Don Anderson: Relative Cohomology and Hopf Algebras.

Examples: Let \( A = \text{mod } 2 \) Steenrod algebra.

1) \( C = H^* (\text{MSpin}) = C_{\text{Spin}} \) is the direct sum of cyclic modules isomorphic to \( A, A/A(Sq^1, Sq^2) \) and \( A/A.Sq^3 \). MSpin is the Thom space of bundles with \( w_1 = w_2 = 0 \).

2) \( C_{SO} = H^* (\text{MSO}) \) \( \langle w_1 = 0 \rangle \). Wall showed \( C_{SO} \) is the direct sum of cyclic modules isomorphic to \( A \) or \( A/A.Sq^1 \).

What we want is a natural way of extracting relations. Let \( H \) be a Hopf algebra, that is \( \mu: H \otimes H \to H \) and \( \Delta: H \to H \otimes H \) maps of algebras. \( C \) is a coalgebra over \( H \); a left \( H \)-module with a map of modules \( r: C \to C \otimes C \). Now \( H \otimes H \) acts on \( C \otimes C \), so \( H \) acts on \( C \otimes C \) via \( \Delta \).

\[
\begin{array}{ccc}
H & \to & H \otimes H \\
\gamma & : & H \otimes H \\
\eta & : & \gamma \\
\end{array}
\]

Weak form of theorem: If \( C \) is a coalgebra over \( H \) and \( A = H \otimes_\mathbb{Z} \mathbb{Z}_2 \) \( = \mathbb{Z}_2 \otimes C H \), then \( \gamma \) induces a monomorphism \( H \otimes_\mathbb{Z} \mathbb{Z}_2 \to C \), and if this map is \( H \)-split, \( C \) is projective as an \( H \)-module relative to \( A \). [The splitting is the troublesome condition].

Let \( M \) be a module over \( H \), \( A \subset H \) a subalgebra.

Dfn. \( M \) is relatively free with respect to \( A \) if \( M = H \otimes_\mathbb{Z} N \) for some \( A \)-module \( N \).

If \( A \) is a free right \( A \)-module, \( H \otimes_\mathbb{Z} \) is exact.

\[
\text{Ext}_H^*(M, \mathbb{Z}_2) = \text{Ext}_A^*(N, \mathbb{Z}_2)
\]

Dfn. \( P \) is relatively projective if it is a direct summand in a relatively free module. Some equivalent properties:

- an \( A \)-linear splitting exists.

\[
M \longrightarrow M' \quad \quad P \quad \quad H\text{-linear}
\]
relatively free $\Rightarrow$ relatively projective

\[ M \longrightarrow M' \]
\[ \otimes_A N \leftarrow \quad N \]
\[ \otimes_A N \]

$P$ is relatively projective if the restriction map

\[ \text{Ext}_H(P, M) \longrightarrow \text{Ext}_A(P, M) \]

is injective for all $M$.

\[ P \longrightarrow H \otimes_A P \]
\[ \text{Ext}_H(P, M) \longrightarrow \text{Ext}_A(H \otimes_A P, M) \cong \text{Ext}_A(P, M) \]

\[ 0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0 \]

A-linear

H-linear

\[ \text{Hom}_H(P, N) \longrightarrow \text{Hom}_H(P, N'') \longrightarrow \text{Ext}_H^1(P, N') \]
\[ \text{Hom}_A(P, N) \longrightarrow \text{Hom}_A(P, N'') \longrightarrow \text{Ext}_A^1(P, N') \]

onto $\Leftrightarrow$ lifting.

\textbf{Proof} Leave out simple Hopf Algebra theory.

\[ C \longrightarrow C \otimes C \overset{\otimes 1}{\longrightarrow} (H \otimes_A Z_2) \otimes C \]
\[ \otimes 1 \quad \text{onto} \quad \text{Eomega} \]

\[ C \] is an \textit{H}-summand in $(H \otimes_A Z_2) \otimes C = H \otimes_A C$.

\[ C \otimes_\text{H} \text{Ext}_H^1(C, M) \text{ Hom}_A(C, M) \text{ is injective and can be split, canonically} \]

with respect to $M$. $\nLeftrightarrow$

If $C = H^*(M, S_0)$, $M = Z_2$, $A = E(S_q)$, $H =$ Steenrod algebra.

This is a nice theory which doesn't do the job.

\textbf{Thm} If $H$ is finite-dimensional and $Z_2 \otimes_A H \otimes_A Z_2 \longrightarrow Z_2 \otimes_A C$ is injective, then the map $\phi: C \longrightarrow H \otimes_A Z_2$ exists ($H$-linear)

\textbf{Proof} (Anderson in special cases, Larson in general) $\nLeftrightarrow$

\textbf{Thm 1} If $H$ is a finite-dimensional Hopf algebra, $H_n$ the highest
non-vanishing group, then $H^n$ is $\mathbb{Z}_2$. Map $H \to H^*$ by evaluation (grade $H^*$ negatively) on dual generator in $H^*_n$; it is an isomorphism of left and right $H$-modules.

Thm 2 $\chi(H\Theta_A Z_2)^* \cong H\Theta_A Z_2$. 
Vincent Giambalvo: Odd Torsion in $<8>$-cobordism

Thm A. $\Omega^*_<8> = <8>$-cobordism, has no $q$-torsion for $q \geq 3$.

Given a sequence of spaces and maps $\{Y_k, f_k, j_k\}$ with $j_k: Y_k \to Y_{k+1}$ and $f_k: Y_k \to BO_k$ such that $Y_k \to Y_{k+1}$ commutes, $BO_k \to BO_{k+1}$.

we say a manifold $M$ has a $Y$-structure if $M \to BO_{n+k}$ lifts.

We use this to define a bordism relation $M \sim N$, where $(M, \partial M) \sim (N, \partial N)$ if there is a manifold with boundary $M \cup N$, and a $Y$-structure compatible with those of $M$ and $N$.

Thm. $\Omega^*_n \cong \Pi_{n+k}^* ( M^*_k \oplus Y_k )$ for large $k$ (see notes of Lashof).

where $Y_k \to BO_k$ is the canonical bundle.

Reasonable algebraic structures are obtained by limiting $Y$'s. One case is the $n$-connected covering $BO_{<n+1}> \to BO_{<n>}$ (killing first $n$ homotopy groups), with $f*: \Pi_i ( BO_{<n+1>}) \to \Pi_i (BO_{<n>})$ an isomorphism for $i \geq n$, and $BO_{<n+1>}$ $n$-connected. Then $BO_{<1>} = BO$, $BO_{<2>} = BSO$, $BO_{<3>} = BO_{<4>} = BSpin$. In our case $BO_{<8>}$ we have a fibration $K(\mathbb{Z}, 3) \to BO_{<8>} \to BO_{<4>} = BSpin$; we have the Whitney sum in this space, for the following lifting exists:

$$
\begin{array}{ccc}
BO_{<8>} \times BO_{<8>} & \to & BO_{<8>} \\
\downarrow & & \downarrow \\
BSpin \times BSpin & \to & BSpin,
\end{array}
$$

by obstruction theory. Thus in $\Omega_{<8>}$, $[M] \times [N] = [M \times N]$. There is a Thom space $MO_{<8>} = M^*_k \oplus Y_k$. In proving Thm A we use the steps:

1. $H^* (MO_{<8>}, \mathbb{Z})$ has no $q$-torsion.
2. $H^* (MO_{<8>}, \mathbb{Z}_q)$ is a free $\Lambda$-module.

One completes the proof using the Brown-Peterson spectrum.

$H^k (BSpin) = \mathbb{Z}_q [P_i^*]$, $i = 1, 2, \ldots$ Pontrjagin classes.

$H^k (\mathbb{Z}, 3; \mathbb{Z}_q) = \mathbb{Z}_q [\mathbb{L}] \otimes E(1) (E = exterior algebra)$. 

\[\text{MIT Topology Seminar}\]

\[\text{10/2/67}\]
\[ H^*(BO\langle 8 \rangle) = \mathbb{Z}_2 \left[ i \mathcal{P}_i \mid i \neq \frac{1}{2}(q^i + 1) \right] \otimes \mathbb{Z}_2 \left[ \mathcal{F}_i \mid i = \frac{1}{2}(q^i - 1) \right] \]

Proof of \( \Box \) apply universal coefficient theorem.

Remark \( H^*(BO\langle 8 \rangle) \) does not have a splitting over \( \mathbb{Q} \); the cohomology of the fiber enters.

Proof of \( \Rightarrow \) use \( e' : a \rightarrow H^*(MO\langle 8 \rangle) \) is free if \( e(\alpha) = \alpha \cdot U \) is a monomorphism and if \( e \left[ \text{primitive elements of } \mathcal{A} \right] \) is a monom.

(from Milnor-Moore Hopf Algebra theory), where \( \alpha = \chi / \chi \beta \).

Use of spectrum: there is a spectrum \( X \) with \( H^*(X) = \mathcal{A} \).

\[ H^*(MO\langle 8 \rangle) = \sum \alpha \cdot b_{\alpha} \]

\( MO\langle 8 \rangle \rightarrow \text{product of } S^1 \times X \) (suspension).

This map has \( b_{\alpha} \leftarrow 1_{\alpha} \) in cohomology; the homotopy of \( (\text{prod. } S^1 X) \) is a free abelian group. This map is an isomorphism in homology, so modulo torsion one would have an embedding of torsion groups in free abelian groups.

This could also be done using the Adams spectral sequence.
We want to study immersions of closed differentiable manifolds \( f: M^m \to N^n \) with \( k = n - m > 0 \). One method is to classify them up to regular homotopy.

**Def.** Let \( f, g: M \to N \) be 2 immersions. Then \( f \sim g \) if there is a homotopy \( f_t: M \to N \) such that each \( f_t \) is an immersion and the induced map of tangent bundles \( df_t: TM \to TN \) is a homotopy.

This classification is delicate and difficult. Another approach is based on Thom's cobordism construction.

**Def.** Let \( f: M \to N, g: M' \to N' \), \( \dim M = \dim M', \dim N = \dim N' \). Say \( f \) is immersion cobordant to \( g \), \( f \equiv g \), if there are manifolds \( V, W \) with \( \dim V = \dim M + 1, \dim W = \dim N + 1 \) such that \( \partial V = M - M' \), \( \partial W = N - N' \), and there is an immersion \( F: V \to W \) giving a commutative diagram:

\[
\begin{array}{c}
 M \subset V \to M' \\
 f \quad M, \quad N, \quad g \\
 N \subset W \to N'.
\end{array}
\]

**Example 1**

In this case \( M \) and \( M' \) are also regularly homotopic.

**Example 2** In general \( \equiv \) is weaker than \( \sim \). \( N \) is a torus and \( M \) is a circle on \( N \) with 2 loops. A tangent winding number is defined on \( N \) and the T WN of \( M \) is 2. However by putting \( N \) inside a 3-sphere and doing surgery, \( (N, M) = (S^3, S^1) \).

We can assume various structures on \( M, N \), e.g. orientability.

**Def.** \( \Omega \) (m,n), \( \Pi (m, n) \) denote equivalence classes of orientable, resp. arbitrary, immersion pairs of dimensions \( m \) and \( n \), mod \( \equiv \).
To study these we use bordism groups $\Omega_n(x)$ and $\pi_n(x)$ defined using bordism classes of maps $N^n \to X$.

Thm $\Omega^\text{imm} (m,n) \simeq \Omega_n \left( \Omega^r \Sigma^r \text{MSO} (k) \right)$, $k = n - m$, where $r$ is large (in stable range; $r > m - k$), and

$$\pi_n (m,n) \simeq \pi_n \left( \Omega^r \Sigma^r \text{MO} (k) \right).$$

We can get rid of the dependence on $r$ by the following device: let $Q$ be the functor $QX = \bigcup_r \Omega^r \Sigma^r X$; we have inclusions $\Omega^r \Sigma^r X \subseteq \Omega^{r+1} \Sigma^{r+1} X$ obtained inductively; $X \subseteq \Omega X$, $\Sigma X \subseteq \Omega^2 X$, $\Omega^2 X \subseteq \Omega^2 X$, etc.

II History. Similar theorems have been considered by Wall, Stong, and R. Wells. Wall (Comm. Math. Helv. 35 (1961)) showed for embeddings that $\Omega^\text{emb} (m,n) \simeq \Omega_n (\text{MSO} (k))$. Stong showed for arbitrary maps that $\Omega^\text{map} (m,n) \simeq \Omega_n (\Omega^r \text{MSO} (k+r))$, $r$ large. Thus the results fit together according to

$$\Omega^\text{emb} (m,n) \to \Omega^\text{imm} (m,n) \to \Omega^\text{map} (m,n),$$

induced from $\text{MSO} (k) \subseteq \Omega^r \Sigma^r \text{MSO} (k)$ and $\Sigma \text{MSO} (k) \to \text{MSO} (k+1)$.

Wells considers $\Omega^\text{imm, sph}$, equivalence classes of immersions where $f: M \subseteq S^n$, $f': M' \subseteq S^n$, and $F: V \subseteq S^n \times I$, and shows $\Omega^\text{imm, sph} (m,n) \simeq \pi_n (Q \text{SO} (k))$. We also have $\Omega^\text{imm, sph} (m,n) \to \Omega^\text{imm} (m,n)$.

III Proof of theorem We use a theorem of Hirsch (cf. Phillips Top. 6 (1966) and Hirsch TAMS 1959). Let $\text{Max} (TM, TN)$ be the space of bundle maps having maximal rank on each fiber, in the compact open topology. Give $\text{Imm} (M^n, N^n)$ the $C^\infty$ topology.

Thm There is a weak homotopy equivalence $\text{Imm} (M^n, N^n) \to \text{Max} (TM, TN)$, $m < n$.

Cor There is a 1-1 correspondence between regular homotopy classes of immersions $f: M \subseteq N$ and regular homotopy classes of immersions $g: M \subseteq N \times \mathbb{R}^r$ with $r$ everywhere independent vector fields.
Thm: $E_0 = E_1$. Going from $f \to g$ is trivial. Given $g : M \to N \times R$ (enough 
to do for $r=1$) with its normal vector field $X$. Take $dg : TM \to T(N \times R)$.
Modify $dg$ and $X$ continuously to $dg'$ and $X'$ such that $X'$ is parallel
to $IR^r$, then get map $h : TM \to TN$ by sliding down on $X'$. Then
apply theorem. //

In proving our theorem we give only the oriented case; the non-oriented
is essentially the same. We define maps $\Omega_1^* (m, n) \xrightarrow{\varphi} \Omega_n (\Sigma^r \Sigma^r MSO(k))$
which are inverses.

1. Definition of $\varphi$ Given $f : M \to N$, let $*$ be the basepoint of $N$.

   1) modify via a regular homotopy $f \sim f_0 : M \to N - *$.
   2) $i : N - * \subset (N - *) \times IR^r$. Now if $f_0$ is regular homotopic
to $f_1 : M \subset (N - *) \times IR^r$ such that $f_1(M)$ has a normal $r$-frame
from $IR^r$, $\nu_{f_1}$ normal bundle.
   3) look at the classifying map of unit disk bundles $E(\nu_{f_1}) \to E(\nu_k \oplus E)$
   Extend (by Thom construction) to a map $f_2 : \Sigma N \to \Sigma^r MSO(k) =$
   $E(\nu_k \oplus E) \cup \{\infty\}$.
   4) Take adj. $f_3 : N \to \Sigma^r \Sigma^r MSO(k)$; $\varphi(f) = [f_3]$.

Repeating steps 1-4 on $F : V \to W$, a cobordism $f = f'$, shows $\varphi$ is
well-defined. If $k = 1$ one may have to do some surgery to avoid the
following situation:

Want $F(V)$ to miss an arc in $W$ between $*_{N'}$ and $*_{N'}$. Then argument is same with
$W - I$ replacing $N - *$.

2. Definition of $\varphi$ Given $f_3 : N \to \Sigma^r \Sigma^r MSO(k)$

   1) Take adjoint $f_2 : \Sigma^r N \to \Sigma^r MSO(k)$.
   2) Make $f_2$ transverse regular and let $M = f_2^{-1}(BSO(k))$, 

obtaining $f_1 : M \to (N - *) \times IR^r$ with normal $r$-frame.
3) Apply Hirsch Thm. to obtain $f_0: M \simeq N$; $\psi(f_3)=[f_0]$.

The same argument shows $\psi(f_3)$ is an invariant of the bordism class of $f_3$, so $\psi$ is well-defined; $\psi$ and $\psi$ are clearly inverses. 

It is interesting to compare this argument with that of Wall for $\Omega^{emb}(m,n)$. Given $f: M \simeq N$, let $\nu(M,N)$ be the normal bundle.

$E(\nu) \to CN$ is a tubular neighborhood of $M \subset N$. $E(\nu) \to E(S_k)$ is the classifying map. Extend to $N \to M_{DR} = E(S_k) \cup \infty$, where $\psi$ gives the bordism class.

Example: $\pi^1_{imm}(1,2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by $\phi \in \mathbb{C}P^2$, $P^1 \subset \mathbb{C}P^2$, and $S^1 \subset S^2 \subset \infty \subset \mathbb{C}P^2$.

IV. Work to be done. $QX$ is an infinite loop space, $QX = \Omega QEX$.

There exist extended Steenrod squares $Q_i$ ($Sq_k$ is analog),

$$Q_i: H_{2k+i}(QX; \mathbb{Z}_2) \to H_{2k+i}(QX; \mathbb{Z}_2).$$

Say $Q_i, \ldots, Q_{ir}$ is admissible $\iff 0 \leq i_1 \leq i_2 \leq \ldots \leq i_r$. Then $H_k(QX; \mathbb{Z}_2)$ is an exterior algebra on generators given by applying all admissible monomials to a basis for $H_k(X, \mathbb{Z}_2)$.


Mod 2: Browder?
Postnikov Systems for U & BU - Bill Singer

For a CW complex $X$, define

$X(r,\alpha) \xrightarrow{\pi} X$ so that $\pi$ induces an iso. of homotopy in dimensions $\geq r$ and $X(r,\alpha)$ is $(r-1)$-connected

$X \xrightarrow{j} X(\alpha,\alpha)$ s.t. $j$ induces an iso. of homotopy in dim. $1-r$ and $\pi_k(X,\alpha) = 0$ if $k > r$.

Given a spectrum $\{Y_f\}$ with homotopy equivalences $Y_f \xrightarrow{\Delta} \Sigma Y_{f+1}$

define $\bar{H}^p(X) = [X, Y_f]$. There is a spectral sequence

$E_2^{p,q} = H^p(X, \pi_q(Y_f)) \Rightarrow \text{grad} \bar{H}^p(X)$ (some filtration).

The differentials are related to Postnikov systems of $Y_f$. A cohomology class of $E_2$ corresponds to a map $f$ (below). If $f$ lifts to $\bar{f}$, there is an obstruction $\bar{r}(f, p, r+2) \in H^{r+2}(Y_f(p, r))$ to lifting once more; $\bar{f}$ lifts $\iff \bar{f}^* \bar{r} = 0$. Roughly, $\bar{r}$ is the differential of $f$. If $Y$ is an $H$-space, the order of any $\bar{r}$ is finite. Ask: what smallest $\alpha$ has $\alpha \cdot d = 0$ if $d$ is a differential?

To get complex K-theory, $Y_{2f} = BU \times \mathbb{Z}$, $Y_{2f+1} = U$. Use Bott periodicity to get $\Sigma$-maps. Compute

$H^*(BU(\mathbb{Z}_n, \ldots, \infty), \mathbb{Z}_p)$ \{Stable case: Adams 1961, $p=2$: Stong 1963\}

$H^*(U(\mathbb{Z}_n, \ldots, \infty), \mathbb{Z}_p)$

Use spectral sequence of Eilenberg-Moore.
Given \( \xymatrix{ F \ar[r] & E_0 \ar[r] & B_0 } \) a fibration get induced fibration \( \xymatrix{ F \ar[r] & F \ar[d] \ar[r] & E \ar[r] & E_0 \ar[d] \ar[r] & B } \).

If
1) all cohomology of finite type
2) \( B_0 \) simply conn.
3) ring is noetherian (I use \( \mathbb{Z}_p \) = field)

then there is a spectral sequence

\[
E_2 = \text{Tor}_2^{\mathbb{Z}_p} \left( H^*(E_0), H^*(B) \right) \Rightarrow E_{\infty} = \text{grad. } H^*(E).
\]

which
1) is a s.seq. of algebras; each differential a derivation
2) is natural (another fibre square gives maps)
3) if \( B = \text{pt} \), gives s.seq. converging to \( H^*(F) \).

By induction assume \( B_0(2n-2, \infty) \) is known. Have fibration

\[
KZ,2n-3 \to BU(2n-1,0) \to BU(2n-2,0). \text{ Regard this as induced from } KZ,2n-3 \to L(Z,2n-2) \to KZ,2n-2.
\]

To compute first fibration look first at the second fibration: \( KZ,2n-3 = KZ,2n-3 = KZ,2n \).

See Larry Smith, Ill. Journal June '67, "Cohomology of 2-stage Postnikov systems."

By Cartan,

\[
H^*(KZ,n, Z_p) = F[V_n] = Z_p[V_n^+] \otimes E[V_n^-] \quad \text{BU}(2n-2, \infty) \to KZ,2n-2 \to \text{pt},
\]

where

\[
\begin{align*}
\text{free commutative} & \quad \text{algebra on } V_n^+ \quad \text{even dim.} \\
\text{free algebra on } I & \quad \text{odd dim.}
\end{align*}
\]

for some vector space \( V_n \). Suspension \( \sigma : \mathbb{Q}^*H(KZ,n, Z_p) \to P^{n-1}H(KZ,n-1; Z_p) \). To state kernel define \( \beta^{P^t} : V_n^{-} \to \text{ker } \sigma \)

\[
\beta^{P^t}(P^tI_n) = \beta^{P^t}P^tI_n \quad 2t+1 = \text{deg } P^tI_n.
\]

Then \( \sigma(P^tP^tI_n) = P^{t+1}P^tI_{n-1} = (P^tI_{n-1})^b \). Cartan shows \( \beta^{P^t} \) is iso. \( (V_n = \text{Steenrod monomials operating on } I_n) \).

\[
E_2 = \text{Tor}_{H^*(KZ,n, Z_p)} \left[ Z_p, Z_p \right] = \Gamma[sV_n^-] \otimes E_1[sV_n^+] \]

\( \Gamma \) is the loop space.
\( \ker \sigma \subset \text{even dimensions hence the spectral seq. must kill } E[sv_n^+] \).

**Thm (Smith)** All differentials are 0 except
\[
d_{p-1} \left[ s(P^1_{i,n}) \right] \cong s(P^1_{i,n}).
\]

**Thm** Given a Hopf fiber square (i.e. \( E,B \) homotopy commutative, homotopy associative \( H \)-spaces & \( p,f \) \( H \)-maps)

\[
\begin{array}{ccc}
K(\mathbb{Z},n) & \to & \mathbb{E} \\
\downarrow & & \downarrow p \\
B & \to & K(\mathbb{Z},n)
\end{array}
\]

Where \( H^*(B,\mathbb{Z}) \) is free commutative as an algebra (not necessarily as a Hopf algebra), there is a subspace \( M_n \subset V_n \) such that \( \ker f^* = \mathbb{F}[M_n] \) and

\[
H^*(E,\mathbb{Z}) = H^*(B,\mathbb{Z}) \otimes A(\sigma M_n) \otimes \mathbb{E} \left[ s \frac{\beta^p}{\beta^p M_n^-} \right]
\]

The category of bicommutative Hopf algebras is abelian, and \( \ker f^* \) is taken in this category.

Inductively determine \( \ker f^* \) and \( \ker f^* \):

\[
\begin{align*}
H^k(U(2n-2,0)) & \leftarrow f^* \leftarrow H^k(K(\mathbb{Z},2n-2)) \\
H^k(U(2n-1,\infty)) & \leftarrow g^* \leftarrow H^k(K(\mathbb{Z},2n-1)) \\
H^k(BU(2n,\infty)) & \leftarrow f^* \leftarrow H^k(K(\mathbb{Z},2n))
\end{align*}
\]

\( \ker f^* = \tilde{\alpha} [\beta P^1_{2n}] \), \( \ker g^* = \tilde{\alpha} [\beta P^1_{2n-1}] \).
Answers: there are indecomposable cohomology classes $\theta_{2i} \in H^{2i}(BU, \mathbb{Z}_p)$

$\mu_{2i} \in H^{2i+1}(U, \mathbb{Z}_p)$

such that $H^{*}(BU(\mathbb{Z}_n, \infty), \mathbb{Z}_p) = \mathbb{Z}_p \left[ \theta_{2i} \mid \theta_{2i}(i-1) \leq n \right] \otimes \prod_{t=0}^{p-2} F[M_{2n-2t}]$

as Hopf algebras where for primes $p$,

$\sigma_p(m) = \sum$ coefficients in $p$-ary expansion of $m$.

Define $\mathbb{Z}_p$-modules $M_n$ such that $F[M_n] \cong \tilde{A}(\beta P_{1n})$

(subalgebra of cohomology of Eilenberg-MacLane space). This $\mathbb{Z}_p$ result determines $\mathbb{Z}$ case since the Bockstein spectral sequence is known.

Let $d_{n,r}$ be differentials in Atiyah-Hirzebruch spectral seq. above. (Bott periodically takes out 1 index). Then $\alpha_{n,r} d_{n,r} = 0$ where

$$\alpha_{n,r} = \frac{(r-1)!}{\prod_{p \mid \text{plim } n} p^\text{plim } \left[ \frac{(n-1) - \sigma_p(r-1)}{p-1} \right]}$$

where $\text{plim } n = \text{ least integer } \geq n$ and $0$. 
For example the tangent microbundle to $M$ is $\mathbb{R}^n \times M$.

**Thm (Kister–Masur)** Every microbundle has an equivalent representative which is an $(\mathbb{R}^n)$ bundle.

**Def** Given $f: M^n \to W^{n+k}$, $f$ is locally flat if for each $x \in M$ there is $(U, h)$ such that $h: (\mathbb{R}^k, \mathbb{R}^n) \to (U, \text{Unf}(M))$ is homeo.
(2-2)

**Def:** \( \nu \) is a normal bundle for \( f \) if \( T_m \oplus \nu \approx f_*(T_N) \).

**Def:** \( \nu \) is a tubular nbd. if \( E_\nu \subset N \) is commutative.

\[ \begin{array}{c}
\circ \downarrow \\
\nu \downarrow \\
M \\
\end{array} \]

and \( E_\nu \) is an open nbd. of \( f(M) \).

**Prop:** Tubular nbd. \( \Rightarrow \) normal bundle.

**Thm (Browder 1963):** There is a (top.) tubular nbd containing no disk bundle.

**Thm (Hirsch 1965):** PWL: there is a \( \Sigma^8 \subset W^{12} \) with no disk tubular nbd.

**Thm (Haefliger & Wall):** PWL: \( \eta < k \Rightarrow \) there is a tubular nbd.

Hence multiplying by \( R^n \), it is always possible to get a stable normal bundle.

**Thm (Sanderson - Rourke):** There is a \( \Sigma^{19} \times I \to \Sigma^{29} \) with no topological tubular nbd.

**Nash-Fadell-Spivak:** Define \( \text{NSP}(M) \overset{ev_0}{\to} M \) to be normal paths in \( M, ev_0 \) = evaluation at 0 (non-singular). Then \( ev_0^{-1}(p) \) is isotopy equivalent to \( S^{n-1} \).

For normal bundle of a locally flat imbedding use paths not returning to \( f(M) \). Fibre has homotopy type of \( S^{k-1} \).

To simplify, look at locally flat imbeddings. If \( \nu_\pm \otimes T_M = f_*(T_N) \) there is also a pair \( (\nu_\pm \otimes T_M, T_M) \approx (f_*(T_N), T_M) \). To get normal bundle, get representative up to fiber homotopy type by

\[ E_{\nu_\pm} = E \underset{f_*(T_N)}{\leftarrow} E_{T_M} \quad \text{(- remove 0-section).} \]
The fibre has homotopy type of $S^k$.

Can extend the Kister-Mazur theorem to $(R^{n+k}, R^n)$-bundles.

Use $M \times N$ as microbundle $= (f^*\nu, t_M)$.

This yields $(f^*\nu, t_M)$, a bundle. Group: homeom

\[ \text{homeom} (R^{n+k}, R^n) \]  

\[ \text{homeom} (R^{n+k} \setminus R^n) \]

Thm $f^*\nu$ is fiber homotopy equivalent to $t_M \otimes \nu$ as sphere bundles (not equivalent as bundles, remove 0-section).

This gives characteristic classes since for sphere bundles one has them and Wu classes.

Def locally flat imbeddings $f_0, f_1$ are concordant if there is a proper locally flat embedding $F: M \times I \to N \times I$ which is $f_0$ on $M \times 0$, $f_1$ on $M \times I$.

Thm If $f_0$ and $f_1$ are concordant, $\gamma_{f_0} \sim \gamma_{f_1}$.

Def An isotopy is a homeom. $F: M \times I \to N \times I$.

\[ \begin{array}{c}
\downarrow p_2^* \\
I \end{array} \quad \begin{array}{c}
\downarrow p_2^* \\
I \end{array} \]

A C-isotopy is an isotopy which is a concordance (loc. flat).

Def A locally flat isotopy: require loc. flat with respect to $I$ also.

Thm If $f_0, f_1$ are locally flat isotopic, $(f_0^*\nu, t_M) \simeq (f_1^*\nu, t_M)$. 
Def \( f_0, f_1 \) are ambient isotopic if there is an isotopy \( F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} \) such that \( F(0, 0) = 1 \) and \( F(1, 1) \circ f_0 = f_1 \).

Thm (Hudson, Ziemann) PWL: isotopy \(\Rightarrow\) ambient isotopy, \( k \geq 3 \).

Topological case conjecture 1) iso \& \( k \geq 3 \Rightarrow \) amb. iso

2) loc. flat iso \& \( k \geq 3 \Rightarrow \) amb. iso.

Thm iso. \& \( n + k > \frac{3}{2}(n + 1) \), \( n > 7 \Rightarrow \) amb. iso.

A conjectured proof of 1: \( f_0 \) iso \( f_1 \) (up)

\[ \text{Hudson} \quad \text{PWL isotopy} \]

\( (H, Z) \quad (k \geq 3) \quad \text{ambient PWL iso} \).

The speaker gave a counterexample to this proof.
Stable Secondary Cohomology Operations — John Harper

I. A primary cohomology operation is a function $f: H^n(X, G) \rightarrow H^{n+i}(X, G')$ which is a natural transformation of functors.

Example Steenrod operations: $G = G' = \mathbb{Z}_2$. satisfying

1) $Sq^i : H^n(X) \rightarrow H^{n+i}(X)$ a homomorphism of groups
2) $Sq^0 = \text{identity}$, $Sq^n x = x \cup x$ if $\dim x = n$.
3) $Sq^i x = 0$ if $i > \dim x$
4) Let $\Delta : H^n(X) \rightarrow H^{n+i}(S^k X)$ be the suspension isomorphism. Then $\Delta Squ = Sq^i \Delta$ (Stability)
5) $Sq^i (xuy) = \sum_{j+k=i} Sq^j x \cup Sq^k y$.
6) $A$ = algebra generated by the $Sq^i$ under composition.
   (the Steenrod algebra).
   a) a vector space basis for $A$ is all monomials
      $Sq^{i_1} Sq^{i_2} \ldots Sq^{i_k}$ with $i_k \geq 1$, $i_j \geq 2 \cdot i_{j+1}$.
      Such a monomial is called admissible.
   b) for $a < 2b$, $Sq^a Sq^b = \sum_{j=0}^{\lfloor a/2 \rfloor} (-1)^{a-2j} Sq^{a-2j} Sq^{b-j}$

7) If $\dim x = 1$, $Sq^i x^k = \binom{k}{i} x^{k+i}$
   If $\dim x = 2$ and $Sq^1 x = 0$
   $Sq^2 x^k = \binom{k}{2} x^{k+i}$, $Sq^2 x^k = 0$.

8) $Sq^i$ can be extended to operations on $H^*(X, Y)$ commuting with the coboundary $d$.

II. Operations and Eilenberg–MacLane (EM) spaces.

If $\pi$ is a finitely generated abelian group and $n \geq 1$ there is a space $K(\pi, n)$ unique up to homotopy type, with homotopy groups all $0$ except $\pi_n (K(\pi, n)) = \pi$. 

For any space \( X \), \( H^n(X, \pi) \) corresponds 1-1 with homotopy classes of maps \( X \to K(\pi, n) \): we write
\[
H^n(X, \pi) \leftrightarrow [X \to K(\pi, n)]
\]
where \( f \to f^*(\zeta_n) \) and \( \zeta_n \) is the cohomology class corresponding to the identity map \( \pi \to \pi \) under the isomorphism
\[
H^n(X, \pi) \cong \text{Hom} \left( H_\pi \left( X, \pi \right), \pi \right) = \text{Hom}(\pi, \pi).
\]

Think of \( H^\ast(K(\pi, n), \pi) \) as cohomology operations acting on \( \zeta_n \).

When \( \pi = \mathbb{Z}_2 \),

**Thm (Cartan-Serre, etc.)** \( H^\ast(\mathbb{Z}_2, n; \mathbb{Z}_2) = \mathbb{Z}_2 \left[ \{ Sq^I \zeta_n \} \right] \)

for those \( I = (i_1, \ldots, i_k) \) which are admissible and \( 2i_1 - \deg I < n \).

(The most generators it can have subject to 2 & 3.)

The suspension \( \sigma: H^r(X) \to H^{r-1}(\Sigma X) \) is given by the fibration \( \Sigma X \to PX \to X \)

\[
H^r(\Sigma X) \xrightarrow{\sigma} H^r(PX, \Sigma X) \xrightarrow{p^*} H^r(X, \ast).
\]

Now \( \Omega K(\pi, n) = K(\pi, n-1) \) so \( \operatorname{Im} \sigma^0 \) in \( H^r(\pi, n-1; \pi) \) is called the stable operations. One shows by adjointness of \( S \) and \( \Omega \) that stable in this sense corresponds to stable in \( A \).

(See Adams 1960, Hopf Invariant 1.)

**III Secondary operations and relations in \( A \).**

Suppose \( \sum a_\ast b_\ast = 0 \) where \( a_\ast, b_\ast \in A \), \( \deg b_\ast = n \) and \( \deg a_\ast + n = k \) for all \( i \). There is a cohomology operation \( \Phi \) defined on all \( x \) of dimension \( n \) with \( b_\ast(x) = 0 \) for all \( i \). Its value is a coset in \( \frac{H^{n+k-1}(x)}{\sum a_\ast H^{n+k-1} - \deg a_\ast} \) (the denominator is known
as the indeterminacy of \( \bar{f} \), \( \bar{f} \) is not a function but it is natural. To construct \( \bar{f} \) let \( V \) be a \( \mathbb{Z}_2 \)-vector space of dimension \( m \), \( H = (n_1, n_2, \ldots, n_m) \) where \( n_i \geq k \).

For any \( J = (j_1, \ldots, j_s) \) let \( K(\pi, J) = K(\pi, j_1) \times \cdots \times K(\pi, j_s) \).

Let \( I = (1, \ldots, 1) \). Let \( B_n = K(\mathbb{Z}_2, n) \) and \( \pi_i : K(V, n) \to K(V, n_i) \) the projection on the \( i \)-th factor. Choose a map \( f : B_n \to K(V, n) \) such that \( (\pi_i f)_* \lambda_{n+n_i} = b_i \lambda_n \). One has a fibration as in the diagram, and \( f \) induces a fibration over \( B_n \). If \( g : K \to B_n \) has \( g^* \lambda_n = 0 \), then \( g \) lifts to \( \tilde{g} \). By Serre, there is an exact sequence

\[
H^q(B_n) \xrightarrow{p^*} H^q(\mathbb{R}_n) \xrightarrow{i^*} H^q(F_n) \xrightarrow{\tau} H^{q+1}(B_n)
\]

for \( q \) connectivity \( B_n \). If \( n_i \) is large and \( q = k-1 \), then \( \tau(\lambda_{n+n_i} - 1) = b_i \lambda_{n_i} \), it turns out. Since \( \tau \) commutes with \( A \),

\[
\tau(\sum a_i \lambda_{n_i}) = \sum a_i b_i \lambda_n = 0
\]

thus there is an element of \( H^{k-1}(\mathbb{R}_n) \) which hits \( \sum a_i \lambda_{n_i} \).

1) There exists \( \alpha_n \in H^k(\mathbb{R}_n) \) such that \( i^*(\alpha_n) = \sum a_i \lambda_{n+n_i-1} \).

There is no unique lifting \( \tilde{g} \) \((g^* \mathbb{R}_n \neq \mathbb{R}_n)\) but different liftings differ by an element of \( \sum a_i H^k(\mathbb{R}_n) \). One defines \( \tilde{g}(x) = g^*(\alpha_n) \# x \). Now \( \tilde{g} \) corresponds to a coset in \( H^{k-1}(\mathbb{R}_n) \). Apply \( \Omega \) to the diagram above; find that \( \bar{f} \) is stable and \( E_n = S E_{n+1} \).
2) \( H^*(E_n) \) is a Hopf algebra and \( e_n \) is primitive. Let
\[ x_n = p^* e_n : (E_n, x_n, e_n) = (Q E_{n+1}, \sigma x_{n+1}, \sigma e_{n+1}). \]
The argument depends on \( q < n \); this can be dispensed with by using the suspension many times. Frequently \( H^*(E_n) \) splits as an\( \mathbb{Z}/2 \) algebra into a tensor product - but not as a Hopf algebra.

Stable secondary cohomology operations correspond 1-1 with relations in \( A_n \) via this procedure.

The operations \( \delta \) Adams considered were associated to relations in \( A_n = \text{subalgebra generated by } 1, S^1_q, S^2_q, \ldots, S^n_q \). I have studied operations associated to relations in the ideal \( B(n) = \{ S^{2i} - 1 \mid 2i, -\deg I > n \} \), a left \( A_n \)-module.

IV Homological algebra and relations.

To find generating relations (the minimal number) one calculates \( \text{Ext} \) and \( \text{Tor} \). For an \( A \)-module \( M \), choose a resolution \( 0 \rightarrow M \rightarrow A \oplus V_0 \rightarrow A \oplus V_1 \rightarrow \cdots \) as follows:

\( M/A \cdot M \) is called the "indecomposables" of \( M \). Let \( V_0 \) be a graded vector space with generators corresponding to \( M/A \cdot M \).

Let \( \Lambda = \kappa e_j \); \( \frac{L}{\Lambda \cdot L} \) yields by the same process \( V_2 \), etc.

The basic relations sit in \( L/\Lambda \cdot L \). One gets a free resolution of the \( A \)-module \( M \). To compute \( \text{Ext}^A (M, N) \) for a left \( A \)-module \( N \) look at

\[ \text{Hom} (M, N) \rightarrow \text{Hom} (A \oplus V_0, N) \rightarrow \text{Hom} (A \oplus V_1, N) \rightarrow \cdots. \]

This complex is no longer acyclic. Put \( \text{Ext}^A (M, N) = H^S (\text{Hom}^+ (A \oplus V, N)) \).

The main problem is to compute \( \text{Ext} \) and \( \text{Tor} \).

I have computed \( \text{Ext}^0_A (B(n), \mathbb{Z}) \) which detects generators and \( \text{Ext}^1_A (B(n), \mathbb{Z}) \) which gives basic relations.

(Any relation \( e_j b^i \) in \( A \) gives \( \sum b^i \) on \( x \) if \( \sum b^i x = 0 \). If \( b^i \in B(n) \) and \( \dim x = n \) then \( b^i x = 0 \).)
Armand Wyler - On a generalization of the Hopf inverse homomorphism.


Conjecture: If \( f: X^m \to Y^n \) is algebraically essential (a.e.s.) then \( f^{-1}(y) \) has dimension \( \geq m-n \) (any \( y \in Y \)).

Define \( f \) to be algebraically essential if \( f_*: H_n(X^m, \mathbb{Q}) \to H_n(Y^n, \mathbb{Q}) \).

I use Dold's formulation of Alexander duality (See Husemoller, Fiber Bundles):

**Thm** If \( f: X^m \to Y^n \) is a.e.s. then for any \( A \subset Y^n \) (compact, connected),

\[ f_*: H_i(X^m, X^m - f^{-1}(A)) \to H_i(Y^n, Y^n - A) \]

for all \( i \).

---

Let \( \check{H} \) denote Čech cohomology.

**Cor 1** \[ \psi: \check{H}^{i+m-n}(f^{-1}(A)) \to \check{H}^i(A) \]

\[ \xymatrix{ \check{H}_{n-i}(X, X - f^{-1}A) \ar[r] \ar[d]_{\check{D}} & \check{H}_{n-i}(Y, Y - A) \ar[u]_{\check{D}^{-1}}} \]

defines \( \psi \) by commutativity of the diagram.

(Take \( i = 0 \); then \( \dim f^{-1}(A) \geq m-n \) if \( \psi \neq 0 \)).

**Cor 2** (Hopf) \[ f_*: H_i(X^m) \to H_i(Y^n) \] (cf. Q)

(Proved 1930, Zur Algebra der Abbildungen von Mannigf.)

**Cor 3** If \( f: S^3 \to S^2 \) is not homotopic to \( \bar{0} \),

\[ \check{H}^1(f^{-1}(y)) \neq 0. \]

**Proof** If \( f \neq \bar{0} \), \( f \) factors thru the Hopf map \( \alpha \):

\[ \begin{array}{ccc}
S^3 & \xrightarrow{\bar{f}} & S^3 \\
\downarrow f & \searrow \alpha & \\
S^2 & \end{array} \]

Apply the theorem to \( g \).
Proof of Cor 1.\ Let $Y^n \supset A$ compact, connected, $V$ an open neighborhood of $A$. There is a cup product
\[ \cap : H^i(V) \otimes H_n(Y, V-A) \to H_{n-i}(V, V-A). \]
By excision $H_i(Y, Y-A) \cong H_i(Y-(Y-V), Y-A-(Y-V)) =$ $H_i(V, V-A)$
hence $\cap : H^i(V) \otimes H_n(Y, Y-A) \to H_{n-i}(Y, Y-A)$.

For an oriented manifold $Y$, $H^n(Y^n, Y^n-A; \mathbb{Z}) \cong \mathbb{Z}$.
(all manifolds compact without boundary). If $\Theta_A$ is the orientation class of $Y$ restricted to $A$,
\[ \cap \Theta_A : H^i(V) \to H_{n-i}(Y^n, Y^n-A) \]
\[ \cap \Theta_A : H^i(V') \]

This is a commutative diagram if $V \subset V' \subset A$. So
\[ \lim_{\to} H^i(V) \xrightarrow{\cap \Theta_A} H_n(Y, Y-A) \]
\[ \lim_{\to} H^i(A) \xrightarrow{\cap \Theta_A} \]

This defines $D$. One then proves it is an isomorphism in steps:
1) $A$ a point
2) $Y = \mathbb{R}^n$, $A = \text{cube}$
3) $Y = \mathbb{R}^n$, $A = \text{finite union of cubes in a lattice}$
   (induction on number of cubes; Mayer-Vietoris.)
4) $Y = \mathbb{R}^n$, $A$ any compact connected set.
   Approximate $A$ by polygonal neighborhoods.

Proof of Dold's thrm. let $\eta_{n-i} \in H_{n-i}(Y, Y-A)$. Choose $V \supset A$
and $\eta^i \in H^i(V)$ such that $\eta_{n-i} = \eta^i \cap \Theta_A$. By diagram on the
next page, there is an $\xi_n$ such that $\Theta_A = f_\infty(\xi_n)$. Hence
$\eta^i \cap \Theta_A = \eta^i \cap f_\infty \xi_n = f_\infty(f_\infty(\eta^i) \cap \xi_n) = \eta_{n-i}$.\]
\( (4-3) \)

\[
H_n(x^m) \to H_n(x^p, x^p - A) \\
\downarrow \\
H_n(y^n) \simeq H_n(y^n, y^n - A)
\]

\( \Rightarrow \) onto.

Let \( f: S^q \to S^3 \); if \( [f] \neq 0 \), \( f \sim \Sigma \alpha \).

Hopf: \( f \) has at most 2 injective points; at all others \( \theta^1(f^{-1}y) \neq 0 \).

For: if \( S_1, S_2 \) injective, smooth \( f \) out onto a suspension in a neighborhood \( N \) of \( S_1, S_2 \). On \( bdN \) use \( \Sigma \alpha: S^q \to S^3 \).

Thm: \( x^m \to y^n \) continuous, \( x \& y \) topological manifolds with \( S \in X \) an injective point. Then there exist neighborhoods \( U_1 \supset U_2 \) of \( S \) and \( g: X^m \to Y^n \) such that \( g|_{X^m - U_1} = f|_{X^m - U_1} \), \( g \circ f \), and \( g|_{U_2} \) is the suspension of a map \( \Sigma^{m-1} \to \Sigma^{n-1} \) (\( g \) gives \( \alpha \in \pi_{m-1}(S^{n-1}) \)). \( (f \circ 0 \neq \alpha \circ 0) \).

Freedman (1985) proved \( \pi_{m-1}(S^{n-1}) \to \pi_m(S^n) \)

If \( m < 2n - 2 \).

This gives a necessary condition for existence of an injective point:

\[
S^m \xrightarrow{f} S^n \} \Rightarrow \pi_{m-1}(S^{n-1}) \neq 0.
\]

a.e.s.
Edgar Brown (Brandeis): Recent work of William Browder on the Arf invariant.

This material will be published by Browder.

Let \( n \) be an odd integer, not 1, 3, 7. Let \( \tau(S^n) \), stand for the disk bundle of the tangent bundle to \( S^n \). Glue two copies together: let \( L = \tau(S^n) \cup \tau(S^n) \), as in the picture. Smooth out the corners (\( \Xi \)).

One knows \( DL \) is homeomorphic to \( S^{2n-1} \).

**Conjecture** \( DL \) is not diffeomorphic to \( S^{2n-1} \).

This was proven for \( n = 5 \) by Kervaire; for all \( n \equiv 1 \mod 4 \) by Peterson and Brown.

**Thm (W. Browder)** The conjecture is true for \( n \neq 2^i - 1 \), false for \( n = 15 \), and false for \( n = 2^i - 1 \) iff \( h_1^2 \) in Adams spectral sequence persists to \( E_\infty \).

This talk will concentrate on the proof of the first statement: true for \( n \neq 2^i - 1 \). It is a greatly edited version of Browder's first 20 pages.

There is an element \( \eta_2 \in H^2(BO) \) such that for the classifying map of the normal bundle of any manifold \( M^n \), \( \nu_M: M \to BO_k \) (large \( k \)), \( Sq^\ell_2: H^{m-\ell}(M^n) \to H^m(M^n) \) is given by \( Sq^\ell_2 u = u \cdot (\nu_M^* \eta_2) \). We say \( M \) admits a \( \eta \)-structure if, in the following diagram, \( \nu_M \) has s-
lifting to \( B(n) \).

One gets a cobordism theory of such \((M, \alpha)\) denoted \( \Omega^B_m \).

There is an obvious map \( \Omega^F_m \to \Omega^B_m \) (the classifying map of a framed manifold is constant).

Lemma Suppose \( \partial L \) is diffeo. to \( S^{2n-1} \). Then \( K = \partial L \cup D^{2n} \) is a smooth manifold. \( K \) is stably parallelizable and if \( u \in H^0_m(K) \) is any class it is represented by an imbedded sphere \( i: S^n \to K \).

Further, the normal bundle of such \( i \) is \( \nu(S^n; K) \simeq \nu(S^n) \).

In fact \( H^n(K) = Z_2 \oplus Z_2 \) (\( Z_2 \) coefficients).

Proof of the theorem of Browder is in 2 parts.

Prop. 1 Suppose \( \partial L \) is diffeo. to \( S^{2n-1} \). Then \( \{K\} \in \Omega^F_m \).

Proof of Prop. 2 \( \eta: \Omega^F_m \to \Omega^B_m \) is 0 for \( n \neq 2^i - 1 \).

Proof of Prop. 2 \( \eta \) is described as follows:

\[
\begin{array}{ccc}
\Omega^F_m & \overset{\eta}{\longrightarrow} & \Omega^B_m \\
\Pi_{2n+k}(S^k) & \overset{\text{inclusion of fiber}}{\longrightarrow} & \Pi_{2n+k}(T(\gamma))
\end{array}
\]

To show: \( T(\gamma) \) through dimension \( 2n + 2 + k \) has a 3-stage Postnikov system. This is not hard; it uses secondary cohomology operations as described by Adams.

Now we give a proof of Prop. 1.
Let \((M, x) \in \Omega^{B(n)}_{2n}\). We define an operation \(\phi\) as follows: there exists a closed \(C^\infty\)-manifold \(N\) of large dimension with a \(B(n)\)-structure such that \(N \to B(n)\) is a homotopy equivalence in a certain range of dimensions. (To construct \(N\) take a CW-complex reproducing \(B(n)\)'s homotopy groups up to a certain dimension. Imbed it in \(\mathbb{R}^p\) for large \(p\). Take the induced disk bundle from \(\gamma\) under \(\begin{cases}\text{bundle} & \to \gamma \\
\text{Skeleton} & \to B(n)\end{cases}\); and glue 2 copies of it together.)

Thus we can now assume \(B(n)\) is a large dimensional closed manifold. Notice that in it, \(\alpha_{n+1} = 0\). For \((M, x)\) let \(f_\alpha : M \to B(n)\). Define \(\phi : \ker ((f_\alpha)_* : H_n(M) \to H_n(B(n))) \to \mathbb{Z}_2\)
and prove a theorem about it: \(\phi(x + y) = \phi(x) + \phi(y) + x \cdot y\), where \(\phi\) is defined: let \(\alpha\) be chosen to be a \(C^\infty\) embedding.

By the Thom construction, (collapsing everything in \(B(n)\) outside a tubular neighborhood to a point) we get a map \(B(n) \to S^\ell(M)\), the \(\ell\)-fold suspension of \(M\), where \(\ell + 2n = \dim B(n)\). Now the normal bundle of \(M\) in \(B(n)\), \(\nu(M, B(n))\), is trivial (for \(\alpha : \nu(M, B(n)) \to \gamma\) induces \(\chi\) from \(\gamma\) hence \(f_\alpha^{-1} \tau B(n) = TM\)).

Let \(u \in H_n(M)\) with \((f_\alpha)_* u = 0\). Let \(r \in H^n(M)\). The map \(B(n) \to S^\ell(M)\) sends \(S^\ell(r) : H_n(M) \to \text{dual}(w.r.t. B(n))\) by standard properties of cap and cup products. Hence \(S^\ell(r) \cdot \phi_0\).

\[
\begin{array}{ccc}
B(n) & \xrightarrow{\phi} & S^\ell(M) \\
\phi_0 & \xrightarrow{S^\ell(r)} & S^K(\mathbb{Z}_2, n))
\end{array}
\]

Define \(\phi(u) = \left[Sq^{n+1}(S^\ell r) \cdot \phi_0\right](B(n)) \in \mathbb{Z}_2\) (functional cohomology operation). There is no indeterminacy; indeed, the indeterminacy is \(\phi_0 \cdot H_n(B(n)) + (S^\ell(r) \cdot \phi_0) \cdot H^{n+1}(S^K(\mathbb{Z}_2, n))\), and the first term vanishes since \(\alpha_{n+1} = 0\) in \(B(n)\) while the second
vanishes because every element of \( H^{l + 2n}_\ast (K(Z_2, n)) \) is a Steenrod operation acting on \( B_n \); since \( S^l (u_0 \circ g)(n) = 0 \), and Steenrod operations commute with \( S^l (u_0 \circ g) \), the whole image is 0.

Now we show \( \eta (K) \neq 0 \), by contradiction. Assume \( K = \mathbb{Z} \mathbb{N} \) where \( N \) has a \( \mathbb{N} \)-structure. Then the classifying map for \( K \) can be constant (or contained in a disk if an embedding is desired.)

\[
\begin{align*}
H^n(N) & \rightarrow H^n(K) \xrightarrow{\nu} H^{n+1}(N, K) \\
\nu & \rightarrow u_1 \; ; \; u_2
\end{align*}
\]

By a standard Poincare duality argument, choose a basis \( u_1, u_2 \) of \( H^n(K) \) so that \( \nu \rightarrow u_1 \). We want to show \( \eta (S^l (\nu)) \) is both 0 and non-0. We use that \( u_1 \) is dual to a homology class with non-trivial embedded normal bundle to show \( \eta S^l (u_1) \neq 0 \).

Assume \( K \rightarrow B(n) \) is an embedding, contained in a disk. We have a trivialization of a tubular neighborhood of \( N \). By the Thom construction we get a map \( B \rightarrow S^{l-1}(N/K) \).

Now collapse the tube: \( S^{l-1}(N/K) \rightarrow S^l(K) \), which is the \( (l-1) \)-fold suspension of \( N/K \rightarrow S(K) \).

\[
\begin{align*}
B(n) & \rightarrow S^{l-1}(N/K) \rightarrow S^l(K) \xrightarrow{S^l(\nu)} S^l(N) \rightarrow S^l(K(Z_2, n)) \\
S^l u_1 & \xleftarrow{S^l(\nu)} S^l(N) \xleftarrow{S^l(\nu)} S^l(K(Z_2, n))
\end{align*}
\]

Now \( S^{l-1}(N/K) \rightarrow S^l(K) \) is trivial, hence the map you use to apply the functional cohomology operation is trivial; so \( \eta (S^l u_1) = 0 \). We have used the fact that the class coming from a boundary gives 0 under \( \eta \). Next use the fact that \( K \) is the Kervaire manifold.

Now \( u_1 \) is dual to \( i \); \( S^n \rightarrow K \); \( \nu (S^n, K) \cong T(S^n) \).
If one takes any homology class, realizes it as a manifold, and does the Thom construction, then the Thom class goes to the dual of the homology class: 

\[ S^d \mathbb{U} \to S^d N. \]

Notice \( K \) lies within a disk. Factor \( \mathbb{B}_n \to S^d \mathbb{K} : \)

\[
\begin{align*}
\mathbb{B}_n &\quad \xrightarrow{e} \quad S^d \mathbb{K} \\
&\quad \xrightarrow{\alpha} \quad S^d \mathbb{U} \\
&\quad \xrightarrow{\beta} \quad S^d (\mathbb{K}(\mathbb{Z}_2, n))
\end{align*}
\]

Now all the numbered maps induce isomorphisms on \( H^{d+2n} \).

Hence computing the functional operation of \( \alpha \) is the same as computing it for \( \beta \).

**Lemma**

Given \( S^{d+2n} \xrightarrow{\gamma} S^d \mathbb{T}(\mathbb{K}(\mathbb{Z}_2, n)) \) with \( \gamma \) an iso. on \( H^{d+2n} \),

and \( S^d \mathbb{T}(\mathbb{K}(\mathbb{Z}_2, n)) \xrightarrow{\delta} S^d \mathbb{U} \xrightarrow{\phi} S^d \mathbb{K}(\mathbb{Z}_2, n) \), then

\[ S^{d+1} \mathbb{U} \mathbb{U} (S^d \mathbb{K}(\mathbb{Z}_2, n)) \neq 0 \]

**Proof**

Everything in sight desuspends to \( l = 1 \). Then, \( T(\mathbb{T}(\mathbb{K}(\mathbb{Z}_2, n))) = S^{n+1} \vee S^{2n+1} \). Hence assume \( l = 1 \).

**Case 1**

\[ S^n \mathbb{U} e^{2n} \xrightarrow{\mathbb{T}(\mathbb{K}(\mathbb{Z}_2, n))} S^n \mathbb{U} e^{2n} \xrightarrow{\mathbb{T}(\mathbb{K}(\mathbb{Z}_2, n))} K(\mathbb{Z}_2, n) \]

Identifies \((x, *) \) with \((*, x) \) as a Thom space.

**Perform Hopf construction**

\[ S^{2n+1} \xrightarrow{g_0} S S^n \xrightarrow{\mathbb{K}(\mathbb{Z}_2, n))} S \mathbb{K}(\mathbb{Z}_2, n) \]

It is known \( g_0 \) is \( 15 \) in top dim. homology. This is an example of \( g \).

One shows \( h \) gives a non-zero \( S^{n+1} \xrightarrow{h} S \mathbb{K}(\mathbb{Z}_2, n) \neq 0 \). Now \( S(S^n \mathbb{U} e^{2n}) = S^{n+1} \mathbb{U} S^{2n+1} \).
Looking at $\pi_{2n+1}(S^{n+1} \vee S^{2n+1})$, an arbitrary $g$ is given by $g = \beta_k$ for $k: S^{2n+1} \to S^{n+1}. The functional operation is additive with respect to homotopy-group addition of maps; hence

$$\Sigma^{n+1}_k(\beta_g) = \Sigma^{n+1}_k(\beta_{g_0}) + \Sigma^{n+1}_k(\beta_k)$$

But you can't detect a $k: S^{2n+1} \to S^{n+1}$ by a functional operation unless $n=1, 3, 7$. 
Tony Armstrong: *Piecewise-Linear Transversality*

This subject was begun by E.C. Zeeman. One tries to move embedded submanifolds so they are transversal. $Q$ is ambient manifold. I describe here only the results for closed manifolds.

**Def.** $M^n, N^r \subset Q^q$ are transversal at $g \in M \cap N$ if there is an embedding $e : D^{q-n} \times D^{q-m} \times D^{m+n-q} \to Q$ onto a nbd. of $g$ such that $e^{-1} M = D^{q-n} \times 0 \times D^{m+n-q}$

$$e^{-1} N = 0 \times D^{q-m} \times D^{m+n-q}.$$

Let $X$ be a polyhedron, $x \in X$. $X$ has a natural local product structure at $x$. Define the intrinsic dimension $I(X, x)$ of $X$ at $x$ to be the largest integer $t$ such that there is an embedding $D^t \times V \to X$ onto a nbd. of $x$, where $V$ is a cone with vertex $x$ and $0 \times x \to x$. (See picture).

It turns out $V$ is well-defined up to PWL homeomorphism.

**Def.** Let $X, Y \subset Q$, $g \in X \cap Y$. Assume $I(X, g) = s$, $I(Y, g) = t$. $X$ and $Y$ are transversal at $g$ if there is an embedding

$$e : D^{q-t} \times D^{q-s} \times D^{s+t-q} \to Q$$

onto a neighborhood of $g$, such that $e^{-1} X = D^{q-t} \times V \times D^{s+t-q}$

$$e^{-1} Y = W \times D^{q-s} \times D^{s+t-q},$$

where $V = \text{subcone of } D^{q-s}$ = join of $0$ to sum subpolyhedron of $D^{q-t}$.

\[ V = \ldots D^{q-t}. \]
Thm \( Y \times C \subseteq \mathbb{Q} \) and both of codimension \( \geq 3 \) in \( \mathbb{Q} \). Then \( X \) can be moved transversal to \( Y \) by an arbitrarily small ambient isotopy of \( \mathbb{Q} \).

(No proper definition of local flatness for codimension \( < 3 \) has been given).

Relative problem \( M, N \subset \mathbb{Q}, D^m \text{ and } D^n \text{ given transversal in } D^Q \). Can you move them transversal? No.

Example \( S^m, S^n \subset S^k \) transversal: can \( S^m \) and \( S^n \) be spanned by transversal balls \( D^{m+1}, D^{n+1} \subset D^{k+1} \)? No. To show why the obvious methods for trying to prove a Yes do not work, first consider the joins of \( S^m \) and \( S^n \) with \( 0 \). Here transversality fails at \( 0 \). Next try join of \( S^m \) with one point, of \( S^n \) with another point. However this also fails as in the following picture: Let \( L \) be a line transversal to a folded disk \( D^1 \) in \( E^3 \). Then \( D \times I \) and \( L \times I \) are transversal in \( E^4 \). Now \( L \times I \) is a square. Tilt the square slightly; this corresponds to taking cones on two different points. Then \( D \times I \cap L \times I \) is 3 lines meeting at \( 0 \), not transversal. (Note this does not yield a counterexample). To get a counterexample note \( S^m \cap S^n \) must bound if \( S^m \) & \( S^n \) are spanned by transversal balls - it would bound \( D^{m+1} \cap D^{n+1} \).

Consider now only the manifold case. Suppose \( M \subset \mathbb{Q} \) and given a triangulation (Ation) of \( \mathbb{Q} \). Can \( M \) be moved so it cuts across all the simplices of \( \mathbb{Q} \) nicely? If so, make \( N \) a subcomplex, then the transversality theorem would be proven.

Def. Let \( x \in \text{ int } A \), \( A \) a subcomplex of \( K = \text{ Ation of } \mathbb{Q} \). Let \( L \) be the link of \( A \) in \( K \).

Write join as juxtaposition.
Let \( s : AL \rightarrow xL \) be the natural simplicial map.

Def. \( M \) is transsimplicial to \( K \) at \( x \) if there is a nbd. \( U \) of \( x \) in \( xL \) and an embedding \( e : U \times (D^a, D^{a+q}) \rightarrow (AL, M \cap AL) \) onto subd. of \( x \) with \( se = p_1: U \times D^a \rightarrow U \).

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{diagram.png}
\end{array}
\]

Lemmas
1. If \( K' \) is a subdivision of \( K \) then \( M \) trans\(^q\) to \( K' \)
   \[\Rightarrow M \text{ trans}^q \text{ to } K\]

2. \( M \) trans\(^q\) to \( K \), \( N \) subcomplex of \( K \) \[\Rightarrow M, N \text{ transversal} \]

Thm. Given \( M \subset Q \), \( K \) a \( A \)-tion of \( Q \), \( M \) can be ambient isotoped trans\(^q\) to \( K \).

Cor. (use Lemma 2) transversality theorem.

Proof of thm. Step I. Divide so \( M \) is a subcomplex

II. subdivide this \( A \)-tion so that it becomes Brouwer

(namely any \( x \) can be linearly imbedded in \( E^q \) for a vertex \( x \)).

III. Make \( M \) trans\(^q\) by operating one simplex at a time (in order of decreasing dimension).
Edgar Brown (Brandeis): Recent work of Wm. Browder on the
Arf Invariant, II

In this talk consider only the stable range. If \( f: X \to Y \) let \( E_f = \{ (x, \alpha) \in X \times Y \mid \alpha(x) = *, \alpha(1) = f(x) \} \) and \( E_f \to X \) by projection on the first factor. If \( f \circ g : Z \to Z \) and \( g \circ f \cong 0 \) we get a diagram

\[
\begin{array}{ccc}
E_f & \xrightarrow{f} & E_g \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z
\end{array}
\]

There is a map \( E_f \xrightarrow{i} \Sigma Y \) namely \((x, \alpha) \mapsto g \circ \alpha\), and a map \( \Sigma Y \to E_f \) namely \( \beta \mapsto (x, \beta) \) such that \( g \circ i(\beta) = g \circ \beta \), hence \( g i = \Sigma g \)

\[
\begin{array}{ccc}
E_f & \xrightarrow{i} & \Sigma Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z
\end{array}
\]

**Lemma** \( E_f = E_f \)

Let \( A_2 \) be the Steenrod algebra mod 2; \( X = K(\mathbb{Z}, e), Y = K(\mathbb{Z}, s), Z = \Pi K(\mathbb{Z}, j) \) (some finite product).

\[
\begin{array}{ccc}
H^*(Z) & \to & H^*(X) \\
\downarrow & & \downarrow \\
\Sigma b_i y_i & \to & a_i x
\end{array}
\]

So \( y_i \to \Sigma a_i b_i x \)

**Def** \( \Sigma a_i b_i = 0 \) detects \( x \in T_{2-} (S^2) \) if there is a map \( h: S^2 \to E_f \) with \( h_x \alpha = 0 \).
Theorem (Adams) Let \( h_i : a_2 \rightarrow \mathbb{Z}_2 \) be \( h_i(S^2_t) = 1 \), \( h_i(S^3_t) = 0 \) for all other admissible \( t \). Then \( \Sigma a_i h_i \) detects \( \ell(\pi_{n}(s)) \) for some \( k, l \) with \( k \geq 2, 0 \) and \( k \neq l + 1 \), \( \Sigma h_{k_l}(a_i) h_{k} h_{l} \neq 0 \) and \( h_{k_l} h_{l} \) persists to \( E_\infty \) in Adams' spectral sequence. //

Let \( N_{n+1} \in H^{n+1}(B_{n+1}, \mathbb{Z}_2) \) be the \( n+1 \) class associated to \( S^n_{n+1} \). In the diagram let \( \gamma \) be the canonical \( k \)-plane bundle and \( N^{*}_n(i_{n+1}) = N_{n+1} \).

\[ B_{n+1} = E_{n+1}, \ \bar{\gamma} \] the induced bundle.

Let \( i : S^k \rightarrow T(\bar{\gamma}) \) (the Thom space) be the "injection of the fiber."

**Prop 2** \[ i^*: \Pi_{2n+k}(S^k) \rightarrow \Pi_{2n+k}(T(\bar{\gamma})) \] is trivial if \( n \not= 2^i - 1 \).

**Proof** We find a 3-stage Postnikov system for \( T(\bar{\gamma}) \) in dimension \( 2n + k + 1 \). Here \( k \) is large and statements are to hold for dimension \( \leq 2k \).

Choose \( B_{n+k} \) so that \( E \cong B_{n+k} \). Let \( E \) be the fiber, \( T \) the total space, \( B \) the base.

\[ \Omega(T(\bar{\gamma})/\Pi_{2n+k}(T(\bar{\gamma}))) \rightarrow T(\bar{\gamma}) \rightarrow T(\gamma) \rightarrow \Omega(T(\bar{\gamma})/\Pi_{2n+k}(T(\bar{\gamma}))) \rightarrow ST(\gamma) \rightarrow \cdots \]

\( T(\gamma) \) is a product of \( K(\mathbb{Z}_2, i) \)’s. Find a 2-stage Postnikov system for \( \Omega(T(\gamma)/\Pi_{2n+k}(T(\gamma))) \). Write \( K_{n+1} = K(\mathbb{Z}_2, n+1) \).

**Lemma** \( (N_{n+1,x} : H^i(K_{n+1} \times B_{n+1}, x) \times B_{n+1}) \) is iso for \( i \leq 2n+1 \) and kernel is \( \{ \xi_{n+1,2n+1} \} \) in dim. \( = 2n+2 \).
Proof Consider the spectral sequences for the 2 fibrations

\[ E_\infty^{pq} = E_2^{pq} = H^p(K_{n+1} \times T(y)) \otimes H^q(B_0) \]

\[ E_2^{pq} = H^p(K_{n+1} \Lambda) \otimes H^q(B_0) \]

\( q \leq n \) i.e. \( p+q \leq 2n+1 \) and \( p+q = 2n+2 \) and \( q < n+1 \).

Kernel in \( 2n+2 \):

1) \( \eta_{n+1} \otimes w_{n+1} + \alpha_n \otimes 1 \rightarrow 0 \)

2) \( \bar{E}_\infty^{pq} \rightarrow \bar{E}_2^{pq} \) iso, \( p+q = 2n+2 \) \& \( q \neq n+1 \).

And has kernel \( \{ \xi_{n+1} \otimes w_{n+1} \} \), \( p = q = n+1 \).

Let \( \gamma_1 = \text{disc bundle}, \gamma_0 = \text{sphere bundle} \).

\[
\begin{align*}
H^k(K_{n+1} \times B_{q_k}, \kappa \times B_{q_k}) &\longrightarrow H^k(B_0, B) \\
H^{q+b}((\gamma_1, \gamma_0) \cup [\gamma_1 \times B_{q_k}]) &\longrightarrow H^{q+b}((\gamma_1, \gamma_0) \cup \gamma_1) \\
H^{q+b}(K_{n+1} \Lambda T(y)) &\longrightarrow H^{q+b}(T(y) / T(\gamma)) \\
&\text{iso, } \quad q + b = 2n+2, \quad q \leq 2n+1,
\end{align*}
\]

\[ \ker = \eta_{n+1} \otimes U_k + \eta_{n+1} \otimes w_{n+1} U_k \quad (U_k = \text{Thom class}) \]

\[ E_w \rightarrow K_{n+1} \Lambda T(y) \rightarrow K_{2n+2+b} \]

\[ T(y) / T(\gamma) \]

\[ T(y) = E_j \rightarrow E_k \rightarrow E_{w} \]

\[ T(y) / T(\gamma) \rightarrow K_{n+1} \Lambda T(y) \rightarrow K_{2n+2+b} \]

\[ \pi_{2n+2+b}(T(\gamma)) \cong \pi_{2n+2+b}(E_{k+j}) \]

Does \( K_k \rightarrow K_{n+1} \Lambda T(y) \rightarrow K_{2n+2+b} \) detect anything in \( \pi_{2n+2+b}(S_k) \)?

\[ H^*(T(y)) \]

\[ H^*(K_{2n+2+b}) \]

\[ H^*(K_{n+1}) \otimes H^*(T(y)) \rightarrow H^*(K_K) \rightarrow H^*(K_{n+1}) \otimes H^*(T(y)) \rightarrow H^*(K_K) \]

\[ \eta_{2n+2+b} \rightarrow \eta_{n+1} \otimes U_k + \eta_{n+1} \otimes w_{n+1} U_k \]
Let $\{x_i\}$ generate $H^\ast(T(G))$ over $A_2$, $x_0 = U_k$. Then $\{u \circ x_i\}$ generate $E$ over $A_2$.

$$\begin{align*}
&\varphi_{n+1} \circ \varphi_{n+1} U = \varphi_{n+1} \circ \chi(S_q^{n+1}) U_k \\
&= \sum_{i=0}^{n+1} \chi(S_q^i) (S_q^{n+1-i}) x_0 \\
&\varphi_{n+1} \circ \varphi_{n+1} x_0 ightarrow (S_q^{n+1-i}) x_0 = \sum b_i x_i, \text{ where } b_n = \chi(S_q^{n+1})
\end{align*}$$

and $\dim b_i = 2n+2-i$. \[\Rightarrow \sum \chi(S_q^i) b_i = 0\] is the form of the relation coming from diagram $(\pm)$. But by theorem 1, this relation cannot detect an element of $\Pi_{2n+k}(S^k)$ if $n \neq 2^i - 1$. 
F. P. Peterson: \( H^*(BSF; Z_p) \)

We are interested in the groups \( O(n) = \text{real orthogonal group} \), \( PL(n) = \text{piecewise linear homeomorphisms of} \ \mathbb{R}^{n-1} \), \( H(n) = \text{homeomorphisms of} \ S^{n-1} \), and \( G(n) = \text{homotopy equivalences of} \ S^{n-1} \). Each is included in the next, naturally; for each there is a classifying space:

\[
BO(n) \to BPL(n) \to BH(n) \to BG(n)
\]

and these spaces classify respectively real \( n \)-vector bundles, \( PW \)-microbundles, \( ? \), and spherical fibrings = fibrations whose fiber has the homotopy type of \( S^{n-1} \). Let \( n \to \infty \); one then writes \( O, PL, H, \) and \( F \) (not \( G \)) for the groups obtained. There are also the orientation-preserving subgroups \( SO, SPL, SM, SF \). This defines \( BSF \).

In each classifying space there is an analogue of the canonical \( n \)-bundle over \( BO(n) \), so that for each group one gets a Thom space; these fit together in spectra. We are interested in

\[
\text{IMS} \to \text{IMSPL} \to \text{IMSF}
\]

The Thom isomorphism \( H^*(BSF) \cong H^*(\text{IMSF}) \) has degree \( d \). Define \( w_i = \phi^{-1} \Sigma^i \phi \). This shows \( w_i \)'s are defined all the way back on \( G(n) \), not just on \( O(n) \). Let \( p \) be an odd prime and \( r = 2p-2 \) in the sequel. One has Wu classes

\[
w_i = \phi^{-1} \phi^i \phi(1) \in H^i(\text{BSF}).
\]

Using \( G(n) = \text{homotopy equivalences of} \ S^{n-1} \) it is easy to see

\[
\Pi_n(\text{BSF}) = \text{stable} \ \Pi_{n-1}(S)
\]

This shows \( \Pi_n(\text{BSF}) \) is all finite groups \( \Rightarrow H^*(\text{BSF}) \) is also \( \Rightarrow \).
are no Pontrjagin classes. Now \( p \neq 0 \). There is a map

\[ \Theta : Z_p [q_i] \otimes E(q_i) \rightarrow H^*(\text{BSF}). \]

Milnor conjectured \( \Theta \) is a monomorphism.

Thm 1: \( H^*(\text{BSF}) \cong Z_p[q_i] \otimes E(q_i) \otimes C \) where \( C \) is a Hopf algebra over the Steenrod algebra \( A \) and the \( \otimes \) holds as Hopf algebras over \( A \).

(Peterson, Toda).

Milnor showed \( \Theta \) is a monomorphism in dimensions \( \leq p-1 \) by using known homotopy groups of spheres and constructing BSF using \( k \)-invariants. Stasheff showed \( \Theta \) is mono in dim. \( \leq (2p+1)r \) and showed \( C \) is \( (pr-2) \)-connected.

Cor: IMSF has the homotopy type of a wedge of Eilenberg-MacLane spaces (wedge over many primes).

A corresponding theorem holds for \( Z_2 \) coefficients:

\[ \theta : BO \rightarrow BF, \quad H^*(BO) \rightarrow H^*(BF) \]

\[ Z_2[w_i] \]

\[ C = H^*(BF) / \Theta(\wedge H^*(BO)) \cdot H^*(BF) \]

one has \( H^*(BF) = H^*(BO) \otimes C \) (using Hopf algebra arguments like those in Milnor-Moore). In \( Z_p \) coefficients much difficulty comes from \( p \) which is quite unlike \( 2g^i = \beta \mod 2 \).

Next we show \( \Theta \) is a monomorphism. Let \( L \) be a lens space (identification space of \( S^{2n+1} \) under the action of \( Z_p \)).

Lemma: There exists a map \( h : SL \rightarrow \text{BSF} \) such that \( h^*(\beta q_i) \neq 0 \) for all \( i \).

Suspension of \( L \).
Proof

\[ L = \frac{S^{2n+1}}{Z_p} \xrightarrow{\lambda} \frac{S^{2n+1}}{S^1} = CP \xrightarrow{i} C_\lambda \xrightarrow{\pi} SL \]

We want a vector bundle \( E \) \( CP \xrightarrow{\pi} BSO \) such that \( i^*j^* \) is trivial. For given such a bundle one can factor (dotted arrows above) to get \( h \). Let \( E \) be the canonical complex line bundle on \( CP \).

One knows \( K(L) \), \( K(CP) \), and \( \lambda^* \); \( \text{Ker } \lambda^* \) is ideal generated by \( \xi^p - 1 \). We take the bundle \( (p+1)^*E(\xi^{p+1} - \xi) \in \text{Ker } \lambda^* \) where \( \xi \) is a number such that \( (p+1)^*E(\xi^{p+1} - \xi) \), made real and pushed to \( BSO \), becomes 0. Adams proves the existence of \( \xi \) in \( J(X) - \pi \) (the bundle in question equals \( \psi^{p+1}(\xi) \)). Make this bundle real (neglect complex structure). Now compute \( h^*(\beta q_i) \) and show it is not 0. Use Chern classes on \( CP \) and the fact that \( i^* \) multiplies by \( p \).

To prove \( \theta \) is 1-1 define \( Z_p[\psi_i] \otimes E(\beta q_i) \xrightarrow{\psi} H^*(BSF) \)

a coalgebra structure \( \psi \) on range \( \theta \) (abbreviated hereafter \( Z_p \otimes E \) ) by \( \psi(q_i) = \sum q_j \otimes q_{i-j} \).

To prove \( \theta \) is 1-1 it is enough to prove \( \theta \) is 1-1 on the primitive elements of \( Z_p \otimes E \). There are primitive elements in \( Z_p \otimes E \) of degree \( t \) only if \( t = ir \) : polynomial in \( q_i \) is primitive or \( t = ir + 1 \): \( \beta q_i \) is decomposable is primitive.

In the first case the map \( H^*(BSF) \xrightarrow{\theta} H^*(BSO) \) carries the
polynomial in $q_i$'s into a non-trivial polynomial in $q_i$'s in $H^*(BSO)$ which must therefore be non-zero. For $r = r_1 + 1$ use $h^*: H^*(BSF) \to H^*(BSF)$. Now $h^*$ of decomposables is 0 since $SL$ is a suspension $\Rightarrow$ all products are 0, and $h^*(\beta q_i)$ to by construction. 

As before define $C = H^*(BSF) \theta (\mathbb{Z}_p \otimes E) \cdot H^*(BSF)$. 

Once we have a map of Hopf algebras back, the proof of Theorem mimics $\mathbb{Z}_2$ proof. The map is:

$$H^*(BSF) \xrightarrow{\psi} H^*(BSF) \otimes H^*(BSF) \xrightarrow{\otimes p} (\mathbb{Z}_p \otimes E) \otimes C$$

Once we have $f$ we have an isomorphism by Milnor–Moore.

Idea: Take $BSO \times SL \times \cdots \times SL \xrightarrow{\text{standard map}} BSF$ by standard map on $BSO$ and $h$ on each $SL$; multiply ($BSF$ is an $H$-space). Show the image is $\mathbb{Z}_p \otimes E$.

Outline: Let $M$ be an algebra over $A$ (module with products an $\otimes M = \bigotimes_{N} S_{N}(M) = \text{symmetric algebra on } N \text{ letters.}$ Let $n \to \infty$ and get $S(M)$. Now $\otimes M$ gets $\infty$-dimensional, but $S_{N}(M)$ does not. One has $S_{2N}(M) \to S_{N}(M) \otimes S_{N}(M)$ from the map $N \cup N \to 2N = \{1, \ldots, 2n\}$. This yields as $n \to \infty$ a diagonal map $S(M) \to S(M) \otimes S(M)$, thus a Hopf algebra over $A$. Use this infinite diagonal map $H^*(BSF) \xrightarrow{\text{diag}} S(H^*(BSF))$.

The diagonal $A$ of $H^*(BSF)$ is coassociative and cocommutative.

Let $M = H^*(SL) / (-)$ where $(-) = \text{everything but } H^0$, $H^{ir}$ and $H^{ir+1}$. Then $(-)$ is a sub-$A$-module so $M$ is an $A$-module.
\[
\begin{align*}
H^* (BSF) & \xrightarrow{\psi} H^* (BSF) \otimes H^* (BSF) \\
& \xrightarrow{\otimes com_i} H^* (BSF) \otimes \mathcal{L} (H^* (BSF)) \\
\downarrow & \downarrow \mathcal{L} (H^*) \\
\end{align*}
\]

\[
\begin{align*}
\mathbb{Z}_p [q_i] \otimes \mathcal{L} (M) & \leftarrow H^* (BSO) \otimes \mathcal{L} (H^* (SL)) \\
\text{quotient of } H^* (BSO) & \text{quotient of } \mathcal{L} (H^* (SL))
\end{align*}
\]

Let this composite be denoted by \( f \). Every map is a map of Hopf algebras over \( A \), so \( f \) is also. (We would like to throw away \( H_{ir} \) but then the map would not preserve \( A \)-structure since \( H_{ir} \) and \( H_{ir} \) are connected by \( \beta \).) Show \( \text{Im} f = \text{Im} f \theta \) and \( f \theta \) is a monomorphism.

About \( C \): \( e_i \in C^{p+r-1} \) is defined by Gitler - Stasheff. The \( q_i \) are defined by primary operations on the Thom class; \( e_i \) is defined by a twisted secondary operation on the Thom class. For a range \( (up \ to \ (2p+1)r) \) \( C \) is a free commutative algebra (exterior algebra on odd dim., polynomial alg on even dim.) on elements \( \{ a(e_i) \} \) where \( \{ a \} \) is a basis of \( A \)

\[
A (\beta^1, 2 \beta^{r+2}, \beta^r \beta^r)
\]

This is false for higher dimensions since \( (\beta e_i)^p = 0 \).

Assuming some results of Cohen and Christensen, one gets more secondary operations and finds \( e_t \) (for each \( t \)) \( \in C^{p+r-1} \)

Conjecture \( C \) is generated as an algebra over \( A \) and \( \beta_s \) (higher order Bocksteins) by \( e_t \).

So far we can only get \( e_t \)'s by getting \( E_* (S) \) and working backwards.

Applications \( p = 3 \): Sullivan's results on \( E/PL \).
The map $BSPL \to BSF$ gives $H^*(BSPL; \mathbb{Z}_3) \to H^*(BSF)$

$\cong \mathbb{Z}_3[q_1] \otimes C \to \mathbb{Z}_3[q_1] \otimes E(q_1) \otimes C$

where the left $\cong$ holds in a certain range of dimensions using the Serre spectral sequence (only for $p=3$ is $r=4$). One shows $BSPL \cong BSO \times B\text{Coher} J$; conjecture $H^*(B\text{Coher} J) = C$, so the map kills $E(q_1)$ and gives other Pontryagin classes. Use Thom isomorphism and get $H^*(MSPL)$. Try to compute (with methods of cobordism theory) $\pi_*(MSPL)$. The last group has $p$-torsion as follows: $H^*(MSO) = \Sigma A$ where $A$ is generated by $\sigma^i$ but not $\beta$. For each copy of $A$, $\pi_*(MSPL)$ contains $p$-torsion

$\mathbb{Z}_p$ in dimension $pr-1$

$\mathbb{Z}_p + \mathbb{Z}_p$ $2pr-1$

$\mathbb{Z}_p + \mathbb{Z}_p^2$ $(2p+1)r-1$

The $\mathbb{Z}_p^2$ is not detectable by characteristic numbers.
John Harper: Cohomology of Massey-Peterson Algebras

Levievics wrote a paper of this title and one called "Multicomplexes" (to appear). Massey-Peterson Algebras (MPA henceforth) were defined in a paper "The Semi-tensor Product" (Topology 1965) and used by Stasheff under the name "split extension algebra" in a paper in Topology 1965. A Multicomplex is an algebraic object necessary to study change of rings in homological algebra.

Study of MPA's is motivated by the problem of lifting \( f \) (at right) where \( \tau \) is a fibration inducing \( \tilde{\tau} \) and \( Y \to B \) isomorphisms in low dimensional homology. Milgram and Becker (to appear) give an Adams-like spectral sequence with

\[
E_2 = \text{Ext}^*_B(A) \left( H^*(B;E), H^*(Y) \right) \quad A = \text{steenrod alg. all mod } 2 \text{ or mod } p.
\]

which converges to the set of homotopy classes of liftings of \( f \).

The MPA \( (X)A \) combines the ring structure of \( H^*(X) \) and its \( A \)-module structure as follows:

**Def.** The MPA \( (X)A \) corresponding to \( H^*(X) \) and the Steenrod algebra \( A \) satisfies

1) \( (X)A \cong H^*(X) \otimes A \) as a vector space

2) \( (X)A \) is a free left \( H^*(X) \)-module by multiplication on left and a free right \( A \)-module right.

3) \( (X)A \) is a left \( A \)-module via \( \psi : A \to A \otimes A \).

4) If \( f : Y \to X \), \( H^*(Y) \) is of course an \( H^*(X) \)-module.

It is also an \( (X)A \) module via \( (x \otimes a) \otimes y \to f^*(x) \otimes ay \).

5) \( (X)A \) is an algebra under the product

\[
(w \otimes b)(x \otimes a) = \sum (-1)^{\deg b_i \deg x} (w - b_i \cdot x) \otimes b_i \cdot a
\]

where \( \psi(b) = \sum b_i \cdot \otimes b_i \). (This is the natural definition in view of 4).
Remark \((\mathbb{Z}_2)A = A\) so there is an augmentation map \((X)A \to A\).

To compute \(\text{Ext}_{(X)A}^n\) from \(\text{Ext}_A^n\) via change of rings we must first replace \((X)A\) by an isomorphic object which is free as a left \(A\)-module.

Theorem There exists an algebra \(A(x)\) and an isomorphism \(A(x) \cong (X)A\) such that
1) \(A(x) \cong A \otimes H^*(X)\) as a vector space.
2) \(A(x)\) is a free left \(A\)-module and a free right \(H^*(X)\)-module.
3) If \(f: Y \to X\) then \(H^*(Y)\) is a left \(A(x)\)-module with
   \[
   (a \circ x) \circ y \to a(f^*(x) \circ y)
   \]
4) \((b \circ w) (a \circ u) = \sum (-1)^j ba_i^j \otimes (x a_i^-w) w \circ u\)

Idea of proof \(T = \text{Twist} \circ h\) where \(h: (A \otimes M) \to (A \otimes M)\)

\[
\begin{aligned}
&\text{left } A\text{-action} = \\
&\{\begin{array}{ll}
\text{multiplication on } A, & \\
\text{action on } A\text{ and } M
\end{array}
\}
\end{aligned}
\]

where \(M\) is any \(A\)-module and \(h\) is the composite

\[
A \otimes A \otimes M \xrightarrow{1 \otimes 1 \otimes \text{action}} A \otimes A \otimes M
\]

shown by Milnor-Moore to be an isomorphism. \(\quad \parallel\)

Theorem \(f: Y \to X\), as a left module over \(A(x)\) (not as an algebra)

\[
A(Y) \cong A(X) \otimes_{H^*(X)}^{} H^*(Y)
\]

\[
a \circ y \to [a \circ 1 \circ y]
\]

This turns resolutions of \(H^*(Y)\) over \(H^*(X)\) into resolutions of \(A(Y)\) over \(A(x)\) since the functor \(A(X) \otimes_{H^*(X)}^{} \) is exact.

Theorem Let \(R, S\) be rings with \(1\), \(h: R \to S\) a ring homomorphism, \(M\) a left \(S\)-module; then \(M\) is a left \(R\)-module via \(h\). Let \(Y_0\) be an \(S\)-free resolution of \(M\) with differential \(d'\) and augmentation \(\epsilon'\). Let \(X_{q0}\) be an \(R\)-free resolution of \(Y_{q0}\), with differential \(d''\) and augmentation \(\epsilon''\): \(X_{q0} \to Y_{q0}\). Let \(C = \sum_{q \geq 0} X_{q0}\), \(C_k = \sum_{f + r = k} X_{q1r}\). There exist
maps \( d^i : C \rightarrow C \) of degree \(-1\) (bidegree \((-i, i-1)\)) for \( i = 0, 1, 2, \ldots \) such that 1) \( d^0 = d^\infty \), 2) \( d^i \) covers \( d^\infty \), and 3) \( \sum_{i=0}^{\infty} d^i d^{k-i} = 0 \).

Further there exists a differential \( d : C \rightarrow C \) such that \( \{C, d, e\} \) is an \( R \)-free resolution of \( M \) characterized by \( d = \sum d^k \).

Remark: Draw a picture to understand the thm. This is the algebraic analogue of the Serre spectral sequence; given a resolution of \( M \) over \( R \), use \( R \otimes \_ \) to get each \( X_\mathfrak{m} \) over \( Y_\mathfrak{m} \) — it plays the part of the fiber and \( Y_\mathfrak{m} \) is the base. If \( S \) is \( R \)-free, \( R \subseteq S \), this collapses.

Thm 4: Suppose \( R \rightarrow S \) as in Thm 3, and \( N \) a left \( S \)-module, \( G \) a left \( R \)-module.

There exists a spectral sequence

\[
E_2^{p,q} = \text{Ext}^p_M \left( M, \text{Ext}^q_R(S, G) \right) \Rightarrow \text{Ext}^R_M(M, G)
\]

which preserves Yoneda products.

Remark: In interesting cases action of \( S \) on \( \text{Ext}^q_R(S, G) \) is trivial so \( E_2^{p,q} \) splits as a tensor product.

Applications: Let \( C \xrightarrow{h} D \) be an epimorphism of algebras over \( A \) and \( D \cong C \otimes \mathbb{Z}_2[w] \mathbb{Z}_2 \). This occurs for example as follows:

\[
\mathbb{H}^*(BSO(n)) \cong \mathbb{H}^*(BO(n)) \otimes_{\mathbb{Z}_2[w_1]} \mathbb{Z}_2
\]

\[
\mathbb{Z}_2 \cong \mathbb{H}^*(BO(1)) \otimes_{\mathbb{Z}_2[w_1]} \mathbb{Z}_2
\]

\[
\mathbb{Z}_2 \cong \mathbb{H}^*(BSO(2)) \otimes_{\mathbb{Z}_2[w_2]} \mathbb{Z}_2
\]

Define \( x \in C \) is \( A \)-\( C \) indecomposable if it determines a nonzero coset in \( Q(C)/A \cdot Q(C) \) (\( \Leftrightarrow \) \( 1 \otimes x \) is indecomposable in \( A(C) \)).

Thm 5: \( C, D \) as above, \( u \in A \)-\( C \) indecomposable. Then

\[
\text{Ext}^R_{A(C)}(Z_2, Z_2) \cong \text{Ext}^R_{A(D)}(Z_2, Z_2) \otimes A(w)
\]

as a vector space, where \( w \) is the class of \( u \) in \( \text{Ext}^R_{A(C)}(Z_2, Z_2) \).
Thus we find
\[ \operatorname{Ext}_{A(BSO(n))} \left( \mathbb{Z}_2, \mathbb{Z}_2 \right) \cong \operatorname{Ext}_{A(BSO(n))} \left( \mathbb{Z}_2, \mathbb{Z}_2 \right) \otimes \Lambda \langle w_i \rangle \]
and it turns out \[ \langle w_i \rangle^2 = h_0 \otimes \langle w_i \rangle. \]

\[ \operatorname{Ext}_{A(BSO(2n))} \left( \mathbb{Z}_2, \mathbb{Z}_2 \right) \cong \operatorname{Ext}_{A(\mathbb{Z}_2, \mathbb{Z}_2)} \otimes \Lambda \langle w_2 \rangle \]
where \[ \langle w_2 \rangle^2 = h_1 \otimes \langle w_2 \rangle. \]

Also \( BSO(n-1) \subset BSO(n) \) so \( A(BSO(n)) \rightarrow A(BSO(n-1)) \).

Stasheff uses a twisted secondary cohomology operation corresponding to \( h_i^2 + h_i \langle w_2 \rangle \).

\[ 0 = \left( \sigma_i^2 \otimes 1 + 1 \otimes w_2 \right)^2 + \left( \sigma_i^2 \otimes 1 \right)\left( \sigma_i^2 \otimes 1 \right) \text{ in } A(BSO(2)) \]

In \( A(BSO(3)) \), \( \operatorname{Ext}_{A(BSO(3))} \left( \mathbb{Z}_2, \mathbb{Z}_2 \right) \) has generators:

\[ s = 1 \]
\[ s = 2 \]

\[ h_i, w_2 \]
\[ h_i; h_j \quad j \neq i+1 \]
\[ h_i \langle w_2 \rangle \]

\[ a = \langle h_1, h_0 \langle w_2 \rangle \rangle \quad (\text{Massey product}) \]
\[ b = \langle h_0, (w_2, h_i), (w_2) \rangle \quad (\text{''}) \]

and certain relations occur.

Advertisement computations are long, but easy.
John Moore (Princeton): Homological Algebra and Topology

We wish to explain the difference between differential and ordinary homological algebra. We talk about complexes. Let \( A \) be an abelian category, \( C(A) = \) complexes over \( A \). We want to embed a complex in a cone over it, or take the product of a complex and a unit interval. Let \( \mathbb{A} \in C(A) \).

\[
\begin{array}{c}
A_n \xrightarrow{d_n(A)} A_{n-1} \xrightarrow{d_{n-1}(A)} A_{n-2} \quad \text{composite } 0
\end{array}
\]

If \( R = \) modules over some ring \( R \), the "unit interval" is the complex \( I \) given by

\[
\begin{cases}
\dim 1 & R \quad \text{basis } [0,1] \\
\dim 0 & R \oplus R \quad \text{basis } [0],[1]
\end{cases}
\]

where \( d_1(I)([0,1]) = [1] - [0] \). We take \( I \otimes R A \) for "\( A \times I \)."

Now \( (I \otimes R A)_n = A_n + A_n + A_{n-1} \) and the differential is

\[
(x,y,z) \rightarrow (d_n x + y, d_n y + z, -d_z).
\]

In any abelian category we can use these formulas (*) to define "\( A \times I \)." To get a cone collapse one copy of \( A \). Map \( A \rightarrow I \times A \) where \( A \) injects into the first summand; let the cokernel be \( C(A) \). Or directly, \( C(A) = A_n + A_{n-1} \xrightarrow{0 + \text{id}_{A_{n-1}}} A_{n-1} + A_{n-2} \). Map \( A \rightarrow C(A) \) by \( A_n \xrightarrow{\text{id} + d_n} A_n + A_{n-1} \).

Often in topology one has what is called a "pushout diagram," namely given maps \( gh \) with \( h_i = gf \), \( X \cup_f C(Y) \) is a complex such that \( k \) always exists. This universal property can be used in any category \( C(A) \). In particular,
\[ \xrightarrow{f_{x}} X \otimes C(Y) \] and take cokernel \( X_{\alpha}C(Y) \).

**Derived functors** Let \( T: A \to B \) be a functor. Assume enough projectives. Take a projective resolution \( \cdots \xrightarrow{p_{2}} p_{1} \xrightarrow{p_{0}} A \) of \( A \) and let \( T \) act on it, getting \( T(p_{2}) \xrightarrow{T(p_{1})} T(p_{0}) \to A \). The homology of this complex is called the derived functors of \( T \). (Assume \( T \) is additive.) A different view is possible: Suppose \( T: A \to C(B) \).

**Example:** Hyperhomology. Given \( \beta \to B \) get \( C(A) \to C(B) \) where \( A \to T(A) \). This is a special case where the functor comes from extension of a functor \( \beta \to B \).

**Example** (not by extension): Let \( R \) be a commutative ring, \( A \) a supplemented differential graded \( R \)-algebra. (model: \( G \) = topological group, \( A = C_{*}(G, R) \) = singular chains on \( G \)). Let the category be \( M \), with objects \( (M, \text{map} \Lambda^{n}M \to M) \) satisfying the usual differential and unitary conditions. Take a functor \( \Lambda^{n}M \to G_{R}(M) \) where \( R^{n}M \) = left \( R \)-modules and \( M \to R^{n}M \). This functor does not come about by extension.

We want derived functors of a functor \( T \). Intuitively we take a projective resolution in \( A \) and do something on the right, then take homology. But almost always you don't want a projective resolution; instead use some class of relative projectives. Look at \( C(B) \), where \( B \) has enough projectives. The projectives are cones \( C(A) \) such that each \( A_{n} \) is projective; but cones have trivial homology, so this is the wrong class of objects. If we restrict ourselves to positive complexes, there are also projectives \( C(A)+P \) where \( P \) is a projective in degree 0. Similarly the projectives among negatively cochain complexes are cones surjective in each degree plus an injective.
One wants any object $C(A) + P$ to be projective, provided $A_n, P$ are projective for $n \in \mathbb{Z}$ and $P$ has 0 differential. More abstractly, let $P$ be any class of objects in $\mathbb{R}$. Consider the class of morphisms $f \in E(P)$ such that the dotted arrow always exists when $P \in P$, making the $\Delta$ commute. Conversely, one can take a class of morphisms $E$ and consider the class of objects $P(E)$ such that $\Rightarrow$ exists whenever $f \in E$ and $P \in P(E)$.

Now $P(E(P)) > P$ and "$p^2 = p$" hence for so-called closed classes the class of morphisms and the class of objects determine each other. Now "enough projectives" translates as the existence of $P \to A$ for any $A$, where $P \in P$ and $f \in E(P)$. Then $f \in E(P)$ means $A \xrightarrow{f} A'$ and $Z(A) \xrightarrow{\text{epi}} Z(A')_m$ (cycles).

This is the correct setting for hyperhomology.

Let $X \in M$ and take $\Lambda \otimes_R X$. If $P = \text{all such objects (a closed family)}$, $E(P) = \text{all morphisms } M \to M''$ which have as $R$-modules a differential inverse (not a $\Lambda$-map back), that is, $M \to M''$ is split as an $R$-module map. Then as $P$ one comes out with all sums of modules $\Lambda \otimes_R X$.

In general if $\mathbb{R} \to C(T)$ and given $(P, E)$ classes of objects and morphisms in the above relation, such that there are enough $(P, E)$-projectives in $\mathbb{R}$, then for $A \in \mathbb{R}$ let $\cdots P_2 \to P_1 \to P_0 \to A$ be a resolution (an exact sequence where $P_i \in P$ and all $d_i, e \in E$). Apply $T$; since $C(T)$ is not abelian one cannot take kernels and cokernels. Instead let $F_0 T(P) = T(P_0)$. Then $T(d_i)$ is a morphism in $C(T)$, the category of complexes. Next put $F_0 T(P) = T(P_0)$.
One then has $\cdots \to \text{ST}(P_3) \to \text{ST}(P_2) \to \text{ST}(d_2) \to \cdots$, hence we let $\text{ST}(d_2)$.

Let $F_2 T(P) = F_1 T(P) \cup \text{ST}(P_2)$. (Assume $A$ has at least countable coproducts.) Then we have $\cdots \to \text{ST}(P_4) \to \text{ST}(P_3) \to F_2 T(P)$, so proceed inductively, and get $F_n T(P)$ for all $n$. Let $F_\infty T(P) = \lim_n F_n T(P)$ and define $H(F_\infty T(P)) = \text{derived functor of } T \text{ relative to } (P,E)$. This is a filtered thing; what is the spectral sequence?

There is another functor $C(B) \overrightarrow{H(-)} \overrightarrow{\mathbb{Z}(B)} = \text{graded objects (complexes with 0 differential).}$ We have $0 \to F_0 T(P) \to F_1 T(P) \to \text{ST}(P) \to 0$; pass to homology:

$$
\begin{array}{ccc}
H(T(P_0)) & \to & H(F_1 T(P)) \\
H(\text{ST}(P_2)) & \searrow & \to \quad & \\
& \nearrow & \text{exact } \Delta.
\end{array}
$$

Next $0 \to F_1 T(P) \to F_2 T(P) \to \text{ST}(P_2) \to 0$ gives a homology exact triangle

$$
\begin{array}{ccc}
& \to & \\
\uparrow & \nearrow & \downarrow \\
H(\text{ST}(P_2)) & \to & H(S^2 T(P_2)).
\end{array}
$$

The differentials $H(T(P_0)) \xleftarrow{d_1} H(\text{ST}(P_1)) \leftarrow H^2 T(P_2) \leftarrow \cdots$ of the spectral sequence come from the composite $HT$ (with some suspensions), hence $E^2 = LHT(A) = \text{left derived functor of } HT \text{ with respect to } (P,E)$. This generality contains the classical case; when $T(R)$ is all trivial complexes, But if $T(R)$ is non-trivial, the spectral sequence may be highly non-trivial.
(n-5)

In general one cannot find another name for \( \text{LHT}(A) \), that is it is not very computable. To illustrate, a particular case is

\[ \begin{align*}
\text{Example} & \quad R, \Lambda; \quad \mathcal{O} \cong M_\Lambda \oplus \Lambda \otimes_R X = \mathcal{P} \cdot \mathcal{O} \to E_R(M) \quad \text{by} \\
M & \to R \otimes_\Lambda M. \quad \text{For simplicity assume } R \text{ is a field. Then}
\end{align*} \]

\[ H(\Lambda \otimes_R X) = H(\Lambda) \otimes_R H(X) \quad \text{is projective in the category of left } \Lambda \text{-modules}. \]

One has a functor \( \Lambda \rightarrow M \rightarrow H(\Lambda) \rightarrow M \rightarrow R \otimes H(\Lambda) \)

\[ E^2 = \text{Tor}_2^H(\Lambda) (R, H(M)) \Rightarrow \text{Tor}^\Lambda (R, M). \]

In this situation your derived functor of the complex \( R \otimes_{H(\Lambda)} M \).

Assume lower indices and positive grading; then the spectral sequence is in the third quadrant, or with luck in an octant. This shows you don't want to stick to positive complexes since your algebra may have to be e.g. the cochain algebra (look at chains as the cochains of some space).

By duality the corresponding things for injectives follow.

\[ \text{Example: class of injectives; } \Lambda = C_*(X, R) \text{ = differential coalgebra. One has left differential comodules over } \Lambda. \text{ Take extended things to be injectives; because the cotensor product is or ought to be (depending on how much you assume) a left exact, not right exact, functor.} \]

\[ \text{Note: Convergence. In general } F_\infty T(P) \text{ is a colimit hence one has cocompleteness of filtration. Classical convergence does not follow in general unless you assume the coproduct of monomorphisms is a monomorphism ( } \text{AB}A^* \text{ of Grothendieck).} \]
D. Anderson: Complex Cobordism

We study operations in $\Omega^2_{u} (x, A) = \lim_{k \to \infty} [ (S^{2k}, S^{2k}, \ast), M^u \ast_{k} ]$ or $\Omega^u_{n} (x, A) = \text{maps } f: (M, \partial M) \to (x, A)$ where M has a complex structure on its normal bundle (all spaces compact) under the equivalence relation: two such are equivalent if a manifold fits between them and a map of it into $(x, A)$ extends $f$ on one end and (say) $f'$ on the other.

See Topology 1-2 years ago for an article by Brown and Peterson on a spectrum $BP_p$ such that $H^*(BP_p, Z_p) \cong A/(\beta)$ where A is the mod p steenrod algebra and $\beta$ the Bockstein. Milnor studied MU and showed

$$\Omega^u(MU, Z_p) \cong A/(\beta)$$

while Brown-Peterson showed $MU = \bigvee BP_p$ mod $p$, i.e. there is a map giving an injection on homotopy with cokernel consisting of all torsion prime to $p$. $BP_p$ is simpler than $MU$ so we study it instead; cohomology operations in $MU$ are only those in the factors $BP_p$, plus factor permutations. However $MU$ has a natural product structure and $BP_p$ does not.

$\Omega^u$ is not divisible by every prime but $p$, while $H^*(BP_p)$ is.

To study $\Omega^u$ by $BP_p$, we thus use instead

$$\Omega^{u}_{\ast} (x)_{(p)} = \Omega^u_{\ast} (x) \otimes \mathbb{Z} \left\{ \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{p}, \ldots \right\} = \Omega^u_{\ast} (x) \otimes \mathbb{Z}(p)$$

Note $\mathbb{Z} \otimes \mathbb{Z}(p) = 0$ if $p \parallel q$ and $\mathbb{Z}_p \otimes \mathbb{Z} (p) = \mathbb{Z}_p$; thus we are simply throwing away torsion prime to $p$. Define a projection...
$B : \Omega^n_u(X) \rightarrow \Omega^n_u(X)$ called the bordism projection, with

1) $B$ is natural and stable
2) $B^2 = B$
3) $B (\Omega^+_u (pt) (p)) = \mathbb{Z} \left[ \frac{M_{p-1}}{p^2-1}, \frac{M_{p-1}}{p^2-1}, \ldots \right]$
   complex dimension.
4) $B$ is multiplicative.

Of these properties, 4) is what study of $BP_p$ alone does not give.

Given $G \rightarrow G$ a projection operator, one has a split exact sequence $\ker (B) \rightarrow G \leftarrow \text{im} (B)$, thus we have (from now on assume all $\Omega$ are localized at $p$)

$\Omega^n_u (X) \cong \text{im} (B) \oplus \ker (B)$ natural and stable.

Check this result with the long exact sequence for $(X, A)$, and find $\text{im} (B)$ is a new cobordism theory called $BP^n (X) = B (\Omega^n_u (X))$.

By 4), $B (x \cdot y) = B (x) \cdot B (y)$, hence $\text{im} (B)$ is closed under multiplication; therefore $BP^* (X) \rightarrow \Omega^*_u (X)$ is a subring (and quotient ring) of $\Omega^*_u (X)$.

Define $B$ as follows: let $T (n) \rightarrow U (n)$ be the torus in the unitary group, as the set of diagonal matrices with entries of absolute value 1: $T (n) = S^1 \times \ldots \times S^1 = T (1)^n \times \ldots \times T (1)$. Then

$$H^* (BT (1)) \otimes \ldots \otimes H^* (BT (1)) = H^* (BT (n)) \xrightarrow{B \circ (n)} H^* (BU (n))$$

$$\mathbb{Z} (t_1 \ldots t_n)$$

where $B$ stands for "classifying space of"; $BT (1) = \mathbb{C} P^\infty$, and $H^* (BT (1)) = \mathbb{Z} [t]$ with $\dim t = 2$. One shows $\text{im} B \circ (n)$ is the symmetric functions in $t_1 \ldots t_n$, and $B \circ (n)$ is injective; using
the Cartan formula and the inclusion, one deduces Steenrod operations on $BU(n)$. This gives much information on $MU(n)$, for

$$\tilde{H}^*(BT(n)) \overset{\text{family}}{\longrightarrow} \tilde{H}^*(MT(n)) \overset{\text{family}}{\longrightarrow} \tilde{H}^*(MU(n))$$

$$t_1 \cdots t_n \cdot \theta$$

This works the same replacing $H^*$ with $\Omega^*$:

$$\Omega^*(BT(n)) \overset{\text{family}}{\longrightarrow} \Omega^*(MT(n)) \overset{\text{family}}{\longrightarrow} \Omega^*(MU(n))$$

$$t_1 \cdots t_n \cdot \sigma$$

$\Omega^*[t_1, \ldots, t_n]$ where $t_i \in \Omega^2(BT(n))$ is the usual homeomorphism $BT(n) \cong MT(n) = MU(n) = CP^\infty$. Once again $\Delta_n(Bo(n)^*)$ is symmetric power series, and $\theta \mapsto t_1 \cdots t_n \cdot \theta$.

The Cartan formula shows all operations in $\hat{\otimes}$, once you know there is no torsion in $MU(n)$. Now $\Omega^*(MU(n)) = \{\text{operations in complex cobordisms}\}$ of course, where an operation corresponds to the image under itself of the identity class in $\Omega^*(MU(n))$. Let $w = (i_1, \ldots, i_n)$, $i_1 \geq \cdots \geq i_n$. Then $c_w \in \Omega^k(BU(n))$ is defined by $Bo(n)^*(c_w) = \text{smallest symmetric function containing } t_1^{i_1} \cdots t_n^{i_n}$ (or 0 if $w$ has more than $n$ entries).

Define $S_w$ by

$S_w(\text{identity}) = \Phi(c_w) \in \tilde{\Omega}^*(MU(n))$

(we now drop the $n$ since we are in a stable situation). Degree $S_w = 2i_1 + \cdots + 2i_n = 2|w|$. There is a map related to the bundle sum,
\[ \mathbb{B}U(m) \times \mathbb{B}U(n) \xrightarrow{\mu_{m,n}} \mathbb{B}U(n+m) \] (and a corresponding map for tori) \( \mu_{m,n}(c_\omega) = \sum_{\omega' + \omega'' = \omega} c_{\omega'} \otimes c_{\omega''} \)

where the sum over \( \omega' + \omega'' \) is all entries of \( \omega' \) and \( \omega'' \) rearranged in decreasing order. Also, \( s_\omega \circ s_\omega \) is a linear combination of \( s_{\omega''} \), by a complicated formula. Hence \( s_\omega(ab) = \sum s_{\omega'}(a) s_{\omega''}(b) \) since multiplication is the Thomification of \( \mu_{m,n} \). The \( s_\omega \) are a basis for all symmetric functions and are closed under composition. One must figure out an Adam formula. To get all operations, one formal sums \( \sum_{\text{finite}} a_\omega s_\omega \) where \( a_\omega \in \text{group of a point and deg } a_\omega s_\omega = \text{constant}. \) To every operation corresponds a symmetric function. Let \( S \mathcal{F}(\theta) \) be the symmetric function associated to \( \theta \) for \( \theta \in \Omega^* (MU) \) any operation.

Milnor showed there are manifolds \( M_i \in \mathbb{S}^{2i} \) such that

\[ \mathcal{S}_i \left( \theta \right)(M_i) = \begin{cases} 1 & \text{if } i+1 \text{ not a power of a prime } p, \\ p & \text{if } i+1 = p^k \text{ for some } k, p. \end{cases} \]

\( \in \Omega^0 = \mathbb{Z} \)

(this takes place localized in \( p \), remember). Further, a collection of manifolds will be a set of generators of \( \Omega^*(pt) \) iff this condition is satisfied (see Conner & Floyd and their Milnor reference).

Let \( \theta(z) \) be described by

\[ S \mathcal{F}(\theta(z)) = \prod_{i=1}^{n} (1 - M_i \cdot t_i^L) \quad \text{large } n. \]
There is no indeterminacy in \( M_1 \); there is only one generator of \( \Omega^2 \).
Fix it by the conditions of Milnor. This product \( \prod_{i=1}^{n} \) is multiplicative since its form in \( n+m \) variables is that of \( n \) variables times that of \( m \) variables. (It is the multiplicative sequence associated to \((1 - M_1 t)\)). Look at \( \Theta^{(i)} | \Omega^* \equiv \text{id} \mod (M_1) \) (ideal).
So \( \Theta^{(i)} \) is the identity on all indecomposables but \( M_1 \). Check that \( \Theta^{(i)^2} = \Theta^{(i)} \). Pick arbitrarily \( M_2' \) by Milnor's condition.

\((*)\) Let \( M_2 = \Theta^{(i)}(M_2') \). Then \( M_2 \) is indecomposable and invariant under \( \Theta^{(i)} \). Put

\[
\text{SF}(\Theta^{(i)}) = \prod_{i=1}^{n} (1 - M_1 t_i)(1 - M_2 t_i^2).
\]

The condition \((*)\) is exactly what you need to get a projection operator. There is no indeterminacy in \( M_2 \) since one can only add a multiple of \( M_1^2 \) to \( M_2' \) and \( \Theta^{(i)} \) kills \( M_1^2 \). Thus

\[
\text{SF}(\Theta^{(i)}) = \prod_{i=1}^{n} (1 - M_1 t_i) \cdots (1 - M_k t_i^k)
\]
for \( k < p-1 \) (\( \sigma^{(i)}(M_1) = 1 \) is necessary to have a projection).

In dimension \( 2(p-1) \) put \( \text{SF}(\Theta^{(p-1)}) = \text{SF}(\Theta^{(p-2)}) \). In dimension \( 4p-2 \), one can vary the generating manifold by the square of a \( 2p-2 \) manifold, thus \( \Theta^{(i)} \) stops being unique.

Thus

\[
\text{SF}(\Theta) = \prod_{1 \leq i \leq n \atop k+1 \neq \text{power of } p} (1 - M_k t_i^k).
\]

One checks this is a projection operator, and that it is multiplicative by \((*)\).

This reduces to the question: what are the cohomology operation on the \( BP \)-theory? Let \( \text{SF}(\sigma^i) = \sigma^i(t_{2^{(p-1)}}, \ldots, t_{n^{(p-1)}}) \). Then

\[
\mathbf{P}_k^{(i)}(\cdots) = \sum \sigma^i(a) \mathbf{P}_k^i(a) \quad \text{and one can show-}
\]
\[ \{ \text{B}_0 \text{(admissible monomials)} \circ \text{B} \} \text{ form a basis over } \mathbb{B}^*(\mathbb{P}) \text{ (left multiplication) for the operations in } \mathbb{B}^* \text{ theory} \]

Questions to be answered: what are the corresponding Adem formulas? The Adem formulas mod p are correct; the binomial coefficients as integers do not work; one must evaluate \( P^i \) on some well-chosen manifolds. However, \( P^i \cdot P^j = 2 \ P^2 \mod \bar{\mathbb{B}}^* \)

= negative dimensional groups of a point.
MIT Topology Seminar

Armand Wyler: Actions of Spinor groups on spheres

Hurwitz-Radon show there are just \( k \) orthogonal vector fields on \( S^{n-1} \) where if \( n = 2 \), \( 2^k \leq c(n) \leq 3 \), then \( k = 2^{c(n)} + 8 \cdot d(n) \). These are cross-sections of the fibration \( O(n) / O(n-k) \rightarrow O(n) / O(n-1) \). Consider the finite group \( \Gamma_{k-1} \) generated by \( e_0, e_1, \ldots, e_{k-1} \) with relations \( e_i e_j = e_0 e_j e_i \) if \( k \leq i \neq j \geq 1 \) and \( e_i^2 = e_0, e_0^2 = 1 \). The order of \( \Gamma_{k-1} \) is \( 2^k \). To get orthonormal vector fields represent \( \Gamma_{k-1} \) as a set of \( n \times n \) orthogonal matrices where \( e_0 \rightarrow -E, e_i \rightarrow A_i \). Then \( X_i = A_i \cdot x \) (\( x \in S^{n-1} \subset \mathbb{R}^n \)) gives \( k-1 \) orthonormal vector fields on \( S^{n-1} \).

Compute the Lie Algebra of these vector fields: \( [X_1, X_2] = -2A_1 A_2 x \), and so on; to see this, associate to \( x = (\alpha_1, \ldots, \alpha_n) \) the symbol \( \sum \alpha_i \frac{\partial}{\partial x_i} \). Then \( X_i \rightarrow \sum \alpha_i^{(r)} \frac{\partial}{\partial x_i} \), where \( \alpha_i^{(r)} = \sum_{j=1}^{n} \alpha_{ij} x_j \).

According to Chevalley, the symbol of the bracket is then \( \sum_{i,j} \left( \alpha_i^{(2)} \frac{\partial \alpha_j}{\partial x_i} - \alpha_j^{(2)} \frac{\partial \alpha_i}{\partial x_j} \right) \frac{\partial}{\partial x_j} \), so \( [X_1, X_2] = (A_2 A_1 - A_1 A_2) x = -2A_1 A_2 x \).

(We need \([,] \) to study the manifolds tangent to these vector fields).

Further \( [[X_1, X_2], X_i] = 0 \) if \( i \neq 1, 2 \) and \( :=-2A_2 \) if \( i = 1 \), etc.

Thus we remain in the region of products of \( \leq 2 \) matrices.

Consider in the algebra of \((n \times n)\) matrices the space \( M \) spanned by \( A_1, \ldots, A_{k-1} \) and that \( M_2 \) spanned by the products \( A_i A_j \). One computes these and finds both are equal to the Lie algebra of \( SO(k-1) = \text{the Lie algebra of Spin} (k-1) \).

Now find manifolds tangent to these vector fields:

Thm There is an integral manifold tangent to these \((k-1)\) vector...
fields, \( \Leftrightarrow \) the vector space \( V(x) \) spanned by \( x, \ldots, x_{k-1} \) and all their brackets, has constant dimension \( q \).

Look at the problem from the point of view of actions of Lie groups on spheres. \( M_2 = \) Lie algebra of \( SO(k-1) \subseteq (n \times n) \) matrices \( \subseteq SO(n) \).

The simply connected Lie group having this as Lie algebra is \( \text{Spin}(k-1) \subseteq SO(n) \). (this is an embedding since the kernel would be a central discrete subgroup; but the only central discrete subgroup of \( \text{Spin}(k-1) \) is carried non-trivially, to \( e_0 \)). Now \( SO(n)/\text{SO}(n-1) = S^{n-1} \) so the non-trivial embedding gives an action of \( \text{SO}(k-1) \) on \( S^{n-1} \). Note \( \text{Spin}(k-1) \cap \text{SO}(n-1) = \) isotropy group at \( \text{Ti}(e) \) \( (SO(n) \xrightarrow{\pi} S^{n-1}) \) thus the orbit of \( \text{Spin}(k-1) \cap \text{SO}(n-1) \) is \( \text{Spin}(k-1)/\text{Spin}(k-1) \cap \text{SO}(n-1) \).

The last time this action is transitive is \( \text{Spin} 9 \) on \( S^1 \), and the orbit is \( \text{Spin} 9/\text{Spin} 7 \). (but the inclusion \( \text{Spin} 7 \subseteq \text{Spin} 9 \) is not the usual one). For this case \( 15 + 1 = 16 = 16' \), \( \beta(n) = 2 + 8 \cdot 1 = 9 \).

Thus there are 8 vector fields; \( \text{Spin} 9 \supset \text{Spin} 8 \) and \( \text{Spin} 8 \) is not transitive but \( \text{Spin} 9 \) is.

Next we discuss free action of \( S^1 \) or \( S^3 \) on homotopy spheres. (see a paper by Hsiang with that title). According to Borel, \( S^1 \) and \( S^3 \) are the only compact connected Lie groups acting freely on spheres (Sur le cohomologie des espaces fibres principaux, Ann. Math. 1953). Hsiang proved that from the point of view of homotopy there are nothing but Hopf fibrations as follows: Take a free action of \( S^1 \) on \( \Sigma^n \). Hsiang states such an action is always a principal fibration. One checks easily that \( n = 2k+1 \) (for \( S^3 \), \( n = 4k+3 \)) for some integer \( k \).

Thm The principal fibration \( S^1 \rightarrow \Sigma^{2k+1} \rightarrow \Sigma^{2k+1}/\text{action} \)

is homotopically equivalent to the Hopf bundles
\[ \nu: S^1 \to \Sigma 2k+1 \to \mathbb{C}P^k \]
(or \[ S^3 \to \Sigma 4k+3 \to HP^k \])

Proof The homologies of \( \Sigma 2k+1 / \phi \) and \( \mathbb{C}P^k \) are identical by Serre; thus we get a classifying map \( \Sigma 2k+1 / \phi \to \mathbb{C}P^\infty \). Since \( \mathbb{C}P^k = 2k \)-skeleton of \( \mathbb{C}P^\infty \), this map deforms to \( \Sigma 2k+1 / \phi \to \mathbb{C}P^k \). But the Hopf map is universal in this dimension \( (\leq 2k) \) thus \( \nu = \xi \).

Hsiang's main point is that up to homotopy one has only the Hopf bundle. This is far from true including differentiable structure however. Let \( X^k \) be a manifold of homotopy type of \( \mathbb{C}P^k \) and \( \tilde{g}: X^k \to CP^k \) a homotopy equivalence. The total space of the induced bundle \( \tilde{g}^*\nu \) is a homotopy sphere with free action of \( S^1 \). But there are infinitely many different differentiable manifolds \( X^k \) (see Browder-Novikov) so there are infinitely many inequivalent actions of \( S^1 \) on \( \Sigma 2k+1 \).
MIT Topology Seminar

Sandy Blank (Northeastern Univ.): Immersions of $S^1$ in $\mathbb{R}^2$

The interesting question about an immersion $f: S^1 \to \mathbb{R}^2$ is whether it extends to the disk.

Two invariants are associated with an immersion $f$:

1) Tangent winding number $\text{TWN}(f) = \text{RN}(f)$, invariant w.r.t. (with respect to) regular homotopy

   $\text{degree} [ x \mapsto \frac{f_x'(t)}{|f_x'(t)|}]$

2) Winding number $\text{WN}_p(f)$ for any $p \in \mathbb{R}^2$, invariant w.r.t. homotopy in $\mathbb{R}^2 - \{p\}$, or w.r.t. change of $p$ within one connected component of $\mathbb{R}^2 - f$.

A sufficient condition that $f$ extend is that $f$ be an imbedding.

This is not necessary [1]. Consider an extension $F: \mathbb{D}^2 \to \mathbb{R}^2$ of $f = F|_{S^1}$, where $F$ is an immersion.

Then there is a $s > 0$ with $F|_{|z| \leq s}$ an imbedding. [1]

(since $F$ immersion). Let $\tilde{F} = F|_{s \leq |z| \leq 1}$ a regular homotopy between $F|_{|z| \leq s}$ and $f$. Therefore it is necessary, for an extension to exist, that $f$ be regular homotopic to an imbedding; thus $\text{TWN}(f) = \pm 1$. We assume now all $f$ have $\text{TWN}(f) = \pm 1$.

Consider an immersion $f$, extension $F$ which is an immersion, and $f_2$ as above, with $f_0 = F|_{|z| \leq s}$ an imbedding. Choose $p \not\in \text{Im} f_0$. Either $\text{WN}_p(f_0) = 0$, $p$ outside $f_0$; or $\text{WN}_p(f_0) = 1$, $p$ inside $f_0$. Thus $\text{WN}_p(f) = \# \text{ of inverse images of } p \text{ by an extension } F_0 \text{ of } f_0$. This is in fact invariant under regular homotopy. So for any immersion $f$ which extends to $F$, $\text{WN}_p(f) = \#(F^{-1}(p))$.

[2] is an example showing $\text{WN} = 1$ is not sufficient.
Prop. If $f$ extends, $\text{WN}(f) \geq 0$ for all $p \not\in \text{Im} f$. (recall $\text{TNW}(f) = +1$). This condition is again not sufficient [3].

Think back about the extension homotopy. It remains in regions where $\text{WN} \neq 0$, so there is another general condition which I won't state.

Either Milnor or Stallings gave an example [4] of a curve with more than one extension, where $F_1, F_2 : D^2 \to \mathbb{R}^2$ are equivalent if there exists $d : D^2 \to D^2$ with $F_1 = F_2 \circ d$.

One shows that $F_1^{-1}(f)(S')$ and $F_2^{-1}f(S')$ are not diffeomorphic. To get a curve with 4 extensions one can join 2 of these [5] and in general joining $n$ curves [4] gives a curve with $2^n$ different extensions, for any $n$.

There exists a curve with any number of extensions (see below). We now construct machinery to deduce from a curve whether it extends, how many extensions it has, and how they are constructed. Suppose $f$ is in general position (no self tangencies, no triple points). Then $f$ divides $\mathbb{R}^2$ into an unbounded region plus finitely many bounded regions $P_1, \ldots, P_n$. Choose $p_i \in P_i$ for each $i$. Note $\pi_1(\text{Im} f) = \pi_1(\mathbb{R}^2 - \cup P_i)$ by retraction. From each $p_i$ draw a ray at from $p_i$ to $\infty$ [6].
such that $a_i \cdot a_j = a_k$ and $a_i$ is transversal to $\text{im} f$ (not tangent, not through any double point). With each intersection of $a_i$ with $\text{im} f$ we associate an orientation $\pm 1$ according to [7].

Let $w(f) \in \mathcal{F}(a_i)$ be a word in the free group on the $a_i$, obtained by listing the $a_i$'s with proper exponent as they intersect $\text{im} f$ (traverse $f$ once around).

**Example**

\[ w(f) = abc^{-1}bc \]

Then $w(f) \in [f] \in \pi_1(\text{im} f) \cong \mathcal{F}(a_i)$.

We have 2 operations on words:

- Operation **C** consists of deleting any positive letter. A cancellation is any sequence of $C$'s such that the resulting word is the identity.
- Two cancellations are different if the letters deleted are different (including position but not order of deletion).

Operation **G** is deleting any sequence of the form $a^pa^{-1}$ where $p$ is a string of positive letters. A grouping is a sequence of **G**'s such that the resulting word is a string of positive letters.

Two groupings differ if the sequences grouped are different.

The following are equivalent:

1. $f$ extends
2. $f$ has a cancellation
3. $f$ has a grouping.

Note one can ignore those letters which never occur with a negative exponent. One gets a reduced word by this and also omitting any pair $a^- a^+ a^- a^+$; also if $w(f) = a^+ \cdot \cdots \cdot a$ or $a^- \cdot \cdots \cdot a^-$ one can reparametrize $f$ so that the $a, a^-$ drop out.

Then let $w(f)$ be the reduced word. The following are equivalent:
1) $f$ has $n$ extensions
2) $f$ has $n$ cancellations
3) $w(f)$ has $m$ groupings.

Further from the groupings (cancellations) one can construct the extensions.

$w(f) = a b a^{-1} b a b^{-1}$

A curve with $n$ extensions: one loop left, $(n-1)$ loops right

[10]
John Harper: Stable homotopy of Eilenberg-MacLane Spaces.

Th (Freudenthal) \([X, Y] \rightarrow [\Sigma X, \Sigma Y]\) is an isomorphism if \(\dim X \leq 2\cdot\text{connectivity}(Y)\).

Define \(\pi_i^S(Y) = \lim_{n \to \infty} [S^{n+i}, \Sigma^n Y]\). The idea is that \(\pi_i^S\) is a good first approximation to \(\pi_i\), and may be easier to compute.

There is an (Adams) spectral sequence \(E^r_{s,t}\) with

1) \(E^r_{2,s,t} = \text{Ext}^{s,t}_A(\tilde{H}^s(Y), \mathbb{Z})\)

2) \(E_{\infty} = \text{graded object associated to a filtration of } \pi_{t-s}^S(Y)\)

3) \(d_r\) (\(r \geq 2\)) corresponds to \(r\)-th-order cohomology operations.

4) Spectral sequence for a sphere acts on \(E^r_{2,s,t}(Y)\) for any \(Y\); there is a composition \(E^r_2(S) \otimes E^r_2(Y) = \text{Ext}^{s,t}_A(Z_2, Z_2) \otimes E^r_{r+s-t, r+t}(Y, Z_2)\)

\(\rightarrow E^r_{r+s-t, r+t}(Y, Z_2)\), called the Yoneda product, which corresponds to composition of maps in \(E_{\infty} = g_2(\pi^S(Y))\).

There is an element \(b_0 \in \text{Ext}^{s,1}_A(Z_2, Z_2)\) corresponding to \(S^1 \to S^n\) map of degree 2; and in \(E_{\infty}\), multiplication by \(b_0\) corresponds to multiplication by 2. Also, \(d_r\) is a derivation.

5) The spectral sequence is natural with respect to maps of spaces. \(d_r : E^r_{s,t} \rightarrow E^r_{s+r, t+r-1}\).

Let \(Y = K(Z_2, n)\) and \(n \geq 10\). I computed \(\pi_{2n+k}^S(K(Z_2, n))\), \(0 \leq k \leq 7\).

In this range, a \(Z_2\)-basis of the cohomology is independent of \(n\). The groups are 0 up to \(\pi_{2n-1}^S\) except the fundamental class in dimension \(n\), since the cohomology in that range is a free \(A\)-module. On the next page appears a table of results. The notation \(G_q\) means an unknown group of order \(q\); various \(G_q\) are not necessarily the same. Also, ? means a group about which something is known but a complete description is not.
This table is periodic (a conjecture, which is almost certain) of period $2^i$ in $\mathbb{Z}_n$, for rows $k = 0, 1, \ldots, 2^i - 1$; for each $i$. Horizontal periodicity has not been observed.

**Adams' Edge Theorem**

Let $ECA$ be the subalgebra generated by $1$ and $S^1$. Let $i : E \to A$. Then $i^* : \text{Ext}^*_E(M, \mathbb{Z}_2) \to \text{Ext}^*_A(M, \mathbb{Z}_2) = H^{t-s}(M, S^1)$. Adams shows $i^*$ is an isomorphism for $t < m + f(s)$ where $N_t = 0$ for $t < m$ and $f(s)$ is Adams' function:

$$f(s) = \{ \Theta \in \mathbb{A} | \Theta(c_n) = 0 \}$$

$$0 \to A_{B(n)}(n) \to H^*(\mathbb{Z}_2, n) \xrightarrow{A(c_n)} H^*(\mathbb{Z}_2, n) \to 0.$$  

Let the right-hand module be $M$. It is $(2n\mathbb{Z})$-connected and $B(n)(n)$ is $(2n\mathbb{Z})$-connected. The long exact sequence in $\text{Ext}$ shows $\text{Ext}(A_{B(n)}(n))$ = $\text{Ext}(B(n))$ so $H^*(\mathbb{Z}_2, n)$ is $2n$-connected also.

In the following table $n \equiv 5 \mod 8$. 

<table>
<thead>
<tr>
<th>$n \mod 8$</th>
<th>0</th>
<th>1</th>
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<td>$\mathbb{Z}_2 + ?$</td>
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</table>
\( S_q^{1-n} + S_q^{1-p} = 0 \) where \( p \) is the operation associated with \( \eta \circ S_q^1 \).

An interesting case is \( e_{2,q} \leftrightarrow (S_q^{7}S_q^{1} + S_q^{5}S_q^{2}S_q^{1})S_q^{6} + S_q^{7}(S_q^{6}S_q^{1} + S_q^{0}S_q^{p}) \) + \( S_q^{3}(S_q^{8}S_q^{3} + S_q^{5}S_q^{0} + S_q^{3}S_q^{1})(\theta) \) where \( \theta = \eta \circ S_q^2 \).

A relation of 4th order in \((4,2n+6)\) actually decomposes a secondary operation; hence it does not give a quaternion operation.

**Universal Example for \( e_{2,q} \):**

\[ E_m \quad \text{fibration} \]

\[ \Sigma_k K(Z_2,5) \rightarrow K(Z_2,m) \times K(Z_2,m+6) \times K(m+7) \rightarrow K(Z_2,m+6) \times K(Z_2,m+7) \times K(Z_2,m+11) \]

\[ \Sigma_q S_q^1 \]

\[ l_m \quad S_q^6 \quad l_m+6 \]

\[ l_m+6 \quad S_q^6 \]

\[ l_m+7 \quad S_q^6 \quad l_m+11 \]

we must identify some class in \( \Sigma_k K(Z_2,5) \) (some \( r \)). We choose \( m \) so large that \( \Pi^S(E_m) = \Pi(E_m) \), which is known.
MIT Topology Seminar

John Milnor (Princeton): Curvature and the fundamental group

The problem is to relate the differential geometry of a manifold with its topology.

Let $G$ be a finitely generated group throughout. Let $g_1, \ldots, g_k$ be preferred generators of $G$. Let $\gamma(s)$, the growth function of $G$, be defined as the number of distinct group elements of the form $g_1^{\pm r_1} g_2^{\pm r_2} \cdots g_k^{\pm r_k}$ where $r \leq s$, for each natural number $s$. We investigate the asymptotic behavior of $\gamma(s)$ as a measure of how big $G$ is.

**Lemma** $\gamma(s+t) \leq \gamma(s) \gamma(t)$

So in particular for any natural number $k$, $\gamma(k \cdot s) \leq \gamma(s)^k$; $\gamma(k) \leq \gamma(1)^k$.

**Examples**

1. $G$ finite; $\gamma$ bounded.
2. $G$ infinite cyclic; $\gamma(s) = 2s+1$
3. $G$ free abelian on 2 generators $\gamma(s) = 2s^2 + 2s + 1$
   - $\gamma(s) = a_0 s^k + a_1 s^{k-1} + \cdots + a_r$; rational
4. $G$ a free group on 2 generators: $\gamma(s) = 2 \cdot 3^s - 1$.

**Question**: To what extent is $\gamma$ an invariant of the group?

**Lemma** If $\{g_1, \ldots, g_s\}$ is a second set of generators, yielding a growth function $\gamma'(t)$ for $G$, then there exists $k > 0$ such that $\gamma(s) \leq \gamma'(k \cdot s)$ and $\gamma'(s) \leq \gamma(k \cdot t)$. Thus if $\gamma$ increases as a polynomial (resp. exponential), so does $\gamma'$.

All examples found so far have been polynomial or exponential. Are all groups thus?

**Lemma** $1 \leq \lim_{s \to \infty} \gamma(s)^{1/s} < \infty$.

**Proof** Let $t_0$ and $s$ be integers $> 0$ and choose $k$ such that $kt_0 \leq s \leq (k+1) t_0$. For fixed $t_0$,

\[ \gamma(s) \leq \gamma((k+1) t_0) \leq \gamma(t_0)^{k+1} \leq \gamma(t_0)^{\frac{s}{t_0} + 1} \quad (k < \frac{s}{t_0}). \]

raise to $1/s$ power:

\[ \gamma(s)^{1/s} \leq \gamma(t_0)^{\frac{k+1}{t_0}}. \]

\[ \Rightarrow \limsup_{s \to \infty} \gamma(s)^{1/s} \leq \gamma(t_0)^{1/t_0} \quad \text{for all } t_0. \]

\[ \Rightarrow \limsup_{s \to \infty} \gamma(s)^{1/s} \leq \inf \gamma(t)^{1/t} \leq \liminf_{t \to \infty} \gamma(t)^{1/t}. \]

hence $\gamma(s)^{1/s}$ converges to a finite number.
How is this connected with curvature? Consider a Riemannian manifold with distance $d$. Then $\{y \mid d(y, x_0) \leq r\}$ is the ball of radius $r$ about $x_0$; this ball has Riemannian volume $\omega_n \cdot r^n +$ higher terms. Loosely speaking, "positive curvature" means the higher terms are negative; the ball has less than normal volume; negative curvature means they are positive.

**Thm. (Günter)** In a complete, 1-connected Riemannian manifold suppose all sectional curvatures $K$ have $K_0 \leq K \leq K_1$ (constants). Then

$$V_{K_1}(r) \leq V(r) \leq V_{K_0}(r)$$

where $V_{K_i}(r) =$ value of $V(r)$ in a space of constant curvature $K_i$ (e.g., a sphere has constant positive curvature).

**Thm. (R.L. Bishop)** It suffices to assume

$$\text{Mean curvature} \geq K_0 \cdot (n-1)$$

to conclude $V(r) \leq V_{K_0}(r)$.

Combine these theorems with the universal covering space:

1. If mean curvature $\geq 0$ everywhere, then $\gamma(s) \leq \text{constant}, S^n$, where $n =$ dimension. ($\pi_1 = $ fundamental group of $M =$ manifold)

2. If the sectional curvature $K < 0$ always, $\gamma(s) \geq (1+e)^s$.

There exists a theorem that if the curvature is always $> 0$, $\gamma(s)$ is bounded.

The first statement is easy to prove and more interesting; the second harder, and much is already known about that case.

**Proof of (1)** Let $\tilde{M} \rightarrow M$ be the simply connected covering space of the Riemannian manifold $M$. Mean curvature $\geq 0 \Rightarrow V(r) =$ Volume of $N_r(x_0) \leq V($ euclidean space, $r) \leq \omega_n \cdot r^n$ by Bishop & Günter.

Let $g \in \pi_1(M) =$ covering transformations of $M$. Think of the points $g(x_0)$ (where $x_0 \in \tilde{M}$) as a lattice in $\tilde{M}$; we want to say the number of points $g(x_0)$ in a volume is proportional to the volume, $\sigma$ (Think of the torus $T^2$ : $\tilde{M} = R^2$)

$T^2$
Then let $\mu = \max d(x_0, y_0)_{\text{gi}}$ as $\text{gi}$ runs over preferred generators for $\pi_1(M)$. Let $N_{r}(x_0) = \{ y \in M \mid d(y, x_0) \leq r \}$. Look at $N_{5\mu}(x_0)$:
It contains at least $y(s)$ translates $g_i(x_0)$, since $d(x_0, g_{i_1}^{l_1} \cdots g_{i_s}^{l_s} x_0) \leq 5 \cdot \mu$. Choose $E$ such that the balls of radius $E$ about lattice points are all disjoint. Consider $N_{5\mu + E}(x_0) = V_{(5\mu + E)} \geq y(s) V(E)$
since for each lattice point in $N_{5\mu}(x_0)$ there is contained in $N_{5\mu}(x_0)$ an entire $E$-neighborhood. Hence

$$w_n (5\mu + E)^n \geq V_{(5\mu + E)} \geq y(s) V(E),$$
or

$$w_n V(E) \geq y(s)$$
or writing $SE$ for $E$, $(w_n V(E))^{5(1+e)^n} \geq (w_n V(E)) (5\mu + E)^n \geq y(s)$$

The proof of (2) is a little harder but no less elementary.

**Examples** An orientable surface is described by its genus

- genus 0 (sphere) $\gamma(s) = 1$
- genus 1 $K \equiv 0$ if you choose metric correctly; $\gamma(s) = 2s^2 + 2s + 1$
- genus $\geq 2$ has metric with constant negative $K < 0$;

$$\pi_1(M) = \text{free group on } \{ a_1, \ldots, a_j ; b_1, \ldots, b_j \}/\langle [a_1, b_1], \ldots, [a_j, b_j] \rangle.$$  

So $\pi_1(M)$ free group on $2j - 1$ generators $\implies$ free group on 2 generators, thus $\gamma$ has exponential growth.

These spaces have a metric with $K$ of constant sign. A manifold $M$ without $K$ of constant sign is the nil manifold: the set of matrices

$$\{ (0 \ 1 \ c) \mid a, b, c \in \mathbb{R} \}$$
is a nilpotent Lie group (topologically, $\mathbb{R}^3$). Let $\Pi = \{ (0 \ 1 \ c) \mid a, b, c \in \mathbb{Z} \}$ is discrete subgroup. Then $N/\Pi$ is a compact 3-manifold with fundamental group $\Pi$. This group satisfies neither conclusion; in fact $\gamma(s)$ increases like a polynomial of degree 4 (e.g. $c_1 s^4 \leq \gamma(s) \leq c_2 s^4$). If so, $M$ has no metric with a curvature of constant sign, since $\gamma$ is neither exponential nor of degree 3.

Take $S^3$, remove knot: the resulting manifold has trivial higher homotopy groups, hence the universal covering space is contractible. Does this admit a complete metric with $K \equiv 0$? This question is quite open.
Wm. Cockcroft: Whitehead Torsion

J.H. Whitehead (hereafter JHW) wrote a paper 'Simple Homotopy Types' in the Amer. J. Math in 1950 in which was defined what is now known as Whitehead torsion. This talk presents a new way of looking at that.

1. Whitehead Group

In the following, A is a ring with unit 1, such that finitely generated modules over A have well-defined rank, i.e. $A^n \cong A^m \iff n = m$. We consider all modules which arise as submodules of a module $M = \{ (r_1, r_2, \ldots) : r_i \in A$ and almost all $r_i = 0 \}$, in fact as submodules on a set of generators which are chosen from among the preferred generators $m_1 = (1, 0, \ldots)$, $m_2 = (0, 1, 0, \ldots)$, \ldots \ of M. Any two finitely generated modules over A may be embedded disjointly in M and a natural direct sum within M is thus defined.

Endomorphisms $F$ of M are called admissible if $f m_i = m_i$ for almost all $i = 1, 2, \ldots$. Admissible endomorphisms of M form a group $\text{Gl}(A)$. An endomorphism $f$ is elementary if $f m_i = m_{i+j}$ and $f m_k = m_k$ for all other $k$, where $i$ is any index and $a \in A$. The subgroup of $\text{Gl}(A)$ generated by elementary endomorphisms is called $E(A)$, and (Lemma) it equals $[\text{Gl}(A), \text{Gl}(A)]$ (note—this would be false if M were cut off after a finite number of generators). We call $K, A = \text{Gl}(A)/E(A)$ the Whitehead Group of A. Let N be the subgroup of $K/A$ of order two generated by $[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}]$. Call $\overline{K}_A = K, A/N$ the reduced Whitehead group of A. There is an obvious matrix formulation for these definitions.

Lemma $[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}] = [\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}]$ in $\overline{K}_A$. (a,b square matrices)

Proof $[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}] = [\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}] = [\begin{smallmatrix} a & 0 \\ -a & b \end{smallmatrix}] = [\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}] = [\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}]$.

* The modules considered will be free.
$$\begin{bmatrix} a & b \\ 0 & I \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & I \end{bmatrix}$$

2. Modules and Torsion

Now we associate elements of $K, A$ to mappings of modules. Let $B, B' \subseteq M$ be modules and $\alpha : B \to B'$. Then $B, B'$ have the same rank. There is a permutation of the $m_i$; $P : M \to M$, such that $P \alpha : B \to B'$ defines an admissible endomorphism, hence an element $\tau(P \alpha) \in K, A$. (recall $B, B'$ have preferred sets of generators among the $m_i$).

Lemma: $\tau(\alpha) = \tau(P \alpha)$ is independent of choice of $P$.

We say $\alpha$ is simple if $\tau(\alpha) = 0$.

Prop. 1 $\tau(\alpha \beta) = \tau(\alpha) + \tau(\beta)$

Prop. 2 If $\gamma : A + B \to A' + B'$ (all within $M$) satisfies $\gamma(a+b) = \alpha a + (\beta a + \gamma b)$ where $\alpha : A \to A'$, $\beta : B \to B'$, then if $\alpha$ and $\beta$ are (so, so is $\gamma$ and $\tau(\gamma) = \tau(\alpha) + \tau(\beta)$).

3. Chain systems $C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \cdots$

A chain system $C$ is a finite set of finitely generated $A$-modules indexed by $0, 1, 2, \ldots, n$ for some $n$, together with maps $d$ as above. Again $C_i \subseteq M$ for all $i$; we assume $C_i \cap C_j = \emptyset$.

A chain mapping $f$ between chain systems $C, C'$ is a sequence of $A$-homomorphisms $f_i : C_i \to C_i'$ satisfying $f d = d f$. A chain homotopy $\eta$ between two chain mappings $f, g$ is a sequence of $A$-homomorphisms $C_i \xrightarrow{\eta_i} C_{i+1}$ with $f - g = \eta d + d \eta$. If $f : C \to C'$ is a chain mapping and $f_i : C_i \cong C_i'$ for each $i$, and $\tau(f_i) = 0$ for all $i$, say $f$ is a simple isomorphism.

An elementary complex $B : 0 \to \cdots \to B_p \xrightarrow{d_p} B_{p-1} \to 0 \to \cdots \to 0$ is such a $B$ satisfying $d : B_p \cong B_{p-1}$ and $\tau(d) = 0$. A collapsible complex is a direct sum of elementary complexes. A chain mapping $f : C \to C'$ is a simple chain equivalence if $f$ factors

$$C \to C + D \xrightarrow{h} C' + D' \xrightarrow{k} C'$$

where $h$ is a simple isomorphism and $D, D'$ are collapsible. Write then $f : C \equiv C'(\Sigma)$. 
Lemma. If $C$ is acyclic, then $C \xrightarrow{f} C'$ is simply equivalent to a chain system $\cdots \to 0 \to 0 \to P \xrightarrow{f} Q \to 0 \to 0 \to \cdots$.

Proof. We give the first step only.

Since $C_1 \xrightarrow{d} C_0$ is onto a free module, there exists $\eta: C_0 \to C_1$ with $d\eta = 1$.

Chain equivalences are defined at right; the map of $C_1 + C_0$ to itself has matrix $\begin{bmatrix} 1 & \eta \\ 0 & 1 \end{bmatrix}$. Hence it is simple since each of the factors is $0$.

This is the start of an inductive process. ($C_0 \xrightarrow{f} C_0$ is collapsible).

Define $\tau(C) = \tau(\alpha)$.

Lemma. $\tau(C)$ is independent of the choice of $\eta$.

Lemma. If $C, C' \equiv 0$, then $\tau(C) = \tau(C') \iff C \equiv C'(\Sigma)$.

Cor. $C \equiv 0(\Sigma) \iff \tau(C) = 0$.

Lemma. If $C = C' + C''$ then $\tau(C) = \tau(C') + \tau(C'')$ (all acyclic).


Let $f: C \equiv C'$. Define the mapping cylinder $C_f$ of $f$ by

$$(C_f)_i = C_i + C_{i-1}, \quad d_f C' = d' C', \quad d_f C_i = (f - 1)C_{i-1}.$$

One shows $C_f \equiv 0$; put $\tau(f) = \tau(C_f)$.

Prop. $f \equiv g: C \equiv C' \Rightarrow \tau(f) = \tau(g)$ (because the mapping cylinders are chain equivalent).

Prop. $f: C \equiv C'$, $f': C' \equiv C'' \Rightarrow \tau(f'f) = \tau(f) + \tau(f')$.

If $f: C \equiv C'(\Sigma)$ then $\tau(f) = 0$.

Prop. $f: C \equiv C'$, then $\tau(f) = 0 \iff f: C \equiv C'(\Sigma)$.

5. Homotopy equivalences and their torsion.

Let $K, L$ be finite dimensional complexes. If $K \xrightarrow{\sim} L$ is a homotopy equivalence, take the universal covering $\tilde{K}, \tilde{L}$ and
the associated chain complexes $C(K), C(L)$. Let $A = \mathbb{Z}(\pi_1(L))$, the group ring. To $\phi$ corresponds an isomorphism $\Theta: \pi_1(K) \cong \pi_1(L)$; also to get $\tilde{K}, \tilde{L}$ one must choose base points. One gets a chain map (associated with choice of $\Theta$) $C(K) \to C(L)$. Use the group $\tilde{K}(A) / \{ \text{units of } A = \pi_1 \subset A^2 \}$ to get the torsion $\tau(\phi)$ of the homotopy equivalence $\phi$. If $\tau(\phi) = 0$, $\phi$ is called simple.

**Formal moves.** A formal move is a map of simplicial complexes like that indicated in the picture:

If one complex can be obtained from another by a series of formal moves, they have the same homotopy type, i.e., $K \xrightarrow{\text{formally}} K'$.

$\Rightarrow K \equiv K'(\Sigma)$. Whitehead asks how much of this can be done in the infinite case.

An infinite series $K \to K \cup e^n e^s e^{-s}$ is possible, that is, the geometry makes sense, but the algebra apparently doesn't generalize.

**Conjecture.** If $K$ and $L$ have the same homotopy type, then $K \equiv L(\Sigma)$ if an infinite number of formal moves is allowed. (Of bounded dimension.)

The following supports this: Let $[C]$ be all complexes of the same simple homotopy type as $C$. Let $F$ be the free abelian group on all such $[C]$. Let $G = F / \{ \text{all relations } [C + C'] = [C] + [C'] \}$.

Map $G \to \tilde{K}, A$ by $[C] \to \tau(C)$; then the theorems about $\tau$ simply show this map is an isomorphism, (or start with chain equivalences; use $[fg] = [f] + [g]$ etc.). Now $[C] + \sum_c C = \sum_c [C]$ but $\sum_c \sum_0^\infty = \sum_0^\infty$, hence $[C] = 0$. (A corresponding fact is that in the category of modules over a ring, if infinitely generated modules are allowed the Grothendieck group is 0).

**Problem.** How can you carry out the algebra in the infinite case?
Kenneth C. Millett: Normal Structures

Let \( f: M^m \rightarrow N^n \) be a topological embedding.

**Defn** \( f \) is locally flat if for every \( x \in M \) there is a neighborhood \( U \) of \( f(x) \) and a homeomorphism of pairs \( h: (U, U \cap f(M)) \rightarrow (f(U), f(M)) \).

**Remark** Piecewise linear and codimension \( \geq 3 \) \( \Rightarrow \) loc. flat.

There are various normal structures available.

**Defn 1.** Euclidean bundle group is \( H(n) \), homeomorphisms of \( \mathbb{R}^n \); bundle is \( \mathbb{R}^n \rightarrow E \xrightarrow{p} M \), a fibration.

**Defn 2.** Euclidean bundle with o-section group \( H_0(n) = \text{homeomorphisms of } \mathbb{R}^n \).

(\( \text{In Euclidean bundles a o-section is a weak condition as any two sections are equivalent - there is an isomorphism of bundles carrying one into the other} \).)

1. A tubular neighborhood is an \( H_0(n-m) \)-bundle \( V \) over \( M \) and an embedding \( g \) in a commutative diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{g} & \mathbb{R}^n \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & N
\end{array}
\]

This does not always exist; see Rourke and Sanderson in Invent. Math., 1968.

2. Homotopy normal bundle. Let \( T_M \) be the \( H_0(n) \)-bundle coming from \( M \times M \xrightarrow{p_2} M \) (the tangent microbundle). Then \( f^*T_M \) contains \( T_M \) as a subbundle. (i.e. the group of \( f^*T_M \) can be reduced to those homeomorphisms of \( \mathbb{R}^n \) leaving \( \mathbb{R}^n \) invariant setwise). The Homotopy normal bundle \( \nu \) is

\[
E_{\nu} = (E_{f^*T_M} \setminus E_{TM}) \cup \text{0-section}.
\]

3. Normal block bundle \((P_n L_n)^*(\text{Rourke-Sanderson})\).

4. A normal bundle \( \eta \) is any \( H_0(n-m) \)-bundle with \( \eta \oplus T_M/T_M \).

5. A Homotopy normal euclidean bundle \( \eta_f \) is any \( H_0(n-m) \)-bundle fiber homotopy equivalent to \( \nu_f \).

**Remark** \( 1 \Rightarrow 4 \Rightarrow 5 \Rightarrow 2 \) (existence).

**Main Result.** There is an embedding for which not even a \( 5 \) \( (\text{HNEB}) \) exists.
Cor. There is a Euclidean bundle pair \((\alpha, \beta)\) which does not split off, even up to fiber homotopy equivalence, a Euclidean bundle.

We describe below Rourke and Sanderson's example. Let \(q+1 \geq 3\).

**Def.** A link is an embedding \(f : \Sigma^n \times S^1 \to \Sigma^{n+q+1}\) \(\{ I = [0,1] \}

A ribbon link is an embedding \(f : \Sigma^n \times I \to \Sigma^{n+q+1} \{ I = [-1,1] \}\).

By a theorem of Zeeman, \(q+1 \geq 3 \Rightarrow f\) is unknotted: there is an isotopy of \(\Sigma^{n+q+1}\) such that \(f \to \) standard embedding, when restricted to \(\Sigma^n \times 0\). In that case we arrive at an \(f_1\) - the end result of the isotopy, restricted to \(\Sigma^n \times 1\) - with \(f : \Sigma^n \times 1 \to \Sigma^{n+q+1} \setminus \Sigma^n\) which is homotopy equivalent to \(S^q\). Hence we have a class \([f_1] \in \Pi_n(S^q)\), the second linking class of \(f\). Similarly define the first linking class; show both are well-defined and invariant under isotopy of \(f\).

**Def.** \(\Sigma^f : \Sigma^{n+1} \times S^1 \to \Sigma^{n+q+3}\), for any link \(f\). (see picture.)

**Prop.** Any link \(\Sigma^n \times S^1 \to \mathbb{R}^{n+q+1}\) extends to \(f : \Sigma^n \times I \to \mathbb{R}^{n+q+2}\) + .

**Prop.** \(f : \Sigma^n \times S^1 \to \mathbb{R}^{n+q+1}\) extends to \((\Sigma f)' : \Sigma^{n+1} \times I \to \Sigma^{n+q+3}\).

Let \(L_n^q\) be the set of isotopy classes of \(\Sigma^n \times I\) in \(\Sigma^{n+q}\), and \(N_n^q\) the set of classes in \(L_n^q\) whose first linking class is 0.

Then (Haefliger) there is an exact sequence

\[
\begin{align*}
N_n^q \xrightarrow{2^n\text{class}} \Pi_n(S^{q-1}) & \to \Pi^{n+q+1} \to N_{n-1}^q.
\end{align*}
\]

There are natural suspension maps \(\Sigma : L_n^q \to L_{n+1}^q\), \(E : \Pi_n(S^{q-1}) \to \Pi_{n+1}(S^q)\) which fit in a commutative diagram

\[
\begin{array}{cccc}
N_n^q & \to & \Pi_n(S^{q-1}) & \to & \Pi^{n+q+1} \\
\downarrow & & \downarrow E & & \downarrow I \\
N_{n+1}^q & \to & \Pi_{n+1}(S^q) & \to & \Pi^{n+q+1}
\end{array}
\]
Let $\alpha$ generate $\mathbb{Z}_{29} \subset \pi_{19}(S^9)$. Then $6\alpha \to 0$ in $\mathbb{Z}_6 = \mathbb{Z}/6\mathbb{Z}$; by  

a diagram chase there exists an $f$ with $[f] \to 6\alpha$ where $f$ has a 2nd class of order 4. But:

Then $f: \Sigma^n \times I^1 \to \Sigma^{n+q+1}$, codimension $\geq 3$. Then $f$ has H.N.E.B., $f|\Sigma^n \times 0 =$ std. inclusion $\begin{cases} 2^{nd} \text{class of } f|\Sigma^n \times I^1 \text{ has order 2.} \end{cases}$ (conjecture: each such 2nd class must be 0). \(\Box\)

The contradiction shows that an $f$ for which $[f] \to 6\alpha$ has no H.N.E.B. \(\Box\)
H. Toda (Kyoto Univ.) Recent progress in homotopy theory

1. Stable homotopy groups of spheres

\[ \pi_t = \pi_{n+t}(S^n) \text{ for } n \text{ large. For an odd prime } p, \text{ let } \]
\[ \pi_t^p = p\text{-primary component of } \pi_t. \]

\[ \exists \alpha_i \in \pi_{2i(p-1)+1}, i = 1, 2, 3, \ldots \] \text{ of order } p

\[ \exists \beta_s \in \pi_{2 (sp+s-1)(p-1) - 2}, 1 \leq s \leq p \]
\[ \text{ } \beta_p = ? \]

\[ \exists \alpha_i' \text{ s.t. } i \alpha_i' = \alpha_i \text{ for } i \leq p^3 \]

For \( t < 2p^2(p-1)-3 \), \( \pi_t^p \) is determined with generators \( \alpha_i', \beta_s \) and some compositions \(( \sum \pi_t^p = \pi_t^p, \text{ a ring, generated up to } 2p^2(p-1)-3 \text{ by } \alpha_i', \beta_s )\).

The next problem is \( t = 2p^2(p-1)-3 \). Now \( \pi_t^p \) is generated by \( \alpha, \beta \), of order at most \( p \), i.e.

\[ \pi_t^p = \begin{cases} 
\mathbb{Z}_p & \text{if } \alpha, \beta \neq 0 \\
0 & \text{if } \alpha, \beta = 0 
\end{cases} \]
(Cohen (1966), Yamamoto, Gray (p=3)

(Toda (1967)).

**Thm.** \( p, y = 0, y \in \pi_t \Rightarrow \alpha, y^p = 0 \)

**Cor.** \( \alpha, \beta^p = 0 \)

**Thm.** \( \alpha, y = 0 \Rightarrow \beta, y^p = 0 \)

**Cor.** \( p, y = 0 \Rightarrow \beta, y^p = 0 \), hence \( \beta^{p^2+1} = 0 \)

(since \( \alpha, y \in \pi_n \Rightarrow (\alpha, y)^2 = 0 \))

**Thm.** \( \beta_s, 1 \leq s \leq p \) are all nilpotent

Conjecture. every \( \pi_t \) is nilpotent.

2. Non-associativity or non-commutativity in mod q theories.

Let \( q \) be an integer \( \geq 1 \). Let \( M^{q^n} = S^n \cup q \cdot e^{n+1} \), a Moore
Space of type \((\mathbb{Z}_q, n)\).

\[ T_\ast (\mathbb{Z}_q) \overset{\text{def}}{=} T_{n+t} (M_{q, n}) \text{ if } n \text{ large.} \]

Given \(h\), a generalized cohomology theory, define \(h^*(\ ; \mathbb{Z}_q)\) by

\[ h^i (x, \mathbb{Z}_q) = h^i (x \wedge M_{q, 1}) \]

\[ T_\ast = \mathbb{Z}[T^t_\ast] \text{ ring; } h^* \text{ has an associative and commutative multiplication.} \]

\[ T_\ast (\mathbb{Z}_q) \otimes T_\ast (\mathbb{Z}_q) = T_{n+t} (M_{q, n}) \otimes T_{n+t} (M_{q, n}) \overset{\text{Smash}}{\longrightarrow} \overset{\text{product}}{\longrightarrow} T_{2n+t} (M_{q, 2n} \wedge M_{q, 2n}) \overset{\psi^*}{\longrightarrow} T_{2n+t} (M_{q, 2n}) = T_{\ast+t} (\mathbb{Z}_q). \]

\(\psi^*\) might come from

\[ \xymatrix{ M_{q, 2n} \wedge M_{q, 2n} \ar[r]^-{\psi} \ar[d]^-{s^t \times s^t} & M_{q, 2n} \ar[r]^-{\text{id}} \ar[u]^-{\text{id}} & \mathbb{S}^n \times \mathbb{S}^n \ar[u]^-{s^t} } \]

(if \(\mu_t, \mu_t'\) are multiplications on \(h^*(\ ; \mathbb{Z}_q)\) then \(\mu_q (x \wedge y) - \mu_q' (x \wedge y) = \pm b \cdot s^t \wedge s^t y\); be \(k^2 (s^t \wedge \mathbb{Z}_2)\), \(s^t \wedge s^t y = 0\); this is the uniqueness theorem. So it is enough to look at such a \(\psi\).

By a theory of Araki-Toda, the following table concerning multiplication in \(h^*(\ ; \mathbb{Z}_q)\) was obtained:

<table>
<thead>
<tr>
<th>(q)</th>
<th>exists acommutative</th>
<th>exists an associative</th>
</tr>
</thead>
<tbody>
<tr>
<td>(q \equiv 2(4))</td>
<td>No ((K\Omega (\mathbb{Z}_4)))</td>
<td></td>
</tr>
<tr>
<td>(q \equiv 0\ (8))</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(q \equiv 4\ (8))</td>
<td>Yes</td>
<td>No ((K\Omega (\mathbb{Z}_4)))</td>
</tr>
<tr>
<td>(q \equiv \text{odd; } q \neq 0\ (3))</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(q \equiv 0\ (4))</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(q \equiv 3\ (6))</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(q \equiv 0\ (8))</td>
<td>Yes</td>
<td>Yes (Anderson)</td>
</tr>
<tr>
<td>(q \equiv 4\ (8))</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(q \equiv \text{odd; } q \neq 0\ (3))</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(q \equiv 0\ (4))</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(q \equiv 3\ (6))</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

No (Gray).
In case \( q = 3,6 (4) \), Gray showed "no" was equivalent to \( \alpha, \beta = 0 \).

3. \( \delta^n \) in \( Q(x) = \lim_{k \to \infty} S^{2k} S^k x \) or \( S^{2k} S^{2k+2} \), \( n \geq 0 \).

Equivalently, study the dual operation \( \delta^* \) in homology. Assume \( X \) connected.

\[ H_* (Q(X); \mathbb{Z}_p) = \text{free commut. ring over } \Delta^{E_i} Q^* \ldots \Delta^{E_r} Q^* x^j, \]

where \( \{ x_j \} \) is basis of \( H_* (X; \mathbb{Z}_p) \) (cf. \( H_* (Q(X); \mathbb{Z}_p) = \text{Dyer-Lashof} \)). In the Dyer-Lashof paper \( Q_i^* ; H_2 (Q(X); \mathbb{Z}_p) \to H_{2p+1} (Q(X); \mathbb{Z}_p) \),

with \( \Delta Q_{2i} = Q_{2i-1} \) (D = Homology Bockstein), \( Q_{2i} x = 0 \) if \( 2i \neq \#(p-1) \) (mod \( 2p-2 \)). So write instead

\[ Q^n (x) = (-1)^j + 2m p^{-1} (p+1)/2 \binom{m}{j} \delta^* Q_{2j-1} Q_n (x) \] (\( m = \frac{r-1}{2}, p > 2 \)).

Then

\[ \delta^* : H^p \to H^p + 2j \] \( (p-1) \)

\[ \delta^* : H^p \to H^p - 2u (p-1) \]

**Nishida formula**

\[ \delta^* Q^n x = \sum \binom{-1}{n-p} \frac{1}{n-p+i} Q^{n+i} \delta^* x_i \]

\[ \delta^* \Delta Q^{n+i} = \sum \binom{-1}{n-p} \frac{1}{n-p+i} \Delta Q^{n+i} \delta^* x_i + \sum \binom{-1}{n-p+i} \frac{1}{n-p+i} Q^n \delta^* \Delta \]

Or for \( p > 2 \), \( \delta Q^n : Q^{n+i} = \sum \{ S^p \} Q^{n+i} \delta^* x_i \).

In \( \Omega^k S^{2n+k} (n > 0) \), \( \Omega^k S^{2n+k} \sim S^{2n-1} \times \Omega^k S^{2n+1} (p > 2) \).

To prove "p, y \Rightarrow \alpha, \beta = 0"

\( S^{2n+1} = \text{units of } C^{n+1} ; S^i \) acts on \( S^{2n+1} \), \( W = \bigcup_{n} S^{2n+1} \) so \( S^i \) acts on \( W \). \( \pi = \mathbb{Z}_p \subset S^1 \) so \( \pi \) acts on \( W \). \( (\text{cyclic grp. of order } p) \).

\( W \) is a \( \pi \)-free acyclic (regular) CW-complex. \( W^r = r \)-skeleton of \( W \).

\( W^r / W^r-1 = e_r \cup \sigma e_r \cup \ldots \cup \sigma^{r-1} e_r \) where \( 1 \neq \sigma \in \pi \).

Let \( X \) be a finite CW-complex \( \Rightarrow \) base \( p^t \).

\[ X(p) = X \times L \] \( p \times X \)

\( \pi \) acts on \( X(p) \), permuting factors, \( W \times \pi X(p) \Rightarrow W \times \pi \{x(p)\} \) and \( W^r \times \pi X(p) \Rightarrow W^r \times \pi \{x(p)\} = W^r / \pi \) (r-skeleton of lens space).

Let \( \epsilon_p (x) = (w \times \pi x / \pi) / (w / \pi) \). Consider \( X \longrightarrow \epsilon_p (x) \).
This a functor from the category of spaces (finite CW-complexes) and continuous maps to itself; it preserves homotopy. Also \( e_p^r(f) \mid e_p^r(x) = e_p^r(f) \) \((s \leq r)\) for \( f: X \to Y \), and \( e_p^r(0) = 0 \).

\( e_p^0(f) = f^* = f \times \cdots \times f : e_p^0(x) \to e_p^0(y) \), since \( \omega = \pi \).

Let \( X = S^n, p \geq 2 \). \( e_p^r(S^n) = S^{np} v e_{p^{n+1}} v e_{p^{n+2}} \cdots v e_{p^{n+r}} \) (if \( r \) is even). If \( r \leq 2p - 4, n \geq 2 \), then

\[
\begin{align*}
\text{h. eq.} & \\
\Rightarrow & \quad p^{t \times t} = 0, t < 2p - 3.
\end{align*}
\]

\( C_2 : e_p^r(S^n) \xrightarrow{r} S^{np} = e_p^0(S^n) \).

Let \( X = M_{p,n} \) \( e_p^r(M_{p,n}) = S^{np} \cdots \).

**Lemma** \( \beta_0^\ell : H^\ell(e_p^r(M_{p,n}), \mathbb{Z}_p) \to 0 \) if \( r \geq p - 1 \).

So \( \exists j : C_2 = S^{np} v C_2, S^{np+2p-2} \to e_p^r(M_{p,n}) \).

Assume \( p : y = 0 \).

Then \( \exists \widetilde{y} : M_{p,n} = S^n v e_n \to S^{n-t} \) with \( \widetilde{y} \mid S^n = y \).

There is a diagram:

\[
\begin{align*}
&e_p^{-1}(M_{p,n}) \xrightarrow{e_p^{-1}(y)} e_p^{-1}(S^{n-t}) \\
&\quad \uparrow i \\
&(M_{p,n})^{(p)} \xrightarrow{i} S^{np} \\
&\quad \uparrow i
\end{align*}
\]

which commutes.

We build a larger diagram on it:
This yields a commutative diagram:

\[ S^{n+2p-2} \xrightarrow{\pi} S^{n+2p-2} \xrightarrow{\gamma} S^{n+2p-2} \]

but such a diagram exists if \( i \gamma = 0 \)

\( \iff \alpha_i \gamma^p = 0 \).

\[ \beta \gamma = 0 \Rightarrow \beta_\gamma^p = 0 \]

We have maps

\[ S^{n+2p-3} \xrightarrow{\alpha_i} S^n \xrightarrow{\gamma} S^{n+2p-3} \]

\[ \Rightarrow \alpha_i \gamma^p = 0 \]

Lemma: Functional \( \beta^p \)-operation for \( \beta = \gamma_0 \alpha_i \gamma^p \) is not trivial.

Thus we can get a \( \overline{\beta} \) in the following diagram, which has a non-trivial \( \beta^p \)-operation.

But \( \overline{\beta} = 0 \Rightarrow \beta = 0 \).

[Contradicts triviality of the mod p Hopf invariant], hence

\( \beta \neq 0 \), \( \beta = \star \beta \in H^2_\pi(p) \).

where \( \star \neq 0 \mod p \); hence \( \beta \gamma^p = 0 \).

Remark

\[ \delta^* : H_* (\pi; \mathbb{Z}_p) \Rightarrow 1 \]

\[ \delta^* = \delta_* : H_* (W X, X^\pi; \mathbb{Z}_p) \Rightarrow 1 \]

\[ \delta_* : H_* (W X, X^\pi; \mathbb{Z}_p) \Rightarrow 2 \]

Nishida's formula.
H. Toda (Kyoto Univ.): The Nishida Formulas

\[ \Theta : \pi_\ast x \rightarrow x = \lim_{\rightarrow n} \Omega^* S^n x \]

\[ \gamma_n \in H_\ast (\pi \ast x \pi; Zp) \]

\[ \gamma_n \in H_\ast (\pi \ast x \pi; Zp) \]

**Definition of \( \gamma_n \)**

Let \( \gamma \in H_i^f (x; Z_p) \) [all coefficients \( Z_p \) hereafter], and let \( c : C(x) \rightarrow Z_p \) be a cochain representing \( u \). The external reduced \( p \)-th power \( P(u) \in H^{pq} (\pi \ast x \pi) \) is given by \( C(x) \otimes C(x) \xrightarrow{\otimes u} C(x \pi \pi) \xrightarrow{c^{-1}} Z_p \), an equivariant cocycle. There is a diagonal map

\[ \pi \ast x \pi \xrightarrow{\pi \ast x \pi} \pi \ast x \pi \]

\[ \pi \ast x \pi \]

commuting with projections \( p_0, p_1 \), \( d^* P \) is a homomorphism; and

\[ d^* P(u) \in H^k (\pi \ast x \pi) \otimes H^q (x) \]

by Kunneth formula

\[ \Sigma_{i=0}^n w_{i} \times D_{i} u \quad \text{where} \quad w_{i} \in H^i (\pi \ast x \pi), \quad w_{i+1} = (w_{i})^i \quad \text{and} \quad \Sigma_{i=0}^n w_{i+1} = w_{i} w_{i+1} \]

\[ \beta w_{i} = w_{i} \quad \text{One shows} \quad D_{i-1} u = \beta D_{i} u \quad \text{and} \quad D_{i} u = 0 \quad \text{if} \quad 2i \neq q (mod 2(p-1)) \]

Define \( \nu(q) = (-1)^{q(q-1)/2} (m!)^{-m} \in Z_p \) and \( m = \frac{p-1}{2} \), and put

\[ \delta^{q}(u) = (-1)^{-q(q-1)/2} \nu^{q} \left( P_{q-2i}^q - 2^{q-2i}(q-1)^{p-1} \right)(u) \in H^{p-2i}(x) \]

\[ = H^{p-2i}(x, \pi \ast x \pi) \]

**Thm** \( \delta^n P(u) = \Sigma_{i=0}^n (-1)^i \nu^{q} \left( P_{q-2i}^q - 2^{q-2i}(q-1)^{p-1} \right)(u) \in H^{p-2i}(x, \pi \ast x \pi) \)

**Proof**

1. \( d^* : H^k (\pi \ast x \pi) \rightarrow H^k (\pi \ast x \pi) \)

\[ w_i \times x = \rho^* w \pi \times x = \rho^* w \pi \times x \]

\[ d_i^* \text{ is a } H^k (\pi \ast x \pi) \text{-homomorphism} \]
\[ \nu(q) \equiv \nu(q + 2i(p-1)) \quad \text{since} \quad \nu(q + k) = \nu(q) \Leftrightarrow (m!)^2 \equiv (-1)^{m+1} \pmod{p}. \]

Thus

\[ \nu(q) \Delta * \left( \sum_{i} \left( -\frac{1}{2} \sum_{i} \right) \frac{\partial \partial i}{\partial \partial i} \right)^{m+1} \sum_{i} \left( \frac{q - 2i}{n - pi} \right) w_{2(n - pi)(p-1)} x \]

\[ = \sum_{i} (-1)^{i} \left( \frac{(q - 2i)}{n - pi} \right) w_{z(n - pi)(p-1)} \cdot x \sum_{i} (-1)^{i} \left( \frac{(q + 2i)(p-1)}{n - pi} \right) x \theta^{j} \beta^{i} u + w_{...} \beta^{j} \beta^{i} u \]

\[ = \sum_{i} (-1)^{i} \left( \frac{(q - 2i)}{n - pi} \right) w_{(2n + q - 2i - 2j)(p-1)} \cdot x \theta^{j} \beta^{i} u + w_{...} \beta^{j} \beta^{i} u \]

\[ + \sum_{i} (-1)^{i} \left( \frac{(q - 2i)}{n - pi} \right) w_{(2n + q - 2i - 2j)(p-1)} \cdot x \theta^{j} \beta^{i} u \]

\[ = \sum_{i} (-1)^{i} \left( \frac{(q - 2i)}{n - pi} \right) w_{(2n + q - 2i - 2j)(p-1)} \cdot x \theta^{j} \beta^{i} u \]

\[ = \sum_{i} (-1)^{i} \left( \frac{(q - 2i)}{n - pi} \right) w_{(2n + q - 2i - 2j)(p-1)} \cdot x \theta^{j} \beta^{i} u \]

\[ \text{(note: } w_{odd}, w_{odd} = w_{1}^2, \text{even } w_{even} = 0.) \]

\[ \nu(q) \Delta * \mathfrak{B} \mathfrak{A} \mathfrak{A} \mathfrak{P}(u) = \theta^{n} \sum_{i} (-1)^{i} \left( \frac{(q - 2i)}{n - pi} \right) x \theta^{j} \beta^{i} u \]

\[ \text{Carter formula: } \theta^{j} w_{2k} = \theta^{j} (w_{2})^{k} = \left( \frac{k}{j} \right) (w_{2})^{k} (w_{2})^{k} = \left( \frac{k}{j} \right) w_{2k + 2j(p-1)} \]

\[ \theta^{j} w_{2k} = \left( \frac{k}{j} \right) w_{2k + 2j(p-1)} \]

\[ = \sum_{i} \sum_{j} (-1)^{i} \left( \frac{(q - 2i)}{n - pi} \right) w_{(q - 2i + 2n - 2j)(p-1)} \cdot x \theta^{j} \beta^{i} u \]

\[ = \sum_{i} (-1)^{i} \left( \frac{(q - 2i)}{n - pi} \right) \theta^{j} \beta^{i} u \]

\[ \text{(with } k = i + j \)
These are proved in Steenrod "Cohom. oper." VIII p. 120 (4), p. 121 (5).

\[ \delta^n (w_s \times P_u) = \sum_i \left( \text{mod } p \right) \left( \frac{[S/2] + (q-2i)m}{n-\pi} \right)^{w_{s-2(n-\pi)}(p-1)} \cdot P(u) \]

\[ = \frac{v(q+1)}{v(q)} E(s) \sum_i \left( \frac{[S/2] + (q-2i)m}{n-\pi} \right)^{w_{s-2(n-\pi)}(p-1)} \cdot P(u) \]

where \( E(s) = \begin{cases} 0 & s \text{ even} \\ 1 & s \text{ odd} \end{cases} \)

**Proof**

\( s=0 \) is the theorem

\( s=1 \)

\[ \delta^n(w_s) = \begin{cases} w_s & n=0 \\ 0 & n \neq 0 \end{cases} \]

so by Cartan formula,

\[ \delta^n(w_s \times P_u) = \delta^n(w_s) \cdot \delta^n(P_u) = \sum_{i=1}^{\infty} (-1)^{i} \cdot w_s \cdot w_2(x^i) \cdot \delta^{i} \cdot \frac{w_{2x}(x^i)}{w_0} \]

(1)

Then by induction on \( s \), if \( s > 2 \)

\[ \delta^n(w_s \times P_u) = w_2 \cdot \delta^n(w_{s-2} \times P_u) + w_2 \cdot \delta^{n-1}(w_{s-2} \times P_u) \]

one uses

\[ \left( \frac{[S/2] + (q-2i)m}{n-\pi} \right) \cdot \left( \frac{[S/2] + (q-2i)m-1}{n-\pi} \right) \equiv \left( \frac{[S/2] + (q-2i)m}{n-\pi} \right)^{\text{mod } p} \]

In the coefficient of \( w_{s-2(n-\pi)}(p-1) \) use \( = w_2 \cdot w_{s-2} + w_{s-2} + z(n-\pi) \).

To study \( \delta^n \), let \( \{x_j\} \) be a \( Z_p \)-basis of \( H_*(X) \) such that

\[ \Delta x_j = 0, \quad \Delta x_j = Z_j \quad \text{for some } Z_j \]

let \( \{y_i\} \) be the dual basis of \( H^*(X) \).

**Lemma**

\( H_*(w \times \Pi \times \Pi') \) has the following \( Z_p \)-basis: Let \( e_i \) be a basic cell in \( C_i(W) \), dual to \( x_i \). The basis will contain \( \{e_i \otimes \Pi \cdot x_j \} \), \( x_i \cdot x_j \), \( x_j \cdot e_i \), \( x_j \cdot x_{i+1} \), \( x_j \cdot x_i \), \( x_j \cdot x_{i+p} \), \( x_j \cdot x_{i+p} \).

for some \( s, t \); choose a representative from each class of cyclic permutations of \( (j_1 \cdots j_p) \), take one basis elt. for each.
(22-4) (this is proved in Dyer-Lashof); and further
\[ H^*(w \times X^r) \cong \text{ker } d^* \cong \mathbb{Z}_k \] such that \( \{ d_k, w_i \times P(u_j) \} \)
is a basis for \( H^*(w \times X^r) \); \( w_i \times P(u_j) \) is dual to \( e_i \otimes x^r \) (proved by Steenrod).

\[ \delta^n \text{ ker } d^n \subset \text{ ker } d^*. \]
\[ \delta^n w_i \cong \sum_k a_{k,j}^{(n)} u_k \]
\[ \leftrightarrow \delta^*_n x^i = \sum_j a_{i,j}^{(n)} x^j \] defines coefficients \( a_{i,j}^{(n)} \in \mathbb{Z}_p \).

Thus (Nishida's formula)
\[ \delta^*_n (e^*_c \otimes x_i \otimes x^p) = \sum_i \left( \frac{\Delta_i}{i^2} \right) a_{i,j}^{(n)} (e_c \otimes x_{i,j} \otimes x_i \otimes x^p) \]
\[ + \frac{\delta_i (q-p)}{\delta_i (q+1)} \]
\[ \left( e \left( c+1 \right) \right) \left( e \left( c+1 \right) \right) \left( x_i \right) (n-p, i) \] \( a_{i,j}^{(n)} \left( e_c \otimes x_{i,j} \otimes x_i \otimes x^p \right) \)

where \( q = \deg x_i \), \( x_i = \Delta x_i \) if \( \Delta x_i \neq 0 \),
if \( \Delta x_i = 0 \) let \( a_{i,j}^{(n)} (i) = 0 \). Note \( C \) may be \( < 0 \).

This formula is also true for \( H_*(w \times X^r) \), \( H_*(e^r(x)) \), and true for \( H_*(e^r(x)) \) mod \( \text{ ker } \left( i_* : H_*(e^r(x)) \to H_*(e^r(x)) \right) \). This kernel is generated by \( \delta (e_{p+1} \otimes x_{i,j} \otimes \cdots \otimes x_{p,r}) \).

Let \( X = QX' = \lim S^n X' \). Then \( \exists \Theta : w \times X^r \to X \) which defines (Dyer-Lashof) \( Q_i (x) = \Theta_\ast (e_i \otimes x^r) \in H_{p+1} (x) \); \( Q_i \) is a homomorphism and has properties similar to those of \( D_i : \Delta Q_{2i} = Q_{2i-1} \)
and \( Q_{2i} = 0, q \neq 2i (2p-2) \). Let
\[ \nu (q) Q_j (x) = (-1)^j Q (z_j - n) (p-1) (x) \in H_{p-n} (z_j - n) (p-1) \]

Then \( Q_j \) is a natural homomorphism, commutes with suspension, and satisfies a Cartan formula \( Q^{1/2} = \nu (q) Q_j \) if \( q \) is even \( < \deg x \); \( Q^j = 0 \) if \( q > 2j \).
However \( Q^j ) \) is not true. Apply \( \Theta_\ast \delta^* \) and get 2 formulas of last lecture.
H. Toda (Kyoto Univ.): Mod $q$ cohomology theories.


Let $X, Y$ be finite CW-complexes throughout, and $q$ an integer $>1$. Let $M = M_{fin} = S^n \cup e^{n+1}$ where $n$ is large.

$S^n \to S^{n+1} \to M \to S^{n+1}$

$S^n = X \times S^m \to X \times S^{m+1} \to X \times M \to X \times S^{m+1}$

$0 \to [S^n, X] \otimes \mathbb{Z}_q \to [X, X, M] \to [S^n, X] \to \cdots$

$0 \to [Y, S^n] \otimes \mathbb{Z}_q \to [Y, X, M] \to [Y, S^n, X] \to \cdots$

A.T. Thm 1.1 $[M, M] \in [M, M]$ is of order $q$ if $q \neq 2 \mod 4$

We consider only the first case, $q \neq 2 \mod 4$.

\[ \mathbb{Z}_q \cong [X, M, W], [Y, X, M] \text{ are } \mathbb{Z}_q \text{-modules.} \]

A.T. Thm 2.9 $q \neq 2 \mod 4$

\[ 0 \to [S^n, X] \otimes \mathbb{Z}_q \to [X, X, M] \to \text{Tod} (\mathbb{Z}_q) \to 0 \] (Split)

\[ 0 \to [Y, S^n] \otimes \mathbb{Z}_q \to [Y, X, M] \to \text{Tod} (\mathbb{Z}_q) \to 0 \]

If $X = M$,

\[ [M, M, S^n M] \xrightarrow{(\text{ini})^*} [S^n M, S^n M] \xrightarrow{q} \]

\[ \exists \phi \xrightarrow{\text{order } q} 1 \to 0 \]

Thus $q: M_{M^M} \to S^n M = M \cup s^n M$ s.t. $\phi(\text{ini}) = 1_{S^n M}$

\[ [S^n M, S^n M] \xrightarrow{(\text{ini})^*} [M, M, S^n M] \xrightarrow{(\text{ini})^*} [S^n M, S^n M] \xrightarrow{q} \]

\[ \{ 0 \quad q \text{ odd} \}
\[ Z_2 + Z_2 + Z_2, q \equiv 0 \mod 4 \] so if $q$ is not odd, $\phi$ may not be unique.
\[ p + (1 \pi) \colon M \times M \xrightarrow{\text{F, inverse}} S^n \cup S^{n+1} \]

The map \( \psi \) induces an isomorphism of homology groups, hence is a homotopy equivalence; so there exists \( \psi \) (depending on \( p \)) such that

\[ \begin{align*}
(1 \pi) \psi &= \psi & \quad & \\
\psi \circ \psi &= 0
\end{align*} \]

Thus \( 1 \colon M \times M \xrightarrow{\text{def}} (1 \psi) + \psi \circ (1 \pi) \). Now \( \psi \) defines a multiplication:

\[ T \colon \pi \times \pi \rightarrow \pi \]

If \( f \colon S^m \rightarrow M_{n, q} \), \( g \colon S^s \rightarrow M_{n, q} \),

\[ f \circ g \colon S^{m+s} \xrightarrow{\text{def}} \varphi \colon M_{n, q} \times M_{n, q} \rightarrow M_{n, q} \]

Also \( \psi \) defines a multiplication in \( H^*(\pi; \mathbb{Z}_q) \) (See A.T.).

**Def.** \( \psi \colon M \times M \rightarrow S^n M \) is a multiplication if \( \psi(1 \pi) \circ \psi \) exists only if \( q \not\equiv 2 \pmod{4} \). \( \psi \) is commutative if the commutator

\[ c(p) = \psi - (-1)^m \psi \circ T \]

vanishes (\( T \colon M \times M \rightarrow M_{n, q} \), (xy) \rightarrow (yx)) \( \psi \) is associative if the associator

\[ a(p) = S^n \psi (1 \pi) - S^n \psi (-1)^m \pi (1 \pi) \psi \]

vanishes. Commutativity (or associativity) of \( \psi \Rightarrow \) of the multiplication defined by \( \psi \).

**Suppose** \( c(p) = c_1 S^n \eta^2 (1 \pi) ; M \times M \rightarrow S^{n+2} \eta^2 \rightarrow S^{n+2} \subset S^n M \)

Then \( \begin{align*}
xy &= (-1)^m \eta x d y y \\
x &= c_1 \eta^2 (d_x x)(d_y y)
\end{align*} \)

\[ x(yz) - (xy)z = c_1 \alpha_3 (d_x x)(d_y y)(d_z z) \]

Now \( M \times M \cong S^n M \cup S^n M \); \( T \in [M \times M, M \times M] \).
\[
(23-3)
\]
\[
[MAM, MAM] = \left(1+i\right)\left[S^M, S^M\right]\psi + \psi \left[\left[S^{M+1}, S^{M+1}\right] + 4 \left[S^M, S^{M+1}\right]\right] \psi + \left(1+i\right)\left[S^M, S^M\right]\psi
\]
\[
q \text{ odd} \\
\frac{22}{Z_2} Z_2 = 1 \\
\frac{22}{Z_2} Z_2 = 1 \\
\frac{22}{\mathbb{Z}_4} \cdot \mathbb{Z}_4 = 0
\]
\[
q = 0 (4)
\]
\[
\frac{Z_6 + Z_6}{1+i} = \frac{Z_8 + Z_4}{1+i} = \frac{Z_6}{1+i}
\]

where \( S = (S^{q+1}) (S^{q+1}) \) and \( S^M \rightarrow S^{2q+1} \rightarrow S^{2q} \rightarrow C^S \).

\[ T_+ : H_+ (MAM; \mathbb{Z}_4) \rightarrow H_+ (MAM; \mathbb{Z}_4). \]

One shows
\[ T = (-1)^n (1+i)\psi + (-1)^{n+1} (1+i)\psi (1+i) + (-1)^n \psi \delta \psi \mod \mathbb{Z}_4 \]
\[ q \text{ odd} \]
\[ q = 0 (4), \]
If \( q \) is odd this is an equality, so \( (-1)^n \psi T = \psi \) since \( \psi \delta \psi = 0 \).
Hence \( c(\phi) = 0 \) if \( q \) odd; multiplication is commutative.

\[ \psi \] is unique \mod \( (1+i)^k \left[ S^{q+1} (M) S^q (M) \right] \); this has 3 generators \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) where \( \gamma \) is the Hopf map. If we alter \( \phi \) \[ \phi \rightarrow \phi + a \gamma_1 + b \gamma_2 \]

Thm (c.f. A.T. Theorem 10) \[ \exists_2 \psi \]
\[ T = (-1)^n (1+i)\psi + (-1)^{n+1} (1+i)\psi (1+i) + (-1)^n \psi \delta \psi + \psi (1+i) \gamma^2 (1+i) \gamma \]
where \( \psi \) \( = 0 \) or 1, independent of \( \psi \). \( (\iff T = 1_{MAM}) \)
\[ \psi - (-1)^n \psi T = E_0 \left( S^q (1+i) \gamma^2 (1+i) \gamma \right) \]
\[ E_0 = ? \]

Problem.

Now let \( q \) be odd. \( a(\phi) = S^q (1+i) - S^q (-1)^n T (1+i) \); the second term can be altered.

\[
\begin{array}{c}
\text{P an even permutation} \\
M^\dagger M^\dagger M^\dagger M^\dagger M^\dagger M^\dagger M^\dagger M^\dagger \\
P \downarrow \quad \circ \quad \downarrow P \\
M^\dagger M^\dagger M^\dagger M^\dagger M^\dagger M^\dagger M^\dagger M^\dagger M^\dagger M^\dagger \\
\text{(-1)}^n \psi T = \psi \text{ if } q \text{ is odd.}
\end{array}
\]
\[ a(p) = (s^n \varphi(1 \varphi))(1 - p) \quad \text{since} \quad \varphi(-1)^n = \varphi(1(-1)^nT); \quad \text{take } n \text{ even.} \]
\[ e_{p'}(M) = \begin{cases} W_{\pi}^1(M \wedge M) / w_{\pi}^1 & p = 2 \\ W_{\pi}^1(M \wedge M \wedge M) / w_{\pi}^1 & p = 3 \end{cases} \]
\[ = \begin{cases} \overline{e}^1_{\pi} (\pi) \big/ \\ \overline{e}^1_{\pi} (\pi) \big/ \end{cases} \]
\[ = \begin{cases} C_{1 - T} = (M \wedge M) \cup_{1 - T} C(M \wedge M) \\ C_{1 - P} = (M \wedge M \wedge M) \cup_{1 - P} C(M \wedge M \wedge M) \end{cases} \]

If \( \varphi \) is odd, \( a(\varphi) \in [M \wedge M, S^{2n}M] \), then
\[ \begin{align*}
\left[ S^{2n}M, S^{2n}M \right] & \rightarrow [S^{2n+1}M, S^{2n}M] \\
& \cup \left[ S^{2n+2}M, S^{2n}M \right]
\end{align*} \]

One checks \( H_{3n}^{\varphi}(\mathbb{Z}_2) \) to show the coefficient in \( \mathbb{Z}_2 \) is trivial; so
\[ a(\varphi) = a_\varphi(s^{2n} ; \alpha_i(\pi \pi \pi \pi \pi \pi)) \quad \text{for odd } q \]
\[ = 0 \quad \text{if } q \text{ is not a multiple of } 3. \]

Hence \( a(q) = 0 \) if \( q \notin \{0, 3\} \) \( \Rightarrow \) associativity.

Assume \( q \) odd, \( q \neq 0(3) \). A \( \mathbb{Z}_3 \)-basis of \( H_k(C_{1 - P} ; \mathbb{Z}_3) \) is
\[ e_0 \omega_{\pi}^3, \ldots, e_1 \omega_{\pi}^3 y^3 \]
where \( x \in H_k(M, \mathbb{Z}_3) \), \( y \in H_{k+1}(M, \mathbb{Z}_3) \). By
\[ \text{Nichida's formula } \delta_3^1(e_i \omega_{\pi}^3 y^3) = \pm e_0 \omega_{\pi}^3 (dy^3) \]

\[ \delta^1 H^{3n}(C_{1 - P}, \mathbb{Z}_3) \]
\[ \begin{cases} \neq 0 & \text{if } q = 3, 6 \text{ (q)} \\ = 0 & \text{if } q \notin 0(3). \end{cases} \]
1. \( P \in \{M, M, M, M, M, M, M\} \) which has 16 direct summands.

\[ C(1-P) \cong C_a(p) \]


\[ S^{2n} \cup C(M, M, M, M, M, M, M, M) \]

\[ \text{id} \]

\[ \text{const} \]

\[ \alpha(p) = S^n \cap (1-P) \]

\[ (S^{2n}, (a_q, x_i)) \]

\[ C_{S^{2n}}, (a_q, x_i) = S^{2n} \cup C(S^{2n+2}) \]

\[ C_{\alpha(q)} = S^{3n} \cup C(S^{3n+3}) \]

On \( C_q \times 1 \), \( q' = 0 \) \( \Leftrightarrow \) \( a \equiv 0 \mod{3} \); on \( 0 \), \( q' = 0 \) \( \Leftrightarrow \) \( q \equiv 0 \mod{q} \). Hence

\[ a(p) = \left\{ \begin{array}{ll} 0 & \text{if } q \equiv 0 \mod{3} \Rightarrow \text{assoc.} \\ \pm (S^{2n}, a_q, (x_i)) & \text{if } q \equiv 3, 6 \mod{q}. \end{array} \right. \]

So

\[ x(jy) - (xy)z = \pm \alpha_q (S^{2n}, x_i, (x_j, y), (x_i, z)) \]

*nontrivial example:* \( \beta_1, \beta_1, \beta_2 \neq 0 \)

Hence \( \pi_0(M_3) \) is not associative.

If \( q \equiv 0 \mod{4} \), \( C(p) = p (U-T) = \varepsilon_q (S^n, \gamma^2 \pi_1 \pi_1) \); \( n \text{ even} \)

\( q^2 \) is not affected by primary operations.

\[ H_0(C_{1-T}; Z) \]

\[ X \xrightarrow{f} Y \]

\[ \text{ep}^r(M_{\pi 1}) \]

\[ \text{ep}^r(x) \xrightarrow{\text{ep}^r(y)} \text{ep}^r(C_{X}) \]

\[ \Rightarrow \exists D_f: \text{C}_e^r(C) = \text{C}_e^r(Y) \cup C(\text{C}_e^r(x)) \xrightarrow{\text{ep}^r(C)} \]

given by a diagonal map

\[ W^r \times X^r \times [0, 1] \]

\[ W^r \times (x \times [0, 1]) \]

\[ w_3(x_1, \ldots, x_r) \]

(\text{even if } f \text{ is cellular, } D_f \text{ may not be}). Assume \( f \) is cellular and

\[ f = 0: C_{\pi_1}^r(X) \circ Z_p \longrightarrow C_{\pi_1}^r(Y) \circ Z_p. \]
\[(23-6)\]

\[E_i(C_f) \otimes Z_p = C_{i-1}(x) \otimes Z_p + C_i(\gamma) \otimes Z_p \]

split as chain complex.

\[\Rightarrow H_i(C_f, Z_p) = H_{i-1}(x, Z_p) + H_i(\gamma, Z_p)\]

\[\psi \quad \chi\]

Thm. \[D_{p, n} (e_i \otimes \pi \otimes x^p) = -p (\deg x + 1) \left( e_{i-\gamma+1} \otimes \pi \right) (\pi)^p\]

Proof: essentially the same as \[\delta \phi = \delta \phi \delta\]

Remark. This is the lemma for \[e_p^r(x, \gamma) \Rightarrow \alpha, \gamma = 0 \Rightarrow \beta, \gamma^p = 0\]

namely

\[
edp^r(s^n) \xrightarrow{e_p^r(x)} e_p^r(s^n) \xrightarrow{\delta^r_{e_p^r(x)}} e_p^r(s^n \cup e_{n+2\gamma-2})
\]

\[
edp^r(x) \xrightarrow{\delta^r_{e_p^r(x)}} e_p^r(s^n \cup e_{n+2\gamma-2})
\]

If \(q\) is even (maybe 2), \(n = 0\) (a), \(S^n \xrightarrow{e_p^r(x)} S^n\)

\[e_p^2(s^n) \xrightarrow{e_p^2(x)} e_p^2(s^n)\]

Decompose the map \(e_p^2(q):\)

\[
2n+2
\]

\[
2n+1
\]

\[
2n
\]

In cohomology \(\text{Sq}_q^1(w, x P(w)) = \sum_{k \geq 0} w^{i-k-j} x P(S^k u)\)

In \(\text{ep}_2^2(M_p, x)\) \(\text{Sq}_q^2(w, x P(w)) = \sum_{k \geq 0} w^{i-k-j} x P(w) + w^{n+2j} x P(S^k u)\)

Hence \(x = \frac{1}{2} \text{ if } q = 2\)

Hence \(x = \frac{1}{2} \text{ if } q = 2\)
Henry Laufer: Some Link Invariants

A knot is an embedding $K: S^1 \to S^3$. Is $K$ isotopic to a trivial knot $\mathbb{O}$? (An isotopy is a homotopy $f(s, x)$ where $f(s, t_0)$ is also a knot for each $t_0$). If $K$ is smooth, it is isotopic to a trivial knot; e.g.:

\[ \begin{align*}
\text{pull} & \quad \text{pull} \\
\end{align*} \]

A link $L: \mathbb{S}^1 \to S^3$ is an embedding. If $L$ has 2 components $(L_1, L_2)$, the linking number $\lambda(L_1, L_2)$ is defined to be the homology class of $L_2$ in $S^3 - L_1$ (use Alexander duality). An equivalent definition: Bound by $L_1$, an oriented 2-manifold $M$. Count the number of intersections of $L_2$ with $M$ ($+n$). The knots $\mathbb{O}$ have linking no. 1.

Linking no. is preserved under homotopy if at each level the images of the knots remain disjoint. Hence linking no. is certainly an isotopy invariant (topological category). In fact it is the only invariant under that kind of homotopy.

The Borromean rings (2) are trivial pairwise; so linking numbers do not characterize them, since they are non-trivial under isotopy.

The 2-component link (3) has 0 linking number but is non-trivial.

Proof: We take a branched cyclic covering of $S^3$ branched over $L_1$. This is defined as follows:

\[ \pi_1(S^3 - L_1) \to H_1(S^3 - L_1) \xrightarrow{\rho_p} \mathbb{Z} \text{ has kernel } \mathbb{Z} \]

equal to paths with linking number in $L_1$ divisible by $p$. This defines $p$-sheeted covering surface of $S^3 - L_1$. One then inserts the knot $L_2$ (surround $L_1$ by a solid tube; lift it to the covering, where it remains a solid tube; replace $L_1$ in its center).
In our case we take a 3-fold covering, \( p=3 \). The covering surface turns out to be \( S^3 \). \( L_2 \) can be lifted in 3 ways to this surface; the 3 knots thus obtained are pairwise linked, hence \( L_i \) and \( L_j \) were non-trivially linked \((4)\). \[\begin{array}{cc}
(4) \text{ redraw:} & \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1.png}
\end{array} \\
(3) \text{ lifting to} & \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram2.png}
\end{array} \\
3\text{-sheeted} & \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram3.png}
\end{array} \\
\text{covering:} & \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram4.png}
\end{array}
\end{array} \]

In the branched covering space different liftings of \( L_2 \) may have non-0 linking numbers; these are preserved under isotopy of \( L_2 \) because such isotopies lift to the covering space.

In order to isotop \( L_1 \) (then the 2 kinds of isotopies of \( L_i \) and of \( L_2 \) can be put together as often as needed and any isotopy of the link reproduced) one shows any P.L. isotopy (proofs given here only for P.L. category) looks locally like either an ambient isotopy, or an isotopy which is trivial except at a single vertex at any given level. (An isotopy is not a P.L. map on the product \( U \times I \).) At such a vertex one may have a situation like \((1)\). \[\begin{array}{cc}
(5) \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram5.png}
\end{array} & \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram6.png}
\end{array} \\
\text{covering and lift } L_2, L_3 \text{ (6).} & \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram7.png}
\end{array}
\end{array} \]

Once again the linking numbers between liftings of \( L_2, L_3 \) are non-trivial.

Note that the invariants thus obtained have many parameters, e.g., \( p \).

It is possible that all these liftings are trivial but the link is not. The isotopy \((6)\) of a trivial link to the link at right shows it is non-trivial; however one gets invariants which are not 0, by the above

\[\begin{array}{cc}
(6) & \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram8.png}
\end{array} \\
& \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram9.png}
\end{array}
\end{array} \]
branched covering process. (In fact, liftings of $L_2$ look pairwise like 
(4): one takes then another covering, of the covering space, now
lifting over $S^3 - L_2$, and finds non-zero linking numbers of various liftings.)

Another example is shown in (7) in 2 forms. Take a branched
covering corresponding to the
commutator subgroup of $\pi_1(S^3 - L_1)$, and lift $L_2$ to each sheet (8).
Any finite subset of these is
trivial, but a single lifting $L_2^*$
is a non-trivial element in the first homotopy group of $\Sigma - \text{the rest}$
of the liftings, where $\Sigma$ is the covering space.

In (6), $L_2$ is non-trivial as a homotopy
class of $\pi_1(S^3 - L_1)$. One wishes to finesse the
problem by approximating all (continuous) links
by smooth ones and showing all sufficiently
close smooth (or P.H.) links give the same in-
variants. But (6) cannot be smoothed, of course.
The solution is to let $p$ be a prime. Take
$p$-fold coverings $\mathbb{Z}_p$ and prove $H_1(\Sigma, \mathbb{Z}_p) = 0$ for
a smooth knot (note coefficients $\mathbb{Z}_p$). Define
linking numbers mod $p$ only. These are isotopy invariants, and the
smoothing procedure works.

Usually $\pi_1(S^3 - L_1 - L_2)$ is the source of link invariants. But there
is an example (9) of 2 links with homeomorphic complements, which
are essentially different as links. (For other link invariants see Milnor's
1954 paper in the Annals, and the Lefschetz symposium). Their 2-fold
coverings (10) show the difference;

Taking a second covering over one of
the liftings of $L_2, L_2'$ one gets unequal invariants.
This work was done in collaboration with Mahowald and Milgram.

Frank Adams in "Chern Characters and the Structure of Unitary groups proved for a stable complex bundle E over a CW-complex X, with E trivial over the $(2g-1)$-skeleton, $ch^q(E)$ is integral and $2^r ch^{q+r}(E)$ is integral mod 2. Atiyah-Hirzebruch proved non-embedding theorems using divisibility properties of characteristic numbers in "Quelques theoremes de non-plongement." We get theorems like this assuming divisibility conditions stronger than the above universal ones.

**Theorem** If $2^{2r-s} ch^{s+2r}(S^s)$ is integral mod 2 and its reduction mod 2 is non-0, then $X \not\subseteq S^m$ if $m \leq 2(2g-5+4r-1)$.

**Remark** Specializing to complex projective spaces, get essentially Atiyah-Hirzebruch's results.

**Proof** If $2^{2r-s} ch^{s+2r}(S^s)$ is integral mod 2 and its reduction mod 2 is not 0, Adams proved $c(S^2)^2 P_2(ch^s) = P_2(2^r ch^{s+2r}) \neq 0$. If $X \subseteq S^n$ ($s=0$) $S^{2r}$ must be non-0 in the dual complex in $S^n$. But by dimensions this cannot occur. The idea of the proof is essentially to generalize this to higher-order operations.

Let $\alpha(n) = \text{no. of 1's in the dyadic expansion of } n$.

**Theorem** Let $\Theta(n) = \{ 2\alpha(n) \alpha(n) = 1, 2 \mod 4, 2\alpha(n)+1 \alpha(n) = 0 \mod 4, 2\alpha(n)+2 \alpha(n) = 3 \mod 4 \}$. Then if $n \equiv 3 \mod 4$, $\mathbb{R}P^n \not\subseteq R^{2n-\Theta(n)-1}$

and if $n \equiv 7 \mod 8$, $\mathbb{R}P^n \not\subseteq R^{2n-\Theta(n)}$.

**Remark** In case $n = 2^k-1$, $k \geq 4$ this is the embedding result of Ian James, which is known to be best possible. We conjecture our result is best possible.