Top. Seminar 1, Prof. J. R. Munkres

Concordance of diff. structures – two approaches.

Problem: I) given a piecewise-linear manif. K, find for it a compatible, diff. structure α.

II) Classify such structures up to diffeo, or some other suitable equivalence rel.

Def. PL-manifold = complex K with loc. PL'ly homeomorphic to R^n structure

Compatibility: for some subdivision of K, each simplex σ of the subdivision inherits its original diff structure.

Concordance: (convention: small greek letters denote diff. structures if this makes sense in the context)

α, β on K are concordant if $\exists\, \gamma$ on $K \times I \ni$

$\gamma|_{K\times 0} = \alpha$ & $\gamma|_{K\times 1} = \beta$.

Notation: γ = concordance between α & β; $\alpha \sim_c \beta$

Construction of a concordance = extension of α given on $K\times 0$ & β given on $K \times 1$ to γ given on all of $K \times I$.

$\sim_c$ is an equivalence relation.

Strong concordance: $K \times t$ is a diff submanifold of $(K\times I)_\gamma$ $\forall\, t \in I$.

Munkres' approach to problem II:

K: PL manif., α fixed structure on K.

Consider all alternative structures β on K & try to construct homeo

$K_\alpha \longrightarrow K_\beta$

by stepwise extension over cellular skeleta. Obstruction to this extension process gives some measure of the # of distinct diff. structures that K admits.

we start with the id-map $K_\alpha \xrightarrow{f} K_\beta$ & try to smooth it to a diffeo. Going to suitable subdivisions, can assume that each cell of K inherits original structure under both α & β. Using the dual cell decomposition of K f is already a diffeo in some nbhd of the dual 0-skeleton of K.

Assume f has already been modified to a diffeo f' on some nbhd of the $p-1$ skeleton (dual) of K. For simplicity take the general p-cell c_p to be a smooth ball in K_α which f' maps onto itself. $f'|\partial c_p$ is a diffeo of the $p-1$ sphere with itself. If this diffeo is extendable to the ball c_p then (as will be shown) f' may be smoothed to a diffeo in a nbhd of c_p.

Def. Γ_n = group of diffeo's of S^n / subgroup of those extendable to diffeos on D^n

Def. obstruction cochain $\mathcal{O}^p f'$ assigns to each dual cell c_p the elt of Γ_p represented by $f'|\partial c_p$.

Obstruction in dim. p depends on choices made in lower dimensions.

Alteration 1 step back alters the cocycle $\mathcal{O}^p f'$ within its class in the cohom. group $H^p(K;\Gamma_p)$

Alteration 2 steps back alters $\mathcal{O}^p f'$ by an elt in the image of some homo.

$$\Lambda^2 : H^{p-2}(K;\Gamma_{p-1}) \longrightarrow H^p(K;\Gamma_p).$$

Alteration 3 steps back alters $\mathcal{O}^p f'$ by an elt in the image of some homo.

$$\Lambda^3 : H^{p-3}(K;\Gamma_{p-2}) \cap \ker \Lambda^2 \longrightarrow H^p(K;\Gamma_p)/\operatorname{Im}(\Lambda^2)$$

The class of $\mathcal{O}^p f'$ in $(\ldots((H^p(K;\Gamma_p)/\operatorname{Im}(\Lambda^2))/\operatorname{Im}(\Lambda^3))/\ldots)/\operatorname{Im}(\Lambda^p)$ depends only on f. It is denoted by $O^p(f)$ and is called the obstruction in dimension p to smoothing the <u>identity map f</u> to a diffeo in a nbhd of the dual p-skeleton of K. It exists (the smoothing of the id.) iff all obstruction $O^p(f)$ vanish.

This construction gives new equivalence relation: $K_\alpha \sim_s K_\beta$ iff the id can be smoothed to a diffeo $K_\alpha \to K_\beta$.

Denote $K_\alpha \sim_d K_\beta$ iff $\exists$ some diffeo $K_\alpha \to K_\beta$.

We shall have $\sim_d \Leftarrow \sim_s \not\Leftarrow \sim_c$

<u>I-cobordism theorem</u> (J. Munkres, obstructions to extending diffeos. Proc. Am. Math. Soc. 15 (1964), 297-299)

If γ is a concordance between α & β then $\exists$ a diffeo $g : (K\times I)_\gamma \to K_\beta \times I$

"Proof": $id : (K\times I)_\gamma \to K_\beta \times I$ is already diffeo when restricted to $K\times 1$.

Hence $\delta^p(f)_{C_p} \in H^p(K\times I, K\times 1; \sim; \Gamma_p) = 0$.

So the diffeo g not only exists but is the smoothing of the identity. qed.

Considering the bottom face we get

<u>Corollary</u>. $\alpha \sim_c \beta \Rightarrow id : K_\alpha \to K_\beta$ can be smoothed to a diffeo.

Concordance is strictly stronger than $\sim_d$. (Consider $\Sigma^i \times R^+$, J. Munkres, Higher obstruction to smoothing, Topology 4 (1965) 27-45 & M. Hirsch, obstruction theories for smoothing manifolds & maps, Bull. Am. Math. Sc. 69 (1963) 352-356.)

The other direction of the equivalence follows from the

<u>Strong concordance Theorem</u>

If $id : K_\alpha \to K_\beta$ may be smoothed to diffeo then α is strongly concordant to β

These results establish that the obstruction classes $\delta(f)$ obtained from a smoothing process of the id. are actually obstructions to the existence of a concordance between α & β.

$\Rightarrow$ <u>upper bound</u> for the # of distinct concordance classes of diff structures on K.

Notation: $\Gamma(K) (= \mathbb{C}(K))$ = set of these concordance classes.

$$\text{order } \Gamma(K) \leq \text{order } \sum_p H^p(K;\Gamma_p)/\mathrm{Im}(\Lambda^i)$$

Equality? $\to$ realizability of certain cohomology classes as obstructions.

E.g. $\quad$ order $\Gamma(\Sigma^n)$ = order Γ_n.

$\Gamma(S^n)$ is a group under connected sums (cf M. Kervaire, J. Milnor; Groups of homotopy spheres I, Ann. of Math. 77 (1963), 504-537) & $\exists$ a mono $\Gamma_n \to \Gamma(S^n)$ defined by assigning to each elt in Γ_n, represented by an orientation-preserving diffeo ϕ on S^{n-1}, the diff structure on S^n obtained by pasting two balls together along their boundary by ϕ. So by our equality this mono is an iso.

Remark: If $n \geq 5$ then $\Gamma_n \cong \Theta_n$ = group of h-cobordism classes of homotopy spheres.

(cf S. Smale, On the structure of manifolds, Am. J. of Math. 84 (1962) 387-399. also J. Milnor, Lectures on the h-cobordism theorem, Princeton Math notes, Princeton Univ. Press. 1965).

another example: order $\Gamma(S^i \times S^j)$ = order $\Gamma_i \oplus \Gamma_j \oplus \Gamma_{i+j}$.

<u>Bundle theoretic approach</u>. (Milnor).

(cf. J. Milnor, Microbundles I, Topology 3, suppl. 1 (1964) 53-80 & J. Milnor, Microbundles & diff. structures (mimeographed) Princeton Univ. 1961)

Def a <u>microbdle</u> is a fiber bdle whose fiber is R^n & whose group is PL_n
PL_n = group of homeom. of $(R^n, 0)$ which are piece wise linear.

Facts: 1) any vector bdle η over a complex has an underlying PL microbdle $|\eta|$ (obtained from η by extending the bdle group O_n to PL_n)
2) any PL manifold K has a tangent PL microbdle t_K.
3) any diff manifold K_α has a tangent vector bdle T_α whose underlying PL microbdle is t_K.

$\Longrightarrow$ the tangent PL microbdle of K_α must be the underlying microbdle of some vectorbdle over K_α.

The converse of this is also true ...

Theorem (Milnor).

If the tangent PL microbundle of K is the underlying microbundle of some vector bdle η over the complex K, then K has a diff. structure α.
(all that is required is that t_K is stably equivalent to $|\eta|$ in the sense of Whitney sum; & the conclusion can be strengthened to: T_α & η are stably equivalent).

"Proof": Construction of a diff manif $E_{(\eta)}$ $\Rightarrow$ $K^n \times R^q \xrightarrow[1]{\mathcal{X}} E_{(\eta)}$

PD-embedding
i.e $\mathcal{X}$ is a piecewise diff. homeo of $K^n \times R^q$ onto an open subset in E_η.
$\Rightarrow$ $K \times R^q$ inherits diff structure from $E_{(\eta)}$ via $\mathcal{X}$. Now apply

Product Theorem (Hirsch) (on combinatorial submanifolds of diff manif. Comment. Math. Helv. 36 (1961) 103-111)
If $K \times R^q$ has a diff structure then so has K.

By this result of Milnor, Problem I becomes one of microbdles.
W.r.t Problem II Hirsch and Mazur have shown that the concordance class of the diff structure α asserted in Milnor's theorem depends only on the stable class of the vector bdle η. From this there follows.

Theorem (Hirsch, Mazur)

$\exists$ a (homotopy-) commutative & associative H-space Γ $\Rightarrow$
a) for any PL manif. K, $\exists$ a bdle ξ_K having fiber Γ, $\Gamma(K)$ is in 1-1 correspondence with the homotopy classes of crosssections of ξ_K.
b) if ξ_K has a crossection, then it is fiber-homotopically trivial, so that $\Gamma(K)$ is in 1-1 correspondence with the set $[K, \Gamma]$.

Immediate consequences for the concordance problem:
α) if $\Gamma(K)$ is nonempty it may be given a product structure since Γ is H-space
β) $\pi_i(\Gamma) = [S^i, \Gamma]$ is in 1-1 correspondence with $\Gamma(S^i)$, moreover the group structures on both sets coincide. $\Rightarrow$ actually $\pi_i(\Gamma) \cong \Gamma(S^i) = \Gamma_i$

finally from mere homotopy theory we get.

$$\Gamma(\Sigma^i \times \Sigma^j) \overset{1-1}{\longleftrightarrow} \Gamma_i \oplus \Gamma_j \oplus \Gamma_{i+j}.$$

Connection between the two approaches:

It stands on the corresponding obstruction theories.
Obstruction to the construction of a homotopy from a map $\phi : K \to \Gamma$ to the constant map, if it appears belongs to $H^p(K; \pi_p(\Gamma)) = H^p(K; \Gamma_p)$.
By homotopy-theoretic means one can construct homos Λ^i and prove them to be equivalent to the ones defined in the earlier theory. Then

$$\text{order}(\Gamma(K)) \leq \text{order}\left(\sum_p H^p(K; \Gamma_p)/\text{Im}(\Lambda^i)\right) \quad . \quad \text{as before.}$$

Further, for any complex K one can construct homos.

$$\Phi^2 : H^p(K; \Gamma_p) \longrightarrow H^{p+2}(K; \Gamma_{p+1})$$
$$\Phi^3 : H^p(K; \Gamma_p) \cap \ker(\Phi^2) \longrightarrow H^{p+3}(K; \Gamma_{p+2})/\text{Im}(\Phi^2)$$

etc.

$\Rightarrow$ the realizable obstructions are precisely those elts in $H^p(K; \Gamma_p)$ lying in the kernel of Φ^k $\forall k$. So the realizable obstructions form a subgroup of $H^p(K; \Gamma_p)$ & we have:

$$\Gamma(K) \overset{1-1}{\longleftrightarrow} \sum_p \left(H^p(K; \Gamma_p) \cap \ker(\Phi^i)\right)/\text{Im}(\Phi^i)$$

The Λ^i & Φ^i are naturally related in the following way.

e.g.

$$\begin{array}{ccc} H^{p-2}(K; \Gamma_{p-1}) & \xrightarrow{\Lambda^2} & H^p(K; \Gamma_p) \\ \cong \downarrow & & \cong \downarrow \\ H^{p-1}(\Sigma K; \Gamma_{p-1}) & \xrightarrow{\Phi^2} & H^{p+1}(\Sigma K; \Gamma_p) \end{array} \quad \text{suspension isos.}$$

commutes.

For an example of computation carried out in the earlier theory see J. R. Munkres, concordance of diff. structures – two approaches to appear in

Top. Seminar 2 Prof. F. P. Peterson

Spin Bordisms.

Review of Bordismtheory: (M, f) where M is n-dim C^∞ manif. $f: M \to X$, X arb. fixed space

$(M, f) \sim (M', f')$ (are bordant) iff $\exists$ $n+1$ dim. C^∞ manif W $\ni$

$$\partial(W) = M \cup M' \quad \text{(disjoint union)}$$

& $\exists\, F : W \to X$, extending f & f' over W.

Notation: $\mathfrak{N}_*(X)$ = bordism groups of X.
defines a generalized homology theory
which is representable.

$$\pi_n(X \wedge MO) \overset{\text{def}}{=} H_n(X, \underset{\substack{\uparrow \\ \text{Thom spectrum}}}{MO}) = \mathfrak{N}_n(X).$$

Notation: $\Omega_*^{Spin}(X)$ maps of spin manifolds into spaces.

$$\| $$

$$\underline{H_*(X, MSpin) = \pi_*(X \wedge MSpin)}.$$

Study of $H^*(X \wedge MO) = H^*(X) \otimes \underset{\substack{\uparrow \\ \text{free } \mathfrak{A}\text{-module}}}{H^*(MO)} = \sum H^*(X) \otimes \mathfrak{A} = ?$

Result: $H^*(MSpin) = \sum \mathfrak{A} \oplus \sum \mathfrak{A}/\mathfrak{A}(Sq^1, Sq^2) \oplus \sum \mathfrak{A}/\mathfrak{A}(Sq^3)$

What are $H^*(X) \otimes \mathfrak{A}/\mathfrak{A}(Sq^1, Sq^2)$ & $H^*(X) \otimes \mathfrak{A}/\mathfrak{A}(Sq^3)$?

Prop. If N is a left $\mathcal{A}$-module, then $N \otimes \mathcal{A}$ is a free left-$\mathcal{A}$-module.

Proof. Start with a basis of N as a vectorspace. Then define a map $\phi : \Sigma \mathcal{A} \to N \otimes \mathcal{A}$ which is onto & finally use a counting argument to show that ϕ is an iso.

Cor: The cohomology of $X \wedge MO$ is a free $\mathcal{A}$-module.

also $\pi_n(X \wedge MO) \cong \mathfrak{N}_n(pt) \oplus H_n(X, \mathbb{Z}_2)$.

Geometric case: $X = \mathbb{R}P^\infty$ (infinite dim. real projective space)

$$\Omega_*^{SPin}(\mathbb{R}P^\infty) = \Omega_*^{Pin}$$

Construct C^∞ manifold $M^n \ni \bar{w}_2$ vanishes & then develop cobordism theory.

Steenrod Algebra:

$$\mathcal{A} \oplus \mathcal{A} \xrightarrow{R_{Sq^1} + R_{Sq^2}} \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{A}(Sq^1, Sq^2) \longrightarrow 0$$

right multiplication by Sq

dualise to get $\quad \mathcal{A}^* \oplus \mathcal{A}^* \xleftarrow{L_{Sq^1} + L_{Sq^2}} \mathcal{A}^* \longleftarrow \left(\mathcal{A}/\mathcal{A}(Sq^1, Sq^2)\right)^* \longleftarrow 0$

or $\quad \mathcal{A}^* \oplus \mathcal{A} \xleftarrow{R_{Sq^1} + R_{Sq^2}} \mathcal{A}^* \longleftarrow \chi\left(\left(\mathcal{A}/\mathcal{A}(Sq^1, Sq^2)\right)^*\right) \longleftarrow 0$

Theorem: $\mathcal{A}^* = \mathbb{Z}_2[\xi_1, \xi_2, \ldots]$ where $Sq(\xi_k) = \xi_k + \xi_{k-1}^2$, $\quad Sq = Sq^0 + Sq^1 + \cdots$

$\dim \xi_k = 2^k - 1$

& $(\xi_k) Sq = \xi_k + \xi_{k-1}$

One can check that $\ker(R_{Sq^1} + R_{Sq^2}) = A = \mathbb{Z}_2[\xi_1^4, \xi_2^2, \xi_3, \xi_4 \cdots]$

& $\mathcal{A}^* = A\cdot 1 \oplus A\xi_1 \oplus A\xi_1^2 \oplus A\xi_1^3 \oplus A\xi_2 \oplus A\xi_1\xi_2 \oplus A\xi_1^2\xi_2 \oplus A\xi_1^3\xi_2$

& $\chi\left(\left(\mathcal{A}/\mathcal{A}(Sq^3)\right)^*\right) = A\cdot 1 \oplus A\cdot\xi_1 \oplus A\cdot\xi_1^2 \oplus A(\xi_1^3 + \xi_2) \oplus A\xi_1\xi_2$.

Let $\mathcal{A}_1 = \{Sq^0, Sq^1, Sq^2\} \subset \mathcal{A}$.

heuristic conjecture: If we know N as an $\mathcal{A}_1$-module then we can compute $N \otimes \mathcal{A}/\mathcal{A}(Sq^1, Sq^2)$ & $N \otimes \mathcal{A}/\mathcal{A}(Sq^3)$ as $\mathcal{A}$-modules.

Examples:

a) $\mathcal{A}_1/\mathcal{A}_1(J) \otimes \mathcal{A}/\mathcal{A}(Sq^1, Sq^2) = \mathcal{A}/\mathcal{A}(J)$ for any ideal J.

b) $\mathcal{A}_1/\mathcal{A}_1(Sq^3) \otimes \mathcal{A}/\mathcal{A}(Sq^3) = \underset{0}{\mathcal{A}/\mathcal{A}(Sq^5, Sq^1)} \oplus \underset{1}{\mathcal{A}}$ in dim: 0, 1

c) $\mathcal{A}_1/\mathcal{A}_1(Sq^3) \otimes \mathcal{A}/\mathcal{A}(Sq^3) = \underset{0}{\mathcal{A}} \oplus \underset{1}{\mathcal{A}} \oplus \underset{2}{\mathcal{A}} \oplus \underset{4}{\mathcal{A}/\mathcal{A}(Sq^1, Sq^2)}$ in dim 0, 1, 2, 4

Prop. If $\bar{H}^*(\mathbb{R}P^\infty) = N \supset \cdots \supset N^{4k-1} \supset \cdots \supset N^3 \supset N^1$ is a filtration of N as an $\mathcal{A}_1$-module then

$$N^1 = \mathcal{A}_1/\mathcal{A}_1(Sq^2)$$

& $$N^{4k-1}/N^{4k-5} = \begin{cases} \mathcal{A}_1/\mathcal{A}_1(Sq^1) & \text{if } k \text{ is even} \\ \mathcal{A}_1/\mathcal{A}_1(Sq^1, Sq^5 + Sq^4Sq^1) & \text{if } k \text{ is odd. (may-be)} \end{cases}$$

Prop. $N \otimes \mathcal{A}/\mathcal{A}(Sq^1, Sq^2) \supset \cdots \supset F^{4k-1} \supset \cdots \supset F^3 \supset F^1$ where

$F^1 = \mathcal{A}/\mathcal{A}(Sq^2)$

$F^{4k-1}/F^{4k-5} = \mathcal{A}/\mathcal{A}(Sq^1)$

& $Sq^1(\sigma_{4k-1}) = (Sq^5 + Sq^4Sq^1)(\sigma_{4k-5})$

$Sq^1(\sigma_3) = Sq^2Sq^1\sigma_1$

also $N \otimes \mathcal{A}/\mathcal{A}(Sq^3) \supset \cdots \supset G^{4k-1} \supset \cdots \supset G^3 \supset G^1$ where

$G^1 = \underset{g_1}{\mathcal{A}/\mathcal{A}(Sq^5Sq^1)} \oplus \underset{g_2}{\mathcal{A}}$

$G^{4k+3}/G^{4k-1} = \underset{g_{4k+3}}{\mathcal{A}} \oplus \underset{g_{4k+5}}{\mathcal{A}/\mathcal{A}(Sq^1)} \oplus \underset{g_{4k+6}}{\mathcal{A}}$

$\& \ Sq^1(g_{4k+5}) = (Sq^5 + Sq^4 Sq^1) g_{4k+1} \oplus Sq^3 Sq^1 g_{4k+2}$ (4)

These relations allow the computation of certain homotopy groups.
Write down $Ext_{\mathcal{O}l}(\bar{H}^*(\mathbb{R}P^\infty) \otimes \mathcal{O}l/\mathcal{O}l(Sq^3), \mathbb{Z}_2) = E^2$ term of Adams spectral sequence. For alg.-reasons $E_2 = E_\infty$ & one can read off $\pi_*(\mathbb{R}P^\infty \wedge MSpin)$.

Recall: $H^*(MSpin) = \underset{0}{\mathcal{O}l/\mathcal{O}l(Sq^1, Sq^2)} \oplus \underset{8}{\mathcal{O}l/\mathcal{O}l(Sq^1, Sq^2)} \oplus \underset{10}{\mathcal{O}l/\mathcal{O}l(Sq^3)} \oplus \underset{16}{\cdots}$

& for $\pi_*(\mathbb{R}P^\infty \wedge \mathcal{O}l/\mathcal{O}l(Sq^1, Sq^2))$ we get.

$\mathbb{Z}_2$ in dim. $\equiv 0, 1$ mod 8		dim	2	6	10	14	18	22
$\mathbb{Z}_{2^r}$ in dim. $\equiv 2, 6$ mod 8								
0 otherwise		$(\mathbb{Z}_{2^r})$: r	3	4	7	8	11	12

Similar (but more complicated) we also could represent $\pi_*(\mathbb{R}P^\infty \wedge \mathcal{O}l/\mathcal{O}l(Sq^3))$

e.g. $\Omega_{10}^{Pin} = \mathbb{Z}_{2^1} \oplus \mathbb{Z}_{2^3} \oplus \mathbb{Z}_2$

Note: for any $r \exists n \ni \mathbb{Z}_{2^r}$ appears in the representation of $\Omega_n^{Pin} = \Omega_n^{Spin}(\mathbb{R}P^\infty)$.

Remark: The product of Pin manifolds is not necessarily again a Pin manifold.

Let $f: X \to Y$ be a map of spectra where $H^*(Y) = \Sigma \mathcal{O}l$.
Let $G_* \subset \pi_*(Y)$ be the set of elements represented by maps $g: S \to Y$ $\ni g^*(u) = 0 \ \forall u \in \ker f^*$ where $f^*: H^*(Y) \to H^*(X)$. ($\operatorname{Im} f_* \subset G_*$)

Def. X has property P if for every $m \in H^*(X)$, $m \neq 0$
$m \in H^*(X)/\mathfrak{a}H^*(X)$, $\exists\ g: S \to X \ni g^*(m) \neq 0$

Theorem: If X has property P, then $\operatorname{Im} f_* = G_*$
If $\operatorname{Im} f_* = G_*$ & f^* is epi, then X has property P.

Apply this to $Y = X \wedge MO$, $X = X \wedge MSO$ etc.

Cor. $\operatorname{Im}(\Omega_*(X) \longrightarrow \mathfrak{N}_*(X)) =$ all $(M, f) \ni$ their Stiefel-Whitney #s involving w_1 vanish iff $H_*(X)$ has no 4-torsion.

Cor. $\operatorname{Im}(\Omega_*^{Pin}(X) \longrightarrow \mathfrak{N}_*(X)) =$ all $M \ni$ their Stiefel-Whitney #s involving w_2 vanish.

Cor. $\operatorname{Im}(\Omega_*^{Spin}(X) \longrightarrow \mathfrak{N}_*(X)) =$ all $(M, f) \ni$ their Stiefel-Whitney #s involving w_1 & w_2 vanish. iff $X \wedge MSpin$ has property P.

Theorem BSO has property P.

Cor. $\operatorname{Im}(\Omega_*^{PL} \to \mathfrak{N}_*^{PL}) \subsetneq$ all $M \ni$ all Stiefel-Whitney #s involving w_1 vanish.

Top. Seminar 3

Polynomial maps from spheres to spheres.

Problem: Which elts of $\pi_r(S^n)$ can be represented by polynomial maps $S^r \longrightarrow S^n$.

Examples of polynomial maps from spheres to spheres:

a) $S^1 \longrightarrow S^1$: $(x_1^2 - x_2^2, 2x_1x_2)$ map of top. degree 2; double-covering of the circle.

b) $S^3 \longrightarrow S^2$: $(x_1^2 + x_2^2 - x_3^2 - x_4^2, 2(x_1x_3 + x_2x_4), 2(x_1x_4 - x_2x_3))$
$= (\pm)$ Hopf-map.

Condition for a form f to be a map from the sphere to the sphere:

$$\underline{|f(x)|^2 = |x|^{2k}} \qquad k = \text{degree of } f.$$

Any polynomial can be represented in the form $f+g$ where f has even & g has odd degree. In this representation the above condition becomes:

$$|f(x)|^2 + |g(x)|^2|x|^2 = |x|^{4k} \qquad 2k - \text{degree of } f$$

$$\langle f(x), g(x)\rangle \equiv 0 \qquad 2k-1 = \text{degree of } g.$$

Unsolved problem: can $S^2 \xrightarrow{\times 2} S^2$ be represented by a polynomial map?

(Spheres of odd dimensions lend themselves more readily to this polynomial-map-approach than those of even dimension).

Two constructions:

1) given $f, g: S^r \longrightarrow S^n$

define $\varphi(f,g): S^r \longrightarrow S^n$ by:

$$\varphi(f,g)(\underline{x}) = f(\underline{x})|g(\underline{x})|^2 - 2g(\underline{x})\langle g(\underline{x}), f(\underline{x})\rangle$$

$\varphi(f,g)$ depends only on the homotopy classes of f & g.

when $r = n$ is odd then $\varphi([f],[g]) = -[f] - 2[g]$

when $r = n$ is even then $\varphi([f],[g]) = -[f]$

2) J-construction:

let $l: S^n \longrightarrow O_{t,r}$ be a linear form, where $O_{t,r}$ is the set of rectangular, orthogonal $t \times r$ matrices, i.e. $A_i^T A_i = I$ & $A_i^T A_j + A_j^T A_i = 0$ for $i \neq j$.

$l(x) = \sum x_i A_i \quad A_i \in O_{t,r}$.

Then $J(l): S^{n+r} \longrightarrow S^t$ is defined by:

$$J(l)(\underline{x}, y) = (|x|^2 - |y|^2, 2l(x)y).$$

i.e. the image of J can be represented by quadratic forms.

Prop. 1. A map $S^{2n+1} \longrightarrow S^{2n+1}$ of top. degree k can be represented by a form of alg. degree $|k|$.

(this follows from the 1st construction)

Remark: for even spheres this construction doesn't work.

Prop 2. Every even form $S^{2n} \longrightarrow S^{2n}$ is null-homotopic.

Prop 3. Every odd form $S^r \longrightarrow S^r$ has odd degree.

(this is a consequence of Borsuk's Theorem.)

Prop. 4. There are no forms $S^n \longrightarrow S^r$ of odd degree if $n > r$

Theorem Every polynomial map of 2^{nd} degree is homotopic to a suspension of a quadratic form.
Every quadratic form is homotopic to a form arising from the J-construction.

There are no non-constant quadratic forms $S^{2n} \longrightarrow S^n$, so a fortiori
" " " " " " " $S^{2n+k} \longrightarrow S^n$)

Hence, quadratic forms can only exist in the stable range.

The only non-constant quadratic forms $S^{2n-1} \longrightarrow S^n$ exist when $n = 1, 2, 4$.

Conjectures:

1) every homotopy class can be represented by a polynomial map.
2) all polynomial maps are in the image of J.

Remarks: 1) is rather doubtful
both conjectures can be reformulated for only the stable range.

Reduction of the non-homogeneous to the homogeneous case:

Given a quadratic form A, there always exist two orthogonal matrices O_1 & O_2 $\ni$

$$O_1 A O_2 = \begin{pmatrix} \lambda_1 & \cdot & 0 & \\ & \cdot & \lambda_r & \\ 0 & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}$$

Given a linear map $S^r \longrightarrow R^t$, then $\exists$ an orthogonal frame $v_1, \dots v_{r+1}$ on S^r $\ni$ $\langle f(v_i), f(v_j) \rangle = 0$ if $i \neq j$.

Does such a framing of S^r exist if we only require $f(-x) = -f(x)$?

Given a form, can one diagonalize the pure part?

e.g. form $\underbrace{x^2 + y^2}_{\text{pure part}} + \underbrace{xy}_{\text{mixed part}}$.

Can one go

from $\left\{ \begin{matrix} x^3 & y^3 & \text{mixed} \\ x^3 & y^3 & \text{mixed} \end{matrix} \right\}$ to $\left\{ \begin{matrix} x^3 & & \text{mixed} \\ & y^3 & \text{mixed} \end{matrix} \right\}$?

<u>Similar problems for rational functions.</u>

only the entire rational functions are of interest.

<u>Problem</u>: Can every elt of $\pi_r(S^n)$ be represented by an entire rational function $S^r \to S^n$? (by an entire analytic function respectively)?

<u>Theorem</u>: Every map $S^r \to S^n$ is homotopic to an entire rational function $S^r \to S^n$ iff the entire rational functions $S^r \to S^n$ are dense in the continuous functions $S^r \to S^n$.

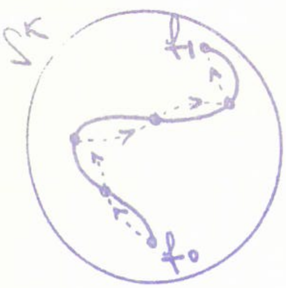
(Any map of Hopf invariant k; $S^3 \to S^2$ can be represented by a form of alg. degree $|2k|$);

Hence the entire rational functions $S^3 \to S^2$ are dense in the cont. fct. $S^3 \to S^2$.

<u>Theorem</u>: X be a compact subspace of $\mathbb{R}^{m+1}$. Then every elt in $[X, S^k]$ can be represented by an entire rational (entire analytic) function iff the entire rational (entire analytic) functions are dense in the cont. functions $X \to S^k$ w.r.t. the C.O top.

Outline of a proof:

Given a map $f: X \to S^k$ & a homotopy $f_t: X \to S^k$, $f = f_0$



break the path $(f_0 \to f_1)$ up in small steps each of which is geodesic, i.e. achieved by a rotation.

<u>Lemma 1</u> $\exists$ matrices $A_i(x) : X \to U$, where U is an open nbhd of I in $O(k+1)$ $\ni$
$f_1(x) = A_r(x) A_{r-1}(x) \cdots A_1(x) f_0(x)$.

<u>Lemma 2</u> (Cayley) Every matrix $A \ni (A - I)$ is non-singular can be written in the form $A = I \pm S$ where S is symmetric and uniquely defined by A.

By Weierstrass, $S_i(X)$ can be approximated by polynomial functions $P_i(X)$; so $B_i(X) = I \pm S_i(X)$ is an entire rational approximation to $A_i(X)$.
etc.

Topology Seminar 4 Prof. A. Dold

Homology intersection in topological manifolds.

Methods of pairing pairs of subsets of an oriented manifold in the terms of various (co-) homology theories, generalizing a result in Seifert-Threlfall about the intersection properties of such pairs.

Let $M = M^n$ be an oriented topological manifold.
& (X,A), (Y,B) be pairs of subsets in M, $A \cap Y = X \cap B = \emptyset$.
Then $\exists$ a map

$$H_k(X,A) \times H_j(Y,B) \longrightarrow \varprojlim H_{k+j-n}\, U = \underset{\wedge}{H}_{k+j-n}(X \cap Y)$$

where the inverse limit is taken over the inverse system of nbhds U of $X \cap Y$ in M directed by inclusion.

Remarks: 1) nbhds taken in X (or Y respectively) would do as well.
2) If $X \cap Y$ is locally closed or a nbhd retract $\underset{\wedge}{H}$ is the Čech homology.

Let $L \subset K \subset M$, L, K closed subsets of M.
& $W \subset V$ nbhds of $L \subset K$.
Then,

$$\begin{tikzcd}
H^*(V,W) \times H(M,M-K) \arrow[r] \arrow[rdd, dashed] & H^*(V-L,W-L) \times H(M,(M-K) \cup W) \\
& H^*(V-L,W-L) \times H(V-L,(V-K) \cup (W-L)) \arrow[u, "\cong"', "\text{excision}"] \arrow[d, "\cap\ \text{cap product}"] \\
& H(V-L,V-K) \cong H(M-L,M-K)
\end{tikzcd}$$

Taking the limit over all such nbhds $(V,W) \supset (K,L)$ we get

$$\chi : \check{H}(K,L) \times H(M,M-K) \longrightarrow H(M-L,M-K)$$

<u>more general</u>: Let $L_1, L_2 \subset K$; L_1, L_2 closed subsets of K.
$(L = L_1 - L_2)$.
replace M by $M - L_2$.

$$\begin{tikzcd}
\check{H}(K-L_2, L_1-L_2) \times H(M-L_2, M-K) \arrow[r, "\times"] & H(M - L_1 \cup L_2, M-K) \\
& \check{H}(K,L) \times H(M-L_2, M-K) \arrow[u, "\times"'] \arrow[ul]
\end{tikzcd}$$

(V,S), (W,T) be open pairs in M. Then

$$\begin{tikzcd}
\check{H}(M-S, M-V) \times H(W,T) \arrow[r] \arrow[ddddd, dashed] & \check{H}(M-S\cup T, M-V\cup T) \times H(W\cup S, T\cup S) \arrow[d, "\times"] \\
& H((V\cup T)\cap(S\cup W), S\cup T) \arrow[d, "="] \\
& H((V\cap W)\cup(S\cup T), S\cup T) \arrow[d, "\cong\ \text{excision.}"] \\
& \\
& \\
H(V\cap W, (S\cap W)\cup(V\cap T)) \arrow[r, phantom, "="] & H(V\cap W, (V\cap W)\cap(S\cup T))
\end{tikzcd}$$

Remark: this construction could also be carried out using Čech cohomology with compact support.

Then $\check{H}_c(M-S, M-V) \cong H(V,S)$.

& $H_k(V,S) \times H_j(W,T) \longrightarrow H_{k+j-n}(V\cap W, (V\cap T)\cup(S\cap W))$

This is a generalization of the result in Seifert-Threlfall which was obtained for $(V\cap T)\cup(S\cap W) = \emptyset$.

Generalization to arbitrary pairs.

(X,A), (Y,B) be arbitrary. Take nbhds (V,S), (W,T) & go to the limit. Here a difficulty enters:

$(W\cap V, (V\cap T)\cup(S\cap W))$ is a nbhd of $(X\cap Y, (X\cap B)\cup(A\cap Y))$.
However the nbhds so obtained do not form a cofinal system in the general system of nbhd, so that the limit over this special system of nbhd does not necessarily give the (co-) homology we want.

Sufficient conditions:

$\{(V\cap W, (V\cap T)\cup(S\cap W))\}$ is cofinal in the system of nbhds of $(X\cap Y, (X\cap B)\cup(A\cap Y))$ if

$X\cap Y$	seperates	$X\cup Y$
$A\cap Y$	"	$A\cup Y$
$X\cap B$	"	$X\cup B$.

where $X\cap Y$ seperates $X\cup Y$ iff $(X\cup Y)-(X\cap Y)$ = topological sum of the parts, i.e $(X\cup Y)-(X\cap Y) = (X-Y)\cup(Y-X)$.

Remark: In Čech theory this seperation condition is sufficient for excisive pairs.

Examples: $X\cap Y$ seperates $X\cup Y$. if X & Y are open in $X\cup Y$.

Let us assume that we are in the seperating case. Then we get:

$$\begin{tikzcd}
\check H(X,A)\times \check H(Y,B) \arrow[rr] & & H(X\cap Y, (X\cap B)\cup(A\cap Y)) \\
H(X,A)\times H(Y,B) \arrow[u] \arrow[urr] \arrow[rr, dashed] & & \varinjlim H(U) \arrow[u, dashed]
\end{tikzcd}$$

U = nbhd of $X\cap Y$ in Y.

Generalisation to arbitrary sets by taking limits over the compact subsets.

$$\begin{array}{ccl} (X,A) & (Y,B) & \text{arbitrary pairs.} \\ \cup & \cup & \\ (X',A') & (Y',B') & \text{compact pairs.} \end{array}$$

$$\begin{tikzcd} H(X',A') \times H(Y',B') \arrow[r] \arrow[dr] & \underset{\wedge}{H}(X'\cap Y', (X'\cap B')\cup(A'\cap Y')) \arrow[d, "\text{(since } \underset{\wedge}{H} \text{ is functorial w.r.t. } \subset)"] \\ & \underset{\wedge}{H}((X\cap Y), (X\cap B)\cup(A\cap Y)) \end{tikzcd}$$

Take direct limits over the directed system of compact pairs ordered by inclusion, to get:

$$\underline{H(X,A)\times H(Y,B) \longrightarrow \underset{\wedge}{H}(X\cap Y, (X\cap B)\cup(A\cap Y))}.$$

<u>Summarizing</u>: for arbitrary pairs (X,A), (Y,B) in an oriented manif M

$$\begin{tikzcd} \underset{\wedge}{H}(X,A) \times \underset{\wedge}{H}(Y,B) \arrow[r, dashed, "\text{exists e.g. if } X, Y \text{ separates}"] & \underset{\wedge}{H}(X\cap Y, (X\cap B)\cup(A\cap Y)) \\ H(X,A)\times H(Y,B) \arrow[u] \arrow[ur, "\text{exists always.}"'] & \end{tikzcd}$$

the results still holds for arbitrary manifolds, if we take the coeff. to be $\mathbb{Z}_2$.

Remark: The orientability of M enters where we use duality in our construction of the pairing

Properties of the pairing :

1) naturality w.r.t inclusion & $f_!$ if f is a proper map.
2) locality (if $X \cap Y$ decomposes into parts well apart then so does the pairing i.e it can be computed componentwise)
3) commutativity, associativity
4) transitivity: let N^k, N^j be oriented submanif. of M^n & assume that $N^k \cap N^j = N^{k+j-n}$ is compact conn.

Then $H_k(N^k, N^k - N^{k+j-n}) \ni \sigma_k$ = fundamental cycle of N^k

also $H_j(N^j, N^j - N^{k+j-n}) \ni \sigma_j$ = " " N^j.

If N^k, N^j intersect transversally at one pt $P \in N$ then $\sigma_k \circ \sigma_j = \pm \sigma_N$.

5) duality to $\vee$ in some sense

Another method of pairing : Intersection numbers.

$(X,A), (Y,B)$ arbitrary pairs in M. Assume $X \cap B = \emptyset = A \cap Y$.

$$H_k(X,A) \times H_{n-k}(Y,B) \longrightarrow \hat{H}_0(X \cap Y) \longrightarrow H_0(M) \xrightarrow{\eta} \mathbb{Z} .$$

augmentation.

In case $M = \mathbb{R}^n$ we have the following simple way of defining intersection numbers:

$$H_k(X,A) \times H_{n-k}(Y,B) \xrightarrow{\times} H_n((X \times Y), (X \times B) \cup (A \times Y)) \xrightarrow{(-1)^{n-k} d_*} H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \longrightarrow \mathbb{Z}$$

where $X \times Y \xrightarrow{d} \mathbb{R}^n$ is the difference i.e $d(x,y) = x - y$.

by our assumption d is not 0 on $X \times B \cup A \times Y$; that's why d_* maps into $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \neq 0$.

In the Euclidean space these two def. of $\cap$-#'s coincide.

Another possibility to define $\cap$-#'s. (displayed for open sets).

(V,S), (W,T) open pairs in M.

$$\begin{array}{c}
H_k(V,S) \times H_j(W,T) \longrightarrow H_{k+j}(V\times W,\ V\times T \cup S\times V) \\
\big\downarrow \\
H_{k+j}(V\times W \cup (M\times M - \Delta M),\ V\times T\cup S\times W \cup (M\times M-\Delta M)) \\
\cong \big\downarrow \text{ Thom iso (transfer) corr. to duality before.} \\
H_{k+j-n}(V\times W \cap \Delta M,\ (V\times T\cup S\times W)\cap \Delta M) \\
\cong \big\downarrow \\
H_{k+j-n}(V\cap W,\ (V\cap T)\cup(S\cap W)).
\end{array}$$

gives the same intersection # as above up to sign $((-1)^{kj-j})$.

P.M. Thom isomorphism.

$N^n \subset M^{n+k}$ both oriented manif.

$B\subset A\subset N$, A, B closed in N.

$$\begin{tikzcd}
H(M-B, M-A) \arrow[r, phantom, "\cong"] \arrow[rr, bend right, "\text{Thom iso. or transfer.}"'] & \check{H}_c(A,B) \arrow[r, phantom, "\cong"] & H(N-B, N-A)
\end{tikzcd}$$

Poincaré duality in M Poincaré duality in N

Thom iso. or transfer.

Ag.

Topology Seminar 5 Prof. V. Puppe

Steenrod Squares in symmetric products.

Let X be a top. space.

define $SP^n(X) = X \times \cdots \times X / S(n)$ = n'th symmetric product of X

= orbit space of the group $S(n)$ acting on $X \times \cdots \times X$ by permutation of the factors.

more general, let $\Gamma \subset S(n)$ & define

$$\Gamma(X) = X \times \cdots \times X / \Gamma = \Gamma\text{- product of } X.$$

(for $\Gamma = 1$ this specializes to the Cartesian product.)

Theorem (Dold).

If X is a CW-complex of finite type (in each dimension only finitely many cells) then $C_*(X)$ & $C^*(X)$ are completely dual & $H^*(X)$ determines $H_*(X)$ (universal coeff Th). determines $H_*(\Gamma(X))$ & $H^*(\Gamma(X))$, as groups or modules respectively.

Question: Are also the cup product structure and the action of the Sq^i's on $H^*(\Gamma(X))$ determined by the respective structural properties of $H^*(X)$.

In general no! eg. $H^*(X, \mathbb{Z})$ doesn't determine the ring structure of $H^*(X \times X, \mathbb{Z})$ (consider Klein bottle & $P_2 \vee S^1$)

Definition: Cohomology spectrum (also Bockstein spectrum) $Sp^*(X)$ = system of all rings $H^*(X, \mathbb{Z}_n)$ $n = 0, 1, 2, \ldots$ together with coefficient homomorphisms $k_{n,m}: H^*(C, \mathbb{Z}_m) \longrightarrow H^*(C, \mathbb{Z}_n)$ $m, n \geq 0$ & $n \neq 0$.

Bockstein homomorphisms $\beta_n : H^*(C, \mathbb{Z}_n) \longrightarrow H^*(C, \mathbb{Z})$ $n > 0$.

Theorem (Palermo).

If X, Y are CW-complexes of finite type then $Sp^*(X)$ & $Sp^*(Y)$ together determine $Sp^*(X\times Y)$.

(best possible result one can expect. e.g. $X = L(5,1)$ & $Y = L(5,2)$ lense spaces then $H^*(X,\mathbb{Z}_n)\cong H^*(Y,\mathbb{Z}_n)$ $\forall n$ but $H^*(X\times X,\mathbb{Z}_n)\not\cong H^*(Y\times Y,\mathbb{Z}_n)$).

Theorem 1: If X is CW-complex of finite type then $Sp^*(X)$ determines $Sp^*(\Gamma(X))$.
(it would be enough to assume that $H_*(X,\mathbb{Z})$ is of finite type.)

Theorem 2: under the same conditions as in Th. 1, $Sp^*_\pi(X)$ determines $Sp^*_\pi(\Gamma(X))$.
for $\pi = 1, \mathbb{Z}_p, S(n,p), S(n)$ p prim.

($Sp^*_\pi(X) = Sp^*(X)$ including all Steenrod operations derived from π.)

Th. 2 still holds if we replace $\mathbb{Z}, \mathbb{Z}_n$ by $\Lambda, \Lambda_\lambda = \Lambda/\lambda\Lambda$ where Λ is a principal ideal domain.

Example: $Sp^*_\pi(X,\mathbb{Z}_p)$ with $\pi = \mathbb{Z}_p$, p prim reduces to $H^*(X,\mathbb{Z}_p)$ with Sq^i for $p=2$ & with $\mathcal{P}^i_p$ & β if $p>2$.

Theorem: $H^*_{\mathbb{Z}_p}(X,\mathbb{Z}_p)$ determines $H^*_{\mathbb{Z}_p}(\Gamma(X),\mathbb{Z}_p)$.

Proof: elaboration of an idea of Dold (in terms of categories & functors).

$$\begin{tikzcd}
{[Top]} \arrow[r] \arrow[d, "\Gamma_{Top}"] & {[FD]} \arrow[r, Leftrightarrow] \arrow[d, "\Gamma_{FD}"] & {[Ch]} \arrow[r, dashed, leftrightarrow] \arrow[d, dashed, "\Gamma_{Ch}"] & H_* \arrow[d, dashed] \\
{[Top]} \arrow[r] & {[FD]} \arrow[r, Leftrightarrow] & {[Ch]} \arrow[r] & H_* \arrow[l, bend left]
\end{tikzcd}$$

does not exist since the Künneth formula is not natural.

not homotopy preserving, not additive.

commutative for good spaces (e.g. if X is a geometric realization of a css complex.

Adaptation to our situation, carrying along the additional structure.

$$\begin{tikzcd}
{[Top]} \arrow[r] \arrow[d, "\Gamma_{Top}"] & {[(FD,\Delta)]} \arrow[r, dashed, leftrightarrow] \arrow[d, "\Gamma_{(FD,\Delta)}"] & {[(Ch,d)]} \arrow[r, dashed, leftrightarrow] & Sp^* \\
{[Top]} \arrow[r] & {[(FD,\Delta)]} \arrow[r] & {[(Ch,d)]} \arrow[r] & Sp^*.
\end{tikzcd}$$

or more general

$$\begin{tikzcd}
{[\mathrm{Top}]} \arrow[r] \arrow[d, "\Gamma_{Top}"] & {[(FD, \Delta^{\pi})]} \arrow[r, leftrightarrow, dashed] \arrow[d, "\Gamma(FD,\Delta^{\pi})"] & {[(Ch, \varphi_{\pi})]} \arrow[r, leftrightarrow, dashed] & Sp^{*}_{\pi} \\
{[\mathrm{Top}]} \arrow[r] & {[(FD', \Delta^{\pi})]} \arrow[r] & {[(Ch, \varphi_{\pi})]} \arrow[r] & Sp^{*}_{\pi} .
\end{tikzcd}$$

more explicitely by:

1. Step| Lemma 1: If. $Sp^{*}(X) \cong Sp^{*}(\bar{X})$ where $X, \bar{X}$ are CW-complexes of finite type. then $\exists$ homotopy equivalence $f: C(X) \to C(\bar{X})$

$$\Rightarrow \begin{tikzcd}
C(X) \arrow[r, "d"] \arrow[d, shift left, "f"] & C(X)\otimes C(X) \arrow[d, shift left, "f\otimes f"] \\
C(\bar{X}) \arrow[r, "\bar{d}"] \arrow[u, dashed, shift left, "g"] & C(\bar{X})\otimes C(\bar{X}) \arrow[u, dashed, shift left, "g\otimes g"]
\end{tikzcd}$$

(J.H.C Whitehead). is homotopy commutative

This is a special case of

Lemma 2 If $Sp^{*}_{\pi}(X) \underset{\pi}{\cong} Sp^{*}_{\pi}(\bar{X})$ where $X, \bar{X}$ are CW-complexes of finite type then $\exists$ homotopy equivalence $f: X \to \bar{X} \Rightarrow$

$$\begin{tikzcd}
W\otimes_{\pi} C^{*}(X)^{(n)} \arrow[r, "\varphi^{*}_{\pi}"] & C^{*}(X) \\
W\otimes_{\pi} C^{*}(\bar{X})^{(n)} \arrow[r, "\bar{\varphi}^{*}_{\pi}"] \arrow[u, "id\otimes_{\pi} f^{*(n)}"] & C^{*}(\bar{X}) \arrow[u, "f^{*}"]
\end{tikzcd}$$

where $\pi = 1, S(n), S(n,p), \mathbb{Z}_p$. is homotopy commutative.

Proof only necessary for 1 & $\mathbb{Z}_p$. Lemma 1 takes care of the case $\pi = 1$.

2nd Step| Translation to FD-complexes:

K = FD-complex , $\pi \in S(n)$.

$\Delta : K \longrightarrow K^{(n)\pi}$ — generalization of Eilenberg-Zilber.

then $W\otimes_{\pi} K \xrightarrow{id\otimes\Delta^{(n)}} W\otimes_{\pi} K^{(n)} \xrightarrow[\text{natural}]{F'} K^{(n)}_{\pi}$

or $W \underset{\pi}{\otimes} K^{(n)*} \underset{G}{\overset{F^{*}/\pi}{\rightleftarrows}} K^{n*}/\pi \underset{\bar{\Delta}^{*}/\pi}{\overset{\Delta^{*}/\pi}{\rightrightarrows}} K^{*}$ $\quad \pi = 1, \mathbb{Z}_p, S(n,p), S(n)$.

more generally:

$$\mathrm{Hom}_\pi(Sp^*_\pi(X), Sp^*_\pi(\bar{X})) \xleftrightarrow{1-1} [C(\bar{X}), C(X)]_\pi$$

this indicates, that we only consider those homotopy classes of chain maps f^* for which the following diagram is homotopy commutative:

$$\begin{array}{ccc} W \otimes C(\bar{X})^{*(n)} & \xrightarrow{\varphi^*/\pi} & C(\bar{X})^* \\ \Big\uparrow{\scriptstyle id \otimes f^{*(n)}} & & \Big\uparrow{\scriptstyle f^*} \\ W \otimes C(X)^{*(n)} & \xrightarrow{\varphi^*/\pi} & C(X)^* \end{array}.$$

3rd Step | define $\Gamma_{FD} : [(FD, \Delta^\pi)] \longrightarrow [(FD, \Delta^\pi)]$.

K = FD-complex. $\quad T'(K) \overset{def}{=} K^m/_{T'}$

$$K \xrightarrow{\Delta^{(n)}} K^n$$

$$\Gamma(K) \xrightarrow{\Gamma(\Delta^{(m)})} \Gamma(K^m) \xrightarrow{\ \rho\ } (\Gamma(K))^{(m)}.$$

$$\begin{array}{ccc} [Top] & \xrightarrow{S} & [FD] \\ \Big\downarrow{\scriptstyle \Gamma_{Top}} & & \Big\downarrow{\scriptstyle \Gamma_{FD}} \\ [Top] & \xrightarrow{S} & [FD] \end{array}$$

commutes for geometric realizations of c.s.s. complexes. & for CW-complexes of finite type.
$\Gamma_{Top} : Top \to Top$ preserves homotopy.

---

other examples of functors:

$$\begin{array}{ccccc} (T, T') : & [Top] & \xrightarrow{S} & [(FD, \Delta^\pi)] & \longrightarrow & [(Ch, \varphi_\pi)] \\ & \Big\downarrow{\scriptstyle T} & & \Big\downarrow{\scriptstyle T'} & & \Big\downarrow{\scriptstyle T''} \\ & [Top] & \xrightarrow{S} & [(FD, \Delta^\pi)] & \longrightarrow & [(Ch, \varphi_\pi)] \end{array}.$$

generalization of Γ: let $\bar{X}$ be a geom. realization of a c.s.s.-complex V, on which Γ operates; $\Gamma \subset S(m)$.

a) $T(X) = (\bar{X} \times X^m)/\Gamma$, $T'(K) = (K(V) \times K^m)/\Gamma$.

b) $T(X, x_0) = X + X \times X + \dots + X^n + \dots / R$; $R : (x_1, \dots, x_n) \sim (x_1, \dots, \hat{x}_k, \dots, x_n)$ if $x_k = x_0$.

base pt.

$T(X, x_0) = J(X)$ = reduced product space of James.

If X is CW-complex of finite type then $J(X) \cong \Omega \Sigma (X)$.

$T(K, P) = \bigotimes K^0 = P \oplus K^0 \oplus K^0 \otimes K^0 \oplus \dots \oplus (K^0)^n \dots$

where $K^0 = \ker (K \to P)$

FK - construction of Milnor.

c) $T(X, x_0) = SP^\infty(X) = J(X)/S(\infty)$.

$T'(K, P) = \bigotimes K^0 / S(\infty) = P \oplus K^0 \oplus K^0 \times K^0/S(2) \oplus \dots \oplus (K^0)^n/S(n) \oplus \dots$

Dold - Thom: this gives nothing new if X is a CW-complex of finite type.

Ag

Topology Seminar 6, Prof. D. Anderson.

Chern Characters.

Let $f: X \longrightarrow Y$ be a map & $R = \begin{cases} \mathbb{Z}_p & , p \text{ prime} \\ \mathbb{Z} \\ \mathbb{Q} \end{cases}$

Assume: $H_*(X, R)$ is of finite type over R, i.e finitely generated in each dimension.

$$f_* : \tilde{H}(X; R) \longrightarrow \tilde{H}_*(Y; R)$$

hence $f_* \in \mathrm{Hom}(\tilde{H}_*(X;R), \tilde{H}_*(Y;R)) \xleftarrow{\quad} \tilde{H}^*(X, \tilde{H}_*(Y;R)) \ni Ch(f)$

onto for R as specified above.

This defines "the" Chern character $Ch(f)$ of f up to indeterminacy.

Properties of Ch:

a) The definition given here is equivalent to the usual one.

b) $\exists \quad \beta : \Sigma^2 BU \longrightarrow BU \Rightarrow$ all decomposable elts are killed.

& $\beta^*(c_n) = (n-1)c_{n-1}$ where c_n is the n'th Chern class.

also $\varinjlim H_{2i}(BU, \mathbb{Z}) = \mathbb{Q}$

and $\varinjlim H_{2i+1}(BU, \mathbb{Z}) = 0$

where the projections of the direct system are β_* followed by Σ_*^{-1}.

c) $K^*(X) \xrightarrow{Ch.} H^*(X; \varinjlim \tilde{H}_*(BU, \mathbb{Z}))$

d) suppose $\mathcal{M}$ = spectrum. i.e. we have a sequence M_n & maps $\Sigma M_n \longrightarrow M_{n+1}$

define $H_i(\mathcal{M}; R) = \varinjlim H_{i+n}(M_n; R)$.

If $f \in [X, M_n] = H^n(X, \mathcal{M})$

then $Ch(f) \in H^*(X; H_*(M_n, R)) \longrightarrow H^*(X, H_*(\mathcal{M}; R))$

& $Ch : H^n(X; \mathcal{M}) \longrightarrow \sum H^i(X; H_{n-i}(\mathcal{M}; R))$

is an <u>additive</u> homomorphism (even if there is an indeterminacy).

e) Let $f : X \longrightarrow Y$ & $f' : X' \longrightarrow Y'$

then $f \wedge f' : X \wedge X' \longrightarrow Y \wedge Y'$

we have : $\tilde{H}_*(Y; R) \otimes \tilde{H}_*(Y'; R) \longrightarrow \tilde{H}_*(Y \wedge Y'; R)$.

& $\tilde{H}^*(X; R) \otimes \tilde{H}^*(X'; R) \longrightarrow \tilde{H}^*(X \wedge X'; R)$.

or more generally for two different R's

$$\tilde{H}^*(X, R_1) \otimes \tilde{H}^*(X, R_2) \longrightarrow \tilde{H}^*(X \wedge X', R_1 \otimes R_2)$$

from these maps we get the pairing :

$$\begin{array}{ccccc} \tilde{H}^*(X, \tilde{H}_*(Y, R)) & \otimes & \tilde{H}^*(X', H_*(Y', R)) & \longrightarrow & \tilde{H}^*(X \wedge X', \tilde{H}_*(Y \wedge Y', R)) \\ \Downarrow & & \Downarrow & & \\ Ch(f) & \otimes & Ch(f') & \longmapsto & Ch(f \wedge f'). \end{array}$$

or short : $\underline{Ch(f \wedge f') = Ch(f)\, Ch(f')}$.

f) Suppose we have a <u>ring spectrum</u> $\mathcal{M}$, then we have a map $\mu : M_n \wedge M_m \longrightarrow M_{n+m}$

This map induces one on cohomology

$$\tilde{H}^n(X, \mathcal{M}) \otimes \tilde{H}^m(Y, \mathcal{M}) \longrightarrow \tilde{H}^{n+m}(X \wedge Y, \mathcal{M}).$$

by applying μ to the values of maps into M_n, M_m respectively representing cohomology classes in $\tilde{H}^n(X, \mathcal{M})$, $\tilde{H}^m(Y, \mathcal{M})$ respectively.

From $\quad Ch(x \otimes y) = Ch(x)\, Ch(y)$

& $\quad \mu_* : \Pi_*(\mathcal{M}) \otimes \Pi_*(\mathcal{M}) \longrightarrow \Pi_*(\mathcal{M})$

we get that

$$Ch : \tilde{H}^*(X; \mathcal{M}) \longrightarrow \tilde{H}^*(X, H_*(\mathcal{M}))$$ is <u>multiplicative</u>

<u>Applications and computations</u>

a) $$\begin{array}{ccc} Ch : H^i(pt, \mathcal{M}) & \longrightarrow & \sum H^{i+j}(pt, H_j(\mathcal{M})). \\ \parallel & & \parallel \\ \Pi_{-i}(\mathcal{M}) & \xrightarrow[\text{Hurewicz}]{} & H_{-i}(\mathcal{M}) \end{array}$$

If $\Pi_*(\mathcal{M}) \longrightarrow H_*(\mathcal{M})$ is injective, then the characteristic numbers in the $H^*(\cdot, \mathcal{M})$-theory can be computed from ordinary characteristic numbers (i.e integral or mod 2 theory).

b) M be a manifold, orientable for $\mathcal{M}$, i.e. $\exists$ a Thom-class $U \in \tilde{H}^k(M^\nu)$ where ν is the k-dimensional normal bdle

$\Rightarrow \quad \cup U : H^i(M) \longrightarrow H^{i+k}(M^\nu)$ is an iso. onto.

Let $\chi : S^{m+k} \longrightarrow M^\nu$ be the collapsing map, which collapses everything outside the normal bdle.

Then if $x \in H^i(M)$ we get $x \cap [M] = \sigma^{-m-k} \chi^*(x \cup U) \in H^{i-m}(pt, \mathcal{M})$

& $\underline{Ch(x \cap [M]) = \chi^*(Ch(x \cup U)) = \chi^*(Ch(x)\, Ch(U))}$.

If x is a Chern class of ν, then $Ch(x \cup U)$ is a polynomial in chern classes of ν.

c) Connective K-theory:

Adams $\quad bu^{2n} = BU\langle 2n, 2n+1 \cdots \rangle \longrightarrow BU$.

$$\begin{array}{l} H_*(bu, \mathbb{Z}_2) \cong \mathbb{Z}_2[\xi_1^2, \xi_2^2, \xi_3, \xi_4, \ldots]. \\ \Big\downarrow \; \text{iso. of algebras.} \\ H_*(K(\mathbb{Z}_2), \mathbb{Z}_2) \end{array}$$

d) Hopf-Invariant-1 Problem

$S^{2^{i+1}-1} \longrightarrow S^{2^i} \longrightarrow C \longrightarrow S^{2^{i+1}}$

$a \in k^{2^i}(C)$
$b \in k^{2^{i+1}}(C)$

$$\star \quad ch(a) = [a] \otimes 1 + Sq^{2^i}[a] \otimes \hat{S}q^{2^i} \qquad (\text{mod } 2).$$

$$\in H^*(C, H_*(bu, \mathbb{Z}_2)).$$

we have cohomology operations of the following form:

$\varphi_i = \psi^{2^{i+1}} - \psi^{2^i - 1}$ (raising filtration) defined on $K(X) \otimes \mathbb{Z}_2$.

If $\star$ holds, then it is the case that $\varphi_i(a) = b$.

For $i > 1$, $(\varphi_i)^2 = \varphi_{i+1}$ + higher filtration terms.

$\varphi_i(a) = b$

$= \varphi_{i-1}(\varphi_{i+1}(a)) = \varphi_{i-1}(\kappa b) = \kappa \varphi_{i-1}(b) = 0$, contradiction.

Qq.

Topology Seminar 7 Prof. J. Levine

Self-equivalences of $S^n \times S^k$

We consider the categories $\mathcal{H}$ = homotopy cat.

$\mathcal{P}$ = cat. of piecewise lin. manif.

$\mathcal{D}$ = cat. of differentiable manif.

$\mathfrak{O}$ means one of these if we don't distinguish between them.

Problem: Given an object $O \in \mathfrak{O}$ we are interested with group of self equivalences

As equivalence relations we consider: homotopy equivalence
piecewise lin. homeomorphisms
diffeomorphisms.

on the respective cats. or the weaker equivalence relations

a) isotopy

b) concordance.

The latter two are both the homotopy equivalence in $\mathcal{H}$ and in $\mathcal{P}$ & $\mathcal{D}$ they are equal if the object O is 1-connected and of dimension ≥ 6.

Example: Let $O = S^n$. Then the groups of selfequivalences for the three categories w.r.t. their strongest equivalence relation are:

$\mathcal{H}$: $\mathbb{Z}_2$

$\mathcal{P}$: $\mathbb{Z}_2$

$\mathcal{D}$: $\mathbb{Z}_2 \times \Gamma^{n+1}$

where $\Gamma^{n+1} = \begin{cases} 1) \text{ group of orientation preserving diffeos of } S^{n+1} \text{ onto itself} \\ 2) \text{ group of concordance classes of orientation preserving diffeos.} \end{cases}$

$\Gamma^r = 0$ if $r < 7$.

$\Gamma^r \neq 0$ in general for $r \geq 7$. (Kervaire-Milnor)

We study the case $O = S^n \times S^k$ $n \geq k$.

$\mathfrak{A}^{n,k}$ = group of self-equivalences. (stands for $H^{n,k}$, $P^{n,k}$, $D^{n,k}$ in turn).

$$\phi : \mathcal{A}^{n,k} \longrightarrow \mathrm{Auto}\,[H^*(S^n \times S^k)] = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } n > k \\ SL(2,\mathbb{Z}) & \text{if } n = k. \end{cases}$$

$\mathrm{Im}(\phi)$ is independent of the particular $\mathcal{O}$.

Let $\mathcal{A}_0^{n,k} = \ker(\phi)$.

We consider 3 subgroups of $\mathcal{A}_0^{n,k}$.

$\mathcal{A}_1^{n,k}$ = subgroup of self-equiv. which extend over $S^n \times D^{k+1}$

$\mathcal{A}_2^{n,k}$ = subgroup " " " " over $D^{n+1} \times S^k$

$\mathcal{A}_3^{n,k}$ = subgroup " " $f \ni f \neq 1$ outside some disc $\Delta \subset S^n \times S^k$ & $f(\Delta) \subset \Delta$.

$(\mathcal{A}_3^{n,k} = \mathcal{A}_0^{n+k})$ also $\mathcal{A}_3^{n,k}$ = 0 in $\mathcal{H}$

= 0 in $\mathcal{P}$

= Γ^{n+k+1} in $\mathcal{D}$.

<u>Geometric facts</u> ($n+k \geq 5$ in $\mathcal{P}$ & $\mathcal{D}$).

(1) using a concordance we can get $f(D_+^n \times S^k) = D_+^n \times S^k$ (because $f|D_+^n \times S^k$ is homotopic to an inclusion & $n \geq k$)

(2) f extends over $(S^n \times D^{k+1} - \text{disk})$ iff $f|_{D_+^n \times S^k} = 1$.

f " " $(D^{n+1} \times S^k - \text{disk})$ iff $f|_{D_+^k \times S^n} = 1$.

(If $f|_{D_+^n \times S^k} = 1$ extend to 1 on $D_+^n \times D^{k+1}$)

(3) If f is 1 on $S^n \times D_+^k$ & extends over $S^n \times D^{k+1}$, then $f \simeq 1$.

If f is 1 on $D_+^n \times S^k$ & " " $D^{n+1} \times S^k$, then $f \simeq 1$.

(f is concordant to one which is 1 on a nbhd of $S^n \times D_+^k$ in $S^n \times D^{k+1}$ hence on $(S^n \times D_+^{k+1}$ − concentric tube$)$, tube in $S^n \times D_-^{k+1}$, & this gives concordance to 1)

These facts yield the following

Information about the group structure:

Theorem:

a) A_i are abelian $\quad i = 0,1,2,3$.

b) $A_3 \subset \text{center}(A_0)$

c) $A_1 \cdot A_3$ normal subgroup of A_0

d) $A_0 = A_1 \times A_2 \times A_3$ cartesian prod. as sets.

Further if $\underline{f_1 f_2 f_3 \cong g_1 g_2 g_3}$

then $\underbrace{g_1^{-1} f_1}_{\substack{\text{extends to} \\ S^n \times D^{k+1}}} = (g_2 f_2^{-1})\underbrace{(f_3^{-1} g_3)}_{= 1 \text{ on } S^n \times D_+^k}$ by (h).

hence $\underline{g_1 \sim f_1}$.

also $(\alpha_1\alpha_2\alpha_3)(\beta_1\beta_2\beta_3) = (\alpha_1\alpha_2\beta_1\beta_2)(\alpha_3\beta_3)$ by (h)

set $\alpha_2\beta_1 = \gamma_1\gamma_2\gamma_3$ then by (c) we get $\alpha_2 = \gamma_2$.

thus γ_1, γ_3 give the following functions:

$T : A_1 \times A_2 \longrightarrow A_1$, $\quad M : A_1 \times A_2 \longrightarrow A_3$

T = group action of A_2 on A_1; M satisfies some relation involving T.

Conversely: given A_i, abelian, with T & M, we can compute the group structure on $A_1 \times A_2 \times A_3$.

If $A_3 = 0$ we get the ordinary semi-direct product

Specialization to the three categories $\mathcal{H}, \mathcal{P}, \mathcal{D}$.

$\underline{\mathcal{H}}$: Let G_{r+1} = space of maps $S^r \to S^r$ of degree $+1$.

Then $H_1 = \pi_n(G_{k+1})$ if $n > k$

$= \ker(\pi_n(G_{n+1}) \longrightarrow \pi_n(S^n))$ if $n = k$.

$H_2 \cong \pi_k(G_{n+1})$

$H_3 = 0.$

<u>Description of T</u>: from the fibration $\Omega^n S^n = F_n \xrightarrow{i} G_{n+1} \xrightarrow{p} S^n$

$$\begin{tikzcd}
\overset{\alpha}{\pi_n(G_{k+1})} \times \overset{\beta}{\pi_k(G_{n+1})} \arrow[r, "T"] \arrow[d, "p_*"'] & \pi_n(G_{k+1}) \\
\pi_n(S^k) \qquad \pi_k(F_n) \arrow[u] & \pi_{n+k}(S^k) \arrow[u, "i_*"']
\end{tikzcd}$$

$$\pi_k(F_n) = \underset{\bar\beta}{\pi_{n+k}(S^n)}$$

hence $\underline{\beta\cdot\alpha = \alpha + i_*((p_*\alpha)\circ\bar\beta)}$.

<u>P</u>: assume $n+k \geq 5$.

let $\Gamma_f^{r,s}$ = concordance classes of framed submanifolds.

we get the <u>exact sequence</u>:

$$\cdots\cdots \longrightarrow I_{r+1} \longrightarrow \Gamma_f^{r,s+1} \xrightarrow{\sigma} \pi_r(G_{s+1}) \xrightarrow{\tau} I_r \longrightarrow \cdots\cdots$$

where $I_r = \begin{cases} 0 & r \text{ odd} \\ \mathbb{Z} & r \equiv 0 \bmod 4 \\ \mathbb{Z}_2 & r \equiv 2 \bmod 4. \end{cases}$

τ is known to be 0 if $r \equiv 6 \bmod 8$.

τ " " " $\neq 0$ if $r = 6, 14$. (usually)

σ coincides with the obvious homo $P_i \to H_i$.

piecewise lin. homeo is sent into a homotopy equivalence

$P_1 = \begin{cases} \Gamma_f^{n,k+1} & k > 1 \\ 0 & k = 1 \end{cases}$

$P_2 = \Gamma_f^{k,n+1}$

$P_3 = 0.$

Applications:

a) in $S^6 \times S^4$ $\exists$ a homotopy equivalence which is not homotopic to a piece-wise linear homeomorphism.

b) in $S^5 \times S^2$ we have 2 piece-wise lin. homeomorphisms which are homotopic but not concordant.

§. assume again $n+k \geq 5$

let. $C_f^{r,s}$ = concordance classes of orthogonally framed embeddings $S^r \xrightarrow[C]{} S^{r+s}$ (the embeddings have to be smooth)

we get the exact sequence:

$$\cdots \longrightarrow \Gamma^{r+1} \longrightarrow C_f^{r,s} \xrightarrow{\sigma} \Gamma_f^{r,s} \longrightarrow \Gamma^r \longrightarrow \cdots$$

$$D_1 = \begin{cases} C_f^{n,k+1} & k>1 \\ 0 & k=1 \end{cases}$$

$$D_2 = C_f^{k,n+1}$$

$$D_3 = \Gamma^{n+k+1}.$$

$D_i \longrightarrow P_i$ corresponds to σ.

Applications. $\alpha \in \mathcal{J}^{n,k}$, define $X_\alpha = D^{n+1} \times S^k \cup_\alpha D^{n+1} \times S^k$

Theorem: $\alpha_1, \alpha_2 \in \mathcal{J}^{n,k}$, then X_{α_1} isomorphic to X_{α_2} iff α_1, α_2 belong to the same orbit of T.

Example: $\exists$ a smooth 8-manifold which is homotopy-equivalent to $S^6 \times S^2$ but not piece-wise linearly homeomorphic to it

Ag

Topology Seminar 8 Prof. D. Quillen

Rational homotopy type.

X : 1-conn. top. space.
The following products can be defined:

$\pi_p(X) \times \pi_q(X) \longrightarrow \pi_{p+q-1}(X)$ Whitehead product

$\pi_p(X) \times \pi_q(X) \longrightarrow \pi_{p+q}(X)$ Samuelson product.

$\alpha \qquad \beta \longmapsto [\alpha, \beta]$

$\pi_*(\Omega X)$ is a graded Lie alg. over $\mathbb{Z}$ w.r.t. the Samuelson product.

e.g. $\pi_*(\Omega S^{n+1}) \otimes \mathbb{Q}$ = graded Lie alg. over $\mathbb{Q}$ with one generator of degree n.

$[x_n, x_n]$ generates $\pi_{2n}(\Omega S^{n+1})$ if n is odd.
$[x_n, x_n] = 0$ if n is even.

Categorical localization:

$\mathcal{C}$: category, $S \subset \mathrm{Mor}(\mathcal{C})$ closed under composition.
In this situation, $\exists$ a cat. $S^{-1}(\mathcal{C})$ (growing set theory) & a functor $Q: \mathcal{C} \longrightarrow S^{-1}(\mathcal{C})$ $\ni$ $Q(s)$ is an iso. $\forall s \in S$ & for any functor $F: \mathcal{C} \longrightarrow \mathcal{O}$ which has the same property $\exists$ a functor θ $\ni$ the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Q"] \arrow[dr, "F"'] & S^{-1}\mathcal{C} \arrow[d, "\theta"] \\ & \mathcal{O} \end{tikzcd}$$

$\alpha \in \mathrm{Hom}_{S^{-1}\mathcal{C}}(X, Y)$, then
$\alpha = \alpha_1 \circ \cdots \circ \alpha_n$ where
$\alpha_i = Q(f)$ or $Q(s)$
$f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $s \in S$.

Examples:

a) $\mathcal{C}$ = cat of top spaces & cont. maps.
S = weak homotopy equivalences.
then $S^{-1}(\mathcal{C})$ = homotopy cat in the usual sense.

b) (CW) = cat. of CW-complexes.
$Ho(CW)$ = cat. of CW-complexes with homotopy classes of maps.
then $Ho(CW) = S^{-1}(CW)$

c) $\mathcal{C}$ = 1-conn. pted spaces with cont. base pt. preserving maps
P = some set of prime numbers.
$\mathbb{Z}[\frac{1}{P}] = \{ \frac{k}{p_1^{a_1} \cdots p_n^{a_n}} \mid p_i \in P \} \subset \mathbb{Q}$.

S_P = maps $f : X \longrightarrow Y$ in $\mathcal{C}$ $\ni$
$$\pi_*(f) \otimes \mathbb{Z}[\tfrac{1}{P}] : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{P}] \xrightarrow{\cong} \pi_*(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{P}].$$
or equivalently: $H_*(f, \mathbb{Z}[\frac{1}{P}])$ is an iso.
the cat. $S_P^{-1}(\mathcal{C})$ realizes working mod p-torsion $\forall p \in P$.
analogous statement:
CW_P = those pted, 1-conn. CW-complexes which satisfy
$$\pi_*(X) \cong \pi_*(X) \otimes \mathbb{Z}[\tfrac{1}{P}].$$
we then have: $\underline{Ho(CW_P) \cong S_P^{-1}(CW)}$.

If P = all prime numbers then we get the <u>rational homotopy category $Ho\mathbb{Q}$</u>.

Def. <u>DGL+</u> : diff. graded Lie algebras over $\mathbb{Q}$ which are 0-connected.

$L = \bigoplus_{n=1}^{\infty} L_n$: graded $\mathbb{Q}$-vector space on which a bracket operation:

$x \in L_p,\ y \in L_q \Rightarrow [x,y] \in L_{p+q}$.

$[x,y] = [y,x]\cdot(-1)^{pq}$

& $[x,[y,z]] = [[x,y],z] + (-1)^{pq}[y,[x,z]]$ Jacobi identity.

differential $d : L_p \longrightarrow L_{p-1}$ satisfies the relation:

$$d[x,y] = [dx,y] + (-1)^{pq}[x,dy].$$

$H_*(L)$ is a differential graded Lie alg. whose differential is trivial.

Let $S = \{ f : L \longrightarrow L' \mid H_*(L) \xrightarrow[f_*]{\cong} H_*(L') \}$.

Def. $S^{-1}(DGL+) = Ho(DGL+)$.

Def <u>DGC+</u> : diff. graded, 1-conn. cocommutative coalgebras over $\mathbb{Q}$.

$C = \mathbb{Q} \oplus C_1 \oplus C_2 \oplus \cdots$

differential $d : C_q \longrightarrow C_{q-1}$
coproduct $\Delta : C \longrightarrow C \otimes C$
augmentation $\varepsilon : \mathbb{Q} \longrightarrow C$

What is $Ho(DGC+)$?

<u>Theorem</u>. $\exists$ equivalences of categories:

$$Ho\,\mathbb{Q} \xrightarrow{\lambda} Ho(DGL+) \xrightarrow{\mathcal{C}} Ho(DGC+)$$

& canonical isomorphisms of graded Lie algebras $\pi_*(\Omega X) \otimes \mathbb{Q} \cong H_*(\lambda X)$

& " " of graded coalgebras $H_*(X,\mathbb{Q}) \cong H_*(\mathcal{C}\lambda X)$.

Corollary (Conjecture of Hopf)

Any 1-connected, graded, locally finite, commutative algebra over $\mathbb{Q}$ is the rational cohomology ring of some space.

Sketch of a proof: results of Milnor, Kan & Curtis will be used.

$$(\text{spaces}) \longrightarrow (\text{s.s sets}) \longrightarrow (\text{s.s. groups})$$

formal Lie Theory.

we want to associate a Lie alg. to a free group

Def. A complete augmented $\mathbb{Q}$-algebra R is an augmented $\mathbb{Q}$-alg. (augmentation $\varepsilon : R \to \mathbb{Q}$) with a filtration $R = F_0R \supset F_1R \supset \cdots \supset$

a) $F_pR \cdot F_qR \subset F_{p+q}R$

b) $F_1R = \ker(\varepsilon)$.

c) $F_nR = \bigcap_{m>n} ((F_1R)^n + F_mR)$

d) $R \xrightarrow{\cong} \varprojlim_n R/F_nR$.

Examples:

1) $R = \mathbb{Q}\langle x_1, \dots, x_n \rangle$ formal power series (in non-commutative indeterminates).

F_pR = those power series having no terms of degree $\le p-1$.

2) G: group

$\mathbb{Q}(G)$: group ring of G = all formal finite sums $\sum a_g g$, $a_g \in \mathbb{Q}$, $g \in G$.

$I(G) = \ker(\varepsilon : \mathbb{Q}(G) \to \mathbb{Q})$ $(\varepsilon(g) = 1)$.

$$\hat{\mathbb{Q}}(G) = \varprojlim_n \mathbb{Q}(G)/_{I(G)^{n+1}}$$

filtration: $F_q \hat{\mathbb{Q}}(G) = \overline{I(G)^q}$

3) <u>complete tensor product</u>:

$$R \hat{\otimes} R' = \varprojlim_n R/_{F_n R} \otimes R/_{F_n R'}$$

$$\mathbb{Q}\langle x_1, \ldots x_n\rangle \hat{\otimes} \mathbb{Q}\langle y_1, \ldots y_n\rangle = \mathbb{Q}\langle x_1, \ldots x_n\rangle /_{x_i y_j = y_j x_i}.$$

<u>complete Hopf alg</u> = complete, augmented, $\mathbb{Q}$-coalg. R

coproduct $\Delta : R \longrightarrow R \hat{\otimes} R$.

commutative, coassociative, compatible with ε.

also $\mathbb{Q}(G) \xrightarrow{\Delta} \mathbb{Q}(G) \otimes \mathbb{Q}(G)$ gives rise to:

$$\hat{\mathbb{Q}}(G) \xrightarrow{\hat{\Delta}} \hat{\mathbb{Q}}(G) \hat{\otimes} \hat{\mathbb{Q}}(G)$$

4) $\mathfrak{g}$: Lie alg. over $\mathbb{Q}$.

$\mathcal{U}(\mathfrak{g})$: universal enveloping alg.

$I(\mathfrak{g})$: aug ideal.

$\hat{\mathcal{U}}(\mathfrak{g}) = \varprojlim_m \mathcal{U}(\mathfrak{g})/_{I(\mathfrak{g})^{m+1}}$.

exp : $e^q = 1 + q + \frac{q^2}{2!} + \cdots$

log : $\log(1+z) = z - \frac{z^2}{2!} + \cdots$

$$\begin{tikzcd} F_1 R \arrow[r, bend left, "\exp"] & 1 + F_1 R \arrow[l, bend left, "\log"] \end{tikzcd}$$

Lie alg   Lie group

$\exp(q) = e^q$ converges for $q \in F_1 R$ in any complete augmented alg. R. If R is commutative then exp is a homomorphism; otherwise not.

$\hat{R}$: complete Hopf algebra.

$\mathcal{G}(R) = \{ x \in R \mid \Delta x = x \hat{\otimes} x, \ \varepsilon(x) = 1 \}$ = group of "grouplike" elts in R.

$\mathcal{P}(R) = \{ x \in R \mid \Delta x = x \hat{\otimes} 1 + 1 \hat{\otimes} x, \ \varepsilon(x) = 0 \}$ = Lie alg of primitive elts in R.

we have : $\underline{\exp : \mathcal{P}(R) \xleftrightarrow{\cong} \mathcal{G}(R) : \log}$.

$$\begin{tikzcd}
\begin{pmatrix} 1\text{-conn.} \\ \text{ptd spaces} \end{pmatrix} \arrow[d, shift right, "\substack{\text{Milnor on} \\ \text{realization}}\ \mathrm{Sing}^1"'] \\
\begin{pmatrix} 1\text{-trivial} \\ \text{s.s. sets} \end{pmatrix} \arrow[u, dashed, shift right, "|\ |"'] \arrow[d, shift right, "\text{Kan}\quad G"'] \\
\begin{pmatrix} 0\text{-trivial} \\ \text{s.s. groups} \end{pmatrix} \arrow[u, dashed, shift right, "\overline{W}"'] \arrow[d, shift right, "\hat{\mathbb{Q}}"'] \\
\begin{pmatrix} 0\text{-triv. s.s compl.} \\ \text{Hopf algs over } \mathbb{Q} \end{pmatrix} \arrow[u, dashed, shift right, "\mathcal{G}"'] \arrow[d, shift left, "\mathcal{P}"] \\
\begin{pmatrix} 0\text{-triv. s.s} \\ \text{Lie-algs over } \mathbb{Q} \end{pmatrix} \arrow[u, dashed, shift left, "\hat{\mathcal{U}}"] \arrow[d, shift left, "N"] \\
(DGL+) \arrow[u, dashed, shift left, "N^*"] \arrow[d, shift left, "\mathcal{C}"] \\
(DGC+) \arrow[u, dashed, shift left, "\mathcal{L}"]
\end{tikzcd}$$

$\mathrm{Sing}^1(X)$ = subcomplex of $\mathrm{Sing}(X)$ consisting of those sing. simplices whose 1-skeleton is at the base pt of X.

G : Kan's functor.

$\hat{\mathbb{Q}}$: completion.

N : normalization.

$$(NX)_q = \bigcap_{i<q} \ker(d_i : X_q \to X_{q-1})$$

if $x \in (NG)_p$ & $y \in (NG)_q$ we have a new bracket operation:

$$[\![x, y]\!] = \sum_{\substack{\mu,\nu\ pq \\ \text{shuffles}}} (-1)^{\varepsilon(\mu,\nu)} [s_\nu x, s_\mu y]$$

this turns a s.s. Lie alg into a differential graded Lie alg.

---

any closed "loop" of functors in this diagram is equivalent to the identity functor on the cat at the "base pt of the loop" up to homotopy equivalence.

Theorem: If G is a free s.s. group, trivial in dim 0, then

$$\begin{array}{ccccc} \pi_*(G)\otimes\mathbb{Q} & \xrightarrow{\cong} & \pi_*(\mathcal{G}\hat{\mathbb{Q}}(G)) & \overset{\log}{\cong} & \pi_*(\mathcal{P}\hat{\mathbb{Q}}(G)) \\ \text{curtis.}\Uparrow & & \Uparrow & & \\ E^1=\pi_*(L_r(G_{ab}\otimes\mathbb{Q})) & \longrightarrow & \pi_*(L_r^{\mathbb{Q}}\underbrace{I(G)/I(G)^2}_{G_{ab}\otimes\mathbb{Q}.}) & & \end{array}$$

spectral sequence of the lower central series.

map of spectral sequences.

Remark: $\log:\pi_*(\mathcal{G}\hat{\mathbb{Q}}(G))\longrightarrow\pi_*(\mathcal{P}\hat{\mathbb{Q}}(G))$ preserves the productstructure i.e. Samuelson product $\xrightarrow{\log}$ ordinary product.

lg.

Topology Seminar 9 Prof. Vasquez.

Spectral sequence of a groupextension.

Situation: Φ : group.

M : Φ-module. i.e. we have $\varphi : \Phi \longrightarrow \mathrm{Aut}(M)$.

$\alpha \in H^2(\Phi, M) \underset{\text{canonical}}{\cong} H^2(\Phi, H_1(M))$

classical fact: to any such α there corresponds a $G_\alpha \ni$
$\exists$ an epi $G_\alpha \xrightarrow{\chi} \Phi$
$\ni 0 \longrightarrow M \longrightarrow G_\alpha \xrightarrow{\chi} \Phi \longrightarrow 1$ is exact

Problem: Cohomology of G_α in terms of what determines G_α?

Remark: G_α appears as the fundamental group of flat manif.

$\exists$ the Hochschild-Serre spectral sequence associated with any group extension.

In particular for $G_\alpha \underset{\chi}{\xrightarrow{\text{epi.}}} \Phi$

$$^{(2)}E^2_{p,q} = H_p(\Phi, H_q(M))$$

homology as an abelian group.

This depends on Φ, M & φ but not on α.
However the differential will depend on α.

So $d_2(\alpha) : H_p(\Phi, H_q(M)) \longrightarrow H_{p-2}(\Phi, H_{q+1}(M))$

Theorem 1: $d_2(\alpha) - d_2(0)$ is the composit of

differential for the split case.

$$H_p(\Phi, H_q(M)) \xrightarrow{\alpha \cap} H_{p-2}(\Phi, H_q(M) \otimes H_1(M)) \xrightarrow{p_*} H_{p-2}(\Phi, H_{q+1}(M))$$

where p_* is induced by the Pontryagin pairing $H_q(M) \otimes H_1(M) \xrightarrow{p} H_{q+1}(M)$

Remark: If φ is trivial then $d_2(0) = 0$.

Dually we have

Theorem 2 (cohomology version)

"$\cap$" is replaced by "$\cup$"

"p" is replaced by $H_1(M) \otimes H^q(M) \longrightarrow H^{q-1}(M)$

where this pairing crucially depends on the fact that we have abelian groups.

Application:

Kervaire: The fundamental group of the homology sphere is finitely presented and its 1st & 2nd homology groups over $\mathbb{Z}$ vanish.

Algebraic Problem: How to construct groups whose 1st & 2nd homology groups over $\mathbb{Z}$ vanish.

Theorem 3 (Kervaire) $\exists$ a 1-1 correspondence

$$\left\{ G \;\middle|\; \begin{array}{l} G \text{ has no centre} \\ \& \; H_1(G) = 0 \end{array} \right\} \longleftrightarrow \left\{ G \mid H_1(G) = H_2(G) = 0 \right\}.$$

$(\xleftarrow{\text{trivial}})$

<u>Lemma</u> If G is such that $H_1(G)=0$ then $\exists$ an essentially unique covering $\tilde{G}$ of $G \ni H_1(\tilde{G}) = H_2(\tilde{G}) = 0$.

Here covering means that $\exists$ an exact sequence

$$1 \longrightarrow C \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

where $C \subset$ center of $\tilde{G}$.

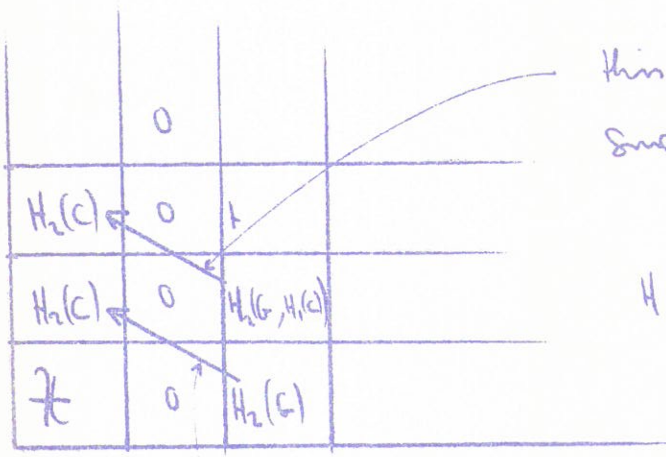
By the universal coeff. theorem we have $H^2(G, C) \cong \mathrm{Hom}(H_2(G), C)$

Take $C = H_2(G)$ & pick an iso $\alpha \in H^2(G, H_2(G))$

then $1 \longrightarrow H_2(G) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$

& α corresponds to the identity.

Now the E^2-term looks like:



this differential maps onto since

this has to be an iso if $\tilde{G}$ exists.

$$\begin{array}{c} H_2(G, H_1(C)) \cong H_2(G) \otimes H_1(C) \\ \downarrow {\scriptstyle \alpha \otimes id} \\ H_1(C) \otimes H_1(C) \\ \downarrow {\scriptstyle p_*} \\ H_2(C) \end{array}$$

composite is onto. & equal to the differential in question

<u>Proof of Th. 1.</u> $\alpha, \beta \in H^2(\Phi, M)$. We want spectral sequences

$G_\alpha \longrightarrow$ Hochschild-Serre s.s

$G_\beta \longrightarrow$ some other type of s.s. associated with the extension G_β of G

$G_{\alpha+\beta} \longrightarrow$ Hochschild-Serre s.s.

& a pairing from the 1st & 2nd to the 3rd.

$\exists$ such a pairing of the E^2-terms.

H.S. s.s. assoc. to G_α — s.s. assoc. to G_β — H.S. s.s assoc. to $G_{\alpha+\beta}$.

$$H_p(\Phi, H_q(M)) \otimes H^s(\Phi, H_t(M)) \longrightarrow H_{p-s}(\Phi, H_{q+t}(M))$$

$\cap$-prod. in Φ-variable
& Pontryagin pairing in $H_t(M)$-variable

$\exists$ id. elt $\in H^0(\Phi, H_0(M))$ then $d_2(id) = \beta$.

& we get the usual rule

$$d_2(\alpha+\beta)(x) = d_2(\alpha+\beta)(x \cdot id) = d_2(\alpha)(x) \cdot id \pm x \cdot \underbrace{d_2(id)}_{\beta}.$$

Remaining Problem: Pairing of spectral sequences.
pull back so that diagram commutes.

$$\begin{tikzcd} G \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, "\varphi"] & G_1 \arrow[d, "\varphi_1"] \\ G_2 \arrow[r, "\varphi_2"] & \Phi \end{tikzcd}$$

$G = \{(g_1, g_2) \in G_1 \times G_2 \mid \varphi_1(g_1) = \varphi_2(g_2)\}$.

let. $N_i = \ker \varphi_i$ $\quad N = \ker \varphi = N_1 \times N_2$.

$\Lambda_i = K(N_i)$
$\Gamma_i = K(G_i)$
$\Gamma = K(G)$

group rings with coeff in K for N_i, G_i, G resp.

$\Omega = K(\Phi) = \Gamma_i /\!/ \Lambda_i = \Gamma /\!/ \Lambda_1 \otimes \Lambda_2$.

Theorem 4. There are pairings in this situation as follows:

assoc. to Γ_1 — assoc. to Γ_2 — assoc to Γ.

$$\mathrm{Tor}^{\Omega}_{m}\left(\mathrm{Tor}^{\Lambda_1}_{n}(A_1,K),\, C_1\otimes C_2\right)\otimes \mathrm{Ext}^{t}_{\Omega}\left(C_2, \mathrm{Tor}^{\Lambda_2}(A_2,K)\right) \longrightarrow \mathrm{Tor}^{\Omega}_{m-t}\left(\mathrm{Tor}^{\Lambda_1\otimes\Lambda_2}(A_1\otimes A_2),\, C_2\right)$$

the C's are left Ω-modules, the A's are right Γ_i-modules.

The map on pairs immediately using the free resolutions for the A's & C's used to define the Tor's & Ext!

$(\Gamma_1\otimes\Gamma_2)\otimes_{\Gamma_1\otimes\Gamma_2} \Omega\otimes\Omega = (\Gamma_1\otimes\Gamma_2)\otimes_{\Gamma}\Omega$ makes pairing work.

For the cohomological situation we similarly get

Theorem 4' $\exists$ a pairing of spectral sequences in the given situation as follows.

$$\mathrm{Ext}^{p}_{\Omega}\left(A_1, \mathrm{Ext}^{q}_{\Lambda\otimes\Lambda_2}(K, \mathrm{Hom}(C_1,C_2))\right)\otimes \mathrm{Ext}^{s}_{\Omega}\left(A_2, \mathrm{Tor}^{\Lambda_1}_{t}(C,K)\right) \longrightarrow \mathrm{Ext}^{p+s}_{\Omega}\left(A_1\otimes A_2, \mathrm{Ext}^{q-t}_{\Lambda_2}(K,C_2)\right)$$

The proofs are functorial in both cases.

Ag.

Topology Seminar 10   Dr. D. P. Kraines

Massey Matrix Products, loop Products & $H^*(\Omega X)$.

X : n-connected space

$H^*(X)$: sing. cohomology with coeff. in a PID. (usually $\mathbb{Z}_2$).

$\mathcal{A}$: Steenrod algebra

Problem: We want to determine $H^*(\Omega X)$ as an $\mathcal{A}$-module, given $H^*(X)$ as module, algebra, $\mathcal{A}$-module, etc.

old result: If $X = SY$ then the module structure of $H^*(\Omega SY)$ is determined by that of $H^*(Y) \cong H^*(X)$.
also the $\mathcal{A}$-module structure of of $H^*(S\Omega Y)$ is determined by that of $H^*(X)$.

Massey triple product of a space: $\langle u, v, w \rangle$.

Let u v, $w \in H^*(X)$. $\ni$ $u \cdot v = v \cdot w = 0$.
let $a, b, c \in Z^*(X)$ be cocycle representatives of u v, w respectively.
then $\exists$ $\alpha . \beta \in C^*(X)$, $\delta\alpha = ab$ & $\delta\beta = bc$.
Then $\alpha c + a\beta$ is a cocycle mod 2 and represents $\langle u, v, w \rangle$ by def.

This notion can be generalized to a k-fold product $\langle u_1, \ldots, u_k \rangle$ ($k-1$-order cohomology operation)
or to matrix products defined on relations $u_1 v_1 + \cdots + u_n v_n = 0$.

e.g to get a secondary operation assume $u_1 v_1 + u_2 v_2 = 0$
$v_1 w = v_2 w = 0$

then $\left\langle (u_1, u_2), \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, w \right\rangle$ is defined and represented by a cocycle

$\alpha c + a_1\beta_1 + a_2\beta_2$ where $a_1 \in h_1$, $b_1 \in N_1$, $c \in W$.
$a_2 \in h_2$, $b_2 \in N_2$
are representatives

& $\alpha, \beta_1, \beta_2 \in C^*(X)$ exist so that

$$\delta\alpha = a_1 b_1 + a_2 b_2, \quad \delta\beta_1 = b_1 c, \quad \delta\beta_2 = b_2 c.$$

In general: let M_i be matrices over $H^*(X)$. Then under certain conditions $\exists$ a k-fold product

$\langle \underset{\sim}{M_1}, \dots, \underset{\sim}{M_k} \rangle$ which defines a $(k-1)$-order cohomology-operation.

Theorem X: n-conn. $n > 0$.

Then $H^*(\Omega X)$ is determined as module (& also as $\mathcal{O}$-module at least up to $3n+6$) by the Massey matrix product structure in $H^*(X)$ (& the $\mathcal{O}$-module-structure of $H^*(X)$).

i.e. If X_1, X_2 are two n-conn. spaces & $\varphi: H^*(X_1) \to H^*(X_2)$ is an iso. of modules (& $\mathcal{O}$-modules) $\ni$

$$\varphi\langle \underset{\sim}{M_1}, \dots, \underset{\sim}{M_k} \rangle = \langle \varphi \underset{\sim}{M_1}, \dots, \varphi \underset{\sim}{M_k} \rangle \quad \text{whenever defined}$$

then $H^*(\Omega X_1) \cong H^*(\Omega X_2)$ as modules (& $\mathcal{O}$-modules at least up to $3n+6$).

Method of Proof:

Start with dual to the cobar-construction of Adams.

Assume $C^*(X) \ni C^0(X) \cong \mathbb{Z}_2$, $C^1(X) = 0$ (this is no essential restriction).

Call $\bar{C}^*(X) = C^*(X)/C^0(X)$: reduced cochains.

Form the bigraded complex

$$F(C^*) = \mathcal{F}_2 + \bar{C}^* + \bar{C}^*\otimes\bar{C}^* + \cdots + (\bar{C}^*)^n + \cdots$$

a basis elt can be written as $[a_1|\ldots|a_k]$

Then 1st degree $= p = \sum_i \deg a_i$

2nd degree $= q = k$

total degree $= p-q$.

2 differentials:

internal differential $\delta' : ([a_1|\cdots|a_k]) \longmapsto \sum [a_1|\cdots|\delta a_i|\cdots|a_k]$

$(p,q) \longrightarrow (p+1,q)$

external differential $\delta'' : ([a_1|\cdots|a_k]) \longmapsto \sum [a_1|\cdots|a_i a_{i+1}|\cdots|a_k]$

$(p,q) \longrightarrow (p,q-1)$.

Adams : $(F(C^*(X)), \delta'+\delta'')$ is a cochain complex whose cohomology is iso to $H^*(\Omega X)$ as a module.

In this situation one can define the Eilenberg-Moore spectral sequence

$$E_r^{*,*} \Longrightarrow H^*(\Omega X).$$

where

$$E_0^{**} = F(C^*(X))$$

$$E_1^{**} = F(H^*(X)) = \mathcal{F}_2 + \bar{H}^* + \cdots + (\bar{H}^*)^n + \cdots$$

$$E_2^{**} = \mathrm{Tor}_*^{H^*(X)}(\mathcal{F}_2, \mathcal{F}_2).$$

& $d_r : E_r^{p,q} \longrightarrow E_r^{p-r+1,\, q+r}$

So as soon as we know the differential, we know $H^*(\Omega X)$ up to group extensions.

<u>Theorem:</u> $\theta = [m_1 | \cdots | m_k] \in E_1^{p,k}$ survives to $E_{k-1}^{p,k}$ iff $\langle m_1, \ldots, m_k \rangle$ is defined &
$$d_{k-1}\{\theta\} = \{\langle m_1, \ldots, m_k\rangle\} \in E_{k-1}^{p-k+2,1}$$

Proof. Would be done if θ was the general elt in $E_1^{p,k}$. However we also have elt of the form say

$$[u_1|v_1|w] + [u_2|v_2|w].$$

without loss of information this can be written as $[(u_1,u_2)|\binom{v_1}{v_2}|w]$.
In general, any elt in $E_1^{p,k}$ can be written as $[\underset{\sim}{m}_1|\underset{\sim}{m}_2|\cdots|\underset{\sim}{m}_k]$.

<u>Theorem:</u> The differentials in the Eilenberg-Moore spectral sequence (from d_1 on) are given by the Massey-matrix product. (over $\mathbb{Z}_2$).

<u>Corollary:</u> If $\sigma : H^n(X) \longrightarrow H^{n-1}(\Omega X)$ is the loop suspension then $\sigma w = 0$ iff $w \in \langle \underset{\sim}{u}_1, \underset{\sim}{u}_2, \ldots, \underset{\sim}{u}_n\rangle$

Proof $\quad H^n(X) \cong E_1^{n,1} \xrightarrow{\text{epi}} E_1^{n,1}/\ker d_2 = E_\infty^{n,1} \xrightarrow{\text{mono}} H^{n-1}(\Omega X)$.
this composition is just σ. Hence something can only be killed by the epimorphism.

<u>Steenrod algebra:</u>
Assume $0 \in \langle \underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_n\rangle$
Then one constructs in a functorial way another set of cohomology operations:
$$\text{Loop products} : H^*(X) \longrightarrow H^*(\Omega X).$$
If $[\underset{\sim}{u}_1| \cdots |\underset{\sim}{u}_n] \in E_1^{p,k}$ the <u>loop product</u> of $\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_n$ is denoted by $\langle \underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_n\rangle_\Omega \subset H^*(\Omega X)$.

Theorem: All ∞ cycles in $E_1^{p,k}$ can be written as loop products if we include $\sigma n = \langle n \rangle_\Omega$ as primary loop product.
Further $H^*(\Omega X)$ is generated, as module, by loop products.

Since this is functional

$$Sq^n \langle u, v \rangle_\Omega = \left\langle (Sq^0 u, \dots, Sq^n u), \begin{pmatrix} Sq^n v \\ \vdots \\ Sq^0 v \end{pmatrix} \right\rangle_\Omega$$

$$= \sum_i \langle Sq^i u, Sq^{n-i} v \rangle_\Omega \quad \text{if defined}$$

Similarly

$$Sq^n \langle u, v, w \rangle_\Omega = \sum_{i+j+k=n} \langle Sq^i u, Sq^j v, Sq^k w \rangle_\Omega \text{ if defined.}$$

Want to define operation similar to Toda bracket.

$$A \xrightarrow{a} B \xrightarrow{b} C \quad \ni \quad ba \simeq * \quad \& \ \Omega b \simeq *.$$

then $\langle a, b, \Omega \rangle \in [\Omega A, \Omega^2 C] / \Omega[A, \Omega C] + (\Omega a)_\# [\Omega B, \Omega^2 C]$.

Take $A = X$

$B =$ universal example for $\langle \underset{\sim}{u}_1, \dots, \underset{\sim}{u}_n \rangle$

$C = K(\mathbb{Z}_2, n)$

assuming $0 \in \langle \underset{\sim}{u}_1, \dots, \underset{\sim}{u}_k \rangle$

$a : A \longrightarrow B$ will rep. $\langle \underset{\sim}{u}_1, \dots \underset{\sim}{u}_n \rangle$

$b : B \longrightarrow C$ " " $0 \in \langle \underset{\sim}{u}_1, \dots \underset{\sim}{u}_n \rangle$.

$\Omega b : \Omega B \to \Omega C$ corresponds to $\sigma 0 = 0$.

$$\sum_{\text{all } a} \langle a, b, \Omega \rangle \subset \langle \underset{\sim}{u}_1, \dots, \underset{\sim}{u}_n \rangle_\Omega .$$

$$
\begin{tikzcd}
 & \Omega C \arrow[r, equal] \arrow[d, "j"] & \Omega C \arrow[d, dashed] \\
 & E \arrow[r, dashed] \arrow[d, "\pi"] & PC \arrow[d, dashed] \\
A \arrow[ur, dashed, "\alpha"] \arrow[r, "a"'] & B \arrow[r, "b"'] & C
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
 & \Omega^2 C \arrow[r, equal] \arrow[d, "\Omega j"] & \Omega^2 C \arrow[d, dashed] \\
 & \Omega E \arrow[r, dashed] \arrow[d, "\Omega\pi"] \arrow[u, dashed, bend left, "\rho"] & \Omega PC \arrow[d, dashed] \\
\Omega A \arrow[ur, dashed, "\Omega\alpha"] \arrow[r, "\Omega a"'] & \Omega B \arrow[r, "\Omega b"'] & \Omega C
\end{tikzcd}
$$

$\cong *$

$$\Omega E \cong \Omega B \times \Omega^2 C$$

$\exists \quad \rho : \Omega E \longrightarrow \Omega^2 C \quad \ni \quad \rho \circ \Omega j = id_{\Omega^2 C}$.

$\{\rho \circ \Omega\alpha\} \in [\Omega A, \Omega^2 C]$ is representative of $\langle a, b, \Omega \rangle$

independent of choices for α & ρ.

Ag.

Topology Seminar II Prof. F. P. Peterson.

Statements about $H^*(BSPL, \mathbb{Z}_p)$

$BSO \longrightarrow BSPL \longrightarrow BSF$ classifying spaces.

$MSO \longrightarrow MSPL \longrightarrow MSF$ Thom-spaces.

$U \in H^*(MSF, \mathbb{Z}_p)$ Thom-class.

$H^*(BSF, \mathbb{Z}_p) \xrightarrow{\phi} H^*(MSF, \mathbb{Z}_p)$ then

$q_i = \phi^{-1} p^i(U) \in H^{ir}(BSF, \mathbb{Z}_p)$ Wu-class.

Statements:

1) $\mathbb{Z}_p[q_i] \otimes E(\beta q_i) \subset H^*(BSF)$ (Milnor conjecture)

2) $J: \pi_i(BSO) \longrightarrow \pi_i(BSF)$. Can construct space $B\,\mathrm{Im}\,J \ni$
$BSO \longrightarrow BSO \longrightarrow B\,\mathrm{Im}\,J$ is a fibration & $\ni$
$\pi_i(BSO) \twoheadrightarrow \pi_i(B\,\mathrm{Im}\,J) \rightarrowtail \pi_i(BSF)$ is J. (not induced by a map)

3) $\exists$ a map $B\,\mathrm{Im}\,J \longrightarrow BSF \ni BSO \longrightarrow B\,\mathrm{Im}\,J \longrightarrow BSF$ is the trivial map.

4) $\exists\, B\,\mathrm{Coker}\,J \ni BSF \underset{p}{\simeq} B\,\mathrm{Im}\,J \times B\,\mathrm{Coker}\,J$. (i.e. their mod p cohomology is isomorphic).

5) a) $H^*(B\,\mathrm{Im}\,J) = \mathbb{Z}_p[x_i] \otimes E(\beta x_i)$
   b) can choose $x_i = q_i$.

6) $H^*(BSF) \cong H^*(B\,\mathrm{Im}\,J) \otimes C$ where C is a $(pr-2)$-connected Hopf alg. over $\mathcal{A}$.

7) $H^*(BSPL) \cong H^*(BSO) \otimes C$ as alg. over $\mathfrak{A}$. Same C as in 6)

8) $BSPL \underset{p}{\simeq} BSO \times B\,\mathrm{Coker}\,J$

9) first p-torsion in Ω_*^{PL} is $\mathbb{Z}_p$ in dim $pr-1$.

10) $MSF \underset{p}{\simeq} K(\mathbb{Z},0) \vee \bigvee K(\mathbb{Z}_{p^r}, n)$

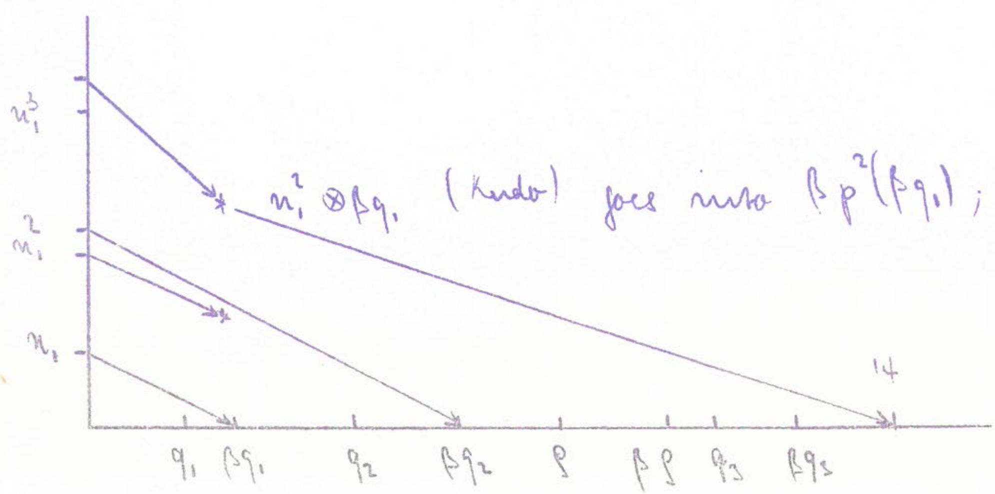
<u>Theorem</u>. All above statements hold in dim $< 2pr$.
(1, 4, 5, 6 proved by Stasheff)

<u>Theorem</u> The following statements are true:
2; 5a; 9; $3 \Longrightarrow 1 \Longrightarrow 6$; $3 \Longrightarrow 5b$, $6 \Longrightarrow 10$
also 7 a partial outline of $6 \Longrightarrow 7$,

<u>Study of 7)</u>: we have a fibration where the fibre is something like a polynomial ring on Pontriagin classes.

$$\begin{array}{ccc} F/_{PL} & \longrightarrow & BSPL \\ & & \downarrow \\ & & BSF \end{array}$$

by Sullivan: $F/PL \underset{p}{\simeq} BSO$.
Consider the S.S. of this fibration (take $p=3$, $r=4=2p-2$)



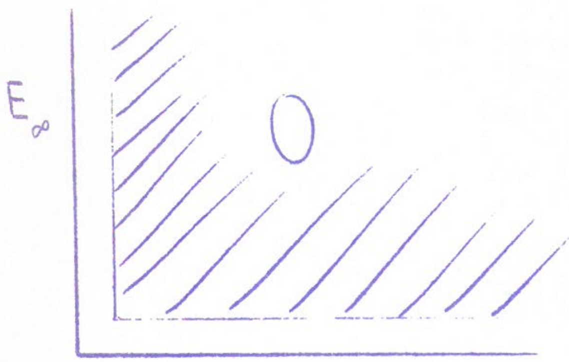
$u \in H^4(F)$

$\beta q_1 \in H^5(B)$

$\tau(u) = \beta q_1$

$\beta p^1(u) = 0 = \beta p^1(\beta q_1)$

$\exists$ unstable sec. operation $\Phi(u) \in H^{12}(F) \ni d_5(\Phi(u)) = u^2 \otimes \beta q_1$.



all that's left : $Z_p[q_i] \otimes C$

want to show $\beta q_i \longmapsto 0$ in $H^*(BSPL)$

sufficient to do this for $i = p^s + p^{s-1} + \dots + 1$

diagonal map: $\psi(q_i) = \sum_{t+s=i} q_t \otimes q_s$

$\exists$ a map $Z_p[q_i] \otimes E(\beta q_i) \longrightarrow H^*(BSF)$

<u>Study of 2</u>: (suggested by Bromfield).

given p; $\exists$ q (p, q prime) $\ni$

$$BSO \xrightarrow{4^q - 1} BSO \longrightarrow B\,\mathrm{Im}\,J \longrightarrow SU/SO \longrightarrow SU/SO$$

$$\quad BSF$$

does this exist? would give Adam's conjecture.

<u>Cor.</u> $0 \longrightarrow \pi_i(BSO) \longrightarrow \pi_i(BSPL) \longrightarrow \Gamma_{i-1} \longrightarrow 0$ exact.

p-torsion in Γ_{i-1} never splits in this sequence if $i < pr - 1$.

<u>Remark</u>: $_3\Omega_*^{PL} = \begin{cases} Z_3 & \text{in dim } 11, 19, 22 \\ Z_3 + Z_3 + Z_3 & \text{in dim } 23. \end{cases}$

Topology Seminar 12 Prof. R. Bott.

Singularities of vectorfields

Consider a C^∞ vectorfield X on a diff. manif. W.
Locally, on a coord. patch (U,φ) X has the representation

$$X|_U = \sum_\alpha a_\alpha \frac{\partial}{\partial x_\alpha}$$

At $p \in W$ where $X_p = 0$ $\left(\frac{\partial a_\alpha}{\partial x_\beta}\right)$ is an invariant of the vectorfield.
Define the lin. operator $L_p : W_p \longrightarrow W_p$ on the tangent space of W at p.

$$\text{by } L_p(X)\, Y_p = [X, \tilde{Y}]_p$$

where Y is any other vectorfield on W besides X & $\tilde{Y}$ is an extension of Y_p to a nbhd of p in W.

If $\det(L_p) \neq 0$ $\forall p \in \text{zero}(X)$ we call X nondegenerate.

Theorem (Hopf)

If W is a compact manifold & X a nondegenerate vectorfield on W then

$$\sum_{p \in \text{zero}(X)} \operatorname{sig} \det(L_p) = \chi(M) \quad \text{(Euler characteristic)}.$$

Remark: If X satisfies additional diff. equations we get a more intrinsic relationship between the topology of W & zero(X).

two cases: a) W: complex manif.
X: analytic vectorfield

b) W: Riemannian manif
X: C^∞ vectorfield.

let W be a complex manifold $\Rightarrow \dim_{\mathbb{C}} W = n$.
φ be a symmetric function in n variables.

If L is an endomorphism we get a representation

$$\varphi(L) = \varphi(\lambda_1, \dots, \lambda_n) \qquad \lambda_i : \text{eigenvalues of } L.$$

& $\varphi(W) = \varphi(x_1, \dots x_n)[W]$. where the x_i's are the formal roots of Chern polynomial of W

$$\prod (1+x_i) = 1 + c_1(W) + \dots + c_n(W).$$

e.g. $\varphi = x_1 \cdot x_2 \cdot \dots \cdot x_n$ then $\varphi(W) = c_n(W) \cdot W$
$\varphi = (x_1 + \dots + x_n)^n$ then $\varphi(W) = c_1(W)^n \cdot W$.

Theorem

If W is a compact complex manif of dim n & X an analytic vector field on W & if φ has degree $\leq n$, then

$$\varphi(W) = \sum_{p \in \text{zero}(X)} \frac{\varphi(L_p)}{\det(L_p)}$$

Geometric def of Chern classes of a vector bundle over W.

$$\begin{array}{c} E \\ \downarrow \pi \\ W \end{array} \qquad \text{let } \Gamma(E) \text{ be the cross-sections of } E$$

one can construct a diff. operator $D : \Gamma(E) \longrightarrow \Gamma(E \otimes T^*)$ (T^* = cotangent bdle of W) $\Rightarrow$

$$D(f \cdot s) = df \otimes s + f \otimes Ds.$$

Over a coordinate patch (U, φ) let s_i be generators for $\Gamma(E|_U)$. Then

$$Ds_i = \sum \theta_{ij} \otimes s_j$$

The transformations of the θ's which describe their change on the overlaps of two coord. patches going from one coord system to the other is very difficult to describe, not so for

$$K_{ij} = d\theta_{ij} - \sum_k \theta_{ik} \wedge \theta_{kj}$$

which match up nicely.

If $s' = As$ then $K' = AKA^{-1}$.

In general, anything which is locally invariant under inner automorphisms is actually defined globally.

By applying φ locally to the matrix K we so gain something which is globally defined. In particular with a special choice for φ

$$\underline{\det\left(1 + \frac{1}{2\pi i} K\right) = 1 + c_1(E) + c_2(E) + \cdots + c_n(E)}.$$

~~For~~ $E = TW$ we get the $c_i(W)$'s.

<u>Proof</u>.

Idea: consider $\varphi(W)$ as boundary & construct a vector-field with singularities $\frac{\varphi(L_p)}{\det(L_p)}$). Then apply Stokes formula.

A vectorfield X on W acts on any other Y by means of the bracket

$$Y \longrightarrow [X, Y]$$

another way of producing a new vectorfield from X & Y is

$$i(X) DY.$$

Define $L(Y) = [X, Y] - i(X) DY$.

Properties of L:

1) L defines a section of the endomorphisms of TW.
2) $L|_p = L_p$ if $p \in \text{zero}(X)$.

From K & L we now attempt to construct a new form.

Notation: Projector π for X on $W - \text{zero}(X)$ is a 1-form of type $(1,0)$ so that $\pi(X) = 1$.

Then $d\pi = \omega$ is well defined.

Lemma: If φ is of degree $n = \dim W$, then

$$\varphi(K) = d\left(\frac{\pi}{1-\omega}\,\varphi(K+L)\right)_{n-1}.$$

where $\frac{\pi}{1-\omega}$ stands for its formal power expansion in ω.

Cor. $\varphi(W)$ - integral of $\varphi(K)$ over W.

$$\varphi(W) = \sum \lim_{\varepsilon \to 0} \int_{V_\varepsilon} \left\{ \frac{\pi}{1-\omega}\,\varphi(K+L) \right\}.$$

where $\sum$ indicates distinct components & V_ε is an ε-nbhd of $\text{zero}(X)$.

Generalization to submanifolds.

Assume that V is a nice non singular manifold of zero X.
$\Rightarrow X$ has 1st order behaviour in normal direction.

$$\text{then} \quad \lim \int_{\partial V_\varepsilon} \varphi(K+L)\frac{\Pi}{1-W} = \frac{\varphi(\lambda_1+x_1, \dots, \lambda_k+x_k, x_{k+1}, \dots, x_n)}{\Pi(\lambda_i+x_i)}(V)$$

where the λ's are the eigenvalues along the linebundles (eigenspaces) & the x's are the characteristic classes there.
$k = \operatorname{codim}(V)$.

Remark: consider $\begin{array}{c} V_\varepsilon \\ \downarrow \sigma \\ V \end{array}$ as sphere bdle.
σ = integration over the fibres

$$\text{then} \quad \sigma_*\left(\frac{\Pi}{1-W}\right) = 1/\det(K+L).$$

Historical remark:
Our formula was suggested by the talk of V. Guillemin
X, W as before. Integrate the action of X on W to get an analytic diffeo. This one acts on the sheaf of holomorphic functions Ω. Then an interpretation of the Lefschetz formula is:

$$\underbrace{\sum (-1)^i \operatorname{Tr} H^i(\exp tX)}_{\text{regular function in } t} = \underbrace{\sum_{p \in \text{zero}(X)} \frac{1}{\det(1-e^{tL_p})}}_{\text{meromorphic fct in } t.}$$

expanding the right hand side in powers of t yields

$$\frac{\det(L)}{\det(1-e^{tL})} = t^{-n}\phi_0(L) + \quad + \varphi_n(L) + t \cdots\cdots$$

So it is true that $\quad '' \quad = \quad \varphi_n(L) + t \cdots\cdots$

Rg.

Topology Seminar 14   Prof E. Curtis

Adams' Spectral Sequence

Starting point: s.s. set X or s.s. spectrum X
(for the latter we can substitute a s.s. set along with its suspensions)

FX : free s.s. group associated to X

Filtration of FX : (lower central series mod p)

$$\cdots \subset \Gamma_n FX \subset \Gamma_{n-1} FX \subset \cdots \subset \Gamma_2 FX \subset FX$$

where $\Gamma_n FX$ = subgroup of FX generated by the elements $[x_1, \dots, x_s]^{p^i}$ & $sp^i \geq n$.

$\bigoplus \Gamma_n / \Gamma_{n+1} F$ = free restricted Lie algebra generated by $F/\Gamma_2 F$

due to the restriction each summand is a vector space over $\mathbb{Z}_p$.

If V is a vector space & $L(V)$ denotes the free restricted Lie algebra generated by V we can write

$$\bigoplus \Gamma_n FX / \Gamma_{n+1} FX \cong L\left(FX/\Gamma_2 FX\right).$$

P.M. The tensor alg. $T(V)$ generated by a vector space V is a free rest. Lie algebra. It has a bracket relation $[\ ,\ ]$ satisfying the Jacobi identity & a p-power relation:
$x^{[p]} = x \otimes x \otimes \cdots \otimes x$ & $[x, y]^{[p]} = [x^{[p]}, y^{[p]}]$.

If L is any free restricted Lie algebra (mod p) then $\exists$ a vector space V over $\mathbb{F}_p$ & an imbedding $L \xrightarrow{1-1} T(V)$ preserving bracket & p-power operation.

Passing from the lower central sequence to the homotopy exact couple we get a spectral sequence E whose E^1 term is

$$E^1 = \pi_*\left(\Gamma_r FX / \Gamma_{r+1} FX\right) \Longrightarrow E^\infty = \int \pi_*(FX).$$

Remark : $\pi_*(FX) \cong \pi(\Omega S X)$.

<u>Theorem</u> : (mod 2) $E^1(X) = H_*(X) \otimes \Lambda$.

where Λ is a differential graded ring generated by $\{1, \lambda_0, \lambda_1, \dots\}$.

degree $(\lambda_i) = i$, degree $(1) = 0$.

& the differential d^1 is given by:

$$d^1(x \otimes \lambda) = \sum_i x Sq^i \otimes \lambda_{i-1}\lambda + x \otimes d\lambda.$$

where $d\lambda_n = \sum_j \binom{n-1-j}{j+1}_{\text{mod } 2} \lambda_{n-1-j}\lambda_j$.

& the λ_i's satisfy the following relations:

$$\lambda_i \lambda_{2i+1+n} = \sum_{j\geq 0} \lambda_{i+n-j}\lambda_{2i+1+j}\binom{n-1-j}{j+1}_{\text{mod } 2}.$$

A basis for Λ is given by so called admissible sequences $\lambda_{i_1}\cdots\lambda_{i_k}$ where $2i_j \geq i_{j+1}$

Remark: a similar theorem holds mod p.

Special case : $X = S^n$: s.s. n-sphere.

Then $E^1(FS^n)$ is a subvector space of Λ generated by admissible sequences $\lambda_{i_1} \cdots \lambda_{i_k} \ni i_1 \leq n$ & this E^1-term converges to $E^\infty = \int \pi_*(\partial S^{n+1})$.

Further

$$\cdots \subset E^1(FS^n) \subset E^1(FS^{n+1}) \subset \cdots \subset \Lambda \quad \text{where } \Lambda = \bigcup_i E^1(FS^i)$$

Sketch of a proof of the Theorem :

$\exists$ a bilinear composition of

$$\pi_s\left(\Gamma_r FX/\Gamma_{r+1}FX\right) \times \pi_t\left(\Gamma_s FS/\Gamma_s FS\right) \longrightarrow \pi_{s+t}\left(\Gamma_{rs}FX/\Gamma_{r\cdot s+1}FX\right)$$

this induces a pairing

$$E^1(FX) \otimes E^1(FS) \longrightarrow E^1(FX)$$

$$\& \quad H_*(X, \mathbb{Z}_2) \otimes \Lambda \xrightarrow{\cong} E^1(FX)$$

$$\| $$

$$\pi_*(FX/\Gamma' FX).$$

For a sphere :

$$\pi_*\left(\Gamma_2 FS^n/\Gamma_3 FS^n\right) = \pi_*(L_2(\partial S^n)) \quad \text{where } \partial S^n = FFS^n/\Gamma_2 FS^n$$

$\lambda_0, \ldots, \lambda_i, \ldots, \lambda_r, \ldots$ are generators for $\pi_*(L_2(\partial S^n))$ where $\lambda_i \in \pi_{n+i}(L_2(\partial S^n))$.

& $\lambda_0 = \iota_n \otimes \iota_n$

$\lambda_1 = s_1\iota_n \otimes s_0\iota_n + s_0\iota_n \otimes s_1\iota_n$

$\lambda_n = \sum_{(\alpha,\alpha')} s_\alpha\iota_n \otimes s_{\alpha'}\iota_n$ where α,α' are shuffles of $0,1,2,n$.

Dold-Puppe-Suspension $L_r(AS^n) \longrightarrow L_r(AS^{n+1})$

induces $\pi_* L_r(AS^n) \longrightarrow \pi_* L_r(AS^{n+1})$

also $\exists$ an exact sequence

$$0 \longrightarrow T \longrightarrow W \xrightarrow{\lambda} L(AS^n) \longrightarrow 0$$

with $T \cong LAS^{n-1}$ — induces 0 on homotopy level.
$W \cong LAS^{2n}$

& a corresponding exact sequence

$$0 \longrightarrow \pi\left(L_{\frac{r}{2}}AS^{2n}\right) \longrightarrow \pi\left(L_r(AS^n)\right) \underset{\partial}{\overset{\text{suspension}}{\longrightarrow}} \pi\left(L_r(AS^{n-1})\right) \longrightarrow 0$$

composition with λ_n.

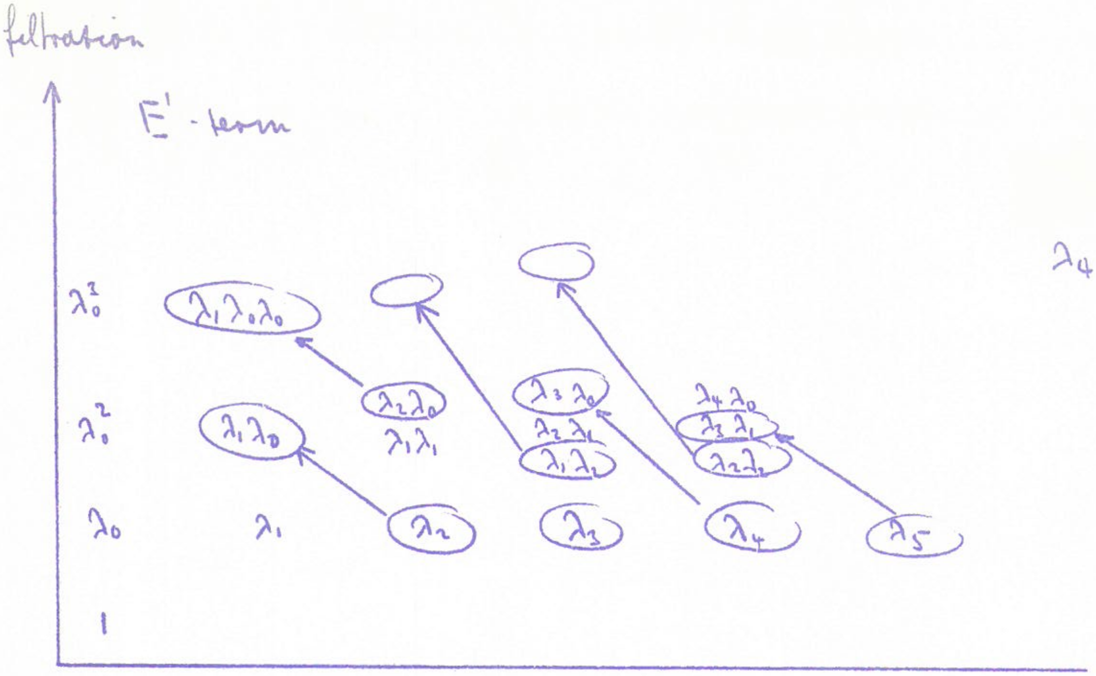
i.e. $\lambda \longmapsto \lambda_n\lambda$

<u>Computation of the E^2-term for a sphere of dim n.</u>

It is enough to consider the coarse filtration

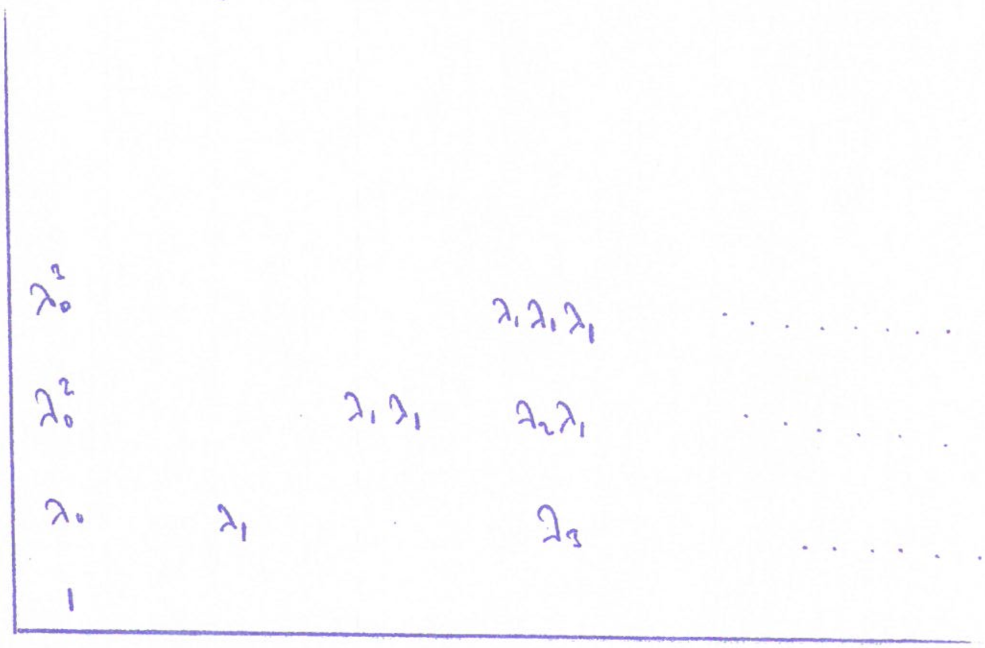
$$\cdots \subset \Gamma_{2^n} \subset \Gamma_{2^{n-1}} \subset \cdots \subset \Gamma_8 \subset \Gamma_4 \subset \Gamma_2 \subset FX$$

since $\pi\left(\Gamma_{2^n}FX/\Gamma_{2^n+1}FX\right) \cong \pi\left(\Gamma_{2^n}FX/\Gamma_{2^{n+1}}FX\right)$



λ_4 goes to $\lambda_3\lambda_0 + \lambda_2\lambda_1$

remaining nontrivial elts of E^2 :



Topology Seminar 15. Mr. G. Brumfield.

Integrality theorem for P.L. characteristic classes.

Def. $\exists$ a fibration $PL/O \to BSO$
$$\downarrow$$
classifying space for micro bdles: $\to BSPL$

Then by Serre:

$$H^*(BSO, \mathbb{Q}) \xleftarrow{\cong} H^*(BSPL, \mathbb{Q}).$$

$$p_n \longleftarrow\!\!| \; p_n \in H^{4n}$$

Properties of $p_n \in H^*(BSO, \mathbb{Q})$.

1) $\Delta(p_n) = \sum_{i+j=n} p_i \otimes p_j$ i.e. $p_n(\xi \times \eta) = \sum_{i+j=n} p_i(\xi)\, p_j(\eta)$.

2) $\exists$ a multiplicative sequence by Hirzebruch: $(1, L_1, L_2, \dots)$ which gives a characteristic class formula for the index of smooth $4n$-manifolds:

$$\langle L_n(M), [M] \rangle = I(M).$$

e.g. $L_1 = \frac{1}{3} p_1$

$L_2 = \frac{1}{45}(7p_2 + p_1^2)$

$L_3 = \frac{1}{357}(62 p_3 - 13 p_1 p_2 + 2 p_1^3)$

$\vdots$

by multiplicative we mean: $\Delta(L_n) = \sum L_i \otimes L_j$.

Claim: The index formula also holds for PL manifolds.
Immediate from:

$$\Omega_*^{SO} \otimes \mathbb{Q} \cong \Omega_*^{SPL} \otimes \mathbb{Q}.$$

<u>Problem</u>: Which alg. combinations of Pontrjagin classes are integral, i.e. are in the image of:

$$H^*(BSPL, \mathbb{Z}) \longrightarrow H^*(BSPL, \mathbb{Q})?$$

Facts: p_1 is integral
Milnor has an 8-manifold with $p_2 = \frac{360}{7}$.
($7p_2$ is integral)

write $L_n = \frac{1}{\mu_n} \bar{L}_n(p_1, \dots, p_n)$ with $\bar{L}_n \in \mathbb{Z}[p_1, \dots, p_n]_{4n}$.

$$\mu_n = \prod_{\text{odd primes}} p^{\left[\frac{4n}{2(p-1)}\right]}$$

<u>Theorem</u> $\mu_n L_n \in H^{4n}(BSPL, \mathbb{Q})$ is integral.

$$\Big(\mathbb{Z}[\bar{L}_1, \bar{L}_2, \dots] \subseteq H^*(BSPL, \mathbb{Z})/_{\text{Torsion}} \subset \mathbb{Z}[R_1, R_2, \dots]$$

$$H^*(BSPL, \mathbb{Z})/_{\text{torsion}} \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \mathbb{Z}[R_1, R_2, \dots] \otimes \mathbb{Z}[\tfrac{1}{2}] \Big)$$

Outline of a proof:

$M^{N+4n}, \partial M^{N+4n}$, $N > 4n$; be a PL manif. with bdry

Theorem reduces to showing the following:

If $x' \in H^N(M, \partial M; \mathbb{Z})$ let $x = \mu_n x'$, then $\langle \mu_n L_n(M) \cup x', [M,\partial M]\rangle \in \mathbb{Z}$.

$$\langle \mu_n L_n(M) \cup x', [M,\partial M]\rangle = \langle L_n(M) \cup x, [M,\partial M]\rangle.$$

1) by Poincaré duality: $\mu_n L_n(M)$ is integral.

2) let $K = K(\mathbb{Z}, N)^{N+4n+1}$ skeleton

then $\exists\ f : K \longrightarrow MSO(N) \ni f^*(u) = \mu_n \iota$

where $u \in H^N(MSO(N), \mathbb{Z})$ is the Thom class

& $\iota \in H^N(K, \mathbb{Z})$ is the fundamental class.

(this proved using obstruction methods, stable homotopy theory & a result by Peterson - Brown).

3)
$$\begin{tikzcd}
 & K \arrow[dr, "f\,:\ \text{map in 2)}"] & \\
M/\partial M \arrow[ur, "g"] \arrow[r, "d"] & T(\nu_L) \arrow[r, "h"] \arrow[d] & MSO(N) \arrow[d] \\
 & L^{4n} \arrow[ul, "i"] \arrow[r] & BSO(N)
\end{tikzcd}$$

$g^*(\iota) = x'$

$g^* f^*(u) = x$.

$T(\nu_L)$: Thom space

ν_L : normal bdle.

Thom iso : $H^*(L, \mathbb{Q}) \xrightarrow[\phi]{\cong} H^*(T(\nu_L))$

Thom class : $\phi(1) = h^*(u)$.

so $\underline{d^*(\phi(1)) = x}$.

4) Evaluate: $\langle L_n(M) \cup x, [M, \partial M] \rangle$

$$= \langle L_n(M) \cup d^* \phi(1), [M, \partial M] \rangle.$$

Fact: If $y \in H^*(M)$ & $a \in H^*(L)$ then

$$y \cup d^*(\phi(a)) = d^*(i^* y \cup \phi(a)) - (\quad d^*(\phi(i^* y \cup a)) \quad)$$

$$L_n(M) \cup d^*(\phi(1)) = d^*(i^* L_n(M) \cup \phi(1)) = d^*\Big(\sum_{i+j=n} L_i(L) L_j(\nu_L) \cup \phi(1)\Big)$$

since $i T_M = T_L + \nu_L$

$$= \underbrace{d^*(L_n(L) \cup \phi(1))}_{= I(L^{4n})} + \underbrace{\sum_{j>0} d^*(L_i(L) L_j(\nu_L) \cup \phi(1))}_{\text{each term is zero}} \in \mathcal{H}.$$

$$H^*(K, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } * = N \\ 0 & \text{if } N < * \leq N + 4n. \end{cases}$$

Hence $d^*(\phi(\text{monomials in } p_j(\nu_L))) = 0$

so $d^*(L_n(\nu_L) \cup \phi(1)) = 0$.

for another term, say $L_1(L) L_{n-1}(L) \cup \phi(1)$ we get:

write $i^* L_1(M) = L_1(L) + L_1(\nu_L)$

so
$$d^*(L_1(L) L_{n-1}(\nu_L) \cup \phi(1)) = d^*((i^* L_1(M) - L_1(\nu_L)) L_{n-1}(\nu_L) \cup \phi(1))$$
$$- d^*(L_1(\nu_L) L_{n-1}(\nu_L) \cup \phi(1)) = 0.$$

$$\text{so } d^*(i^* L_1(M) L_{n-1}(\nu_L) \cup \phi(1)) = L_1(M) \cup \underbrace{d^*(L_{n-1}(\nu_L) \cup \phi(1))}_{= 0} = 0.$$