be seen by looking at the known homotopy groups of spheres.

HOMOTOPY GROUPS OF SPHERES SEMINAR
THE ADAMS SPECTRAL SEQUENCE - REVISITED
7-21-65

Let $A(p)$ = Steenrod Algebra of "stable" Cohomology operations $H^*(X, Z_p) \to H^*(X, Z_p)$

$A = A(2)$ is generated by $Sq^i : H^n(X, Z_2) \to H^{n+i}(X, Z_2)$

A Basis is given by $\{ Sq^{a_1} \circ \cdots \circ Sq^{a_k} \mid a_i \geq 2a_{i+1} \}$

with The Adem Relations $Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$

for $a < 2b$ determining The algebra structure.

__Thm__ (Adams) There is a Spectral Sequence $E_r^{s,t}$ $d_r : E_r^{s,t} \to E_r^{s+r,\, t+r-1}$ satisfying

1) $E_2^{s,t} = \mathrm{Ext}_{A(p)}^{s,t}(Z_p, Z_p)$

2) $\bigoplus_s E_\infty^{s,\, s+k}$ = Graded Group associated to a filtration of ${}_p\pi_k^S$

3) Products can be introduced in each E_r such that E_2 = The Algebra $\mathrm{Ext}_{A(p)}(Z_p, Z_p)$

The d_r are derivations and the multiplication in E_r induces The one in E_{r+1}.

In E_∞ The product from The E_r coincides (to sign) with That induced by The ring π_*^S with The composition product.

Computation of $\mathrm{Ext}_A(Z_2, Z_2)$

Let $\cdots C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \to Z_2 \to 0$ be an exact sequence of left A-modules, with The C_i free, and d of degree 0. Form The Chain Complex

$$\cdots \xleftarrow{\delta} \mathrm{Hom}_A(C_2, Z_2) \xleftarrow{\delta} \mathrm{Hom}_A(C_1, Z_2) \xleftarrow{\delta} \mathrm{Hom}_A(C_0, Z_2) \leftarrow 0 \qquad (?)$$

Then $\mathrm{Ext}_A^{s,t}(Z_2\, Z_2) = \mathcal{H}^s(\mathrm{Hom}_A^t(C_*, Z_2))$

where $\mathrm{Hom}_A^t(M,N) = \{A\text{-maps } f: M \to N \text{ lowering degree by } t\}$

$\mathrm{Hom}_A(C, Z_2)$ has a very simple structure. As A-module it is trivial because this is the case for Z_2. It is just the graded Z_2 vector space with one copy of Z_2 for each free generator of C. More precisely:
$\mathrm{Hom}_A(C, Z_2) \simeq \mathrm{Hom}_{Z_2}\left(\frac{C}{I(A)C}, Z_2\right)$ where $I(A) = \bigoplus_{i>0} A^i$

We have

$$\begin{array}{ccc}
\mathrm{Hom}_A(C_s, Z_2) & \xrightarrow{\delta} & \mathrm{Hom}_A(C_{s+1}, Z_2) \\
\wr\wr & & \wr\wr \\
\mathrm{Hom}_{Z_2}\left(\frac{C_s}{I(A)C_s}, Z_2\right) & \dashrightarrow & \mathrm{Hom}_{Z_2}\left(\frac{C_{s+1}}{I(A)C_{s+1}}, Z_2\right)
\end{array}$$

so $\delta = 0 \iff d_{s+1}: C_{s+1} \to I(A)C_s$
$\iff \mathrm{Ker}\, d_s \subset I(A)\, C_s$

<u>Def</u> $\{x_i\}$ form an <u>indecomposable</u> <u>basis</u> for an A module M if $\{\bar{x}_i\}$ is a Z_2-basis for $\frac{M}{I(A)M}$ where $\bar{x}_i$ means projection $M \to \frac{M}{I(A)M} \to 0$

If $C = \bigoplus A e_i \xrightarrow{d} M$ is given by $d(e_i) = x_i$ one shows by induction on the components M^q of M that d is onto. Also, since
$\frac{C}{I(A)C} \simeq \frac{M}{I(A)M}$, it follows $\mathrm{Ker}\, d \subset I(A)C$.

One now constructs $C_s \xrightarrow{d_s} \mathrm{Ker}\, d_{s-1} \to 0$ such that $\mathrm{Ker}\, d_s \subset I(A)C_s$ inductively, beginning of course with $A \to Z_2 \to 0$, to obtain a resolution $C_* \to Z_2 \to 0$

with the differential $\delta = 0$ in $Hom_A(C_*, Z_2)$.
Such a resolution is called "minimal".
For example, $\operatorname{Ker} d_0 = I(A)$ and it is well known $\{Sq^{2^i}\}_{i \geq 0}$ form an indecomposable basis of the A-module $I(A)$. Hence C_1 has generators in dimension $t = 2^i$ and

$$Ext_A^{1,t}(Z_2, Z_2) = \begin{cases} Z_2 & t = 2^i \text{ with generator } h_i \\ 0 & \text{otherwise} \end{cases}$$

It is necessary also to know the multiplicative structure of $Ext_A(Z_2, Z_2)$. When computing with a minimal resolution as above, the Yoneda product can be used. This coincides (to sign) with the product induced by $A \xrightarrow{\Delta} A \otimes A$. It is defined as follows:

$$\begin{tikzcd}
C_{n+m} \arrow[r] \arrow[d, dashed, "\alpha_n"] & \cdots \arrow[r] & C_{m+1} \arrow[r] \arrow[d, dashed, "\alpha_1"] & C_m \arrow[r] \arrow[d, dashed, "\alpha_0"] \arrow[dr, "\alpha"] & \cdots \arrow[r] & C_0 \arrow[r] & Z_2 \arrow[r] & 0 \\
C_n \arrow[r] \arrow[dr, "\beta"] & \cdots \arrow[r] & C_1 \arrow[r] & C_0 \arrow[r] & Z_2 \arrow[r] & 0 & & \\
& Z_2 & & & & & &
\end{tikzcd}$$

Let α, β be cocycles in $Hom_A(C_*, Z_2)$ representing $\bar{\alpha}, \bar{\beta} \in Ext_A(Z_2, Z_2)$. Lift α by $\alpha_0 \cdots \alpha_n$ using $\alpha \circ d_{m+1} = 0$ and that the C_i are free. Define $\bar{\alpha} \cdot \bar{\beta}$ to be the class of $\beta \circ \alpha_n$. This is well-defined.

By other methods, Adams has computed

$$\mathrm{Ext}_A^{2,\tau}(Z_2, Z_2) = \begin{cases} Z_2 & \text{corresponding to } h_i h_j \quad j \neq i+1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ext}_A^{3,\tau}(Z_2, Z_2) = \begin{cases} Z_2 & \text{corresponding to } h_i h_j h_k \text{ subject to } h_i^3 = h_{i-1}^2 h_{i+1} \text{ and } h_i h_{i+2}^2 = 0 \\ Z_2 & \text{in dims } 2\cdot(2^{i+2}-1) + \cancel{3}\cdot 2^i - 1 \text{ with generator } c_i \\ 0 & \text{otherwise} \quad (\text{Conjecture}) \end{cases}$$

A Crude Vanishing Thm

Let M be an A-module. $Sq^1 \cdot Sq^1 = 0$ so Sq^1 acts as a differential operator on M. Let $H^*(M)$ be the cohomology.

Lemma 1 If $0 \to M' \to M \to M'' \to 0$ is exact and H^* of any two $= 0$ then H^* of the third $= 0$.

Pf The long exact sequence

Lemma 2 $H^*(A) = H^*\left(\frac{I(A)}{A\, Sq^1}\right) = 0$. Hence $H^*(C) = 0$ for any free A module C.

Pf $Sq^1 Sq^n = \begin{cases} Sq^{n+1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$. The Lemma follows by looking at the action of Sq^1 on the basis $\{Sq^I\}$.

Thm $H^*(M) = 0 \Rightarrow \mathrm{Ext}_A^{s,t}(M, Z_2) = 0$ for $t - s < s$

Pf By induction on s. For $s = 0$ there is nothing to prove. Suppose $\cdots C_s \xrightarrow{d_s} C_{s-1} \xrightarrow{d_{s-1}} \cdots \xrightarrow{d_0} M \to 0$ is a minimal resolution. By induction and Lemmas 1 and 2

$H^*(\operatorname{Ker} d_r) = 0 \quad \forall r \geq 0$. Assume the generators of C_s are in dimensions $\geq 2s$. We get a diagram:

$$\begin{tikzcd}
0 \arrow[r, dashed] & C_s^{(2s+1)} \arrow[r] & \operatorname{Ker}(d_{s-1})^{(2s+1)} \\
& C_s^{(2s)} \arrow[u, "Sq^1"] \arrow[r, phantom, "\cong"] & \operatorname{Ker}(d_{s-1})^{(2s)} \arrow[u, "Sq^1"'] \\
& 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Since the resolution is minimal there are no relations in $\operatorname{Ker}(d_{s-1})^{(2s+1)}$ and hence $(\operatorname{Ker} d_s)^t = 0$ for $t < 2s+2$. The same then holds for C_{s+1} QED

To apply this to $\operatorname{Ext}_A(Z_2\, Z_2)$ we use Lemma 2 and

$$0 \to \frac{I(A)}{A\, Sq^1} \to \frac{A}{A\, Sq^1} \to Z_2 \to 0 \qquad \text{inducing}$$

$$\operatorname{Ext}_A^{s,t}(Z_2, Z_2) \cong \operatorname{Ext}_A^{s,t}\left(\frac{A}{A\,Sq^1}, Z_2\right) \oplus \operatorname{Ext}_A^{s-1,t}\left(\frac{I(A)}{A\,Sq^1}\, Z_2\right)$$

$$\begin{array}{ccc} \| & & \| \\ \begin{cases} Z_2 & t=s \\ 0 & \text{otherwise} \end{cases} & & \{ 0 \quad t-s < s \end{array}$$

This result $\Rightarrow$ the higher stable homotopy groups are finite. In fact if $\alpha \in {}_2\pi^S_{k_{S-1}}$, $k > 0$ then $2^k \alpha = 0$. It can also be used to prove convergence of the Adams Spectral Sequence in the spectal case $[S^0, S^0]^S_*$ under consideration.

Adams' Vanishing and Periodicity Theorems

Let $\quad A_r =$ Subalgebra of A generated by $Sq^{2^i} \quad i \leq r$

$\quad A_\infty = A$

The injections $A_p \to A_r$, $p < r$ induce $\mathrm{Ext}_{A_r}(M,N) \to \mathrm{Ext}_{A_p}(M,N)$ where M and N are A_r-modules. Assume $H^*(M) = 0$, $M^t = 0$ $t < q$.

<u>Thm 1</u> $\mathrm{Ext}_{A_r}^{s,t}(M, Z_2) = 0$ for $t-s < q + T(s)$

$$\text{where } T(s) = \begin{cases} 2s & s \equiv 0 \\ 2s-1 & 1 \\ 2s-2 & 2 \\ 2s-2 & 3 \end{cases} \mod 4$$

<u>Thm 2</u> $\mathrm{Ext}_{A_r}^{s,t}(M, Z_2) \to \mathrm{Ext}_{A_p}^{s,t}(M, Z_2)$, $r > p$, is an isomorphism for $t-s < q + T(s-1) + 2^{p+1}$, $s \geq 1$.

<u>Thm 3</u> There are elements $p_r \in \mathrm{Ext}_{A_r}^{2^r, 3\cdot 2^r}(Z_2, Z_2)$, $r \geq 2$ such that

i) $x \rightsquigarrow x \cdot p_r : \mathrm{Ext}_{A_r}^{s,t}(M, Z_2) \to \mathrm{Ext}_{A_r}^{s+2^r, t+3\cdot 2^r}(M, Z_2)$

is an isomorphism if $t-s < q + U(s)$ for some function $3s \leq U(s) \leq 5s$

ii) $i^* p_r = (p_{r-1})^2$ where $i : A_{r-1} \to A_r$

iii) In the range where $\mathrm{Ext}_{A}^{s,t}(M, Z_2) \simeq \mathrm{Ext}_{A_r}^{s,t}(M, Z_2)$ $\cdot p_r$ on the ~~left~~ right corresponds to the Massey secondary operation $\langle x, h_0^{2^r}, h_{r+1} \rangle$ on left.

<u>Indications of Proofs</u> A_0 = Exterior Algebra generated by Sq^1
A_1 = 8-dim. algebra over Z_2 generated by Sq^1, Sq^2. It is easy to compute $\mathrm{Ext}_{A_1}^{s,t}(A_0, Z_2)$ completely by the minimal resolution method and get Thm 1 in

This case. Then Thm 1 for $r=1$ follows for any M which can be expressed by extensions of A_0, using long exact sequences for Ext.

Thm 2 supports Thm 1 for higher r.

Thm 1 applies to Thm 2 as follows. Consider

$$0 \to K \to A_r \otimes_{A_p} M \to M \to 0 \qquad \text{giving}$$

$$\begin{array}{ccccccc}
 & & \mathrm{Ext}_{A_p}^{s,t}(M, Z_2) & & & & \\
 & & \wr\wr & \nwarrow i^* & & & \\
\mathrm{Ext}_{A_r}^{s,t}(K, Z_2) & \leftarrow & \mathrm{Ext}_{A_r}^{s,t}(A_r \otimes_{A_p} M, Z_2) & \leftarrow & \mathrm{Ext}_{A_r}^{s,t}(M, Z_2) & \leftarrow & \mathrm{Ext}_{A_r}^{s-1,t}(K, Z_2) \cdots
\end{array}$$

Now $K^t = 0$ for $t < q + 2^{p+1}$ so $\mathrm{Ext}_{A_r}^{s-1,t}(K, Z_2)$ $= \mathrm{Ext}_{A_r}^{s,t}(K, Z_2) = 0$ if $t - s < q + T(s-1) + 2^{p+1}$, and i^* becomes an isomorphism in this range. One needs $H^*(K) = 0$ and the vertical "change of rings" isomorphism which depend on the structure of the algebras A_r. Using these techniques Thm's 1 and 2 can be proved simultaneously by induction on s.

For Thm 3, Thm 1 $\Rightarrow h_0^{2^r} h_{r+1} = h_0^{2^r} x = 0$ for x in a certain range, so $\langle x, h_0^{2^r}, h_{r+1}\rangle$ is defined. For $i: A_r \to A$, $i^*(h_{r+1}) = 0$ so if $\delta c = h_0^{2^r} h_{r+1}$ then $i^* c$ is a cocycle representing $P_r \in \mathrm{Ext}_{A_r}^{2^r, 3\cdot 2^r}(Z_2, Z_2)$. Explicit computation shows $x \mapsto x\cdot(i^* P_2) : \mathrm{Ext}_{A_1}^{s,t}(A_0, Z_2) \to \mathrm{Ext}_{A_1}^{s+4, t+12}(A_0, Z_2)$

is an isomorphism in a certain range. The proof continues by using K and an inductive argument as above.

The function T of Thm 1 does not quite give the edge of $\mathrm{Ext}_A(Z_2, Z_2)$. The "best possible" edge may be read off the Table because of periodicity. This vanishing Theorem gives a bound on the order of elements in ${}_2\pi_k^S$ of approximately $2^{k/2}$.

Table and Differentials

The Table gives $\mathrm{Ext}_A^{s,t}(Z_2, Z_2)$ for $t-s \leq 27$, all non-zero differentials in this range, and the value of ${}_2\pi_{t-s}^S$.

$d_2(h_4) = h_0 h_3^2$ holds because the composition product in π_*^S is anti-commutative $\Rightarrow 2\sigma^2 = 0$ where $h_3 \leftrightarrow \sigma$ $\Rightarrow h_0 h_3^2$ can't survive to E_∞. Using Secondary cohomology operations Adams has shown $d_2(h_i) = h_0 h_{i-1}^2$ $\forall i \geq 4 \Rightarrow \nexists$ elements of Hopf Invariant 1 except in $\pi_3(S^2)$, $\pi_7(S^4)$, $\pi_{15}(S^8)$.

The differentials $d_3(h_0^i h_4) = h_0^i d_0$ $i = 1, 2$ and $d_2(e_0) = h_1^2 d_0$ follow from Toda's calculations of ${}_2\pi_k^S$ $k \leq 19$. In Adams Berkeley Notes the listed differentials $d_3(h_0^i h_2 h_4) = h_0^i h_2 d_0$ $i = 0, 1, 2$ is incorrect. The other non-zero differentials and proof that these are all in the range $t-s \leq 27$ follow using the fact the d_r are derivations, Toda's results, and that $P^i xy = xP^i y = yP^i x$ if $P^i x, P^i y \neq 0$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	Z_2	Z_2	Z_8	0	0	Z_2	Z_{16}	$(Z_2)^2$	$(Z_2)^3$	Z_2	Z_8	0	0	$(Z_2)^2$	$Z_2 + Z_{32}$

$d_2(h_4) = h_0 h_3^2$

$d_3(h_0^i h_4) = h_0^i d_0$

$i = 1, 2$

s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
8															$h_0^7 h_4$
7											$h_0^2 p^1 h_2$				$h_0^6 h_4$
6										$p^1 h_1^2$	$h_0 p^1 h_2$			$h_0^2 d_0$	$h_0^5 h_4$
5									$p^1 h_1$		$p^1 h_2$			$h_0 d_0$	$h_1 d_0$ $\cancel{h_0^4 h_4}$
4							$h_0^3 h_3$		$h_1 c_0$					d_0	$h_0^3 h_4$
3			$h_1^3 = h_0^2 h_2$				$h_0^2 h_3$	c_0	$h_2^3 = h_1^2 h_3$					$h_0 h_3^2$	$h_0^2 h_4$
2		h_1^2	$h_0 h_2$			h_2^2	$h_0 h_3$	$h_1 h_3$						h_3^2	$h_0 h_4$
1	h_1		h_2				h_3								h_4

$t - s$

$d_2(e_0) = h_1^2 d_0$

$d_2(f_0) = h_0 h_2 d_0$

$d(h_0 f_0) = h_0^2 h_2 d_0$

$d_2(i) = h_0 p^1 d_0$

$d(h_0 i) = h_0^2 p^1 d_0$

$d_2(p^1 e_0) = p^1 h_1^2 d_0$

$d_2(h_0^i j) = h_0^i p^1 h_2 d_0$

$i = 0, 1, 2$

	16	17	18	19	20	21	22	23	24	25	26	27
	$(Z_2)^2$	$(Z_2)^4$	$Z_2 + Z_8$	$Z_2 + Z_8$	Z_8	$(Z_2)^2$	$(Z_2)^2$	$(Z_2)^2 + Z_4 + Z_{16}$	$(Z_2)^2$	$(Z_2)^2$	$(Z_2)^2$	Z_8
15												$h_0^2 p^3 h_2$
14											$p^3 h_1^2$	$h_0 p^3 h_2$
13										$p^3 h_1$		$p^3 h_2$
12								$h_0^5 i$		$h_1 p^2 c_0$		
11				$h_0^2 p^2 h_2$				$h_0^4 i$	$p^2 c_0$	$h_0^2 p^1 h_2 d_0$		
10			$p^2 h_1^2$	$h_0 p^2 h_2$			$h_0^2 p^1 d_0$	$h_0^3 i$	$p^1 h_1^2 d_0$	$h_0 p^1 h_2 d_0$		
9		$p^2 h_1$		$p^2 h_2$			$h_0 p^1 d_0$	$\cancel{h_1 p^1 d_0}$ $\cancel{h_0^2 i}$		$p^1 h_2 d_0$	$h_0^2 j$	
8		$h_1 p^1 c_0$					$p^1 d_0$	$h_0 i$		$p^1 e_0$	$p^1 f_0 = h_2 i = h_0 j$	
7	$p^1 c_0$	$h_0^2 h_2 d_0$						i			j	
6	$h_1^2 d_0$	$h_0 h_2 d_0$			$h_0^2 g$			$h_0 h_2 g$			$h_2^2 g$	
5		$h_2 d_0 = h_0 e_0$	$h_1 e_0 = h_0 f_0$		$h_2 e_0 = h_0 g$	$h_2 f_0 = h_1 g$		$h_2 g$	$h_1 h_4 c_0$			
4		e_0	$\cancel{f_0}$ $\cancel{h_0^2 h_2 h_4}$		g		$h_2 c_1$	$h_4 c_0$				
3		$h_1^2 h_4$	$h_0 h_2 h_4$	c_1		$h_3^3 = h_2^2 h_4$						
2	$h_1 h_4$		$h_2 h_4$									
1												

A.2

$H^*(A)$

$p=2$

s \ $t-s$	17	18	19	20	21	22	23	24	25	26	27	28	29
15											$P^3h_0^2h_2$		
14										$P^3h_1^2$	$P^3h_0h_2$		
13									P^3h_1		P^3h_2		
12							$P^2h_0^3h_3$		$P^2h_1c_0$				
11			$P^2h_0^2h_2$				$P^2h_0^2h_3$	P^2c_0	$P^1h_0^2h_2d_0$				
10		$P^2h_1^2$	$P^2h_0h_2$			$P^1h_0^2d_0$	$P^2h_0h_3$	$P^1h_1^2d_0$	$P^1h_0h_2d_0$			$P^1h_0^2g$	
9	P^2h_1		P^2h_2			$P^1h_0d_0$	P^2h_3 / $P^1h_1d_0$		$P^1h_2d_0$	h_0^2j		P^1h_0g	h_0^2k
8	$P^1h_1c_0$					P^1d_0	h_0i		P^1e_0	h_0j		P^1g	h_0k
7	$h_0^2h_2d_0$						i			j			k
6	$h_0h_2d_0$			h_0^2g			h_0h_2g			h_2^2g			
5	h_2d_0	h_0f_0		h_0g	h_1g		h_2g	$h_1h_4c_0$					
4	e_0	f_0 / $h_0^2h_2h_4$		g		h_2c_1	h_4c_0						
3	$h_1^2h_4$	$h_0h_2h_4$	c_1		h_3^3								
2		h_2h_4											
δ_r	$\delta_2(e_0)=h_1^2d_0$; $\delta_2(f_0)=h_0h_2d_0$					$\delta_2(i)=P^1h_0d_0$		$\delta_2(P^1e_0)=P^1h_1^2d_0$		$\delta_2(j)=P^1h_2d_0$		$\delta_2(k)=P^1h_0g$	
$\pi_{t-s}(S)$	$(Z_2)^4$	Z_8+Z_2	Z_8+Z_2	Z_8	$(Z_2)^2$	$(Z_2)^2$	$(Z_2)^2+Z_4+Z_{16}$	$(Z_2)^2$	$(Z_2)^2$	$(Z_2)^2$	Z_8	Z_2	$\le Z_2$

s \ $t-s$	0	1	2	3	6	7	8	9	10	11	14	15	16
8	h_0^8											$h_0^7h_4$	
7	h_0^7									$P^1h_0^2h_2$		$h_0^6h_4$	P^1c_0
6	h_0^6								$P^1h_1^2$	$P^1h_0h_2$	$h_0^2d_0$	$h_0^5h_4$	$h_1^2d_0$
5	h_0^5							P^1h_1		P^1h_2	h_0d_0	h_1d_0 / $h_0^4h_4$	
4	h_0^4					$h_0^3h_3$		h_1c_0			d_0	$h_0^3h_4$	
3	h_0^3			$h_0^2h_2$		$h_0^2h_3$	c_0	h_2^3			$h_0h_3^2$	$h_0^2h_4$	
2	h_0^2		h_1^2	h_0h_2	h_2^2	h_0h_3	h_1h_3				h_3^2	h_0h_4	h_1h_4
1	h_0	h_1		h_2		h_3						h_4	
0	1												
δ_r											$\delta_2(h_4)=h_0h_3^2$; $\delta_3(h_0h_4)=h_0d_0$		
$\pi_{t-s}(S)$	Z	Z_2	Z_2	Z_8	Z_2	Z_{16}	$(Z_2)^2$	$(Z_2)^3$	Z_2	Z_8	$(Z_2)^2$	$Z_{32}+Z_2$	$(Z_2)^2$

A. 3

$H^*(A)$

$p = 2$

~~Tentative (may be incomplete)~~

s \ $t-s$	30	31	32	33	34	35	36	37	38	39	40	41	42
22													$P^5h_1^2$
21												P^5h_1	
20										$P^4h_0^3h_3$		$P^4h_1c_0$	
19						$P^4h_0^2h_2$				$P^4h_0^2h_3$	P^4c_0	$P^3h_0^2h_2d_0$	
18					$P^4h_1^2$	$P^4h_0h_2$			$P^3h_0^2d_0$	$P^4h_0h_3$	$P^3h_1^2d_0$	$P^3h_0h_2d_0$	
17				P^4h_1		P^4h_2			$P^3h_0d_0$	P^4h_3 / $P^3h_1d_0$		$P^3h_2d_0$	$P^2h_0^2j$
16		$h_0^{15}h_5$		$P^3h_1c_0$					P^3d_0	P^2h_0i		P^3e_0	P^2h_0j
15		$h_0^{14}h_5$	P^3c_0	$P^2h_0^2h_2d_0$						P^2i			P^2j
14	$P^2h_0^2d_0$	$h_0^{13}h_5$	$P^2h_1^2d_0$	$P^2h_0h_2d_0$			$P^2h_0^2g$			$P^1h_0^2d_0e_0$			$P^1h_0^2e_0^2$
13	$P^2h_0d_0$	$h_0^{12}h_5$ / $P^2h_1d_0$		$P^2h_2d_0$	$P^1h_0^2j$		P^2h_0g	$P^2h_0^2k$		$P^1h_0d_0e_0$	$P^1h_0^2l$		$P^1h_0e_0^2$
12	P^2d_0	$h_0^{11}h_5$		P^2e_0	P^1h_0j		P^2g	P^1h_0k		$P^1d_0e_0$	P^1h_0l		$P^1e_0^2$
11	h_0^4w	$h_0^{10}h_5$			P^1j			h_0^6x / P^1k			P^1l	h_0z	
10	h_0^3w	$h_0^9h_5$, $h_0^2d_0e_0$			$h_0^2e_0^2$			h_0^5x	P^1r		h_1u	z	
9	h_0^2w	$h_0^8h_5$, $h_0d_0e_0$	h_0^2l		$h_0e_0^2$	h_0^2m		h_0^4x	h_0f_0g	u			v
8	h_0w	$h_0^7h_5$, d_0e_0	h_0l		e_0^2	h_0m		e_0g, h_0^3x	$f_0g = h_0^2y$		g^2		$P^1h_0^2h_2h_5$
7	w	$h_0^6h_5$	l	h_1q		m		h_2^2n, h_0^2x	h_0y	c_1g		$P^1h_1^2h_5$	$P^1h_0h_2h_5$
6	r	$h_0^5h_5$	q		h_2n		t	h_0x	y / $h_2^2d_1$		$h_0^2f_1$, $P^1h_1h_5$		$P^1h_2h_5$
5	$h_0^3h_4^2$	$h_0^4h_5$, n		h_0p		h_2d_1		x	$h_0^3h_3h_5$	h_3d_1	$h_1h_5c_0$, h_0f_1	$h_0^2c_2$	
4	$h_0^2h_4^2$	$h_0^3h_5$	d_1	p	$h_0^2h_2h_5$				$h_0^2h_3h_5$, e_1	h_5c_0	$h_2^3h_5$, f_1	h_0c_2	
3	$h_0h_4^2$	$h_0^2h_5$, $h_1h_4^2$		$h_1^2h_5$	$h_0h_2h_5$			$h_2^2h_5$	$h_0h_3h_5$	$h_1h_3h_5$		c_2	
2	h_4^2	h_0h_5	h_1h_5		h_2h_5				h_3h_5				
1		h_5											

$\delta_2(h_5) = h_0h_4^2$

$\delta_3(h_0^3h_5) = w$

$\delta_4(h_0^8h_5) = P^2h_0d_0$

$\delta_2(l) = h_0d_0e_0$

$\delta_2(P^1j) = P^2h_2d_0$

$\delta_2(m) = h_0e_0^2$

$\delta_2(P^1k) = P^2h_0g$

$\delta_2(P^2i) = P^3h_0d_0$

$\delta_2(P^1l) = P^1h_0d_0e_0$

$\delta_2(P^3e_0) = P^3h_1^2d_0$

$\delta_2(P^2j) = P^3h_2d_0$

Homotopy theory of Spheres Seminar 8/1/65

Toda brackets and Massey Products in the Adams Spectral Sequence

Speaker: Kahn

References:

① Adams: "Non-existence of elements of Hopf Invariant One" Annals of Math, 1960

② Adams: "On the structure and applications of the Steenrod Algebra", Comm. Math. Helv., 1958

③ Adams: Berkeley Notes

④ Barratt: Seattle Notes

⑤ M. Moss: Cambridge Thesis

⑥ Toda: "Composition Methods in the homotopy groups of spheres"

⑦ May: Princeton Thesis

The results discussed here are reported to occur in a more general form in ⑤. Here we restrict our attention to the mod 2 Adams spectral sequence for the 2-component of $\pi_*(\underline{S})$; A = mod 2 Steenrod Algebra.

Let (K,d) be a chain algebra which is associative. If a, b, c are cycles and $\bar{a}, \bar{b}, \bar{c}$ their respective homology classes with $\bar{a}\bar{b} = 0$ and $\bar{b}\bar{c} = 0$, the Massey product $\langle \bar{a}, \bar{b}, \bar{c} \rangle$ is defined as follows:

$$\bar{a}\bar{b} = 0 \Rightarrow \exists\, u \ni du = ab$$
$$\bar{b}\bar{c} = 0 \Rightarrow \exists\, v \ni dv = bc.$$

Then, using the associativity of K, $uc - (-1)^{\dim a} av$ is a cycle. The set of homology classes of all such cycles is called the Massey product $\langle \bar{a}, \bar{b}, \bar{c} \rangle$. As a result of the leeway in choosing u and v, it turns out to be a coset of $\bar{a} \cdot (-) + (-) \cdot \bar{c}$.

Examples: the terms (E_r, d_r) $(r \geq 2)$ of the Adams spectral sequence for $\pi_*(S)$ are associative chain algebras and hence have Massey products defined in their homology. (E_1, d_1) is not invariant and need not be an associative chain algebra. However, the bar construction (see ① sections 2.1, 2.2) may be realized (see ② and ③) yielding an associative chain algebra (E_1, d_1) and hence yielding Massey products in $E_2 = H(E_1, d_1)$. Many such Massey products are calculated in ①.

As in ②, we use the "smash" product to define composition in $\pi_*(\underline{S})$ and relate it to the products in the Adams spectral sequence. Our purpose is to relate (as far as possible), the Toda brackets (defined in the first lecture of this series) and Massey products in the Adams spectral sequence.

Let $\alpha \in \pi_{p+h}(S^p)$, $\beta \in \pi_{q+k}(S^q)$, $\gamma \in \pi_{r+l}(S^r)$ be $\ni$ $\alpha \wedge \beta = 0$, $\beta \wedge \gamma = 0$. Let a, b, c be representatives of α, β, γ respectively and let A_t and B_t be null-homotopies of $a \wedge b$ and $b \wedge c$ respectively. Let ι_n be the identity map $S^n \to S^n$ for each n. Consider $H: S^{s+1} \to S^{p+q+r}$, $s = p+h+q+k+r+l$, given by

$$H(d_s(x,t)) = \begin{cases} (a \wedge \iota_{q+r})(\iota_{p+h} \wedge B_{2t-1})(x), & \frac{1}{2} \leq t \leq 1 \\ (A_{1-2t} \wedge \iota_r)(\iota_{p+h+q+k} \wedge c)(x), & 0 \leq t \leq \frac{1}{2} \end{cases}$$

where $d_s: S^s \times I \to S^{s+1}$ in the usual fashion. This differs from the usual stable Toda bracket $\langle \alpha, \beta, \gamma \rangle$ by a sign discussed in Chapt. III of ⑥ (see especially p. 26). Since we are using the mod 2 Adams spectral sequence, we may ignore this sign difference. Note that for $\frac{1}{2} \leq t \leq 1$, H may be thought of as $a \wedge B: (S^{p+h} \wedge TS^{q+k+r+l}) \to S^{p+q+r}$ and for $0 \leq t \leq \frac{1}{2}$ H may be thought of as $A \wedge c: (TS^{p+h+q+k} \wedge S^{r+l}) \to S^{p+q+r}$, and $H_{\frac{1}{2}}$ as $a \wedge b \wedge c: S^s \to S^{p+q+r}$.

The multiplicative structure of the Adams spectral sequence will be treated as follows. Let B_* be the unreduced bar construction on A, namely

$$B_n = B_n(A) = A \otimes \underbrace{\bar{A} \otimes \cdots \otimes \bar{A}}_{n} \qquad (\otimes \text{ over } \mathbb{Z}_2)$$

where $\bar{A} = \sum_{q>0} A_q$. Then

$$\mathbb{Z}_2 \longleftarrow B_0 \longleftarrow B_1 \longleftarrow \cdots$$

is known to be a free A-resolution of $\mathbb{Z}_2$ (see ①: 2.1, 2.2). $B_* \otimes B_*$ can be made into an A-module using the diagonal map $\psi: A \longrightarrow A \otimes A$. $0 \leftarrow Z_2 \longleftarrow B_* \otimes B_*$ is an <u>acyclic</u> complex over $\mathbb{Z}_2 \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$. Hence there exists a mapping $\Delta: B_* \longrightarrow B_* \otimes B_*$ covering $\mathbb{Z}_2 \xrightarrow{\cong} \mathbb{Z}_2 \approx \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

Let

$$X_0 = S^p \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots$$

$$Y_0 = S^q \longleftarrow Y_1 \longleftarrow Y_2 \longleftarrow \cdots$$

$$Z_0 = S^r \longleftarrow Z_1 \longleftarrow Z_2 \longleftarrow \cdots$$

be realizations of $\bar{B}_*$. Let

$$S^p \wedge S^q \longleftarrow U_1 \longleftarrow U_2 \longleftarrow \cdots$$

$$S^q \wedge S^r \longleftarrow V_1 \longleftarrow V_2 \longleftarrow \cdots$$

$$S^p \wedge S^q \wedge S^r \longleftarrow W_1 \longleftarrow W_2 \longleftarrow \cdots$$

$$(S^p \wedge S^q) \wedge S^r \longleftarrow Q_1 \longleftarrow Q_2 \longleftarrow \cdots$$

$$S^p \wedge (S^q \wedge S^r) \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$$

be given by $U_m = \bigcup_{k+l=m} X_k \wedge Y_l$

$$V_m = \bigcup_{k+l=m} Y_k \wedge Z_l$$

$$W_m = \bigcup_{j+k+l=m} X_j + Y_k + Z_l$$

$$Q_m = \bigcup_{k+l=m} \overline{U}_k \wedge Z_l$$

$$R_m = \bigcup_{k+l=m} X_k \wedge \overline{V}_l .$$

Then $\{U_i\}$ is a realization of $B_* \otimes B_*$ as are $\{V_i\}$, $\{Q_i\}$, $\{R_i\}$ and $\{W_i\}$ is a resolution of $B_* \otimes B_* \otimes B_*$. (see ②). Also let

$$S^p \wedge S^q \leftarrow \overline{U}_1 \leftarrow \overline{U}_2 \leftarrow \cdots$$
$$S^q \wedge S^r \leftarrow \overline{V}_1 \leftarrow \overline{V}_2 \leftarrow \cdots$$
$$S^p \wedge S^q \wedge S^r \leftarrow \overline{W}_1 \leftarrow \overline{W}_2 \leftarrow \cdots$$

be realizations of B_*. Let

$$f_i : U_i \longrightarrow \overline{U}_i$$
$$g_i : V_i \longrightarrow \overline{V}_i$$
$$h_i : Q_i \longrightarrow \overline{W}_i$$
$$k_i : R_i \longrightarrow \overline{W}_i$$

be realizations of Δ. Then $h_i(f_i \wedge 1) : W_i \longrightarrow \overline{W}_i$ realizes $\Delta(\Delta \otimes 1)$ and $k_i(1 \wedge g_i) : W_i \longrightarrow \overline{W}_i$ realizes $\Delta(1 \otimes \Delta)$. In ② it is proved that $h_i(f_i \wedge 1)$ and $k_i(1 \wedge g_i)$ are homotopic <u>after</u> composition with the map $\overline{W}_i \longrightarrow \overline{W}_{i-1}$. It follows that the E_r are differential rings for $r > 1$ with the ring structure in E_2 the usual one for $\mathrm{Ext}_A(\mathbb{Z}_2, \mathbb{Z}_2)$.

Let $X_0 \leftarrow X_1 \leftarrow \cdots$ be a realization of a free A-resolution over $H^*(X_0)$, X_0 a finite complex, and let $\{E_r^{s,t+s}\}$ be the Adams spectral sequence for $\pi_*(X_0)$. For our use here, the most convenient description of $E_r^{s,t+s}$ is $\mathrm{Im}\left[\pi_t(X_s, X_{s+r}) \to \pi_t(X_{s-r+1}, X_{s+1})\right]$.

<u>Definition</u> In E_r, $d_r^{s-r,t+1+s-r} x = y \in E_r^{s,t+s}$ is called an honest relation if $d_{r+k+l}^{s-r-k,t+1+s-r-k} = 0$ for $k \geq 0$, $l \geq 1$.

<u>Proposition 1</u> If $\alpha \in {}_2\pi_t(X_s)$ and $\alpha \to 0$ in $\pi_t(X_0)$ and $\alpha \to a$ in $E_r^{s,t+s}$ and $d_r u = a$ is an honest relation, then $\exists\ \theta \in \pi_{t+1}(X_{s-r}, X_s) \ni$. $\mathrm{Im}\,\theta$ in $E_r^{s-r,t+1+s-r}$ is u and $\partial_\# \theta = \alpha$.

<u>Proof</u> u is represented by $\theta_0 \in \pi_{t+1}(X_{s-r}, X_s)$ such that

$(\alpha - \partial_{\#}\theta_0) \longrightarrow 0$, $\in E_r^{s,t+s}$ (all elements can be assumed to be in the 2-component). Consider

$$\pi_t(X_s) \xrightarrow[i_s^{s-r+1}]{} \pi_t(X_{s-r+1}) \xrightarrow[j_{s+1}^{s-r+1}]{} \pi_t(X_{s-r+1}, X_{s+1})$$

Now $j \circ i(\alpha - \partial_{\#}\theta_0) = 0$.

<u>Case I</u> $i_s^{s-r+1}(\alpha - \partial_{\#}\theta_0) = 0 \Rightarrow \exists\, \theta_1 \in \pi_{t+1}(X_{s-r+1}, X_s) \ni$ $\partial_{\#}\theta_1 = \alpha - \partial_{\#}\theta_0$. Then $\partial_{\#}(\theta_0 + \theta_1') = \alpha$ and $\operatorname{Im}(\theta_0 + \theta_1')$ in E_r is u, where θ_1' is the image of θ_1 in $\pi_{t+1}(X_{s-r}, X_s)$. Then $\theta = \theta_0 + \theta_1'$ satisfies the conclusion of the proposition.

<u>Case II</u> $i_s^{s-r+1}(\alpha - \partial_{\#}\theta_0) \neq 0$. Then $j_{s+1}^{s-r+1}\, i_{s+1}^{s-r+1}(\alpha - \partial_{\#}\theta_0) = 0$ $\Rightarrow \exists\, \alpha_1 \in \pi_t(X_{s+1}) \ni i_{s+1}^{s-r+1}\alpha_1 = i_s^{s-r+1}(\alpha - \partial_{\#}\theta_0)$, and α_1 has as image in $E_r^{s+1,t+s+1}$ the element a_1. Since $(\alpha - \partial_{\#}\theta_0) \to 0$ in $\pi_t(X_0)$, $\alpha_1 \to 0$ in $\pi_t(X_0)$. Hence a_1 must be 0 in some E_p. By the hypothesis of honesty, $\alpha_1 \to 0$ in $\pi_t(X_{s+r-1}, X_{s+2})$. We can now apply cases I and II to α_1 and if necessary to $\alpha_2, \alpha_3, \ldots$. If Case II repeats indefinitely, then by the convergence of the Adams spectral sequence, $i_s^{s-r+1}(\alpha - \partial_{\#}\theta_0) = 0$ and $\theta = \theta_0$ satisfies the conclusion of the proposition. Otherwise, Case I eventually occurs and we have $\theta_g \in \pi_{t+1}(X_{s-r+1}, X_{s+g})$ $\ni \partial_{\#}\theta_g = \alpha_g$ and $i_{s+g}^{s-r+1}(\alpha_g) = i_s^{s-r+1}(\alpha - \partial_{\#}\theta)$. Let $H:$ $S^t \times I \to X_{s-r+1}$ be a map $\ni [H_0] = (\alpha - \partial_{\#}\theta)$ and $[H_1] = \alpha_g$. Let $H': (TS^t, S^t) \to (X_{s-r+1}, X_{s+g})$ represent θ_g with $H'_0 = H_1$. Define $H'': (TS^t, S^t) \to (X_{s-r+1}, X_s)$ by

$$H''(x\wedge t) = \begin{cases} H'(x\wedge 2(t-\frac{1}{2})), & \frac{1}{2}\le t\le 1 \\ H(x\wedge 2t), & 0\le t\le \frac{1}{2}. \end{cases}$$

Let ψ be the class of this map in $\pi_{t+1}(X_{s-r}, X_s)$. Then $\partial_\#(\psi+\theta_0) = \alpha$ and the image of $(\psi+\theta_0)$ in $E_r^{s-r,\, t+1+s-r}$ is u. Q.E.D.

Before proceeding to the main theorem, let us record an easy technical lemma.

<u>Lemma 2</u> Suppose $A\subset X$ and $\pi_{t+1}(X,A) \xrightarrow{\approx} \pi_{t+1}(X/A)$. If $f,g: (TS^t, S^t) \to (X,A)$, $f|S^t = g|S^t$, and $h: S^t\times I \to A$ ∋ $h|S^t = f|S^t$ $(=g|S^t)$ then $F: S^{t+1} \approx T_+X \cup S^t\times I \cup T_-X$ by $f\cup h\cup g$ agrees in $\pi_{t+1}(X)$ with $d(f,g)$ modulo the image of $\pi_{t+1}(A) \to \pi_{t+1}(X)$.

<u>Remark 3</u> The hypothesis on (X,A) of Lemma 2 is satisfied if "every thing" is in the stable range (Using the Blakers-Massey triad theorem).

<u>Theorem 4</u> Let the 2-component of $\pi_*(\underline{S})$ contain α,β,γ such that $\alpha\beta=0$, $\beta\gamma=0$. Let $a\in E_r^{s_1,t_1+s_1}$, $b\in E_r^{s_2,t_2+s_2}$, $c\in E_r^{s_3,t_3+s_3}$ converge to α,β,γ respectively, $r>1$. Given that $d_r u = ab$ and $d_r v = bc$ are honest relations, then the cycle $(uc+av)$ of (E_r, d_r) converges to $\langle\alpha,\beta,\gamma\rangle$.

<u>Proof</u> We may choose representatives $f\in\alpha$, $g\in\beta$, $h\in\gamma$ ∋ $f: S^{t_1+p} \to X_{s_1}$, $g: S^{t_2+q} \to Y_{s_2}$, $h: S^{t_3+r} \to Y_{s_3}$. The hypothesis of honesty says that

$$f_{s_1+s_2}\circ(f\wedge g): S^{t_1+p}\wedge S^{t_2+q} \to \overline{U}_{s_1+s_2}$$

extends to $F_{s_1+s_2}: T(S^{t_1+p}\wedge S^{t_2+q}) \to \overline{U}_{s_1+s_2-r}$ so that $F_{s_1+s_2}$ represents u. Similarly,

$$g_{s_2+s_3} \circ (g \wedge h) : S^{t_2+q} \wedge S^{t_3+r} \longrightarrow \overline{V}_{s_2+s_3}$$

can be extended to $G_{s_2+s_3} : T(S^{t_2+q} \wedge S^{t_3+r}) \to \overline{V}_{s_2+s_3-r}$ so that $G_{s_2+s_3}$ represents v. Now $H = h_{s_1+s_2-r} \circ (F_{s_1+s_2} \wedge h)$ represents uc and $K = k_{s_2+s_3-r} \circ (f \wedge G_{s_2+s_3})$ represents av. We may assume that $\bigcup_i W_i$ is a compound mapping cylinder of a sequence of fibrations. Hence we may assume that

$$h_{s_1+s_2-r} \circ (F_{s_1+s_2} \wedge h) \,\big|\, S^{t_1+p} \wedge S^{t_2+q} \wedge S^{t_3+r}$$

and $\quad k_{s_2+s_3-r} \circ (f \wedge G_{s_2+s_3}) \,\big|\, S^{t_1+p} \wedge S^{t_2+q} \wedge S^{t_3+r}$

project to the same map in $S^p \wedge S^q \wedge S^r$. Call the projection $\pi : \overline{W}_{s_1+s_2+s_3-r} \longrightarrow S^p \wedge S^q \wedge S^r$. Then $d(\pi \circ H, \pi \circ K)$ represents $\pm \langle \alpha, \beta, \gamma \rangle$. Now

$$H' = H \,|\, S^{t_1+p} \wedge S^{t_2+q} \wedge S^{t_3+r} \quad \text{and}$$

$$K' = K \,|\, S^{t_1+p} \wedge S^{t_2+q} \wedge S^{t_3+r}$$

are homotopic in $\overline{W}_{s_1+s_2+s_3-1}$. Let $D : S^{\bar{t}} \times I \to \overline{W}_{s_1+s_2+s_3-1}$ where $\bar{t} = t_1+p+t_2+q+t_3+r$ $\ni$ $D_1 = H'$ and $D_0 = K'$. Then $(H \cup D \cup K) : T_+ S^{\bar{t}} \cup S^{\bar{t}} \times I \cup T_- S^{\bar{t}} \longrightarrow \overline{W}_{s_1+s_2+s_3-r}$. By Lemma 2, $\pi \circ (H \cup D \cup K)$ represents $\langle \alpha, \beta, \gamma \rangle$ modulo elements of filtration $s_1+s_2+s_3-1$.

It is also clear that the image of $H \cup D \cup K$ in $E_r^{s_1+s_2+s_3-r,\, t+1+s_1+s_2+s_3-r}$ is $(av + uc)$.

<u>Example</u> (Barratt) $\eta\{q\} \in \langle \nu, 2\nu, K \rangle$ where $K = \{d_0\}$ in May's notation (see ⑦).

<u>Proof</u> $d_2(\circ) = h_2(h_0 h_2)$ is clearly honest and $d_2 f_0$ is honest since $h_0^2 h_2 h_4$, $h_0 h_2 h_4$, $h_2 h_4$ are all permanent cycles.

Hence $h_2 f_0 \in \langle h_2 g, h_0 h_2, d_0 \rangle$ converges to an element in $\langle \nu, 2\nu, \kappa \rangle$. But, in E_2 (according to ⑦), we have $h_2 f_0 = h_1 g$.
Q.E.D.

<u>Remarks</u> In the case $r = 1$, the Massey product in E_2 is defined in terms of the bar construction (see ①, 2.1 + 2.2). In order for the present approach to apply for $r = 1$, we need that the homotopy $D : S^t \times I \to \overline{W}_{s_1+s_2+s_3}$ rather than $\to \overline{W}_{s_1+s_2+s_3-1}$. It is reported that Moss proves the case $r = 1$ using a definition of the Massey product based on the Yoneda product in Ext. It seems likely that with this approach he uses the original definition of Toda brackets rather than the one used here.

Homotopy Theory of Spheres Seminar

VII : Adam's Spectral Sequence : Computations

8/5/65 Speaker J.P. May

The main purpose of this talk is to simplify the computation of $\mathrm{Ext}_A(Z_2, Z_2)$, the E_2 term of the Adams Spectral Sequence for computing 2-primary components of π_n^S. We do have a 'classical' acyclic resolution of Z_2 over A from the bar construction,

$\bar{B}(A) = T(SI(A))$

'T' - tensor algebra

'A' - 'suspension'

but in practice this is unwieldy.

<u>Milnor - Moore</u> thm on structure of Hopf alg.

filter: $F_p A = I(A)^{-p}, \quad p < 0$

$F_p A = A \quad p \geq 0$

$E^0 A$ is primitively generated

By Milnor-Moore Theorem 5.18

$E^0 A \simeq V P E^0 A$

'P' primitive elements
V universal enveloping algebra of restricted Lie algebra.

Filter $\overline{B}(A)$ by $\otimes$ filtration (essentially)
Look at $E^0\overline{B}(A) = \overline{B}(E^0A) = E^1$ of resulting spectral sequence.

$E^2 = H_*(E^0A)$. Is easy to compute differentials

Now $VPE^0A \simeq E^0A$ calculate H^* by dual spectral sequence. We can get much smaller ~~spectral sequence~~ resolution for VPE^0A then for A.

Milnors results on structure of A (see Annals of Math. Vol. 67)
Let $x_i = Sq^{2^i}Sq^{2^{i-1}} \cdots Sq^1$ admissible monomial V_i have dual x_i^*; A^* = polynomial alg. on $\{x_i^*\}$.
Dualize back and get new basis for A.

If $R = (r_1, \ldots r_n)$ let $P(R) = ((x_1^*)^{r_1} \cdots (x_n^*)^{r_n})^*$

New basis is written: Consider all matrices of integers > 0

$$\begin{array}{c|c|c}
* & x_{0,1} & x_{0,2} \\ \hline
x_{1,0} & x_{1,1} & \cdots \\ \hline
x_{2,0} & \vdots & \ddots
\end{array} = X$$

Define $R(x) : r_i = \sum_j 2^j x_{i,j}$ $\qquad B(x) = \dfrac{\prod t_n!}{\prod_{i,j} x_{i,j}!}$

$S(x) : s_j = \sum_i x_{i,j}$

$T(x) \quad t_n = \sum_{i+j=n} x_{ij}$

Then $$P(R) \cdot P(S) = \sum_{\substack{X \ni \\ R(x)=R \\ S(x)=S}} B(X)\, P(T(x))$$

Consider those sequences R with $r_i = 2^j$, $r_k = 0$ $k \neq i$
Those elts. only project to primitive elts in the associated grading. Call $P(R) = P_i^j$ basis for Lie algebra in associated grading.

Now $[P_j^i, P_\ell^k] = \delta_{i,k+\ell} P_{j+\ell}^k \qquad i \geq k$

$$(P_j^i)^{[2]} = 0$$

- much simpler product than that in A.

$P_i^j = ((X_i^*)^{2^j})^*$ projected to assoc. grading.

Hence (by results on Lie algebras, etc.) have reduced $\mathrm{Ext}_A(\mathbb{Z}_2, \mathbb{Z}_2)$ problem to computing

$H^*(V(L) \otimes \Gamma(SL))$

Γ = divided polynomials

Give this a differential $d\gamma_r(x) = x\gamma_{r-1}(x)$

$$d(\gamma_{r_1}(x_1)\cdots\gamma_{r_n}(x_n)) =$$

$$\sum_{i<j} \gamma_1[x_i, x_j]\cdots\gamma_{r_i-1}(x_i)\cdots\gamma_{r_j-1}(x_j)\cdots$$

$$+\sum_i x_i\cdots\gamma_{r_i-1}(x_i)\cdots$$

We can prove that this is a complex and a free resolution of $\mathbb{Z}_2$.

Then form $\overline{X}(L) = Z_2 \otimes_{V(L)} V(L) \otimes \Gamma(SL)$.

Consider $\overline{X}(L)^*$. Give Γ its natural coalgebra structure, $\gamma_r(x) \to \sum_{i+j=r} \gamma_i(x)\otimes\gamma_j(x)$ dual of coalgebra structure on

$\overline{X}(L)^*$ = polynomial algebra on SL^*.

If $R^i_j = (P^i_j)^*$, then $\delta(R^i_j) = \sum_{k=1}^{j-1} R^i_k R^{i+k}_{j-k}$

This is a polynomial alg. $P(R^i_j)$. Its coh. is coh. of E^2 of the spectral sequence which converges to coh. of A.

Look at $\overline{X}^*_n = P\{R^i_j \mid j \le n\}$, closed under diff. and is subalgebra

$\|$

$\overline{X}^*_{n-1} \otimes Z_n$ where $Z_n = P\{R^i_n\}$.

Filter $\bar{X}_n^*$ by homological degree on $\bar{X}_{n-1}^*$. (This is a decreasing filtration). We get a spectral sequence. $\partial_0 = 0$.

$$E_2 = H^*(\bar{X}_{n-1}^*) \otimes Z_n$$ because of filtration and formula for δ.

$$E_3 = E_\infty = E^0 H^*(\bar{X}_n^*)$$

To calculate differentials imbed $V(L) \otimes P(SL)$ in $\bar{B}(E^0A)$. $\bar{B}$ has shuffle product

$$[x_1 | \cdots | x_m] * [x_{m+1} | \cdots | x_{m+n}]$$
$$= \sum_\pi [x_{\pi(1)} | \cdots | x_{\pi(m+n)}] \quad \pi \in \text{all } (m+n)\text{-shuffles}.$$

Imbedding given by $\gamma_r(x) \to x^r$ where $x^r = [x | \cdots | x]$ <u>r-factors</u>.

Also $Sq^i : E_r \to \underline{E_{r+i}}$

$(R^i_j)^2$ is a cocycle in $(\bar{X})^*$

Call $(R^i_2)^2 = b^i_2$. One has $\delta_{2^n}(b^0_2)^{2^n} = h_0^{2^{n+1}} h_{n+1}$

where $h_0 = R^0_1$

$h_i = R^i_1$ corresponding to h_i in coh. of A

Near $3t = s$, if $x \in E_{2^n}$, then $h_0^{2^{n+1}} x = 0$ in E_{2^n}

Compute Massey products in E_{2^n}. $\langle x, h_0^{2^{n+1}}, h_{n+2} \rangle$ will be represented by $(b_2^0)^{2^n} x = p^n x$ (in E_2)

We know $\delta_1 R_2^0 = R_1^0 R_1^1$ knowing that

$Sq^1 R_2^0 = (R_2^0)^2 = b_2^0 \Rightarrow \delta_2 Sq^1 R_2^0 = \delta_2 b_2^0 =$

$Sq^1(R_1^0 R_1^1) = (R_1^0)^2 R_1^2 + (R_1^1)^3$

<u>Charts</u>

$h_1 e_0 = h_0 f_0$

now $\delta_2 h_1 e_0 = h_1 \delta_2 e_0 = h_1^3 d_0 = h_0^2 h_2 d_0$

$\therefore \delta_2 f_0 = h_0 h_2 d_0$

Have j in 26-stem

$p' h_1 e_0 = h_0^2 j$

$\parallel$

$h_1 p' e_0$

$\parallel$

$e_0 p' h_1$ $\qquad \delta^2(e_0 p' h_1) \neq 0$ hence $\delta(j) \neq 0$

$h_2 i = h_0 j$ hence $\delta(i) \neq 0$.

The $h_0^i h_j$ are in the image of the J-homomorphism.

also $P^i c_0$, $p^i h_1 c_0$

$p^i h_0^j h_2$

$p^k h_0^i h_j$

⑦

Suppose α_i in $H^*(A)$, $a_{i,i}$ rep. cocycles, and $a_{i,j}$ s.t. $\delta a_{i,j} = \sum_{k=i}^{j-1} a_{i,k}\, a_{k+1,j}$

$j-1 \neq n-1$

Easy to verify $\sum_{k=1}^{n-1} a_{1,k}\, a_{k+1,n}$ is a cocycle.

$\{$ coh. classes of such $\} = \langle \alpha_1, \ldots, \alpha_n \rangle$

Massey n-tuple product.

Define homology ops. whose images necessarily generate $H^*(A)$ (A any augmented algebra) these generalize Massey products.

Consider $\bar{B}(A)$ differential coalgebra with coproduct $[a_1|\ldots|a_n] \to \sum_i [a_1|\ldots|a_i] \otimes [a_{i+1}|\ldots|a_n]$.

Hence dual $\bar{C}(A)$ (close to cobar construction) is a differential algebra. Consider $\bar{B}(\bar{C}(A))$.

Basis of $H_1^*(A) \longleftrightarrow$ gens. of A

Basis of $H_2^*(A) \longleftrightarrow$ all relations in A

Hence H_1^*, H_2^* "determine" A

Take $H_1^*, H_2^*(H^*(A))$ – get $H^*(A)$!

$\bar{B}(\bar{C}(A))$ has 2 - differentials:

$\partial'[\alpha_1|\cdots|\alpha_n] = \sum [\alpha_1|\cdots|\alpha_i\alpha_{i+1}|\cdots|\alpha_n]$

$\partial''[\alpha_1|\cdots|\alpha_n] = \sum [\alpha_1|\cdots|\partial''\alpha_i|\cdots|\alpha_n]$.

They commute; let $\partial = \partial' + \partial''$. Grade by total degree.

Filter in 2 ways

1) by homological degree in $\overline{B}(\overline{C}(A))$

Then ∂ comes from ∂' only. Hence

$$E^1 = H_*(T(SI(A))^*)$$

$$\text{"="}\ A^* \text{ essentially}$$

Using this see $H_*(\overline{B}(\overline{C}(A))) = A^*$

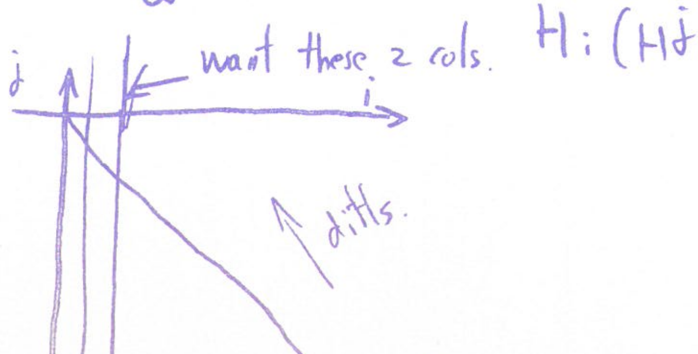
Now filter by homological degree in $\overline{B}$. Then in E_0 ∂ comes from ∂''.

Hence $E^1 = \overline{B}(H^*(A))$

$E^2 = H_*(H^*(A))$ which is what we want.

We know E^∞ additively. Comultiplicatively it is

$(E^0 A)^* = E_\infty$.



$H_i(H^j(A))$

Products of $[[h_i]]$ gen. all diagonal:

$$[[h_{i_1}|h_{i_2}|\cdots|\cdots]]$$

hence differentials are cohomology operations — defined on $\otimes$ products of $H^*(A)$.

But all the first two columns are killed except such products since E_∞ is on diagonal.

There generalize Massey products, If $\alpha_i\alpha_{i+1}=0$,

$$x = [[\alpha_1|\cdots|\alpha_n]] \text{ will satisfy}$$

$$\delta_{n-1}x = [[\langle\alpha_1|\ldots|\alpha_n\rangle]] \text{ if } \delta_i x = 0,\ i<n-1.$$

These generalize useful + common op's. - the complete solution may have to be expressed in terms of them.

Adams s.s. for generalized cohomology theories

Prof. D. W. Anderson 8/11/65
8/17/65

[0] Motivation + General Philosophy (see Adams Seattle notes).

Given $f: X \to Y$ + given H^* a cohomology theory, $\exists$ induced maps $f^* \in \mathrm{Hom}(\tilde{H}^*(Y), \tilde{H}^*(X))$. If $f^* = 0$, let $Z = Y \cup_f CX$ & we have

$0 \to \tilde{H}^*(X) \underset{+1}{\to} \tilde{H}^*(Z) \to \tilde{H}^*(Y) \to 0$ $\therefore$ f determines $\in \mathrm{Ext}^{1,1}(\tilde{H}^*(X), \tilde{H}^*(Y))$

Let $\mathcal{a}$ be an algebra acting as cohomology operations, then f determines as above elements of $\mathrm{Hom}^0_{\mathcal{a}}(\tilde{H}^*(Y), \tilde{H}^*(X))$ and $\mathrm{Ext}^{1,1}_{\mathcal{a}}(\tilde{H}^*(Y), \tilde{H}^*(X))$

As an example let $X = S^{2n+2m-1}$ $Y = S^{2m}$ and $H^* = KU^*$

then $\tilde{K}^0(Z) = J \oplus J$ $J =$ integers

f^* is clearly 0 $\therefore$ f determines an element of $\mathrm{Ext}^{1,1}_{\mathcal{a}}(\tilde{K}^*(S^{2m}), \tilde{K}^*(S^{2m+2n-1}))$

Let $\mathcal{a}$ be the algebra of the Adams operations. These comprise all of the additive operations in K-theory

They have the following properties

1) $\forall k \in Z^+$, $\exists$ $\psi^k: K^0 \to K^0$ a natural ring homomorphism
2) $\psi^1 =$ identity
3) $\psi^k: \tilde{K}(S^{2m})$ is mult by k^m
4) $\psi^k \circ \psi^l = \psi^{kl} = \psi^l \psi^k$
5) E a complex line bundle $\psi^k(E) = E^k$

The ψ^k are characterized by 1 and 5

Choose generators a and b for $\tilde{K}^*(Z)$ such that b restricts from $K^{-1}(X)$ and a goes into a generator of $\tilde{K}(Y)$. Then we get

$\psi^k(b) = k^{m+n} b$ $\psi^k a = k^m a + \lambda_k b$

$\psi^k \psi^l = \psi^l \psi^k \Rightarrow l^m(1-l^n)\lambda_k = k^m(1-k^n)\lambda_l$

$$\lambda_l = \frac{l^m}{k^m} \cdot \frac{(1-l^n)}{(1-k^n)} \cdot \lambda^k$$

If λ_k is integer then all λ_l are + extension is trivial since a can be varied by a multiple of b + hence λ's are determined mod an integer.

This number determines the extension & it can be shown that

$\mathrm{Ext}(\tilde{K}^*(Y), \tilde{K}^*(X)) =$ cyclic finite gp. of order the same order as the image of the complex J-homomorphism.

λ_e is the same invariant as Adams $e(f)$

[I] We now proceed to sketch construction of spectral sequence

Let $\lim^0$ = inverse limit, this functor is

1) left exact 2) has rt. derived functors $\lim^i$

3) If $G_1 \leftarrow G_2 \leftarrow \dots$ G_i are Abelian gps. $\lim^i(G_n) = 0$ $i > 1$

4) If $\{G_i \mid i \in I\}$ is an inverse system of compact Hausdorff top gps & cont. maps then $\lim^n(G_i) = 0$ $n > 0$.

General reference for $\lim^i$ 1) Nöbeling - Topology I 47-63

2) Gothendieck - Complements to Chapter 0 in Chapter III of the Elements.

Def: A Top. gp. G will be special iff $\exists$ a decreasing sequence $\supset G_i \supset G_{i+1} \supset \dots$ of s/gps. which define the topology of G and $\lim^n(G_i) = 0$ $n = 0, 1$

$\lim^0 = 0 \Rightarrow \bigcap G_i = 0 \Rightarrow$ Hausdorff.

We get the following general exact sequence.

$$0 \to \lim^0 G_i \to G \to \lim^0 (G/G_i) \to \lim^1(G_i) \to 0$$

$\uparrow$ completion of G.

$\lim^1 = 0 \Rightarrow G$ complete.

Let C' = category of special top. gps. and closed cont. homomorphisms

Lemma: C' is Abelian

Proof: Note that image & coimage have same top.

If condition on category is changed to compact then condition of closed is unnecessary

This category has countable inverse limits.

We define a filtration on H^* by $F^n\tilde{H}^i = \ker\{\tilde{H}^i(X) \to \tilde{H}^i(X^{n+i-1})\}$

$F^n \tilde{H}^*(X) = \bigoplus F^n \tilde{H}^i(X)$

Lemma: If $H^i(pt) = 0$ $i > 0$ then

1) $\tilde{H}^i(X) \subset F^{i+1}\tilde{H}^i(X)$

(A) 2) $gr^i H^*(X)$ is finitely generated if $H^i_{cc}(X; H^*(pt))$ is of finite type $\forall i$.
[We will assume $H^i(pt)$ is finite $\forall i$]

then $\lim^0 \dfrac{\widehat{H}^*(X)}{F^*(H^*(X))}$ is compact

$\prod \widehat{H}^i(X) = \widehat{H}^{**}(X)$ by condition 1

Each gp $\widehat{H}^i(X)$ is compact Hausdorff and $\widehat{H}^i(X) = \lim_n^0 \dfrac{\widehat{H}^i(X)}{F^n H^i(X)}$

by Milnor's theorem.

[II] Ring of Cohomology operations.

For definition of spectrum + associated concepts see Whitehead - Generalized Homology Theories.

$\widehat{H}^i(\eta) = \lim^0 \widehat{H}^{i+k}(\eta_k)$ $\quad H$ any generalized coh. theory.

If $\mathfrak{M} \xrightarrow[\varphi]{} \eta$ is a map of spectra +

$\varphi_* : \{S^0, \mathfrak{M}\}^* \to \{S^0, \eta\}^*$ is $\cong$ then $\varphi^* : \widehat{H}^i(\eta) \to \widehat{H}^i(\mathfrak{M})$ is also.

Set $\mathcal{a}^{**} = \widehat{H}^{**}(\eta)$ where H in this definition is the cohomology theory associated to η.

$\mathcal{a}^{**}$ is a compact Hausdorff top ring.

We get a map $\mathcal{a}^{**} \otimes \widehat{H}^{**}(X) \to \widehat{H}^{**}(X)$ + hence $\mathcal{a}^{**}$ acts as a ring of continuous maps on $\widehat{H}^{**}(X)$. Furthermore $F^n \mathcal{a}^{**} \otimes H^{**}(X) \to F^n H^{**}(X)$

Def: An $\mathcal{a}^{**}$ module M^{**} of finite type is a top. $\mathcal{a}^{**}$ module $\ni$:

1) $\mathcal{a}^{**} \otimes M^{**} \to M^{**}$ is cont. giving M^{**} the discrete topology

2) every homogeneous element acts continuously on M^{**}

3) The topology of M^{**} is given by a homogeneous filtration such that the two conditions (A) hold.

Prop: $\mathcal{a}^{**}$ modules of finite type + continuous degree preserving $\mathcal{a}^{**}$ module maps form an Abelian category.

Def: A free $\mathcal{a}^{**}$ module is one of the form $\oplus \Sigma^i \mathcal{a}^{**}$

Prop: A countable inverse limit of finitely generated free $\mathcal{a}^{**}$ module is projective.

(To be continued)

Homotopy Theory of Spheres Seminar 8/25/65

Adams spectral sequence for generalized cohomology theories

Speaker: Prof. D.W. Anderson

Let C^* be a cohomology theory. We assume $C^i(\text{one point}) = 0$ for $i \gg 0$, and is finite for all i. In fact by simply shifting dimensions we may assume $C^i(\text{point}) = 0$ for $i > 0$. Let $\mathcal{A}^{**}$ be the algebra of cohomology operations for C^{**}. It has a topology given by a countable decreasing sequence of groups such that the quotient of any 2 adjacent groups in the series is finite. On $C^{**}(X)$ we have a decreasing filtration $F^i C^{**}(X)$ satisfying $F^{i_0}(C^{**}(X)) = C^{**}(X)$ for some i_0, and $F^i C^{**}(X) / F^{i+1} C^{**}(X)$ is a finite group for all i. Again we assume $i_0 = 0$. We wish to find a resolution of $C^{**}(X)$ as an $\mathcal{A}^{**}$-module. We have a diagram

$$\begin{tikzcd}
0 & C^{**}(X) \arrow[l] \arrow[d] & \mathcal{F}_\infty \arrow[l] \arrow[d] & K_\infty \arrow[l] \arrow[d] & 0 \arrow[l] \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & C^{**}(X)/F^2C^{**}(X) \arrow[l] \arrow[d] & \mathcal{F}_2 \arrow[l] \arrow[d] & K_2 \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & C^{**}(X)/F^1C^{**}(X) \arrow[l] \arrow[d] & \mathcal{F}_1 \arrow[l] & K_1 \arrow[l] & 0 \arrow[l] \\
0 & C^{**}(X)/F^0C^{**}(X) \arrow[l, "\simeq"'] & & &
\end{tikzcd}$$

where $\mathcal{F}_i$ is a free $\mathcal{A}^{**}$-module with one generator for each element of (the finite group) $C^{**}(X)/F^iC^{**}(X)$. Taking the $\lim^0$ of the continuous surjections $\mathcal{F}_i \longrightarrow \cdots$ of compact groups we get $\mathcal{F}_\infty$ and a map onto $C^{**}(X)$. The construction is canonical and the resolutions $\mathcal{F}_i$ map onto each other, given an "$\mathcal{A}^{**}$-map of a similar diagram into (1)." The K_i are kernels (in the abelian category) and are also topologized nicely (= filtered by groups with all adjacent quotients finite).

Now construct for each K_i a diagram like (1), replacing with K_i, the group $C^{**}(X)$. Iterate this construction over the kernels of the new infinite sequence of diagrams so obtained, and continue the process. In this way we get a family of objects $\mathcal{F}(\nu_1, \cdots, \nu_k)$ (ν_k an integer ≥ 1), with $\mathcal{F}_i = \mathcal{F}(i)$. If $\mu_k > \nu_k$ for $k = 1, 2, \cdots, n$ we have a map $\mathcal{F}(\mu_1, \cdots, \mu_n) \longrightarrow \mathcal{F}(\nu_1, \cdots, \nu_n)$. This gives us a resolution

$$0 \longleftarrow C^{**}(X) \longleftarrow \mathcal{F}(\infty) \longleftarrow \mathcal{F}(\infty, \infty) \longleftarrow \cdots$$

where $\mathcal{F}(\infty, \infty, \cdots \infty)$ is an n-fold $\lim^0$, and all are projective.

The $\mathcal{F}(\nu_1, \cdots)$ can all be represented by CW-spectra. In the diagram corresponding to (1) of spectra and maps the horizontal arrows (only) are turned around. Thus X maps into each X_i (for $\mathcal{F}_i$) and hence into $X_\infty = \lim^0 X_i$. The K_i map into C^{**} of the mapping cone of $X \longrightarrow$ [Spectrum for $\mathcal{F}_i$]. Now apply the functor $\{Y, \cdot\}^*$ This gives a series of spectral sequences which are described in terms of the path taken from $C^{**}(X)$ to the particular $\mathcal{F}(\nu_1, \cdots, \nu_k)$:

$$E_1^{s,t}(\text{path}) = \mathrm{Hom}^t(s^{\text{th}}\ \mathcal{F}(\ )\ \text{in that path}, C^{**}(Y))$$

for any path out from $C^{**}(X)$ in the diagram (1). This Hom^t will be compact since each $\mathcal{F}(\ )$ is finitely generated and $C^{**}(Y)$ is in our category. Hence we can take inverse limits over all the paths. It will be an exact functor and will commute with taking cohomology.

$$E_2^{s,t}(\text{inverse limit}) = \mathcal{E}xt^{s,t}\left(\tilde{C}^{**}(X), \tilde{C}^{**}(Y)\right)$$

Look at $KC^* =$ complex K-theory <u>mod p</u>. Groups of a point continue indefinitely in positive dimensions, periodic with period 2. Note ψ^k acts on KC^* as an operation of degree 0 for k prime to p.

$$(\psi^k - k^{n_1})(\psi^k - k^{n_2}) \cdots (\psi^k - k^{n_r}) = 0$$

for some collection $(n_1, \cdots, n_r)$. There are only finitely many different powers of k mod p, hence we can pick k a generator of $\mathbb{Z}_p$ and divide up $KC^0(X)$ according to the eigenvalues of ψ^k.

$$KC^0(X) = KC^0(X;k) \oplus KC^0(X;k^2) \oplus \cdots \oplus KC^0(X, k^{p-1})$$

Periodicity map carries $KC^0(X, k^i) \to KC^0(X, k^{i+1})$ etc. Now $KC^*(X,k)$ is a cohomology theory which is periodic; the period is $2(p-1)$. Thus $KC^r(\text{point}; k)$ is given by

r	0	1	2	$\cdots$	$2(p-1)-1$	$2(p-1)$	$2(p-1)+1$	$\cdots$
group	$\mathbb{Z}_p$	0	0	$\cdots$	0	$\mathbb{Z}_p$	0	$\cdots$

Thus (renaming) $KC^*(X)$ is a cohomology theory periodic with period $2(p-1)$. Define a new theory kC^* by

$$kC^i(pt.) = \begin{cases} 0 & i > 0 \\ KC^i(pt), & i \leq 0. \end{cases}$$

(Take a representing spectrum for K, an $(i-1)$-connected covering for i^{th} term). Call the spectrum $b\mathcal{U}$. Then

$$b\mathcal{U}^0 = B\mathcal{U} \times \mathbb{Z}_p \qquad b\mathcal{U}^{+2} = B\mathcal{U}$$
$$b\mathcal{U}^{\pm 1} = \mathcal{U} \qquad b\mathcal{U}^{+4} = BS\mathcal{U}.$$

Adams showed that for $H^* =$ ordinary cohomology, $H^*(b\mathcal{U}; \mathbb{Z}_p) \simeq a_* / a_* \cdot (Q')$ where a_* is the Steenrod Algebra and $= [\beta_p, \mathcal{P}'] = Q'$ (see paper on Chern characters).

We want to compute $kC^{**}(b\mathcal{U}) = a^{**}$.

We may change terminology to avoid confusion, calling the a^{**}, R^{**} instead

$$E_2^{p,q} = H^p(bu ; kC^q(pt))$$

$$E_\infty^{p,q} = gr^p \, kC^{p+q}(bu) \qquad \left\{ \begin{array}{l} p^{th} \text{ graded} \\ \text{associated thing} \end{array} \right.$$

The kC^* theory looks as follows:

If K theory has only 1 differential (true for bu), all the boundaries of $\underline{K}$-theory will be non-0 in $\underline{k}$-theory and will be left.

Next we compute KC, mod 2 only. The cohomology operations in integral KC-theory are half those of mod-2 theory, only the Bockstein operation is missing. So we may compute mod 2 theory in part by proceeding with integral theory.

$$H^*(bu_{\text{integral}} ; \mathbb{Z}_2) \cong a_* / a_*(Sq^1, Sq^{0,1})$$

Compute the cohomology of $a_* / a_*(Sq^1, Sq^{0,1})$ with respect to $\underline{\text{left}}$ multiplication by $Sq^{0,1}$, the first differential of the spectral sequence. This is (additively) isomorphic to the homology of $a_* / (Sq^1, Sq^{0,1}) \cdot a_*$ (applying the canonical antiautomorphism χ to everything) with respect to $\underline{\text{right}}$ multiplication by $Sq^{0,1}$. Now dualize: $\mathcal{a}^*_* = \mathbb{Z}_2[\xi_1, \xi_2, \ldots]$

$$\left(\mathcal{a}_* / (Sq^1, Sq^{0,1})\mathcal{a}_*\right)^* = \mathbb{Z}_2[\xi_1^2, \xi_2^2, \xi_3, \xi_4, \xi_5, \ldots]$$

(this takes some computation). Remember $\varphi(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i$.

Fact: d_3^* is a derivation: $d_3(\xi_k) = \xi_{k-2}^4$. Therefore the homology of $\mathcal{a}_*/\mathcal{a}_*(Sq^1, Sq^{0,1})$ with respect to left multiplication by $Sq^{0,1}$ is (additively) $\simeq E(\xi_1^2, \xi_2^2, \xi_3^2, \ldots)$ where E means "exterior algebra". Therefore $H^*(\ ) = 0$ in odd dimensions, and so the spectral sequence stops at E_4. We have a periodicity map π lowering dimension by $2(p-1)$; $\pi \in \mathcal{R}^{-2(p-1)}$. The left annihilators of π and the right annihilators of π are identical and are equal to the K-theory boundaries of dimension $=0$ in k-theory.

<u>Theorem</u> $\mathcal{R}^{**}/(\text{annihilators of } \pi)$ is abelian.

Call this algebra Ψ^*. We can show Ψ^* is generated by elements $\psi^{2^{r+1}-1} - \psi^{2^r-1} = \varphi_r$ together with the Bockstein operation; the indeterminacy of these is wiped out by dividing out $\mathrm{ann}(\pi)$. We can also show φ_r corresponds to ξ_r^2.

HOMOTOPY THEORY OF SPHERES SEMINAR

8/20/'65 Speaker: J. C. Moore

1)

Let $\mathcal{A}$ be a category, A, B objects. The product of A and B, if it exists, is denoted by $A \times B$ with morphisms $A \xleftarrow{p} A\times B \xrightarrow{q} B$ and with the universal property that given $f: C \to A$ and $g: C \to B$ there exists $f \wedge g: C \to A \times B$ such that the following diagram commutes.

$$\begin{tikzcd} A & A\times B \arrow[l, "p"'] \arrow[r, "q"] & B \\ & C \arrow[ul, "f"] \arrow[u, "f\wedge g"'] \arrow[ur, "g"'] & \end{tikzcd}$$

It is easily seen that the product is unique up to equivalence.

Similarly, the coproduct, if it exists, is $A \oplus B$ with morphisms $A \xrightarrow{i} A\oplus B \xleftarrow{j} B$ and the analagous universal property represented by the diagram:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & A\oplus B \arrow[d, "f\vee g"] & B \arrow[l, "j"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

Again, $A \oplus B$ is unique up to equivalence.

A category $\mathcal{A}$ is pointed if there exists an object $*$ in $\mathcal{A}$ such that if A is any object in $\mathcal{A}$ then there is a unique morphism $*: A \to *$ and a unique morphism $*': * \to A$. $*$ is therefore unique up to equivalence.

One says that a morphism $f: A \to B$ is trivial if it factors so:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & * \arrow[ur] & \end{tikzcd}$$

For example, the category of abelian groups and homomorphisms is pointed, the point being the zero group.

On the other hand, the category of topological spaces and continuous maps is not pointed, for although there is a natural projection from any space onto a point, there is no natural inclusion of a point in an arbitrary space. Again, there is a natural inclusion of the empty set in any topological space, but no projection.

A category has an initial point if there is an object P and a unique morphism $P \to A$ for any object A. A category has a terminal point if there is an object Q and a unique morphism $A \to Q$ for every object A.

Thus the category of topological spaces and continuous maps has an initial point, the empty set, and a terminal point, the point.

If a category is pointed, there is a unique morphism $A \oplus B \to A \times B$. This is because one has $A \oplus B \xrightarrow{f} A$ by

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr] & A \oplus B \arrow[d, "f"] & B \arrow[l, "j"'] \arrow[d] \\ & A & * \arrow[l] \end{tikzcd}$$

and similarly $A \oplus B \xrightarrow{g} B$

and so $f \wedge g : A \oplus B \to A \times B$.

For example, in the category of abelian groups and homomorphisms this map is an isomorphism.

In the category of topological spaces with base points and continuous base point preserving maps (a pointed category) $\times$ equals the cross product and $\oplus$ equals disjoint union with base points identified and the natural map $A \oplus B \to A \times B$ amounts to the inclusion $A \times b \cup a \times B \to A \times B$ where a is the base point of A and b is the base point of B.

Now we will define an additive category. Let $\mathcal{A}$ be a category with product and coproduct (that is, product and coproduct exist for any two objects of the category). Let $\mathcal{A}$ be pointed. Let the map $A\oplus B \longrightarrow A\times B$ be an isomorphism (equivalence).

There is always a natural diagonal map $\Delta: A \to A\times A$ and similarly a map $\phi: A\oplus A \to A$.

If $f,g: A\to B$, define the sum of f and g by
$A \xrightarrow{\Delta} A\times A \xrightarrow{f\times g} B\times B \xleftarrow{\cong} B\oplus B \xrightarrow{\phi} B$.

It can easily be shown that under this operation $\mathrm{Hom}_{\mathcal{A}}(A,B)$ is a monoid with identity $A\to * \to B$.

$\mathcal{A}$ is said to be an additive category if this monoid is a group for all A,B.

2)

Let K be the field $\mathbb{R}$ (reals) $\mathbb{C}$ (complexes), or $\mathbb{H}$ (quaternions). Let X be a topological space. Consider the category of vector bundles over X with field K and vector bundle maps.

There is a special vector bundle over X, the one where the fibre is zero. Denote this by $*$. Given any bundle E there are natural maps $* \to E \to *$, and in this way the category is pointed.

The Whitney sum of P,Q, bundles over X, is those (p,q) in $P\times Q$ such that $\pi_P(p) = \pi_Q(q)$, this set with the subspace topology, etc. The Whitney sum provides a coproduct for the Category.

$K^n\times X$ with the obvious projection is the canonical bundle over X for any integer $n\geq 0$, where $K^n = K\times K\times\ldots\times K$ n times. $K^n\times X$ is also called the n-dimensional trivial bundle.

A bundle E over X is said to be invertible if there exists a bundle F over X such that $E\oplus F = K^n\times X$.

If X is compact, every vector bundle is invertible.

Let $\mathcal{A}_K(X)$ be the category of invertible vector bundles with field K over X. $\mathcal{A}_K(X)$ is pointed, has coproduct as before.

A map $f: X \to Y$ induces a map $f^*: \mathcal{A}_K(Y) \to \mathcal{A}_K(X)$. For any P in $\mathcal{A}_K(Y)$ is contained in a trivial bundle $K^n \times Y$. Extend f to $f': K^n \times X \to K^n \times Y$. $f'^{-1}(P)$ is an invertible bundle over X.

The association $X \to \mathcal{A}_K(X)$ is therefore a contravarient functor from topological spaces and maps into pointed categories.

3)

Let $\mathcal{A}$ be any category with coproducts and $\mathcal{S}$ a distinguished collection of objects such that the isomorphism classes of $\mathcal{S}$ form a set, and if A,B are objects in $\mathcal{S}$, $A \oplus B$ is in $\mathcal{S}$.

Let F be the free abelian group generated by the set of isomorphisms of elements of $\mathcal{S}$. Let $\langle A \rangle$ be the isomorphism class of A. Let R be the subgroup of F generated by elements of the form $\langle A \oplus B \rangle - \langle A \rangle - \langle B \rangle$.

Then the Grothendieck group $G^{\mathcal{S}}_{\mathcal{A}} = F/R$ and $[A]$ represents the equivalence class ofb$\langle A \rangle$ in this group. If the equivalence classes of $\mathcal{A}$ form a set, $G(\mathcal{A})$ denotes $G^{\mathcal{A}}_{\mathcal{A}}$.

For instance, let K be a field and consider the category of finite dimensional vector spaces over the field. Isomorphism classes of these are uniquely determined by a single integer $n \geq 0$, the dimension. Then $G(\mathcal{A}) = Z$, the integers.

Now we had a functor from topological spaces to pointed categories given by $X \to \mathcal{A}_K(X)$. We define the Grothendieck group of X, for K equal to the reals, complexes, or quaternions, by $G(\mathcal{A}_K(X))$ and denote them respectively by KO(X), KC(X), KSp(X), or just K(X) where the field is understood.

Now we consider just this sort of situation in a slightly different context. Let R be a ring. Let P be the category consisting of all finitely generated projective left R modules and R homomorphisms.

This category has a Grothendieck group G(R) and there is a canonical map $Z \to G(R)$ given by $1 \to [R]$. Note that in the previous example where K was a field and everything was projective this map was an isomorphism.

Now $Z \to G(R) \to P(R) \to 0$ defines P(R), the cokernel, as the projective class group of R.

Returning to the vector bundles and topological spaces we have a natural map $Z \to K(X)$ given by $1 \to$ [trivial bundle]. The cokernel, denoted by $\tilde{K}(X)$, is called the projective class group.

Let $\mathcal{R}$ denote the category of rings and reing homomorphisms. If $f: R \to S$, S and all S modules may be considered R modules by the action $r\#s = f(r)s$. If P is any finitely generated projective left R module, then $S \otimes_R P$ is a finitely generated projective left S module. The assignation $P \to S \otimes_R P$ induces the map $G(f): G(R) \to G(S)$, and G is easily seen to be a functor on $\mathcal{R}$, and so is P, the projective class operator.

4)

In reality, the range categories of G and P are not the same. Let $\mathcal{A}$ be the category of abelian groups and homomorphisms. $P: \mathcal{R} \to \mathcal{A}$.

Let $\mathcal{A}_0$ be the category whose objects A are abelian groups togather with maps $Z \xrightarrow{\eta_A} A$. (i.e. A has a distinguished element, the image of 1.) And whose morphisms are homomorphisms f such that

$$\begin{tikzcd} Z \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commutes.

$\mathcal{A}_0$ too has a coproduct, but not quite the usual one (direct sum).

Define $A \oplus_0 B = A \oplus B$/subgroup generated by $(\eta_A(1) - \eta_B(1))$.
Define $Z \to A \oplus_0 B$ by $z \to [\eta_A(z) \oplus 0] = [0 \oplus \eta_B(z)]$. $[\ ,\ ]$ = equivalence class.

$\mathcal{A}_0$ is not pointed. $\mathcal{A}_0$ has the ordinary product, and $Z \longrightarrow A\mathrm{x}B$ is given by $z \longrightarrow \eta_A(z)\mathrm{x}\eta_B(z)$.

Now $\mathcal{A}_0$ is the proper range for G. $G: \mathcal{R} \longrightarrow \mathcal{A}_0$.

There is a natural functor $\mathcal{A}_0 \longrightarrow \mathcal{A}$, which takes the cokernel of $\eta_A: Z \longrightarrow A$, and one of course has commutative.

$$\begin{tikzcd} \mathcal{R} \arrow[r, "P"] \arrow[dr, "G"'] & \mathcal{A} \\ & \mathcal{A}_0 \arrow[u, "\text{cokernel map}"'] \end{tikzcd}$$

5)

It is natural to ask whether the contravariant functor K and the covariant functor G behave nicely with respect to coproducts and products.

$\mathcal{R}$ has a product. The product of rings R and S is simply RxS, the cross product, with termwise addition and multiplication.
$R \xleftarrow{p} R\mathrm{x}S \xrightarrow{q} S$ becomes on application of G $G(R) \longleftarrow G(R\mathrm{x}S) \longrightarrow G(S)$ and one can easily check that G(RxS) is the product of G(R) and G(S) in $\mathcal{A}_0$.

Analagously, the contravariant functor K takes the coproduct of two topological spaces, which is their disjoint union, into a product in $\mathcal{A}_0$. This follows easily from the fact that the union is disjoint and so the bundles above the union are simply sums of bundles over the two original spaces.

6)

Now assume the rings are commutative. If P,Q are finitely generated projective left modules over R, $[P],[Q] \to [P \otimes_R Q]$ gives a well-defined bilinear map $G(R) \times G(R) \longrightarrow G(R)$ which makes G(R) a ring.

If K is an arbitrary commutative ring, let R(K) denote G(K) with ring structure. Thus R is a functor from commutative rings to commutative rings. The map $Z \to R(K)$ is a ring homomorphism.

If P is a finitely generated projective module over K, a commutative ring, let E(P) denote the exterior algebra of P.

$E(P)_0 = K$, $E(P)_1 = P$, and $E(P)_r = 0$ for r large.

$E(K^m)_k = K^{\binom{m}{k}}$. $E(P \oplus Q)_k = \bigoplus_{i+j=k} E(P)_i \otimes E(Q)_j$.

Let $R(K)[[t]]$ denote the ring of formal power series in one indeterminate t. $R(K)^+[[t]]$ is the formal power series with leading coefficeints 1, and is a group under multiplication.

We define a map $\lambda : R(K) \longrightarrow R(K)^+[[t]]$ by

$$\lambda([P]) = \sum_{j=0}^{\infty} \lambda^k [P] \, t^k, \quad \text{where} \quad \lambda^k[P] = [E(P)_k].$$

Note that $\lambda^k[P \oplus Q] = \sum_{i+j=k} \lambda^i[P]\lambda^j[Q]$. With these definitions $\lambda^0[P] = 1$ and $\lambda^1[P] = [P]$.

Let $X_1, \ldots X_n, \ldots$ be a countable set of things which can be added and multiplied. Then the Newton formulas are:

$$P_1 = X_1$$

$$P_2 = P_1X_1 - 2X_2$$

$$P_3 = P_2X_1 - P_1X_2 + 3X_3$$

$$\vdots$$

$$P_{n+1} = \sum_{j=1}^{n} (-1)^{j+1} P_{n+1-j}X_j + (-1)^{n+1}(n+1)X_{n+1}$$

The original significance of the Newton formulas was this: Consider $Y_1, \ldots Y_n$ as indeterminates for a polynomial ring. The symmetric polynomials form a subring. Any symmetric polynomial can be expressed in terms of the so called elementary symmetric polynomials. In particular:

$Y_1 + \ldots + Y_n = X_1$ is the first elementary symmetric polynomial.

$Y_1^2 + \ldots + Y_n^2$ is not elementary, but is symmetric. P_2 is the formula for expressing it as the sum of the first and second elementary polynomials.

Similarly P_3 is the formula for expressing $Y_1^3 + \ldots + Y_n^3$ as the sum of $Y_1^2 + \ldots + Y_n^2$ and the first three elementary symmetric polynomials, and so on.

In our situation, the Adams operations can be defined by:

$$\psi^1 = \lambda^1$$

$$\psi^2 = \psi^1\lambda^1 - 2\lambda^2$$

$$\vdots$$

$$\psi^{n+1} = \sum_{j=1}^{n} (-1)^{n+1}\psi^{n+1-j}\lambda^j + (-1)^{n+1}(n+1)\lambda^{n+1} .$$