Modules over Steenrod Algebra

Prof. F. P. Peterson

10-4-65

This talk is the first of a series which have the ultimate goal of computing Ω_*^{Spin}. This work is in conjunction with D. W. Anderson and E. Brown

Let $\mathcal{A} = \mathcal{A}(2)$, the Steenrod Algebra mod 2

$\mathcal{A}_n$ will be the sub(Hopf) algebra of $\mathcal{A}$ generated by $Sq^1, Sq^2, \ldots, Sq^{2^n}$

Milnor has introduced elements $Q_i \in \mathcal{A}_n$ $0 \le i \le n$ defined inductively by $Q_i = [Q_{i-1}, Sq^{2^i}]$ $Q_0 = Sq^1$. He has shown that

$$Q_i Q_i = 0$$

$$Q_i Q_j = Q_j Q_i$$

Let M be a locally finite graded left $\mathcal{A}_1$ module with $M_i = 0$ for $i < 0$. Then Q_i $(i = 0, 1)$ acts as a differential on M so that we can define $H(M, Q_i)$.

<u>Def</u> $H(M, Q_0) = H(M, Q_1)$ will mean that

$$\frac{\ker Q_0 \cap \ker Q_1}{\operatorname{im} Q_0 \cap \operatorname{im} Q_1} \xrightarrow{\eta_i} H(M, Q_i) \quad i = 0, 1$$

is an isomorphism

Theorem (Wall). If $H(M, Q_i) = 0$ $i = 0, 1$ then M is a free $\mathcal{Q}_1$ module.

For the proof of this theorem (which will be helpful in understanding the present discussion) see C.T.C. Wall, "A characterization of simple modules over the Steenrod Algebra Modulo 2" Topology Vol 1. pp 249-254.

We now state the main result of this presentation:

Theorem If $H(M, Q_0) = H(M, Q_1)$

then
$$M = \Sigma \mathcal{Q}_1 \oplus \Sigma \mathcal{Q}_1/\mathcal{Q}_1(Sq^3) \oplus \Sigma \mathcal{Q}_1/\mathcal{Q}_1(Q_0, Q_0) \oplus \Sigma \mathcal{Q}_1/\mathcal{Q}_1(Sq^1, Sq^2)$$

Proof. This is similar to the proof of the above theorem (q.v.) but with complications;

Let M_i be the submodule of M generated by elements of grading $\leq i$. Set

$$M_0 = A_0 + B_0 + C_0 + D_0$$

by
$$A_0 = \ker Sq^1 \cap \ker Sq^2$$
$$A_0 + B_0 = \ker Q_0 \cap \ker Q_1 \;(= \ker Q_0 = \ker Q_1)$$
$$A_0 + B_0 + C_0 = \ker Sq^3$$

We claim that

A_0	will contain	the generators of copies	of $\mathcal{Q}_1/\mathcal{Q}_1(Sq^1, Sq^2)$
B_0	"	"	" $\mathcal{Q}_1/\mathcal{Q}_1(Q_0, Q_1)$
C_0	"	"	" $\mathcal{Q}_1/\mathcal{Q}_1(Sq^3)$
D_0	"	"	" $\mathcal{Q}_1$

For example, suppose $x \in C_0 \oplus D_0$
Then we show that x cannot satisfy a relation

$$Sq^2 x = 0$$

Suppose it did. $\{Sq^1 x\} = 0 \in H(M, Q_0)$
But $Q_1 Sq^1 x = Sq^3 Sq^1 x = Sq^3 Sq^2 x = 0$ under our assumption so $0 = \{Sq^1 x\} \in H(M, Q_1)$, by the hypotheses. But this implies $Sq^1 x = 0$
Then for $x \in C_0 \oplus D_0$, $Sq^1 x = 0$ we must have $Q_1 x \neq 0$. But $Q^1 x = Sq^3 x$.

$$\{Sq^3 x\} = 0 \in H(M, Q_0)$$

and $Q_1 Sq^3 x = Sq^5 Sq^1 x = 0$

$$\text{so } 0 = \{Sq^3 x\} \in H(M, Q_1)$$

$$\therefore Sq^3 x = 0.$$

Having proved the theorem for M^0 we prove the theorem true for M^n by induction on n and so for M. Assuming the theorem true for M^{n-1} we take the short exact sequence

$$0 \longrightarrow M^{n-1} \longrightarrow M \longrightarrow M/M^{n-1} \longrightarrow 0$$

We see that the generators of $(M/M^{n-1})_n$ will be of one of our four types. We then show that these generators pull back to $M^n \subset M$
This is a fairly complicated procedure.

Example.

$$M = H^*(BSO) = Z_2[W_2, W_3, \ldots \quad]$$

$$Q_0 W_{2i} = W_{2i+1}$$

$$Q_0 W_{2i+1} = 0$$

$$H(M, Q_0) = Z_2[\{(W_2)^2\}, \{(W_4)^2\}, \ldots \quad]$$

$$Q_1 W_3 = W_3^{\,2}$$

$$Q_1 W_{2i+1} = 0 \qquad i > 1$$

$$Q_1(W_{2i}) = W_{2i+3} + \{\text{products}\}$$

Changing the polynomial basis in the odd dimension, we see that

$$H(M, Q_1) = Z_2[\{(W_2)^2\}, \{(W_4)^2\}, \ldots \quad]$$

Conjecture 1 If M is an $\mathcal{Q}_n$-module and $H(M, Q_i) = 0$ $\quad i \le n$ then M is a free $\mathcal{Q}_n$-module.

A corollary would be

Conjecture 2 If M is an $\mathcal{Q}$ module $H(M, Q_i) = 0$, all i, then M is a free $\mathcal{Q}$-module

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If M is a left (right) $\mathcal{Q}$-module
M^* is a right (left) $\mathcal{Q}$ module;
there is a map

$$M^* \otimes \mathcal{Q} \to M^*, \text{ etc}$$

<u>Proposition</u> If M is a left $\mathcal{Q}$-module
~~[illegible]~~ coalgebra with unit over $\mathcal{Q}$, then
M^* is a right algebra over $\mathcal{Q}$

$$(m_1^* \otimes m_2^*)a = \sum m_1^* a_i' \otimes (m_2^*) a_i''$$

$$\psi(a) = \sum a_i' \otimes a_i''$$

Taking $M = \mathcal{Q}$, a left & right $\mathcal{Q}$-module
$\mathcal{Q}^*$ is a right & left algebra over $\mathcal{Q}$

By Milnor's results

$$\mathcal{Q}^* = Z_2[\xi_1, \ldots, \xi_i, \ldots]$$

$$\xi_i \in \mathcal{Q}^*_{2^i-1}$$

Let $Sq = Sq^0 + Sq^1 + \cdots$

$$Sq(\xi_k) = \xi_k + (\xi_{k-1})^2$$

$$(\xi_k)Sq = \xi_k + \xi_{k-1}$$

Consider $M = \mathcal{A}/\mathcal{A}(Sq^1, Sq^2)$

$$\mathcal{A} \oplus \mathcal{A} \xrightarrow{R_{Sq^1}+R_{Sq^2}} \mathcal{A} \to M \to 0$$

$$0 \to M^* \to \mathcal{A}^* \xrightarrow{L_{Sq^1}+L_{Sq^2}} \mathcal{A}^* \oplus \mathcal{A}^*$$

Let χ be the cannonical anti automorphism of Steenrod Algebra.

$\chi(M^*) = Z_2[\xi_1^4, \xi_2^2, \xi_3, \xi_4,)$

Theorem. An additive basis for M is

$$\chi(Sq^R) \qquad R = (r_1, r_2, r_3 \ldots)$$
$$r_1 \equiv 0 \pmod 4$$
$$r_2 \equiv 0 \pmod 2$$
$$Sq^R = (\prod \xi_i^{r_i})^*.$$

We claim that $H(M, Q_0)$, $H(M, Q_1)$ can be found from $H(\chi(M)^*, Q_0)$, $H(\chi(M)^*, Q_1)$ (and will be equal)

Theorem. An additive basis for

$$H(\mathcal{A}/\mathcal{A}(Sq^1, Sq^2), Q_0)$$
$$\chi(Sq^{4k})$$

An additive basis for

$$H(\mathcal{A}/\mathcal{A}(Sq^1, Sq^2), Q_1)$$

is

$$\psi(Sq^2(\Delta_{i_1} + \cdots + \Delta_{i_n}))$$

$$1 < i_1 < \cdots < i_n$$

$\psi(M^*)$ is a free $Z_2[\xi_1^4, \xi_2^2, \xi_3, \ldots]$ module on $1, \xi_1, \xi_1^2, \xi_1^3 + \xi_2, \xi_1\xi_2$.

$$M = \mathcal{A}/\mathcal{A}(Sq^3)$$

Differences between the Extrinsic Geometries of Polyhedral and Smooth Surfaces

by Dr. Thomas Banchoff, Harvard University

10-11-65

We consider three sorts of hypotheses on a 2-manifold M (which will usually be embedded in 3-space).

1) Let K be the Gaussian curvature; we have always that

$$\int_M |K|\,dA \geq 4\pi .$$

We may take as a hypothesis that

$$\int_M |K|\,dA = 4\pi$$

In particular this will be the case if M is smooth (i.e. C^∞) and convex.

2.) $\int_{K>0} |K|\,dA = 4\pi$

A manifold satisfying this hypothesis is said to be tight.

3) $x \in M$ is a critical point for the height function in the direction $\vec{\xi}$, i.e., the tangent plane to M at x is orthogonal to $\vec{\xi}$.

Def. $i(x, \vec{\xi}) = 1 - \frac{1}{2} p$ where p is the number of pieces into which the plane orthogonal to $\vec{\xi}$ through x is divided into locally at x by M; $p = 0$ by convention if the plane is tangent to M at x and does not intersect the manifold in a deleted neighbourhood of x.

A plane is locally a plane of support to M at x if it is tangent to M at x and does not intersect a deleted neighbourhood of x in M.

Theorem. A smooth manifold is tight $\Leftrightarrow$ every local plane of support at any point x intersects the manifold only at x.

This theorem leads us to generalize the concept of tightness to polyhedra. (in an obvious fashion)

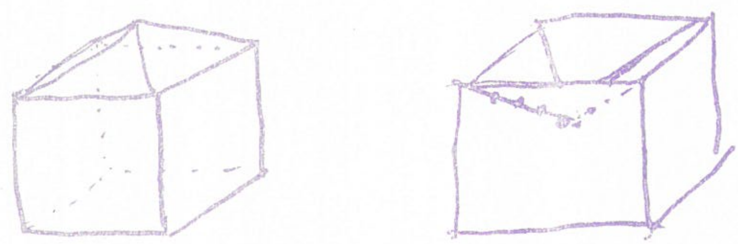
If M is convex and C^∞ we have the Global Rigidity Theorem,

Theorem. If M is isometric to M', M is congruent to M'.

To generalize to the polyhedral case we must add a hypothesis on M':

Theorem. If M, M' are both convex polyhedra and are isometric than they are congruent.

To see why we need the additional hypothesis consider these two figure



Existence Theorem: Given a convex smooth analytic metric on the sphere, ∃ unique manifold whose Gauss-map is 1-1 onto the sphere (the manifold is convex) such that the inverse map pulls the metric of the manifold (imbedded in 3-space) onto the sphere to produce the given metric.

Similarly given polyhedral convex metric on S^2 there is a unique polyhedron giving rise to it. The Leningrad geometers used this result and polyhedral approximations to show that C^∞ convex metrics on the sphere were induced by unique convex C^∞ 2-manifolds

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We give a global rigidity theorem for tight C^∞-manifolds:

M isometric to M', M tight $\Rightarrow M \cong M'$

Alexandroff originally required analyticity. Nirenberg has reduced the condition to C^5 + 2 auxillary conditions.

The analogue for polyhedra is false: We can find M, M' isometric, <u>both</u> tight, but not congruent.

<u>Critical Points</u>

Vertices are of course the analogues of critical points. The index $i(x, \vec{\xi})$ can be defined obviously for polyhedra.

Thm. $\sum_{x \text{ critical}} i(x, \vec{\xi}) = \chi(M)$ $\vec{\xi}$ fixed

Thm. $\sum_{v \text{ vertex}} i(v, \vec{\xi}) = \chi(M)$

$\chi(M)$ is the Euler characteristic of M

It is a theorem that that for any differentiable embedded torus there exists no direction with exactly 3 critical points. This is not true for the polyhedral case. There does exist a triangulation of the torus for which we can find a direction with exactly three critical points

2-manifolds in n-space.

Tightness. We say that a 2-manifold in n-space is tight if any hyperplane H^{n-1} which is a hyperplane of local support at an $x \in M$ touches M only at x.

A manifold is said to be substantially imbedded in E^n if it is not contained in a hyperplane.

The torus can be substantially and tightly imbedded in 4-space

Theorem. If a 2-manifold is substantially and tightly imbedded in E^n, then $n \leq 5$. This will not carry over to polyhedra.

Specifically

A) For any $n \geq 3$ $\exists$ $M(n)$ polyhedron tightly and substantially imbedded in E^n

B) If M^2 is tightly and substantially imbedded in E^n, then $n < \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)})$

This result is the best possible for the torus and for the projective plane

Secondary Cohomology Operations Induced by The Diagonal Mapping

Dr. P. A. Schweitzer

10-18-65

[See Topology, Vol. 3, pp. 337-355 for detailed presentation]

We will work in the category of CW-complexes with base-point $*$.

Cohomology will be with $\mathbb{Z}_p$ coefficients.

$$d_n : (X, *) \longrightarrow (X^n, T^n X)$$

will be given by

$$d_n : (x) \longmapsto (x, \ldots, x)$$

$$T^n X = \{ (x_1, \ldots x_n) \mid \text{some } x_i = * \}$$

We will be working with cohomology operations in the Steenrod algebra $\mathcal{A}_p$. Let $a \in \mathcal{A}_p$ be of degree k.

Suppose $f : X \to Y$ and $u \in H^q(Y)$ such that

$$au = 0$$

$$f^* u = 0 .$$

We can define a functional cohomology operation

a_f by

$$
\begin{tikzcd}
H^q(Y) & H^q(\Psi, X) \arrow[l, "j^*"'] \arrow[r, "a"] & H^{q+k}(Y, X) & H^{q+k-1}(X) \arrow[l, "\delta"'] \arrow[from=1-1, to=1-4, bend right, "a_f"']
\end{tikzcd}
$$

if f is an injection; if f is not an injection we 'make' it one by replacing Ψ by the mapping cylinder $\Psi \cup_f CX$.

Note that $a_f u$ is a class in

$$H^{q+k-1}(X) \big/ aH^{q-1}(X) + f^* H^{q+k-1}(Y).$$

One may think of $a_f u$ as being the effect of a on u when u has been 'extended' to $\Psi \cup_f C X$.

Given $u \in H^q(\Psi, *)$
$v \in H^r(\Psi, *)$

we get $u \times v \in H^{q+r}(\Psi^2, T^2\Psi)$

In order to define $a_d(u \times v)$ we must have:

(*)

a) $d^*(u \times v) = 0$
This means just that $u \cup v = 0$

b) $a(u \times v) = 0$.
By the Cartan formula $a(u \times v) = \sum_{j \in I} b^j u \times c^j v$
Let us assume that for each $j \in J$ either $b^j u = 0$ or $c^j v = 0$.

Then
$$a_d(u\times v)\in H^{q+r+k-1}(Y)\big/\mathrm{Im}\, a+\mathrm{Im}\, d^*$$

Clearly the a_d are natural, i.e. give $g: X\to Y$ then
$$g^* a_d(u\times v)= a_d(g^*u\times g^*v) \quad \text{modulo indeterminacy}$$

Let us recall the defintion of Liusternik-Schnirelman category.

Def. $\mathrm{Cat}\, X=\max n \ni d_n:(X,*)\to(X^n, T^nX)$ is not homotopically trivial.

If $a_{d_n}(u_1\times\cdots\times u_n)\neq 0$ mod indeterminacy then $\mathrm{Cat}\, X\geq n$.

Theorem 1: Assume $(*)$, $f: X\to Y$ and let $f^*u=0$.
Then $f^*a_d(u\times v)=\sum_{j\in I}(b_f^j u)\cup f^*c^j v - a(u\cup_f v)$

modulo total indet. $=\mathrm{Im}\, af^*+\mathrm{Im}\, ad^*+\mathrm{Im}\, f^*d^*$
$$+\sum_{j\in J}\mathrm{Im}\, b^j\cup f^*c^j v.$$
where J is the set of indices j with $c^j v\neq 0$.

(The Functional cup product $u \cup_f v$ is defined, for $f^* u = 0$, $u \cup v = 0$, by

$$u \cup_f v = \delta^{-1}(j^{*-1} u \cup v) \in H^*(X,*)/H^*(X,*) \cup f^* v + \mathrm{Im} f^*)$$

We look at cohomology sequences associated to the diagram below for proof $(Y_d = d_2(Y) \subseteq (Y^2, T^2 Y))$

$$\begin{tikzcd}
 & (Y_d ; X_d, *) \arrow[dl] \arrow[d, "a(u \cup_f v) d"] \arrow[dr] & \\
(Y^2, X^2, T^2 Y) & (Y^2, X_d, T^2 Y) \arrow[l] \arrow[r] & (Y^2, Y_d, T^2 Y)
\end{tikzcd}$$

Application: $p = 2$

$$S^{2q+k-1} \xrightarrow{f} S^{2q-1} \xrightarrow{g} S^q$$

$$0 \neq u \in H^q(S^q, * ; \mathbb{Z}_2)$$

Suppose $u \cup_g u \neq 0$ $\quad$ ($\therefore q = 1, 2, 4$ or 8; we don't consider $q = 1$)

Assume $Sq^k_f (u \cup_g u) \neq 0$ $\quad$ ($k = 0, 1, 3.$ or 7)

$$S_q^{2q-1} \cup_f E^{2q+k} \xrightarrow{F} S^q \cup_{gf} E^{2q+k} \quad (E \text{ a cell})$$

Suppose $0 \neq v \in H^q(S^q \cup_{gf} E^{2q+k})$

Claim: $F^* Sq^k_{b\,d}(v \times v) \neq 0$

$\therefore$ above Theorem applies

$$F^* Sq^{k+1}_{b\,d}(v \times v) = (Sq^{k+1}_{b\,f} v) \cup F^* v - Sq^{k+1}_b (v \cup_F v)$$

but $F^* v = 0$ and $v \cup_F v \neq 0$

$\therefore \quad - Sq^{k+1}_b (v \cup_F v) \neq 0$

$\therefore \quad F^* Sq^k_b$

This shows

(a) gf essential

(b) $S^q_b \cup_{gf} E^{2q+k}_b$ has cat ≥ 2 ...

(c) ... and is not a suspension

(d) gf is not a suspension

Take

$u \in H^q_b(\overline{Y})$

$v \in H^r(\overline{Y})$

$w \in H^s(\overline{Y})$

Let $a \in \mathcal{Q}_p$ satisfy the Cartan formula

$$a(u \times v) = \sum_{j \in I} b^j u \times c^j v$$

Theorem: Assume

(i) $u \cup v = 0$

(ii) $b^j(u \times v) = 0$ for $j \in J$

(iii) $c^j(w) = 0$ for $j \in I - J$

Then $a_{d_3}(u \times v \times w) = \sum_{j \in J} b^j d_2(u \times v) \cup c^j w$ mod total indet.

This follows from Lemma:

Lemma: Let maps

$f : X \to Y$

$f' : X' \to Y'$

be given together with

$y \in H^q(Y)$

$y' \in H^{q'}(Y')$

and $a \in \mathcal{Q}_p$ satisfying Cartan formula as above.

Assume (i) $f^* y = 0$

(ii) $b^j y = 0$ for $j \in J$

(iii) $c^j y' = 0$ for $j \in I - J$

Then $a_{f \times 1}(y \times y') = \sum_{j \in J} b^j_f y \times c^j y'$ mod indet.

This Lemma follows from definition chasing and fact that cohomology coboundary of $(Y, X) \times (Y', \phi)$ is $\delta \times 1$ where δ is coboundary for (Y, X).

Proof of Theorem from Lemma:

$$d_3 = (d_2 \times 1) d_2$$

$$a_{d_3}(u \times v \times w) = d_2^* \, a_{d_2 \times 1}([u \times v] \times w)$$

$$= d_2^* \sum_{j \in J} b^j d_2(u \times v) \times c^j w$$

$$= \sum_{j \in J} b^j_{d_2} (u \times v) \cup c^j w.$$

For every $a \in \mathcal{Q}_p$ and every n, a_{d_n} is non-trivial for suitable spaces. E.g. for $a = Sq^2$

$$Sq^2_{d_n}(v \times \cdots \times v) \neq 0$$

where

$$0 \neq v \in H^q((S^q)^n - E^{nq} \cup_f E^{nq+1})$$

where f is the composition

$$S^{nq} \xrightarrow{g} S^{nq-1} \longrightarrow (S^q)^n - \operatorname{int} E^{nq+1}$$

where g is such that Sq^2 is non trivial in the mapping cylinder of g.

On a Theorem of Kervaire

Prof. A.T. Vasquez, Brandeis University

10-25-69.

The Following Theorem is due to Kervaire.
Let X be a finite simplicial complex, connected.

<u>Thm</u> If (1) $\exists \mu \in H_n(X, Z)$ such that

$$H^i(X, Z) \xrightarrow[\sim]{\cap \mu} H_{n-i}(X, Z)$$

(Poincaré Duality)

and if (2) $\exists$ vector bundle ξ^k over X such that

$$\pi_{n+k}(T(\xi^k)) \longrightarrow H_{n+k}(T(\xi^k))$$

is epimorphic

and if furthermore

(3) n is odd and large and X is simply connected, then:

X has the homotopy type of a compact differentiable manifold.

It is our purpose here to point out the crucial role of the condition of simple connectedness. We will construct combinatorial manifolds satisfying all other conditions which are not differentiable.

Theorem: $\exists$ odd dimensional orientable combinatorial manifolds M^{2n+1} $(n>1)$

such that

1) $\pi_i(M) \simeq \pi_i(S^1)$ $i<n$

2) $H_*(M) \simeq H_*(S^1 \times S^{2n})$

3) $\pi_{2n+2}(\Sigma M) \longrightarrow H_{2n+2}(\Sigma M)$ epimorphism

4) As a module over $Z[\pi_1(M)] = Z[J]$ $\pi_n(M)$ has a presentation of the following form:

$$(x_1, x_2;\ \lambda_1(t-1)x_1 + (\lambda(t-1)+1)x_2,\ (-t+\lambda(t-1))x_1 + (\lambda_2(t-1)\ x_2))$$

(λ_1 odd integer, λ integer, t generator of J)

5) For $n=3,7$ these manifolds have a differentiable structure and for $n=4k+1$ they do not have the homotopy type of a differentiable manifold.

Outline of the Proof:

Start with a sphere Σ^{2n-1} embedded non-differentiably in S^{2n+1}

Take a nice imbedding of $S_1^n \vee S_2^n$ in S^{2n+1}

Now trivialize the normal bundle of each sphere and define a normal vector on each of them by a nice differentiable function

$$r_i : S_i^n \to (R^{n+1} - 0)$$

V^{2n} will be the union of two neighborhoods of S_i^n $i=1,2$ obtained by exponentiating those vectors normal to S_i^n but $\perp$ to r_i.

<u>Lemma</u> If the maps r_i are of odd degree then the normal bundle to $S_i^n \subset V^{2n}$ is equivalent to the tangent bundle of the sphere (n odd).

The assertions about H_* and π_i of M reduce easily to assertions about H_* and π_i of $S^{2n+1} - \text{nbd}(\partial V)$.

Thus corresponding to 1) above, $i=1$ we have

$$\pi_1(S^{2n+1} - \text{nbhd}\,\partial V) \simeq J$$

which is proven by taking the linking number of a loop with V.

The other facts about $(S^{2n+1} - \text{nbhd}(\partial V))$ are obtained by a complicated argument involving the construction of that space's universal covering space.

Filtrations in K-Theory

Prof. Donald Anderson, M.I.T.

11-1-65

Let

K^* denote (at first, any) generalized cohomology theory

X a CW complex

X^n the n-skeleton of X

Definition. $x \in K^*(X)$ has filtration at least n if

$$x \in \ker \left(K^*(X) \to K^*(X^{n-1}) \right)$$

$$F^n K^*(X) = \ker \left(K^*(X) \to K^*(X^{n-1}) \right)$$

$$gr^n K^*(X) = F^n K^*(X) / F^{n+1} K^*(X)$$

We know that there is a spectral sequence with

$$E_2^{p,q} = H^p(X; K^q(pt))$$

$$E_\infty^{p,q} = gr^p K^{p+q}(X)$$

Problem: Compute the filtration in $KO^0(BSO)$.

Now

$$KO^0(BSO) = \mathbb{Z}[[\pi^1, \pi^2, \dots]]$$

$$K^0(BSO) = \mathbb{Z}[[c(\pi^1), c(\pi^2) \dots]]$$

First we compute $gr\, K^*(BSO; Z_2)$ by means of the spectral sequence associated to $K^*(BSO; Z_2)$

$$E_2^{p,q} = H^p(BSO; K^q(pt; Z_2))$$
$$= H^p(BSO; Z_2) \quad q \text{ even}$$
$$= 0 \quad q \text{ odd} \qquad -\infty < q < \infty$$

∂^2 is clearly trivial

∂^3 is both a derivation and a stable homology operation

$$\partial^3 : H^p(BSO; Z_2) \longrightarrow H^{p+3}(BSO; Z_2)$$

and can easily be shown to be non-trivial

$$\therefore \partial^3 = Sq^3 + Sq^2 Sq^1 .$$

$\therefore E_4^{*,q}$ is the cohomology of $H^*(BSO; Z_2)$ with respect to $Sq^3 + Sq^2 Sq^1$ (q even, fixed).

$$H^*(BSO; Z_2) = Z_2[w_2, w_3, \ldots \quad]$$

$$\partial^3(w_3) = w_3^2$$
$$\partial^3(w_{2i+1}) = 0 \qquad i > 1$$
$$\partial^3(w_{2i}) = w_{2i+3} + \text{decomposables}$$

$$\therefore H^*(BSO; Z_2) = Z_2[w_2, w_3, w_4, \partial_3(w_2), w_6, \partial_3(w_4) \ldots]$$
$$= Z_2[w_3] \otimes Z_2[w_2, \partial_3(w_2)] \otimes Z_2[w_4, \partial_3(w_4)]$$

Now $Z_2[W_3]$ is acyclic since $\partial^3(w_3) = W_3^2$.

But $Z_2[W_{2i}, \partial^3(W_{2i})] = Z_2[W_{2i}^2] \otimes E(W_{2i}) \otimes Z_2[\partial^3(W_{2i})]$

with $E(W_{2i}) \otimes Z_2[\partial^3(W_{2i})]$ acyclic

and $Z_2[(W_{2i})^2]$ isomorphic to its homology w.r.t. ∂^3

$$\therefore E_4^{*,0} = Z_2[W_2^2, W_4^2, \ldots]$$

But since $E_4^{*,0}$ has all elements of even total degree all higher ∂^r must be trivial

$$\therefore E_4^{*,0} = E_\infty^{*,0}$$

We conclude that

$$C((\pi^1)^{i_1}(\pi^2)^{i_2} \ldots)$$

has filtration $4\sum_j j i_j$

Note that $H^*(BSO; Z_2)$ has isomorphic cohomology w.r.t. Sq^1 and $Sq^3 + Sq^2 Sq^1$

It follows that as an Q_1-module $H^*(BSO; Z_2)$ is a direct sum of copies of

$$Q_1,\ Q_1/Q(Sq^3),\ Q_1/Q(Sq^1, Sq^2),\ Q/Q(Sq^1, Sq^3)$$

where $Q_1 \subset Q = Z_2[Sq^1, Sq^2]$.

Actually, $\mathcal{Q}/\mathcal{Q}(Sq^1, Sq^3)$ cannot occur for the reason that its Sq^1 cohomology has generators differing by 2 in dimension while all generators of the Sq^1 cohomology of $H^*(BSO; Z_2)$ are of dimension divisible by 4.

One is now in a position to compute the $KO^*(BSO; Z)$ spectral sequence for each of the three types by separate analyses of each of the three types of summands of $H^*(BSO)$ (which make up the E_2 term.) The result will be an E_∞ term where rows are polynomial algebras with generators p^i in dimension $4i$.

Now we employ a series of inequalities and the Chern character and counting argument, to prove that

$$\text{in}\quad \begin{aligned} K^0(BSO; Z_2) &= Z_2[[c(\pi^1), c(\pi^2) \ldots]] \\ K^0(BSO) &= Z[[c(\pi^1), c(\pi^4) \ldots]] \\ KO^0(BSO) &= Z[[\pi^1, \pi^2 \ldots]] \end{aligned}$$

the filtration is the same essentially.

The Riemann-Roch Theorem

Prof. Shih, Brandeis University 11-8-65

The Riemann-Roch Theorem gives the relation between the degree, the dimension of a divisor of meromorphic funtions on a Riemann surface with its genus. Hirzebruch generalizes to algebraic manifolds (Atiyah-Singer to analytic manifolds.)

ξ holomorphic vector bundle over M then

$$\sum_q (-1)^q \dim H^q(V, \Omega(\xi)) = [ch(\xi)\cdot Todd(V)][V]$$
$$= g_*(ch(\xi)\cdot Todd(V)]$$

Grothendieck generalizes this to a problem of covariance of functors:

(1) For all algebraic manifold non singular and rational maps $f: X \to Y$ there exist $f_!: K(X) \to K(Y)$ such that

(2) $f_!(x \cdot f^!(y)) = f_!(x)\cdot y \qquad x \in K(X),\ y \in K(Y)$

(3) If f is an embedding then $f^!(f_!(x)) = \lambda_{-1}(\eta)\cdot x$

and

(4) $f_!(x \cdot x' \cdot \lambda_{-1}(\eta)) = f_!(x)\cdot f_!(x')$

(5) (Riemann-Roch) $Todd(X)\cdot ch(f_!(x)) = f_*(Todd(Y)\cdot ch(x))$

On the other hand we have the results of Hirzebruch-Atiyah-Bott-Shapiro.

(1) For Spin$_{\mathbb{C}}$- differentiable manifold map $f: X \to Y$ there exist $g(=f_!): K^*(X) \to K^*(Y)$ such that (2) and (5) above are verified when

the Todd genus is replaced by $\mathfrak{R}$-class depending on f

Note: f is Spin$_{\mathbb{C}}$ map if $w_2(x) - f^* w_2(y) \in H(X;\mathbb{Z})$
(f not necessarily differentiable)

We try to fill in the difference between Grothendieck and Hirzebruch-Atiyah-Bott-Shapiro

Let $A = \mathbb{Z}[\frac{1}{2}] \subset \mathbb{Q}$, $K_A^*(x) = K^*(X) \otimes A$

Replacing $K^*(X)$ by $K_A^*(X)$ then Grothendieck (1), (2), (3), (4), (5) hold.

Notation

$\xi_{\mathbb{R}}(X)$ = semigroup of (real) oriented vector bundles over X

$KSO(X) = \widehat{\xi_{\mathbb{R}}(X)}$

$L(\cdot) = K_A(\cdot)$ on $H^{2*}(\cdot)$

$L^*(\cdot) = K_A^*(\cdot) = K_A^0(\cdot) \oplus K_A^1(\cdot)$ or $H^*(\cdot)$

$Hom(\xi_{\mathbb{R}}, L) \supseteq Hom(KSO, L)$

(where Hom consists of characteristic class maps taking $\oplus$ onto $\otimes$)

$Fun(\xi_{\mathbb{R}}, L) \supseteq Hom(\xi_{\mathbb{R}}, L)$

B C.W. complex, finite, category $\mathcal{C}_B$ of fiber bundles $X \xrightarrow{p} B$,
$p^{-1}(b) = M$ compact orientable manifold without boundary.

$T(p) = T(X)$ tangent bundle along the fibres of p.

$\tilde{\rho} : \mathrm{Hom}(\xi_{\mathbb{R}}, L) \to \mathrm{Fun}(\xi_{\mathbb{R}}, L^*)$

$Wu_r : \mathrm{Hom}(KSO, K_\Lambda) \times \mathrm{Hom}(KSO, H^{2*}_{\mathbb{Q}}) \to \mathrm{Hom}(KSO, H^{2*}_{\mathbb{Q}})$ $r \in \mathbb{Q}$

$Ad_k : \mathrm{Hom}(KSO, K_\Lambda) \times \mathrm{Hom}(KSO, K_{\mathbb{Q}}) \to \mathrm{Hom}(KSO, K_{\mathbb{Q}})$ $k \in \mathbb{Z}$

Statement

$\forall \alpha \in \mathrm{Hom}(KSO, L)$ $f : X \to Y \in \mathcal{C}_B$

$\exists !$ (a) $f^\alpha_! : L^*(X) \to L^*(Y)$

$\ni$ (i) $(f \cdot g)^\alpha_! = f^\alpha_! \cdot g^\alpha_!$, $1^\alpha_! = 1$

(ii) If f is an imbedding then $f^\alpha_!(x) = x \cdot (\tilde{\rho}(\alpha)(\eta))$ η normal bundle of f

(iii) If $f : X \times S^n \to X$ is projection
then $f^\alpha_! f^!(x) = 0$.

The morphism $f_!^\alpha$ in 1 satisfies the conditions of Grothendieck under appropriate modification. Replace $f_!$ by $f_!^\alpha$, $\lambda_{-1}(\eta)$ by $\tilde{\rho}(\alpha\cdot\mu)(\eta)$

$\mu = \lambda_0^{-1}$ if $L^* = K^*$

$\quad\ \psi$ if $L^* = H^*$

In (3) and (4) $f : X \to Y$ must be embedding of even codimension.

For (5) we have "Riemann-Roch formula for $\mathcal{C}_B$.

$\alpha, \delta \in \mathcal{H}om\,(KSO, K_\Lambda)$; $\beta \in \mathcal{H}om(KSO, H^{2*}_Q)$.

$f : X \to Y \in \mathcal{C}_B$, $\gamma \in Q$, $h \in Z$

$$Wu_\gamma(\alpha,\beta)(Y)\cdot ch^*(f_!^\alpha(x)) = f_!^\beta(Wu_\gamma(\alpha,\beta)(X)\cdot ch^\gamma(x))$$

$$Ad_h(\alpha,\delta) : \psi_h(f_!^\alpha(x)) = f_!^\delta(Ad_h(\alpha,\delta)(X)\cdot\psi_h(x))$$

<u>Remark</u>

1) $B \in \mathcal{C}_B$, hence $\exists\, p_!^\alpha : L^*(X) \to L^*(Y)$ and

$$\begin{tikzcd} L^*(X) \arrow[rr, "f_!^\alpha"] \arrow[dr, "p_!^\alpha"'] & & L^*(Y) \arrow[dl, "p_!^\alpha"] \\ & L^*(B) & \end{tikzcd}$$ commutes

In case $L^* = H^*$, $\alpha = 1$, $p_!^\alpha$ is the dual of Thom's map

$B = pt.$ (Poincaire-Duality Gysin)

2) Let $\mathcal{C}'_B$ be category similar to $\mathcal{C}_B$ but with restriction that fibre be manifold without boundary relaxed to allow manifold with boundary, maps to respect boundary.

there is a functor $C'_B \xrightarrow{\delta} C_B$

so

$$\begin{tikzcd}
L^*(W,\partial W) \arrow[rr, "f_!^\alpha"] \arrow[dr, "p_!^\alpha"'] & & L^*(W',\partial W') \arrow[dl, "p_!^\alpha"] \\
& L^*(B) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
L^*(\partial W) \arrow[rr, "\delta"] \arrow[dr, "p_!^\alpha"'] & & L^*(W,\partial W) \arrow[dl, "p_!^\alpha"] \\
& L^*(B) &
\end{tikzcd}$$

commutes

Massey Higher Products

11-22-65

Dr. D.P. Kraines

We generalize the triple product $\langle u,v,w\rangle$ which was first used to prove the Jacobi identity for Whitehead Products. We define a k-fold product

$$\langle u_1,\ldots,u_k\rangle$$

for certain k-tuples of cohomology classes.

Let (X,A) be a topological pair, R a comm ring with 1.

$C^* = C^*(X,A;R)$ singular cochains (associative cup product structure)

$H^* = H^*(X,A;R)$

Let a k-tuple $\{u_i\}$ be given, $u_i \in H^{p_i}$

Set $p(i,j) = \sum_{n=i}^{j}(p_n-1)$

Definition. A set of cochains $A=\{a_{ij}\}_{1\le i\le j\le k,\,(i,j)\ne(1,k)}$ is said to be a defining system for $\langle u_1,\ldots,u_k\rangle$ if

a_{ii} is a cocycle representative of u_i $(\forall i)$

$\delta a_{ij} = \sum_{n=i}^{j-1}(-1)^{(j+1-n)\,p(i,n)}\,a_{in}a_{n+1,j}$

$\deg a_{in} = p(i,n)+1$

We set

$$C(A) = \sum_{m=1}^{k-1} (-1)^{(k+1-m)p(1,k)} a_{1m} a_{m+1,k}$$

$$\delta C(A) = \sum_m \sum_s a_{1s} a_{s+1,m} a_{m+1,k} + \sum_m \sum_s a_{1m} a_{m+1 s} a_{s+1,k} = 0$$

Def. The k-fold product $\langle u_1, \ldots, u_k \rangle$ is said to be defined if there exists an A which is a defining system for it;

$$\langle u_1, \ldots, u_k \rangle \subset H^{p(1,k)+2}$$

is then $\{C(A)\}$ for all A which are defining systems.

Examples $k=2$ $\quad A = (a_{11}, a_{22})$ $\quad \{a_{11}\} = u$, $\{a_{22}\} = v$

$$C(A) = (a_{11} a_{22})$$

$$\langle u_1, u_2 \rangle = u_1 \cup u_2$$

$k=3$ $\quad A = (a_{11}, a_{22}, a_{33}, a_{12}, a_{23})$ $\quad \{a_{ii}\} = u_i$, $\delta a_{12} = a_{11} a_{22}$, $\delta a_{23} = a_{22} a_{33}$

$(\therefore u_1 u_2 = u_2 u_3 = 0)$

$$C(A) = (-1)^{p_1 - 1} a_{11} a_{23} + a_{12} a_{33}$$

$k = 4$ As above with a_{13}, a_{24}, a_{44}

$$\delta a_{13} = (-1)^{p_1} a_{11} a_{23} + a_{12} a_{33}$$
$$\delta a_{24} = (-1)^{p_1} a_{22} a_{34} + a_{23} a_{44}$$

$$C(A) = a_{11} a_{24} + a_{12} a_{34} + a_{13} a_{44}$$

In general

$\langle u_1, \dots u_k \rangle$ defined $\Rightarrow$

$\langle u_1, \dots u_{k-1} \rangle$, $\langle u_2 \dots u_k \rangle$ defined.

The converse is false

In general $\langle u_1, \dots, u_k \rangle$ is a $k-1$ order coh. op. of k-variables. It is natural under maps of spaces or maps of chain complexes which preserve cup product.

Loop Suspension.

Let $PX \xrightarrow{\pi} X$ be the canonical fibration. Set $E_A = \pi^{-1}(A)$. We define a composition

$$\sigma : H^n(X, A) \xrightarrow{\pi^*} H^n(PX, E_A) \xleftarrow[\sim]{-1} H^{n-1}(E_A)$$

called loop suspension.

If $\langle u_1, \dots, u_k\rangle$ is defined as a subset of $H^n(X,A)$ then $\sigma\langle u_1,\dots,u_k\rangle = \{0\} \subset H^{n-1}(E_A)$

(Massey Products $\subset \ker \sigma$)

If $\langle v_1, \dots, v_k\rangle$ is defined in $H^n(Y,B)$ with $H^*(Y) = 0$, then

$$\langle v_1, \dots, v_k\rangle = \{0\}.$$

$$\langle u_1, \dots, u_k\rangle = (-1)^n \langle u_k, \dots, u_1\rangle$$

$$n = \sum_{1\leq r<s\leq k} p_r p_s + \frac{(k-1)(k-2)}{2}$$

A nice conjecture would be

$$\langle u_1, \dots, u_k\rangle = 0 \text{ if } \operatorname{cat}(X,A) < k \pm \varepsilon$$

But in fact there is a 2 dim. space X_{p^k} such that $\langle \overbrace{u, \dots, u}^{p^k}\rangle \neq 0$.

If $u_1 = u_2 = \dots u_k = u$ then we can restrict the indeterminacy in $\langle u_1, \dots u_k\rangle$ getting an operation $\langle u\rangle^k \subset \langle u_1, \dots, u_k\rangle$.

The Defining system $\underline{\langle u\rangle^k}$ for $\langle u\rangle^k$ is

$$A^* = (a_n) \qquad n = 1, \dots, k-1$$

$$\{a_i\} = u$$

$$\delta a_n = \sum_{r=1}^{n-1} (-1)^r a_r a_{n-r}$$

Example
$k=3$

$$A^* = (a_1, a_2) \qquad \{a_i\} = u$$

$$\delta a_2 = a_1 a_1 \qquad (u^2 = 0)$$

$$C(A^*) = (-1)^{m-1} a_1 a_2 + a_2 a_1$$

Let b be a cocycle of dimension $2m-1$

Then $A_1^* = (a_1, a_2 + b)$

$$C(A_1^*) - C(A^*) = (-1)^{m-1} a_1 b + b a_1 \sim 0$$

$\therefore$ $\langle u\rangle^3$ is a single class

Let p be an odd prime and q any integer

$$p^q : S^1 \to S^1$$

$$X = S^1 \cup_{p^q} e^2$$

$$H^1(X; Z_p) \simeq H^2(X; Z_p) \simeq Z_p$$

If u generates $H^1(X; Z_{p^q})$

$\beta_q u$ generates $H^2(X; Z_p)$

β_q coboundary associated to $0 \to Z_p \to Z_{p^{q+1}} \to Z_{p^q} \to 0$

Theorem

$\langle u \rangle^{p^q}$ is defined as the single class
$$-\beta_q u \in H^{(\Sigma(p_n^{i}-1)+c)}(X; Z_p).$$

It suffices to prove this for $v \in H^1(Z_{p^q}, 1; Z_p) \cong H^1(Z_{p^q}, Z_p)$

Theorem
Let $u \in H^{2m+1}(X, A; Z_p)$

Then $\langle u \rangle^p$ is defined as a single class

$$-\beta P^m u$$

$$\sigma(-\beta P^m u) = -\beta P^m(\sigma u) = -\beta(\sigma u)^p = 0$$

We have specialized our discussion of the Massey Product; now we consider a generalization. Let X be a topological space, (X_i, A_i) pairs of nice subspaces
$$u_i \in H^{p_i}(X_i, A_i; R)$$

We define
$$\langle u_1, \dots, u_p \rangle \subset H^{p(1,k)+2}(\cap X_i, \cup A_i, R)$$

as above.

Another more interesting generalization:

$$\mathcal{U} = \{u_{ij}\}$$
$$V = \{v_{jk}\}$$
$$W = \{w_{kl}\}$$

$\mathcal{U} V = 0$
$VW = 0$
define $\langle \mathcal{U}, V, W \rangle$

Conjecture: Knowledge of the generalized Massey Products in $H^*(X)$ completely determines structure of $H^*(\Omega X)$

Conjecture: $\ker \sigma$ = module generated by Generalized Massey Products.

Remark. Peter May has constructed spectral sequence in which the differentials give generalized Massey Products.

A Generalization of Ext

11 · 29 · 65

Prof. P. J. Hilton

This talk is based on current work with Pressman and arises from an observation of Hilton and Eckmann that Spectral Sequences arise in Topology because the usual functors ($H_*, H^*, \pi_*, \ldots$) can be defined on maps $f: X \to Y$ as well as spaces. We may ask about the functors Ext, Tor, Π_n, Π^n.

Note: $\Pi_n(A,B) = H^n(\mathrm{Hom}(A, \underline{X}))$ $(n \geq 0)$ where $\underline{X}$ is a complete resolution of B. Π^n has a similar definition but we will not make use of it.

$\underline{X}$:

$$
\begin{tikzcd}
P\Omega^2 X \arrow[r, "\eta_2"] & P\Omega X \arrow[r, "\eta_1"] & PX \arrow[rr, "\iota\kappa"] \arrow[dr] & & IX \arrow[r, "\xi_1"] & I\Sigma X \arrow[r, "\xi_2"] & I\Sigma^2 X \\
& & & X \arrow[ur] & & &
\end{tikzcd}
$$

Now given $\phi: X \to Y$, we may extend to a map $\underline{\Phi}: \underline{X} \to \underline{Y}$ of complete resolutions and take the cochain mapping cone $M\underline{\Phi}$

Note: If $\psi: C \to D$ is a cochain complex map

$(M\psi)^n = C^{n+1} \oplus D^n$, $\delta(c,d) = (-\delta c, \phi c + \delta d)$

There is a short exact sequence, splitting as modules,

$$\Psi \to M\Phi \to X$$

We get a splitting exact sequence

$$\mathbb{H}om(A, \Psi) \rightarrowtail \mathbb{H}om(A, M\Phi) \twoheadrightarrow \mathbb{H}om(A, X)$$

So

<u>Definition</u> $H^n(\mathrm{Hom}(A, M\Phi)) = \begin{cases} \mathrm{Ext}^n(A, \phi) & n \geq 1 \\ \Pi_{-n}(A, \phi), & n \leq 0 \end{cases}$

(defining the things on the right).

This definition of $\mathrm{Ext}^n(A, \phi)$, $\Pi_n(A, \phi)$ gives the right exact sequences. It can be shown to be independent of choices of resolutions.

One can define an algebraic analogue to the topological mapping cone: Given $\phi : X \to Y$ define $C\phi$ as making the square

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail] & Y \arrow[d] \\ IX \arrow[r] & C\phi \end{tikzcd}$$

bicartesian (IX arbitrary injective containing X)

In fact we have also a $K\phi$ by dual construction,

$$\begin{tikzcd} K\phi \arrow[r] \arrow[d, two heads] & PY \arrow[d, two heads] \\ X \arrow[r, "\phi"] \arrow[d, tail] & Y \arrow[d, tail] \\ IX \arrow[r, "\sim"] & C\phi \end{tikzcd}$$

all squares bicartesian.

We have
$$\underline{\Pi}_n(A,\phi) \simeq \underline{\Pi}_{n-1}(A, K\phi), \quad n \geq 1.$$

The bicartesian diagram above gives rise to a short exact sequence of a sort

$$K_\phi \rightarrowtail IX \oplus PY \twoheadrightarrow C\phi$$

which can be used to get long sequences

$$\underline{\Pi}_0(A, IX) \to \underline{\Pi}_0(A, C\phi) \xrightarrow{w} Ext^1(A, K\phi) \to Ext^1(A, PY)$$

$$\underline{\Pi}_0(A,\phi) = \mathrm{Im}\, w.$$

Im w is invariant even though the sequence is highly non-invariant.

From the previous characterizations

$$Ext^n(A,\phi) \cong Ext^n(A, C\phi), \quad n \geq 1$$
$$\Pi_n(A,\phi) \cong \Pi_{n-1}(A, K\phi)$$
$$\Pi_0(A,\phi) \cong \mathrm{Im}\, w$$

One gets functoriality.

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "\phi'"] & Y' \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, tail, "\iota"] & IX \arrow[r, two heads, "\xi"] & \Sigma X \\ X \arrow[u, "\wr"] \arrow[r, tail, "\{\phi,\iota\}"] \arrow[d] & Y \oplus IX \arrow[u, two heads, "\pi"'] \arrow[r, two heads, "\langle -\lambda, \psi \rangle"] \arrow[d, "\beta \oplus I\alpha"] & C\phi \arrow[u, two heads, "\tau"'] \arrow[d, "\sigma"] \\ X' \arrow[r, tail, "\{\phi',\iota'\}"] & Y' \oplus IX' \arrow[r, two heads, "\langle -\lambda', \psi' \rangle"'] & C\phi' \end{tikzcd}$$

Theorem: $\sigma_* : \mathrm{Ext}^n(A, C\phi) \to \mathrm{Ext}^n(A, C\phi')$
independent of the choice of $I\alpha$.

Proof. $\bar{I}\alpha$, $(\bar{I}\alpha - I\alpha)\iota = 0$

$$\bar{\sigma}, \; (\bar{\sigma} - \sigma)\langle -\lambda, \psi\rangle = \langle -\lambda', \psi'\rangle (0 \oplus \bar{I}\alpha - I\alpha))$$
$$= \psi'(\bar{I}\alpha - I\alpha)\pi$$
$$= \psi'\gamma\varepsilon\pi = \psi'\gamma\tau\langle -\lambda, \psi\rangle$$

$\bar{I}\alpha - I\alpha = \gamma\varepsilon$

$\langle -\lambda, \psi\rangle$ epi so can be cancelled,

and got $\bar{\sigma} - \sigma = \psi'\gamma\tau$ so first is functorial.

$$\begin{tikzcd}
\underline{\Pi}_0(A, C\phi) \arrow[r, two heads] \arrow[rr, bend left, "w"] \arrow[d, "\sigma_*"] & {} \arrow[r, tail] \arrow[d] & \mathcal{E}xt^1(A, K\phi) \arrow[d, "T_*"] \\
\underline{\Pi}_0(A, C\phi') \arrow[r, two heads] \arrow[rr, bend right] & {} \arrow[r, tail] & \mathrm{Ext}^1(A, K\phi')
\end{tikzcd}$$

arrow down the middle is invariant

---

1) $\mathrm{Ext}^n(A, B) \simeq \mathrm{Ext}^{n-1}(\Omega A, B) \simeq \mathrm{Ext}^{n-1}(A, \Sigma B)$
$n \geq 2$

2) $\underline{\Pi}_n(A, B) \simeq \underline{\Pi}_{n-1}(A, \Omega B)$, $n \geq 1$

Do those formulas hold true when B is replaced by ϕ.

Clearly $Ext^n(A,B) \simeq Ext^{n-1}(\Omega A, B)$.

$Ext^n(A,\phi) \simeq Ext^{n-1}(A, \Sigma\phi)$

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "K"'] & Y \arrow[d, "3"] \\ Z \arrow[r, "\psi"'] & T \end{tikzcd} \qquad \begin{tikzcd} IX \arrow[r, "I\phi"] \arrow[d, "IK"] & IY \arrow[d, "I3"] \\ IZ \arrow[r, "I\psi"'] & IT \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail] & Y \arrow[d, tail] \\ IX \arrow[r] & C_\phi \end{tikzcd} \qquad I_0X \xrightarrow{I_0\phi} I_0Y$$

(I_0 means fixed arbitrary I, etc.)

$$\begin{tikzcd} \Sigma_0 X \arrow[r, "\Sigma_0\phi"] \arrow[d] & \Sigma_0 Y \arrow[d] \\ \Sigma I X \arrow[r] & \Sigma C_\phi \end{tikzcd}$$

bicartesian.

monic.

So can write $\Sigma C_\phi = C_{\Sigma\phi}$

(Suspend any map, it becomes monic)

$$\begin{tikzcd} IX \arrow[r, "I\phi"] \arrow[d, "IK"] & IY \\ IZ & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d, tail] & Z \arrow[d, tail] \\ IX \arrow[r] & C_Z \end{tikzcd}$$

(push out)

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail] & Z \arrow[d, tail] \\ IX \arrow[r] & IZ \end{tikzcd}$$

still pull back
since followed by monic.
(IZ injective container of C_Z?)

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pull back}"] & Z \arrow[d] \\ IX \arrow[r] \arrow[d, two heads] & IZ \arrow[d, two heads] \\ \Sigma X \arrow[r, tail] & \Sigma Z \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, "c_X"'] \arrow[dr, phantom, "\text{pull back}"] & Y \oplus Z \arrow[r, two heads] \arrow[d, tail, "c_Y \oplus c_Z"] & T \arrow[d] \\ IX \arrow[r, tail] & IY \oplus IZ \arrow[r, two heads] & U \end{tikzcd}$$

So can write U as IT

Take a pull back compare with another square with same left hand edge, then still pull back.

Cocommutative Hopf Algebras

12-6-65

Dr. Moss Sweedle.

This discussion of graded Hopf Algebras is largely applicable to the case of ungraded Hopf Algebras.

We are interested in the situation

$$H \simeq H^e \otimes \Gamma(G)$$

H^e connected Hopf algebra, $\Gamma(G)$ of grading 0.

Let d, m be the diagonal, multiplication maps associated to a Hopf algebra H. We define a monoid

$$G(H) = \{x \overset{\in H}{} \mid dx = x \otimes x\}$$

Assume H cocommutative. There is an antipode

$$S : H \to H$$

such that

$$S * I = \varepsilon = I * S$$

($*$ = convolution ; $f * g = m \circ f \otimes g \circ d$)

Given $x \in G = G(H)$ set $x^{-1} = S(x)$.

H is said to be split if and only if all simple subcoalgebras are 1-dimensional.

If H is split there is 1-1 correspondence between its simple subcoalgebras and the elements of $G(H)$.

Let $\Gamma(G) = k\cdot G$, the group ring of G over the ground field k.

If

1) H is cocommutative
2) There is an antipode $S: H \to H$
3) H split

then
$$H \simeq H^e \otimes \Gamma(G),$$
H^e = unique maximal subcoalgebra such that
$$H^e \cap G = \{e\}.$$
The isomorphism will be essentially
$$H^e \otimes \Gamma(G) \xrightarrow{m} H.$$

Ungraded Case.

Let $H^* = \mathrm{Hom}_k(H, k)$

Let $J \subset H^* = G^\perp$

$(J^{i+1})^\perp = H_i$

$H = \cup H_i$

$H_i H_j \subset H_{i+j}$

$H_i = \bigoplus_{g\in G} H_i^g, \quad H^g \cap G = \{g\}.$

H^e cocommutative, split

$\Gamma = G$ char $k = 0$

$H = U(L)$

$\therefore$ H gradable

Extended Birkhoff-Witt Theorems

If H is a Hopf algebra over k char $k = 0$

let $L^E = \{ l_\alpha \}$, $L^O = \{ l_\beta \}$

be the primitives of even, odd dimension respectively

then $\{ \pi\, l_\alpha^{e_\alpha} l_\beta^{\varepsilon_\beta} \}$ e_α integers, ε_β $0, 1$.

give a vector space basis for H.

In case k is a perfect field of characteristic p we have

$$L^E = L_0 \supset L_1 \supset L_2 \cdots$$

with $x \in L_n \iff x$ has $p^{n\text{th}}$ divided powers

The basis becomes

$$\pi\, {}^{e_\alpha}l_\alpha\, l_\beta^{\varepsilon_\beta}$$

where each e_α is an integer for which the divided power of l_α exists and the ${}^{e_\alpha}l_\alpha$ are assigned fixed values. (A priori divided power are not well defined).

Note that

$$d^e l = \Sigma_{i=0}^{e} {}^{i}l \otimes {}^{e-i}l$$

Note that if all primitives are odd then H is primitively generated

A Conjecture

Let A = cocomm. Hopf algebra
H = comm. algebra $/A$
$A \otimes H \xrightarrow{\psi} H$

$$\mathrm{Hom}_R(A, H) = \oplus_i \mathrm{Hom}_R(A_i, H_i)$$

$\mathrm{Reg}(A, H)$ = { invertible elements of $\mathrm{Hom}(A, H)$

$$\mathrm{Reg}^n(A, H) = \mathrm{Reg}(A \overset{n \text{ times}}{\otimes \cdots \otimes} A, H)$$

$$\mathrm{Reg}^n(A, H) \xrightarrow{\delta^n} \mathrm{Reg}^{n+1}(A, H)$$

is defined by

$$\delta^n f = \psi \circ I \otimes f * \prod_{i=1}^{n} f^{(-1)^i} * I^{[i-1]} \otimes m \otimes I^{[n-i]} * f^{(-1)^{n+1}} \otimes \epsilon.$$

(giving us cohomology $H(A, H)$

Suppose $A = \Gamma(G)$, H = algebra, H^r = regular elements of H

$$H = k + \Sigma I^2 = 0$$

$$H^n(H) = H^n(k) \oplus \mathrm{Ext}_A^n(k, I).$$

Conjecture: $H^s(H(S^1)) = H^s(k) \oplus \mathrm{Ext}_{a(p)}^{s,t}(Z_p, Z_p)$.

Stability of Differentiable Maps

John Mather, Princeton U.

12-13-65

Note: All mappings will be C^∞ even if this is not explicitly stated.

Definition. Given $f: M \to N$, $f': M' \to N'$ C^∞ mappings we say that f is equivalent to f' if and only if $\exists$ C^∞ diffeomorphisms $h: M \to M'$, $h': N \to N'$ such that

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ h\downarrow & & \downarrow h' \\ M' & \xrightarrow{f'} & N' \end{array}$$

commutes.

Definition. Given $f: M \to N$ C^∞, M compact, we say f is C^∞-structurally stable (stable for short), if there exists a neighborhood U of f in the C^∞ topology such that every $g \in U$ is equivalent to f.

Problem. Find necessary and sufficient conditions for stability of mappings.

Definition M, N C^∞ manifolds, $S \subseteq M$, $T \subseteq N$ subsets of M, N. By a C^∞-map germ

$$f : (M, S) \dashrightarrow (N, T)$$

we mean an equivalence class of maps

$\tilde{f} : U \to N$ where U is a neighborhood of S in M and $f[S] \subseteq T$; two maps $\tilde{f}_1$ and $\tilde{f}_2$ are equivalent if they agree on some neighborhood V of S.

We use $f : (M, S) \dashrightarrow N$ as an abbreviation of $f : (M, S) \dashrightarrow (N, T)$.

We compose map germs with other map germs or with mappings in the obvious way.

If $f : (M, S) \dashrightarrow (N, T)$ has an inverse with respect to composition, then we say that f is a diffeomorphism germ.

Definition Given M, N, C^∞ manifolds, $S \subseteq M$, $T \subseteq N$ subsets of M, N, we mean by a map-k-jet $f : (M, S) \xrightarrow{k} (N, T)$ an equivalence class of maps $\tilde{f} : U \to N$ where U is a neighborhood of S and $f[S] \subseteq T$. Two maps $\tilde{f}_1 : U \to N$

$f_2 : U \dashrightarrow N$

are equivalent if they and their derivatives up to order k coincide on S.

We use $f : (M,S) -k\to N$ as an abbreviation for $f : (M,S) \to k \to (N,T)$.

We compose map k-jets with map germs or with mappings in the obvious way.

If $f : (M,S) -k\to (N,T)$ has an inverse (with respect to composition) we say that f is a <u>diffeomorphism - k - jet</u>.

<u>Notation</u> $\Sigma_p M$ = manifold of all subsets of M with p-points. ${}_pJ^k(M,N)$ is a bundle over $\Sigma_p M$; given $S \in \Sigma_p M$, the fibre of ${}_pJ^k(M,N)$ over S is the set of k-jets $f : (M,S) -k\to N$.

Given a C^∞ map $f : M \to N$ we define

${}_pj^k(f) : \Sigma_p M \to {}_pJ^k(M,N)$ by letting

${}_pj^k(f)(S)$ be the k-jet of f at S.

(Then ${}_pj^k(f)(S) : (M,S) -k\to N$.)

Theorem. M,N C^∞ manifolds $f : M \to N$ a C^∞-mapping (M compact) Than the following conditions are equivalent:

a) f is stable

b) ${}_p j^k(f)$ is tranversal to all minimal invariant submanifolds of ${}_p J^k(M,N)$.

c) f is infinitesimally stable

d) f is homotopy stable

e) f is infinitesimally$_I$ stable.

(See definitions below for meanings of conditions b) – e).)

<u>Definition</u>. Two k-jets $y_i : (M, S_i) -k\to N \quad i=1,2$ are <u>equivalent</u> if there exist diffeomorphism k-jets

$$z : (M, S_1) -k\to (M, S_2)$$
$$z' : (N, f[S_1]) -k\to (N, f[S_2])$$

such that

$$y_2 = z' \circ y_1 \circ z$$

<u>Definition</u> A subset X of ${}_p J^k(M,N)$ is invariant $\Leftrightarrow$ $z \in {}_p J^k(M,N)$, $z' \in X$ and z equivalent to $z' \Rightarrow z \in X$.

<u>Lemma</u> If X is a minimal invariant subset of ${}_p J^k(M,N)$ then X is a submanifold.

<u>Proof</u>. By local triviality it suffices to prove that if $\varphi : S \to N$ is any map of a subset S of M (with p points) into N then $X \cap J^k_{S,\varphi}(M,N)$ is a submanifold where $J^k_{S,\varphi}(M,N)$ is the set of all

map-k-jets

$z:(M,S)\xrightarrow{k} N$ such that $z|S=\varphi$ (where $z|S$ is defined to be $f|S$ for any representative map f for z.) (Note that $J^k_{S,\varphi}(M,N)$ is an algebraic manifold, isomorphic to Euclidean space.) But the group of all diffeomorphism-k-jets $z:(M,S)\xrightarrow{k}(M,S)$ such that z leaves S pointwise fixed acts on $J^k_{S,\varphi}(M,N)$ on the right and the group of all diffeomorphism-k-jets

$z':(N,f[S])\xrightarrow{k}(N,f[S])$

leaving $f[S]$ pointwise fixed acts on

$J^k_{S,\varphi}(M,N)$ on the left. They are both Lie groups and their actions on $J^k_{S,\varphi}(M,N)$ are differentiable, so any orbit of their combined action is a C^∞-submanifold. But it is easily seen that $X\cap J^k_{S,\varphi}(M,N)$ is a finite union of orbits, and therefore a C^∞-submanifold.

From the lemma, it follows that condition (b) makes sense.

Condition (c) is given by the following definition

<u>Definition</u> We say f is infinitesimally stable if, given any map $s: M \longrightarrow TN$ such that $\pi\circ s=f$, where $\pi: TN\to N$ is the projection, there exist sections ξ of TM over M and ξ' of TN over N such that

$$f_*(\xi) = \tau f \circ \xi + \xi' \circ f$$

where τf is the tangent map to f. This condition may be expressed by a non-commutative diagram

$$\begin{tikzcd} \tau M \arrow[r, "\tau f"] & \tau N \\ M \arrow[u, "\xi"] \arrow[r, "f"] \arrow[ur, "S"] & N \arrow[u, "\xi'"'] \end{tikzcd}$$

For example if f is an immersion f is infinitesimally stable if and only if f is in <u>general position</u> i.e. if $y \in N$, $x_1, \ldots, x_p \in M$ and $f(x_i) = y$ $i = 1, \ldots p$ then

$$\operatorname{codim}\left(\bigcap_{i=1}^{p} \tau_{x_i} f(T_{x_i} M)\right) = p(n-m)$$

where $n = \dim N$, $m = \dim M$.

We say $F: M \times I \to N \times I$ is <u>level preserving</u> if the diagram

$$\begin{tikzcd} M \times I \arrow[rr] \arrow[dr, "\pi"'] & & N \times I \arrow[dl, "\pi"] \\ & I & \end{tikzcd}$$

commutes. If $H: M \times I \to M \times I$ is a level preserving diffeomorphism and H_0 $(= H|M \times \{0\}: M \to M)$ is the identity, then we say that H is an <u>isotopy</u>.

Condition (d) is given by the following definition.

<u>Definition</u> We say f is homotopy stable, if,

given any level preserving map,

$F: M\times I \to N\times I$ (I = unit interval $[0,1]$), such that $F_0 = f$, there exist isotopies

$H: M\times I \to M\times I$

$H': N\times I \to N\times I$

such that for some $\varepsilon > 0$

$$f_0 \times id \,|\, M\times[0,\varepsilon] = H' \circ F \circ H \,|\, M\times[0,\varepsilon]$$

Condition (e) is given by the following definition.

<u>Definition</u> We say f is infinitesimally$_I$ stable if given any level preserving map $F: M\times I \to N\times I$ such that $F_0 = f$, and any map $S: M\times I \to TN\times I$ such that $\pi \cdot S = F$ where $\pi: TN\times I \to N\times I$ is the projection, there exists a C^∞ section ξ' of $TN\times I$ over $N\times I$ such that for some $\varepsilon > 0$ and all $t \in [0,\varepsilon]$,

$$S_t = TF_t\,\xi_t + \xi'_t \circ F_t : M \to TN$$

<u>Proof of Theorem 1.</u> a) ⇒ b) follows from a generalization of Thom's transversality theorem. The generalization states that if $f: M\to N$ is any C^∞ map and U is any submanifold of ${}_pJ^k(M,N)$ then there exist C^∞ maps $g: M\to N$, arbitrarily

close to f in the C^∞-topology, such that ${}_\rho j^k(g)$ is transversal to U. If we assume that g is stable then for g close enough to f, g is equivalent to f. If U is an invariant submanifold, then g equivalent to f and transversal to U imply that f is transversal to U.

To show (b) $\Leftrightarrow$ (c) we need to introduce some further terminology. We let $C^\infty(M)$ denote the ring of C^∞ functions on M. We define $C^\infty(N)$ similarly. We let $\Phi(M)$ denote the module of C^∞-sections of τM over M. We define $\Phi(N)$ similarly. We let $\Phi(f)$ denote the module of C^∞-maps $\xi : M \to \tau N$ such that $\pi \circ \xi = f$, where $\pi : \tau N \to N$ is the projection. We define $f_* : \Phi(M) \to \Phi(f)$ by $f_*(\xi) = \tau f \circ \xi$. We define $\omega(f) : \Phi(N) \to \Phi(f)$ by $\omega(f)(\xi') = \xi' \circ f$. Then, by definition (c) holds if and only if

$$(1) \quad f_* + \omega(f) : \Phi(M) \oplus \Phi(N) \longrightarrow \Phi(f)$$

is onto.

Note that $\Phi(M)$ and $\Phi(f)$ are $C_*(M)$ modules, $\Phi(N)$ is a $C(N)$ module, f_* is a $C(M)$ module homomorphism, and $\omega(f)$ is a homomorphism over the ring homomorphism $f^* : C(N) \to C(M)$

For any finite set S in M and any non-negative integer k, let $\mathfrak{M}_S^{k+1}(M)$ denote the ideal in $C(M)$ of functions vanishing with their derivatives of order $\leq k$ on S.

Lemma 1 Let $S \in \Sigma_p M$. Then ${}_pj^k(f)$ is transversal to minimal invariant submanifold of ${}_pJ^k(M,N)$ containing ${}_pj^k(f)(S)$ if and only if

$$(2) \quad f_*[\Phi(M)] + \omega(f)[\Phi(N)] + R_s^{k+1} \cdot \Phi(f) = \Phi(f)$$

This lemma is proven by a calculation which we omit here.

This (6) is equivalent to (2) holding at each subset S of M having p points where $p \geq n+1$ and $k \geq n$ then (1) holds. This implication is a consequence of Malgrange's preparation theorem and Nakayama's lemma, and some fiddling with partitions of unity.

(b) ⟺ (e) is proved in the same way as (b) ⟺ (c)

To show (e) ⟹ (d) one needs the following lemma due to Thom (Bonn seminar, 1958, notes by H. Levine) If $F: M \times I \to N \times I$ we let $DF: M \times I \to TN \times I$ be the C^∞-maps defined by

$$DF(x, t_0) = \left(\frac{dF_t(x)}{dt}\Big|_{t=t_0}, t_0 \right)$$

where $dF_t(x)/dt\,|_{t=t_0}$ denotes the tangent vector at $t=t_0$

to the curve $t \to F_t(x)$ in N.

Lemma Let $f: M \to N$ be a C^∞ map. Let $F: M\times I \to N \times I$ be a C^∞ level preserving map such that $F_0 = f$. Let $H: M\times I \to M\times I$, $H': N\times I \to N\times I$ be isotopies. Then

$$f = H'_t \circ F_t \circ H_t \qquad \text{for } 0 \le t \le \varepsilon$$

if and only if

$$(DF)_t = -\tau F_t \circ ((DH)_t \circ H_t^{-1}) + ((DH'^{-1})_t \circ H'_t) \circ F_t$$

for $0 \le t \le \varepsilon$

We show (e) $\Rightarrow$ (d) as follows. Since $\pi \circ DF = F$ where $\pi: \tau N \times I \to N\times I$ is the projection, the assumption that F is infinitesimally stable implies that there exists sections ξ of $TM \times I$ over $M\times I$, ξ' of $TN \times I$ over $N\times I$ such that for some $\varepsilon > 0$ and all $t \in [0, \varepsilon]$,

$$(DF)_t = \tau F_t \circ \xi_t + \xi'_t \circ F_t : M \to \tau N$$

It is easily seen that ξ' may be chosen so as to vanish outside a compact subset of $M\times I$. Then, by the ordinary existence and continuity theorem for solutions of ordinary differential equations, there exist isotopies

$$H : M \times I \to M \times I$$
$$H' : N \times I \to N \times I$$

such that

$$(DH)_t \circ H_t^{-1} = -\xi_t \qquad t \in I$$
$$(DH'^{-1})_t \circ H'_t = \xi'_t \qquad t \in I$$

From lemma 2, it follows immediately that

$$f = H'_t \circ F_t \circ H_t \qquad 0 \leq t \leq \epsilon$$

showing that f is indeed homotopy stable.

The proof that (d) ⟹ (c) is easy, using lemma 2.

We have indicated the proofs of (a) ⟹ (b), (b) ⟺ (c), (b) ⟺ (e), (e) ⟹ (d) and (d) ⟹ (c). In particular these implications show that (b), (c), (d), (e) are equivalent. To show (b) ⟹ (a) we follow a procedure analogous to the proof of (b) ⟹ (d) only somewhat more technical.

Tangent Bundles over Homogeneous Space

2-14-66

Prof D.W. Anderson, M.I.T.

G = connected Lie Group
H = closed subgroup

$G/H = G \times_H pt.$ is a differentiable manifold

$\mathfrak{g}$ = Lie algebra of G = tangent space to $\{e\}$
$\mathfrak{h}$ = Lie algebra of H

If $g \in G$ $\quad L_g : G \to G$
by $L_g(x) = gx$;
$R_g(x) = xg$

$$L_{g*} : T_e \to T_g$$
$$R_{g*}^{-1} : T_g \to T_e$$

$$Ad(g) : \mathfrak{g} \to \mathfrak{g}$$

is defined to be $L_{g*} R_{g*}^{-1}$

$Ad : G \to GL(\mathfrak{g})$
$Ad|H$ has $\mathfrak{h}$ as an invariant subspace

$\therefore$ we have a representation

$$Ad_G : H \to GL(\mathfrak{g}/\mathfrak{h})$$

Theorem: The Tangent bundle of G/H is

$$G \times_H (\mathfrak{g}/\mathfrak{h})$$

Define

$$L : G \times \mathfrak{g} \to T(G)$$

by

$$L(g, k) = L_{g*}(k)$$

If

$$\pi : G \to G/H$$

the projection we have

$$\pi_* L : G \times \mathfrak{g} \to T(G/H)$$

The inclusion $H \times \mathfrak{h} \to G \times \mathfrak{g}$ composed with $\pi_* L$ is constant.

$\therefore$ $\pi_* L$ factors throug

$$G \times \mathfrak{g} \to G \times_H (\mathfrak{g}/\mathfrak{h}) \to T(G/H)$$

<u>Theorem</u> The normal bundle of G/H is $G \times_H \mathfrak{h}$.

<u>Proof</u>. The sequence

$$0 \to G \times_H \mathfrak{h} \to G \times_H \mathfrak{g} \to G \times_H (\mathfrak{g}/\mathfrak{h}) \to 0$$

is exact. Thus it suffices to see that $G \times_H \mathfrak{g}$ is trivial over G/H.

Define $G/H \times \mathfrak{g} \to G \times_H \mathfrak{g}$ by

$$([g], k) \to [g, k]$$

Λ = field

$R_\Lambda(G)$ = the group generated by isomorphism classes of the finite dimensional Λ-linear representation of G.

$$\alpha : R_\Lambda(G) \to R_\Lambda(BG)$$

$$\rho : G \to GL(\Lambda, n)$$

give

$$\alpha(\rho) : BG \to BGL(\Lambda, n)$$

We have a fibration
$$\begin{array}{ccc} G/H & \xrightarrow{i} & BH \\ & & \downarrow \pi \\ & & BG \end{array}$$

Let EG be a contractible space upon which G acts freely. Then H acts freely on EG,

so $BG \leftarrow EG \times_H pt$.

Thus G acts on BH; $BH \times_G pt = EG \times_G pt = BG$

Suppose $\rho : H \to GL_\Lambda(V)$

Then $i^* \alpha(\rho) = G \times_H V$

in $K_\Lambda(G/H)$

Notice that the action of H on G extends to G. Thus $\alpha(Ad\ g) \in K_R(BH)$ lies in the image of $K_R(BG)$.

Since πi is constant

$$i^* \pi^* (\alpha(\rho)) = \dim(\rho)$$

$$\underline{\alpha(\operatorname{Ad} \mathfrak{g}) \in \operatorname{Im} \pi^* \Rightarrow i^* \alpha(\operatorname{Ad} \mathfrak{g}) = \text{constant}}$$

From now on G compact, connected

T = maximal abelian subgroup.

__Theorem__. (Well known) $R_\Lambda(G) \to R_\Lambda(T)$ is injective.

__Theorem__ There are representations

$$\rho_1, \dots, \rho_k : T \to \mathcal{U}(\ )$$

such that $R_{\mathbb{C}}(T) = \mathbb{Z}[\alpha(\rho_1), \dots, \alpha(\rho_k)]$

$k = \dim T = \operatorname{rank}(G)$.

__Theorem__. G/T has a complex structure and is the sum $\sum_{i : (K_i) = \text{root}} \alpha(K_i)$, K_i 1-dimensional.

__Proof.__ $\operatorname{Ad}_G : T \to GL_{\mathbb{R}}(\mathfrak{g}/\mathfrak{t})$

as a T-module, $\mathfrak{g}/\mathfrak{t}$ is the direct sum of 2-dim invariant subspace, K_r

The only connected abelian subgroup of $SO(2)$ is $\mathcal{U}(1)$.

$$e^{i\theta} \to \begin{pmatrix} \cos\theta, & \sin\theta \\ -\sin\theta, & \cos\theta \end{pmatrix}$$

Note: The elements $c_1(\alpha(K_1)) \in H^2(BT; Z)$ are called the roots of G.

$H^2(BT; Z)$ is the group of weights of T.

Remark. If τ = tangent bundle of G/T with the indicated complex structure, $\tau \oplus \bar{\tau}$ is trivial.

Proof $\quad \mathfrak{g}/\tau \oplus \overline{\mathfrak{g}/\tau} \oplus t + \bar{t}/\tau$

$$= \mathfrak{g} \oplus \bar{\mathfrak{g}} / T$$

$$= \mathfrak{g} \otimes C / T$$

which extends to G.

Let X be the principal $SO(n)$ bundle over $SO(n+m)/SO(n) \times SO(m)$

The tangent bundle of $SO(n+m)/SO(n) \times SO(m)$ is $\lambda \otimes \mu$.

Lie algebra of $SO(n+m)$ = all $(n+m)\times(n+m)$ matrices $A \ni A + A^t = 0$.

An Unstable Adams Spectral Sequence

2-28-66

David Rector, M.I.T.

This discussion will compare two methods of approximating to homotopy groups from homology information: The Adams Spectral Sequence and the spectral sequence now being studied by Profs. Kan, Curtis, et. al..

Recall that Adams Spectral Sequence is

(i) defined only in the stable range

(ii) converges to $\pi_*(X;p)$, a quotient of $\pi_*(X)$ which is roughly the p-primary and integral component of $\pi_*(X)$.

(iii) E^2 is a Z_p module determined by $H^*(X;Z_p)$ and the action thereon of the Steenrod Algebra $\mathcal{Q}(p)$.

Curtis's spectral sequence is:

(i) defined unstably as well as stably

(ii) converges to $\pi_*(X)$ if X is simply connected

(iii) E^1 depends only on $H^*(X)$.

Recall the category of semi-simplicial group-complexes, the combinatorial analogue of the category of topological groups. There is a group-homotopy relation on group complexes which is the analogue of the group homotopy relation on topological groups.

If T is a functor on groups, the induced functor T on group-complexes will preserve the group homotopy relation.

Let X be a connected s.s complex with base-point — for example, the singular complex of a path connected space. X has a s.s loop complex GX with the following properties

(i) GX is a free group complex

(ii) $\pi_n GX = \pi_{n+1} X$

(iii) If $AX = GX/[GX,GX]$ is the abelianization of GX, then $\pi_n AX = H_{n+1}(X)$; the projection $GX \to AX$ induces the Hurewicz homomorphism.

(iv) If X, Y have the same homotopy type then GX and GY have the same group homotopy type.

Let G be a group, D the functor such that

$$DG = \{[a,b], a^p \mid a, b \in G\}$$

GX may be filtered by iterated applications of D;

$$D^n GX \subseteq D^{n-1} GX \subseteq \ldots \subseteq D^2 GX \subseteq D^1 GX \subseteq GX$$

This filtered group complex gives rise to a homotopy exact couple; the spectral sequence derived from this exact couple is, in the stable range, the Adams Spectral Sequence.

The homotopy groups of the successive quotients of this filtration are determined only by $H_*(X; Z_p)$ in the stable range; in the unstable range they are, unfortunately, not easily accessible

Curtis' spectral sequence is defined in a similar manner using the lower central series of GX

$$\Gamma_r^u GX \subseteq \Gamma_{r-1}^u GX \subseteq \dots \subseteq \Gamma_2^u GX \subseteq \Gamma_1^u GX \subseteq GX$$

where $\Gamma_r^u G$ is the r-commutator subgroup of G. This sequence has the property that

$$[\Gamma_r^u G, \Gamma_s^u G] \subseteq \Gamma_{r+s}^u G$$

and is the smallest sequence with this property.

As above, this filtration defines a spectral sequence. A formidable result of Curtis is that this sequence converges to $\pi_*(X)$. The homotopy groups of the quotients of this filtration depend unstably as well as stably on $H_*(X)$.

Now since the integral sequence is difficult to handle we should like to have a mod-p version converging in the sense of the Adams spectral sequence. We will such a sequence which

(i) converges (unitably) to $\pi_*(X;p)$
(ii) E^1 depends only on $H_*(X;Z_p)$
(iii) A stable version coincides with the adams spectral sequence.

We will use the lower p-central series of GX

$$\subseteq \Gamma_r GX \subseteq \Gamma_{r-1} GX \subseteq \dots \subseteq \Gamma_2 GX \subseteq \Gamma_1 GX \subseteq GX.$$

where
$$\Gamma_r G = \{ [a_1, \dots, a_k]^{p^i} \mid a_i \in G,\ k \geq 1,\ kp^i \geq r \}$$

($[a_1, \dots, a_k]$ ~~the~~ k-commutator)

This sequence has the property that
$$[\Gamma_r G, \Gamma_s G] \subseteq \Gamma_{r+s} G \quad \text{and} \quad (\Gamma_r)^p \subseteq \Gamma_{pr} G$$
and is the smallest sequence with this property. Let us denote by $\{E^r X\}$ the spectral sequence of the exact homotopy couple of GX filtered by its lower p-central series.

To show that $E^1 X$ depends only on $H_*(X, Z_p)$ we shall need the free restricted Lie algebra functor.

A restricted Lie algebra is a Lie algebra with a (not necessarily linear) operation
$$a \to a^{[p]}$$
satisfying
$$[a, b^{[p]}] = [a, \underbrace{b, \dots, b}_{p \text{ times}}]$$

Example. Let A be an associative algebra; define $[a, b] = ab - ba$, $a^{[p]} = a^p$. Then A becomes a restricted Lie algebra. In fact all rest. Lie-algebras may be obtained as sub-objects of the rest. Lie algebra associated with an associative algebra.

If M is a Z_p-module, there is a free rest. Lie alg.

LM on M. L is a covariant functor of M.

We have that

$$\Sigma_{r\geq 1}\Gamma_r G/\Gamma_{r+1}G$$

is a rest. Lie algebra and, if G is free,

$$\Sigma_{r\geq 1}\Gamma_r G/\Gamma_{r+1}G = L(G/\Gamma_2 G)$$

Now $\Gamma_2 GX = \{[a,b], a^p \mid a,b \in G\}$,

so that $GX/\Gamma_2 GX \simeq AGX \otimes Z_p$.

Hence $\pi_n(GX/\Gamma_2 GX) \simeq H_{n+1}(X;Z_p)$.

Now the group homotopy type of a Z_p-module complex depends only on its homotopy groups. Hence the group homotopy type of $GX/\Gamma_2 GX$ depends only on $H_*(X;Z_p)$. By a theorem of Dold induced functors such as L preserve group homotopy type. Hence $\pi_* L(GX/\Gamma_2 GX)$ depends only on $H_*(X;Z_p)$; the same may be said of $E^1 X$.

We will now examine this sequence in the stable range and consider its relation to the Adams spectral sequence. The following results are due to various members of the group studying this spectral sequence and are proved in a forthcoming paper. For technical simplicity we consider only $p=2$; analogous but more complicated results will hold for p odd. We will confine our interest to the stable range; that is, we will assume X is a spectrum.

Prop. I. $\pi_*(\Gamma_r GX/\Gamma_{r+1} GX) = 0$ for $r \neq 2^i$

Note that this is not true unstably. This permits us to modify the spectral sequence in order to obtain a better E^2 term. We use the filtration

$$\subseteq \Gamma_{2^i} GX \subseteq \ldots \subseteq \Gamma_4 GX \subseteq \Gamma_2 GX \subseteq GX.$$

The resulting E^1 term is the same as with the full filtration except for an index change. The E^2 term is the same as for the Adams Spectral Sequence.

The following are mostly soon to be published results of the group studying this spectral sequence

Let S be the sphere spectrum.

E^1S has the properties that

(i) E^1S is an algebra under composition.

(ii) d^1 is a derivation

(iii) E^1S is generated under composition by elements λ_i $i = 0, 1, 2 \ldots$ with stable degree $\lambda_i = i$.

(iv) A free additive base is given by monomials

$$\lambda_{i_1} \cdots \lambda_{i_m} \quad \text{with} \quad i_j \leq 2 i_{j+1}$$

(v) $\lambda_{n-1} \xrightarrow{d_1} \sum_{i+j=n} (i, j)\, \lambda_{i-1} \lambda_{j-1}$ $\quad (n \geq 2)$

(vi)

for $m \geq 1$, $n \geq 0$

$$0 = \sum_{i+j=n} (i\ j)\, \lambda_{i-1+m}\, \lambda_{j-1+2m}$$

E^1X is an E^1S module and d^1 is a derivation

$E^1X = H_*(X; Z_2) \otimes E^1S$

To know d^1 it is necessary only to know it on $H_*(X; Z_2)$. For $\alpha \in H_*(X; Z_2)$.

$$d'\alpha = \Sigma_i\, \alpha Sq^i \otimes \lambda_{i-1}$$

where Sq^i is dual to corresponding Steenrod operations.

Thus E^2 depends only on $H_*(X;Z_2)$ and actions of the Steenrod Algebra. And in fact

$E^2X, E^3X \ldots$ correspond with the corresponding terms of the Adams Spectral Sequence.

Unstably the situation is more complicated. Prop. I no longer holds, so the modified sequence used above is no longer suitable. For $X = S^n$, the n-sphere, Prop. I does hold, and the modification may still be used. In that case

$$E'S^n \subseteq E'S$$

and there is a filtration

$$E'S^2 \subset E'S^3 \subseteq \ldots \subseteq E'S^n \subseteq \ldots E'S$$

and d' preserves this filtration so that

$$d' \cdot S^n = d'S \,|\, E'S^n$$

For X general Prop. I fails;

$$E'X = \pi_* L(AGX \otimes Z_2)$$

The restricted Lie algebra structure on L gives $E'X$ the structure of a graded restricted Lie algebra. The differentials of the spectral sequence respect the Lie operation. That operation gives the Whitehead product on $E^\infty X$. The p-th power operation is not carried to E^2X

The following results due to A. K. Bousfield

Let $S^{-1}H_*(X;Z_2)$ denote $H_*(X;Z_2)$ with gradation reduced by one. There is a natural imbedding

$$S^{-1}H_*(X;Z_2) \to E^1X$$

as a direct summand. If L^G denotes the free graded restricted Lie algebra then there is also a natural imbedding

$$L^G S^{-1}H_*(X;Z_2) \to E^1X$$

a direct summand. E^1X is a kind of unstable E's module under suspension and E^1X is the free such object on $L^G S^{-1}H_*(X;Z_2)$, that is, if $\lambda_{i_1}\cdots\lambda_{i_n} \in E$'s is admissible then it operates linearly on all elements of E^1X of dim $> i_1$. Each $\lambda_{i_1}\cdots\lambda_{i_n}$ maps the elements of dim $> h$ of $L^G S^{-1}H_*(X;Z_2)$ isomorphically onto a direct summand of E^1X. E^1X has a natural direct sum decomposition as these images for all $\lambda_{i_1}\cdots\lambda_{i_n}$. The Lie product of two elements of E^1X is non-zero if and only if both elements are in the summand $L^G S^{-1}H_*(X;Z_2)$. d^1X is also zero on all other summands of $L^G S^{-1}H_*(X;Z_2)$. d^1X is the sum of a map resembling closely the stable d^1 of $H_*(X;Z_2)$ and a map defined as follows:

Let HX denote $S^{-1}H_*(X;\mathbb{Z}_2)$. The diagonal map induces a map

$$HX \xrightarrow{\Delta} HX \otimes HX$$

We may form a chain complex

$$(*) \quad HX \xrightarrow{\Delta} HX \otimes HX \xrightarrow{\Delta\otimes id + id\otimes\Delta} HX \otimes HX \otimes HX \longrightarrow$$

This makes Tens HX a cochain complex.

$L^G HX \subseteq HX$ as a subcomplex. The maps $(*)$ then give $d'X$ in $L^G S^{-1}H_*(X;\mathbb{Z}_2)$

The above conjecture has been substantially proved by Bousfield. The diagonal Δ is dual to the cup product. Hence $d'X$ is determined by the Steenrod operations and the cup product

Derived Functors

R. Knighten Topology Seminar 7 & 14 March 1966

Derived functors were introduced in Cartan & Eilenberg's book Homological Algebra published a decade ago. Since that time derived functors have amply shown there worth. And so, of course, they have been generalized and extended in a variety of ways.

The first generalization was in an appendix to Cartan-Eilenberg where Buchsbaum indicated how functors could be derived in what are now called abelian categories.

The most important derived functors were the Ext's, the derived functors of Hom, and in 1956 Hochschild had already started generalizing them. He was interested in the situation where K is a ring with identity and Λ is a K-algebra. In particular he was interested in classifying exact sequences of Λ-modules which split over K. These gave rise to the relative Ext groups and relative homological algebra was born.

This too was abstracted to a more general context by Heller (1958), Buchsbaum (1959, 1960), Butler & Horrocks (1961) and finally by Eilenberg and Moore (1965).

Another direction of generalization was the attempt to eliminate the need for projectives or injectives. This started with Yoneda's (1954) definition of $\mathrm{Ext}^n(A,B)$ as equivalence classes of exact sequences of length n starting at A and ending at B. A suggestive step was Godement's (1958) use of "flasque" resolutions in his discussion of the cohomology of sheaves.

The first real hint of a systematic theory was in Cartier's (1958) account of Yoneda's work and Gabriel's (1962) offering of a "morsel" of Cartier's ideas. This particular line of development finally achieved reasonable success in Verdier's (1963) invention of the derived category of an abelian category.

In these two talks I want to briefly sketch a theory which combines the relative approach of Eilenberg and Moore with the derived category approach.

I begin by sketching some of the ideas from Eilenberg and Moore's memoir entitled "Foundations of relative homological algebra".

Let $\mathcal{A}$ be an additive category. I.e. for all A and B in $\mathcal{A}$, $\mathcal{A}(A, B)$, the set of morphisms from A to B, is an abelian group in a functorial way and for any pair of object A and B in $\mathcal{A}$ there sum and product exist (and, of course, are canonically isomorphic.) Notation: $A \underset{p_A}{\overset{i_A}{\rightleftarrows}} A \oplus B \underset{p_B}{\overset{i_B}{\leftrightarrows}} B$.
Also $\mathcal{A}$ has a zero object, 0.

$A_3 \to A_2 \to A_1$ in $\mathcal{A}$ is a <u>sequence</u> if the composition is the zero map. The notion of a sequence extends to longer diagrams. In particular a sequence which is infinte in both directions is a <u>complex</u>.
$A_\cdot = \cdots \to A_n \to A_{n-1} \to \cdots$ is a <u>positive complex</u> if $A_n = 0$ for $n \ll 0$. $A_\cdot$ is a <u>negative complex</u> if $A_n = 0$ for $n \gg 0$. $A_\cdot$ is a <u>bounded complex</u>

if $A_n = 0$ for all but a finite number of n.

If $E.$ is a sequence in $\mathcal{A}$ and P is an object in $\mathcal{A}$, then $\mathcal{A}(P, E.)$ is a sequence of abelian groups.

Let $\mathcal{E}$ be a class of sequences in $\mathcal{A}$. Define

$$\underline{\mathcal{P}(\mathcal{E})} = \left\{ P \text{ in } \mathcal{A} \;\middle|\; \begin{array}{l} \mathcal{A}(P, E.) \text{ is an exact sequence} \\ \text{for all } E. \text{ in } \mathcal{E} \end{array} \right\}.$$

For $\mathcal{P}$ a class of objects in $\mathcal{A}$, define

$$\underline{\mathcal{E}(\mathcal{P})} = \left\{ E. \;\middle|\; \begin{array}{l} E. \text{ is a sequence in } \mathcal{A} \text{ and } \mathcal{A}(P, E.) \\ \text{is an exact sequence for all } E. \text{ in } \mathcal{E} \end{array} \right\}.$$

Write $\underline{\bar{\mathcal{E}}} = \mathcal{E}(\mathcal{P}(\mathcal{E}))$ and $\underline{\bar{\mathcal{P}}} = \mathcal{P}(\mathcal{E}(\mathcal{P}))$.

Then $\mathcal{E} \le \bar{\mathcal{E}} = \bar{\bar{\mathcal{E}}}$ and $\mathcal{P} \le \bar{\mathcal{P}} = \bar{\bar{\mathcal{P}}}$.

Called $\mathcal{E}$ a <u>closed class</u> if $\mathcal{E} = \bar{\mathcal{E}}$; similiarly call $\mathcal{P}$ a <u>closed class</u> if $\mathcal{P} = \bar{\mathcal{P}}$.

If $\mathcal{P} = \mathcal{P}(\mathcal{E})$ and $\mathcal{E} = \mathcal{E}(\mathcal{P})$, call the sequences in $\mathcal{E}$ <u>$\mathcal{P}$-exact</u> and the objects in $\mathcal{P}$ <u>$\mathcal{E}$-projectives.</u>

This notation is inspired by the "classical" situation: If $\mathcal{A}$ is an abelian category and $\mathcal{E}$ is the class of all exact sequences, then $\mathcal{P}$ is the class of all projectives. Also $\mathcal{E} = \mathcal{E}(\mathcal{P})$ in this case.

Note the following: If A is a retract of P, P in $\mathcal{P}$ with $\mathcal{P}$ closed, then A is in $\mathcal{P}$.
If $A = \coprod_\alpha A_\alpha$ (direct sum), then A is in $\mathcal{P}$ iff each A_α is in $\mathcal{P}$.

A class $\mathcal{E}$ of sequences is called a <u>projective class</u> iff (i) $\mathcal{E}$ is a closed class, and
(ii) If $A_2 \to A_1$ is in $\mathcal{A}$, then there is a $P \to A_2$ in $\mathcal{A}$ with P in $\mathcal{P} = \mathcal{P}(\mathcal{E})$ and $P \to A_2 \to A_1$.

In the case where $\mathcal{A}$ is an abelian category and $\mathcal{E}$ is the class of all exact sequences, $\mathcal{E}$ is a projective class iff $\mathcal{A}$ has enough projectives.

Now if $\mathcal{E}$ is a projective class in $\mathcal{A}$ the basic machinery for derived functors carries over just as in Cartan and Eilenberg. Define a <u>projective complex</u> (or <u>$\mathcal{P}$-complex</u>) $P_\cdot$ as one with P_n in $\mathcal{P}$ for all n. A <u>left complex</u> $A_\cdot$ <u>over</u> an object <u>A</u> consists in the complex $A_\cdot$ where $A_n = 0$ for $n < 0$ and a map of complexes $\epsilon_\cdot : A_\cdot \to A$ where A is $\cdots \to 0 \to \underset{(\text{deg. } 0)}{A} \to 0 \to \cdots$.
$A_\cdot \xrightarrow{\epsilon_\cdot} A$ is called an <u>$\mathcal{E}$-acyclic resolution</u> if $\cdots \to A_1 \to A_0 \xrightarrow{\epsilon_0} A \to 0 \to \cdots$ is in $\mathcal{E}$.
An $\mathcal{E}$-acyclic resolution is a <u>projective</u> (or <u>$\mathcal{P}$-</u>) <u>resolution</u> if $A_\cdot$ is a $\mathcal{P}$-complex.

The basic results are:

I. If $\mathcal{E}$ is a projective class, then every A in $\mathcal{A}$ has a projective resolution.

II. If $f: A \longrightarrow B$ in $\mathcal{A}$, $A_\circ \longrightarrow A$ a left p-complex over A and $B_\circ \longrightarrow B$ an $\mathcal{E}$-acyclic resolution, then there exists $F: A_\circ \longrightarrow B_\circ$ so that

$$\begin{array}{ccc} A_\circ & \xrightarrow{F} & B_\circ \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

commutes.

Further F is unique up to homotopy.

From this point on derived functors are handled just as in Cartan & Eilenberg.

What will be done next is to place this in a context where derived functors make sense with respect to an arbitrary closed class in an additive category.

Since the technicalities in the next section are fierce let me begin by sketching the observation which led up to it.

If $\mathcal{A}$ is an abelian category with enough projectives, let $C(\mathcal{A})$ be the category of complexes of $\mathcal{A}$ where morphisms are homotopy classes of chain maps. The zero-th homology functor H_0 maps $C(\mathcal{A})$ to $\mathcal{A}$. It has a left adjoint $P: \mathcal{A} \longrightarrow C(\mathcal{A})$, the projective resolution functor. If $F: \mathcal{A} \longrightarrow \mathcal{B}$, $\mathcal{B}$ another abelian category, is an additive functor, then F extends to $C(F): C(\mathcal{A}) \longrightarrow C(\mathcal{B})$. The composition $\mathcal{A} \xrightarrow{P} C(\mathcal{A}) \xrightarrow{C(F)} C(\mathcal{B}) \xrightarrow{H_0} \mathcal{B}$ is just L_0F, the 0-th derived functor of F. To get L_iF usually we simply replace H_0 by H_i. An equivalent but more relevant method is to define $S: C(\mathcal{B}) \longrightarrow C(\mathcal{B})$ by: $S(B_.)_n = B_{n-1}$ and $d_n^{S(B_.)} = -d_{n-1}^{B_.}$. This is an automorphism.

$$L_iF = \mathcal{A} \xrightarrow{P} C(\mathcal{A}) \xrightarrow{C(F)} C(\mathcal{B}) \xrightarrow{S^{-i}} C(\mathcal{B}) \xrightarrow{H_0} \mathcal{B}.$$

Now if $\mathcal{A}$ does not have enough projectives $C(\mathcal{A})$ does not carry all the structure necessary to define derived functors. The additional structure is contained in the sequences $A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} C(a_1) \longrightarrow S(A_1)$, $S(A_1) \cong C(a_2)$, where $C(a_1)$ is the mapping cone of a_1.

This leads to the study of triangulated categories. Let $\mathcal{A}$ be an additive category and $S: \mathcal{A} \to \mathcal{A}$ and additive automorphism. Write $\underline{\mathcal{A}^i(A_1, A_2)}$ for $\mathcal{A}(S^i(A_1), A_2)$ $(i \in \mathbb{Z})$. If $a_1 \in \mathcal{A}^i(A_1, A_2)$ and $a_2 \in \mathcal{A}^j(A_2, A_3)$, define $a_2 \circ a_1 \in \mathcal{A}^{i+j}(A_1, A_2)$ to be $a_2 \circ S^j(a_1)$. This defines a new category, $\underline{\mathcal{A}^\circ}$, with the same objects as A and $\mathcal{A}^\circ(A_1, A_2)$ a $\mathbb{Z}$-graded abelian group. $\mathcal{A}^\circ$ is called the <u>(S-)graded category over $\mathcal{A}$</u>.

Now let $\mathcal{A}^{\cdot}$ be a graded additive category over $\mathcal{A}$. Note that $\mathcal{A}$ is the subcategory of $\mathcal{A}^{\cdot}$ consisting of morphisms of degree 0.
A morphism in $\mathcal{A}^{\cdot}$ will <u>always</u> be of degree 0 unless otherwise specified.

A <u>triangle</u> in $\mathcal{A}^{\cdot}$ is a diagram of the form:

$$\begin{tikzcd} A_1 \arrow[dr, "a_1"'] & & A_3 \arrow[ll, "a_3"'] \\ & A_2 \arrow[ur, "a_2"'] & \end{tikzcd} \quad \text{with } \deg a_3 = -1.$$

A <u>morphism of triangles</u> is a diagram

$$\begin{tikzcd} A_1 \arrow[dr, "a_1"] \arrow[dddd, "f_1"] & & A_3 \arrow[ll, "a_3"'] \arrow[dddd, "f_3"] \\ & A_2 \arrow[ur, "a_2"'] \arrow[dd, "f_2"] & \\ & & \\ & B_2 \arrow[dr, "b_2"] & \\ B_1 \arrow[ur, "b_1"] & & B_3 \arrow[ll, "b_3"] \end{tikzcd} \quad \begin{array}{l} \deg a_3 = -1 \\ \\ \\ \\ \deg b_3 = -1 \end{array}$$

where the three squares commute.

A <u>triangulated category</u> is a graded additive category $\mathcal{A}^{\cdot}$ together with a distinguished class of triangles satisfying:

(TR 0) $\begin{tikzcd} A \arrow[dr, "1_A"'] & & 0 \arrow[ll, "\deg=-1"'] \\ & A \arrow[ur] & \end{tikzcd}$ is a distinguished triangle.

Every triangle isomorphic to a distinguished triangle is distinguished.

Every morphism $a_1 : A_1 \longrightarrow A_2$ fits into a triangle $\begin{tikzcd} A_1 \arrow[dr, "a_1"'] & & A_3 \arrow[ll, "a_3"'] \\ & A_2 \arrow[ur, "a_2"'] & \end{tikzcd}$ $\deg a_3 = -1$.

(TR 1) $$\begin{tikzcd} A_1 \arrow[dr, "a_1"'] & & A_3 \arrow[ll, "a_3"'] \\ & A_2 \arrow[ur, "a_2"'] & \end{tikzcd}$$ $(\deg a_3 = -1)$ is a distinguished triangle iff

$$\begin{tikzcd} A_2 \arrow[dr, "a_2"'] & & SA_1 \arrow[ll, "-Sa_1"'] \\ & A_3 \arrow[ur, "a_3"'] & \end{tikzcd}$$ is a distinguished triangle.

(TR 2) $$\begin{tikzcd} A_1 \arrow[dr, "a_1"] \arrow[ddd, "f_1"'] & & A_3 \arrow[ll, "a_3"'] \arrow[ddd, dashed, "f_3"] \\ & A_2 \arrow[ur, "a_2"'] \arrow[d, "f_2"] & \\ & B_2 \arrow[dr, "b_2"] & \\ B_1 \arrow[ur, "b_1"] & & B_3 \arrow[ll, "b_3"] \end{tikzcd}$$ $\deg a_3 = -1$, $\deg b_3 = -1$

If the top and bottom are distinguished and the square commutes, then f_3 exists (not uniquely!) giving a morphism of triangles.

(TR 3) Let $$\begin{tikzcd} A \arrow[dr, "a"'] & & C' \arrow[ll, "(\deg=-1)"'] \\ & B \arrow[ur, "b'"'] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[dr, "b"'] & & A' \arrow[ll, "a'"', "(\deg a'=-1)"] \\ & C \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[dr, "ba"'] & & B' \arrow[ll, "(\deg=-1)"'] \\ & C \arrow[ur] & \end{tikzcd}$$

be three distinguished triangles. Then there exist morphisms $f: C' \longrightarrow B'$ and $g: B' \longrightarrow A'$ such that:

(i) $$\begin{tikzcd} A \arrow[dr, "a"] \arrow[rrr, "1_A"] & & & A \arrow[dl, "ba"'] \\ & B \arrow[r, "b"] \arrow[dl, "b'"] & C \arrow[dr] & \\ C' \arrow[uu, "(\deg=-1)"] \arrow[rrr, "f"] & & & B' \arrow[uu, "(\deg=-1)"'] \end{tikzcd}$$ is a morphism of triangles.

(ii) $$\begin{tikzcd} A \arrow[dr, "ba"] \arrow[rrr, "a"] & & & B \arrow[dl, "b"'] \\ & C \arrow[r, "1_C"] \arrow[dl] & C \arrow[dr] & \\ B' \arrow[uu, "(\deg=-1)"] \arrow[rrr, "g"'] & & & A' \arrow[uu, "a'\ (\deg=-1)"'] \end{tikzcd}$$ is a morphism of triangles.

(iii) $$\begin{tikzcd} C' \arrow[dr, "f"'] & & A' \arrow[ll, "S(b')\circ a'"'] \\ & B' \arrow[ur, "g"'] & \end{tikzcd}$$ $(\deg(S(b')\circ a') = -1)$ is a distinguished triangle.

These axioms are included for completeness and to show what properties are actually needed. I won't actually use them.

The basic example of a triangulated category is gotten from $C(\mathcal{A})$. $C(\mathcal{A})$ is graded by the suspension functor S mentioned on p. 6. This graded category is denoted by $\mathcal{A}^\cdot$. Note that $\mathcal{A}$ is a subcategory of $\mathcal{A}^\cdot$ and $\mathcal{A}(A,B) = \mathcal{A}^0(A,B)$ for A and B in $\mathcal{A}$. A distinguished triangle in $\mathcal{A}^\cdot$ is one isomorphic to
$$\begin{tikzcd} A_1 \arrow[dr, "a_1"'] & & C(a_1) \arrow[ll, "a_3"'] \\ & A_2 \arrow[ur, "a_2"'] & \end{tikzcd} \quad (\deg a_3 = -1)$$
where $C(a_1)$ is the mapping cone of a_1 and $a_3 : C(a_1) \longrightarrow C(a_2) \cong S(A_1)$.

Another example is the category of semi-simplicial spectra and homotopy classes of graded maps. The triangles are again those isomorphic to
$$\begin{tikzcd} A_1 \arrow[dr, "a_1"'] & & C(a_1) \arrow[ll, "a_3"'] \\ & A_2 \arrow[ur, "a_2"'] & \end{tikzcd}.$$

The special types of functors of interest in studying triangulated categories are the following. A graded functor F between two graded categories $\mathcal{A}^\cdot$ and $\mathcal{B}^\cdot$ is an additive functor such that for all $i \in \mathbb{Z}$, $F : \mathcal{A}^i(\cdot,\cdot) \longrightarrow \mathcal{B}^i(F(\cdot), F(\cdot))$.

An exact functor between two triangulated categories is a graded functor which carries distinguished triangles into distinguished triangles.

A homological functor F is an additive functor from a triangulated category to an abelian category such that if
$$\begin{tikzcd} A_1 \arrow[dr, "a_1"'] & & A_3 \arrow[ll, "a_3"'] \\ & A_2 \arrow[ur, "a_2"'] & \end{tikzcd} \quad (\deg a_3 = -1)$$
is distinguished, then

$F(A_1) \xrightarrow{F(\alpha_1)} F(A_2) \xrightarrow{F(\alpha_2)} F(A_3)$ is exact.

Now putting $\underline{F_i} = F \circ S^{-i}$ and looking at the axioms gives a long exact sequence:

$$\cdots \longrightarrow F_i(A_1) \longrightarrow F_i(A_2) \longrightarrow F_i(A_3) \longrightarrow F_{i-1}(A_1) \longrightarrow \cdots$$

Examples abound: If A is an object of $\mathcal{A}^\circ$, then $\mathcal{A}^\circ(A, \cdot): \mathcal{A}^\circ \longrightarrow \mathcal{Ab}$ and $\mathcal{A}^\circ(\cdot, A): \mathcal{A}^{\circ\, op} \longrightarrow \mathcal{Ab}$ are homological functors. ("op" refers to the dual category)

Let $P: \mathcal{A}_1 \times \mathcal{A}_2 \longrightarrow \mathcal{A}_3$ be a biadditive functor, $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ being additive categories. P induces a functor $\underline{P_\circ}: \mathcal{A}_1^\circ \times \mathcal{A}_2^\circ \longrightarrow \mathcal{A}_3^\circ$ as follows: If $A_\circ$ is in $\mathcal{A}_1^\circ$ and $B_\circ$ is in $\mathcal{A}_2^\circ$, then $P(A_\circ, B_\circ)$ is a bicomplex of $\mathcal{A}_3$. Define $P_\circ(A_\circ, B_\circ)$ to be the associated simple complex. $P_\circ: \mathcal{A}_1^\circ \times \mathcal{A}_2^\circ \longrightarrow \mathcal{A}_3^\circ$ is an exact bifunctor. In particular consider $\mathcal{A}(\cdot,\cdot): \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathcal{Ab}$ ($\mathcal{Ab}$ = the category of abelian groups). This induces an exact bifunctor $\mathcal{A}_\circ(\cdot,\cdot): \mathcal{A}^{\circ\, op} \times \mathcal{A}^\circ \longrightarrow \mathcal{Ab}^\circ$.

Now if $\mathcal{A}$ is an abelian category, let $\underline{H_\circ: \mathcal{A}^\circ \longrightarrow \mathcal{A}}$ be the <u>0-th homology functor</u>. H_0 is a homological functor. Also note that $\underline{H_i} = H_0 \circ S^{-i}$ is the <u>i-th homology functor</u>.

Also note that $H_i \circ \mathcal{A}_\cdot(\cdot,\cdot) = \mathcal{A}^i(\cdot,\cdot)$.

Now we move on to the notion of a derived category. The derived category of $\mathcal{A}$ should be a category related to $\mathcal{A}$ in such a way that finding an exact functor on the derived category corresponds to finding a derived functor on $\mathcal{A}$. And it should be universal. To get such a thing we first have to look at quotient categories.

Let $\mathcal{A}^\cdot$ be a triangulated category. A subcategory $\mathcal{E}^\cdot$ of $\mathcal{A}^\cdot$ is a <u>triangulated subcategory</u> if it is a triangulated category and the inclusion functor is exact. $\mathcal{E}^\cdot$ is called a <u>thick subcategory</u> iff it is a full triangulated subcategory of $\mathcal{A}^\cdot$ satisfying: If $A_1 \xrightarrow{a_1} A_2$ factors through an object of $\mathcal{E}^\cdot$ and

$$\begin{tikzcd} A_1 \arrow[dr, "a_1"'] & & A_3 \arrow[ll, "(\deg=-1)"'] \\ & A_2 \arrow[ur] & \end{tikzcd}$$

is a distinguished triangle with A_3 in $\mathcal{E}^\cdot$, then A_1 and A_2 are in $\mathcal{E}^\cdot$.

The prototypical example of a thick subcategory is provided by the case where $\mathcal{E}^\cdot$ is the subcategory of $\mathcal{A}^\cdot$, $\mathcal{A}$ being an abelian category, generated by the exact complexes.

What is of interest is the following universal problem: Does there exist a triangulated category $\mathcal{A}^\cdot/\mathcal{E}^\cdot$ and an exact functor $Q: \mathcal{A}^\cdot \longrightarrow \mathcal{A}^\cdot/\mathcal{E}^\cdot$ such that for every exact functor $F: \mathcal{A}^\circ \longrightarrow \mathcal{B}^\circ$ such that $F(E)=0$ for every E in $\mathcal{E}^\circ$, there is a unique exact functor $G: \mathcal{A}^\circ/\mathcal{E}^\circ \longrightarrow \mathcal{B}^\circ$ so that

$$\begin{tikzcd} \mathcal{A}^\circ \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{B}^\circ \\ \mathcal{A}^\circ/\mathcal{E}^\circ \arrow[ur, dashed, "G"'] & \end{tikzcd}$$

commutes?

As might be expected $\underline{\mathcal{A}^\circ/\mathcal{E}^\circ}$ is called the <u>quotient of $\mathcal{A}^\circ$ by $\mathcal{E}^\circ$</u> It exists!

To find it we pose an alternative problem. Let $\underline{M}$ be the class of a morphisms $m: A_1 \longrightarrow A_2$ in $\mathcal{A}^\circ$ such that there is a distinguished triangle

$$\begin{tikzcd} A_1 \arrow[dr, "m"'] & & A_3 \arrow[ll, "(\deg=-1)"'] \\ & A_2 \arrow[ur] & \end{tikzcd}$$

with A_3 in $\mathcal{E}^\circ$. M is an example of a <u>saturated multiplicative system</u> (which I won't define here). They correspond bijectively with thick subcategories. The corresponding universal problem is: Does there exist a triangulated category $\mathcal{A}^\circ_M$ and an exact functor $Q: \mathcal{A}^\circ \longrightarrow \mathcal{A}^\circ_M$ such that if $F: \mathcal{A}^\circ \longrightarrow \mathcal{B}^\circ$ is any exact functor taking every m in M into an isomorphism, then there is a unique exact functor $G: \mathcal{A}^\circ_M \longrightarrow \mathcal{B}^\circ$ so that

$$\begin{tikzcd} \mathcal{A}^\circ \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{B}^\circ \\ \mathcal{A}^\circ_M \arrow[ur, "G"'] & \end{tikzcd}$$

commutes.

$\mathcal{A}^\circ_M$ is called <u>$\mathcal{A}^\circ$ localized at M.</u>

$\mathcal{A}^\circ_M$ exists and is in fact the category which can be constructed. $Q: \mathcal{A}^\circ \longrightarrow \mathcal{A}^\circ_M$ is also $Q: \mathcal{A}^\circ \longrightarrow \mathcal{A}^\circ/\mathcal{E}^\circ$.

The elements of M are called <u>quasi-isomorphisms (w.r.t. $\mathcal{E}^\circ$)</u>.

The basic way of obtaining is the following: let $\mathcal{C}$ be an abelian category and $H: \mathcal{A}^\circ \longrightarrow \mathcal{C}$ a homological functor. Let $\underline{\mathcal{E}^\circ_H}$ be the full subcategory of $\mathcal{A}^\circ$ generated by all A in $\mathcal{A}^\circ$ such that $H_i(A) = 0$ for all i in $\mathbb{Z}_1$. $\mathcal{E}^\circ_H$ is a thick subcategory.

Now we return to our basic example. Let $\mathcal{A}$ be an additive category and $\mathcal{A}^\circ$ the associated triangulated category. Let $\mathcal{E}$ be a closed class of sequences in $\mathcal{A}$. Denote by $\underline{\mathcal{E}^\circ}$ the full subcategory of $\mathcal{A}^\circ$ generated by the complexes in $\mathcal{E}$.

$\mathcal{E}^\circ$ is a thick subcategory of $\mathcal{A}^\circ$. For let $\mathcal{P} = \mathcal{P}(\mathcal{E})$. For each P in $\mathcal{P}$, $H_P = H_\circ \mathcal{A}_\circ(P, \cdot): \mathcal{A}^\circ \longrightarrow \mathcal{A}b$ is a homological functor. Thus $\mathcal{E}_{H_P}$ is a thick subcategory. $\mathcal{E} = \bigcap_{P \in \mathcal{P}} \mathcal{E}^\circ_{H_P}$ is thereby a thick subcategory.

The quotient $\mathcal{A}^\circ/\mathcal{E}^\circ$ is called the <u>derived category of $\mathcal{A}$ relative to $\mathcal{E}$</u> and is denoted by $\underline{D^\circ_{\mathcal{E}}(\mathcal{A})}$. The quotient

functor from $\mathcal{A}^\circ$ to $D^\circ_{\mathcal{E}}(\mathcal{A})$ is called D.
$\mathcal{A}$ is a subcategory of $D^\circ_{\mathcal{E}}(\mathcal{A})$ and for all A and B in $\mathcal{A}$, $\mathcal{A}(A,B) \xrightarrow[\cong]{D} D^\circ_{\mathcal{E}}(\mathcal{A})(D(A), D(B))$.

Now consider the situation where $\mathcal{A}$ and $\underset{\sim}{\mathcal{A}}$ are additive categories with closed classes $\mathcal{E}$ and $\underset{\sim}{\mathcal{E}}$ respectively. Let $F: \mathcal{A} \longrightarrow \underset{\sim}{\mathcal{A}}$ be an additive functor. F extends to give an exact functor $F_\circ : \mathcal{A}^\circ \longrightarrow \underset{\sim}{\mathcal{A}}^\circ$. If F is "exact", i.e. if E in $\mathcal{E}$ implies $F(E)$ in $\underset{\sim}{\mathcal{E}}$, then $F_\circ$ induces a functor $D^\circ_{\mathcal{E}}(\mathcal{A}) \longrightarrow D^\circ_{\underset{\sim}{\mathcal{E}}}(\underset{\sim}{\mathcal{A}})$. If F is not exact this doesn't work and we try to find a universal approximation to such an extension.

An exact functor $F_\circ : \mathcal{A}^\circ \longrightarrow \underset{\sim}{\mathcal{A}}^\circ$ has a <u>total left derived functor</u> $\underline{L}F_\circ : D^\circ_{\mathcal{E}}(\mathcal{A}) \longrightarrow D^\circ_{\underset{\sim}{\mathcal{E}}}(\underset{\sim}{\mathcal{A}})$ if there is a natural transformation of graded functors (i.e. respecting the grading) $\underline{L}F_\circ \circ D \longrightarrow D \circ F_\circ$ such that if $G_\circ \circ D \longrightarrow D \circ F_\circ$ is any other such natural transformation, there is a unique natural transformation of graded functors $G_\circ \longrightarrow \underline{L}F_\circ$ so that

$$\begin{tikzcd} G_\circ \circ D \arrow[r] \arrow[d] & D \circ F_\circ \\ \underline{L}F_\circ \arrow[ur] & \end{tikzcd}$$

commutes.

If $F_\circ$ comes from an additive functor F as in the last paragraph, write $\underline{L}F$ instead of $\underline{L}F_\circ$.

Similiarly let $F: \mathcal{A}^\circ \longrightarrow \mathcal{B}$ be a homological functor. F has a <u>left derived functor</u> $LF: D^\circ_{\mathcal{E}}(\mathcal{A}) \longrightarrow \mathcal{B}$ if there is a natural transformation $LF \circ D \longrightarrow F$ such that if $G: D^\circ_{\mathcal{E}}(\mathcal{A}) \longrightarrow \mathcal{B}$ is a homological functor and $G \circ D \longrightarrow F$ a natural transformation, then there is a unique natural $G \longrightarrow LF$ such that

$$\begin{tikzcd} G\circ D \arrow[r] \arrow[d] & F \\ LF\circ D \arrow[ur] & \end{tikzcd}$$

commutes.

If $f: \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor, then $LH_0 \circ f^\circ : D^\circ_{\mathcal{E}}(\mathcal{A}) \longrightarrow \mathcal{B}$ will be denoted by <u>Lf</u>. $Lf \circ S^{-i}$ will be denoted <u>$L_i f$</u> and called the i-th left derived functor of f.

Now of course we want to know when derived functors exist. For this we need some more notation. Let <u>$\mathcal{A}_+$</u> (resp. <u>$\mathcal{A}_-$, $\mathcal{A}_0$</u>) denoted the full subcategory of $\mathcal{A}^\circ$ generated by the positive (resp. negative, bounded) complexes. If $\mathcal{E}^\circ$ is a thick subcategory of $\mathcal{A}^\circ$, then $\mathcal{E}^\circ \cap \mathcal{A}^\circ_* = \mathcal{E}_*$ $(* = +, -$ or $0)$ is a thick subcategory of $\mathcal{A}^\circ_*$. Further $\mathcal{A}_*/\mathcal{E}_*$ is a full triangulated subcategory of $\mathcal{A}^\circ/\mathcal{E}^\circ$.

THEOREM: Let $\mathcal{A}$ and $\underset{\sim}{\mathcal{A}}$ be two additive categories. Let $F: \mathcal{A}^\circ_* \longrightarrow \underset{\sim}{\mathcal{A}}^\circ_*$ be an exact additive functor. $(* = +, -, 0$ or nothing$)$

suppose there is a triangulated subcategory $\mathcal{B}^\circ_* \subseteq \mathcal{A}^\circ_*$ such that: (1.) Every $A_\circ$ in $\mathcal{A}^\circ_*$ admits an $\mathcal{E}$-quasi-isomorphism $B_\circ \longrightarrow A_\circ$ with $B_\circ$ in $\mathcal{B}^\circ_*$. and (2.) If $B_\circ$ is in $\mathcal{B}^\circ_* \cap \mathcal{E}^\circ_*$, then $F(B_\circ)$ is in $\underset{\sim}{\mathcal{E}}^\circ_*$.

THEN (a.) $\underline{\underline{L}}F : D^\circ_{\mathcal{E}}(\mathcal{A})_* = \mathcal{A}^\circ_*/\mathcal{E}^\circ_* \longrightarrow D^\circ_{\underset{\sim}{\mathcal{E}}}(\underset{\sim}{\mathcal{A}})_*$ exists.

(b.) If $B_\circ \longrightarrow A_\circ$ is a quasi-isomorphism with $B_\circ$ in $\mathcal{B}^\circ_*$, then $\underline{\underline{L}}F \circ D_*(A_\circ) \cong D_* F(B)$.

(c.) If $\underset{\sim}{\mathcal{A}}$ is an abelian category and $\underset{\sim}{\mathcal{E}}$ is the class of exact sequences, then $H_\circ \circ F : \mathcal{A}^\circ_* \longrightarrow \underset{\sim}{\mathcal{A}}$ has a left derived functor which is just $H_\circ \circ \underline{\underline{L}}F$.

<u>Corollary:</u> In theorem take $* = +$ and assume $\mathcal{E}$ a projective class and $\underset{\sim}{\mathcal{A}}$ abelian with $\underset{\sim}{\mathcal{E}}$ the class of exact sequences. Suppose F comes from $F_\circ : \mathcal{A} \longrightarrow \underset{\sim}{\mathcal{A}}$. Then the Theorem holds with $\mathcal{B}^\circ_+ = \mathcal{P}^\circ_+$, the class of positive $\mathcal{E}$-projective complexes.

<u>Corollary:</u> Suppose F comes from $F_\circ : \mathcal{A} \longrightarrow \underset{\sim}{\mathcal{A}}$ with $\underset{\sim}{\mathcal{A}}$ abelian and $\underset{\sim}{\mathcal{E}}$ the class of exact sequences. Suppose $\mathcal{A}$ has <u>finite F-homological dimension</u>, i.e. there is an $N > 0$ so that if $i > N$, then $L_i F_\circ(A) = 0$ for all A in $\mathcal{A}$, then by taking $\mathcal{B}^\circ$ to be the class of all $B_\circ$ in $\mathcal{A}^\circ$ so that $\underline{\underline{L}}F(B_\circ) = 0$ the theorem applies.

Although I have talked of projectively closed classes and left derived functors, the theory dualizes perfectly well and one gets dual results regarding injectively closed classes and right derived functors.

The connection between the two is: If $\mathcal{E}$ is a (projectively) closed class and $\mathcal{E}'$ is an injectively closed class of sequences, they are <u>complementary</u> iff the class of complexes in the two are the same. In such a case $D^{\cdot}_{\mathcal{E}}(\mathcal{A}) = D^{\cdot}_{\mathcal{E}'}(\mathcal{A})$.

APPLICATIONS

(1.) Let $\mathcal{A}$ be an additive category and $\mathcal{E}'$ an injectively closed class in $\mathcal{A}$. For $A.$ and $B.$ in $D^{\cdot}_{\mathcal{E}'}(\mathcal{A})$, define <u>$\mathrm{Ext}^i_{\mathcal{E}'}(A., B.)$</u> $= D^i_{\mathcal{E}'}(\mathcal{A})(A., B.)$. (The <u>i-th hyper-Ext</u>)

In particular this defines $\mathrm{Ext}^i_{\mathcal{E}'}(A,B)$ where A and B are in $\mathcal{A}$. If $\mathcal{E}'$ is an injective class, this is the same as Eilenberg and Moore's definition. If $\mathcal{E}$ is a complementary projective class, $\mathrm{Ext}^i_{\mathcal{E}}(A,B)$ can be achieved by a projective resolution of A. Since $\mathrm{Ext}^i_{\mathcal{E}} = \mathrm{Ext}^i_{\mathcal{E}'}$, this shows the equivalence of using projective and injective resolutions without the usual bicomplex argument.

(2.) If $F: D^{\cdot}_{\mathcal{E}'}(\mathcal{A}) \longrightarrow \mathcal{A}b$ is a homological functor, F defines a map $\mathrm{Ext}^0_{\mathcal{E}'}(A., B.) \longrightarrow \mathcal{A}b(F(A.), F(B.))$ and thus a pairing $F(A.) \times \mathrm{Ext}^0_{\mathcal{E}'}(A., B.) \longrightarrow F(B.)$. Applying the translation functor gives $F^i(A.) \times \mathrm{Ext}^j_{\mathcal{E}'}(A., B.) \longrightarrow F^{i+j}(B.)$. When $F = Rf$, $f: \mathcal{A} \longrightarrow \mathcal{A}b$ additive, this is a direct generalization of the Yoneda pairing. In particular this gives the Yoneda product in $\mathrm{Ext}^{\cdot}_{\mathcal{E}'}(A,A)$.

This is only a tiny indication of the scope of this theory which unifies and streamlines all of homological algebra. In addition, in the hands of Grothendieck it is now being used to give a systematic development of global intersection and duality theory in algebraic geometry.

5

The p-primary components of Stable Homotopy Groups of Spheres.

3-22-66

Joel M. Cohen

We show how to compute ${}_p\pi_n(\underline{S})$ for $n \leq 2(p-1)(p^2+p)-$ for odd prime p.

There is a spectral sequence ('Atiyah-Hirzebruch') such that $E^2_{**} = H_*(X; \pi_*(\underline{S}))$ and E^∞_{**} is a bigraded group associated with $\pi_*^S(X)$; this spectral sequence may be generalised by replacing X by a spectrum $\underline{A}$; $E^2_{*,*} = H_*(\underline{A}; \pi_*(\underline{S}))$, $E^\infty_{*,*}$ a bigraded group associated with $\pi_*(\underline{A})$

Consider this spectral sequence for $\underline{A} = \underline{K}(Z)$, the Eilenberg-MacLane spectrum $(A_n = K(Z,n))$. Then $E^\infty_{0,0} = Z$ $E^\infty_{s,t} = 0$ $(s,t) \neq (0,0)$ since $\pi_*(\underline{K}(Z)) = \pi_0(\underline{K}(Z)) = Z$.

$E^2_{*,*} = H_*(\underline{K}(Z); \pi_*(\underline{S}))$ is a tensor and torsion product of a well known ring $H_*(\underline{K}(Z))$ and a ring about which information is sought, $\pi_*(\underline{S})$. This relation gives us the information needed.

For the actual computations we replace Σ by the spectrum $\underline{L}_p$ where $(\underline{L}_p)_n = M(Z_p, n)$ the Moore space of homology type (Z_p, n) — the mapping cone of $S^n \xrightarrow{p} S^n$ (map of degree p)

It is easily shown that $\pi_n(\underline{L}_p) \approx \pi_n(\underline{S}) \otimes Z_p + \mathrm{Tor}(\pi_{n-1}(\underline{S}), Z_p)$

Thus given any $\theta \in \pi_n(\underline{S})$ not divisible by p we have an element also called $\theta \in \pi_n(\underline{L}_p)$. Given any $\eta \in \pi_n(\underline{S})$ of order p, we have an element $\eta' \in \pi_{n+1}(\underline{L}_p)$; this may be determined by the Toda bracket $\langle \eta, p, m \rangle$ in the diagram

$$S^{n+r} \xrightarrow{\eta} S^r \xrightarrow{p} S^r \xrightarrow{m} M(Z_p, r)$$

(passing to stable range) and is defined up to indeterminacy $\pi_{n+1}(\underline{S}) \otimes Z_p \subset \pi_{n+1}(\underline{L}_p)$

Multiplication can be defined making $\pi_*(\underline{L}_p)$ an algebra over Z_p. Then $E^\infty_{*,*} = E^\infty_{0,0} = Z_p$ and

$$E^2_{*,*} = H_*(\underline{K}(Z); \pi_*(\underline{L}_p)) = H_*(\underline{K}(Z); Z_p) \otimes \pi_*(\underline{L}_p)$$
$$= A_* \otimes \pi_*(\underline{L}_p)$$

where A_* is the Hopf Algebra $E(\tau_1, \tau_2, \ldots) \otimes P(\xi_1, \xi_2, \ldots)$ the exterior algebra on generators τ_i tensored with the polynomial algebra on generators ξ_i where $\dim \tau_i - 1 = \dim \xi_i = 2(p^i - 1)$. A_* is the quotient algebra of $\mathcal{a}_*$, the dual to the Steenrod algebra by $\tau_0 \mathcal{a}_*$ the Hopf ideal generated by the element dual to the Bockstein.

Using the fact that the differential is a derivation and that E^r_{**} is an algebra over Z_p, we are able to deduce that $\beta_1^p \alpha_1 \neq 0$ (the first unresolved problem of Toda). This is equivalent to determining the action of the $(2p-1)$st differential acting on the element γ in the Adams Spectral Sequence. Combining this with the results of May we have the following description of ${}_p\pi_*(S)$.

Theorem. ${}_p\pi_n(S)$ for $0 < n \leq (p^2+2p)q-4$ $(q = 2(p-1))$ is described completely by the following table listing each element, its order, and its dimension: (In this table $j \geq 1$).

Element	Order	Dimension	
α_j	p	$jq-1$	$j \not\equiv 0 \bmod p$
$\alpha^{(2)}_{jp}$	p^2	$jpq-1$	$j \not\equiv 0 \bmod p$
$\alpha^{(3)}_{jp^2}$	p^3	jp^2q-1	$j \not\equiv 0 \bmod p$
$\beta_{m+1}\beta_1^{j-1}$	p	$((j+m)p+m)q-2j$	$0 \leq m \leq p-2$
$\alpha_1\beta_{m+1}\beta_1^{j-1}$	p	$((j+m)p+m+1)q-2j-1$	$0 \leq m \leq p<2$
γ	p	p^2q-2	
$\beta_{m+1}\beta_1^{j-1}\gamma$	p	$(p^2+jp+mp+m)q-2j-2$	$0 \leq m \leq p-2$, $j=1$ if $m=1$, $p=3$
$\alpha_1\beta_{m+1}\beta_1^{j-1}\gamma$	p	$(p^2+jp+mp+m+1)q-2j-3$	$0 \leq m \leq p-2$, $j=1$ if $m=1$, $p=3$
$\alpha_m\gamma$	p	$(p^2+m)q-3$	$1 \leq m \leq p-2$
ε_m	p	$(p^2+m)q-2$	$1 \leq m \leq p-1$
ϕ	p^2	$(p^2+p)q-3$	
$\beta_p\beta_1^{j-1}$	p	$(p^2+jp+p-1)q-2j-2$	$j=1$ if $p=3$

If we consider the above terms as being in $\pi_*(S)\otimes Z_p \subset \pi_*(\underline{L}_p)$, then we have the following relations:

(Let $\alpha = \alpha_1$.)

$$\left.\begin{aligned} \alpha_n &= (\alpha')^{n-1}\alpha \\ \alpha_{np} &= (\alpha')^{np-1}\alpha \\ \alpha_{np^2} &= (\alpha')^{np^2-1}\alpha \end{aligned}\right\} n \not\equiv 0 \pmod p$$

$$\varepsilon_m = (\alpha')^m\gamma + m(\alpha')^{m-1}\alpha\gamma' \qquad 1 \le m \le p-1$$

$$\phi = \beta_1\gamma' - \beta_1'\gamma$$

$$\beta_1'\beta_{p-1}' = \varepsilon_{p-2}$$

Furthermore $\langle \overbrace{\alpha, \dots, \alpha}^{p \text{ times}} \rangle = \beta_1$ and there are homotopy operations T, R such that $T(\beta_i) = \beta_{i+1}$, $1 \le i \le p-2$, $R(\beta_i) = \gamma$ (Equalities hold for a particular choice of basis elements).

By Toda our result is equivalent to the following: Given N consider the sequence of topological spaces

$$K_1 \subset K_2 \subset \dots \subset S^N$$

where $\pi_j(K_k) = 0$ for $j \ge N+k$ and the inclusion

$$i : S^N \to K_k$$

induces $i_* : \pi_j(S^N) \approx \pi_j(K_k)$ for $j < N+k$.

Let $A^i(K_k; Z_p) = H^{N+i}(K_k; Z_p)$ for $0 \le i < N < k$ (This does not depend on N.) There is a generator $b_p \in A^{k+1}(K_k; Z_p)$ for $k = (p^2-1)q-2$. Then $P^1 b_p \in A^{k+2p-1}(K_k; Z_p)$

Toda proves that $P^1 b_p = 0$ if and only if $\beta_1^p \alpha_1 \neq 0$ and $P^1 b_p = 0$ if and only if $\gamma \neq 0$. Thus it follows from our result that $P^1 b_p = 0$.

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Higher Order Suspension Maps for Homology of Non-Additive Functors

4·25·66

A. K. Bousfield

Let R be a commutative ring with identity. Let T: R modules $\to$ R modules be a covariant functor with $T(0) = 0$. A general problem in semi-simplicial topology is to determine $\pi_* TX$, X a semi-simplicial R-module (T = prolongation of T). A special case is the study of homology of Eilenberg-MacLane spaces. Higher order suspension maps, which we will presently discuss, permit a certain transfer of information from this special case to the general one. Later on we shall give some applications which depend on the work of Curtis, Kan, Quillen, Rector, Schlesinger on the Adams Spectral Sequence.

We now generalize the Dold-Puppe suspension map

$$\sigma : \pi_* TX \to \pi_{*+1} TSX$$

<u>Viewpoint:</u> $\sum_{m,n \geq 0} \pi_m TS^n X$ will be made into a bigraded module over the algebra $P(R) = \sum_{m,n\geq 0} H_m(R,n,R)$.

<u>The algebra $P(R)$</u> Let $f: K(R,n) \times K(R,n') \to K(R, n+n')$ be such that the fundamental class $\iota^{n+n'} \in H^{n+n'}(R, n+n', R)$ is mapped by f^* to the cohomology product $\iota^n \times \iota^{n'}$. Passing to homology one gets the multiplication in $P(R)$

up to sign.

$P(R)$ is associative with identity

Multiplication in $P(R)$ is stable:

$$\sigma(\alpha\cdot\beta) = \alpha(\sigma\beta) = (\sigma\alpha)\cdot\beta$$

for $\alpha, \beta \in P(R)$. The induced stable algebra of $P(Z_p)$ is iso. to $\mathcal{Q}^*(p)$, dual of Steenrod alg.

Module structure of $\sum_{m,n\geq 0} \pi_m T S^n X$ over $P(R)$ is induced up to sign by the composition

$$\tilde{R} K(R,n) \otimes_R TX \xrightarrow{h} T(\tilde{R} K(R,n) \otimes_R X) \xrightarrow{j} T(K(R,n) \otimes_R X) \xrightarrow{k} T(S^n X)$$

where i) The functor $\tilde{R}$ assigns to a set with b.p. the free R-module it generates with b.p. set equal to 0.

ii) h is the map of s.s. R-module whose restriction in dimension i to $1.a \otimes TX_i \subset \tilde{R} K(R,n)_i \otimes TX_i$ is induced by the map $X_i \to \tilde{R}K(R,n)_i \otimes_R X_i$ sending $x \to 1.a \otimes x$ where $a \in K(R,n)_i$ and $x \in X_i$.

iii) j is the map of s.s. R-modules induced by the natural map $\tilde{R}K(R,n) \to K(R,n)$

iv) R is induced by a homotopy equivalence of the s.s. R-modules $K(R,n) \otimes_R X$ and $S^n X$.

With $\alpha \in H_m(R,n;R)$ is associated a

pension*) $\alpha: \pi_n TX \to \pi_{n+m} TS^n X$

The canonical element $c^n \in H_n(R,n;R)$ gives $c^n = \sigma^n$, the n-fold suspension map.

If T additive, then the only non-zero pensions are iterated suspensions and their multiples.

Most of our results are for pensions over the rings Z and Z_p

Let us suppose $R = Z$:

*) a coined word to suggest 'generalized suspension'

$H_* K(\mathbb{Z},1)$ contributes nothing new.

Let $\varepsilon_r \in H_{2r}(\mathbb{Z},2)$ be dual to $(\iota^2)^r$

Theorem. Suppose T is of degree r and X trivial above n. Then $\varepsilon_r : \pi_i TX \to \pi_{i+2r} T S^2 X$ is iso for $i > (r-1)n+1$ and mono $i = (r-1)n+1$

Note: $\pi_i TX \xrightarrow{\varepsilon_r} \pi_{i+2r} T S^2 X \to \pi_{i+4r} T S^4 X \to \ldots$ will stabilize.

Pensions are especially useful when there is a product structure about.

The Eilenberg-Zilber theorem induces a map

$$\underset{\alpha}{\pi_i TX} \otimes \underset{\beta}{\pi_j T'X} \longrightarrow \underset{\alpha\cdot\beta}{\pi_{i+j}(T \otimes T')X}$$

Theorem $\varepsilon_r(\alpha\cdot\beta) = \sum_{i=1}^{r-1} \varepsilon_i(\alpha)\cdot\varepsilon_{r-i}(\beta)$

The composition product

$$\underset{\alpha}{\pi_m TX} \cdot \underset{\beta}{\pi_j T'K(\mathbb{Z},m)} \to \underset{\alpha\cdot\beta}{\pi_j T'\cdot TX}$$

is additive in β but not necessarily in α (unless β is a suspension)

$$\sigma(\alpha\cdot\beta) = (\sigma\alpha)\cdot(\sigma\beta)$$

Theorem. For p prime, $r \geq 0$

$$\varepsilon_{p^r}(\sigma\alpha\cdot\sigma\beta) = \sum_{k=0}^{r} \varepsilon_{p^k}(\sigma\alpha)\circ(\varepsilon_{p^{r-k}})^{p^k}(\sigma\beta).$$

Note: For $s \neq p^n$, $\varepsilon_s(\sigma\alpha\cdot\sigma\beta) = 0$ since $\varepsilon_s\circ\sigma = 0$.

Application to E'S mod 2

E'S is the initial term of an Adams spectral sequence for the (mod 2) stable homotopy of spheres.

$E'S = \lim_{n\to\infty} \pi_{*+n} Lk(Z_2, n)$ where L denotes the restricted Lie algebra functor. Furthermore E'S is a D-G algebra generated by $\lambda_0, \lambda_1, \lambda_2, \ldots$ and formulae for the relations and differential are known.

We wish to characterize E'S more in terms of Steenrod algebra notions. The stable mod 2 pensions act on E'S to give the following structure.

Theorem. E'S is an algebra over $\mathcal{A}^*$, the dual to the Steenrod algebra (For $x, y \in E'S$, $\xi_n(x \cdot y) = \sum_{i=0}^n (\xi_i x)(\xi_{n-i}^{2^i} y)$)

Furthermore

1) $\xi_1 \lambda_i = \lambda_{i+1}$ for $i \geq 0$; $\xi_n \lambda_i = 0$, $n \geq 2$

2) Let $\gamma : E'S \to E'S$ be "squaring" operation".

If $\alpha \in \mathcal{A}^*$ and $x \in E'S$, then $\gamma(\alpha x) = \alpha^2 \gamma(x)$.

3) For $x \in E'S$,

$$\partial \xi_n x + \xi_n \partial x = \lambda_0 (\xi_{n-1}^2 x) + (\xi_{n-1} x) \lambda_{2^{n-1}-1}$$

We shall characterize E'S via a larger DG algebra $\overline{E'S}$

Suppose A a Z_2 DG algebra

We form $\bar{A}$ by adjoining an element Δ of degree -1

$$\Delta^2 = 0$$
$$\Delta x + x \Delta = \partial x \quad , x \in A$$

Then A is a subalgebra of $\bar{A}$

E'S is determined by the conditions

1. $\overline{E'S}$ is an algebra over $\mathcal{A}^*$

2. $\overline{E'S}$ has distinguished element λ_{-1} of degree -1 such that $\lambda_{-1}\lambda_{-1} = 0$

$$\xi_n \lambda_{-1} = 0 \quad n \geq 2$$

3. If B is another algebra satisfying 1, 2 with distinguished element Δ, then there is a unique structure preserving map $\overline{E'S} \to B$ sending λ_{-1} to Δ.

$E'S$ is subalgebra of $\overline{E'S}$ generated by products $\alpha \cdot \lambda_{-1}$ where $\alpha \in \mathcal{Q}^*$, $\deg \alpha > 0$.

Another Application

$E'S$ was computed using the Restricted lie algebra functor. Another functor of interest is the restricted symmetric algebra functor

$$SP_p = \sum_{r=1}^{\infty} SP_p^r : Z_p \text{ modules} \to Z_p \text{ modules}$$

If M is a Z_p module then SP_pM is the free symmetric algebra generated by M with $x^p = 0$ for $x \in SP_pM$

If K is a s.s complex with ℓ_p. then there is an isomorphism

$$\sum_{r=1}^{\infty} SP_p^r \tilde{\Sigma}_p K \xrightarrow{\cong} \tilde{Z}_p \tilde{Z}_p K .$$

Let us set $p = 2$.

$$\text{Let } \mathcal{S} = \lim_{n \to \infty} \pi_{*+n} SP_2 K(Z_2, n)$$

with multiplication induced by composition, $\mathcal{S}$ is an algebra generated by $\sigma_1, \sigma_2, \ldots$ σ_i of degree i

The stable mod 2 operations act to make $\mathcal{S}$ an algebra over $\mathcal{A}^*$,

$$\xi_n(xy) = \sum_{i=0}^{n} \xi_i(x)\, \xi_{n-i}^{2^i}(y)$$

Moreover $\xi_1 \sigma_i = \sigma_{i+1}$, $\xi_n \sigma_i = 0$, $n \geq 2$

The relations are generated by $\sigma_1\sigma_1 = 0$, $\sigma_2\sigma_2 = 0$

(e.g. $\xi_2(\sigma_1\sigma_1) = \sigma_2\sigma_3 = 0$ is a relation)

$\mathcal{S}$ has an additive basis given by

$\sigma_{i_1}, \ldots, \sigma_{i_n}$ with $i_{j+1} \geq 2 i_j$

$\mathcal{S}$ seems very closely related to the Steenrod algebra. There are pairings

$\mathcal{A}^* \otimes \mathcal{S} \to \mathcal{S}$ making $\mathcal{S}$ an algebra over $\mathcal{A}^*$

$\mathcal{A}^* \otimes \mathcal{S} \to \mathcal{A}^*$ making $\mathcal{A}^*$ a right module over $\mathcal{S}$.

The module product satisfies

i) for $\alpha \in \mathcal{A}^*$ $x \in \mathcal{S}$ $\xi_n(\alpha \cdot x) = \sum_{i=0}^{n} (\xi_i \alpha)(\xi_{n-i}^{2^i} x)$

ii) $(1)\sigma_1\sigma_2\sigma_4 \ldots \sigma_{2^{n-1}} = \xi_n$ for $n \geq 1$

The above facts permit computation of additive isomorphisms

$\mathcal{S} \cong \mathcal{A}^*$

induced in the stable limit by $\pi_* SP_2 K(Z_2, n) \cong H_*(Z_2, n)$

This isomorphism maps $x \to (1) \cdot x$ using the module products.

I believe that $\mathcal{S} \cong \mathcal{A}^*$ maps $\sigma_{i_1} \cdots \sigma_{i_k} \to (Sq^{i_n} \ldots Sq^{i_1})$ where $i_{j+1} \geq 2 i_j$ for all j and where the dual is taken with respect to the adem basis.

We can get a complete description of unstable homotopy operations on s.s. commutative Z_2 algebras (with or without 1)

These operations for example act on mod 2 homology of commutative topological semi-groups.

R is a ss commutative Z_2 algebra

$\pi_* R$ is a commutative Z_2 algebra with $x^2 = 0$ if $\deg x > 0$

There are operations $\sigma_2, \sigma_3, \sigma_4, \ldots$

$$\sigma_n : \pi_i R \to \pi_{i+n} R \qquad n \leq i$$
$$x \to (x)\sigma_n$$

<u>Theorem</u> Any natural operation $\pi_i R \to \pi_j R$ is expressible using compositions of the σ_n and algebra addition and multiplication of $\pi_* R$

These operations σ_n satisfy:

1. The relations of $\mathcal{S}$ hold for compositions of the σ_i (when they make sense).

2. $x, y \in \pi_i R$ $\quad (x+y)\sigma_n = x \cdot \sigma_n + y \cdot \sigma_n \quad , n < i$
$\quad (x+y)\sigma_i = x\sigma_i + y\sigma_i + xy$

3. $x \in \pi_i R$, $y \in \pi_j R$
$(x \cdot y)\sigma_n = 0$ if $i, j > 0$ $\quad (x \cdot y)\sigma_n = x^2 \cdot (y\sigma_n)$ $i = 0$

A New Proof for the Non Existence of Maps of Hopf Invariant One

4-11-66

J. Wang, Cornell U.

Using a mod p version of the lower central series a group at M.I.T. has constructed a very nice E^1-term of the Adams spectral sequence for spheres. E^1 is nice in the sense that (E^1, d^1) is completely known. My objective is to present some computations of E^2 from (E^1, d^1).

Recall for $p=2$, E^1 is a DGA with unit over Z_2 generated by the symbols λ_i $(i \geq 0)$ such that

i) $\deg \lambda_i = i$

ii) $\sum_{i+j=n} \binom{i+j}{i} \lambda_{i-1+m} \lambda_{j-1+2m} = 0$, $\quad n \geq 0,\ m \geq 1$

iii) $d^1 \lambda_{n-1} = \sum_{\substack{i+j=n \\ i,j \geq 1}} \binom{i+j}{j} \lambda_{i-1} \lambda_{j-1}$ $\quad n \geq 2$

A sequence $I = (i_1, \ldots, i_j)$ of non-negative integers is said to be of length j, i_1, i_j the leading, resp. the ending integer. I is admissible if $2i_\lambda \geq i_{\lambda+1}$ for $1 \leq \lambda \leq r-1$. Let $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_j}$ $(\lambda_\phi = 1)$. Then the additive structure of E^1 is determined by the following.

Theorem. λ_I, I admissible sequence of non-negative integers or ϕ, form a basis for E^1:

When an element is expressed in this basis, we say that it is in admissible expression. Clearly the additive structure enables one to compute $\ker d^1$. However ii), iii) are expressed in symmetric forms; we wish to rearrange it in admissible forms.

Theorem. $\lambda_i\lambda_{2i+1+n} = \sum_{j\ge 0}\binom{n-1-j}{j}\lambda_{i+n-j}\lambda_{2i+1+j}$, with the convention $\binom{u}{v}=0$ for $u<v$.

Proof. Clearly it is obtained by applying the powers of a derivation D which sends λ_i to λ_{i+1} on $\lambda_m\lambda_{2m+1}=0$, $m\ge 0$.

Let $\lambda_i\lambda_{2i+1+n} = \sum_{\substack{n-j,\\ j\ge 0}} a_{n-j,j}\lambda_{i+n-j}\lambda_{2i+1+j}$

with $a_{n-j,j}=1$ or 0, $a_{n-j,j}=0$ for $2(n-j)<1+j$.

Applying D on both sides, we get the identities

$a_{n,i}+a_{n-1,i+1}+a_{n-1,i-1}=a_{n,i+1}$ for $n\ge 1$, $i\ge 1$

$a_{n,0}=a_{n-1,0}$ for $n\ge 2$

$a_{1,0}=1$, $a_{1,1}=0$

Let $F(X,Y)=\sum_{n,i} a_{n,i}X^nY^i$

$F(X,Y)(X+Y+XY^2) = F(X,Y)+X+XY$

$F(X,Y)\cong X/1+(1-Y)X = X+\sum X^{s+1}(1-Y)^s$ (mod 2).

$\Rightarrow a_{n,i}=\binom{n-1}{i}$

—

E^1 is bigraded, $E^1_{r,n}$ generated by λ_I with length $I=r$, $\sum_{j=1}^{r} i_j = n$. Then.

Theorem. There is only one non-trivial derivation of deg $(1,-1)$, that is d^1, which has admissible form

$$d^1\lambda_n = \sum_{j\ge 0}\binom{n-1-j}{1+j}\lambda_{n-1-j}\lambda_j$$

Lemma. $\binom{n-\lambda}{\lambda}=0$ (mod. 2) for $\lambda>0$ iff $n=2^d-1$

Corollary. $E^2_{1,n} = \begin{cases}\{h_j\}, & n=2^j-1\\ 0 & \text{otherwise}\end{cases}$ $h_j = cl(\lambda_{2^j-1})$

Proposition. Given $(n_1,\dots,n_j;m)$ such that $(n_1,\dots,n_j)$ is admissible and $n_1+\dots+n_j+j+n\ge m$, $n\ge n_j$, then the leading integers of the admissible expression of $\lambda_{(n_1,\dots,n_r,m)}$ are $\leq n$.

Proof. This is obviously true for the case $r=1, m=0$. Thus induction on (r,m). Assume $2n_j+1<m$, then

$$\lambda_{(n_1,\dots,n_r,m)} = \sum_{i\geq 0}\binom{m-2n_j-2-i}{i}\lambda_{(n_1,\dots,m-n_j-1-i,2n_j+1+i)}$$

Note: $n_1+\dots+n_{j-1}+(j-1)+n \geq m-(n_j+1+i)$

$\geq n_j+1+i<m$

The proposition follows immediately from double induction.

Proposition. Given an admissible sequence $(n_1,\dots,n_j)$, then the leading integers of $d'\lambda_{(n_1,\dots,n_j)}$ are $\leq n_j$.

Proof. $d'\lambda_{(n_1,\dots,n_j)} = (d'\lambda_{n_1})\lambda_{n_2}\cdots\lambda_{n_j} + \lambda_{n_1}d'(\lambda_{n_2}\cdots\lambda_{n_j})$

First term on right hand side has leading integer $\leq n_j-1$, Second term has leading integers $\leq n_1$.

Corollary. Let $x=\lambda_{n_1}x_1+\dots+\lambda_{n_s}x_s$ be expressed in admissible terms, such that $n_1>\dots>n_s$, then $d'x=0$ implies $d'x_1=0$.

Corollary. Let $E'(n)$ be generated by λ_I, I admissible and $i_1\leq n$, then $E'(n)$ is a sub-DGA of E'.

Corollary. Let E'_{odd} be generated by λ_I with i_j (ending integer) odd, then E'_{odd} is a left-DGA ideal of E'

Proof. $\binom{2n+1-j}{j}\equiv 0 \pmod 2$ for odd j, the ending integer of w_1w_{2n+1}, $d'w_{2n+1}$ are odd.

Proposition. Given $x\in E'_{r,n}$ with $d'x=0$, $n>0$, then there is $y\in E'_{odd}$ such that $x\sim y$.

Proof. In the homology class of x, choose y with minimal s such that

$$y = \Sigma_{i=0}^{s} y_i \lambda_0^i$$

where ending integers of $y_i \neq 0$. We claim $s=0$, otherwise let $y_s = \Sigma \lambda_{(n_1,\dots,n_t)}$ then n_t is odd (otherwise $\lambda_{n_1} \dots \lambda_{n_{t-1}} \lambda_0^{i+1}$ will appear in $d'y$ and $d'y \neq 0$.) Let $z_s = \Sigma \lambda_{(n_1,\dots,n_{t-1},n_{t+1})} \lambda_0^{s-1}$, then $d'z_s + y$ contradicts our choice. Thus $s=0$ and ending integers of y_0 must be odd.

Corollary. $x \in E'_{r,n}$ such that $0<n<r$ and $d'x=0$, then $x \sim 0 \Rightarrow Ext_A^{s,t}(Z_2, Z_2) = 0$ for $0<t-s<s$.

Proposition. $x \in E'_{-,n}$, $d'x = 0$ then

$$x \in \begin{cases} \lambda_{(2^i-1,\,2^j-1)},\ 0,\ d'\lambda_{n+1} & \text{for } n = 2^i+2^j-2, \\ d'\lambda_{n+1},\ 0 & \text{otherwise} \end{cases}$$

Corollary. $x \in E'_{2,n} \wedge E'_{odd}$ and $d'x = 0$, then $x=0$ or $n=2^j-1$, $x = \lambda_0 \lambda_{2^j-1}$

Theorem. There is a unique DA-endomorphism

$$\gamma : E' \to E'$$

sending λ_i to λ_{2i+1}.

Proof. γ is compatible with the defining relation ii) and this follows $\binom{2n}{j} \equiv 0$ for j odd. $d'\gamma\lambda_n = \gamma d'\lambda_n$

γ induces an endomorphism on $H^*(A) \to H^*(A)$ which sends h_i, c_i, e_i, d_i to $h_{i+1}, c_{i+1}, e_{i+1}, d_{i+1}, \dots$

γ stabilizes the defining relations of $H^*(A)$. For example to show $h_i h_{i+1}=0$, $i\geq 0$ it suffices to verify $h_0 h_1 = 0$, the following relations is immediate.

$$h_i h_{i+1}=0, \quad h_i h_{i+2}^2 = 0, \quad h_i^2 h_{i+2} = h_{i+1}^3$$

Proposition. $E^2_{2,\cdot}$ is generated by $h_i h_j$ $(=h_j h_i)$ $|i-j|\neq 1$. $E^2_{3,\cdot}$ is generated by $h_i h_j h_k$, c_l where $c_i = cl(\lambda_{(3\cdot 2^i-1,\, 2^{i+1}-1,\, 2^{i+1}-1)})$

Proposition. $h_0 h_i^3 \neq 0$ for $i\geq 4$.

Proof. Consider $\lambda_{(0,2^i-1,2^i-1,2^i-1)} = \lambda_{(2^i-2,1,2^i-1,2^i-1)}$ + (terms with leading integers $\leq 2^i-3$). If $h_0 h_i^3=0$, we would draw $(1,2^i-1,2^i-1)\sim 0$. But this is impossible for $i\geq 4$.

Theorem. $d^2 h_i = h_0 h_{i-1}^2$ for $i\geq 4$.

Proof. h_3 retains in E^∞, let $\bar{h}_3$ be the element which corresponds to h_3. $2\bar{h}_3^2=0$, by commutativity. Hence $h_0 h_3^2$, corresponding to $2\bar{h}_3^2$, must be killed. Its only chance to be killed is from E^2 to E^3. Hence $d^2 h_4 = h_0 h_3^2$. Now $h_i h_{i+1}=0$, $h_i d^2 h_{i+1} + (d^2 h_i) h_{i+1} = 0$.

$h_i d^2 h_{i+1} = h_0 h_{i-1}^2 h_{i+1} = h_0 h_i^3 \neq 0$ for $i\geq 4$.

But $E^2_{3,2^{i+2}-2} = \{0, h_0 h_i^2\}$ Hence $d^2 h_{i+1} = h_0 h_i^2$.

And the theorem follows by induction.

4-Cobordism and the Arf-Invariant.

5-16-66

Dr. V. Giambalvo, M.I.T.

Let $BO\langle 4\rangle$ be the 4 connected covering of BO. Consider manifolds M (imbedded in Euclidean space) such that the stable normal bundle map $M \to BO$ lifts to $BO\langle 4\rangle$. Let $\langle 4\rangle^n$ be the set of such manifolds of dimension n. $M, N \in \langle \cdot \rangle^n$ are cobordant if $\exists W \in \langle 4\rangle^{n+}$ such that

$$\partial W = M \cup (-N)$$

We denote the cobordism group by $\Omega_n^{\langle 4\rangle}$

$H^q(BO\langle 4\rangle) = 0 \quad q < 8. \; q = 9, 10, 11, 13$

Prop. (Lashof) If $\gamma \in \Omega_n^{\langle 4\rangle}$, $\exists$ $M \in \gamma$ such that

$$\nu^* : H^q(BO\langle 4\rangle) \to H^q(M) \quad \text{is} \cong \text{for } q < [\tfrac{n}{2}]$$

(ν is the classifying map for M)

In particular for n large there is a 7-connected representative.

Prop. $M \in \langle 4\rangle^n$

$$Sq^k : H^{n-k}(M) \to H^n(M) \quad \text{is } 0 \quad k = 1, 2, 4$$

Let Φ be the Arf invariant. We recall that it can be defined on $4k+2$ manifolds that are $2k$-connected and stably parallelizable.

We have

$$\Phi : \Omega^{\text{framed}}_{8k+2} \to \mathbb{Z}_2$$

$\Phi(M) = 0$ iff M framed cobordant to homotopy sphere

<u>$n = 16k+6$</u>

We claim we can lift Φ to Ψ on $\Omega^{\langle 4 \rangle}_{16k+6}$, i.e.

$$\begin{tikzcd}
\Omega^{\langle 4 \rangle}_{16k+6} \arrow[dr, "\Psi"] & \\
\Omega^{\text{framed}}_{16k+6} \arrow[u, "\exists \text{ natural}"] \arrow[r, "\Phi"'] & \mathbb{Z}_2
\end{tikzcd}$$

Ψ, can be defined.

Given M^{2m} m-1 connected, for $S^m \subset M^{2m}$, let $v \in H^m(M^{2m})$ dual to S^m

We have a secondary cohomology operation φ_0,

$$\varphi_0(v) \in H^{2m}(M^{2m})$$

$\varphi_0(v) = 0 \iff S^m$ has trivial normal bundle.

Pick symplectic basis for $H^m(M, \mathbb{Z}_2)$, $\{x_i, y_i\}$,

$$x_i x_j = y_i y_j = 0$$

$$x_i y_j = \delta_{ij}.$$

$$\Psi = \sum_i \varphi_0(x_i)[M]\, \varphi_0(y_i)[M]$$

Define a secondary operation

$\varphi : H^m(M) \to H^{2m}(M)$

From the relation

$$\left[Sq^4 Sq^{m-3} + Sq^2 (Sq^4 Sq^{m-5}) + Sq^1 (Sq^2 Sq^4 Sq^{m-6}) \right] u = 0$$

φ is defined on $H^m(M) \cap \ker Sq^{m-3} \cap \ker Sq^4 Sq^{m-5} \cap \ker Sq^2 Sq^4 Sq^{m-6}$

$$\varphi(x+y) = \varphi(x) + \varphi(y) + xy.$$

If $M \in \langle 4 \rangle^{2m}$, $m-1$ connected, stably parallelizable,

$$\Phi(M) = \varphi(M).$$

1) What is image $\Omega^{\text{framed}} \to \Omega^{\langle 4 \rangle}$

2) Show ψ is 0 on it.

<u>Theorem</u>. Suppose $\gamma \in \Omega^{\langle 4 \rangle}$ such that in the Adams' spectral sequence for $\Omega^{\langle 4 \rangle} \simeq \pi_*(MO\langle 4 \rangle)$ γ has a representative with filtration degree ≥ 0. Then $\exists M \in \gamma$ which is unorientably cobordant to 0.

<u>Theorem</u> $N \in \langle 4 \rangle^{16k}$, $\chi(N) = W_{16k}(N) = 0$

Then $\psi(S^3 \times S^3 \times N) = 0 = \psi(S^3 \times S^3)\chi(N)$.

Let $M = S^3 \times S^3 \times N$

$H^{8k+3}(M) = H^3(S^3) \otimes H^{8k}(N) + H^3(S^3) \otimes H^{8k}(N) + (\ldots)$

<u>Theorem</u> $W \in \langle 4 \rangle^{16k}$, $M \in \langle 4 \rangle^{16k+6}$

$\mathcal{Q}^{8}$ $H^{16l \cdot 6 \cdot 8}(M) \to H^{16l+6}(M) = 0$ φ odd.

Then $\psi(M \times N) = \varphi(M)\,\chi(N)$

$u \in H^{8k}(N)$
$v \in H^{8l+3}(M)$

$\mathcal{Q}(u \otimes v) = \varphi(u) \otimes v^2$

Can show

Φ $\Omega^{\text{framed}}_{22} \to Z_2$ is 0

$\Omega^{\langle 4 \rangle}_{22}$ has 4 elements
2 generators

one of filtration 2, the other of filtration about 8.

In $\Omega^{\text{framed}}_{22}$ no elements of these filtrations

The one of filtration 2 in $\Omega^{\langle 4 \rangle}_{22}$ cannot be in image

The one of filtration 8 in $\Omega^{\langle 4 \rangle}_{22}$ factors to elements of dim. 8 and dim 14. For dimensional reasons ψ must vanish.

Structure of $H^*(MO\langle 4\rangle)$ (for dim < 55)

sum of modules	dim
$A/A(Sq^1, Sq^2, Sq^4)$	0, 16, 32, 32, 48, 48, 48
$A/A(Sq^1, Sq^5, Sq^6, Sq^{13})$	20, 36, 36, 52, 52
$A/A(Sq^1, Sq^9)$	40
$A/A(Sq^1, Sq^5)$	44
$A/A(Sq^1, Sq^2)$	48
$A/A(Sq^2, Sq^{2,1})$	46.

Geometric Characterization of Differentiable Manifolds in Euclidean Space

5·2·66

Prof. H. Gluck, Harvard U.

<u>Prototype Theorem</u> Let M be a 1-dimensional topological manifold $\subset R^n$. Then M is a C^1-manifold if and only if the secant map

$$\Sigma : M \times M - \Delta \longrightarrow P^{n-1}$$

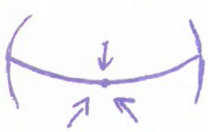
has a continuous extension over $M \times M$.

Let M be a topological manifold in R^n.

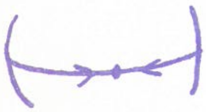
Let Σ be generalized secant map

$$\Sigma (x_0, \ldots, x_n) \longrightarrow G_{n,k}$$

Consider the 2-sphere. If we take three points



approaching a central points from different general directions, the secant plane approaches the tangent plane satisfactorily. But, if we restrict ourselves to the equator



we find that the secant plane is perpendicular to the tangent plane! The difficulty is the bad 'shape' of the approximating 2-simplices 

To overcome this difficulty we will introduce a shape function σ on simplices, vanishing on degenerate simplices and positive on non-degenerate ones.

$(M^{k+1})_{\sigma_0}$ will be set of all $k+1$ tuples of points in M such that $\sigma(x_0, \ldots, x_n) > \sigma_0$.

<u>Main Theorem</u> Let M be a k-dimensional topological manifold in R^n and $\sigma_0 > 0$ a real number $(\le \frac{1}{\sqrt{k}})$. Then M is a C^1-manifold if and only if the generalized secant map

$$\Sigma : (M)^{k+1}_{\sigma_0} \to G_{n,k}$$

has a continuous extension over $(M)^{k+1}_{\sigma_0} \cup \Delta$ (Δ = diagonal, i.e. im $M \to M^k$)

Program of Proof.

A Shapes of Simplices
B. Building up simplices of good shape in R^n
C Building up simplices of good shape in M in R^n
D
E Proof of Main theorem.

A. Let a k simplex $\Delta^k = (x_0, \ldots, x_k)$ $(x_i \neq x_j)$ be given in $R^k \subset R^n$

L_{ij}, the line through 0 parallel to $x_i x_j$ can be regarded as a point in P^{k-1}. Thus we get a set of $\binom{k(k+1)}{2}$ points in P^{k-1}. In a nicely shaped simplex, whose faces go in different directions, these points will be well distributed. A standard way of measuring the distribution of points in a compact metric space is to take the maximum value over the entire space of the function which assigns to each point its (minimum) distance to a point of the distribution. Let $\Theta(\Delta^k)$ be this maxmin distance

$$0 \le \Theta(\Delta^k) \le \pi/2 .$$

Def. $\sigma(\Delta^k) = \cos \Theta(\Delta^k)$

$$0 \le \sigma(\Delta^k) \le 1 .$$

So $\sigma(\text{edge}) = 1$
$\sigma(\text{equilateral } \Delta) = \sqrt{3}/2 = .866$
$\sigma(\text{reg. tetrahedron}) = \sqrt{2}/2 = .707$.

Another possible measure of shape might be

$$\frac{\text{inradius}}{\text{length of longest side}}$$

This would fail to distinguish between

 and 

although from our point of view there will be a crucial difference as the second triangle must be thought of as better shaped since the sides do not <u>all</u> go in the same <u>direction</u>

Let P, Q be k-planes in R^n. We can choose bases u_i, v_i, for P, Q in R^n so that

$$\langle u_i, u_j \rangle = \langle v_i, v_j \rangle = \delta_j^i$$

$$\langle u_i, v_j \rangle = 0 \qquad i \neq j$$

$$\langle u_1, v_1 \rangle \geq \langle u_2, v_2 \rangle \geq \ldots \geq \langle u_k, v_k \rangle \geq 0$$

Let $\cos \theta_i = \langle u_i, v_i \rangle$

$$0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_k \leq \pi/2$$

We call these the principle angles between P and Q

What do we mean when we say ϕ is '<u>the</u>' angle between P and Q. We claim ϕ should be $0 \leq \phi \leq \pi/2$ so that $\cos \phi$ is the reduction factor for k-measure under orthogonal projection.

Theorem. Let Δ^k be a non-degenerate k-simplex in R^n each of whose edges makes an angle $\le \varepsilon$ with n-plane Q ($\varepsilon \le \pi/2$). Then every line L in the plane P of Δ^k makes angle $\phi(L)$ with Q such that

$$\sin \phi(L) \le \frac{\sin \varepsilon}{\sigma(\Delta^k)}$$

Orthogonal Partition

Suppose Δ^r is non-degenerate r-simplex in R^n, $r<n$ $x_0, \dots, \underline{x_r}$ (with one vertex, x_r, distinguished). Let P^r be plane of Δ^r. Through x_r erect P^{n-r} orthogonal to P^r. Choose x_{r+1} in $P^{n-r} - \{x_r\}$.

Let σ_{r+1} be the value obtained by calculating σ on $(x_0, \dots, x_{r+1})$ "Δ^{r+1}" when only those L_{ij} are used for which $0 \le i,j \le r$ or $i=r$, $j=r+1$.

Clearly $\sigma_{r+1} \le \sigma(\Delta^{r+1})$

Claim $$\frac{\sigma(\Delta^r)}{\sqrt{1+\sigma(\Delta^r)^2}} \le \sigma(\Delta^{r+1})$$

Orthogonal Fission

Starting with $x_0, \dots, \underline{x_r}$ as above, take neighborhood of $\underline{x_r}$ (in M). Choose therein x_r', x_{r+1}' so that the line connecting them is perpendicular to P^r. We get simplex $(x_0, \dots, x_{r-1}, x_r', x_{r+1}')$.

We need to employ fission because in partition it is not always possible to choose x_{r+1} to lie in a given M with $x_0, \dots, x_r$.

Theorem. If $x_0, \ldots, x_r$ are inscribed in a manifold it is possible to choose x_r', x_{r+1}' in the manifold arbitrarily close to x_r.

Theorem. M a topological k-manifold in R^n $x_0 \in M$, X a nbhd of x_0 in M. Then there exist $x_1, \ldots, x_r$ in X such that

$$\frac{1}{\sqrt{r}} - \varepsilon < \sigma(\Delta^r) < \frac{1}{r} + \varepsilon \qquad r \le k$$

Theorem M a topological k-manifold in R^n, $k \geq 2$, X a preassigned connected open set in M, $x_0, x_1 \in X$ such that $|x_0 - x_1| < \operatorname{diam} X$. Then there is an $x_2 \in X$ such that $\Delta^2 = x_0 x_1 x_2$ has $\sigma(\Delta^2) > 1/\sqrt{2}$

Theorem M topological k-manifold in R^n, X connected, open, $\subset M$, $x_0, x_1 \in X$ $|x_0 - x_1| < \operatorname{diameter} X$. There are $x_2, \ldots, x_r$ in X such that

$\sigma(\Delta^r) > \frac{1}{\sqrt{r}}$ $r \le k$.

Corollary Δ is in closure of $(M)^{r+1}_{\sigma_0}$ for $\sigma_0 \le \frac{1}{\sqrt{r}}$

D Theorem Let M be k-dimensional topological manifold in R^n. Then M is C^1 if and only if

1) M has tangent k-plane $P(x)$ at each $x \in M$

2) $M \to G_{n,k}$ continuously by $x \to P_0(X)$

3) For each $x \in M$ the orthog. proj.

$$\Pi_x : R^n \to P(x)$$

is 1-1 on some nbhd X of x in M.

E Main Theorem - Proof

Necessity. Suppose M a C^1-manifold.

Set $\Sigma(x, x, \ldots, x) = P_0(x)$.

Show this extension of Σ continuous. Problem is to show that if we have limit of simplices with shape $> \sigma$ approach x Then their planes approach $P_0(x)$, or, given ε there exists δ such that if $\sigma(x_0, \ldots, x_k) > \sigma$, $x_0 = x$, $d(x, x_i) < \delta$ the the angle between $x_i x_j$ and $P_0(x)$ is less than $\sigma\varepsilon$, so $P(x_0, \ldots, x_k)$ has angle less than ε with $P(x_0)$

Sufficiency Must show 1), 2), 3) above

1) $P = \Sigma(x, x, \ldots x)$ is tangent plane. Must show

$$\lim_{x \to x_0} \frac{d(x, P)}{d(x, x_0)} = 0$$

$$\frac{d(x, P)}{d(x, x_0)}$$ is sine of angle θ between $x x_0$ and P.

To make θ small we extend to a small simplex of shape $> \frac{1}{\sqrt{k}} > \sigma$ by a previous Theorem. Then since the Σ is continuous, the angle between the plane of the simplex and $P \to 0$ in particular, $\theta \to 0$.

2) follows from the fact that Σ is continuous on Δ.

3) follows since $\exists$ a neighborhood X of x_0 such that angle between $\Sigma(x_0, \ldots, x_p)$ and P is less than $\pi/2$ so we get a contradiction if x_0 & $x \in M$ and they project onto the same point of P.