MIT Topology Seminar 9/25/67

Don Anderson : Relative Cohomology and Hopf Algebras.

Examples: Let A = mod 2 Steenrod algebra.

1) $C = H^*(MSpin) = C_{spin}$ is the direct sum of cyclic modules isomorphic to A, $A/A(Sq^1, Sq^2)$, and $A/A\cdot Sq^3$. MSpin is the Thom space of bundles with $w_1 = w_2 = 0$.

2) $C_{SO} = H^*(MSO)$ $(w_1 = 0)$. Wall showed C_{SO} = direct sum of cyclic modules isomorphic to A or $A/A\cdot Sq^1$.

What we want is a natural way of extracting relations. Let H be a Hopf algebra, that is $\mu : H\otimes H \to H$ and $\Delta : H \to H\otimes H$ maps of algebras. C is a coalgebra over H; a left H-module with a map of modules $r : C \to C\otimes C$. Now $H\otimes H$ acts on $C\otimes C$, so H acts on $C\otimes C$ via Δ.

$$H \longrightarrow H\otimes H \xrightarrow{1\otimes\eta} H\otimes C \quad , \quad \eta : H \to C = \text{evaluation on unit.}$$

Weak form of theorem: If C is a coalgebra over H and $A = H\,\square_C\,Z_2 = Z_2\,\square_C\,H$, then η induces a monomorphism $H\otimes_A Z_2 \longrightarrow C$, and if this map is H-split, C is projective as an H-module relative to A. [The splitting is the troublesome condition]. //

Let M be a module over H, $A\subset H$ a subalgebra.

Dfn M is relatively free with respect to A if $M = H\otimes_A N$ for some A-module N.

If A is a free right A-module, $H\otimes_A -$ is exact.

$$\mathrm{Ext}_H(M, Z_2) \cong \mathrm{Ext}_A(N, Z_2).$$

Dfn P is relatively projective if it is a direct summand in a relatively free module. Some equivalent properties:

an A-linear splitting $\dashleftarrow$ always exists.

$$\begin{tikzcd} M \arrow[rr, two heads, shift left] & & M' \arrow[ll, dashed, shift left] \\ & P \arrow[ul] \arrow[ur] & \end{tikzcd}$$

H-linear

relatively free $\Rightarrow$ relatively projective

$$\begin{tikzcd} M \arrow[r, two heads] & M' \\ H\otimes_A N \arrow[u] \arrow[ur] & N \arrow[l] \end{tikzcd}$$

$1\otimes_A N$

P is relatively projective if the restriction map
$$\mathcal{E}xt_H(P,M) \longrightarrow \mathcal{E}xt_A(P,M)$$
is injective for all M.

$$P \rightleftarrows H\otimes_A P$$

$$\mathcal{E}xt_H(P,M) \rightleftarrows \mathcal{E}xt_A(H\otimes_A P, M) \cong \mathcal{E}xt_A(P,M)$$

$$0 \to N' \underset{H\text{-linear}}{\longrightarrow} N \longrightarrow N'' \to 0$$

A-linear

$$\begin{tikzcd} \mathrm{Hom}_H(P,N) \arrow[r] & \mathrm{Hom}_H(P,N'') \arrow[r, "0"] \arrow[d] & \mathcal{E}xt^1_H(P,N') \arrow[d, tail] \\ \mathrm{Hom}_A(P,N) \arrow[r] & \mathrm{Hom}_A(P,N'') \arrow[r, "0"] & \mathcal{E}xt^1_A(P,N') \end{tikzcd}$$

onto $\Leftrightarrow$ lifting.

<u>Proof</u> Leave out simple Hopf Algebra theory.

$$\begin{tikzcd} C \arrow[r, "r"] & C\otimes C \arrow[r, "\phi\otimes 1"] \arrow[dr, "\varepsilon\otimes 1"'] & (H\otimes_A \mathbb{Z}_2)\otimes C \arrow[d, "\varepsilon\otimes 1"] \\ & & C \end{tikzcd}$$

$\phi : C \to H\otimes_A \mathbb{Z}_2$ the hypothesized ~~H~~A-linear splitting.

C is an H-summand in $(H\otimes_A \mathbb{Z}_2)\otimes C \simeq H\otimes_A C$.

<u>Cor</u> $\mathcal{E}xt_H(C,M) \longrightarrow \mathrm{Ext}_A(C,M)$ is injective and can be split, canonically with respect to M. //

If $C = H^*(MSO)$, $M = \mathbb{Z}_2$, $A = E(Sq^1)$, $H =$ Steenrod algebra.

This is a nice theory which doesn't do the job.

<u>Thm</u> If H is finite-dimensional and $\mathbb{Z}_2\otimes_A H\otimes_A \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2\otimes_A C$ is injective, then the map $\phi : C \to H\otimes_A \mathbb{Z}_2$ exists (H-linear)

<u>Proof</u> (Anderson in special cases, Larson in general) //.

<u>Thm I</u> If H is a finite-dimensional Hopf algebra, H_n the highest

non-vanishing group, then H_n is $\mathbb{Z}_2$. Map $H \longrightarrow H^*$ by evaluation (grade H^* negatively) on dual generator in H^*_{-n}; it is an isomorphism of left and right H-modules. //

Thm 2 $\chi(H \otimes_A \mathbb{Z}_2)^* \cong H \otimes_A \mathbb{Z}_2$.

10/2/67

MIT Topology Seminar

Vincent Giambalvo: Odd Torsion in $\langle 8 \rangle$-cobordism

Thm A $\Omega_*^{\langle 8 \rangle}$ = $\langle 8 \rangle$-cobordism, has no q-torsion for $q > 3$.

Given a sequence of spaces and maps $\{Y_k, j_k, f_k\}$ with $j_k : Y_k \to Y_{k+1}$ and $f_k : Y_k \to BO_k$ such that
$$\begin{array}{ccc} Y_k & \to & Y_{k+1} \\ \downarrow & & \downarrow \\ BO_k & \to & BO_{k+1} \end{array}$$
commutes, we say a manifold M has a Y-structure if $M^n \xrightarrow{\nu} BO_{n+k}$ lifts. We use this to define a bordism relation $M \underset{Y}{\sim} N$, where $(M, \tilde{\nu}_M) \sim (N, \tilde{\nu}_N)$ if there is a manifold with boundary $M \cup N$, and a Y-structure compatible with those of M and N.

Thm $\Omega_n^Y \simeq \pi_{n+k}(M f_k^* \gamma_k)$ for large k (see notes of Lashof). where $\gamma_k \to BO_k$ is the canonical bundle. //

Reasonable algebraic structures are obtained by limiting Y's. One case is the n-connected covering $BO\langle n+1 \rangle \xrightarrow{f} BO$ (killing first n homotopy groups), with $f_* : \pi_i(BO\langle n+1 \rangle) \to \pi_i(BO)$ an isomorphism for $i > n$, and $BO\langle n+1 \rangle$ n-connected. Then $BO\langle 1 \rangle = BO$, $BO\langle 2 \rangle = BSO$, $BO\langle 3 \rangle = BO\langle 4 \rangle = BSpin$. In our case $BO\langle 8 \rangle$ we have a fibration $K(\mathbb{Z}, 3) \to BO\langle 8 \rangle \to BO\langle 4 \rangle = BSpin$; we have the Whitney sum in this space, for the following lifting exists:
$$\begin{array}{ccc} BO\langle 8 \rangle \times BO\langle 8 \rangle & \longrightarrow & BO\langle 8 \rangle \\ \downarrow & & \downarrow \\ BSpin \times BSpin & \xrightarrow{\oplus} & BSpin, \end{array}$$
by obstruction theory. Thus in $\Omega^{\langle 8 \rangle}$, $[M] \times [N] = [M \times N]$. There is a Thom space $MO\langle 8 \rangle = M f_k^* \gamma_k$. In proving Thm A we use the steps

① $H^*(MO\langle 8 \rangle, \mathbb{Z})$ has no q-torsion

② $H^*(MO\langle 8 \rangle, \mathbb{Z}_q)$ is a free $\mathcal{A}/\text{Bockstein}$-module.

One completes the proof using the Brown-Peterson spectrum.

$H^*(BSpin) = \mathbb{Z}_q[\mathcal{P}_i]$ $i = 1, 2, \cdots$ Pontrjagin classes.

$H^*(\mathbb{Z}, 3; \mathbb{Z}_q) = \mathbb{Z}_q[\quad] \otimes E(\quad)$ (E=exterior algebra).

$$H^*(BO\langle 8\rangle) = Z_q[\{P_i \mid i \neq \tfrac{1}{2}(q^j+1)\}] \otimes Z_q[\{\tilde{P}_i \mid i = \tfrac{1}{2}(q^j+1)\}]$$

Proof of ① apply universal coefficient theorem.

Remark $H^*(BO\langle 8\rangle)$ does not have a splitting over $\mathcal{a}_p$; the cohomology of the fiber enters.

Proof of ② use $e': \mathcal{a} \longrightarrow H^*(MO\langle 8\rangle)$ is free

if $e(\alpha) = \alpha \cdot U$ is a monomorphism

and if $e \,|\, \{\text{primitive elements of } {}'\mathcal{a}\}$ is a monom.

(from Milnor-Moore Hopf Algebra theory). where ${}'\mathcal{a} = \mathcal{a} / \mathcal{a}\cdot\beta$.

Use of spectrum: there is a spectrum X with $H^*(X) = {}'\mathcal{a}$.

$$H^*(MO\langle 8\rangle) = \sum {}'\mathcal{a} \cdot b_\alpha$$

$MO\langle 8\rangle \longrightarrow$ product of $S^{|\alpha|}X$ (suspension).

This map has $b_\alpha \longleftarrow 1_\alpha$ in cohomology; the homotopy of (prod. $S^{|\alpha|}X$) is a free abelian group. This map is an isomorphism in homology, so modulo torsion one would have an embedding of torsion groups in free abelian groups. //

This could also be done using the Adams spectral sequence.

MIT Topology Seminar 2/26/68

Paul Schweitzer: Cobordism of Immersion Pairs.

I We want to study immersions of closed differentiable manifolds $f: M^m \subseteq N^n$ with $k = n-m > 0$. One method is to classify them up to regular homotopy.

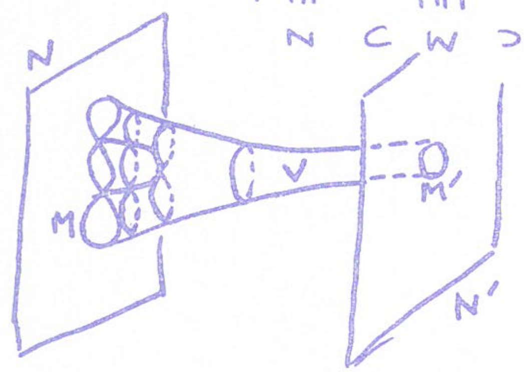
Dfn Let $f, g: M \subseteq N$ be 2 immersions. Then $f \sim g$, f is regularly homotopic to g, if there is a homotopy $f_t: M \subseteq N$ such that each f_t is an immersion and the induced map of tangent bundles $df_t: TM \to TN$ is a homotopy.

This classification is delicate and difficult. Another approach is based on Thom's cobordism construction.

Dfn Let $f: M \subseteq N$, $g: M' \subseteq N'$, $\dim M = \dim M'$, $\dim N = \dim N'$. Say f is immersion cobordant to g, $f \equiv g$, if there are manifolds V, W with $\dim V = \dim M + 1$, $\dim W = \dim N + 1$ such that $\partial V = M - M'$, $\partial W = N - N'$, and there is an immersion $F: V \to W$ giving a commutative diagram

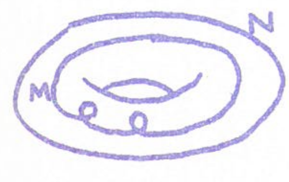
$$\begin{array}{ccccc} M & \subset & V & \supset & M' \\ f \cap\!| & & \cap\!| F & & \cap\!| g \\ N & \subset & W & \supset & N' \end{array}.$$

Example 1



In this case M and M' are also regularly homotopic.

Example 2 In general $\equiv$ is weaker than $\sim$. N is a torus and M is a circle on N with 2 loops. A tangent winding number is defined on N and the TWN of M is 2.



However by putting N inside a 3-sphere and doing surgery, $(N, M) \equiv (S^3, S^1)$. //

We can assume various structures on M, N, e.g. orientability.

Dfn $\Omega^{imm}(m,n)$, $\mathfrak{N}(m,n)$ denote equivalence classes of orientable, resp. arbitrary, immersion pairs of dimensions m and n, mod $\equiv$.

To study these we use bordism groups $\Omega_n(X)$ and $\mathfrak{N}_n(X)$ defined using bordism classes of maps $N^n \to X$.

<u>Thm</u> $\Omega^{imm}(m,n) \simeq \Omega_n(\Omega^r\Sigma^r MSO(k))$ $k = n-m$, where r is large (in stable range; $r > m-k$), and

$$\mathfrak{N}(m,n) \simeq \mathfrak{N}_n(\Omega^r\Sigma^r MO(k)). \quad //$$

We can get rid of the dependence on r by the following device: let Q be the functor $QX = \bigcup_r \Omega^r\Sigma^r X$; we have inclusions $\Omega^r\Sigma^r X \subset \Omega^{r+1}\Sigma^{r+1}X$ obtained inductively; $X \subset \Omega\Sigma X$, $\Sigma X \hookrightarrow \Omega\Sigma^2 X$, $\Omega\Sigma X \hookrightarrow \Omega^2\Sigma^2 X$, etc.

<u>II</u> <u>History</u>. Similar theories have been considered by Wall, Stong, and R. Wells. Wall (Comm. Math. Helv. 35(1961)) showed for embeddings that $\Omega^{emb}(m,n) \simeq \Omega_n(MSO(k))$. Stong showed for arbitrary maps that $\Omega^{map}(m,n) \simeq \Omega_n(\Omega^r MSO(k+r))$, r large. Thus the results fit together according to

$$\Omega^{emb}(m,n) \longrightarrow \Omega^{imm}(m,n) \longrightarrow \Omega^{map}(m,n)$$

induced from $MSO(k) \subset \Omega^r\Sigma^r MSO(k)$ and $\Sigma MSO(k) \longrightarrow MSO(k+1)$. Wells considers $\Omega^{imm\,sph}$, equivalence classes of immersions where $f: M \subseteq S^n$, $f': M' \subseteq S^n$, and $F: V \subseteq S^n \times I$, and shows $\Omega^{imm,sph}(m,n) \simeq \pi_n(QSO(k))$. We also have $\Omega^{imm\,sph}(m,n) \longrightarrow \Omega^{imm}(m,n)$.

<u>III</u> <u>Proof of theorem</u> We use a theorem of Hirsch (cf. Phillips Top. 6 (1966) and Hirsch TAMS 1959). Let $\mathrm{Max}(TM,TN)$ be the space of bundle maps having maximal rank on each fiber, in the compact open topology. Give $\mathrm{Imm}(M^m,N^n)$ the C^1 topology.

<u>Thm</u> There is a weak homotopy equivalence $\mathrm{Imm}(M^m,N^n) \to \mathrm{Max}(TM,TN)$, $m<n$.

<u>Cor</u> There is a 1-1 correspondence between regular homotopy classes of immersions $f: M \subseteq N$ and regular homotopy classes of immersions $g: M \subseteq N \times \mathbb{R}^r$ with r everywhere independent vector fields.

<u>Thm ⇒ Cor</u> Going from $f \to g$ is trivial. Given $g: M \subset N \times R$ (enough to do for $r=1$) with its normal vector field X. Take $dg: TM \to T(N \times R)$. Modify dg and X continuously to dg' and X' such that X' is parallel to $\mathbb{R}$, then get map $h: TM \to TN$ by sliding down on X'. Then apply theorem. //

In proving our theorem we give only the oriented case; the non-oriented is essentially the same. We define maps $\Omega^{imm}(m,n) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \Omega_n(\Omega^r \Sigma^r MSO(k))$ which are inverses.

1. <u>Definition of φ</u> Given $f: M \subseteq N$, let $*$ be the basepoint of N.

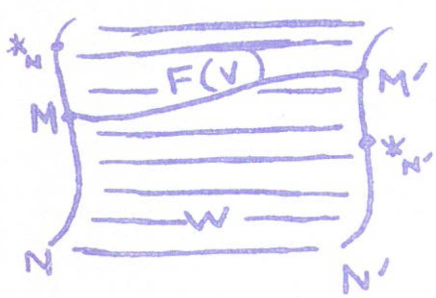
1) modify via a regular homotopy, $f \sim f_0 : M \subseteq N - *$.

2) $i: N - * \subset (N-*) \times \mathbb{R}^r$. Now if$_0$ is regular homotopic to $f_1 : M \subset (N-*) \times \mathbb{R}^r$ such that $f_1 M$ has a normal r-frame from $\mathbb{R}^r$, ν_{f_1} normal bundle.

3) look at the classifying map of unit disk bundles $E(\nu_{f_1}) \to E(\tilde{\xi}_k \oplus r\varepsilon)$. Extend (by Thom construction) to a map $f_2 : \Sigma^r N \to \Sigma^r MSO(k) = E(\tilde{\xi}_k \oplus r\varepsilon) \cup \{\infty\}$.

4) Take adj. $f_3 : N \to \Omega^r \Sigma^r MSO(k)$; $\varphi(f) = [f_3]$.

Repeating steps 1-4 on $F: V \to W$, a cobordism $f \equiv f'$, shows φ is well-defined. If $k=1$ one may have to do some surgery to avoid the following situation:



Want $F(V)$ to miss an arc in W between $*_N$ and $*_{N'}$. Then argument is same with $W - I$ replacing $N - *$.

2. <u>Definition of ψ</u> Given $f_3 : N \to \Omega^r \Sigma^r MSO(k)$

1) Take adjoint $f_2 : \Sigma^r N \to \Sigma^r MSO(k)$.

2) Make f_2 transverse regular and let $M = f_2^{-1}(BSO(k))$, obtaining $f_1 : M \to (N - *) \times \mathbb{R}^r$ with normal r-frame.

3) Apply Hirsch Thm. to obtain $f_0 : M \subseteq N$; $\psi(f_3) = [f_0]$.

The same argument shows $\psi(f_3)$ is an invariant of the bordism class of f_3, so ψ is well-defined; ψ and φ are clearly inverses. //

It is interesting to compare this argument with that of Wall for $\Omega^{emb}(m,n)$. Given $f : M \subseteq N$ let $\nu(M,N)$ be the normal bundle. $E(\nu) \subset N$ is a tubular neighborhood of M in N. $E(\nu) \to E(\xi_k)$ is the classifying map. Extend to $N \xrightarrow{\varphi f} MO_k = E(\xi_k) \cup \{\infty\}$, which gives the bordism class.

<u>Example</u> $\mathfrak{N}^{imm}(1,2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by $\emptyset \subset P^2$, $P^1 \subset P^2$, and $S^1 \subset S^2$ ($\infty \subset$ ).

<u>IV Work to be done</u> QX is an infinite loop space, $QX = \Omega Q \Sigma X$. There exist extended Steenrod squares Q_i (Sq_i is analog),

$$Q_i : H_k(QX; \mathbb{Z}_2) \longrightarrow H_{2k+i}(QX; \mathbb{Z}_2).$$

Say $Q_{i_1} \cdots Q_{i_r}$ is admissible $\Leftrightarrow 0 \le i_1 \le i_2 \le \cdots \le i_r$. Conjecture: Then $H_*(QX; \mathbb{Z}_2)$ is an exterior algebra ——————— on generators given by applying all admissible monomials to a basis for $\tilde{H}_*(X, \mathbb{Z}_2) \subset H_*(QX, \mathbb{Z}_2)$

Mod p: Dyer, Lashof, <u>Iterated Loop Spaces</u>, Amer. J. 1962

Mod 2: Browder?

MIT Topology Seminar 10/9/67

Postnikov Systems for U & BU — Bill Singer

For a CW complex X define

$X(r,\infty) \xrightarrow{\pi} X$ so that π induces an iso. of homotopy in dimensions $\geq r$ and $X(r,\infty)$ is $(r-1)$-connected

$X \xrightarrow{j} X(1,r)$ s.t. j induces an iso. of homotopy in dim. $1-r$ and $\pi_k(X(1,r))=0$ if $k>r$.

$X(r,s)$ as the obvious.

Given a spectrum $\{Y_q\}$ with homotopy equivalences $Y_q \xrightarrow[h_q]{} \Omega Y_{q+1}$ define $\bar{H}^q(X) = [X, Y_q]$. There is a spectral sequence

$$E_2^{p,q} = H^p(X, \pi_p(Y_q)) \Rightarrow \text{grad } \bar{H}^p(X) \quad \text{(some filtration)}.$$

The differentials are related to Postnikov systems of Y_q. A cohomology class of E_2 corresponds to a map f (below). If f lifts to $\bar{f}$ there is an obstruction $k(q,p,r+2) \in H^{r+2}(Y_q(p,r))$ to lifting once more; $\bar{f}$ lifts $\Leftrightarrow$ $\bar{f}^* k = 0$. Roughly, k is the differential of f. If Y is an H-space the order of any k is finite. Ask: what smallest α has $\alpha \cdot d = 0$ if d is a differential?

To get complex K-theory, $Y_{2q} = BU \times \mathbb{Z}$, $Y_{2q+1} = U$. Use Bott periodicity to get Ω-maps. Compute

$H^*(BU(2n,\cdots,\infty), Z_p)$ $\left\{ \begin{array}{ll} \text{Stable case: Adams 1961} \\ p=2 \quad \text{: Stong 1963} \end{array} \right.$

$H^*(U(2n+1,\infty), Z_p)$

Use spectral sequence of Eilenberg-Moore:

Given $F \longrightarrow E_0 \longrightarrow B_0$ ... fibration, $B \xrightarrow{f \text{ ... map}} B_0$, get induced fibration

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ E & \longrightarrow & E_0 \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B_0 \end{array}$$

If
1) all cohomology of finite type
2) B_0 simply conn.
3) ring is noetherian (I use Z_p = field)

then there is a spectral sequence

$$E_2 = \mathrm{Tor}_{H^*(B_0)}\left[H^*(E_0), H^*(B)\right] \Rightarrow E_\infty = \text{grad. } H^*(E).$$

which
1) is a s. seq. of algebras; each differential a derivation
2) is natural (another fibre square gives maps)
3) if B = pt. gives s. seq. converging to $H^*(F)$.

By induction assume $BU(2n-2,\infty)$ is known. Have fibration $K(Z, 2n-3) \longrightarrow BU(2n,\infty) \longrightarrow BU(2n-2,\infty)$. Regard this as induced from $K(Z,2n-3) \longrightarrow L(Z,2n-2) \longrightarrow K(Z,2n-2)$. To compute first fibration look first at the second fibration:

$$\begin{array}{ccccc} K(Z,2n-3) & = & K(Z,2n-3) & = & K(Z,2n \\ \downarrow & & \downarrow & & \downarrow \\ BU(2n,\infty) & \longrightarrow & L(Z,2n-2) & \longrightarrow & K(Z,2n \\ \downarrow & & \downarrow & & \downarrow \\ BU(2n-2,\infty) & \longrightarrow & K(Z,2n-2) & \longrightarrow & pt. \end{array}$$

See Larry Smith, Ill. Journal June '67, "Cohomology of 2-stage Postnikov systems".

By Cartan,

$$H^*(K(Z,n), Z_p) = \underset{\substack{\uparrow \\ \text{free commutative} \\ \text{algebra on [ ]}}}{F[V_n]} = Z_p\,\underset{\substack{\uparrow \\ \text{even} \\ \text{dim.}}}{[V_n^+]} \otimes E\,\underset{\substack{\uparrow \\ \text{odd} \\ \text{dim.}}}{[V_n^-]}$$

for some vector space V_n. Suspension $\sigma : Q^* H(K(Z,n), Z_p) \longrightarrow P^{*-1} H(K(Z,n-1); Z_p)$. To state kernel define $\underline{\beta^{pt}} : V_n^- \xrightarrow{\cong} \ker \sigma$

$$\underline{\beta^{pt}}\,(P^J\iota_n) = \beta P^t P^J \iota_n \qquad 2t+1 = \deg P^J \iota_n .$$

Then $\sigma(P^t P^J \iota_n) = P^t P^J \iota_{n-1} = (P^J \iota_{n-1})^p$. Cartan shows $\underline{\beta P^t}$ is iso. (V_n = Steenrod monomials operating on ι_n).

$$E_2 = \mathrm{Tor}_{H^*(K(Z,n);Z_p)}[Z_p, Z_p] = \Gamma[sV_n^-] \otimes E[sV_n^+]$$

Ker σ $\subset$ even dimensions hence the spectral seq. must kill $E[sV_n^+]$.

<u>Thm</u> (Smith) All differentials are 0 except

$$d_{p-1}[s(P^J\iota_n)] = s\beta P^t P^J \iota_n. \quad //$$

<u>Thm</u> Given a Hopf fiber square (i.e. E, B homotopy commutative, homotopy associative H-spaces & p, f H-maps)

$$\begin{tikzcd} K(Z,n-1) \arrow[d] \arrow[dr] & \\ E \arrow[r] \arrow[d, "p"] & K(Z,n) \arrow[d] \\ B \arrow[r, "f"'] & K(Z,n) \end{tikzcd}$$

where $H^*(B, Z_p)$ is free commutative as an algebra (not necessarily as a Hopf algebra), there is a subspace $M_n \subset V_n$ such that $\ker f^* = F[M_n]$ and

$$H^*(E, Z_p) = \underbrace{H^*(B, Z_p)}_{\text{im } p^*} \otimes \underbrace{\underset{\uparrow \text{free comm. algebra generated by ...}}{A(\sigma M_n)}}_{\text{im } i^*} \otimes \underbrace{E\left[s\frac{\beta^{p^t} V_n^- \cap M_n^+}{\beta^{p^t} M_n^-}\right]}_{\text{elements of ker } f^* \text{ also in ker(suspension of } K \to L \to K)} \quad //$$

The category of bicommutative Hopf algebras is [redacted] abelian; ker f^* is taken in this category. ($f: B \to A$, ker $f = Z_p \square_A B \subset B$).

Inductively determine im f^* and ker f^*:

$$\text{by periodicity} \left\{ \begin{tikzcd} H^*(BU(2n-2,\infty)) & H^*(K(\mathbb{Z},2n-2)) \arrow[l, "f^*"'] \\ H^*(U(2n-1,\infty)) \arrow[u, "\sigma"] & H^*(K(\mathbb{Z},2n-1)) \arrow[l, "g^*"] \arrow[u] \\ H^*(BU(2n,\infty)) \arrow[u] & H^*(K(\mathbb{Z},2n)) \arrow[l, "f^*"] \arrow[u] \end{tikzcd} \right.$$

subalg. generated by

Ker $f^* = \tilde{a}[\beta P^1 \iota_{2n}]$, Ker $g^* = \tilde{a}[\beta P^1 \iota_{2n-1}]$

Answers: there are indecomposable cohomology classes $\theta_{2i} \in H^{2i}(BU, Z_p)$
$\mu_{2i+1} \in H^{2i+1}(U, Z_p)$
such that $H^*(BU(2n,\infty), Z_p) = Z_p[\theta_{2i} \mid \sigma_p(i-1) < n] \otimes \prod_{t=0}^{p-2} F[M_{2n-2t}]$
as Hopf algebras where for primes p,

$$\sigma_p(m) = \sum \text{ coefficients in } p\text{-ary expansion of } m.$$

Define Z_p-modules M_n such that $F[M_n] \simeq \tilde{a}(\beta P^1 \iota_n)$ (subalgebra of cohomology of Eilenberg-MacLane Space). This Z_p result determines Z case since the Bockstein spectral sequence is known. Let $d_{n,r}$ be differentials in Atiyah-Hirzebruch spectral seq. above. (Bott periodicity takes out 1 index). Then $\alpha_{n,r}\, d_{n,r} = 0$ where

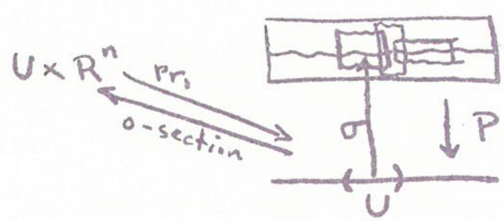
$$\alpha_{n,r} = \frac{(r-1)!}{\prod_p p^{\operatorname{lig}\left[\frac{(n-1)-\sigma_p(r-1)}{p-1}\right]}}$$

where $\operatorname{lig} m$ = least integer $\geq m$ and 0.

MIT Topology Seminar 10/16/67

Smooth	PWL	Topological	
diffeom.	PWL homeom.	homeom.	manifold maps
vector space autom.	PWL homeom. 0 fixed	homeom., 0 fixed	R^n-bundle group
easy	$t_M \to \tau_M$	$t_M \to \tau_M$	tangent bdl
yes	if $k \geq 3$	must be assumed	are imbeddings locally flat?
yes	$n<k$	?	normal bundle
	$\Uparrow$		
yes	$n<k$ ~~in general~~	in ~~general~~	tubular nbd.

Microbundle :



Require: $U\times R^n$ PWL homeom or homeom.

For example the tangent microbundle to M is $M\times M \underset{\Delta}{\overset{pr_1}{\rightleftarrows}} M$, t_M.

Thm (Kister-Mazur) Every microbundle has an equivalent representative which is (R^n) bundle.

Df Given $f: M^n \to W^{n+k}$, f is locally flat if for each $x\in M$ there is (U,h) such that $h: (R^{n+k}, R^n) \to (U, U\cap f(M))$ is homeo.

Df ν is a normal bundle for f if $\tau_M \oplus \nu_f \simeq f^*(\tau_N)$.

Df ν is a tubular nbd. if
$$\begin{array}{ccc} E_\nu & \subset & N \\ {\scriptstyle \sigma_0}\uparrow & \nearrow {\scriptstyle f} & \\ M & & \end{array}$$
is commutative.

and E_ν is an open nbd. of $f(M)$.

Prop. Tubular nbd. $\Rightarrow$ normal bundle.

Thm (Browder 1963) there is a (top.) or PWL tubular nbd containing no disk bundle.

Thm (Hirsch 1965) PWL: there is a $\Sigma^8 \subset W^{12}$ with no disk tubular nbd.

Thm (Haefliger & Wall) PWL: $n < k \Rightarrow$ there is a tubular nbd. Hence multiplying by R^n, it is always possible to get a stable normal bundle.

Thm (Sanderson - Rourke). There is a $\Sigma^{19} \times I \longrightarrow \Sigma^{29}$ with no topological tubular nbd.

Nash - Fadell - Spivak Define $NSP(M) \xrightarrow{ev_0} M$ to be non-singular paths in M, ev_0 = evaluation at 0 (non-singular: never returns to starting point). Then $ev_0^{-1}(p)$ is homotopy equivalent to S^{n-1}. For normal bundle of a locally flat imbedding use paths not returning to $f(M)$. Fibre has homotopy type of S^{k-1}.

To simplify, look at locally flat imbeddings. If $\nu_f \oplus \tau_M = f^*\tau_M$, there is also a pair $(\nu_f \oplus \tau_M, \tau_M) \simeq (f^*\tau_N, \tau_M)$. To get normal bundle, get representative up to fiber homotopy type by

$E_{\tilde{\nu}_f} = E_{f^*(\tau_N)} \setminus E_{\tau_M}$. ($\leftarrow$ remove 0-section).

The fibre has homotopy type of S^k.

Can extend the Kister-Mazur theorem to (R^{n+k}, R^n)-bundles.

Use
$$\begin{tikzcd} M \times N \arrow[d, shift left, "pr_1"] \\ M \arrow[u, shift left, "(1,f)"] \end{tikzcd}$$
as microbundle $= (f^* t_N, t_M)$.

This yields $(f^* \tau_N, \tau_M)$, a bundle. Group:
$$\begin{tikzcd} \text{homeom}_0 (R^{n+k}, R^n) \arrow[d, hook] \\ \text{homeom} (R^{n+k} \setminus R^n) \end{tikzcd}$$

__Thm__ $f^* \tau_N$ is fiber homotopy equivalent to $\tau_M \oplus \nu_f$ as sphere bundles (not equivalent as bundles; remove 0-section).

This gives characteristic classes since for sphere bundles one has thom thm. and Wu classes.

__Df__ locally flat imbeddings f_0, f_1 are concordant if there is a proper locally flat embedding $F: M \times I \to N \times I$ which is f_0 on $M \times 0$, f_1 on $M \times I$.

__Thm__ If f_0 and f_1 are concordant, $\nu_{f_0} \underset{\text{fhe}}{\sim} \nu_{f_1}$.

__Df__ An isotopy is a homeom.
$$\begin{tikzcd} M \times I \arrow[r, "F"] \arrow[d, "pr_2"] & N \times I \arrow[d, "pr_2"] \\ I \arrow[r, equal] & I \end{tikzcd}$$

A C-isotopy is an isotopy which is a concordance (loc. flat).

__Df__ A locally flat isotopy: require loc. flat with respect to I also.

__Thm__ If f_0, f_1 are locally flat isotopic, $(f_0^* \tau_N, \tau_M) \simeq (f_1^* \tau_N, \tau_M)$.

__Df__ f_0, f_1 are ambient isotopic if there is an isotopy $F: N\times I \to N\times I$ such that $f(\ ,0) = \mathbb{1}$ and $F(\ ,1)\circ f_0 = f_1$.

__Thm__ (Hudson, Zieman) PWL: isotopy $\Rightarrow$ ambient isotopy, $k \geq 3$.

__topological case__ conjecture 1) iso & $k \geq 3 \Rightarrow$ amb. iso

2) loc. flat iso & $k \geq 3 \Rightarrow$ amb iso.

__Thm__ iso. & $n+k > \frac{3}{2}(n+1)$, $n \geq 7$ $\Rightarrow$ amb. iso.

A conjectured proof of 1): f_0 iso $f_1 \xrightarrow{(top.)}$ PWL concordance (?)

$\xrightarrow{Hudson}$ PWL isotopy

$\xrightarrow[k\geq 3]{(H,Z)}$ ambient PWL iso.

The speaker gave a counterexample to this proof.

MIT Topology Seminar 10/23/67

Stable Secondary Cohomology Operations - John Harper

I. A Primary Cohomology operation is a function $f: H^n(X,G) \to H^m(X,G')$ which is a natural transformation of functors.

Example Steenrod operations: $G = G' = \mathbb{Z}_2$. satisfying

1) $Sq^i : H^n(X) \longrightarrow H^{n+i}(X)$ a homomorphism of groups
2) Sq^0 = identity, $Sq^n x = x \cup x$ if $\dim x = n$.
3) $Sq^i x = 0$ if $i > \dim x$
4) Let $s: H^n(X) \to H^{n+1}(SX)$ be the suspension isomorphism.

Then $s\, Sq^i = Sq^i s$ (stability)

5) $Sq^i(x \cup y) = \sum_{j+k=i} Sq^j x \cup Sq^k y$.

6) A = algebra generated by the Sq^i under composition. (the Steenrod algebra).

a) a vector space basis for A is all monomials

$Sq^{i_1} Sq^{i_2} \cdots Sq^{i_k}$ with $i_k \geq 1$, $i_j \geq 2 \cdot i_{j+1}$.

Such a monomial is called admissible.

b) for $a < 2b$, $Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$

7) If $\dim x = 1$ $Sq^i x^k = \binom{k}{i} x^{k+i}$

If $\dim x = 2$ and $Sq^1 x = 0$,

$Sq^{2i} x^k = \binom{k}{i} x^{k+i}$, $Sq^{2i+1} x^k = 0$.

8) Sq^i can be extended to operations on $H^*(X,Y)$ commuting with the coboundary δ.

II Operations and Eilenberg-MacLane (EM) spaces.

If π is a finitely generated abelian group and $n \geq 1$ there is a space $K(\pi,n)$ unique up to homotopy type, with homotopy groups all 0 except $\pi_n(K(\pi,n)) = \pi$.

For any space X, $H^n(X,\pi)$ corresponds 1-1 with homotopy classes of maps $X \longrightarrow K(\pi,n)$: we write

$$H^n(X,\pi) \longleftrightarrow [X \to K(\pi,n)]$$

where $f \longleftrightarrow f^*(\iota_n)$ and ι_n is the cohomology class corresponding to the identity map $\pi \longrightarrow \pi$ under the isomorphism

$$H^n(X,\pi) \simeq \mathrm{Hom}(H_n(X,\pi), \pi) = \mathrm{Hom}(\pi,\pi).$$

Think of $H^*(K(\pi,n),\pi)$ as primary cohomology operations acting on ι_n.

When $\pi = \mathbb{Z}_2$,

<u>Thm</u> (Cartan-Serre, etc.) $H^*(\mathbb{Z}_2,n;\mathbb{Z}_2) = \mathbb{Z}_2[\{Sq^I\iota_n\}]$ for those $I = (i_1,\cdots,i_k)$ which are admissible and $2i_1 - \deg I < n$. (the most generators it can have subject to 2) & 3).)

The suspension $\sigma: H^r(X) \longrightarrow H^{r-1}(\Omega X)$ is given by the [redacted] fibration $\Omega X \longrightarrow PX \longrightarrow X$

$$\begin{tikzcd}
H^{r-1}(\Omega X) \arrow[rr, "\simeq", "\text{because } PX \text{ contractible}"'] & & H^r(PX,\Omega X) \\
 & & H^r(X,*) \arrow[ull, "\sigma"] \arrow[u, "p^*"']
\end{tikzcd}$$

Now $\Omega K(\pi,n) = K(\pi,n-1)$ so $\mathrm{Im}\,\sigma$ in $H^*(\pi,n-1;\pi)$ is called the <u>stable</u> operations. One shows by adjointness of S and Ω that stable in this sense corresponds to stable in 4). (See Adams 1960, Hopf Invariant 1.)

<u>III Secondary operations and relations in A.</u>

Suppose $\sum a_i b_i = 0$ where $a_i, b_i \in A$, $\deg b_i = n_i$ and $\deg a_i + n_i = k$ for all i. There is a cohomology operation Φ defined on all x of dimension n with $b_i(x) = 0$ for all i. Its value is a coset in $\dfrac{H^{n+k-1}(X)}{\sum a_i H^{n+k-1-\deg a_i}}$. (the denominator is known

as the indeterminacy of Φ). Φ is not a function but it is natural. To construct Φ let V be a $\mathbb{Z}_2$-vector space of dimension m, $\boxed{\cancel{n=(n_1,n_2,\cdots,n_m)}}$ $\mathcal{n} = (n+n_1, n+n_2, \cdots, n+n_m)$ where $n \gg k$.
For any $J=(j_1,\cdots,j_s)$ let $K(\pi, J) = K(\pi, j_1) \times \cdots \times K(\pi, j_s)$.
Let $I = (\underbrace{1,1,\cdots,1}_{m})$. Let $B_n = K(\mathbb{Z}_2, n)$ and $\pi_i : K(V,\mathcal{n}) \to K(V, n+n_i)$ the projection on the i^{th} factor. Choose a map $f: B_n \to K(V,\mathcal{n})$ such that $(\pi_i f)^* \iota_{n+n_i} = b_i \iota_n$. One has a fibration as in the diagram, and f induces a fibration over B_n. If $g: X \to B_n$ has $g^* \iota_n = 0$, then g lifts to $\tilde{g}$. By Serre, there is an exact sequence

$$\begin{tikzcd}
 & & K(V,\mathcal{n}-I)=F_n \arrow[dll] \arrow[d] \\
 & E_n \arrow[r] \arrow[d] & PK(V,\mathcal{n}) \arrow[d] \\
X \arrow[r, "g"'] \arrow[ur, dashed, "\tilde{g}"] & B_n \arrow[r, "f"] & K(V,\mathcal{n})
\end{tikzcd}$$

$$H^q(B_n) \xrightarrow{p^*} H^q(E_n) \xrightarrow{i^*} H^q(F_n) \xrightarrow[\tau]{} H^{q+1}(B_n)$$

trans-
gression

for $q <$ connectivity $B_n +$ connectivity $F_n - 2$ (roughly). If n is large and $q = k-1$. Then $\tau(\iota_{n+n_i-1}) = b_i(\iota_{n_i})$, it turns out. Since τ commutes with A,

$$\tau(\Sigma a_i \iota_n) = \Sigma a_i b_i \iota_n = 0$$

thus there is an element e of $H^{k-1}(E_n)$ which hits $\Sigma a_i \iota_{n_i-1}$.

1) There exists $e_n \in H^q(E_n)$ such that $i^*(e_n) = \Sigma a_i \iota_{n+n_i-1}$.
There is no unique lifting $\tilde{g}$, $(g^* e_n \neq \Phi(x))$ but different liftings differ by an element of $\Sigma a_i H^{\cdots}(E_n)$. One defines $\Phi(x) = g^*(e_n)$. Now Φ corresponds to a coset in $\dfrac{H^{k-1}(E_n)}{\Sigma a_i H^{\cdots}(E_n)}$. Apply Ω to the diagram above; find that Φ is stable and $E_n = \Omega E_{n+1}$.

2) $H^*(E_n)$ is a Hopf algebra and e_n is primitive. Let $x_n = p^* \iota_n$: $(E_n, x_n, e_n) = (\Omega E_{n+1}, \sigma x_{n+1}, \sigma e_{n+1})$.

The argument depends on $q \ll n$; this can be dispensed with by using the suspension many times. Frequently $H^*(E_n)$ splits as an [redacted] algebra into a tensor product - but not as a Hopf algebra.

Stable secondary cohomology operations correspond 1-1 with relations in A, via this procedure.

The operations Adams considered were associated to relations in A_n = subalgebra generated by $1, Sq^1, Sq^2, \cdots, Sq^{2^n}$. I have studied operations associated to relations in the ideal $B(n) = \{ Sq^I \mid 2i_1 - \deg I > n \}$, a left A-module.

IV Homological algebra and relations.

To find generating relations (the minimal number) one calculates Ext_A and Tor^A. For an A-module M choose a resolution $0 \leftarrow M \xleftarrow{\varepsilon} A \otimes V_0 \leftarrow A \otimes V_1 \leftarrow \cdots$ as follows: $M/A \cdot M$ is called the "indecomposables" of M. Let V_0 be a graded vector space with generators corresponding to $M/A \cdot M$. Let $L = \ker \varepsilon$; $\frac{L}{\bar{A} \cdot L}$ yields by the same process V_1, etc. The basic relations sit in $L/\bar{A} \cdot L$. One Gets a free resolution of the A-module M. To compute $\mathrm{Ext}_A(M,N)$ for a left A-module N look at

$$\mathrm{Hom}(M,N) \to \mathrm{Hom}_A(A \otimes V_0, N) \to \mathrm{Hom}_A(A \otimes V_1, N) \to \cdots$$

This complex is no longer acyclic. Put $\mathrm{Ext}_A^{s,t}(M,N) = H^s(\mathrm{Hom}_A^t(A \otimes V_*, N))$. The main problem is to compute Ext and Tor.

I have computed $\mathrm{Ext}_A^{0,t}(B(n), Z_2)$ which detects generators and $\mathrm{Ext}_A^{1,t}(B(n), Z_2)$ which gives basic relations.

(Any relation $\sum a_i b_i$ in [redacted] A gives Φ on x if [redacted] $b_i x = 0$. If $b_i \in B(n)$ and $\dim x = n$ then $b_i x = 0$.)

MIT Topology Seminar 10/30/67

Armand Wyler - On a generalization of the Hopf inverse homomorphism.

See Hopf, 1962: On singularities of continuous maps of manifolds.

Conjecture: If $f: X^m \to Y^n$ is algebraicly essential (a.es.) then $f^{-1}(y)$ has dimension $\geq m-n$ (any $y \in Y$).

Define f to be algebraicly essential if $f_*: H_n(X^m, \mathbb{Q}) \twoheadrightarrow H_n(Y^n, \mathbb{Q})$

I use Dold's formulation of Alexander duality (See Husemoller, Fiber Bundles):

Thm If $f: X^m \to Y^n$ is a.es. then for any $A \subset Y^n$ (compact, connected), $f_*: H_i(X^m, X^m - f^{-1}(A)) \twoheadrightarrow H_i(Y^n, Y^n - A)$ for all i.

Let h denote Čech cohomology.

Cor 1

$$\begin{tikzcd} h^{i+m-n}(f^{-1}(A)) \arrow[r, "\psi"] \arrow[d, "D", "\cong"'] & h^i(A) \\ H_{n-i}(X, X-f^{-1}A) \arrow[r] & H_{n-i}(Y, Y-A) \arrow[u, "\cong", "D^{-1}"'] \end{tikzcd}$$

defines ψ by commutativity of the diagram. (Take $i=0$; then $\dim f^{-1}(A) \geq m-n$ if $\psi \neq 0$).

Cor 2 (Hopf) $f_*: H_i(X^m) \twoheadrightarrow H_i(Y^n)$ (coef. $\mathbb{Q}$) (Proved 1930, Zur Algebra der Abbildungen von Mannigf.).

Cor 3 If $f: S^3 \to S^2$ is not homotopic to 0,

$$h^1(f^{-1}(y)) \neq 0.$$

Proof If $f \not\simeq 0$, f factors thru the Hopf map α:

$$\begin{tikzcd} S^3 \arrow[r, "g"] \arrow[dr, "f"'] & S^3 \arrow[d, "\alpha"] \\ & S^2 \end{tikzcd}$$

Apply the theorem to g.

<u>Proof of Cor 1</u> Let $Y^n \supset A$ – compact, connected, V an open neighborhood of A. There is a cap product

$$\cap : H^i(V) \otimes H_n(V, V-A) \longrightarrow H_{n-i}(V, V-A).$$

By excision $H_i(Y, Y-A) \simeq H_i(Y-(Y-V), Y-A-(Y-V))$

$$\simeq H_i(V, V-A)$$

hence $\quad \cap : H^i(V) \otimes H_n(Y, Y-A) \longrightarrow H_{n-i}(Y, Y-A).$

For an oriented manifold Y, $H^n(Y^n, Y^n-A; \mathbb{Z}) \simeq \mathbb{Z}$.
(all manifolds compact without boundary). If Θ_A is the orientation class of Y restricted to A,

$$\begin{tikzcd}
\cap\Theta_A : H^i(V) \arrow[r] \arrow[d] & H_{n-i}(Y^n, Y^n-A) \\
\cap\Theta_A : H^i(V') \arrow[ur] &
\end{tikzcd}$$

This is a commutative diagram if $V \supset V' \supset A$. So

$$\begin{tikzcd}
\varinjlim H^i(V) \arrow[r, "\cap\Theta_A"'] \arrow[d, equal] & H_n(Y, Y-A) \\
\check{h}^i(A) \arrow[ur, "\cap\Theta_A"'] &
\end{tikzcd}$$

This defines D. One then proves it is an isomorphism in steps:

1) A = a point
2) $Y = \mathbb{R}^n$, A = cube
3) $Y = \mathbb{R}^n$, A = finite union of cubes in a lattice
   (induction on number of cubes; Mayer-Victoris.)
4) $Y = \mathbb{R}^n$, A any compact connected set.
   <u>Approximate</u> A by bolygonal neighborhoods.

<u>Proof of Dold's thm.</u> let $\eta_{n-i} \in H_{n-i}(Y, Y-A)$. Choose $V \supset A$ and $\eta^i \in H^i(V)$ such that $\eta_{n-i} = \eta^i \cap \Theta_A$. By diagram on the next page, there is an ξ_n such that $\Theta_A = f_*(\xi_n)$. Hence $\eta^i \cap \Theta_A = \eta^i \cap f_* \xi_n = f_*(f^*(\eta^i) \cap \xi_n) = \eta_{n-i}$.

$$\begin{tikzcd} H_n(X^m) \arrow[r] \arrow[d] & H_n(X, X - f^{-1}A) \arrow[d, "\therefore \text{ onto}"] \\ H_n(Y^n) \arrow[r, "\approx"] & H_n(Y^n, Y^n - A) \end{tikzcd}$$

Let $f: S^4 \to S^3$; if $[f] \neq 0$, $f \sim \Sigma\alpha$.

Hopf: f has at most 2 injective points; at all others $\check{H}^1(f^{-1}y) \neq 0$.

For: if δ_1, δ_2 injective, smooth f out onto a suspension in a neighborhood N of δ_1, δ_2. On bd$N$ use $\Sigma\alpha: S^4 \to S^3$.

Thm $X^m \xrightarrow{f} Y^n$ continuous, X & Y topological manifolds with $\delta \in X$ an injective point. Then there exist neighborhoods $\mathcal{U}_1 \supset \mathcal{U}_2$ of δ and $g: X^m \to Y^n$ such that $g|X^m - \mathcal{U}_1 = f|X^m - \mathcal{U}_1$, $g \sim f$, and $g|\mathcal{U}_2$ is the suspension of a map $\Sigma^{m-1} \to \Sigma^{n-1}$ (g gives $\alpha \in \pi_{m-1}(S^{n-1})$). ($f \neq 0 \not\Rightarrow \alpha \neq 0$).

Freudenthal (1935) proved $\pi_{m-1}(S^{n-1}) \twoheadrightarrow \pi_m(S^n)$

if $m < 2n-2$.

This gives a necessary condition for existence of an injective point:

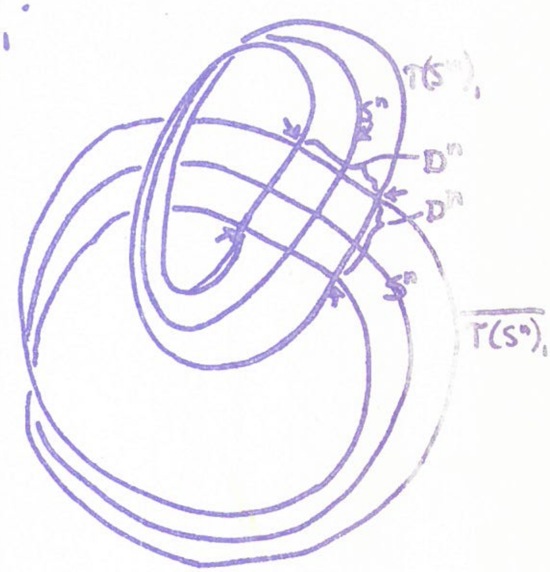
$$\left.\begin{array}{c} S^m \xrightarrow[f]{} S^n \\ \text{a.e.s.} \end{array}\right\} \Rightarrow \pi_{m-1}(S^{n-1}) \neq 0.$$

MIT Topology Seminar 11/6/67

Edgar Brown (Brandeis): Recent work of William Browder on the Arf invariant.

This material will be published by Browder.

Let n be an odd integer, not 1, 3, 7. Let $\tau(S^n)_1$ stand for the disk bundle of the tangent bundle to S^n. Glue two copies together: let $L = \tau(S^n)_1 \cup \overline{\tau(S^n)_1}$, as in the picture. Smooth out the corners (ɤ).



One knows ∂L is homeomorphic to S^{2n-1}.

Conjecture ∂L is not diffeomorphic to S^{2n-1}.

This was proven for $n=5$ by Kervaire; for all $n \equiv 1 \bmod 4$ by Peterson and Brown.

Thm (W. Browder) The conjecture is true for $n \neq 2^i - 1$, false for $n = 15$, and false for $n = 2^i - 1$ iff h_i^2 in Adams spectral sequence persists to E^∞.

This talk will concentrate on the proof of the first statement: true for $n \neq 2^i - 1$. It is a greatly edited version of Browder's first 20 pages.

There is an element $v_\ell \in H^\ell(BO)$ such that for the classifying map of the normal bundle of any manifold M^m, $\nu_M : M \longrightarrow BO_k$ (large k), $Sq^\ell : H^{m-\ell}(M^m) \longrightarrow H^m(M^m)$ is given by $Sq^\ell u = u \smile (\nu_M^* v_\ell)$. We say M admits v-structure if, in the following diagram, ν_M has a

(5-2)

$$\begin{tikzcd}
\nu_M \arrow[ddd] \arrow[rr, dashed, "\alpha"] & & \bar{\gamma} \arrow[dl] \arrow[dd] & \\
 & B_{(n)} \arrow[dd] \arrow[rr, "w_{n+1}^{*}E"] & & E = \text{contractible bundle over} \arrow[dd] \\
 & & \gamma = \text{canonical } k\text{-plane bundle} \arrow[dl] & \\
M \arrow[uur, dashed] \arrow[r, "\nu_M"'] & BO_k \arrow[rr, "w_{n+1}"'] & & K(\mathbb{Z}_2, n+1)
\end{tikzcd}$$

lifting to $B_{(n)}$. One gets a cobordism theory of such (M, α) denoted $\Omega_m^{B(n)}$.

There is an obvious map $\Omega_m^{\text{Framed}} \longrightarrow \Omega_m^{B(n)}$ (the classifying map of a framed manifold is constant).

<u>Lemma</u> Suppose ∂L is diffeo. to S^{2n-1}. Then $K = L \cup D^{2n}$ is a smooth manifold. K is stably parallelizable and if $u \in H_n(K)$ is any class it is represented by an imbedded sphere $i: S^n \subset K$. Further, the normal bundle of such i is $\nu(S^n, K) \simeq T(S^n)$. In fact $H_n(K) = \mathbb{Z}_2 + \mathbb{Z}_2$ ($\mathbb{Z}_2$ coefficients). //

Proof of the theorem of Browder is in 2 parts.

<u>Prop. 1</u> Suppose ∂L is diffeo. to S^{2n-1}. $\{K\} \in \Omega_{2n}^{\text{Framed}}$ by lemma 1. Then $\eta(\{K\}) \neq 0$ where $\eta: \Omega_{2n}^{\text{Framed}} \to \Omega_{2n}^{B(n)}$.

<u>Prop. 2</u> $\eta: \Omega_{2n}^{\text{Framed}} \longrightarrow \Omega_{2n}^{B(n)}$ is 0 for $n \neq 2^i - 1$.

<u>Proof of Prop. 2</u> η is described as follows:

$$\begin{tikzcd}
\Omega_{2n}^{\text{Framed}} \arrow[r, "\eta"] \arrow[d, phantom, "\wr\wr"] & \Omega_{2n}^{B(n)} \arrow[d, phantom, "\wr\wr"] \\
\pi_{2n+k}(S^k) \arrow[r, "\substack{\text{inclusion} \\ \text{of fiber}}"'] & \pi_{2n+k}(T(\bar{\gamma}))
\end{tikzcd}$$

To show: $T(\bar{\gamma})$ through dimension $2n+2+k$ has a 3-stage Postnikov system. This is not hard; it uses secondary cohomology operations as described by Adams.

Now we give a proof of <u>prop. 1</u>.

Let $(M,\alpha) \in \Omega_{2n}^{B(n)}$. We define an operation φ as follows: there exists a closed C^∞-manifold N of large dimension with a $B_{(n)}$ - structure such that $N \longrightarrow B_{(n)}$ is a homotopy equivalence in a certain range of dimensions. (To construct N take a CW-complex reproducing $B(n)$'s homotopy groups up to a certain dimension. Imbed it in $\mathbb{R}^p$ for large p. Take the induced disk bundle from $\bar{\gamma}$ under

$$\left\{\begin{array}{ccc} \text{bundle} & \longrightarrow & \bar{\gamma} \\ \downarrow & & \downarrow \\ \text{Skeleton CW-cx.} & \longrightarrow & B_{(n)} \end{array}\right.$$

; and glue 2 copies of it together).

Thus we can now assume $B_{(n)}$ is a large dimensional closed manifold. Notice that in it, $v_{n+1} = 0$. For (M,α) let $f_\alpha : M \longrightarrow B_{(n)}$. Define $\varphi : \mathrm{Ker}\left((f_\alpha)_* : H_n(M) \longrightarrow H_n(B_{(n)})\right) \longrightarrow \mathbb{Z}_2$ and prove a theorem about it: $\varphi(x+y) = \varphi(x) + \varphi(y) + x \wedge y$. where φ is defined: let ~~f~~ be chosen to be a C^∞ embedding. By the Thom construction, (collapsing everything in $B(n)$ outside a tubular neighborhood to a point) we get a map $B_{(n)} \longrightarrow S^\ell(M)$, the ℓ-fold suspension of M, where $\ell + 2n = \dim B(n)$. Now the normal bundle of M in $B(n)$, $\nu(M, B(n))$, is trivial (for $\alpha : \nu_M \to \bar{\gamma}$ induces γ_M from $\bar{\gamma}$ hence $f_\alpha^{-1}\tau B_{(n)} = \tau M$).

Let $u \in H_n(M)$ with $(f_\alpha)_* u = 0$. Let $v \in H^n(M)$. The map $B(n) \xrightarrow{g} S^\ell(M)$ sends $S^\ell(v) \in H^{n+\ell}(S^\ell(M)) \longrightarrow f_\alpha^*(\text{dual (w.r.t. } B(n)) \text{ of } v)$ by standard properties of cap and cup products. Hence $S^\ell(u) \circ g = 0$.

$$B_{(n)} \xrightarrow{g} S^\ell(M) \xrightarrow{S^\ell(u)} S^\ell(K(\mathbb{Z}_2, n)).$$

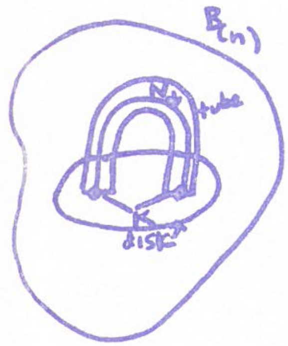
Define $\varphi(u) = \left[Sq^{n+1}_{(Sq^\ell u \circ g)}(S^\ell \iota_n)\right] \frown (B(n)) \in \mathbb{Z}_2$ (functional cohomology operation). There is no indeterminacy; indeed, the indeterminacy is $Sq^{n+1} H^{\ell+n-1}(B(n)) + (S^\ell(u) \circ f)^* H^{\ell+2n}(S^\ell K(\mathbb{Z}_2, n))$, and the first term vanishes since $v_{n+1} = 0$ in $B(n)$ while the second vanishes because every element of $H^{\ell+2n}(K(Z_2,n))$ is a Steenrod operation acting on ι_n; since $(S^\ell(u)\circ g)(\iota_n) = 0$, ~~and~~ Steenrod operations commute with $S^\ell(u)\circ g$, the whole image is 0.

Now we show $\eta(K) \neq 0$, by contradiction. Assume $K = \partial N$ where N has a ν-structure. Then the classifying map for K can be constant (or contained in a disk if an embedding is desired.)

$$\begin{tikzcd} K \arrow[r, "\text{const.}"] \arrow[d, phantom, "\cap"] & B_{(n)} \\ N \arrow[ur] & \end{tikzcd}$$

$$H^n(N) \longrightarrow H^n(K) \xrightarrow{\delta} H^{n+1}(N,K)$$
$$\nu \longrightarrow u_1 \ ; \ u_2$$

By a standard Poincaré duality argument, choose a basis u_1, u_2 of $H^n(K)$ so that $\nu \longrightarrow u_1$. We want to show $\varphi(S^\ell \nu)$ is both 0 and non-0. We use that u_1 is dual to a homology class with non-trivial embedded normal bundle to show $\varphi S^\ell(u_1) \neq 0$.



Assume $K \longrightarrow B_{(n)}$ is an embedding, contained in a disk. We have a trivialization of a tubular neighborhood of N. By the Thom construction we get a map $B \to S^{\ell-1}(N/K)$. Now collapse the tube: $S^{\ell-1}(N/K) \longrightarrow S^\ell(K)$ which is the $(\ell-1)$-fold suspension of $N/K \to S(K)$.

$$B_{(n)} \longrightarrow S^{\ell-1}(N/K) \longrightarrow S^\ell(K) \xrightarrow{S^\ell(i)} S^\ell(N) \longrightarrow S^\ell(K(Z_2,n)).$$
$$S^\ell u_1 \longleftarrow S^\ell \nu \longleftarrow S^\ell \iota_n$$

Now $S^{\ell-1}(N/K) \longrightarrow S^\ell(K)$ is trivial, hence the map you use to apply the functional cohomology operation is trivial: so $\varphi(S^\ell u_1) = 0$. We have used the fact that the class coming from a boundary gives 0 under φ. Next use the fact that K is the Kervaire manifold.

Now u_1 is dual to $i : S^n \longrightarrow K$; $\nu(S^n, K) \cong \tau(S^n)$.

$$B_{(n)} \longrightarrow S^\ell(K) \longrightarrow S^\ell T(\nu(S^n,K)) = S^\ell(T(\tau(S^n))) \xrightarrow{S^\ell U} S^\ell K(\mathbb{Z}_2,n)$$
$$S^\ell(u_1) \longleftarrow S^\ell U$$

If one takes any homology class, realizes it as a manifold & does the Thom construction, then the Thom class goes to the dual of the homology class: $S^\ell U \longrightarrow S^\ell u_1$.

Notice K lies within a disk. Factor $B_{(n)} \longrightarrow S^\ell(K)$:

$$\begin{tikzcd}
B_{(n)} \arrow[rr, "③"] \arrow[dr, "①"'] & & S^\ell(K) \arrow[r, "④"'] & S^\ell T(\nu(S^n,K)) \arrow[r, equal, "⑤"'] & S^\ell(T[\tau(S^n)]) \arrow[r, "S^\ell U", "⑥"'] & S^\ell(K(\mathbb{Z}_2,n)) \\
& S^{\ell+2n} \arrow[ur, "②"'] & & & &
\end{tikzcd}$$

Now all the numbered maps induce isomorphisms on $H^{\ell+2n}$. Hence computing the functional operation of $\bullet\xrightarrow{\approx}\bullet$ is the same as computing it for $\nearrow \approx$.

<u>Lemma</u> Given $S^{\ell+2n} \xrightarrow{g} S^\ell T[\tau(S^n)]$ with g an iso. on $H^{\ell+2n}$, and $S^\ell T[\tau(S^n)] \xrightarrow{S^\ell U} S^\ell(K(\mathbb{Z}_2,n))$, then

$$Sq^{n+1}_{(S^\ell U \circ g)}(S^\ell \iota_n) \neq 0$$

<u>Proof</u> Everything in sight desuspends to $\ell = 1$. Then, $T(\tau(S^n)) = S^n \cup_{[\iota_n,\iota_n]} e^{2n} \Rightarrow ST(\tau(S^n)) = S^{n+1} \vee S^{2n+1}$. Hence assume $\ell = 1$.

<u>Case 1</u> $S^n \times S^n \longrightarrow S^n \cup_{[\iota_n,\iota_n]} e^{2n} \xrightarrow{U} K(\mathbb{Z}_2,n)$

Identifies $(x,*)$ with $(*,x)$

U = thom class of $S^n\cup e^{2n}$ as a thom space

Perform Hopf construction: Steenrod & Epstein ch. I.

$$\begin{tikzcd}
S^{2n+1} \arrow[r, "g_0"] \arrow[rr, bend left, "h"] & S(S^n \cup e^{2n}) \arrow[r] & S(K(\mathbb{Z}_2,n))
\end{tikzcd}$$

It is known g_0 is $\overset{\text{iso}}{\cong}$ in top dim. ~~C~~ohomology. This is an example of a g. One shows h gives a non-0: $Sq^{n+1}_{(h)}(S\iota_n) \neq 0$. Now $S(S^n \cup e^{2n}) = S^{n+1} \cup S^{2n+1}$

Looking at $\pi_{2n+1}(S^{n+1} \vee S^{2n+1})$, an arbitrary g is given by $g_0 + k$ for $k: S^{2n+1} \to S^{n+1}$. The functional operation is additive with respect to homotopy-group addition of maps; hence

$$Sq^{n+1}_{f(g)} = Sq^{n+1}_{f(g_0)} + Sq^{n+1}_{f(k)}$$

But you can't detect a $k: S^{2n+1} \to S^{n+1}$ by a functional operation unless $n = 1, 3, 7$.

MIT Topology Seminar 11/10/67

Tony Armstrong : Piecewise-Linear Transversality

This subject was begun by E.C. Zeeman. One tries to move embedded submanifolds so they are transversal. Q = ambient manifold. I describe here only the results for closed manifolds.

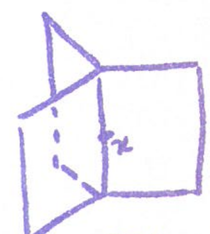
Dfn $M^m, N^n \subset Q^q$ are transversal at $z \in M \cap N$ if there is an embedding $e: D^{q-n} \times D^{q-m} \times D^{m+n-q} \longrightarrow Q$ onto a nbd. of z such that

$$e^{-1} M = D^{q-n} \times 0 \times D^{m+n-q}$$

$$e^{-1} N = 0 \times D^{q-m} \times D^{m+n-q}.$$

Let X be a polyhedron, $x \in X$. X has a natural local product structure at x. Define the intrinsic dimension $I(X, x)$ of X at x to be the largest integer t such that there is an embedding $D^t \times V \longrightarrow X$ onto a nbd. of x, where V is a cone with vertex v and $0 \times v \longrightarrow x$. (See picture).

It turns out V is well-defined up to PWL homeomorphism.



$I(X, x) = 1$

V = cone on 3 points

Dfn Let X, Y be polyhedra $\subset Q$, $z \in X \cap Y$. Assume $I(X, z) = s$, $I(Y, z) = t$. X and Y are transversal at z if there is an embedding

$$e: D^{q-t} \times D^{q-s} \times D^{s+t-q} \longrightarrow Q$$

onto a neighborhood of z, such that

$$e^{-1} X = D^{q-t} \times V \times D^{s+t-q}$$

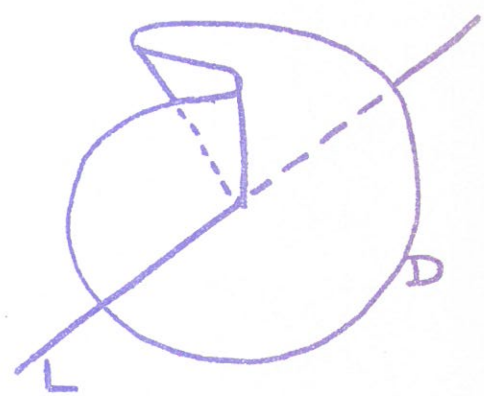
$$e^{-1} Y = W \times D^{q-s} \times D^{s+t-q}$$

where V = subcone of D^{q-s} = join of 0 to sum subpolyhedron of ∂D^{q-s}, W = " " D^{q-t}.

Thm $Y, X \subset Q$ and both of codimension ≥ 3 in Q. Then X can be moved transversal to Y by an arbitrarily small ambient isotopy of Q.
(No proper definition of local flatness for codimension <3 has been given).

Relative problem $M, N \subset Q$, ∂M and ∂N given transversal in ∂Q. Can you move them transversal? No.

Example $S^m, S^n \subset S^q$ transversal: can S^m and S^n be spanned by transversal balls $D^{m+1}, D^{n+1} \subset D^{q+1}$? No. To show why the obvious methods for trying to prove a Yes do not work, first consider the joins of S^m and S^n with 0. Here transversality fails at 0. Next try join of S^m with one point, of S^n with another point. However this also fails as in the following picture: Let L be a line transversal to a folded disk D in E^3. Then $D \times I$ and $L \times I$ are transversal in E^4. Now $L \times I$ is a square. Tilt the square slightly; this corresponds to taking cones on 2 different points. Then $D \times I \cap L \times I$ is 3 lines meeting at 0, not transversal. (Note this does not yield a counter-example). To get a counterexample note $S^m \cap S^n$ must bound if S^m & S^n are spanned by transversal balls — it would bound $D^{m+1} \cap D^{n+1}$.



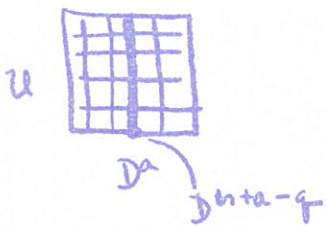
Consider now only the manifold case. Suppose $M \subset Q$ and given a triangulation (Δtion) of Q. Can M be moved so it cuts across all the simplices of Q nicely? If so, make N a subcomplex; then the transversality theorem would be proven.

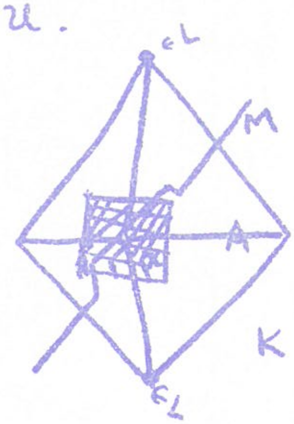
Dfn Let $x \in \operatorname{int} A$, A a subcomplex of $K = \Delta$tion of Q. Let L be the link of A in K.

Write join as juxtaposition.

Let $s: AL \to xL$ be the natural simplicial map.

Dfn M is transimplicial to K at x if there is a nbd. $\mathcal{U}$ of x in xL and an embedding $e: \mathcal{U} \times (D^a, D^{m+a-q}) \to (AL, M \cap AL)$ onto a nbd. of x with $se = p_1 : \mathcal{U} \times D^a \to \mathcal{U}$.





Lemmas 1. If K' is a subdivision of K then M trans^l to K' $\Rightarrow$ M trans^l to K

2. M trans^l to K, N subcomplex of K $\Rightarrow$ M, N transversal.

Thm Given $M \subset Q$, K a Δtion of Q, M can be ambient isotoped trans^l to K.

Cor (use Lemma 2) transversality theorem.

Proof of thm. Step I. Δate so M is a subcomblex

II subdivide this Δtion so that it becomes Brouwer

(namely any St v can be linearly imbedded in E^q for a vertex v).

III Make M trans^l by operating one simplex at a time (in order of decreasing dimension)

MIT Topology Seminar 7 11/20/67

Edgar Brown (Brandeis): Recent work of Wm. Browder on the Arf Invariant, II

In this talk consider only the stable range. If $f: X \to Y$ let $E_f = \{(x,\alpha) \in X \times Y^I \mid \alpha(0) = *, \alpha(1) = f(x)\}$. and $E_f \to X$ by projection on the first factor. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $g \circ f \sim 0$ we get a diagram

$$\begin{tikzcd} E_{\bar f} \arrow[d] & E_g \arrow[d] \\ X \arrow[ur, "\bar f"] \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

There is a map $E_f \xrightarrow{\bar g} \Omega Z$ namely $(x,\alpha) \to g \circ \alpha$, and a map $\Omega Y \to E_f$ namely $\beta \to (*,\beta)$; such that $\bar g i(\beta) = g \circ \beta$, hence $\bar g i = \Omega g$

$$\begin{tikzcd} & E_{\bar g} \arrow[d] \\ \Omega Y \arrow[r, "i"] & E_f \arrow[r, "\bar g"] \arrow[d] & \Omega Z \\ & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Lemma $E_{\bar g} = E_{\bar f}$ //

Let $\mathcal{a}_2$ be the Steenrod algebra mod 2, $X = K(\mathbb{Z}_2, \ell)$, $Y = K(\mathbb{Z}_2, s)$, $Z = \prod K(\mathbb{Z}_2, j)$ (some finite product).

$$\begin{array}{ccccc} H^*(Z) & \longrightarrow & H^*(Y) & \longrightarrow & H^*(X) \\ z & \longrightarrow & \Sigma b_i y_i & & \\ & & y_i & \longrightarrow & a_i x \end{array}$$

free $\mathcal{a}_2$-modules with generators z_j, y, x.

so $z \longrightarrow \Sigma a_i b_i x$

Dfn $\Sigma a_i b_i = 0$ detects $\alpha \in \pi_{s-2}(S^\ell)$ if there is a map $h: S^\ell \to E_{\bar g}$ with $h_* \alpha \neq 0$.

Thm 1 (Adams) Let $h_i : \mathcal{a}_2 \to \mathbb{Z}_2$ be $h_i(Sq^{2^i}) = 1$, $h_i(Sq^I) = 0$ for all other admissible I. Then $\sum a_i b_i = 0$ detects $\alpha \in \pi_n(S) \Leftrightarrow$ for some k, l with $k \geq l \geq 0$ and $k \neq l+1$, $\sum h_k(a_i)\, h_l(b_i) \neq 0$ and $h_k h_l$ persists to E_∞ in Adams' spectral sequence. //

Let $v_{n+1} \in H^{n+1}(BO_k, \mathbb{Z}_2)$ be the Wu class associated to Sq^{n+1}. In the diagram let γ be the canonical k-plane bundle and $v_{n+1}^*(\iota_{n+1}) = v_{n+1}$, $B_{(n)} = E_{v_{n+1}}$, $\bar{\gamma}$ the induced bundle. Let $i : S^k \to T(\bar{\gamma})$ (the thom space) be the "injection of the fiber."

$$\begin{array}{ccccccc}
R^k & \longrightarrow & \bar{\gamma} & \longrightarrow & \gamma & & \\
\downarrow & & \downarrow & & \downarrow & & \\
pt & \longrightarrow & B_{(n)} & \longrightarrow & BO_k & \xrightarrow{v_{n+1}} & K(\mathbb{Z}_2, n+1)
\end{array}$$

Prop. 2 $i_* : \pi_{2n+k}(S^k) \to \pi_{2n+k}(T(\bar{\gamma}))$ is trivial if $n \neq 2^i - 1$.

Proof We find a 3-stage Postnikov system for $T(\bar{\gamma})$ in dimension $2n+k+1$. Here k is large and statements are to hold for dimension $< 2k$. Choose BO_k so that $B_{(n)} \subset BO_k$.

$$\underset{\text{fiber}}{\Omega\left(T(\gamma)/T(\bar{\gamma})\right)} \longrightarrow \underset{\text{total space}}{\overset{E_j \text{ stably}}{T(\bar{\gamma})}} \longrightarrow \underset{\text{base}}{T(\gamma)} \longrightarrow T(\gamma)/T(\bar{\gamma}) \longrightarrow ST(\gamma) \longrightarrow \cdots$$

$T(\gamma) = MO_k$ is a product of $K(\mathbb{Z}_2, i)$'s. Find a 2-stage Postnikov system for $\Omega(T(\gamma)/T(\bar{\gamma}))$. Write $K_{n+1} = K(\mathbb{Z}_2, n+1)$.

$$\begin{array}{ccccc}
 & & (b,\alpha) & \longrightarrow & \alpha(0) \\
\{(b,\alpha) \mid \alpha(0) = *,\ \alpha(1) = v_{n+1}(b)\} & \subset & \{(b,\alpha) \in BO_k \times K_{n+1}^I \mid \alpha(1) = v_{n+1}(b)\} & \longrightarrow & K_{n+1} \\
\| & & \| & & \| \\
\underset{\text{fiber}}{B_{(n)}} & \longrightarrow & BO_k & \underset{\text{fibration}}{\xrightarrow{v_{n+1}}} & K_{n+1}
\end{array}$$

Lemma $(v_{n+1} \times id)^* : H^q(K_{n+1} \times B_{(n)}, * \times BO_k) \to H^q(BO_k, B_{(n)})$ is iso for $q \leq 2n+1$ and kernel is $\{\iota_{n+1} \otimes v_{n+1} + \iota_{n+1}^2 \otimes 1\}$ in dim. $q = 2n+2$.

$\downarrow$ fibrations:
$$\begin{array}{ccc}
(BO_k, B_{(n)}) & \xrightarrow{v_{n+1} \times id} & (K_{n+1} \times BO_k, \{*\} \times BO_k) \\
\downarrow & & \downarrow \\
(K_{n+1}, *) & = & (K_{n+1}, *)
\end{array}$$

<u>Proof</u> Consider the spectral sequences for the 2 fibrations

$$\left.\begin{aligned}\bar{E}_\infty^{pq} &= \bar{E}_2^{pq} = H^p(K_{n+1},*) \otimes H^q(BO_k) \\ E_2^{pq} &= H^p(K_{n+1},*) \otimes H^q(B_{(n)})\end{aligned}\right\} \begin{array}{l} q \le n \text{ i.e. } p+q \le 2n+1 \\ \text{and } p+q = 2n+2 \\ \text{and } q < n+1. \end{array}$$

Kernel in $2n+2$: 1) $\iota_{n+1} \otimes w_{n+1} + \iota_n^2 \otimes 1 \longrightarrow 0$

2) $\bar{E}_\infty^{pq} \longrightarrow E_\infty^{pq}$ iso, $p+q = 2n+2$ & $q \ne n+1$.

and has kernel $\{\iota_{n+1} \otimes w_{n+1}\}$, $p=q=n+1$.

Let γ_1 = disc bundle, γ_0 = sphere bundle.

$$\begin{tikzcd} \bar{\gamma} \arrow[r] \arrow[d] & \gamma \arrow[r] \arrow[d] & \tilde{\gamma} \arrow[r] \arrow[d] & \gamma \arrow[d] \\ B \arrow[r] & BO_k \arrow[r] & K_{n+1} \times BO_k & BO_k \end{tikzcd}$$

$$\begin{array}{ccc} H^q(K_{n+1} \times BO_k, * \times BO_k) & \longrightarrow & H^q(BO, B) \\ \wr\wr & & \wr\wr \\ H^{q+k}(\tilde{\gamma}_1, \tilde{\gamma}_0 \cup [\tilde{\gamma}_1 | * \times BO_k]) & \longrightarrow & H^{q+k}(\gamma_1, \gamma_0 \cup \bar{\gamma}_1) \\ \wr\wr & & \wr\wr \\ H^{q+k}(K_{n+1} \wedge T(\gamma)) & \xrightarrow[\substack{\text{iso,} \\ q \le 2n+2,}]{h^*} & H^{q+k}(T(\gamma)/T(\bar{\gamma})) \end{array}$$

$\ker = \iota_{n+1}^2 \otimes U_k + \iota_{n+1} \otimes w_{n+1} U_k$ (U_k = Thom class)

$$\left.\begin{tikzcd} E_w \arrow[r] \arrow[dr, "k"'] & K_{n+1} \wedge T(\gamma) \arrow[r, "w"] & K_{2n+2+k} \\ & T(\gamma)/T(\bar{\gamma}) \arrow[u] & \end{tikzcd}\right\} \begin{array}{l} k^* \text{ on } H^{q+k} \begin{cases} \text{iso} & q \le 2n+1 \\ \text{1-1} & q = 2n+2 \end{cases} \\ \Rightarrow k_* \text{ on } \pi_q \begin{cases} \text{iso} & q \le 2n+1 \\ \text{onto} & q = 2n+2 \end{cases} \end{array}$$

(±)

$$\begin{tikzcd} & T(\bar{\gamma}) = E_j \arrow[r] \arrow[ddr] & E_{kj} \arrow[dd, "\Omega E_w"] & \frac{T(\gamma)}{T(\bar{\gamma})} \arrow[r, "k"] & E_w \arrow[dd] & \\ & E_{kjU_k} \arrow[ur] \arrow[d] & \Omega\left(\frac{T(\gamma)}{T(\bar{\gamma})}\right) & & & \\ S^k \arrow[uur] \arrow[ur] \arrow[r] & K_k \arrow[r, "U_k"'] & T(\gamma) \arrow[uur, "j"] \arrow[rr] & & K_{n+1} \wedge T(\gamma) \arrow[r, "w"] & K_{2n+2+k} \end{tikzcd}$$

$$\pi_{2n+k}(T(\bar{\gamma})) \approx \pi_{2n+k}(E_{kj}).$$

Does $K_k \longrightarrow K_{n+1} \wedge T(\gamma) \longrightarrow K_{2n+2+k}$ detect anything in $\pi_{2n+k}(S^k)$?

$$\begin{tikzcd} & H^*(T(\bar{\gamma})) \arrow[dr] & & \\ H^*(K_{2n+2+k}) \arrow[r] \arrow[ur] & \underbrace{\tilde{H}^*(K_{n+1}) \otimes H^*(T(\gamma))}_{e} \arrow[r] & H^*(K_k) \end{tikzcd}$$

$$\iota_{2n+2+k} \longrightarrow \iota_{n+1}^2 \otimes U_k + \iota_{n+1} \otimes w_{n+1} U_k$$

Let $\{x_i\}$ generate $H^*(T(\gamma))$ over $\mathcal{a}_2$, $x_0 = U_k$. Then $\{u \otimes x_i\}$ generate $\mathcal{C}$ over $\mathcal{a}_2$.

$$\iota_{n+1} \otimes v_{n+1} U = \iota_{n+1} \otimes \chi(Sq^{n+1}) U_k$$

$$= \sum_{i=0}^{n+1} \chi(Sq^i)\,(Sq^{n+1-i}\,\iota_{n+1} \otimes \overset{x_0}{\overset{\shortparallel}{U_k}})$$

$$w = \sum_{i>0} \chi(Sq^i)\,(Sq^{n+1-i}\,\iota_{n+1} \otimes x_0)$$

$$Sq^{n+1-i}\iota_{n+1} \otimes x_0 \longrightarrow (Sq^{n+1-i} v_{n+1})\,x_0 = \sum b_i x_i \quad \text{where } b_{n+1} = \chi(Sq^{n+1})$$

and $\dim b_i = 2n+2-i$. $\Rightarrow \sum_{i\geq 1}^{n+1} \chi(Sq^i)\,b_i = 0$ is the form of the relation coming from diagram (*). But by <u>theorem 1</u>, this relation can't detect an element of $\pi_{2n+k}(S^k)$ if $n \neq 2^i - 1$.

MIT Topology Seminar 8 11/28/67

F. P. Peterson : $H^*(BSF ; Z_p)$

We are interested in the groups $O(n)$ = real orthogonal group, $PL(n)$ = piecewise linear homeomorphisms of S^{n-1}; $H(n)$ = homeomorphisms of S^{n-1}; and $G(n)$ = homotopy equivalences of S^{n-1}. Each is included in the next, naturally; for each there is a classifying space: $BO(n) \longrightarrow BPL(n) \longrightarrow BH(n) \longrightarrow BG(n)$

and these spaces classify respectively real n-vector bundles, PWL microbundles, ?, and (n-1)spherical fibrings = fibrations whose fiber has the homotopy type of S^{n-1}. Let $n \to \infty$; one then writes O, PL, H, and F (not G) for the groups obtained. There are also the orientation-preserving subgroups SO, SPL, SH, SF. This defines BSF.

In each classifying space there is an analogue of the canonical n-bundle over $BO(n)$, so that for each group one gets a Thom space; these fit together in spectra. We are interested in

$$MSO \longrightarrow MSPL \longrightarrow MSF$$

The Thom isomorphism $H^*(BSF) \xrightarrow{\phi} H^*(MSF)$ has degree 0. Define $w_i = \phi^{-1} Sq^i \phi(1)$. This shows w_i's are defined all the way back on $G(n)$, not just on $O(n)$. Let p be an odd prime and $r = 2p-2$ in the sequel. One has Wu classes

$$q_i = \phi^{-1} P^i \phi(1) \in H^{ir}(BSF).$$

Using $G(n)$ = homotopy equivalences of S^{n-1} it is easy to see

$$\pi_n(BSF) = \text{stable } \pi_{n-1}(S)$$

This shows $\pi_*(BSF)$ is all finite groups $\Rightarrow$ $H^*(BSF)$ is also $\Rightarrow$ there are no Pontrjagin classes. Now $\beta q_1 \neq 0$. There is a map

$$\theta : Z_p[q_i] \otimes E(\beta q_i) \longrightarrow H^*(BSF).$$

Milnor conjectured θ is a monomorphism.

Thm 1 $H^*(BSF) \approx \mathbb{Z}_p[q_i] \otimes E(\beta q_i) \otimes C$ where C is a Hopf algebra over the Steenrod algebra A and the $\approx$ holds as Hopf algebras over A. (Peterson, Toda).

Milnor showed θ is a monomorphism in dimensions $< pr-1$ by using known homotopy groups of spheres and constructing BSF using k-invariants. Stasheff showed θ is mono in dim. $< (2p+1)r$ and showed C is $(pr-2)$-connected.

Cor MSF has the homotopy type of a wedge of Eilenberg-MacLane spaces (wedge over many primes).

A corresponding theorem holds for Z_2 coefficients :

$$BO \longrightarrow BF$$

$$\begin{tikzcd} H^*(BO) \arrow[d, equal] & H^*(BF) \arrow[l, "f"'] \\ Z_2[w_i] \arrow[ur, "\theta"'] & \end{tikzcd}$$

The w_i can be defined on BF so choose θ with $f\theta$ = identity; find θ is a map of Hopf algebras over A. Then with

$$C = \frac{H^*(BF)}{\theta(\bar{H}^*(BO)) \cdot H^*(BF)}$$

one has $H^*(BF) = H^*(BO) \otimes C$ (using Hopf algebra arguments like those in Milnor-Moore). In Z_p coefficients much difficulty comes from β which is quite unlike $Sq^1 = \beta \bmod 2$.

Next we show θ is a monomorphism. Let L be a lens space (identification space of S^{2n+1} under the action of Z_p).

Lemma There exists a map $h: SL \longrightarrow BSF$ such that $h^*(\beta q_i) \neq 0$ for all i.

SL = suspension of L

Proof

$$\begin{tikzcd}
L = S^{2n+1}/\mathbb{Z}_p \arrow[r, "\lambda"] & S^{2n+1}/S^1 = CP \arrow[r, "j"] \arrow[d, "g"] & C_\lambda \arrow[r, "\pi"] \arrow[dl, dashed, "\text{cone on } \lambda"] & SL \arrow[dl, dashed, "h", "\text{cone on } j"'] \\
& BSO \arrow[r, "i"] & BSF &
\end{tikzcd}$$

We want a vector bundle $CP \xrightarrow{g} BSO$ such that ig is trivial; for given such a bundle one can factor (dotted arrows above) to get h. Let ξ be the canonical complex line bundle on CP. One knows $K(L)$, $K(CP)$, and λ^*; $\operatorname{Ker} \lambda^* =$ ideal generated by $\xi^p - 1$. We take the bundle $(p+1)^e(\xi^{p+1} - \xi) \in \operatorname{Ker} \lambda^*$ where e is a number such that $(p+1)^e(\xi^{p+1} - \xi)$, made real and pushed to BSF, becomes 0. Adams proves the existence of e in $J(X) - \mathrm{II}$ (the bundle in question equals $\psi^{p+1}(\xi)$.) Make this bundle real (neglect complex structure). Now compute $h^*(\beta q_i)$ and show it is not 0. Use chern classes on CP and the fact that j multiplies by p. //

To prove Θ is 1-1 define a coalgebra structure ψ on range Θ $\left\{ \mathbb{Z}_p[q_i] \otimes E(\beta q_i) \xrightarrow{\Theta} H^*(BSF) \right\}$ (abbreviated hereinafter $\mathbb{Z}_p \otimes E$) by $\psi(q_i) = \sum q_j \otimes q_{i-j}$.

To prove Θ is 1-1 it is enough to prove Θ is 1-1 on the primitive elements of $\mathbb{Z}_p \otimes E$. There are primitive elements in $\mathbb{Z}_p \otimes E$ of degree t only if

$t = ir$: polynomial in q_i is primitive

or $t = ir+1$: βq_i + decomposable is primitive.

In the first case the map $H^*(BSF) \longrightarrow H^*(BSO)$ carries the polynomial in q_i's into a non-trivial polynomial in q_i's in $H^*(BSO)$ which must therefore be non-0. For $t = ir+1$ use $h^*: H^*(BSF) \to H^*(SL)$. Now h^* of decomposables is 0 since SL is a suspension $\Rightarrow$ all products are 0; and $h^*(\beta q_i) \neq 0$ by construction. //

As before define $C = H^*(BSF) \big/ \theta(\mathbb{Z}_p \otimes E) \cdot H^*(BSF)$.

Once we have a map of Hopf algebras back, the proof of Thm 1 mimics Z_2 proof. The map is:

$$H^*(BSF) \xrightarrow{\psi} H^*(BSF) \otimes H^*(BSF) \xrightarrow{f \otimes \rho} (\mathbb{Z}_p \otimes E) \otimes C$$

Once we have f we have an isomorphism by Milnor-Moore.

Idea Take $BSO \times SL \times \cdots \times SL \longrightarrow BSF$ by standard map on BSO and h on each SL; multiply (BSF is an H-space). Show the image is $Z_p \otimes E$.

Outline Let M be an algebra over $\mathcal{A}$ (module with products an Cartan formula holds). If N is $\{1, 2, \cdots, n\}$,

$$\underset{N}{\otimes} M \supset \mathcal{S}_N(M) = \text{symmetric algebra on } N \text{ letters.}$$

Let $n \to \infty$ and get $\mathcal{S}(M)$. Now $\underset{N}{\otimes} M$ gets ∞-dimensional, but $\mathcal{S}_N(M)$ does not. One has $\mathcal{S}_{2N}(M) \longrightarrow \mathcal{S}_N(M) \otimes \mathcal{S}_N(M)$ from the map $N \cup N \longrightarrow 2N = \{1, \cdots, 2n\}$. This yields as $n \to \infty$ a diagonal map $\mathcal{S}(M) \longrightarrow \mathcal{S}(M) \otimes \mathcal{S}(M)$, thus a Hopf algebra over $\mathcal{A}$. Use this infinite diagonal map $H^*(BSF) \xrightarrow{\Psi} \mathcal{S}(H^*(BSF))$. The diagonal Δ of $H^*(BSF)$ is coassociative and cocommutative.

Let $M = H^*(SL) \big/ (\ \ )$, where $(\ \ )$ = everything but H^0, H^{ir} and H^{ir+1}. Then $(\ \ )$ is a sub-$\mathcal{A}$ module so M is an $\mathcal{A}$ module

$$\begin{tikzcd}
H^*(BSF) \arrow[r, "\psi"] & H^*(BSF)\otimes H^*(BSF) \arrow[r, "1\otimes\Phi"] & H^*(BSF)\otimes \mathcal{S}(H^*(BSF)) \arrow[d, "i\otimes \mathcal{S}(h^*)"] \\
& \mathbb{Z}_p[q_i]\otimes \mathcal{S}(M) & H^*(BSO)\otimes \mathcal{S}(H^*(SL)) \arrow[l]
\end{tikzcd}$$

$\mathbb{Z}_p[q_i]$ quotient of $H^*(BSO)$; $\mathcal{S}(M)$ quotient of $\mathcal{S}(H^*(SL))$

Let this composite be denoted by f. Every map is a map of Hopf algebras over A, so f is also. (we would like to throw away H^{ir} but then the map would not preserve A-structure since H^{ir} and H^{ir+1} are connected by β). Show $\operatorname{Im} f = \operatorname{Im} f\theta$ and $f\theta$ = monomorphism. //

About C: $e_1 \in C^{pr-1}$ is defined by Gitler – Stasheff. The q_i are defined by primary operations on the Thom class; e_1 is defined by a twisted secondary operation on the Thom class. For a range (up to $(2p+1)r$) C is a free commutative algebra (exterior algebra on odd dim., polynomial alg on even dim.) on elements $\{a(e_1)\}$ where $\{a\}$ is a basis of $A \big/ A(\bar{\mathcal{P}}^1, 2\mathcal{P}^{p+1}\beta - \mathcal{P}^p\mathcal{P}^1_\beta)$.

This is false for higher dimensions since $(\beta e_1)^p = 0$.

Assuming some results of Cohen and Christiansen, one gets more secondary operations and finds e_t (for each t) $\in C^{p^t r-1}$

<u>Conjecture</u> C is generated as an algebra over A and β_s (higher order Bocksteins) by e_t.

So far we can only get e_t's by getting $\pi_*(S)$ and working backwards.

<u>Applications</u> $p=3$: Sullivan's results on F/PL.

The map BSPL $\to$ BSF gives $H^*(BSPL; Z_3) \leftarrow H^*(BSF)$

$$\begin{array}{ccc} H^*(BSPL;Z_3) & \leftarrow & H^*(BSF) \\ \cong & & \parallel \\ Z_3[q_i]\otimes C & \longleftarrow & Z_3[q_i]\otimes E(\beta q_i)\otimes C \end{array}$$

where the left $\cong$ holds in a certain range of dimensions, using the Serre spectral sequence (only for $p=3$ is $r=4$). One shows $BSPL \underset{p}{\sim} BSO \times BCoker\, J$; <u>conjecture</u> $H^*(BCoker\, J) = C$, so the map kills $E(\beta q_i)$ and gives other Pontrjagin classes. Use Thom isomorphism and get $H^*(MSPL)$. Try to compute (with methods of cobordism theory) $\pi_*(MSPL)$. The last group has p-torsion as follows: $H^*(MSO) = \Sigma\, 'A$ where $'A$ is generated by P^i but not β. For each copy of $'A$, $\pi_*(MSPL)$ contains p-torsion

Z_p	in dimension	$pr-1$
Z_p+Z_p		$2pr-1$
$Z_p+Z_{p^2}$		$(2p+1)r-1$.

The Z_{p^2} is not detectable by characteristic numbers.

MIT Topology Seminar 9 12/4/67

John Harper: <u>Cohomology of Massey-Peterson Algebras</u>

Liulevicius wrote a paper of this title and one called "Multicomplexes" (to appear). Massey-Peterson Algebras (MPA hereinafter) were defined in a paper "The Semi-tensor Product" (<u>Topology</u> 1965) and used by Stasheff under the name "split extension algebra" in a paper in Topology 1965. A Multicomplex is an algebraic object necessary to study change of rings in homological algebra.

Study of MPA's is motivated by the problem of lifting f (at right) where π is a fibration inducing isomorphisms in low dimensional homology. Milgram-Becker (to appear) give an Adams-like spectral sequence with

$$\begin{array}{ccc} & & E \\ & \nearrow & \downarrow \pi \\ Y & \xrightarrow{f} & B \end{array}$$

$$E_2 = \mathrm{Ext}_{(B)A}\big(\underbrace{H^*(B,E)}_{\text{mapping cylinder}}, H^*(Y)\big)$$

A = steenrod alg.; all mod 2; or mod p.

which converges to the set of homotopy classes of liftings of f. The MPA $(X)A$ combines the ring structure of $H^*(X)$ and its A-module structure as follows:

<u>Dfn</u> The MPA $(X)A$ corresponding to $H^*(X)$ and the steenrod algebra A satisfies

1) $(X)A \cong H^*(X) \otimes A$ as a vector space

2) $(X)A$ is a free left $H^*(X)$-module by multiplication on left and a free right A - module " " " right.

3) $(X)A$ is a left A-module via $\psi : A \to A \otimes A$.

4) If $f: Y \to X$, $H^*(Y)$ is of course an $H^*(X)$-module. It is also an $(X)A$ module via $(x \otimes a) \otimes y \longrightarrow f^*(x) \smile ay$.

5) $(X)A$ is an algebra under the product

$$(w \otimes b)(x \otimes a) = \sum_i (-1)^{\deg b_i'' \deg x} (w \smile b_i' x) \otimes b_i'' a$$

where $\psi(b) = \sum b_i' \otimes b_i''$. (This is the natural definition in view of 4)).

<u>Remark</u> $(\mathbb{Z}_2)A = A$ so there is an augmentation map $(X)A \longrightarrow A$.

To compute $Ext_{(X)A}$ from Ext_A via change of rings we must first replace $(X)A$ by an isomorphic object which is free as a <u>left</u> A-module.

<u>Thm 1</u> There exists an algebra $A(X)$ and an isomorphism $A(X) \xrightarrow{T} (X)A$ such that

1) $A(X) \simeq A \otimes H^*(X)$ as a vector space
2) $A(X)$ is a free left A-module and a free right $H^*(X)$-module.
3) If $f: Y \to X$ then $H^*(Y)$ is a left $A(X)$-module with
$$(a \otimes x) \otimes y \longrightarrow a(f^*(x) \cup y)$$
4) $(b \otimes w)(a \otimes u) = \sum (-1)^{\deg a_i'' \deg w} b a_i' \otimes (\chi a_i'') w \cup u$

<u>Idea of proof</u> $T = \text{Twist} \circ h$ where $h: (A \otimes M)_L \longrightarrow (A \otimes M)_D$

$$\text{left A-action} = \begin{Bmatrix} \text{multiplication} \\ \text{on left} \end{Bmatrix} \begin{Bmatrix} \text{diagonal on } A, \\ \text{action on } A \text{ and } M \end{Bmatrix}$$

where M is any A-module and h is the composite
$$A \otimes M \xrightarrow{\psi \otimes 1} A \otimes A \otimes M \xrightarrow{1 \otimes \text{action}} A \otimes M$$
shown by Milnor-Moore to be an isomorphism. //

<u>Thm 2</u> $f: Y \to X$. As a left module over $A(X)$ (not as an algebra)
$$A(Y) \cong A(X) \otimes_{H^*(X)} H^*(Y)$$
$$a \otimes y \longrightarrow [a \otimes 1 \otimes y] \quad //$$

This turns resolutions of $H^*(Y)$ over $H^*(X)$ into resolutions of $A(Y)$ over $A(X)$ since the functor $A(X) \otimes_{H^*(X)}$ —— is exact.

<u>Thm 3</u> Let R, S be rings with 1, $h: R \to S$ a ring homomorphism, M a left S-module; then M is a left R-module via h. Let $Y_\circ$ be an S-free resolution of M with differential d' and augmentation ε'. Let $X_{q\circ}$ be an R-free resolution of Y_q, with differential d'' and augmentation ε_q: $X_{q,0} \to Y_q$. Let $C = \sum_{q \geq 0} X_{q\circ}$, $C_k = \sum_{q+r=k} X_{q,r}$. There exist

maps $d^i : C_{\bullet} \to C$ of degree -1 (bidegree $(-i, i-1)$) for $i = 0, 1, 2, \cdots$ such that 1) $d^0 = d''$, 2) d^1 covers d', and 3) $\sum_{i=0}^{k} d^i d^{k-i} = 0$. Further there exists a differential $d : C \to C$ such that $\{C, d, \varepsilon\}$ is an R-free resolution of M characterized by $d = \sum d^k$. //

<u>Remark</u> Draw a picture to understand the thm. This is the algebraic analogue of the Serre spectral sequence; given a resolution of S over R use $R \otimes_S$ — to get each $X_{q\bullet}$ over Y_q — it plays the part of the fiber and Y_q is the base. If S is R-free, $R \subset S$, this collapses.

<u>Thm 4</u> Suppose $R \xrightarrow{h} S$ as in <u>thm 3</u> and N a left S-module, G a left R-module There exists a spectral sequence

$$E_2^{pq} = \operatorname{Ext}_S^p(M, \operatorname{Ext}_R^q(S, G)) \Rightarrow \operatorname{Ext}_R(M, G)$$

which preserves Yoneda products. //

<u>Remark</u> In interesting cases action of S on $\operatorname{Ext}_R^q(S, G)$ is trivial so E_2^{pq} splits as a tensor product.

<u>Applications</u> Let $C \xrightarrow{h} D$ be an epimorphism of algebras over A and $D \cong C \otimes_{\mathbb{Z}_2[u]} \mathbb{Z}_2$. This occurs for example as follows:

$$H^*(BSO(n)) \cong H^*(BO(n)) \otimes_{\mathbb{Z}_2[w_1]} \mathbb{Z}_2$$

$$\mathbb{Z}_2 \cong H^*(BO(1)) \otimes_{\mathbb{Z}_2[w_1]} \mathbb{Z}_2$$

$$\mathbb{Z}_2 \cong H^*(BSO(2)) \otimes_{\mathbb{Z}_2[w_2]} \mathbb{Z}_2$$

<u>Dfn</u> $x \in C$ is A-C indecomposable if it determines a nonzero coset in $Q(C) / \bar{A}\, Q(C)$ ($\Leftrightarrow 1 \otimes x$ is indecomposable in $A(C)$).

<u>Thm 5</u> C, D as above, u A-C indecomposable. Then

$$\operatorname{Ext}_{A(C)}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \operatorname{Ext}_{A(D)}(\mathbb{Z}_2, \mathbb{Z}_2) \otimes \underbrace{\Lambda(w)}_{\text{exterior algebra}}$$ as a vector space, where w is the class of u in $\operatorname{Ext}^{1, \deg u}_{\mathbb{Z}_2[u]}(\mathbb{Z}_2, \mathbb{Z}_2)$. //

Thus we find

$$\mathrm{Ext}_{A(BO(n))}(\mathbb{Z}_2,\mathbb{Z}_2) \cong \mathrm{Ext}_{A(BSO(n))}(\mathbb{Z}_2,\mathbb{Z}_2)\otimes\Lambda(w_1)$$

and it turns out $\langle w_1\rangle^2 = h_0 \otimes \langle w_1\rangle$.

$$\mathrm{Ext}_{A(BSO(2))}(\mathbb{Z}_2,\mathbb{Z}_2) \cong \mathrm{Ext}_{A}(\mathbb{Z}_2,\mathbb{Z}_2)\otimes\Lambda(w_2)$$

where $\langle w_2\rangle^2 = h_1 \otimes \langle w_2\rangle$.

Also $BSO(n-1)\subset BSO(n)$ so $A(BSO(n)) \longrightarrow A(BSO(n-1))$.

Stasheff uses a twisted secondary cohomology operation corresponding to $h_1^2 + h_1\langle w_2\rangle$.

$$0 = (Sq^2\otimes 1 + 1\otimes w_2)^2 + (Sq^2\otimes 1)(Sq^2\otimes 1) \quad \text{in } A(BSO(2)).$$

In $A(BSO(3))$, $\mathrm{Ext}_{A(BSO(3))}(\mathbb{Z}_2,\mathbb{Z}_2)$ has generators:

$s=1$	$h_i\,,\ w_2$	
$s=2$	$h_i h_j$	$j\neq i+1$
	$h_i\langle w_2\rangle$	
	$a=\langle h_1, h_0, \langle w_2\rangle\rangle$	(Massey product)
	$b=\langle h_0, (w_2,h_1), \binom{\langle w_2\rangle}{\langle w_2\rangle}\rangle$	(")

and certain relations occur.

<u>Advertisement</u> computations are long — but easy.

MIT Topology Seminar 12/18/68

John Moore (Princeton) : <u>Homological Algebra and Topology</u>

We wish to explain the difference between differential and ordinary homological algebra. We talk about complexes, Let $\mathcal{A}$ be an abelian category, $C(\mathcal{A})$ = complexes over $\mathcal{A}$. We want to imbed a complex in a cone over it, or take the product of a complex and a unit interval. Let $A \in C(\mathcal{A})$.

$$A_n \xrightarrow{d_n(A)} A_{n-1} \xrightarrow{d_{n-1}(A)} A_{n-2} \qquad \text{composite } 0$$

If $\mathcal{A}$ = modules over some ring R, the "unit interval" is the complex I given by

$$\begin{cases} \dim. 1 & R & \text{basis } [0,1] \\ \dim. 0 & R \oplus R & \text{basis } [0], [1] \end{cases}$$

where $d_1(I)([0,1]) = [1] - [0]$. We take $I \otimes_R A$ for "$A \times I$."
Now $(I \otimes_R A)_n = A_n + A_n + A_{n-1}$ and the differential is *

$$(x, y, z) \longrightarrow (d_n x + z, d_n y + z, -dz). \qquad *$$

In any abelian category we can use these formulas (*) to define "$A \times I$." To get a cone collapse one copy of A. Map $A \to I \times A$ where A injects into the first summand; let the cokernel be $C(A)$. Or directly, $C(A)_n = A_n + A_{n-1} \xrightarrow[0 + id_{A_{n-1}}]{} A_{n-1} + A_{n-2}$. Map $A \to C(A)$ by $A_n \xrightarrow{id + d_n} A_n + A_{n-1}$.

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "f"'] & C(Y) \arrow[d] \arrow[ddr, bend left, "h"] & \\ X \arrow[r] \arrow[drr, bend right, "g"'] & X \cup_f C(Y) \arrow[dr, dashed, "k"] & \\ & & Z \end{tikzcd}$$

Often in topology one has what is called a "pushout diagram", namely given maps g h with $hi = gf$, $X \cup_f C(Y)$ is a complex such that k always exists. This universal property can be used in any category $C(\mathcal{A})$. In particular, map $X \xrightarrow{f \wedge - i} X \oplus C(Y)$ and take cokernel $= X \cup_f C(Y)$.

<u>Derived functors</u> Let $T: \mathfrak{A} \to \mathfrak{B}$ be a functor. Assume enough projectives. Take a projective resolution $\cdots P_2 \to P_1 \to P_0 \to A$ of A and let T act on it, getting $T(P_2) \to T(P_1) \to T(P_0) \to A$. The homology of this complex is called the derived functors of T. (assume T is additive). A different view is possible: suppose $T: \mathfrak{A} \to \mathcal{C}(\mathfrak{B})$.

<u>Example</u>: Hyperhomology. Given $\mathfrak{A} \xrightarrow{T} \mathfrak{B}$ get $\mathcal{C}(\mathfrak{A}) \longrightarrow \mathcal{C}(\mathfrak{B})$ where $A \to T(A)$. This is a special case where the functor comes from extension of a functor $\mathfrak{A} \to \mathfrak{B}$.

<u>Example</u> (not by extension): Let R be a commutative ring, Λ a supplemented differential graded R-algebra. (model: G = topological group, $\Lambda = C_*(G, R)$ = singular chains on G). Let the category be ${}_\Lambda M$, with objects (M, map $\Lambda \otimes M \longrightarrow M$) satisfying the usual differential and unitary conditions. Take a functor ${}_\Lambda M \longrightarrow \mathcal{C}({}_R M)$ where ${}_R M$ = left R-modules and $M \longrightarrow R \otimes_\Lambda M$. This functor does not come about by extension.

We want derived functors of a functor T. Intuitively we take a projective resolution in $\mathfrak{A}$ and do something on the right, then take homology. But almost always you don't want a projective resolution; instead use some class of relative projectives. Look at $\mathcal{C}(\mathfrak{B})$, where $\mathfrak{B}$ has enough projectives. The projectives are cones $C(A)$ such that each A_n is projective; but cones have trivial homology, so this is the wrong class of objects. If we restrict ourselves to positive complexes, there are also projectives $C(A) + P$ where P is a projective in degree 0. Similarly the injectives among negative complexes are cones injective in each degree plus an injective in dimension 0.

One wants any object $C(A)+P$ to be projective, provided A_n, P are projective for $n \in \mathbb{Z}$ and P has 0 differential. More abstractly let $\mathcal{P}$ be any class of objects in $\mathfrak{A}$. Consider the class of morphisms $f \in \mathcal{E}(\mathcal{P})$ such that the dotted arrow always exists when $P \in \mathcal{P}$, making the Δ commute. Conversely one can take a class of morphisms $\mathcal{E}$ and consider the class of objects $\mathcal{P}(\mathcal{E})$ such that $\dashrightarrow$ exists whenever $f \in \mathcal{E}$ and $P \in \mathcal{P}(\mathcal{E})$

$$\begin{tikzcd} & A \arrow[d, "f"] \\ P \arrow[r] \arrow[ur, dashed] & A'' \end{tikzcd}$$

Now $\mathcal{P}(\mathcal{E}(\mathcal{P})) \supset \mathcal{P}$ and "$\mathcal{P}^2 = \mathcal{P}$" hence for so-called <u>closed classes</u> the class of morphisms and the class of objects determine each other. Now "enough projectives" translates as the existence of $P \xrightarrow{f} A$ for any A, where $P \in \mathcal{P}$ and $f \in \mathcal{E}(\mathcal{P})$. Then $f \in \mathcal{E}(\mathcal{P})$ means $A \xrightarrow[\text{epi}]{f} A''$ and $Z(A)_n \xrightarrow{\text{epi}} Z(A'')_n$ (cycles). This is the correct setting for hyperhomology.

Let $X \in {}_\Lambda M$ and take $\Lambda \otimes_R X$. If $P =$ all such objects (a closed family), $\mathcal{E}(\mathcal{P})$ = all morphisms $M \longrightarrow M''$ which have <u>as R-modules</u> a differential inverse (<u>not</u> a Λ-map back); that is, $M \longrightarrow M''$ is split as an R-module map. Then as $\mathcal{P}$ one comes out with all <u>sums</u> of modules $\Lambda \otimes_R X$.

In general if $\mathfrak{A} \xrightarrow{T} C(\mathcal{B})$ and given $(\mathcal{P}, \mathcal{E})$ classes of objects and morphisms in the above relation, such that there are enough $(\mathcal{P},\mathcal{E})$ projectives in $\mathfrak{A}$, then for $A \in \mathfrak{A}$ let $\cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A$ be a resolution (an exact sequence where $P_i \in \mathcal{P}$ and all d_i, ε in $\mathcal{E}$). Apply T; since $C(\mathcal{B})$ is not abelian one cannot take kernels and cokernels. Instead let $F_0 T(P) = T(P_0)$. Then $T(d_1)$ is a morphism in $C(\mathcal{B})$, the category of complexes. Next put $F_1 T(P) = T(P_0) \cup_{T(d_1)} C(T(P_1))$

One then has $\cdots ST(P_3) \xrightarrow[ST(d_2)]{} ST(P_2) \xrightarrow[ST(d_1)]{} C(T(P_1))$, hence we let

suspension (abuse of notation)

$F_2T(P) = F_1T(P) \cup_{ST(d_2)} ST(P_2)$. (Assume A has at least countable coproducts). Then we have $\cdots S^2T(P_4) \to S^2T(P_3) \to F_2T(P)$, so proceed inductively, and get $F_nT(P)$ for all n. Let $F_\infty T(P) = \varinjlim F_nT(P)$ and define $H(F_\infty T(P))$ = derived functor of T relative to $(\mathcal{P}, \mathcal{E})$. This is a filtered thing; what is the spectral sequence?

There is another functor $\mathcal{C}(\mathcal{B}) \xrightarrow{H(\ )} \mathbb{Z}(\mathcal{B})$ = graded objects (complexes with 0 differential). We have $0 \to \underset{\underset{T(P_0)}{\parallel}}{\overset{}{F_0T(P)}} \to F_1T(P) \to ST(P_1) \to 0$; pass to homology:

$$F_0T(P) = E_0 = T(P_0), \qquad ST(P_1) = E_1$$

$$\begin{tikzcd}
H(T(P_0)) \arrow[rr] & & H(F_1T(P)) \arrow[dl] \\
& H(ST(P_1)) \arrow[ul] &
\end{tikzcd}$$

exact Δ.

Next $0 \to F_1T(P) \to F_2T(P) \to S^2T(P_2) \to 0$ gives a homology exact triangle

$$\begin{tikzcd}
\cdot \arrow[rr] & & \cdot \arrow[dl] \\
& H(S^2T(P_2)) \arrow[ul] &
\end{tikzcd}$$

The differentials $H(T(P_0)) \xleftarrow{d_1} HST(P_1) \xleftarrow{d_2} HS^2T(P_2) \leftarrow \cdots$ of the spectral sequence come from the composite HT (with some suspensions), hence $E^2 = LHT(A)$ = left derived functor of HT with respect to $(\mathcal{P}, \mathcal{E})$. This generality contains the classical case; when $T(\mathcal{R})$ is all trivial complexes. But if $T(\mathcal{R})$ is non-trivial the spectral sequence may be highly non-trivial.

In general one cannot find another name for LHT(A), that is it is not very computable. To illustrate, a particular case is

Example R, Λ; $\Omega = {}_{\Lambda}M_g \oplus \Lambda \otimes_R X = \mathcal{P}$. $\Omega \longrightarrow \mathcal{C}({}_R M)$ by $M \longrightarrow R \otimes_{\Lambda} M$. For simplicity assume R is a field. Then

$$H(\Lambda \otimes_R X) = H(\Lambda) \otimes_R H(X) = \text{projective in the category of left } H\Lambda\text{-modules.}$$

One has a functor ${}_{\Lambda}M \longrightarrow {}_{H(\Lambda)}M \longrightarrow {}_{R \otimes H(\Lambda)}M$ and

$$E^2 = \mathrm{Tor}^{H(\Lambda)}(R, H(M)) \Longrightarrow \mathrm{Tor}^{\Lambda}(R, M).$$

In this situation your derived functor of the complex $R \otimes_{H(\Lambda)} M$. Assume lower indices and positive grading: then the spectral sequence is in the third quadrant, or with luck in an octant. This shows you don't want to stick to positive complexes since your algebra may have to be e.g. the cochain algebra (look at chains as the cochains of some space).

By duality the corresponding things for injectives follow.

Example classe of injectives; $\Lambda = C_*(X, R)$ = differential coalgebra. One has left differential comodules over Λ. Take extended things to be injectives, because the cotensor product is or ought to be (depending on how much you assume) a left exact, not right exact, functor.

Note Convergence. In general $F_\infty T(P)$ is a colimit hence one has cocompleteness of filtration. Classical convergence does not follow in general unless you assume the coproduct of monomorphisms is a monomorphism (AB4* of Grothendieck).

MIT Topology Seminar

D. Anderson: <u>Complex Cobordism</u>

We study operations in $\Omega^{2n}_{\mathcal{U}}(X,A) = \lim\limits_{k\to\infty} [(S^{2k}X, S^{2k}A), M\mathcal{U}(n+k,*)]$ or $\Omega^{\mathcal{U}}_{n}(X,A)$ = maps $f: (M,\partial M) \longrightarrow (X,A)$ where M has a complex structure on its normal bundle (all spaces compact) under the equivalence relation: two such are equivalent if a manifold fits between them and a map of it into (X,A) extends f on one end and (say) f' on the other.

See Topology 1-2 years ago for an article by Brown and Peterson on a spectrum $\mathcal{BP}_p$ such that $H^*(\mathcal{BP}_p, Z_p) \cong \mathcal{a}/(\beta)$ where $\mathcal{a}$ is the mod p steenrod algebra and β the Bockstein. Milnor studied $M\mathcal{U}$ and showed $H^*(M\mathcal{U}, Z_p) \cong \oplus\, \mathcal{a}/(\beta)$ while Brown-Peterson showed $M\mathcal{U} = \vee\, \mathcal{BP}_p$ mod p, i.e there is a map giving an injection on homotopy with cokernel consisting of ~~all~~ torsion prime to p. $\mathcal{BP}_p$ is simpler than $M\mathcal{U}$ so we study it instead; cohomology operations in $M\mathcal{U}$ are only those in the factors $\mathcal{BP}_p$, plus factor permutations. However $M\mathcal{U}$ has a natural product structure and $\mathcal{BP}_p$ does not.

$\Omega_{\mathcal{U}}$ is ~~not~~ divisible by every prime but p, while $H^*(\mathcal{BP}_p)$ is. To study $\Omega_{\mathcal{U}}$ by $\mathcal{BP}_p$, we thus use instead

$$\Omega^n_{\mathcal{U}}(X)_{(p)} = \Omega^n_{\mathcal{U}}(X) \otimes \mathbb{Z}[\tfrac{1}{2}, \tfrac{1}{3}, \cdots \widehat{\tfrac{1}{p}} \cdots] = \Omega^n_{\mathcal{U}}(X) \otimes \mathbb{Z}_{(p)}$$

Note $\mathbb{Z}_q \otimes \mathbb{Z}_{(p)} = 0$ if $p \nmid q$ and $\mathbb{Z}_{p^r} \otimes \mathbb{Z}_{(p)} = \mathbb{Z}_{p^r}$, thus we are simply throwing away torsion prime to p. Define a projection

$\mathcal{B} : \Omega^n_u(X)_{(p)} \longrightarrow \Omega^n_u(X)_{(p)}$ called the bordism projection, with

1) $\mathcal{B}$ is natural and stable
2) $\mathcal{B}^2 = \mathcal{B}$
3) $\mathcal{B}(\,\Omega_u^*(pt)_{(p)}\,) = \mathbb{Z}_{(p)}[M_{p-1}, M_{p^2-1}, M_{p^3-1}, \dots]$
$\uparrow$ complex dimension.
4) $\mathcal{B}$ is multiplicative.

Of these properties, 4) is what study of $\mathcal{BP}_p$ alone does not give.

Given $G \xrightarrow{\mathcal{B}} G$ a projection operator, one has a split exact sequence $\ker(\mathcal{B}) \underset{\longleftarrow}{\xrightarrow{id}} G \underset{\text{inclusion}}{\overset{\mathcal{B}}{\rightleftarrows}} \mathrm{Im}(\mathcal{B})$, thus we have (from now on assume all Ω are localized at p)

$$\Omega^n_u(X) \simeq \mathrm{Im}(\mathcal{B}) \oplus \mathrm{Ker}(\mathcal{B}) \qquad \text{natural and stable.}$$

Check this result with the long exact sequence for (X,A), and find $\mathrm{Im}(\mathcal{B})$ is a new cobordism theory called $\mathcal{BP}^n(X) = \mathcal{B}(\Omega^n_u(X))$.

By 4), $\mathcal{B}(x\cdot y) = \mathcal{B}(x)\,\mathcal{B}(y)$, hence $\mathrm{Im}(\mathcal{B})$ is closed under multiplication; therefore $\mathcal{BP}^*(X) \hookrightarrow \Omega^*_u(X)$ is a sub<u>ring</u> (and quotient <u>ring</u>) of $\Omega^*_u(X)$.

Define $\mathcal{B}$ as follows: let $T(n) \xrightarrow{\alpha(n)} U(n)$ be the torus in the unitary group, as the set of diagonal matrices with the entries of absolute value 1: $T(n) = S^1 \times \dots \times S^1 = T(1) \times \dots \times T(1)$. Then

$$H^*(BT(1)) \otimes \dots \otimes H^*(BT(1)) = H^*(BT(n)) \xleftarrow{B\alpha(n)} H^*(BU(n))$$
$$\wr\!\wr$$
$$\mathbb{Z}(t_1, \dots t_n)$$

where B stands for "classifying space of"; $BT(1) = CP^\infty$, and $H^*(BT(1)) = \mathbb{Z}[t]$ with $\dim t = 2$. One shows $\mathrm{Im}\, B\alpha(n)^*$ is the symmetric functions in $t_1 \cdots t_n$, and $B\alpha(n)^*$ is injective; using the Cartan formula and the inclusion, one deduces Steenrod operations on $BU(n)$. This gives much information on $MU(n)$, for

$$\begin{tikzcd}
H^*(BT(n)) & \tilde{H}^*(MT(n)) \arrow[l, "\substack{\text{0-section}\\ \text{in bundle}}"'] & \tilde{H}^*(MU(n)) \arrow[l, "M\alpha(n)^*"'] \\
\boxed{t_1\cdots t_n\cdot\Theta} & & \\
 & \boxed{\Theta}\in H^*(BT(n)) \arrow[uu, phantom, sloped, "\cong"] \arrow[ul, bend right] & H^*(BU(n)) \arrow[l] \arrow[uu, phantom, sloped, "\cong"]
\end{tikzcd}$$

This works the same replacing H^* with Ω^*_u:

$$\begin{tikzcd}
 & \Omega^*(BT(n)) & \tilde{\Omega}^*(MT(n)) \arrow[l, "\text{0-section}"'] & \tilde{\Omega}^*(MU(n)) \arrow[l, "M\alpha(n)^*"'] \\
 & \boxed{t_1\cdots t_n\cdot SF(\Theta)} & & \\
\Omega^*(BT(1))\hat{\otimes}_{\Omega^*}\cdots\hat{\otimes}_{\Omega^*}\Omega^*(BT(1)) \arrow[r, phantom, "\cong"] & & \boxed{\Theta}\in\Omega^*(BT(n)) \arrow[uu, phantom, sloped, "\cong"] \arrow[ul, bend right] & \Omega^*(BU(n)) \arrow[l, "B\alpha(n)^*"'] \arrow[uu, phantom, sloped, "\cong"]
\end{tikzcd}$$

completed $\otimes$

$$\Vert$$

$\Omega^*[[t_1,\cdots,t_n]]$ where $t_i\in\Omega^2(BT(1))$ is the usual homeomorphism $BT(1)\cong MT(1)=MU(1)=CP^\infty$. Once again $\operatorname{Im}(B\alpha(n)^*)=$ symmetric power series, and $\Theta\longrightarrow t_1\cdots t_n\cdot\Theta$. The Cartan formula shows all operations in $\hat{\otimes}$, once you know there is no torsion in $MU(n)$. Now $\Omega^*(MU(n))=\{$operations in complex cobordism$\}$ of course, where an operation corresponds to the image under itself of the identity class in $\Omega^*(MU(n))$.

Let $\omega=(i_1,\cdots,i_n)$, $i_1\geq\cdots\geq i_n$. Then $c_\omega\in\Omega^*(BU(n))$ is defined by $B\alpha(n)^*(c_\omega)=$ smallest symmetric function containing $t_1^{i_1}\cdots t_n^{i_n}$ (or 0 if ω has more than n entries).

Define s_ω by $\quad s_\omega(\text{identity})=\Phi(c_\omega)\in\tilde{\Omega}^*(MU(n))$

(we now drop the n since we are in a stable situation). Degree $s_\omega=2i_1+\cdots+2i_n=2|\omega|$. There is a map related to the bundle sum,

$BU(n) \times BU(m) \xrightarrow{\mu_{m,n}} BU(n+m)$ (and a corresponding map for tori) with the property that

$$\mu_{n,m}(c_\omega) = \sum_{\omega'+\omega''=\omega} c_{\omega'} \otimes c_{\omega''}$$

where the sum $\omega' + \omega''$ is all entries of ω' and of ω'' rearranged in decreasing order. Also, $s_\omega \circ s_{\omega'}$ is a linear combination of $s_{\omega''}$, by a complicated formula. Hence $s_\omega(ab) = \sum s_{\omega'}(a)\, s_{\omega''}(b)$ since multiplication is the Thomification of $\mu_{m,n}$. The s_ω are a basis for all symmetric functions, and are closed under composition. One must figure out an Adem formula. To get all operations, one takes formal sums $\sum_{\text{infinite}} a_\omega s_\omega$ where $a_\omega \in$ groups of a point and $\deg a_\omega s_\omega =$ constant. To every operation corresponds a symmetric function. Let $SF(\theta)$ be the symmetric function associated to θ for $\theta \in \Omega^*(MU)$ any operation.

Milnor showed there are manifolds $M_i \in \Omega^{-2i}$ such that

$$\underbrace{\underbrace{s_{(i)}}_{\substack{\text{operation}\\ \text{of degree } 2i}} \underbrace{(M_i)}_{\in \Omega^{-2i}}}_{\in \Omega^0 = \mathbb{Z}} = \begin{cases} 1 & \text{if } i+1 \text{ not a power of a prime, } p. \\ p & \text{if } i+1 = p^k \text{ for some } k, p. \end{cases}$$

(this takes place localized in p, remember). Further, a collection of manifolds will be a set of generators of $\Omega^*(pt)$ iff this condition is satisfied (See Conner & Floyd and their Milnor reference).

Let $\mathcal{B}^{(1)}$ be described by

$$SF(\mathcal{B}^{(1)}) = \prod_{i=1}^{n} (1 - M_1 t_i) \qquad \text{large } n.$$

There is no indeterminacy in M_1; there is only one generator of Ω^{-2}. Fix it by the conditions of Milnor. This product $\prod_1^n$ is multiplicative since its form in $n+m$ variables is that of n variables times that of m variables. (it is the multiplicative sequence associated to $(1-M_1t)$). Look at $\mathcal{B}^{(1)} \big| \Omega^*(pt) \equiv id \quad \mathrm{mod}\,(M_1)$ (ideal). So $\mathcal{B}^{(1)}$ is the identity on all indecomposables but M_1. Check that $\mathcal{B}^{(1)\,2} = \mathcal{B}^{(1)}$. Pick arbitrarily M_2' by Milnor's condition.

(*) Let $M_2 = \mathcal{B}^{(1)}(M_2')$. Then M_2 is indecomposable and invariant under $\mathcal{B}^{(1)}$. Put

$$SF(\mathcal{B}^{(2)}) = \prod_1^n (1-M_1t_i)(1-M_2t_i^2).$$

The condition (*) is exactly what you need to get a projection operator. There is no indeterminacy in M_2 since one can only add a multiple of M_1^2 to M_2' — and $\mathcal{B}^{(1)}$ kills M_1^2. Thus

$$SF(\mathcal{B}^{(k)}) = \prod (1-M_1t_i)\cdots(1-M_kt_i^k)$$

for $k < p-1$ ($s_{(i)}(M_i) = 1$ is necessary to have a projection). In dimension $2(p-1)$ put $SF(\mathcal{B}^{(p-1)}) = SF(\mathcal{B}^{(p-2)})$. In dimension $4p-2$, one can vary the generating manifold by the square of a $2p-2$ manifold, thus $\mathcal{B}^{(i)}$ stops being unique. Thus

$$SF(\mathcal{B}) = \prod_{\substack{1\le i\le n \\ k+1 \ne \text{power of } p}} (1-M_kt_i^k). \qquad (\ddagger)$$

One checks this is a projection operator, and that it is multiplicative by ($\ddagger$).

This reduces to the question, what are the cohomology operations of the $\mathcal{BP}$-theory? Let $SF(\mathcal{P}^i) = \sigma^i(t_1^{(p-1)}, \ldots, t_n^{(p-1)})$. Then

$\mathcal{P}^k(u\,v) = \sum \mathcal{P}^i(u)\,\mathcal{P}^j(v)$ and one can show

$\{ \mathcal{B} \circ (\text{admissible monomials}) \circ \mathcal{B} \}$ form a basis over $\mathcal{BP}^*(pt)$ (left multyplication) for the operations in $\mathcal{BP}^*$ theory

Questions to be answered here: what are the corresponding Adem formulas? The Adem formulas mod p are correct; the binomial coefficients as integers do not work; one must evaluate $\mathcal{P}^i$ on some well-chosen manifolds. However, $\mathcal{P}^1 \mathcal{P}^1 \equiv 2 \mathcal{P}^2 \bmod \overline{\Omega}^*$ = negative dimensional groups of a point.

 2/19/68

MIT Topology Seminar

Armand Wyler: Actions of Spinor groups on spheres

Hurwitz-Radon show there are just $k-1$ orthogonal vector fields on S^{n-1} where if $n = \text{odd} \cdot 2^{c(n)} \cdot 16^{d(n)}$, $0 \le c(n) \le 3$, then $k = 2^{c(n)} + 8 \cdot d(n)$. These are cross-sections of the fibration $O(n)/O(n-k) \to O(n)/O(n-1)$. Consider the finite group Γ_{k-1} generated by $e_0, e_1, \cdots, e_{k-1}$ with relations $e_i e_j = e_0 \cdot e_j e_i$ if $1 \le i \ne j \ge 1$ and $e_i^2 = e_0$, $e_0^2 = \text{identity}$. The order of Γ_{k-1} is 2^k. To get orthonormal vector fields represent Γ_{k-1} as a set of $n \times n$ orthogonal matrices where $e_0 \longrightarrow -E$, $e_i \longrightarrow A_i$. Then $X_i = A_i x$ ($x \in S^{n-1} \subset \mathbb{R}^n$!) gives $k-1$ orthonormal vector fields on S^{n-1}. Compute the Lie Algebra of these vector fields: $[X_1, X_2] = -2A_1A_2x$, and so on; to see this, associate to $x = (\alpha_1 \cdots \alpha_n)$ the symbol $\sum \alpha_i \frac{\partial}{\partial x_i}$. Then $X_r \to \sum \alpha_i^{(r)} \frac{\partial}{\partial x_i}$ where $\alpha_i^{(r)} = \sum_{j=1}^{n} \alpha_{ij}^{(r)} x_j$. According to Chevalley the symbol of the bracket is then $\sum_{ij} \left(\alpha_i^{(2)} \frac{\partial \alpha_0}{\partial x_i} - \alpha_i^{(1)} \frac{\partial \alpha_j^{(2)}}{\partial x_i} \right) \frac{\partial}{\partial x_j}$, so $[X_1, X_2] = (A_2A_1 - A_1A_2)x = -2A_1A_2x$. (We need $[\,,\,]$ to study the manifolds tangent to these vector fields). Further $[[X_1, X_2], X_i] = 0$ if $i \ne 1, 2$ and $= -2A_2$ if $i = 1$, etc. Thus we remain in the region of products of ≤ 2 matrices. Consider in the algebra of $(n \times n)$ matrices the vector space M spanned by $A_1, \cdots, A_{k-1}$; and that M_2 spanned by the products $A_i \cdot A_j$. One computes these and finds both are equal to the Lie algebra of $SO(k-1)$ = the Lie algebra of $Spin(k-1)$.

Now find manifolds tangent to those vector fields:

<u>Thm</u> There is an integral manifold tangent to these $(k-1)$ vector

fields, $\Leftrightarrow$ the vector space $V(x)$ spanned by $X_1 \cdots X_{k-1}$ and all their brackets, has constant dimension q.

Look at the problem from the point of view of actions of Lie groups on spheres. M_2 = Lie algebra of $SO(k-1) \subseteq (n \times n)$ matrices $\subset SO(n)$.

The simply connected Lie group having this as Lie algebra is $\mathrm{Spin}(k-1) \subset SO(n)$. (this is an imbedding since the kernel would be a central discrete subgroup; but the only central discrete subgroup of $\mathrm{Spin}(k-1)$ is carried nontrivially, to e_0). Now $SO(n)/SO(n-1) = S^{n-1}$ so the non-trivial embedding gives an action of $SO(k-1)$ on S^{n-1}. Note $\mathrm{spin}(k-1) \cap SO(n-1)$ = isotropy group at $\pi(e)$ $(SO(n) \xrightarrow{\pi} S^{n-1})$ thus the orbit of $SO(n-1)$ is $\mathrm{Spin}(k-1) / \mathrm{Spin}(k-1) \cap SO(n-1)$.

The last time this action is transitive is Spin 9 on S^{15}, and the orbit is $\mathrm{Spin}\,9 / \mathrm{Spin}\,7$. (but the inclusion $\mathrm{Spin}\,7 \subset \mathrm{Spin}\,9$ is not the usual one). For this case $15+1 = 16 = 16^1$, $\rho(n) = 2^0 + 8 \cdot 1 = 9$. Thus there are 8 vector fields; $\mathrm{Spin}\,9 \supset \mathrm{Spin}\,8$ and Spin 8 is not transitive but Spin 9 is.

Next we discuss free action of S^1 or S^3 on homotopy spheres. (see a paper by Hsiang with that title). According to Borel, S^1 and S^3 are the only compact connected Lie groups acting freely on spheres (Sur le cohomologie des espaces fibrés principaux, Ann. Math 1953). Hsiang proved that from the point of view of homotopy these are nothing but Hopf fibrations as follows: Take a free action of S^1 on Σ^n. Hsiang states such an action is always a principal fibration. One checks easily that $n = 2k+1$ (for S^3, $n = 4k+3$) for some integer k.

Thm The principal fibration $S^1 \hookrightarrow \Sigma^{2k+1} \to \Sigma^{2k+1} / \text{action on } \Sigma^{2k+1} \equiv \varphi$

is homotopically equivalent to the Hopf bundles

$$\nu : S^1 \longrightarrow S^{2k+1} \longrightarrow \mathbb{C}P^k$$

$$\left(\text{or} \qquad S^3 \longrightarrow S^{4k+3} \longrightarrow \mathbb{H}P^k\right)$$

Proof The homologies of Σ^{2k+1}/φ and $\mathbb{C}P^k$ are identical by Serre; thus we get a classifying map $\Sigma^{2k+1}/\varphi \longrightarrow \mathbb{C}P^\infty$. Since $\mathbb{C}P^k = 2k$-skeleton of $\mathbb{C}P^\infty$ this map deforms to $\Sigma^{2k+1}/\varphi \longrightarrow \mathbb{C}P^k$. But the Hopf map is universal in this dimension $(\leq 2k)$ thus $\nu = \xi$. //

Hsiang's main point is that up to homotopy one has only the Hopf bundle. This is far from true including differentiable structure however. Let X^k be a manifold of homotopy type of $\mathbb{C}P^k$ and $g : X^k \longrightarrow \mathbb{C}P^k$ a homotopy equivalence. The total space of the induced bundle $g^{-1}\nu$ is a homotopy sphere with free action of S^1. But there are infinitely many different differentiable manifolds X^k (See Browder-Novikov) so there are infinitely many inequivalent actions of S^1 on Σ^{2k+1}.

(15-1)

3/4/68

MIT Topology Seminar

Sandy Blank (Northeastern Univ.) : Immersions of S^1 in $\mathbb{R}^2$

The interesting question about an immersion $S^1 \xrightarrow{f} \mathbb{R}^2$ is whether it extends to the disk:

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, hook] & \mathbb{R}^2 \\ D^2 \arrow[ur, "F"'] & \end{tikzcd}$$

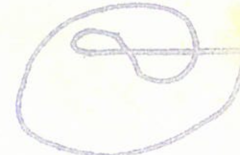
Two invariants are associated with an immersion f:

1) Tangent winding number $TWN(f) = RN(f)$, invariant w.r.t. (with respect to) regular homotopy

$$= \text{degree}\left[x \longrightarrow \frac{f_x'(1)}{|f_x'(1)|} \right]$$

2) Winding number $WN_p(f)$ for any $p \in \mathbb{R}^2$, invariant w.r.t. homotopy in $\mathbb{R}^2 - \{p\}$, or w.r.t. change of p within one connected component of $\mathbb{R}^2 - f$.

A sufficient condition that f extend is that f be an imbedding. This is not necessary [1]. Consider an extension $F: D^2 \to \mathbb{R}^2$ of $f = F|_{S^1}$ where F is an immersion. Then there is a $\delta > 0$ with $F|_{|z| \le \delta}$ an imbedding.



[1]

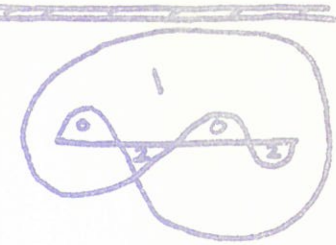
(since F immersion). Let $\tilde{F} = F|_{\delta \le |z| \le 1}$, a regular homotopy between $F|_{|z|=\delta}$ and f. Therefore it is necessary, for an extension to exist, that f be regular homotopic to an embedding; thus $TWN(f) = \pm 1$. We assume now all f have $TWN(f) = +1$.

Consider an immersion f, extension F which is an immersion, and $\tilde{F}$ as above, with $f_0 = F|_{|z|=\delta}$ an imbedding. Choose $p \notin \operatorname{Im} f_0$. Either $WN_p(f_0) = 0$, p outside f_0; or $WN_p(f_0) = 1$, p inside f_0. Thus $WN_p(f_0)$ = # of inverse images of p by an extension F_0 of f_0. This is in fact invariant under regular homotopy. So for any immersion f which extends to F, $WN_p(f) = \#(F^{-1}(p))$.

[2] is an example showing $WN = 1$ is not sufficient.

[2]

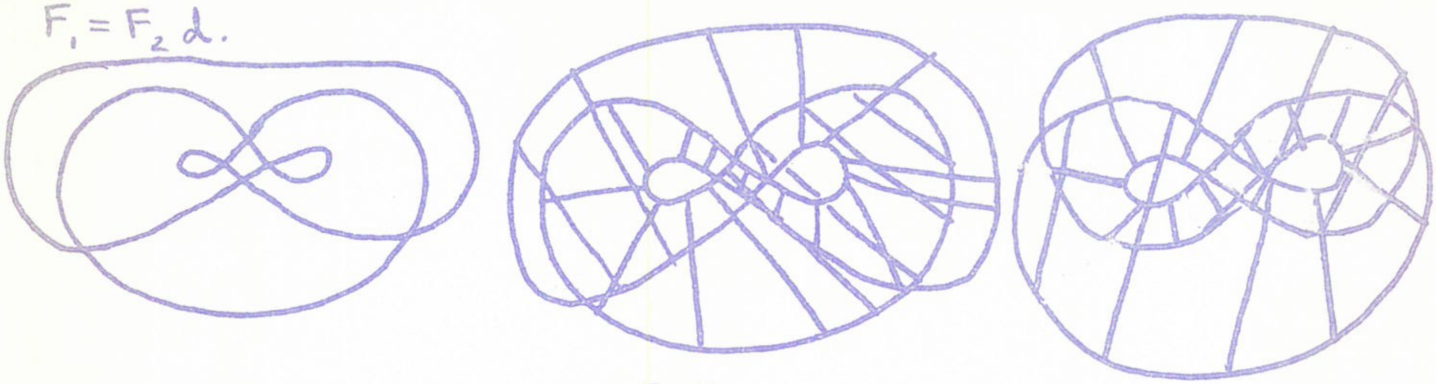
Prop. If f extends, $WN(f) \geq 0$ for all $p \notin \mathrm{Im} f$. //
(recall $TWN(f) = +1$). This condition is again not sufficient [3].



[3]

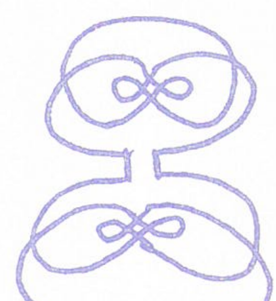
Think back about the extension homotopy. It remains in regions where $WN \neq 0$, so there is another a general condition which I won't state.

Either Milnor or Stallings gave an example [4] of a curve with more than one extension, where $F_1, F_2 : D^2 \longrightarrow \mathbb{R}^2$ are equivalent if there exists $d : D^2 \longrightarrow D^2$ with $F_1 = F_2 d$.



[4]

One shows that $F_1^{-1}(f)(S^1)$ and $F_2^{-1} f(S^1)$ are not diffeomorphic. To get a curve with 4 extensions one can join 2 of these [5] and in general joining n curves [4] gives a curve with 2^n different extensions, for any n.

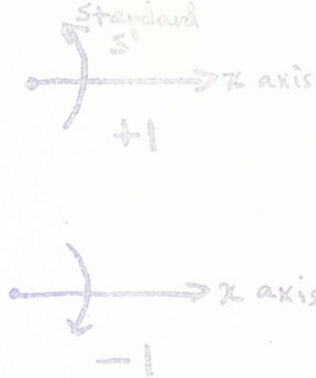


[5]

There exists a curve with any number of extensions (see below). We now construct a machinery to deduce from a curve whether it extends, how many extensions it has, and how they are constructed. Suppose f is in general position (no self tangencies, no triple points). Then f divides $\mathbb{R}^2$ into an unbounded region plus finitely many bounded regions $P_1 \cdots P_n$. Choose $p_i \in P_i$ for each i. Note $\pi_1(\mathrm{Im} f) = \pi_1(R_2 - \bigcup_i p_i)$ by retraction. From each p_i draw a ray a_i from p_i to ∞. [6].

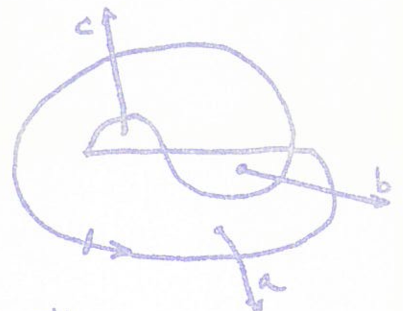
such that $a_i \cap a_j = \emptyset$ and a_i is transversal to $\mathrm{Im} f$ (not tangent, not through any double point). With each intersection of a_i with $\mathrm{Im} f$ we associate an orientation ± 1 according to [7].
Let $w(f) \in FG(a_i)$ be a word in the free group on the a_i, obtained by listing the a_i's with proper exponent as they intersect $\mathrm{Im} f$ (traverse f once around).

[7]



<u>Example</u>

[8]



$w(f) = abc^{-1}bc$

Then $w(f) \in [f] \in \pi_1(\mathrm{Im} f) \simeq FG(a_i)$.

We have 2 operations on words:

<u>Operation C</u> consists of deleting any positive letter. A <u>cancellation</u> is any sequence of C's such that the resulting word is the identity. Two cancellations are different if the letters deleted are different (including position but not order of deletion).

<u>Operation G</u> is deleting any sequence of the form $a^{-1}pa$ or apa^{-1} where p is a string of positive letters. A <u>grouping</u> is a sequence of G's such that the resulting word is a string of positive letters. Two groupings differ if the sequences grouped are different.

<u>Thm</u> The following are equivalent:

1) f extends
2) f has a cancellation
3) f has a grouping. //

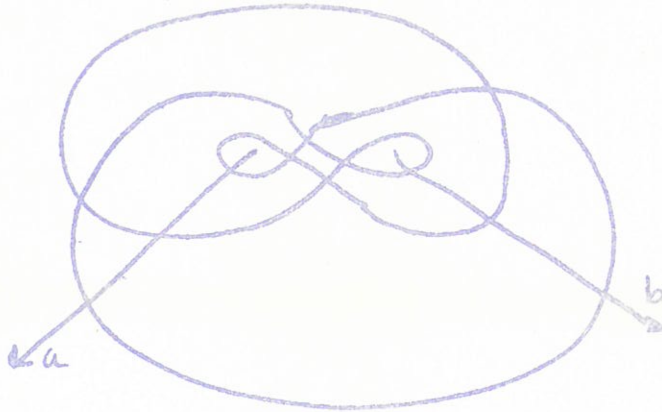
<u>Note</u> one can ignore those letters which never occur with a negative exponent. One gets a <u>reduced word</u> by this and also omitting any pair aa^{-1} or $a^{-1}a$; also if $w(f) = a^{-1} \cdots a$ or $a \cdots a^{-1}$ one can reparametrize f so that the a, a^{-1} drop out.

<u>Thm</u> Let $w(f)$ be the reduced word. The following are equivalent:

1) f has n extensions
2) w(f) has n cancellations
3) w(f) has n groupings.

Further from the groupings (cancellations) one can construct the extensions.

[9]

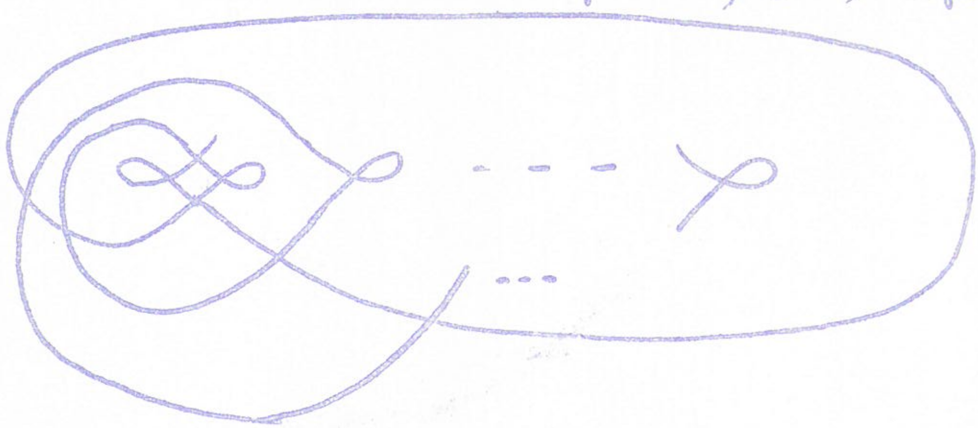


$w(f) = \underbrace{a\,b}\,a^{-1}\,\underbrace{b\,a}\,b^{-1}$

2 groupings

A curve with n extensions : one loop left, (n-1) loops right

[10]



MIT Topology Seminar 3/11/68

John Harper: <u>Stable homotopy of Eilenberg-MacLane Spaces.</u>

<u>Th</u> (Freudenthal) $[X,Y] \to [\Sigma X, \Sigma Y]$ is an isomorphism if dim X $\leq 2 \cdot$ connectivity (Y). //

Define $\pi_i^S(Y) = \varinjlim_{n\to\infty} [S^{n+i}, \Sigma^n Y]$. The idea is that π_i^S is a good first approximation to π_i, and may be easier to compute. There is an (Adams) spectral sequence $E_r^{s,t}$ with

1) $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\tilde{H}^*(Y), Z)$

2) E_∞ = graded object associated to a filtration of $\pi^S_{t-s}(Y)$

3) d_r $(r \geq 2)$ corresponds to r^{th}-order cohomology operations.

4) Spectral sequence for a sphere acts on $E_2^{s,t}(Y)$ for any Y; there is a composition $E_2(S) \otimes E_2(Y) = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2) \otimes \mathrm{Ext}^{u,v}(Y, \mathbb{Z}_2) \to \mathrm{Ext}_A^{s+u,t+v}(Y, \mathbb{Z}_2)$, called the Yoneda product, which corresponds to composition of maps in $E_\infty = gr(\pi^S(Y))$.

There is an element $h_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}_2, \mathbb{Z}_2)$ corresponding to $S^n \to S^n$ map of degree 2; and in E_∞, multiplication by h_0 corresponds to multiplicati[on] by 2. Also, d_r is a derivation.

5) The spectral sequence is natural with respect to maps of spaces.

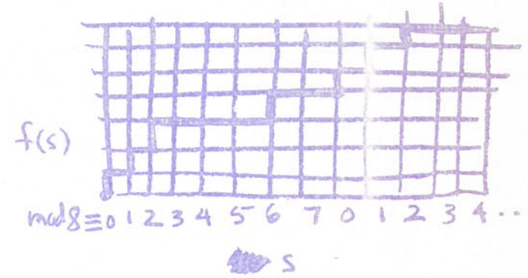
$d_r : E_r^{s,t} \to E_r^{s+r, t+r-1}$.

Let $Y = K(\mathbb{Z}_2, n)$ and $n \geq 10$. I computed $\pi^S_{2n+k}(K(\mathbb{Z}_2,n))$, $0 \leq k \leq 7$. In this range, a $\mathbb{Z}_2$-basis of the cohomology is independent of n. The groups are 0 up to π^S_{2n-1} except the fundamental class in dimension n, since the cohomology in that range is a free A-module. On the next page appears a table of results. The notation G_q means an unknown group of order q; various G_q are not necessarily the same. Also ? means a group about which something is known but a complete description is not.

$\equiv n \bmod 8$ \ k	0	1	2	3	4	5	6	7
0	Z_2	Z_2	Z_2+Z_2	Z_2+Z_2	Z_2	Z_2	Z_2+G_4	$Z_2+Z_2+G_4$
1	Z_2	Z_4	Z_2	Z_2	Z_2	Z_2+Z_2	Z_2+G_4	Z_2+G_4
2	Z_2	Z_2	Z_2	Z_2	G_4	Z_2+G_4	$Z_2+?$	$Z_4+?$
3	Z_2	Z_4	Z_2	G_4	G_8 or G_4	G_4 or G_2	Z_2	Z_2+Z_4
4	Z_2	Z_2	Z_2+Z_2	Z_2+Z_2	Z_2	Z_2	Z_2+G_4	$Z_2+\cancel{Z_2}G_4$
5	Z_2	Z_4	Z_2	Z_2	Z_2	Z_2+Z_2	Z_2+G_4	Z_2+G_4
6	Z_2	Z_2	Z_2	Z_2	G_4	Z_2+G_4	$Z_2+?$	$Z_4+?$
7	Z_2	Z_4	Z_2	G_4	G_8 or G_4	G_4 or G_2	Z_2	Z_2+Z_4

This table is periodic (a conjecture which is almost certain) of period 2^i in n, for rows $k = 0, 1, \cdots, 2^i - 1$; for each i. Horizontal periodicty has not been observed.

<u>Adams' Edge Theorem</u> Let $E \subset A$ be the subalgebra generated by 1 and Sq^1. Let $i : E \hookrightarrow A$. Then $i^* ; \mathrm{Ext}_A^{st}(M, \mathbb{Z}_2) \longrightarrow \mathrm{Ext}_E^{st}(M, \mathbb{Z}_2) = H^{t-s}(M, Sq^1)$. Adams shows i^* is an isomorphism for $t < m + f(s)$ where $M_t = 0$ for $t < m$ and $f(s)$ is Adams' function:

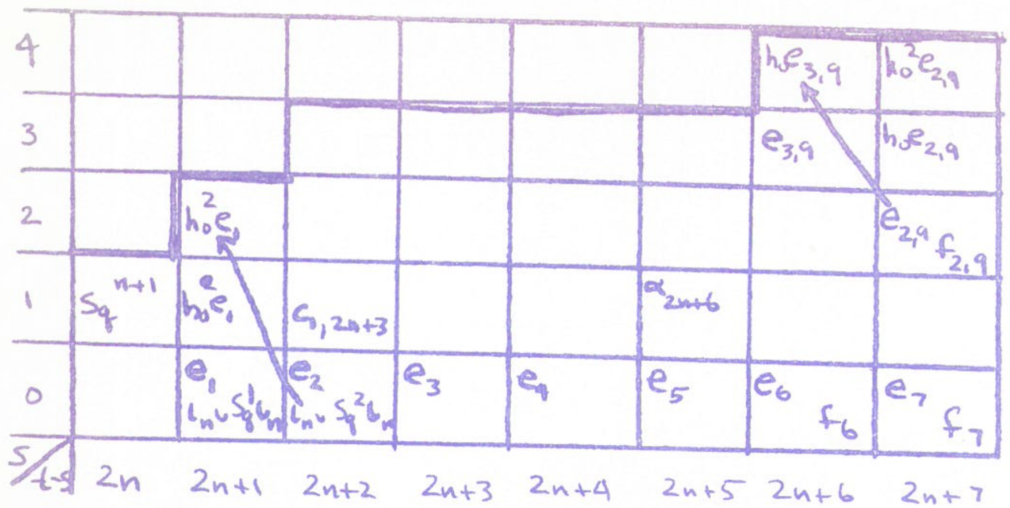


Let $B(n) = \{\theta \in A \mid \theta(i_n) = 0\}$

$$0 \to \frac{A}{B(n)}(n) \longrightarrow H^*(\mathbb{Z}_2, n) \longrightarrow \frac{H^*(\mathbb{Z}_2, n)}{A(\iota_n)} \to 0.$$

Let the right hand module be M. It is $(2n)$-connected and $B(n)(n)$ is $(2n)$-connected. The long exact sequence in Ext shows $\mathrm{Ext}\left(\frac{A}{B(n)}\right) = \mathrm{Ext}(B(n))$ so $H^*(\mathbb{Z}_2, n)$ is $2n$-connected also.

In the following table $n \equiv 5 \bmod 8$.



$Sq^1 Sq^{n+1} + Sq^1 \varphi = 0$ where φ is the operation associated with $\iota_n \cup Sq^1 \iota_n$.

An interesting case is $e_{2,9} \leftrightarrow (Sq^7 Sq^1 + Sq^5 Sq^2 Sq^1) Sq^6 + Sq^7 (Sq^6 Sq^1 + Sq^1 \varphi)$
$+ Sq^3 (Sq^8 Sq^3 + Sq^5 \varphi + Sq^3 Sq^1 \theta)$ where $\theta = \iota_n \cup Sq^2 \iota_n$.

A relation of 4th order in $(4, 2n+6)$ actually decomposes a secondary operation; hence it does not give a quaternary operation.

<u>Universal Example</u> for $e_{2,9}$.

$$
\begin{array}{ccccc}
 & & E_m \ \text{fibration} & & \\
 & & \downarrow & & \\
\Sigma^r K(Z_2,5) & \longrightarrow & K(Z_2,m) \times K(Z_2,m+6) \times K(m+7) & \longrightarrow & K(Z_2,m+6) \times K(Z_2,m+7) \times K(Z_2,m+11) \\
\overset{?}{\Sigma^r}\iota_5 & \longleftarrow & \iota_m & & \\
 & & Sq^6 \iota_m & \longleftarrow & \iota_{m+6} \\
\iota_5 \cup Sq^1 \iota_5 & \longleftarrow & \iota_{m+6} & & \\
\iota_5 \cup Sq^2 \iota_5 & \longleftarrow & \iota_{m+7} \qquad Sq^6 Sq^1 \iota_m + Sq^1 \iota_{m+6} & \longleftarrow & \iota_{m+7} \\
 & & Sq^8 Sq^3 \iota_m + Sq^5 \iota_{m+6} + Sq^3 Sq^1 \iota_{m+7} & \longleftarrow & \iota_{m+11}
\end{array}
$$

We must identify some class in $\Sigma^r K(Z_2,5)$ (some r). We choose m so large that $\pi^S(E_m) = \pi(E_m)$, which is known.

MIT Topology Seminar

John Milnor (Princeton): Curvature and the fundamental group

The problem is to relate the differential geometry of a manifold with its topology. Let G be a finitely generated group throughout. Let $g_1, \cdots, g_k$ be preferred generators of G. Let $\gamma(s)$, the growth function of G, be defined as the number of distinct group elements of the form $g_{i_1}^{\pm 1} g_{i_2}^{\pm 1} \cdots g_{i_r}^{\pm 1}$ where $r \leq s$, for each natural number s. We investigate the asymptotic behavior of $\gamma(s)$ as a measure of how big G is.

Lemma $\gamma(s+t) \leq \gamma(s)\gamma(t)$ //

So in particular for any natural number k, $\gamma(k\cdot s) \leq \gamma(s)^k$; $\gamma(k) \leq \gamma(1)^k$.

Examples ① G finite; γ bounded.

② G infinite cyclic; $\gamma(s) = 2s+1$

③ G free abelian on 2 generators $\gamma(s) = 2s^2 + 2s + 1$

" " " on k " $\gamma(s) = a_0 s^k + a_1 s^{k-1} + \cdots$; a_i rational

④ G a free group on 2 generators: $\gamma(s) = 2\cdot 3^s - 1$.

Question: to what extent is γ an invariant of the group?

Lemma If $\{g_1', \cdots, g_s'\}$ is a second set of generators, yielding a growth function $\gamma'(t)$ for G, then there exists $k > 0$ such that $\gamma(s) \leq \gamma'(ks)$ and $\gamma'(t) \leq \gamma(kt)$. Thus if γ increases as a polynomial (resp., exponential), so does γ'. //

All examples found so far have been polynomial or exponential. Are all groups thus?

Lemma $1 \leq \lim_{s\to\infty} \gamma(s)^{1/s} < \infty$.

Proof Let t_0 and s be integers >0 and choose k such that $kt_0 \leq s \leq (k+1)t_0$. For fixed t_0,

$$\gamma(s) \leq \gamma((k+1)t_0) \leq \gamma(t_0)^{k+1} \leq \gamma(t_0)^{\frac{s}{t_0}+1} \quad \left(k < \frac{s}{t_0}\right).$$

raise to $1/s$ power: $$\gamma(s)^{1/s} \leq \gamma(t_0)^{1/t_0} \cdot \gamma(t_0)^{1/s}.$$

$$\Rightarrow \limsup \gamma(s)^{1/s} \leq \gamma(t_0)^{1/t_0} \quad \text{for all } t_0.$$

$$\Rightarrow \limsup \gamma(s)^{1/s} \leq \inf \gamma(t)^{1/t} \leq \liminf \gamma(t)^{1/t}$$

hence $\gamma(s)^{1/s}$ converges to a finite number.

How is this connected with **curvature**? Consider a Riemannian manifold with distance d. Then $\{y \mid d(y, x_0) \le r\}$ is the ball of radius r about x_0; this ball has Riemannian volume $\omega_n r^n$ + higher terms. Loosely speaking, "positive curvature" means the higher terms are negative; the ball has less than normal volume; negative curvature means they are positive.

<u>Thm</u> (Günther) In a complete 1-connected Riemannian manifold suppose all sectional curvatures K have $K_0 \le K \le K_1$ (constants). Then

$$V_{K_1}(r) \le V(r) \le V_{K_0}(r)$$

where $V_{K_i}(r)$ = value of $V(r)$ in a space of constant curvature K_i (e.g. a sphere has constant positive curvature). //

<u>Thm</u> (R.L. Bishop) It suffices to assume

$$\text{Mean curvature} \geqslant K_0 \cdot (n-1)$$

to conclude $V(r) \le V_{K_0}(r)$. //

Combine these theorems with the universal covering space of a group:

<u>Thm</u> ① If mean curvature $\geqslant 0$ everywhere, then $\gamma(s) \le \text{constant} \cdot s^n$, where n = dimension. (G = fundamental group of M = manifold)

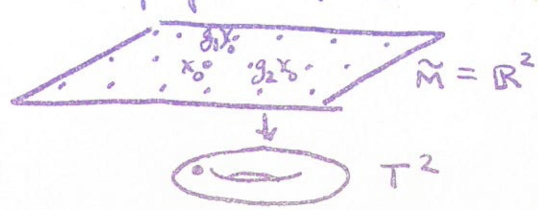
② If the sectional curvature $K < 0$ always, $\gamma(s) \geqslant (1+\varepsilon)^s$. //

There exists a theorem that if the curvature is always >0, $\gamma(s)$ is bounded.

The first statement is easy to prove and more interesting; the second harder, and much is already known about that case.

<u>Proof of ①</u> Let $\tilde{M} \longrightarrow M$ be the simply connected covering space of the Riemannian manifold M. Mean curvature $\geqslant 0 \Rightarrow V(r)$ = Volume of $N_r(x_0) \le V(\text{euclidean space}, r) \le \omega_n \cdot r^n$ by Bishop & Günther.

Let $g \in \pi_1(M)$ = covering transformations of M. Think of the points $g(x_0)$ (where $x_0 \in \tilde{M}$) as a lattice in $\tilde{M}$; we want to say the number of points $g(x_0)$ in a volume is proportional to the volume. (Think of the torus T^2:



).

Then let $\mu = \mathrm{Max}\ d(x_0, g_i(x_0))$ as g_i runs over preferred generators for $\pi_1(M)$. Let $N_r(x_0) = \{y \in \tilde{M} \mid d(y, x_0) \le r\}$. Look at $N_{s\mu}(x_0)$: It contains at least $\gamma(s)$ translates $g(x_0)$, since $d(x_0, g_{i_1}^{\pm 1} \cdots g_{i_s}^{\pm 1} x_0) \le s \cdot \mu$. Choose ε such that the balls of radius ε about lattice points are all disjoint. Consider $N_{s\mu+\varepsilon}(x_0)$: $V(s\mu+\varepsilon) \ge \gamma(s) V(\varepsilon)$ since for each lattice point in $N_{s\mu}(x_0)$ there is contained in $N_{s\mu+\varepsilon}(x_0)$ an entire ε-neighborhood. Hence $\omega_n (s\mu+\varepsilon)^n \ge V(s\mu+\varepsilon) \ge \gamma(s) V(\varepsilon)$,

or
$$\frac{\omega_n}{V(\varepsilon)} (s\mu+\varepsilon)^n \ge \gamma(s)$$

or writing $s\varepsilon$ for ε, $(\omega_n / V(\varepsilon)) \cdot s^n (1+\varepsilon)^n \ge (\omega_n / V(\varepsilon)) (s\mu+\varepsilon)^n \ge \gamma(s)$ ///

The proof of ② is a little harder but no less elementary.

Examples An orientable surface is described by its genus

genus 0 (sphere) $\gamma(s) = 1$

genus 1 $K \equiv 0$ if you choose metric correctly; $\gamma(s) = 2s^2 + 2s + 1$

genus ≥ 2 has metric with constant negative $K < 0$;

$\pi_1(M) =$ free group on $\{a_1, \cdots, a_j; b_1, \cdots, b_j\} / [a_1, b_1] \cdot \cdots \cdot [a_j, b_j] = 1$.

So $\pi_1(M) \supset$ free group on $2j-1$ generators $\supset$ free group on 2 generators, thus γ has exponential growth.

These spaces have a metric with K of constant sign. A manifold M without K of constant sign is the following ~~nil~~ manifold : the set of matrices $\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a,b,c \in \mathbb{R} \right\}$ is a nilpotent Lie group (topologically, $\mathbb{R}^3$). Let $\pi = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a,b,c \in \mathbb{Z} \right\}$ = discrete subgroup. Then N/π is a compact 3-manifold with fundamental group π. This group satisfies niether conclusion; in fact $\gamma(s)$ increases like a polynomial of degree 4 (e.g. $c_1 s^4 \le \gamma(s) \le c_2 s^4$). If so, M has no metric with a curvature of constant sign, since γ is niether exponential nor of degree 3.

Take S^3, remove knot: the resulting manifold has trivial higher homotopy groups, hence the universal covering space is contractible. Does this admit a complete metric with $K \le 0$? This question is quite open.

MIT Topology Seminar 4/5/68

Wm Cockroft: Whitehead Torsion

J.H. Whitehead (hereafter JHW) wrote a paper 'Simple Homotopy Types' in the Amer. J. Math in 1950 in which was defined what is now known as Whitehead torsion. This talk presents a new way of looking at that.

1. Whitehead Group

In the following, A is a ring with unit 1, such that finitely generated modules over A have well-defined rank, i.e. $A^r \simeq A^s \Leftrightarrow r=s$. We consider all modules which arise* as submodules of a module $M = \{(r_1, r_2, \cdots) \mid r_i \in A \text{ and almost all } r_i = 0\}$, in fact as submodules on a set of generators which are chosen from among the preferred generators $m_1 = (1, 0, \cdots)$, $m_2 = (0, 1, 0, \cdots)$, $\cdots$ of M. Any two finitely generated modules over A may be embedded disjointly in M and a natural direct sum within M is thus defined.

Endomorphisms f of M are called admissible if $fm_i = m_i$ for almost all $i = 1, 2, \cdots$. Admissible endomorphisms of M form a group $Gl(A)$. An endomorphism f is elementary if $fm_i = m_i + am_j$ and $fm_k = m_k$ for all other k, where i is any index and $a \in A$. The subgroup of $Gl(A)$ generated by elementary endomorphisms is called $E(A)$ and (Lemma) it equals $[Gl(A), Gl(A)]$ (note – this would be false if M were cut off after a finite number of generators). We call $K_1A = Gl(A)/E(A)$ the Whitehead Group of A. Let N be the subgroup of K_1A of order two generated by $\begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix}$. Call $\bar{K}_1A = K_1A/N$ the reduced Whitehead group of A. There is an obvious matrix formulation for these definitions.

Lemma $\begin{bmatrix} ab & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ in $\bar{K}_1A$. (a, b square matrices)

Proof $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & b \\ -a & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -a & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -a & I \end{bmatrix} =$

* all modules considered will be free

$$= \begin{bmatrix} ab & b \\ 0 & I \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & I \end{bmatrix} \quad //$$

2. Modules and Torsion

Now we associate elements of $\bar{K}_1 A$ to mappings of modules. Let $B, B' \subset M$ be modules and $\alpha : B \approx B'$. Then B, B' have the same rank. There is a permutation of the m_i, $P : M \to M$, such that $P\alpha : B \to B$ defines an admissible endomorphism, hence an element $\tau(P\alpha) \in \bar{K}_1 A$. (recall B, B' have preferred sets of generators among the m_i). Lemma, $\tau(\alpha) = \tau(P\alpha)$ is independent of choice of P.

We say α is simple if $\tau(\alpha) = 0$.

Prop.1 $\tau(\alpha\beta) = \tau(\alpha) + \tau(\beta)$

Prop.2 If $\gamma : A+B \to A'+B'$ (all within M) satisfies $\gamma(a+b) = \alpha a + (ha + \beta b)$ where $\alpha : A \to A'$, $h : A \to B'$, $\beta : B \to B'$, then if α and β are iso., so is γ and $\tau(\gamma) = \tau(\alpha) + \tau(\beta)$.

3. Chain systems $\mathcal{C} : C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} C_{n-2} \to \cdots$

A chain system $\mathcal{C}$ is a finite set of finitely generated A-modules indexed by $0, 1, 2, \cdots, n$ for some n, together with maps ∂ as above. Again $C_i \subset M$ for all i; we assume $C_i \cap C_j = \emptyset$. A chain mapping f between chain systems $\mathcal{C}, \mathcal{C}'$ is a sequence of A-homomorphisms $f_i : C_i \to C_i'$ satisfying $f\partial = \partial f$. A chain homotopy η between two chain mappings f, g is a sequence of A-homomorphisms $C_i \xrightarrow{\eta} C_{i+1}'$ with $f - g = \eta\partial + \partial\eta$. If $f : C \to C'$ is a chain mapping and $f_i : C_i \approx C_i'$ for each i, and $\tau(f_i) = 0$ for all i, say f is a simple isomorphism.

An elementary complex $B : 0 \to \cdots \to 0 \to B_p \xrightarrow{\partial} B_{p-1} \to 0 \to \cdots \to 0$ is such a B satisfying $\partial : B_p \approx B_{p-1}$ and $\tau(\partial) = 0$. A collapsible complex is a direct sum of elementary complexes. A chain mapping $f : \mathcal{C} \to \mathcal{C}'$ is a simple chain equivalence if f factors

$$C \xrightarrow[\text{injection}]{} C + D \xrightarrow{h} C' + D' \xrightarrow[\ker = D']{} C'$$

where h is a simple isomorphism and D, D' are collapsible. Write then $f : C \equiv C(\Sigma)$.

<u>Lemma</u> If C is acyclic, then C is simply equivalent to a chain system $\cdots \to 0 \to 0 \to P \overset{\alpha}{\approx} Q \to 0 \to 0 \to \cdots$.

<u>Proof</u> We give the first step only: Since $C_1 \xrightarrow{\partial} C_0$ is onto a free module, there exists $\eta : C_0 \to C_1$ with $\partial\eta = 1$. Chain equivalences are defined at right; the map of $C_1 + C_0$ to itself has matrix $\begin{bmatrix} 1 & \eta \\ \partial & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ \partial & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \eta \\ 0 & -1 \end{bmatrix}$, hence it is simple since each of the factors is. This is the start of an inductive process. ($C_0 \xrightarrow{1} C_0$ is collapsible).

$$
\begin{tikzcd}
C_3 \arrow[r, "\partial"] \arrow[d, "1"'] & C_2 \arrow[r, "\partial"] \arrow[d, "1"'] & C_1 \arrow[r, "\partial"] \arrow[d, "1"'] & C_0 \arrow[r] \arrow[d, "1"'] & 0 \\
C_3 \arrow[r, "\partial"] \arrow[d, "1"'] & C_2+C_0 \arrow[r, "{(\partial,1)}"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "1"] & C_1+C_0 \arrow[r, "\partial"] \arrow[d, shift right=2, "1"'] \arrow[d, "\eta"] \arrow[d, shift left=2, "\partial"] & C_0 \arrow[r] \arrow[d, "1"'] & 0 \\
C_3 \arrow[r, "\partial"] \arrow[d, "1"'] & C_2+C_0 \arrow[r, "{(\partial,\eta)}"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "1"] & C_1+C_0 \arrow[r, "{(0,1)}"] \arrow[d, "pr_1"'] & C_0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
C_3 \arrow[r] & C_2+C_0 \arrow[r] & C_1 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
$$

Define $\tau(C) = \tau(\alpha)$.

<u>Lemma</u> $\tau(C)$ is independent of the choice of η. //

<u>Lemma</u> If $C, C' \equiv 0$, then $\tau(C) = \tau(C') \Leftrightarrow C \equiv C'(\Sigma)$. //

<u>Cor.</u> $C \equiv 0 \, (\Sigma) \Leftrightarrow \tau(C) = 0$. /

<u>Lemma</u> If $C = C' + C''$ then $\tau(C) = \tau(C') + \tau(C'')$ (all acyclic). /

4. <u>Mapping cylinders.</u>

Let $f : C \equiv C'$. Define the <u>mapping cylonder</u> C_f of f by $(C_f)_i = C_i' + C_{i-1}$, $\partial_f c' = \partial' c'$, $\partial_f c_{i-1} = (f - \partial) c_{i-1}$. One shows $C_f \equiv 0$; put $\tau(f) = \tau(C_f)$.

<u>Prop.</u> $f \simeq g : C \equiv C' \Rightarrow \tau(f) = \tau(g)$ (because the mapping cylinders are chain equivalent). //.

<u>Prop.</u> $f : C \equiv C'$, $f' : C' \equiv C'' \Rightarrow \tau(f'f) = \tau(f) + \tau(f')$.
If $f : C \equiv C' \, (\Sigma)$ then $\tau(f) = 0$.

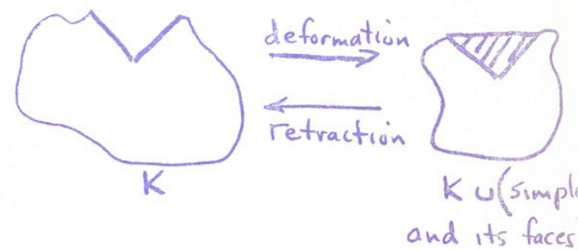
<u>Prop</u> $f : C \equiv C'$; then $\tau(f) = 0 \Leftrightarrow f : C \equiv C'(\Sigma)$.

One also has an analog of the $\begin{bmatrix} \alpha & h \\ 0 & \beta \end{bmatrix}$ proposition.

5. <u>Homotopy equivalences and their torsion</u>

Let K, L be finite dimensional complexes. If $K \underset{\phi}{\xrightarrow{\cong}} L$ is a homotopy equivalence, take the universal coverings $\tilde{K}, \tilde{L}$ and the associated chain complexes $C(\tilde{K})$, $C(\tilde{L})$. Let $A = \mathbb{Z}(\pi_1(L))$, the group ring. To ϕ corresponds an isomorphism $\Theta: \pi_1(K) \simeq \pi_1(L)$; also to get $\tilde{K}, \tilde{L}$ one must choose basepoints. One gets a chain map (associated with choice of Θ) $C(\tilde{K}) \to C(\tilde{L})$. Use the group $\bar{K}_1(A) / \{\text{units of } A = \pi_1 \subset A\}$; get the torsion $\tau(\phi)$ of the homotopy equivalence ϕ. If $\tau(\phi) = 0$, ϕ is called <u>simple</u>.

<u>Formal moves</u> A formal move is ~~a~~ map of simplicial complexes like that indicated in the picture:



If one complex can be obtained from another by a series of formal moves, they have the simple homotopy type, i.e. $K \xrightarrow{\text{formally}} K'$ $\Rightarrow K \equiv K'(\Sigma)$. Whitehead asks how much of this can be done in the infinite case.

An infinite series $K \longrightarrow K \cup \bigcup_\alpha e^n_\alpha \cup e^{n-1}_\alpha$ is possible, that is, the geometry makes sense; but the algebra apparently doesn't generalize.

<u>Conjecture</u> If K and L have the same homotopy type, then $K \equiv L(\Sigma)$ if an infinite number of formal moves* is allowed. (*of bounded dimension).

The following supports this: Let $[C]$ be all complexes of the same simple homotopy type as C. Let F be the free abelian group on all such $[C]$. Let $G = F / \{\text{all relations } [C+C'] = [C] + [C']\}$. Map $G \longrightarrow \bar{K}_1 A$ by $[C] \longrightarrow \tau(C)$; then the theorems about τ, simply show this map is an isomorphism. (or start with chain equivalences; use $[fg] = [f] + [g]$ etc.). Now $[C] + \sum_1^\infty [C] = \sum_0^\infty [C]$ but $\sum_1^\infty = \sum_0^\infty$, hence $[C] = 0$. (A corresponding fact is that in the category of modules over a ring, if infinitely generated modules are allowed the Grothendieck group is 0).

<u>Problem</u> how much can you carry out the algebra in the infinite case?

MIT Topology Seminar 4/8/68

Kenneth C. Millett : Normal Structures

Let $f: M^m \to N^n$ be a topological embedding.

Dfn f is locally flat if for every $x \in M$ there is a neighborhood $\mathcal{U}$ of $f(x)$ and a homeomorphism of pairs $h: (\mathcal{U}, \mathcal{U} \cap f(M)) \to (\mathbb{R}^n, \mathbb{R}^m \times 0)$.

Remark Piecewise linear and codimension $\geq 3 \Rightarrow$ loc. flat.

There are various normal structures available.

Dfn 1. Euclidean bundle group is $H(n)$, homeomorphisms of $\mathbb{R}^n$; bundle is $\mathbb{R}^n \longrightarrow E \xrightarrow[p]{} M$, a fibration.

Dfn 2. Euclidean bundle with 0-section group $H_0(n)$ = homeomorphisms of $(\mathbb{R}^n, 0)$.

(in Euclidean bundles a 0-section is a weak condition as any two sections are equivalent — there is an isomorphism of bundles carrying one into the other).

① A tubular neighborhood is an $H_0(n-m)$-bundle ν over M and an embedding g in a commutative diagram:

$$\begin{tikzcd} E_\nu \arrow[r, hook, "g"] \arrow[d, "p"'] & N \\ M \arrow[ur, "f"'] & \end{tikzcd}$$

This does not always exist: see Rourke and Sanderson in Invent. Math., 1967.

② Homotopy normal bundle. Let τ_M be the $H_0(m)$-bundle coming from $M \times M \xrightarrow[pr_1]{} M$ (the tangent microbundle). Thm $f^*\tau_N$ contains τ_M as a subbundle. (i.e. the group of $f^*\tau_N$ can be reduced to those homeomorphisms of $(\mathbb{R}^n, 0)$ leaving $\mathbb{R}^m$ invariant setwise). The Homotopy normal bundle ν_f is

$$E_\nu = \left(E_{f^*\tau_N} \setminus E_{\tau M}\right) \cup \text{0-section}.$$

③ Normal block bundle (P.L.)^N (Rourke-Sanderson).

④ A normal bundle η is any $H_0(n-m)$-bundle with $(\eta \oplus \tau_M, \tau_M) \cong (f^*\tau_N, \tau_M)$.

⑤ A Homotopy normal euclidean bundle η_f is any $H_0(n-m)$-bundle fiber homotopy equivalent to ν_f (②).

Remark ① ⇒ ④ ⇒ ⑤ ⇒ ② (existence).

Main Result. There is an embedding for which not even a ⑤ (HNEB) exists.

Cor. There is a Euclidean bundle pair (α,β) which does not split off, even up to fiber homotopy equivalence, a Euclidean bundle.

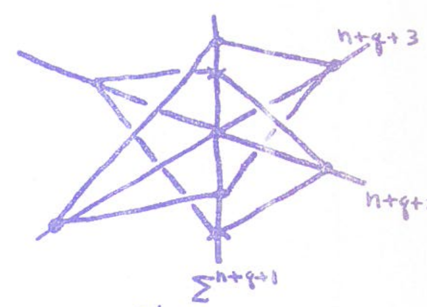
We describe below Rourke and Sanderson's example. Let $q+1 \geq 3$.

Dfn A link is an embedding $f: \Sigma^n \times \partial I \longrightarrow \Sigma^{n+q+1}$ $\begin{cases} I = [0,1] \\ I' = [-1,1]. \end{cases}$

A ribbon link is an embedding $f: \Sigma^n \times I \longrightarrow \Sigma^{n+q+1}$

By a theorem of Zeeman, $q+1 \geq 3 \Rightarrow f$ is unknotted: there is an isotopy of Σ^{n+q+1} such that $f \rightsquigarrow$ standard embedding, when restricted to $\Sigma^n \times 0$. In that case we arrive at an f_1 - the end result of the isotopy, restricted to $\Sigma^n \times 1$ - with $f_1 : \Sigma^n \times 1 \longrightarrow \Sigma^{n+q+1} \setminus \Sigma^n$ which is homotopy equivalent to S^q. Hence we have a class $[f_1] \in \pi_n(S^q)$, the second linking class of f. Similarly define the first linking class; show both are well-defined and invariant under isotopy of f.

Dfn $\Sigma f : \Sigma^{n+1} \times \partial I \longrightarrow \Sigma^{n+q+3}$, for any link f. (see picture).



Prop. Any link $\Sigma^n \times \partial I \to \mathbb{R}^{n+q+1}$ extends to $f: \Sigma^n \times I \longrightarrow \mathbb{R}^{n+q+2}_+$ //.

Prop. $f: \Sigma^n \times \partial I \longrightarrow \mathbb{R}^{n+q+1}$ extends to $(\Sigma f)' : \Sigma^{n+1} \times I \longrightarrow \Sigma^{n+q+3}$. //

Let L^q_n be the set of isotopy classes of $\Sigma^n \times I$ in Σ^{n+q}, and N^q_n the set of classes in L^q_n whose first linking class is 0.

Thm (Haefliger) there is an exact sequence.

$$N^q_n \xrightarrow{2^{nd}\text{ class}} \pi_n(S^{q-1}) \longrightarrow \Pi^{n+q+1} \longrightarrow N^q_{n-1}. \ //$$

There are natural suspension maps $\Sigma: L^q_n \to L^{q+1}_n$, $E: \pi_n(S^{q-1}) \to \pi_{n+1}(S^q)$ which fit in a commutative diagram

$$\begin{array}{ccccc} N^q_n & \longrightarrow & \pi_n(S^{q-1}) & \longrightarrow & \Pi^{n+q+1} \\ \Sigma \downarrow & & -E \downarrow & & -1 \downarrow \\ N^{q+1}_{n+1} & \longrightarrow & \pi_{n+1}(S^q) & \longrightarrow & \Pi^{n+q+1} \end{array}$$

The homotopy groups are all known for $q=9$, $n=18$, giving

$$\begin{array}{ccccc}
N^9_{18} & \longrightarrow & \mathbb{Z}_2+\mathbb{Z}_2+\mathbb{Z}_{24} & \longrightarrow & \mathbb{Z}_6 \\
\downarrow & & \downarrow{\scriptstyle -E} & & \downarrow{\scriptstyle -1} \\
N^{10}_{19} & \longrightarrow & \mathbb{Z}_{24}+\mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_6 . \\
\in & & \alpha \in \quad \downarrow{\scriptstyle \text{Hopf}} & & \\
[f] & & \mathbb{Z}_2 & &
\end{array}$$

Let α generate $\mathbb{Z}_{24} \subset \pi_{19}(S^9)$. Then $6\alpha \to 0$ in $\mathbb{Z}_6 = \Pi^{10}$; by a diagram chase there exists an f with $[f] \to 6\alpha$ where f has a 2^{nd} class of order 4. But:

<u>Thm</u> $f : \Sigma^n \times I^1 \longrightarrow \Sigma^{n+q+1}$, codimension ≥ 3. Then

$$\left.\begin{array}{l}
f \text{ has H.N.E.B.} \\
f\,|\,\Sigma^n \times 0 = \text{std. inclusion} \\
f\,|\,\Sigma^n \times (1) \longrightarrow \Sigma^{n+q+1} \setminus \Sigma^n \text{ is null homotopic}
\end{array}\right\} \Rightarrow \begin{array}{l} 2^{nd} \text{ class of } f\,|\,\Sigma^n \times \partial I^1 \\ \text{has order 2.} \end{array}$$

(conjecture: each such 2^{nd} class must be 0). //.

The contradiction shows that an f for which $[f] \to 6\alpha$ has no H.N.E.B.

MIT Topology Seminar 4/15/68

H. Toda (Kyoto Univ.) <u>Recent progress in homotopy theory</u>

1. Stable homotopy groups of spheres

$\pi_t = \pi_{n+t}(S^n)$ for n large. For an odd prime p, let

${}^p\pi_t$ = p-primary component of π_t.

$\exists\, \alpha_i \in {}^p\pi_{2i(p-1)-1}$ $\quad i = 1, 2, 3, \ldots$ $\Big\}$ of order p

$\exists\, \beta_s \in {}^p\pi_{2(sp+s-1)(p-1)-2}$ $\quad 1 \le s \le p$ $\qquad \beta_p = ?$

$\exists\, \alpha_i'$ s.t. $i\alpha_i' = \alpha_i$ for $i < p^3$

For $t < 2p^2(p-1)-3$, ${}^p\pi_t$ is determined; with generators α_i', β_s and some compositions ($\sum_t {}^p\pi_t = {}^p\pi_*$, a ring, generated up to $2p^2(p-1)-3$ by α_i', β_s).

The next problem is $t = 2p^2(p-1)-3$. Now ${}^p\pi_t$ is generated by $\alpha_1\beta_1^p$, of order at most p, i.e.

$${}^p\pi_t = \begin{cases} Z_p & \text{if } \alpha_1\beta_1^p \neq 0 \\ 0 & \text{if } \alpha_1\beta_1^p = 0 \end{cases}$$

(Cohen (1966), Yamamoto; Gray ($p=3$))
(Toda (1967).)

<u>Thm</u> $p\cdot\gamma = 0,\ \gamma \in \pi_t \Rightarrow \alpha_1 \cdot \gamma^p = 0$

<u>Cor</u> $\alpha_1\beta_1^p = 0$

<u>Thm</u> $\alpha_1 \cdot \gamma = 0 \Rightarrow \beta_1\gamma^p = 0$

<u>Cor</u> $p\cdot\gamma = 0 \Rightarrow \beta_1\gamma^{p^2} = 0$; hence $\beta_1^{p^2+1} = 0$

(since $\alpha_i' \in \pi_{odd} \Rightarrow (\alpha_i')^2 = 0$)

<u>Thm</u> β_s $\quad 1 \le s \le p$ are all nilpotent

<u>Conjecture</u> every $\xi \in \pi_t$, $t \neq 0$ is nilpotent.

2. Non-associativity or non-commutativity in mod q theories.

Let q be an integer > 1. Let $M_{q,n} = S^n \cup_q e^{n+1}$, a Moore

Space of type $(\mathbb{Z}_q, n)$.

$$\pi_t(\mathbb{Z}_q) \overset{\text{def}}{=} \pi_{n+t}(M_{q,n}) \quad n \text{ large.}$$

Given h^*, a generalized cohomology theory, define $h^*(\ ;\mathbb{Z}_q)$ by

$$h^i(X, \mathbb{Z}_q) = \tilde{h}^{i+2}(X \wedge M_{q,1})$$

$\pi_* = \sum_t \pi_t$ ring; h^* has an associative and commutative multiplication.

$$\pi_s(\mathbb{Z}_q) \otimes \pi_t(\mathbb{Z}_q) = \pi_{n+s}(M_{q,n}) \otimes \pi_{n+t}(M_{q,n}) \xrightarrow[\text{product}]{\text{smash}}$$

$$\longrightarrow \pi_{2n+s+t}(M_{q,n} \wedge M_{q,n}) \xrightarrow{\varphi^*} \pi_{2n+s+t}(M_{q,2n}) = \pi_{s+t}(\mathbb{Z}_q).$$

φ^* might come from

$$\begin{tikzcd} M_{q,n} \wedge M_{q,n} \arrow[r, "\varphi"] & M_{q,2n} \\ S^n \wedge S^n \arrow[u, "i \wedge i"] \arrow[r, "id"] & S^{2n} \arrow[u, "i"'] \end{tikzcd}$$

(if μ_q, μ_q' are multiplications on $h^*(\ ;\mathbb{Z}_q)$ then $\mu_q(x \otimes y) - \mu_q'(x \otimes y) = \pm b \cdot Sq\, x\, Sq\, y$, $b \in h^{-2}(S^0; \mathbb{Z}_2)$, $Sq\, b = 0$; this is the uniqueness theorem. So it is enough to look at such a φ).

By a theory of Araki-Toda, the following table concerning multiplacation in $h^*(\ ;\mathbb{Z}_q)$ was obtained:

	exists	exists a commutative	exists an associative
$q \equiv 2(4)$	No ($KO(\mathbb{Z}_2)$)		
$q \equiv 0(8)$	Yes	Yes	?
$\equiv 4(8)$	Yes	No ($KO(\mathbb{Z}_4)$)	?
q odd: $q \not\equiv 0(3)$	Yes	Yes	Yes (Anderson)
$q \equiv 0(9)$	Yes	Yes	Yes
$q \equiv 3,6(9)$	Yes	Yes	No (Gray).

In case $q \equiv 3,6\,(9)$, Gray showed 'no' was equivalent to $\alpha_1\beta_1^{\,3}=0$.

3. $\mathcal{P}^n$ in $Q(X) = \lim_{k\to\infty} \Omega^k S^k X$ or $\Omega^k S^{n+k}$, $n>0$.

Equivalently, study the dual operation $\mathcal{P}^n_*$ in homology. Assume X connected. $H_*(Q(X);\mathbb{Z}_p)$ = free commut. ring over $\Delta^{\varepsilon_1} Q^{j_1}\cdots\Delta^{\varepsilon_r} Q^{\varepsilon_r} x_j$, where $\{x_j\}$ = basis of $H_*(X;\mathbb{Z}_p)$ ($\xrightarrow{\text{mono}} H_*(Q(X);\mathbb{Z}_p)$) (Dyer–Lashof). In the Dyer–Lashof paper $Q_i : H_q(Q(X);\mathbb{Z}_p) \to H_{pq+i}(Q(X);\mathbb{Z}_p)$, with $\Delta Q_{2i} = Q_{2i-1}$ (Δ = Homology Bockstein), $Q_{2i}x = 0$ if $2i \not\equiv q(p-1)$ (mod $2p-2$). So write instead

$$Q^j(x) = (-1)^{j+2mq(p+1)/2}(m!)^q\, Q_{(2j-q)(p-1)}(x) \qquad (m=\tfrac{p-1}{2},\ p>2)$$

Then
$$Q^j : H_q \to H_{q+2j(p-1)}$$
$$\mathcal{P}^n_* : H_q \to H_{q-2n(p-1)}$$

<u>Nishida formula</u> $\mathcal{P}^n_* Q^{n+q} = \sum_i (-1)^{n+i}\binom{s(p-1)}{n-pi} Q^{s+i}\mathcal{P}^i_*$;

$$\mathcal{P}^n_*\Delta Q^{n+q} = \sum(-1)^{n+i}\binom{s(p-1)}{n-pi-1}\Delta Q^{q+i}\mathcal{P}^i_* + \sum(-1)^{n+i+1}\binom{s(p-1)-1}{n-pi-1} Q^{q+i}\mathcal{P}^i_*\Delta$$

Or for $p=2$, $Sq^n_* Q^{n+q} = \sum\binom{s}{n-2i} Q^{s+i} Sq^i_*$.

In $\Omega^k S^{n+k}$ $(n>0)$, $\Omega S^{2n} \simeq_p S^{2n-1}\times\Omega S^{2pn-1}$ $(p>2)$.

To prove <u>"$p\cdot\gamma \Rightarrow \alpha_1\gamma^p = 0$"</u>

S^{2n+1} = units of $\mathbb{C}^{n+1}$; $S^1 \subset \mathbb{C}^1$ acts on S^{2n+1}, $W = \bigcup_n S^{2n+1}$ so S^1 acts on W; $\pi = \mathbb{Z}_p \subset S^1$ so π acts on W. (cyclic gp. of order p). W is a π-free acyclic (regular) CW-complex. W^r = r-skeleton of W. $W^r - W^{r-1} = e_r \cup \sigma e_r \cup\cdots\cup \sigma^{p-1} e_r$ where $1 \neq \sigma \in \pi$.

Let X be a finite CW-complex $\ni *$ base pt.

$$X^{(p)} = X\wedge\cdots\wedge X \qquad p \text{ times}$$

π acts on $X^{(p)}$, permuting factors. $W\times_\pi X^{(p)} \supset W\times_\pi\{*\}^{(p)}$ and $W^r\times_\pi X^{(p)} \supset W^r\times_\pi\{*\}^{(p)} = W^r/\pi$ (r-skeleton of lens space). Let $ep^r(X) \overset{\text{def}}{=} (W^r\times_\pi X^{(p)})/(W^r/\pi)$. Consider $X \rightsquigarrow ep^r(X)$.

This a functor from the category of spaces (finite CW-complexes) and continuous maps to itself; it preserves homotopy. Also $ep^r(f)\big|_{ep^s(X)} = ep^s(f)$ $(s \le r)$ for $f: X \to Y$, and $ep^r(0) = 0$.

$$ep^0(f) = f^{(p)} = f \wedge \cdots \wedge f : \underset{\overset{\shortparallel}{X^{(p)}}}{ep^0(X)} \longrightarrow \underset{\overset{\shortparallel}{Y^{(p)}}}{ep^0(Y)}, \text{ since } W^0 = \pi.$$

Let $X = S^n$, $p > 2$. $\quad ep^r(S^n) = S^{pn} \cup_p e^{pn+1} \cup_p e^{pn+2} \cup_p \cdots \cup_p e^{pn+r}$ (if r is even). If $r \le 2p-4$, $n \ge 2$, then

$$ep^r(S^n) \underset{h.eq.}{\simeq} S^{pn} \vee M_{p,pn+1} \vee M_{p,pn+3} \vee \cdots \vee M_{p,pn+r-1}.$$

$$\Rightarrow \boxed{{}^p\pi_t = 0,\ t < 2p-3.}$$

Cor. $\quad ep^r(S^n) \xrightarrow[\text{retraction}]{r} S^{pn} = ep^0(S^n)$.

Let $X = M_{p,n}$ $\quad ep^r(M_{p,n}) = S^{pn} \cup \cdots$

Lemma $\quad \beta\mathscr{P}^1 H^{pn}(ep^r(M_{p,n}); \mathbb{Z}_p) \ne 0$ if $r \ge p-1$. //

So $\exists$

$$\begin{tikzcd}
j: C_{\alpha_1} = S^{pn} \cup_{\alpha_1} S^{pn+2p-2} \arrow[rr] & & ep^r(M_{p,n}) \\
& S^{pn} \arrow[ul, "i"] \arrow[ur, "i"'] \arrow[u, phantom, "\circlearrowright"] &
\end{tikzcd}$$

Assume $p \cdot \gamma = 0$.

$$\begin{tikzcd}
S^n \arrow[r, "\gamma"] & S^{n-t} \\
S^n \arrow[u, "p"] \arrow[ur, "\simeq 0"'] \arrow[u, phantom, shift left=4, "\circlearrowright"] &
\end{tikzcd} \quad n \text{ large.}$$

Then $\exists\ \bar{\gamma} : M_{p,n} = S^n \cup_p e^{n+1} \longrightarrow S^{n-t}$ with $\bar{\gamma}|_{S^n} = \gamma$.

There is a diagram: which commutes.

$$\begin{tikzcd}
ep^{p-1}(M_{p,n}) \arrow[r, "ep^{p-1}(\bar{\gamma})"] & ep^{p-1}(S^{n-t}) \\
(M_{p,n})^{(p)} \arrow[u, "i"] \arrow[r, "\bar{\gamma}^{(p)}"'] & S^{pn} \arrow[u, "i"']
\end{tikzcd}$$

We build a larger diagram on it:

$$
\begin{tikzcd}
S^{pn} \cup_{\alpha_1} e^{pn+2p-2} \arrow[r] & ep^{p-1}(M_{p,n}) \arrow[r, "ep^{p-1}(\gamma)"] & ep^{p-1}(S^{n-t}) \arrow[dr, "r"] & \\
 & (M_{p,n})^{(p)} \arrow[u, "i"] \arrow[r, "(\bar{\gamma})^{(p)}"'] & S^{pn-pt} \arrow[u] \arrow[r, "id"] & S^{pn-pt} \\
 & S^{pn} \arrow[uul, "i"] \arrow[u, "i"] \arrow[ur, "\gamma^{(p)} = \gamma\wedge\cdots\wedge\gamma = \pm\gamma^p"'] & &
\end{tikzcd}
$$

This yields a commutative diagram:

$$
\begin{tikzcd}
S^{pn} \cup_{\alpha_1} e^{pn+2p-2} \arrow[rr] & & S^{pn-pt} \\
 & S^{pn} \arrow[ul, "i"] \arrow[ur, "\pm\gamma^p"'] &
\end{tikzcd}
$$

but such a digram exists $\Leftrightarrow \pm\gamma^p\alpha_1 = 0$ $\Leftrightarrow \alpha_1\gamma^p = 0$. //

"$\alpha_1\gamma = 0 \Rightarrow \beta_1\gamma^p = 0$"

We have maps $S^{n+2p-3} \xrightarrow{\alpha_1} S^n \xrightarrow{\gamma} S^{n-t} \simeq 0$.

$$
\begin{tikzcd}
ep^{p-1}(S^{n+2p-3}) \arrow[r, "ep^{p-1}(\alpha_1)"] \arrow[d, leftrightarrow, "\simeq"] & ep^{p-1}(S^n) \arrow[r, "ep^{p-1}(\gamma)"] \arrow[d, leftrightarrow, "\wr"] \arrow[dd, bend right, "r"'] & ep^{p-1}(S^{n-t}) \simeq 0 \arrow[d, shift left, "r"] \\
S^{p()} \vee M_{()} \vee \cdots \vee M_{p,\{p(n+2p-3)+p-2\}} & S^{pn} \vee M_{p,pn+1} \vee \cdots & S^{pn-pt} \arrow[u, shift left, "i"] \\
 & S^{pn} \arrow[u, "i"] \arrow[ur, "\pm\gamma^p"'] &
\end{tikzcd}
$$

Lemma Functional $\wp^p$-operation for $\bar{\beta} = \gamma_0\, ep^{p-1}(\alpha_1)\circ j$. is not trivial.

Thus we can get a $\bar{\bar{\beta}}$ in the following diagram, which has a non-trivial functional $\wp^p$-operation.

$$
\begin{tikzcd}
 & S^{pn+2p(p-1)-1} \arrow[dr, "\bar{\bar{\beta}}"] & \\
S^{pn+2p(p-1)-2} \cup_p e^{pn+2p(p-1)-1} \arrow[ur, "\tilde{\pi}"] \arrow[rr, "\bar{\beta}"] & & S^{pn} \\
 & S^{pn+2p(p-1)-2} \arrow[ul] \arrow[ur, "\beta"'] &
\end{tikzcd}
$$

But $\bar{\beta} \simeq 0 \Rightarrow \bar{\bar{\beta}} \simeq 0$.
[Contradicts triviality of the mod p Hopf invariant], hence $\beta \neq 0$, $\beta = * \cdot \beta_1 \in {}^p\pi_{2p(p-1)-2}$

where $* \not\equiv 0 \bmod p$; hence $\beta_1\gamma^p = 0$.

Remark

$$
\begin{array}{lcl}
\wp^n_* : H_*(ep^r(X); Z_p) & \Rightarrow & \underline{\underline{1}} \quad \text{"}p\gamma = 0 \Rightarrow \alpha_1\gamma^p = 0\text{"} \\
\Uparrow & & \\
\wp^n_* : H_*(W^r \times_\pi X^{(p)}; Z_p) & \underset{X = M_{q,n}}{\overset{p=3,2,\ r=1}{\Longrightarrow}} & \underline{\underline{2}} \quad \pi_*(Z_q),\ h^*(Z_q) \\
\Uparrow & & \\
\wp^n_* : H_*(W \times_\pi X^p; Z_p) & \Rightarrow & \underline{\underline{3}} \quad \text{Nishida's formula.}
\end{array}
$$

MIT Topology Seminar 4/17/68

H. Toda (Kyoto Univ.): <u>The Nishida Formulas</u>

$$\Uparrow$$

$$\exists\, \theta : W\times_\pi X^p \longrightarrow X = \lim_n \Omega^n S^n X'$$

$$\Uparrow$$

$$\wp^n_* \text{ in } H_*(W\times_\pi X^p; Z_p)$$

$$\Uparrow$$

$$\wp^n \text{ in } H^*(W\times_\pi X^p; Z_p)$$

<u>Definition of $\wp^n$</u>

Let $u\in H^q(X;Z_p)$ [all coefficients Z_p hereafter], and let $c: C(X)\to Z_p$ be a cochain representing u. The external reduced p^{th} power $P(u)\in H^{pq}(W\times_\pi X^p)$ is given by $C(W)\otimes C(X^p)\xrightarrow{\varepsilon\otimes 1} C(X^p)\xrightarrow{c^p} Z_p$, an equivariant cocycle. There is a diagonal map

$$\begin{array}{ccc} & W\times_\pi * & \\ {\scriptstyle\rho_0}\nearrow & & \nwarrow{\scriptstyle\rho} \\ W\times_\pi X & \longrightarrow & W\times_\pi X^p \\ \| & & \\ W/\pi\times X & & \end{array}, \qquad d(w,x)=(w,(x,\cdots,x))$$

commuting with projections ρ_0, ρ; d^*P is a homomorphism, and

$$d^*P(u)\in H^*(W/\pi)\otimes H^*(X) \qquad \text{by Kunneth formula}$$
$$\|$$
$$\sum_i w_i\times D_i u$$

where $w_i\in H^i(W/\pi)$, $w_{2i}=(w_2)^i$ and $w_{2i+1}=w_1 w_{2i}$, $\beta w_1=w_2$. One shows $D_{2i-1}u=\beta D_{2i}u$, and $D_{2i}u=0$ if $2i\not\equiv q \pmod{2(p-1)}$.

Define $\nu(q)=(-1)^{mq(q-1)/2}(m!)^{-q}\in Z_p$ and $m=\frac{p-1}{2}$, and put

$$\wp^i(u)=(-1)^i\nu(q)^{-1}D_{(2q-2i)(p-1)}(u)\in H^{pq-(2q-2i)(p-1)}(X)$$
$$= H^{q+2i(p-1)}(X).$$

<u>Thm</u> $$\wp^n P(u)\equiv \sum_i \binom{(q-2i)m}{n-pi} w_{2(n-pi)(p-1)}\times P(\wp^i(u))$$
$$-\frac{\nu(q+1)}{\nu(q)}\sum_i \binom{(q-2i)m-1}{n-pi-1} w_{2(n-pi)(p-1)-p}\times P(\beta\wp^i(u)). \pmod{\ker d^*}$$

<u>Proof</u> ① $$\begin{array}{ccc} H^*(W\times_\pi X^p) & \xrightarrow{d^*} & H^*(W\times_\pi X) \\ & {\scriptstyle\rho^*}\nwarrow \quad \nearrow{\scriptstyle\rho_0^*} & \\ & w_i\in H^*(W/\pi) & \end{array}$$

$w_i\times x=\rho^* w\cup x=\rho_0^* w\cup x$; d^* is a $H^*(W/\pi)$-homomorphism

② $\nu(q) \equiv \nu(q+2i(p-1))$ since $\nu(q+4) \equiv \nu(q) \Leftarrow (m!)^2 \equiv (-1)^{m+1} \pmod p$.

③ $q+2i(p-1) = \deg \mathcal{P}^i u$. Thus

$$\nu(q)\, d^*\Big(\sum_i - \frac{\nu}{\nu}\sum_i\Big) \overset{①②③}{=} \sum_i \binom{(q-2i)m}{n-pi} w_{2(n-pi)(p-1)} \times$$

$$\times \sum_j (-1)^j \Big\{ w_{(q+2i(p-1)-2j)(p-1)} \times \mathcal{P}^j\mathcal{P}^i u + w_{\dots -1} \times \beta \mathcal{P}^j \mathcal{P}^i u\Big\}$$

$$- \sum_i \binom{(q-2i)m}{n-pi-1} w_{2(n-pi)(p-1)-p} \times \sum_j (-1)^j \Big\{ w_{(q+2i(p-1)+1-2j)(p-1)} \times \mathcal{P}^j\beta\mathcal{P}^i u$$

$$+ w_{\dots}\ \beta\mathcal{P}^j\beta\mathcal{P}^i u\Big\}$$

$$(*) \quad = \sum_{i,j} (-1)^j \binom{(q-2i)m}{n-pi} \Big\{ w_{(2n+q-2i-2j)(p-1)} \times \mathcal{P}^j\mathcal{P}^i u + w_{\dots -1}\times \beta\mathcal{P}^j\mathcal{P}^i u\Big\}$$

$$+ \sum_{i,j} (-1)^{j+1} \binom{(q-2i)m-1}{n-pi-1} w_{(2n+q-2i-2j)(p-1)-1} \times \mathcal{P}^j\beta\mathcal{P}^i u$$

(note $w_{odd}\cdot w_{odd} = w_1^2 \cdot w_{even}\cdot w_{even} = 0$.)

$$\nu(q)\, d^* \mathcal{P}^n P(u) = \mathcal{P}^n \sum_i (-1)^i \Big\{ w_{(q-2i)(p-1)} \times \mathcal{P}^i u + w_{()()-1} \times \beta\mathcal{P}^i u\Big\}$$

[Cartan formula: $\mathcal{P}^j w_{2k} = \mathcal{P}^j (w_2)^k = \binom{k}{j}(w_{2p})^j (w_2)^{k-j} = \binom{k}{j} w_{2k+2j(p-1)}$

$\mathcal{P}^j w_{2k-1} = \binom{k-1}{j} w_{2k-1+2j(p-1)}$

$$= \sum_j\sum_i (-1)^i \Big\{\mathcal{P}^{n-j} w_{()} \times \mathcal{P}^j\mathcal{P}^i u + \mathcal{P}^{n-j} w_{()()-1} \times \mathcal{P}^j\beta\mathcal{P}^i u\Big\}$$

$$= \sum_{i,j}(-1)^i \Big\{ \binom{(q-2i)\overbrace{\tfrac{p-1}{2}}^{m}}{n-j} w_{(q-2i+2n-2j)} \times \mathcal{P}^j\mathcal{P}^i u \qquad (*)$$

$$+ \binom{(q-2i)m-1}{n-j} w_{(\ )(p-1)-1} \times \mathcal{P}^j\beta\mathcal{P}^i u\Big\}.$$

Comparison of the formulas (*) shows it is sufficient to show (with $k=i+j$)

$$\boxed{\begin{array}{l} ① \ \sum_i (-1)^{k+i}\binom{(q-2i)m}{n-pi} \mathcal{P}^{k-i}\mathcal{P}^i = \sum_i (-1)^i \binom{(q-2i)m}{n-k+i}\mathcal{P}^{k-i}\mathcal{P}^i u \\ ② \ \sum (-1)^{k+i}\binom{(q-2i)m}{n-pi}\beta\mathcal{P}^j\mathcal{P}^i u + \sum (-1)^{k+1+i}\binom{(q-2i)m-1}{n-pi-1}\mathcal{P}^j\beta\mathcal{P}^i u = \sum(-1)^i \binom{(q-2i)m-1}{n-k+i}\mathcal{P}^{k-j}\beta\mathcal{P}^i u \end{array}}$$

These are proved in Steenrod "Cohom. oper." VIII p.120(4), p.121(5).

<u>Cor.</u> $$\mathcal{P}^n(w_s \times Pu) = \sum_i \binom{[s/2]+(q-2i)m}{n-pi} w_{s+2(n-pi)(p-1)} \times P(\mathcal{P}^i u)$$
$$- \frac{\nu(q+1)}{\nu(q)}\, \varepsilon(s) \sum_i \binom{[s/2]+(q-2i)m-1}{n-pi-1} w_{s-p+2(n-pi)(p-1)} \times P(\beta\mathcal{P}^i u)$$

where $$\varepsilon(s) = \begin{cases} 0 & s \text{ even} \\ 1 & s \text{ odd} \end{cases}$$

<u>Proof</u> $s=0$ is the theorem

$s=1$ $\quad \mathcal{P}^n(w_1) = \begin{cases} w_1 & n=0 \\ 0 & n\neq 0 \end{cases}$ so by Cartan formula,

$$\mathcal{P}^n(w_1 \times Pu) = w_1 \times \mathcal{P}^n Pu = \sum(-1)(\ )\, w_1 w_{2(\ )(\ )}\, \mathcal{P}\cdots$$
$$- (\ )\ \underbrace{w_1 w_{2(\ )-p}}_{\overset{\shortparallel}{0} \text{ (both odd)}}$$

Then by induction on s, if $s \geq 2$

$$\mathcal{P}^n(\underset{\underset{w_2 w_{s-2}}{\shortparallel}}{w_s} \times Pu) = w_2 \times \mathcal{P}^n(w_{s-2} \times Pu) + w_{2p} \times \mathcal{P}^{n-1}(w_{s-2} \times Pu)$$

one uses $$\binom{[\frac{s-2}{2}]+(q-2i)m}{n-pi} + \binom{[\frac{s-2}{2}]+(q-2i)m-1}{n-pi-1} = \binom{[\frac{s}{2}]+(q-2i)m}{n-pi} \bmod p$$

in the coefficient of $w_{s+2(n-pi)(p-1)} = w_{2p} \times w_{s-2+2(n-1-pi)(p-1)}$. //

<u>To study $\mathcal{P}^n_*$</u> let $\{x_j\}$ be a Z_p-basis of $H_*(X)$ such that if $\Delta x_j \neq 0$, $\Delta x_j = x_\ell$ for some ℓ; let $\{u_j\}$ be the dual basis of $H^*(X)$.

<u>Lemma</u> $H_*(W \times_\pi X^p)$ has the following Z_p-basis: Let e_i be a basic cell in $C_i(W)$ $\{e_i\}$ dual to w_i. The basis will contain $\{e_i \otimes_\pi x_j^{\,p}\}$ $\quad x_j^{\,p} = x_j \otimes \cdots \otimes x_j$ and $e_0 \otimes_\pi (x_{j_1} \otimes \cdots \otimes x_{j_p})$ for $j_s \neq j_t$, for some s, t; (choose a representative from each class of cyclic permutations of $(j_1 \cdots j_p)$, take one basis elt. for each).

$$H_*(W \times_\pi X^p) = H_*(\mathcal{C}(W) \otimes_\pi \underbrace{H_*(X^p)}_{\overset{\shortparallel}{H_*(X)\otimes\cdots\otimes H_*(X)}})$$

(this is proved in Dyer-Lashof); and further

$H^*(w \times_\pi X^p) \supset \ker d^* \ni z_k$ such that $\{z_k, w_i \times P(u_j)\}$ is a basis for $H^*(w \times_\pi X^p)$; $w_i \times P(u_j)$ is dual to $e_i \otimes_\pi x^p$ (proved by Steenrod).

$\underline{\mathcal{P}^n \ker d^* \subset \ker d^*}$.

$\mathcal{P}^n u_j = \sum_k a_{k,j}(n) u_k \quad \left[\Leftrightarrow \mathcal{P}^n_* x_i = \sum_j a_{ij}(n) x_j\right]$ defines coefficients $a_{k,j}(n) \in \mathbb{Z}_p$.

Thm (Nishida's formula)

$$\mathcal{P}^n_*(e_{c+2n(p-1)} \otimes_\pi x_k{}^p) = \sum_{i,j} \binom{[\frac{c}{2}]+qm}{n-pi} a_{k,j}(i) \left(e_{c+2ip(p-1)} \otimes_\pi x_j{}^p\right)$$

$$- \underbrace{\frac{\nu(q+1)}{\nu(q)}}\, \varepsilon(c+1) \binom{[\frac{c+1}{2}]+qm-1}{n-pi-1} a_{\ell,j}(i) \left(e_{c+p+2ip(p-1)} \otimes_\pi x_j{}^p\right)$$

$\left(\text{or } \frac{\nu(q+2)}{\nu(q+1)}\, ?\right)$ where $q = \deg x_k$, $x_\ell = \Delta x_k$ if $\Delta x_k \neq 0$; if $\Delta x_k = 0$ let $a_{\ell,j}(i) = 0$. Note c may be <0.

This formula is also true for $H_*(w \times_\pi X^{(p)})$, $H_*(ep^\infty(X))$; and true for $H_*(ep^r(X))$ mod $\ker\left(i_*: H_*(ep^r(X)) \to H_*(ep^\infty(X))\right)$. This kernel is generated by $\partial(e_{r+1} \otimes_\pi x_{j_1} \otimes \cdots \otimes x_{j_p})$. //

Let $X = QX' = \lim \Omega^n S^n X'$. Then $\exists\, \theta: w \times_\pi X^p \to X$ which defines (Dyer-Lashof) $Q_i(x) \equiv \theta_*(e_i \otimes x^p) \in H_{qp+i}(X)$; Q_i is a homomorphism and has properties similar to those of D_i: $\Delta Q_{2i} = Q_{2i-1}$ and $Q_{2i} = 0$, $q \neq 2i\ (2p-2)$. Let

$$\nu(q)\, Q^j(x) = (-1)^j Q_{(2j-n)(p-1)}(x) \in H_{\underbrace{pn-(2j-n)(p-1)}_{n+2j(p-1)}}(X).$$

Then Q^j is a natural homomorphism, commutes with suspension, and satisfies a Cartan formula $Q^{q/2} x = x^p$ if q even $< \deg x$; $Q^j x = 0$ if $q > 2j$. However $Q^{\cancel{q}}\cancel{x}\, x$ is not true. Apply $\theta_* \mathcal{P}^n_*$ and get 2 formulas of last lecture.

MIT Topology Seminar 4/22/67

H. Toda (Kyoto Univ.): <u>Mod q cohomology theories.</u>

See Araki - Toda, "Multip. Structures in mod q cohomology theories," I, II, Osaka J. Math (1965-6).

Let X, Y be finite CW-complexes throughout, and q an integer >1. Let $M = M_{q,n} = S^n \cup_q e^{n+1}$ where n is large.

$$S^n \xrightarrow{q} S^n \xrightarrow{i} M \xrightarrow{\pi} S^{n+1}$$

$$S^n X = X \wedge S^n \xrightarrow{1 \wedge q} X \wedge S^n \xrightarrow{1 \wedge i} X \wedge M \xrightarrow{1 \wedge \pi} X \wedge S^{n+1}$$ so

$$0 \xrightarrow{q} [S^{n+1}X, W] \otimes \mathbb{Z}_q \xrightarrow{(1\wedge\pi)^*} [X \wedge M, W] \xrightarrow{(1\wedge i)^*} [S^n X, W] \xrightarrow{q} \cdots$$

$$\xrightarrow{q} [Y, S^n X] \xrightarrow{(1\wedge i)_*} [Y, X \wedge M] \xrightarrow{(1\wedge\pi)_*} [Y, S^{n+1} X] \xrightarrow{q} \cdots$$

A.T. <u>Thm 1.1</u> $1_M \in [M,M]$ is of order $\begin{cases} q & q \not\equiv 2 \bmod 4 \\ 2q & q \equiv 2 \bmod 4 \end{cases}$.

We consider only the first case, $q \not\equiv 2 \bmod 4$.

<u>Cor</u> $[X \wedge M, W]$, $[Y, X \wedge M]$ are $\mathbb{Z}_q$-modules.

A.T. <u>Thm 2.9</u> $q \not\equiv 2 \bmod 4$

$$\left.\begin{array}{l} 0 \to [S^{n+1}X, W] \otimes \mathbb{Z}_q \to [X \wedge M, W] \to \mathrm{Tor}([S^n X, W], \mathbb{Z}_q) \to 0 \\ 0 \to [Y, S^n X] \otimes \mathbb{Z}_q \to [Y, X \wedge M] \to \mathrm{Tor}(\qquad) \to 0 \end{array}\right\} \boxed{\text{Split}}$$

If $X = M$, $[M \wedge M, S^n M] \xrightarrow{(1\wedge i)^*} [S^n M, S^n M] \xrightarrow{q}$

$$\underset{\text{order } q}{\exists \varphi} \longrightarrow 1 \longrightarrow 0,$$

Thus $\varphi : M \wedge M \longrightarrow S^n M = M \wedge S^n$ s.t. $\varphi(1 \wedge i) \simeq 1_{S^n M}$

$$[S^{n+1}M, S^n M] \xrightarrow{(1\wedge\pi)^*} [M \wedge M, S^n M] \xrightarrow{(1\wedge i)^*} [S^n M, S^n M] \xrightarrow{q}$$

$\left(\begin{cases} 0 & q \text{ odd} \\ \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2, & q \equiv 0 \bmod 4 \end{cases} \right\}$ so if q is not odd, φ may not be unique.

$$\begin{tikzcd}
\varphi + (1\wedge\pi) : M\wedge M \arrow[rr, shift left] \arrow[d] & & S^n M \vee S^{n+1} M \arrow[ll, shift left, "\text{inverse}"] \\
M\wedge S^{n+1} = S^{n+1} M \arrow[urr, "i"'] \arrow[u, dashed, "\exists\psi"'] & &
\end{tikzcd}$$

The map $\varphi + (1\wedge\pi)$ induces an isomorphism of homology groups, hence is a homotopy equivalence; so $\exists\psi$ (depending on φ) such that

$$\begin{cases} (1\wedge\pi)\psi = 1_{S^{n+1}M} \\ \varphi\psi = 0 \end{cases}$$

Thus $1_{M\wedge M} = (1\wedge i)\varphi + \psi(1\wedge\pi)$. Now φ defines a multiplication: $\pi_*(M) = \lim_n \pi_{n+*}(M_{n,q})$. If $f: S^{n+t} \to M_{n,q}$, $g: S^{n+s} \to M_{n,q}$,

$$f\cdot g : S^{2n+t+s} \xrightarrow{(-1)^{s(n+t)}} S^{2n+t+s} \xrightarrow{f\wedge g} M\wedge M \xrightarrow{\varphi} S^n M.$$

Also ψ defines a multiplication in $h^*(\ ;Z_q)$ (See A.T.).

<u>Def</u> $\varphi: M\wedge M \to S^n M$ is a <u>multiplication</u> if $\varphi(1\wedge i) \simeq 1$. (exists only if $q \not\equiv 2 \pmod 4$). φ is <u>commutative</u> if the <u>commutator</u>

$$c(\varphi) \overset{\text{def}}{=} \varphi - (-1)^n \varphi T$$

vanishes ($T: M\wedge M \to M\wedge M$, $(x\wedge y) \to (y\wedge x)$). φ is <u>associative</u> if the <u>associator</u>

$$a(\varphi) \overset{\text{def}}{=} S^n\varphi\,(1\wedge\varphi) - S^n\varphi\,\big((-1)^n \wedge T\big)(\varphi\wedge 1)$$

vanishes. Commutativity (or associativity) of $\varphi \Rightarrow$ " of the multiplication defined by φ.

Suppose
$$c(\varphi) = c_q'\ S^n i\ \eta^2(\pi\wedge\pi) : M\wedge M \to S^{2n+2} \xrightarrow{\eta^2} S^{2n} \subset S^n M$$
$$a(\varphi) = a_q'\ (S^{2n} i)\,\alpha_1\,(\pi\wedge\pi\wedge\pi) : M\wedge M\wedge M \to S^{3n+3} \xrightarrow{\alpha_1} S^{3n} \subset S^{2n} M$$

order 3

Then
$$xy - (-1)^{\deg x \deg y}\, yx = c_q'\,\eta^2\,\overbrace{(\partial_q x)(\partial_q y)}^{\neq 0\,?}$$

$$x(yz) - (xy)z = a_q'\,\alpha_1 \cdot \underbrace{(\partial_q x)(\partial_q y)(\partial_q z)}_{\exists x,y,z:\ \neq 0}$$

Now $M\wedge M \simeq S^n M \vee S^n M$; $T \in [M\wedge M, M\wedge M]$.

$$[M\wedge M, M\wedge M] \overset{\text{Th.2.9}}{=} (1\wedge i)[S^nM, S^nM]\varphi + \psi[S^{n+1}M, S^{n+1}M](1\wedge\pi) + \psi[S^nM, S^{n+1}M]\varphi + (1\wedge i)[S^{n+1}M, S^nM](1\wedge\pi)$$

$$\begin{array}{lcccc}
 & \cong & \cong & \cong & \cong \\
q \text{ odd} & \mathbb{Z}_q \ni 1 & \mathbb{Z}_q \ni 1 & \delta \in \mathbb{Z}_q & 0 \\
q \equiv 0\ (4) & 1 \in \mathbb{Z}_q + \mathbb{Z}_q & 1 \in \mathbb{Z}_q + \mathbb{Z}_q & \mathbb{Z}_q & \eta_1 \in \mathbb{Z}_q + \mathbb{Z}_q + \mathbb{Z}_q \ni \eta_2, \ni (S^n i)\eta^2(S^{n+1}\pi) \\
 & \quad \ni (S^n i)\eta(S^n\pi) : S^nM \to S^{2n+1} \xrightarrow{\eta} S^{2n} \subset S^nM & & &
\end{array}$$

where $\delta = (S^{n+1}i)(S^n\pi) : S^nM \xrightarrow[\pi]{} S^{2n+1} \subset S^{n+1}M$ induces ∂_q.

$T_* : H_*(M\wedge M; \mathbb{Z}_q) \to H_*(M\wedge M; \mathbb{Z}_q)$. One shows

$$T \equiv (-1)^n(1\wedge i)\varphi + (-1)^{n+1}\psi(1\wedge\pi) + (-1)^n\psi\delta\varphi \quad \text{mod} \begin{cases} 0 & q \text{ odd} \\ \mathbb{Z}_q^{\,5} & q \equiv 0\ (4). \end{cases}$$

If q is odd this is an equality, so $(-1)^n\varphi T = \varphi$ since $\varphi\psi = 0$. Hence $c(\varphi) = 0$ if q odd; multiplication is commutative.

φ is unique mod $(1\wedge\pi)^*[S^{n+1}(M), S^n(M)]$; this has 3 generators η_1, η_2 and $i\eta^2\pi$ where η is the Hopf map. If we alter φ, $\varphi \to \varphi + a\eta_1 + b\eta_2$,

<u>Thm</u> (c.f. A.T. <u>Th7.10</u>) $\quad \exists_2\, \varphi$

$$T = (-1)^n(1\wedge i)\varphi + (-1)^{n+1}\psi(1\wedge\pi) + (-1)^n\psi\delta\varphi + \varepsilon_q(i\wedge i)\eta^2(\pi\wedge\pi)$$

where $\varepsilon_q = 0$ or 1, independent of φ. $(\Leftarrow T^2 = 1_{M\wedge M})$.

$$\varphi - (-1)^n\varphi T = \varepsilon_q(S^n i)\eta^2(\pi\wedge\pi) \qquad \boxed{\varepsilon_q = ?} \text{ problem.}$$

Now let q be odd. $a(\varphi) = S^n\varphi(1\wedge\varphi) - \underline{S^n\varphi((-1)^n\wedge T)(\varphi\wedge 1)}$; the second term can be altered.

P an even permutation

$$\begin{tikzcd}
M\wedge M'\wedge M'' \arrow[r, "\varphi\wedge 1"] \arrow[d, "P"] \arrow[dr, phantom, "\circlearrowright"] & M^*\wedge S^n\wedge M'' \arrow[r, "1\wedge T"] \arrow[d, dashed, "P"] \arrow[dr, phantom, "\circlearrowright"] & M^*\wedge M''\wedge S^n \arrow[r, "S^n\varphi"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & M\wedge S^{2n} \\
M''\wedge M\wedge M' \arrow[r, "1\wedge\varphi"] & M''\wedge M^*\wedge S^n \arrow[r, "T\wedge 1 = S^nT"'] \arrow[rr, bend right=60, "\simeq (-1)^n S^n\varphi"'] & M^*\wedge M''\wedge S^n \arrow[ur, "S^n\varphi"'] &
\end{tikzcd}$$

since $(-1)^n\varphi T = \varphi$ if q is odd.

$\Rightarrow a(\varphi) = (S^n\varphi(1\wedge\varphi))(1-P)$ since $\varphi - (-1)^n \varphi T = \varphi(1-(-1)^n T)$; take n even.

$$ep'(M) = \begin{cases} w' x_\pi (M\wedge M) / w'/\pi & p=2 \\ w' x_\pi (M\wedge M\wedge M) / w'/\pi & p=3 \end{cases}$$

$$= \begin{cases} \bar{e}^1 x_\pi (\quad) / \\ \bar{e}^1 x_\pi (\quad) / \end{cases}$$

$$= \begin{cases} C_{1-T} = (M\wedge M) \cup_{1-T} c(M\wedge M) \\ C_{1-P} = (M\wedge M\wedge M) \cup_{1-P} c(M\wedge M\wedge M) \end{cases}$$

If q is odd, $a(\varphi) \in [M\wedge M\wedge M, S^{2n}M] \simeq \underset{\mathbb{Z}_q}{\overset{\shortparallel}{[S^{2n}M, S^{2n}M]}} + \underset{0}{\overset{\shortparallel}{[S^{2n+1}M, S^{2n}M]}}$

$$+ \underset{0}{\overset{\shortparallel}{[\quad " \quad]}} + \underset{\mathbb{Z}_{(q,3)}}{\overset{\shortparallel}{[S^{2n+2}M, S^{2n}M]}} \text{ gen. by}$$

$$S^{2n+2}M \xrightarrow{S^{2n+2}\pi} S^{3n+3} \xrightarrow{\alpha_1} S^{3n} \subset S^{2n}M$$

One checks $H_{3n}(\ ;\mathbb{Z}_q)$ to show the coefficient in $\mathbb{Z}_q$ is trivial; so

$$\boxed{a(\varphi) = a_q (S^{2n} i)\, \alpha_1 (\pi\wedge\pi\wedge\pi)} \quad q: \text{odd}$$

$$= 0 \text{ if } q \text{ is not a multiple of } 3.$$

Hence $a(\varphi) = 0$ if $q \begin{cases} \not\equiv 0 \ (3) \\ \text{odd} \end{cases}$ $\Rightarrow$ associativity.

Assume q odd, $q \equiv 0\ (3)$. A $\mathbb{Z}_3$-basis of $H_*(\underset{ep'(M)}{\overset{\shortparallel}{C_{1-P}}}; \mathbb{Z}_3)$ is

$\underset{3n}{e_0 \otimes_\pi x^3}$, $\underbrace{\ \cdots\ }_{6}$, $\underbrace{e_1 \otimes_\pi y^3}_{3n+4}$ where $x \in H_n(M, \mathbb{Z}_3)$, $y \in H_{n+1}(M; \mathbb{Z}_3)$. By

Nishida's formula $\delta'_*(e_1 \otimes_\pi y^3) = \pm e_0 \otimes_\pi (\Delta y^3)$, hence

$$\delta' H^{3n}(C_{1-P}, \mathbb{Z}_3) \begin{cases} \neq 0 & \text{if } q \equiv 3, 6\ (9) \\ = 0 & \text{if } q \equiv 0\ (9). \end{cases}$$

$1-P \in [M\wedge M\wedge M, M\wedge M\wedge M]$ which has 16 direct summands.

$$\begin{tikzcd}
C_{1-P} \arrow[r, "\exists"] \arrow[d, equal] & C_{a(\varphi)} \arrow[d, Rightarrow] & \\
M\wedge M\wedge M \cup_{1-P} C(M\wedge M\wedge M) \arrow[r, bend left, "S^n\varphi(1\wedge\varphi)"] \arrow[r, bend right, "id"'] & S^{2n}M \cup C(M\wedge M\wedge M) \arrow[dr, "id"'] \arrow[dr, bend left, "C(\pi\wedge\pi\wedge\pi)"] & \\
& & C_{S^{2n}i(a_q\alpha_1)} = S^{2n}M \cup C(S^{2n+2}M) \\
& & C_{a_q\alpha_1} = S^{3n} \cup C(S^{3n+3}) \arrow[u]
\end{tikzcd}$$

$a(\varphi) = S^n\varphi\,(1\wedge\varphi)(1-P)$

$\Leftarrow (S^{2n}i)(a_q\alpha_1)(\pi\wedge\pi\wedge\pi)$

On $C_{a_q\alpha_1}$, $\wp' = 0 \Leftrightarrow a \equiv 0\ (3)$; on $\longrightarrow$, $\wp' = 0 \Leftrightarrow q \equiv 0\,(9)$. Hence

$$a(\varphi) = \begin{cases} 0 & q \equiv 0\,(9) \quad \Rightarrow \text{assoc.} \\ \pm(S^{2n}i)\,\alpha_1(\pi\wedge\pi\wedge\pi) \neq 0 & \text{if } q \equiv 3,6\ (9). \end{cases}$$

so $x(yz) - (xy)z = \pm\alpha_1(\partial_q x)(\partial_q y)(\partial_q z)$

nontrivial example: $\beta_1 \quad \beta_1 \quad \beta_2 \quad \neq 0$

hence $\pi_*(\mathbb{M}_3)$ is not associative.

If $\underline{q \equiv 0\,(4)}$: $c(\varphi) = \varphi(1-T) = \varepsilon_q(S^n i)\,\eta^2(\pi\wedge\pi)$; $\underline{n \text{ even}}$

η^2 is not affected by primary operations.

$H_*(C_{1-T}; \mathbb{Z}_2) = ep^1(M_{n,q})$

$X \xrightarrow{f} Y \xrightarrow{i} C_f \simeq 0$, $C_f = Y \cup_f CX$

$ep^r(X) \xrightarrow{ep^r(f)} ep^r(Y) \xrightarrow{ep^r(i)} ep^r(C_f) \simeq 0$

$\Rightarrow \exists\, D_f : C_{ep^r(f)} = ep^r(Y) \cup C(ep^r(X)) \longrightarrow ep^r(C_f)$, $\quad ep^r(Y) \xrightarrow{ep^r(i)} ep^r(C_f)$

given by a diagonal map

$$\begin{array}{ccc} W^r \times X^p \times [0,1] & \longrightarrow & W^r \times (X \times [0,1])^p \\ w, (x_1, \ldots, x_p), t & \longrightarrow & w, (x_1, t, x_2, t, \cdots) \end{array}$$

(even if f is cellular, D_f may not be). Assume f is cellular and $f_\# = 0 : C_\#(X)\otimes\mathbb{Z}_p \longrightarrow C_\#(Y)\otimes\mathbb{Z}_p$.

$\mathcal{C}_i(C_f)\otimes \mathbb{Z}_p = \mathcal{C}_{i-1}(X)\otimes \mathbb{Z}_p + \mathcal{C}_i(Y)\otimes \mathbb{Z}_p$ split as chain complex.

$\Rightarrow \quad H_i(C_f, \mathbb{Z}_p) = H_{i-1}(X, \mathbb{Z}_p) + H_i(Y, \mathbb{Z}_p)$

$\underset{\hat{x}}{\psi} \qquad \underset{x}{\psi}$

<u>Thm</u> $D_{f*}\left(\widehat{e_i \otimes_\pi x^p}\right) = -\mu\,(\underset{\substack{\#\\ 0}}{\deg x+1})\left(e_{i-p+1}\otimes_\pi (\hat{x})^p\right)$

<u>Proof</u>: essentially the same as $\delta \wp^i = \wp^i \delta$.

<u>Remark</u> This $\Rightarrow$ lemma for $ep^r(\alpha_1) \Rightarrow$ "$\underline{\alpha_1\gamma = 0 \Rightarrow \beta_1\gamma^p = 0}$"

namely

$$\begin{array}{ccc} ep^r(S^n) & \xrightarrow{ep^r(\alpha_1)} & ep^r(S^n) \longrightarrow ep^r(S^n \cup_{\alpha_1} e^{n+2p-2}) \\ \text{hence}\Big| & \wp^p_{ep^r(\alpha_1)} \neq 0 & \Big| \\ e_{p-1}\otimes_\pi x^p & \longleftarrow & \underbrace{e_0 \otimes_\pi x^p}_{n} \end{array}$$

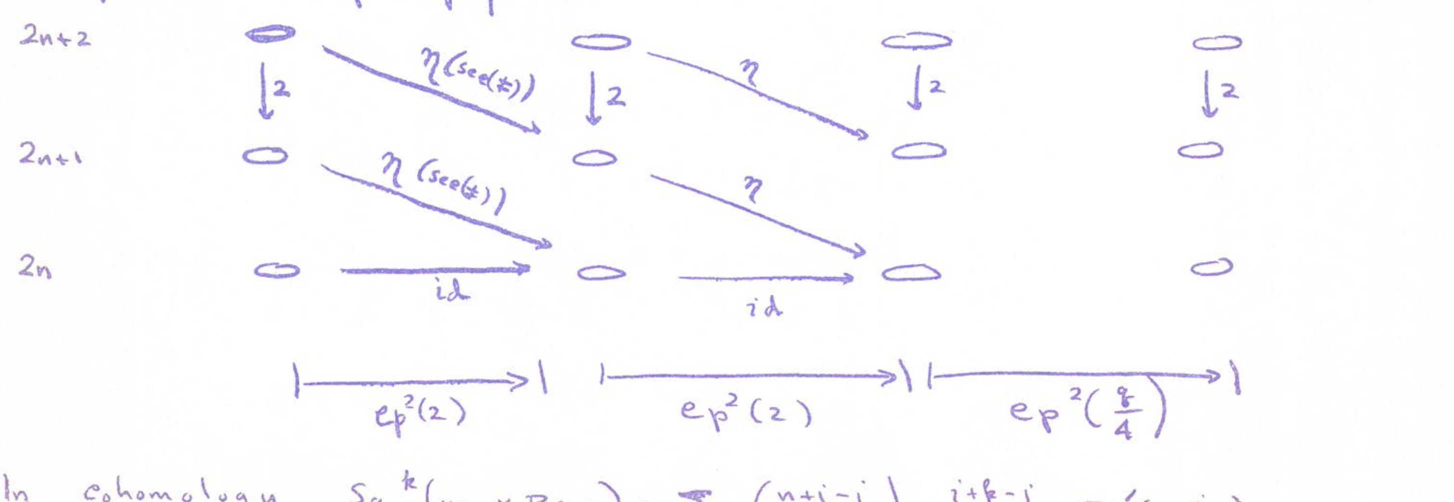
$$\wp^p_*(e_0 \otimes_\pi x^r) = e_0 \otimes_\pi (\underset{\deg n}{\wp^1_* x})^p \qquad n+2p-2$$

If q is even (maybe 2), $n \equiv 0\ (4)$, $S^n \xrightarrow{q} S^n$ so

$$ep^2(S^n) \xrightarrow{ep^2(q)} ep^2(S^n)$$
$$\|$$
$$S^{2n} \cup e^{2n+1} \cup e^{2n+2} \simeq S^{2n} \cup M.$$

Draw the cells as below:

Decompose the map $ep^2(q)$:



In cohomology $Sq^k(w_i \times P(u)) = \sum_j \binom{n+i-j}{k-2j} w^{i+k-j} \times P'(Sq^j u)$.

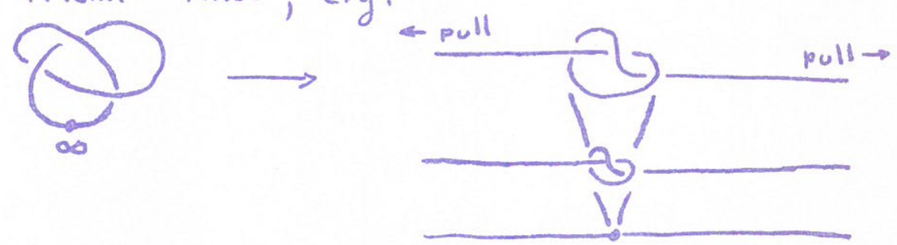
In $ep^2(M_{n,2})$ $\quad Sq^2(w^i \times \underset{\deg n}{P(u)}) = \binom{n+i}{2} w^{n+i+2} \times P(u) + \underbrace{w^{n+i} \times P(Sq^1 u)}_{\neq 0 \text{ if } q=2} \qquad (\#)$

Hence $\varepsilon_q = \begin{cases} 1 & q \equiv 4\ (8) \\ 0 & q \equiv 0\ (8) \end{cases}$

MIT Topology Seminar 4/24/68

Henry Laufer: Some Link Invariants

A knot is an embedding $K: S^1 \to S^3$. Is K isotopic to a trivial knot  ? (an isotopy is a homotopy $f(s,x)$ where $f(s,t_0)$ is also a knot for each t_0). If K is smooth, it is isotopic to a trivial knot; e.g.

(1) 

A link $L: \cup S^i \to S^3$ is an embedding. If L has 2 components (L_1, L_2), the linking number $\ell(L_1, L_2)$ is defined to be the homology class of L_2 in $S^3 - L_1$ (use Alexander duality). An equivalent definition: Bound by L_1 an oriented 2-manifold M. Count the number of intersections of L_2 with M (+ or −). The knots  have linking no. 1. Linking no. is preserved under homotopy if at each level the images of the knots remain disjoint. Hence linking no. is certainly an isotopy invariant (topological category). In fact it is the only invariant under that kind of homotopy.

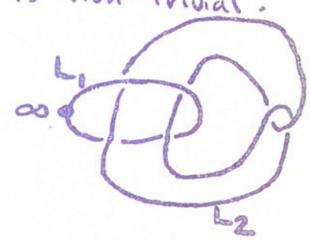
The Borromean rings (2) are trivial pairwise, so linking numbers do not characterize them, since they are non-trivial under isotopy.

(2) 

The 2-component link (3) has 0 linking number but is non-trivial.

<u>Proof</u> We take a <u>branched cyclic covering</u> of S^3, branched over L_1. This is defined as follows:

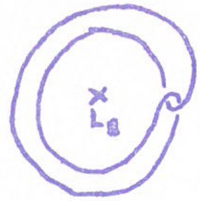
$$\pi_1(S^3 - L_1) \longrightarrow \underset{\underset{\mathbb{Z}}{\parallel}}{H_1(S^3 - L_1)} \xrightarrow{\rho_p} \mathbb{Z}_p \quad \text{has kernel}$$

(3) 

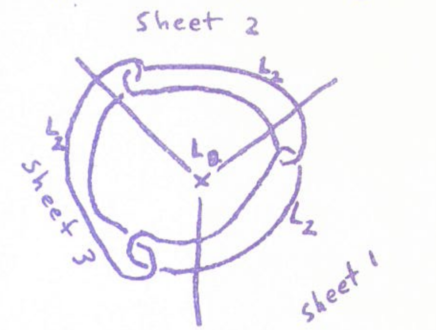
equal to paths with linking number in L_1 divisible by p. This defines a p-sheeted covering surface of $S^3 - L_1$. One then reinserts the knot L_1 (surround L_1 by a solid tube; lift it to the covering, where it remains a solid tube; replace L_1 in its center).

In our case we take a 3-fold covering, $p=3$. The covering surface turns out to be S^3. L_2 can be lifted in 3 ways to this surface; the 3 knots thus obtained are pairwise linked, hence L_1 and L_2 were non-trivially linked (4). //

(4) redraw: (3)



lifting to 3-sheeted covering:

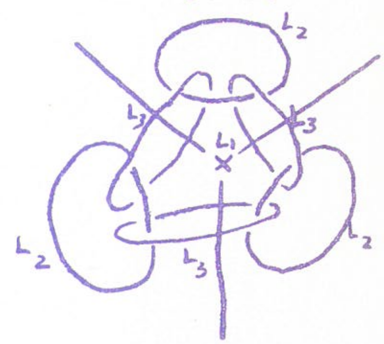


In the branched covering space different liftings of L_2 may have non-0 linking numbers; these are preserved under isotopy of L_2 because such isotopies lift to the covering space.

In order to isotop L_1 (then the 2 kinds of isotopies - of L_1 and of L_2 - can be put together as often as needed and any isotopy of the link reproduced) one shows any P.L. isotopy (proofs given here only for P.L. category) looks locally like either an ambient isotopy, or an isotopy which is trivial except at a single vertex at any given level. (an isotopy is not a P.L. map on the product $US^i \times I$.) At such a vertex one may have a situation like (1). //

The Borromean rings are equivalent to (5). Take a 3-sheeted covering and lift L_2, L_3 (6). Once again the linking numbers between liftings of L_2, L_3 are non-trivial.

(5)



Note that the invariants thus obtained have many parameters, e.g. p.

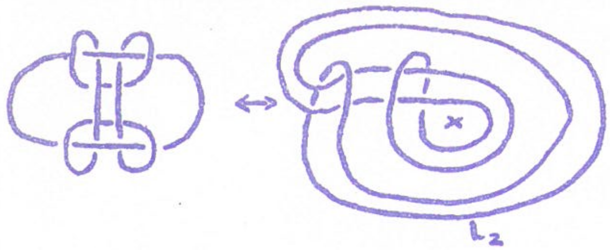
It is possible that all these liftings are trivial but the link is not. The isotopy (6) of a trivial link to the link at right shows it is

(6)

trivial; however one gets invariants which are not 0, by the above branched covering process. (in fact liftings of L_2 look pairwise like (4); one takes then another covering, of the covering space, now over a lifting of L_2; and finds non-0 linking numbers of various liftings.).

Another example is shown in (7) in 2 forms. Take a branched covering corresponding to the commutator subgroup of $\pi_1(S^3 - L_1)$, and lift L_1 to each sheet (8). Any finite subset of these is trivial, but a single lifting L_2^* is a non-trivial element in the first homotopy group of $\Sigma -$ (the rest of the liftings), where Σ is the covering space.

(7)



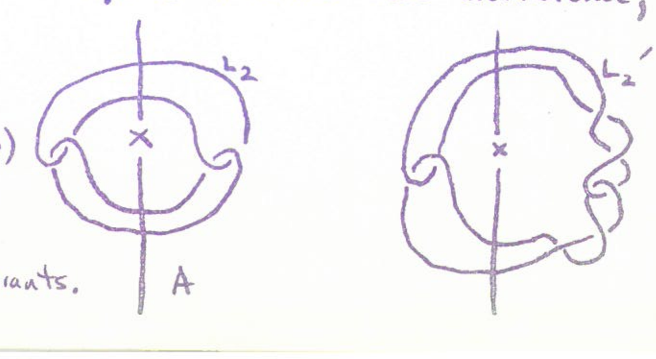
In (6), L_2 is non-trivial as a homotopy class of $\pi_1(S - L_1)$. One wishes to finesse the problem by approximating all (continuous) links by smooth ones and showing all sufficiently close smooth (or P.L.) links give the same invariants. But (6) cannot be smoothed, of course. The solution is to let p be a prime. Take p-fold coverings Σ and prove $H_1(\Sigma, \mathbb{Z}_p) = 0$ for a smooth knot (note coefficients: $\mathbb{Z}_p$). Define linking numbers <u>mod p</u> only. These <u>are</u> isotopy invariants, and the smoothing procedure works.

(8)

Usually $\pi_1(S^3 - L_1 - L_2)$ is the source of link invariants. But there is an example (9) of 2 links with homeomorphic complements, which are essentially different as links. (for other link invariants see Milnor's 1954 paper in the Annals, and the Lefschetz symposium). Their 2-fold coverings (10) show the difference; taking a second covering over one of the liftings of L_2, L_2' one gets unequal invariants.

(9)

(10)



(25-1)

MIT Topology Seminar 5/6/67

Samuel Gitler (Mexico): <u>Higher Order Cohomology Operations</u>

This work was done in collaboration with Mahowald and Milgram.

Frank Adams in "Chern Characters and the Structure of Unitary groups" proved for a stable complex bundle ξ over a CW-complex X, with ξ trivial over the $(2q-1)$-skeleton, $ch_q(\xi)$ is integral and $2^r ch_{q+r}(\xi)$ is integral mod 2. Atiyah-Hirzebruch proved non-embedding theorems using divisibility properties of characteristic numbers in "Quelques theoremes de non-plongement." We get theorems like this assuming divisibility conditions stronger than the above universal ones.

<u>Thm 1</u> If $2^{2r-s} ch_{q+2r}(\xi)$ is integral mod 2 and its reduction mod 2 is non-0, then $X \not\subset S^m$ if $m \leq 2(2q-s+4r-1)$.

<u>Remark</u> Specializing to complex projective spaces, get essentially Atiyah-Hirzebruch's results.

<u>Proof</u> If $2^{2r-s} ch_{q+2r}(\xi)$ is integral mod 2 and its reduction mod 2 is not 0, Adams proved $c(Sq^2)\rho_2(ch_q) = \rho_2(2^r ch_{q+r}) \neq 0$. If $X \subset S^n$ $(s=0)$ Sq^{2r} must be non-0 in the dual complex in S^n. But by dimensions this cannot occur. The idea of the proof is essentially to generalize this to higher-order operations. //

Let $\alpha(n)$ = no. of 1's in the dyadic expansion of n.

<u>Thm 2</u> Let
$$\theta(n) = \begin{cases} 2\alpha(n) & \alpha(n) \equiv 1,2 \mod 4 \\ 2\alpha(n)+1 & \alpha(n) \equiv 0 \mod 4 \\ 2\alpha(n)+2 & \alpha(n) \equiv 3 \mod 4. \end{cases}$$

Then if $n \equiv 3 \mod 4$, $RP^n \not\subset R^{2n-\theta(n)-1}$

and if $n \equiv 7 \mod 8$, $RP^n \not\subset R^{2n-\theta(n)}$

<u>Remark</u> In case $n = 2^k - 1$, $k \geq 4$ this is the embedding result of Ian James, which is known to be best possible. We conjecture our result is best possible.