COHOMOLOGY STRUCTURE OF CERTAIN FIBRE SPACES

Prof. F. P. Peterson. 28th September 1964

This talk represents joint work with W. S. Massey

Problem Given a bundle $\xi = (E, p, B, F, G)$, to compute $H^*(E ; Z_2)$ as an algebra and module over $\mathcal{A}$ (the Steenrod algebra)

I Algebraic Preliminaries

Let R be a graded algebra over Z_2 and an algebra over the Hopf algebra $\mathcal{A}$. i.e. $Sq^i(x.y) = \sum_r Sq^r x \,.\, Sq^{i-r} y$.

Let M be a graded R module and $\mathcal{A}$ module with

$$a(r.m) = \sum_i a_i'(r).\, a_i''(m) \quad \text{where} \quad \psi(a) = \sum_i a_i' \otimes a_i''. \qquad (*)$$

Example (X, A) a pair. $H^*(X, A)$ is a module over $H^*(X)$.

Def $R \odot \mathcal{A}$, the semi-tensor product of R and $\mathcal{A}$.

$R \odot \mathcal{A} = R \otimes \mathcal{A}$ as a vector space and multiplication is given by $(r \otimes a).(s \otimes b) = \sum_i r.\, a_i'(s) \otimes a_i'' \ell$.

This defines an associative algebra and M is an $R \odot \mathcal{A}$ module under the operation $(r \otimes a)(m) = r(a(m))$.

Theorem The category of $R \odot \mathcal{A}$ modules is equivalent to the category of R and $\mathcal{A}$ modules with condition $(*)$.

Def M is called an unstable $\mathcal{A}$ module if $Sq^i(m) = 0$ if $i > \dim m$.

R is called an unstable $\mathcal{A}$ algebra if it is unstable as an

$\mathcal{A}$ module and $Sq^i(r) = r^2$ if $i = \dim r$.

M is called an <u>unstable $R \odot \mathcal{A}$ module</u> if it is unstable as an $\mathcal{A}$ module.

<u>Def</u>. A <u>base pt.</u> for an unstable $R \odot \mathcal{A}$ module M is a linear map $R \longrightarrow M$.

<u>Def</u> $\mathcal{U}_R(M)$ = free (unstable $R \odot \mathcal{A}$ module with base pt.) algebra generated by module M, defined by the diagram

$$\begin{tikzcd}
 & \mathcal{U}_R(M) \arrow[d, dashed] \\
M \arrow[ur] \arrow[r] & A
\end{tikzcd}$$

where A is any other unstable $R \odot \mathcal{A}$ module with base pt.

<u>Note</u> $Sq^i(m) = m^2$ if $i = \dim m$.

<u>Theorem</u> If M has as R basis $b_0, b_1, \dots$ with b_0 = base pt. then $\mathcal{U}_R(M)$ is a free R module on $b_0, b_{i_1} \dots b_{i_r}$ with $i_1 < i_2 < \dots < i_r$.

II To state the main theorem we require

<u>Conditions on F and G.</u>

1. $H^*(F) = \mathcal{U}_{Z_2}(X)$ where X is some $\mathcal{A}$ module and X is transgressive.
2. If $\xi_0 = (E_0, p_0, B_0, F, G)$ is the universal bundle with fibre F and group G then
   (a) local coeffs. are trivial in ξ_0.
   (b) $p_0^* : H^*(B_0) \longrightarrow H^*(E_0)$ is epi.
   and (c) $\operatorname{Ker} p_0^* = (\tau_0(X))$ = ideal in $H^*(B_0)$ generated by $\tau_0(X)$.

<u>Example</u> (i) $F = V_{n,r} = \dfrac{O(n)}{O(n-r)}$ and $G = O(n)$

Univeral bundle is $$\begin{tikzcd} BO(n-r) \arrow[d] \\ BO(n). \end{tikzcd}$$

$H^*(V_{n,r}) = \mathcal{U}_{Z_2}\left(H^*\left(\frac{P_{n-1}}{P_{n-r-1}}\right)\right)$ and so satisfies 1 by a classical theorem.

(ii) $F =$ product of $K(\pi, n)$'s.

<u>Conditions on the bundle ξ.</u>

<u>Def.</u> Let S be a connected commutative graded algebra over Z_2. An <u>E-sequence</u> $x_1, x_2, \ldots \subset S$ is a sequence of homogeneous elements s.t. x_{i+1} is not a zero divisor in $\frac{S}{(x_1, \ldots, x_i)}$

<u>Def</u> I, an ideal in S, is called a <u>Borel ideal</u> if $\exists$ an E-sequence $x_1, \ldots$ (possibly infinite) s.t. $I = (x_1, x_2, \ldots)$

We can now state the

<u>Main Theorem</u> Let F and G be as above and assume $(\tau(X)) \subset H^*(B)$ is a Borel ideal. Then $\exists$ $N \subset H^*(E)$, an $R \odot \mathcal{A}$ module where $R = \frac{H^*(B)}{\operatorname{Ker} p^*} = \operatorname{Im} p^*$, s.t. $H^*(E) = \mathcal{U}_R(N)$. Furthermore

$$0 \longrightarrow R \longrightarrow N \xrightarrow{\text{degree } +1} M \longrightarrow 0$$

is an exact sequence of $R \odot \mathcal{A}$ modules, where $M = \operatorname{Tor}_1^{H_*(B_0)}(H^*(B), H^*(E_0))$ and $\operatorname{Ker} p^* = (\tau(X))$.

<u>Note</u> (i) Construction of M and N.

Let $\xi_T = (E_T, p_T, B, T(F), G)$ be fibre bundle with fibre $T(F) =$ cone on F.

We have the diagram

$$\begin{tikzcd} H^*(E_0) \arrow[r, "\delta"] & H^*(E_{0T}, E_0) \arrow[r] \arrow[d, "h^*"] & H^*(B_0) \arrow[r, "p_0^*"] & H^*(E_0) \\ H^*(E) \arrow[r, "\delta"] & H^*(E_T, E) \arrow[r] & H^*(B) \arrow[r, "p^*"] & H^*(E) \end{tikzcd}$$

where the map h comes by universality.

Then $M' = H^*(B)$-module generated by $h^*(H^*(E_{0T}, E_0))$.

$M = M' \cap \operatorname{Im} \delta$

$N = \delta^{-1}(M)$.

M and N are free R-modules.

(ii) In the proof it is shown that $E_r^{p,q} \cong E_r^{p,0} \otimes E_r^{0,q}$ in the cohomology spectral sequence.

Special Case $(\tau(X)) = 0$, fibre totally non-homologous to zero.

Then $E_2 = E_\infty$ and $H^*(E) \cong H^*(B) \otimes H^*(F)$ as a $\mathbb{Z}_2$-module but not as a ring (by a classical theorem).

III We state some results of

Paul Baum (thesis)

A, a graded commutative connected algebra (finitely generated).

Def. dim A = largest t s.t. $\mathbb{Z}_2[a_1, \dots a_t] \subset A$.

Def deficiency of A = def A = $r - n$ where $A = \dfrac{\mathbb{Z}_2[x_1, \dots x_n]}{I}$

and r = minimum # of generators for I.

Def Presentation index of A = dim A + def A (≥ 0).

Theorem (Baum) $A = \dfrac{\mathbb{Z}_2[x_1, \dots x_n]}{I}$, I a Borel ideal

$\iff$ presentation index of $A = 0$

$\iff 0 \to I \to \mathbb{Z}_2[x_1 \cdots x_n] \to A \to 0 \Rightarrow I$ is a Borel ideal.

Let E be a space s.t. $H^*(E) = R$, finitely generated as an algebra, has presentation index 0.

$$E \xrightarrow{p} \prod_i K(\pi_i, n_i)$$

Then $\exists$ an "Adams - Postnikov" system

$$\begin{tikzcd}
E \arrow[r] \arrow[rr, bend right, "p_q"'] \arrow[rrrrrr, bend right, "p."'] & \cdots \arrow[r] & E^q \arrow[r, "p^q_{q-1}"] & E^{q-1} \arrow[r] & \cdots \arrow[r] & E^1 \arrow[r] & \prod_i K(\pi_i, n_i)
\end{tikzcd}$$

s.t. each $E^q \longrightarrow E^{q-1}$ is a fibre space with fibre a product of $K(\pi,n)$'s, and s.t. $p_q^* \big| \mathrm{Im}\, p^{q}_{q-1}{}^*$ is an isomorphism.

Then $\exists$ an $R \odot \mathcal{A}$ module N^q s.t. $H^*(E^q) = \mathcal{U}_R(N^q)$ = polynomial algebra over R. Further more $N^q = R \oplus M^q$ as $R \odot \mathcal{A}$-modules and M^q is as in main theorem.

<u>Example</u> $E = S^n$.

POWER OPERATIONS IN K-THEORY

Prof. M. F. Atiyah 5th October 1964

As an independent but connected result we first prove the following

Theorem ∄ elements of Hopf invariant 1 in $\pi_{2n-1}(S^n)$ for $n \neq 1, 2, 4$ or 8.

Proof

We merely assume a few basic facts about $K(X)$ and the operation ψ^k defined by Adams.

Let $f : S^{2n-1} \longrightarrow S^n$ represent an element of $\pi_{2n-1}(S^n)$

Let $X_f = S^n \cup_f e^{2n}$ and assume $n = 2k$ (if n is odd the Hopf invariant is zero.)

Now the graded ring associated with $K(X_f)$, $GK(X_f) \cong H^*(X; \mathbb{Z})$ and so $K(X_f)$ is generated by $1, a, b$, where $a \in K_{2k}$, $b \in K_{4k}$ the suffix denoting the filtration.

∴ $a^2 = \alpha b$ for some $\alpha \in \mathbb{Z}$, and so the Hopf invariant is one $\iff$ α is odd.

Now $\psi^p(a) \equiv a^p$ mod. p. In particular $\psi^2(a) \equiv a^2 \equiv \alpha b$ mod 2

Thus $\psi^2(a) = 2^k a + \mu b$ for μ odd

$\psi^3(a) = 3^k a + \nu b$

But $\psi^2 \psi^3 = \psi^3 \psi^2$ $(= \psi^6)$

$$\therefore \ 3^k(2^k a + \mu b) + \nu 2^{2k} b = 2^k(3^k a + \nu b) + \mu 3^{2k} b$$

Equating coeffs. of b we get $3^k(3^k - 1)\mu = 2^k(2^k - 1)\nu$

∴ 2^k divides $3^k - 1 \implies k = 1, 2,$ or 4.

The main results are

Theorem 1 Suppose X is a space with $\dim X \leq 2(q+t)$ with $t < q(p-1)$ where p is a prime, and take $x \in K_{2q}(X)$. Then $\psi^p(x)$ is divisible by p^{q-r} where $r = [t/p-1]$.

Theorem 2 Suppose X is a space without torsion so that $GK^*(X) \cong H^*(X; \mathbb{Z})$. Let $x \in K_{2q}(X)$. Then $\exists$ elements $x_i \in K_{2q+2i(p-1)}(X)$ s.t. $\psi^p(x) = \sum_{i=0}^{q} p^{q-i} x_i$. Moreover, if $\bar{x} \in H^{2q}(X; \mathbb{Z}_p)$ is the element correspondi to x, then $\bar{x}_i = P_p^i \bar{x}$.

Note The second theorem can be stated more generally without conditions on X.

Definition of Operations in $K(X)$

Approach 1 By way of representations. For a vector bundle V over X of dimension n take the associated principal bundle P with group $GL(n; \mathbb{C})$. Each representation $\rho : GL(n : \mathbb{C}) \longrightarrow GL(N : \mathbb{C})$ gives rise to an operation in $K(X)$.

Approach 2 Grothendieck's λ-operations. We can form the exterior powers λ^i of a vector bundle V over X and introduce the formal sum $\lambda_t(V) = \sum \lambda^i(V) t^i$. Then for $x \in K(X)$, $\lambda^p(x)$ is the coeff. of t^p in $\lambda_t(x)$.

Approach 3 Our approach is based on two observations. Firstly, Steenro derivation of the Steenrod powers starting from the symmetric grou Secondly, the classical construction of representations of $GL(n; \mathbb{C}$ If V is a vector space we can form the k-fold tensor product $V^{\otimes k}$ and there is a canonical decomposition under the action of the symmetric group, S_k,

$$V^{\otimes k} = \sum_{\pi} V^{\otimes k}_{\pi}$$ where π runs over all partitions of k in which the number of parts $\leq \dim. V$.

We can also look at $V^{\otimes k}$ as a $GL(V) \times S_k$ module and then $V^{\otimes k}_{\pi} \cong A_{\pi} \otimes B_{\pi}$

where A_{π} = irreducible representation of $GL(V)$

and B_{π} = " " " S_k.

Given B_{π} we can form $\mathrm{Hom}_{S_k}(B_{\pi}, V^{\otimes k})$ which is canonically isomorphic to A_{π}.

We now generalize this to the case of V a vector bundle over
We first form $V^{\otimes k}$, again a vector bundle over X. For π a representati
of the symmetric group take B_{π} = trivial bundle over X and form $\pi(V) = \mathrm{Hom}_{S_k}(B_{\pi}, V^{\otimes k})$, another vector bundle.

Recall that $K(X, Y)$ can be described as equivalence classes of complexes $[E]$ of vector bundles over X, acyclic on Y.

$$E = \left\{ 0 \longrightarrow E_0 \xrightarrow{d} E_1 \longrightarrow \cdots \xrightarrow{d} E_n \longrightarrow 0 \right\}$$

For instance, if $Y = \phi$, then $[E] = \sum (-1)^i [E^i]$.

Define $\pi(E) = \mathrm{Hom}_{S_k}(B_{\pi}, E^{\otimes k})$ and finally

$$\pi[E] = [\pi(E)]$$ and is well defined.

Relations between Filtration and Operations

Suppose A is a space on which S_k acts freely. Then S_k acts freely on $X^k \times A$. Let $E^{\boxtimes k}$ be the tensor product on the cartesian product. Take the induced complex $p^*(E^{\boxtimes k})$

where $p : X^k \times A \longrightarrow X^k$ is the projection.
Then $\dfrac{p^*(E^{\boxtimes k})}{S_k}$ is a complex on $X^k \times^{S_k} A$ $\left(= \dfrac{X^k \times A}{S_k}\right)$

The diagonal map $: X \longrightarrow X^k$ extends to $X \times A/S_k \xrightarrow{\Delta_A} X^k \times^{S_k} A$ and so we get a complex $\Delta_A^*\left(\dfrac{p^*(E^{\boxtimes k})}{S_k}\right)$ on $X \times A/S_k$.

Let this complex be $\mathcal{P}_A^k(E)$.
This defines an operation $\mathcal{P}_A^k : K(X) \longrightarrow K(X \times A/S_k)$ by passing to equivalence classes, and it is such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{P}_A^k : K(X) \arrow[r] \arrow[rdd, "\otimes^k"] & K(X \times A/S_k) \\
& K(X) \otimes K(A/S_k) \arrow[u] \\
& K(X) \otimes R(S_k) \arrow[u]
\end{tikzcd}$$

decomposition of tensor product.

__Theorem__ If $x \in K_{2q}(X)$ then $\mathcal{P}_A^k(x) \in K_{2kq}(X \times A/S_k)$

__Proof__

$\exists\ [E] \in K(X, X_{2q-1})$ representing x. $E^{\boxtimes k}$ is acyclic on Z where $Z \subset X^k$ and $Z = \{ \bar{x} \,/\, x_i \in X_{2q-1} \text{ for some } i \}$.
$\therefore X^k_{2kq-1} \subset Z$. Pulling back as above does not change the filtration and hence $\mathcal{P}_A^k(E)$ is acyclic on $(X \times A/S_k)_{2kq-1}$.

i.e. $\mathcal{P}_A^k(x) \in K_{2kq}(X \times A/S_k)$.

__Definition of ψ^k__

Any homomorphism $: R(S_k) \longrightarrow \mathbb{C}$ gives rise to an operation in $K(X)$. Evaluation at g of the character where g generates a k-cycle in S_k, is such a homomorphism and gives rise to the operation ψ^k.

For E a vector bundle $\psi^k(E) = tr_g(E^{\otimes k}) \in K(X) \otimes \mathbb{C}$, th trace being defined as follows.

For V a vector space $V = \sum\limits_{\substack{\text{corr. to} \\ \text{eigenvalues } \lambda_i}} V_i$ under action of g,

and $tr_g(V) = \sum \lambda_i \dim V_i \in \mathbb{C}$.

For vector bundles, $tr_g([V]) = \sum \lambda_i [V_i] \in K(X) \otimes \mathbb{C}$.

In particular if $k = p$, a prime:

<u>Proposition</u> $$a^{\otimes p} = a^p \otimes 1 + \frac{a^p - \psi^p(a)}{p} \otimes N - p$$

where N = regular representation of $\mathbb{Z}_p$, $N - p \in \frac{R(\mathbb{Z}_p)}{I}$

I = augmentation ideal.

This is used to prove Theorem 1.

IMMERSIONS AND EMBEDDINGS OF DOLD MANIFOLDS.

Dr. J. J. Ucci 19th October 1964

Def. Dold Manifold $P(m, n) = \dfrac{S^m \times CP_n}{\sim}$ where the relation is $(x, z) \sim (-x, \bar{z})$.

This is an analytic manifold of dimension $m + 2n$.

Note $P(m, 0) = \mathbb{R}P_m$
$P(0, n) = CP_n$.

Problem To find the least integer k (ℓ) such that there exists an immersion (embedding) of M^n in $\mathbb{R}^{n+k}$ ($\mathbb{R}^{n+\ell}$).

Whitney has shown that $k \leq n-1$ and $\ell \leq n$ and the latter result has been improved for M^n orientable to $\ell \leq n-1$ by Peterson and Massey and Haefliger and Hirsch.

Let X be a finite connected C.W. complex and let $\mathcal{E}(X)$ be the set of isomorphism classes of real vector bundles over X, a semi-group with zero under the Whitney sum of bundles. Then we have

$$\begin{tikzcd}
\mathcal{E}(X) \arrow[r, "\theta"] \arrow[dr, "\text{dim. of bundle}"'] & KO(X) \arrow[d, dashed, "\text{dim.}"] \\
 & \mathbb{Z}
\end{tikzcd}$$

where θ is the obvious homomorphism.

Put $\widetilde{KO}(X) = \operatorname{Ker}(\text{dim.})$

Dual Stiefel-Whitney classes

Suppose we have an embedding $M^n \subset R^{n+k}$. Then $\tau \oplus \nu = n+k$. Define $\overline{W}(M^n) = W(\nu)$. Then $\overline{W}_i(M^n) = 0$ if $i \geq k$.

Similarly for an immersion $M^n \subseteq R^{n+k}$ we have $\overline{W}_i(M^n) = 0$ if $i > k$.

Operations in $KO(X)$

For $x \in \mathcal{E}(X)$ we can form the exterior powers $\lambda^i(x)$, having fibre dimension $\binom{k}{i}$, $k = \dim . x$. Then

$$\lambda^0(x) = 1$$

$$\lambda^1(x) = x$$

$$\lambda^i(x+y) = \sum_{j=0}^{i} \lambda^j(x)\, \lambda^{i-j}(y)$$

We introduce the formal sum $\lambda_t(x) = \sum_{i=0}^{\infty} \lambda^i(x)\, t^i$ with coeffs. in $\mathcal{E}(X)$. If $A(X)$ is the multiplicative group of formal power series in t with coeffs. in $\mathcal{E}(X)$, we have

$$\begin{tikzcd}
\mathcal{E}(X) \arrow[r, "\lambda_t"] \arrow[dr, "\theta"'] & A(X) \\
& KO(X) \arrow[u, dashed, "\lambda_t"']
\end{tikzcd}$$

Define $\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x)$ so that $\sum_{i=0}^{\infty} \gamma^i(x)\, t^i = \sum_{i=0}^{\infty} \lambda^i(x)\, t^i (1-t)^{-i}$

Equating coeffs. we have

$$\gamma^1 = \lambda^1$$

$$\gamma^2 = \lambda^2 + \lambda^1 \quad \text{etc.}$$

$$\gamma_t : KO(X) \longrightarrow A(X).$$

Now $\lambda^i(x) = 0$ if $i > \dim x$ for $x \in \mathcal{E}(X)$. If $x \in KO(X)$ then this is still true if x is <u>positive</u>, i.e. is in the image of θ.

<u>Def</u>. Let $x \in \widetilde{KO}(X)$. The <u>geometric dimension of x</u> $=$ g. dim x $=$ least k s.t. $x + k$ is positive.

Lemma $\lambda^i(x) = 0$ for $i > k \iff \gamma^i(x - k) = 0$ for $i > k$.

Hence we have our main tool:

Prop. If $x \in \widetilde{KO}(X)$ then $\gamma^i(x) = 0$ for $i > g.\dim.x$.

Theorem $M^n \subset \mathbb{R}^{n+k} \Longrightarrow \gamma^i(-\tau_0) = 0$ for $i \geq k$

$M^n \subseteq \mathbb{R}^{n+k} \Longrightarrow \gamma^i(-\tau_0) = 0$ for $i > k$

where $\tau \oplus \nu = n+k$ and we let $\nu_0 = \nu - k$ = reduced normal bundle

$= -\tau_0 = -(\tau - n)$.

Applications ① $M^m = \mathbb{R}P_m$

Adams has shown that $\widetilde{KO}(\mathbb{R}P_m) = \mathbb{Z}_{2^{\phi(m)}}$ where $\phi(m)$ = number of integers i, $0 < i \leq m$ s.t. $i \equiv 0, 1, 2, 4 \bmod 8$. It has a generator x where $x = \tilde{\xi} - 1$ and $\hat{\xi}$ is the canonical line bundle over $\mathbb{R}P_m$. Then $x^2 = -2x$. Since $\tau \oplus 1 = (n+1)\tilde{\xi}$ we have

$$-\tau_0 = -(n+1)x$$

$$\therefore \gamma_t(-\tau_0) = \gamma_t(x)^{-(n+1)} = (1+xt)^{-(n+1)} = \sum_{i=0}^{\infty} \binom{n+i}{i} x^i t^i.$$

$$\therefore \gamma^i(-\tau_0) = \pm \binom{n+i}{i} 2^{i-1} x$$

Let $\sigma(m)$ = largest i s.t. $\binom{n+i}{i} 2^{i-1}$ is not divisible by $2^{\phi(m)}$.

Thus we have

Theorem $\mathbb{R}P_m \not\subseteq \mathbb{R}^{m+\sigma(m)-1}$

$\mathbb{R}P_m \not\subset \mathbb{R}^{m+\sigma(m)}$

② Dold Manifolds

Theorem $P(m,n) \not\subset \mathbb{R}^{m+2n+\sigma'(m,n)}$ where $\sigma'(m,n) = \max(\sigma(m+n), 2[\frac{n}{2}])$.

Proof

Let ξ = line bundle over $P(m,n)$

η = 2 plane " "

Let $i : P(m,0) = \mathbb{R}P_m \longrightarrow P(m,n)$ Then $i^*(\xi) = \tilde{\xi}$

$i^*(\eta) = 1 \oplus \tilde{\xi}$

Let $j : P(0,n) = \mathbb{C}P_n \longrightarrow P(m,n)$. Then $j^*(\eta) = r(\tilde{\eta})$ the decomplexification.

Then $\xi \otimes \xi = 1$

$\xi \otimes \eta = \eta$

and $\tau(m,n) \oplus \xi \oplus 2 = (m+1)\xi \oplus (n+1)\eta$.

Now $\widetilde{KO}(P(m,n)) = \mathbb{Z}_{2^{\phi(m)}} + (\mathbb{Z})^{[\frac{n}{2}]} + G_0$

(direct sum)

where $\mathbb{Z}_{2^{\phi(m)}}$ has a generator $x = \xi - 1$, and $(\mathbb{Z})^{[\frac{n}{2}]}$ has generators $y, y^2, \dots y^{[\frac{n}{2}]}$ where $y = z - x$ and $z = \eta - 2$.

Then $x^2 = -2x$ and $xy = 0$.

$$-\tau_0 = -mx - (n+1)z = -(m+n+1)x - (n+1)y$$

and so $\gamma_t(-\tau_0) = \gamma_t(x)^{-(m+n+1)} \left(1 + yt + \gamma^2(y)t^2\right)^{-(n+1)}$.

But $\gamma^2(y) = -y$ and $y^i = 0$ for $i > [\frac{n}{2}]$.

$$\therefore \gamma_t(-\tau_0) = \left(\sum_{i=0}^{\infty} \binom{m+n+i}{i} x^i t^i\right)\left(1 + y(t-t^2)\right)^{-(n+1)}$$

Highest non-vanishing term in the second factor is (coeff.) $y^{[\frac{n}{2}]}(t^2)^{[\frac{n}{2}]}$.

Define $\sigma'(m,n) = \max\{\sigma(m+n), 2[\frac{n}{2}]\}$, and the result follows from an earlier theorem.

SECONDARY K - THEORY

Dr. D. W. Anderson

26th October 1964

This theory is developed to solve a specific problem, namely:

Let G be a compact Lie group and B_G its classifying space. We have a canonical bundle

$$\begin{tikzcd} G \arrow[r] & E_G \arrow[d] \\ & B_G \end{tikzcd}$$

Let $\rho : G \longrightarrow GL(n:\mathbb{C})$ be a representation of G. We can associate with this a vector bundle $\alpha(\rho)$ of dimension n over B_G. Then $R(G) \xrightarrow{\alpha} K(B_G)$ is a ring homomorphism where $R(G)$ is the representation ring of G.

Now $K(B_G)$ is a complete topological ring and we let $R(G)^\wedge$ be the completion of $R(G)$ in the topology induced by α. Atiyah and Hirzebruch proved

Theorem $\alpha : R(G)^\wedge \longrightarrow K(B_G)$ is an isomorphism.

We now ask if the corresponding theorem is true in the real case.

We have $K^*(X) = \sum K^n(X)$.

and $K^0(B_G) = R(G)^\wedge$, $K^1(B_G) = 0$.

Let KO^* denote real K-Theory. Then we have the commutative diagram

$$\begin{tikzcd} RO(G) \arrow[r] \arrow[d] & R(G) \arrow[d] \\ KO^0(B_G) \arrow[r, "\text{(complexification)}"] & K^0(B_G) \end{tikzcd}$$

. A real representation is self-conjugate and so we will need a theory whose objects include the self-conjugate bundles.

Definition of $KC(X)$

Define $\psi : K^*(X) \longrightarrow K^*(X)$ by $\psi(x) = x - \bar{x}$.

This gives a stable cohomology operation and hence a map of spectra

$$\psi : B_U \longrightarrow B_U$$

Take the fibring induced over B_U by ψ,

$$\begin{tikzcd}
U \arrow[r] & B_C \arrow[r] \arrow[d] & PB_U \arrow[d] \\
& B_U \arrow[r, "\psi"] & B_U
\end{tikzcd}$$

Define $KC(X) = [X, B_C]$

Then we have an exact sequence

$$K^{-1}(X) \xrightarrow{\psi} K^{-1}(X) \xrightarrow{\gamma} KC^0(X) \xrightarrow{\rho} K^0(X) \xrightarrow{\psi} K^0(X).$$

Alternative definition

$KC(X)$ will consist of equivalence classes of pairs (E, e) where E is a complex bundle and $e : E \cong \bar{E}$ is a bundle isomorphism. The equivalence relation is :

$f : E' \cong E''$. f induces an equivalence $(E', e') \sim (E'', e'')$ if $e'' f \simeq \bar{f} e'$

$$\begin{tikzcd}
E' \arrow[r, "f"] \arrow[d, "e'"'] & E'' \arrow[d, "e''"] \\
\bar{E}' \arrow[r, "\bar{f}"] & \bar{E}''
\end{tikzcd}$$

Note

1. $(E', e') \otimes (E'', e'') = (E' \otimes E'', e' \otimes e'')$ and so KC^0 is a ring.
2. $\rho(E, e) = E$.
3. $\underline{\gamma}$

Suppose $f : X \longrightarrow U(n)$

This defines an automorphism of the n-plane bundle

$$X \times \mathbb{C}^n \xrightarrow{f'} X \times \mathbb{C}^n \quad \text{by}$$

$$(x, v) \longrightarrow (x, f(x)(v))$$

Then we have

$$\begin{tikzcd} X \times \mathbb{C}^n \arrow[r, "f'"] \arrow[d] & X \times \mathbb{C}^n \arrow[r, "u", "\cong"'] & \overline{X \times \mathbb{C}^n} \\ X & & \end{tikzcd}$$

and $\gamma(f) = (X \times \mathbb{C}^n, uf') - (X \times \mathbb{C}^n, u)$ and the sequence is exact.

4. $K^0(X)$ is a module over $KC^0(X)$ and ψ is a module map.

Periodicity Theorem

We know that $K^{2n}(pt.) = \mathbb{Z}$, $K^{2n+1}(pt.) = 0$.

$$\tilde{K}^0(S^2) = K^{-2}(pt.) = \mathbb{Z}.$$

If H is the Hopf bundle over S^2 then $h = H - 1$ is a generator of $K^{-2}(pt.)$

$$\bar{h} = \bar{H} - 1 = \frac{1}{H} - 1$$

$$= \frac{1}{1+h} - 1 = 1 - h - 1 = -h \qquad (h^2 = 0)$$

Hence $\psi = 0 : K^{4n}(pt.) \longrightarrow K^{4n}(pt)$

$$\psi = \times 2 : K^{4n+2}(pt.) \longrightarrow K^{4n+2}(pt.)$$

From a consideration of the exact sequence

$$\cdots \longrightarrow K^{n-1}(X) \xrightarrow{\psi} K^{n-1}(X) \xrightarrow{\gamma} KC^n(X) \xrightarrow{\rho} K^n(X) \xrightarrow{\psi} K^n(X) \longrightarrow \cdots$$

we get

$$KC^0(pt.) = \mathbb{Z}$$

$$KC^{-1}(pt.) = \mathbb{Z}_2$$

$$KC^{-2}(pt.) = 0$$

$$KC^{-3}(pt.) = \mathbb{Z}$$

$$KC^{-4}(pt.) = \mathbb{Z}$$

Further $\rho : KC^{-4}(pt.) \longrightarrow K^{-4}(pt.)$ is an isomorphism.

Let ξ = generator of $KC^{-4}(pt.)$

Then $\cup \xi : K^{n}(X) \longrightarrow K^{n-4}(X)$

$: KC^{n}(X) \longrightarrow KC^{n-4}(X)$ is an isomorphism.

Bott showed that there is an exact sequence

$$\cdots \to KO^{n+1}(X) \longrightarrow KO^{n}(X) \xrightarrow{(\text{complexify})} K^{n}(X) \longrightarrow KO^{n+2}(X) \longrightarrow \cdots$$

Theorem For all X, $\widetilde{K}^{n}(X \wedge S^4) \cong \widetilde{KO}^{n}(X \wedge CP_2)$

or more generally, $\widetilde{K}^{n}(X \wedge HP_m) \cong \widetilde{KO}^{n}(X \wedge CP_m)$

where HP_m = quaternionic projective space.

If T = quaternionic Hopf bundle and $t = T - 2$, then the isomorphism is given by

$x \otimes t \longrightarrow r(x \otimes h)$ (real underlying bundle)

$$\widetilde{K}^{n}(X \wedge S^4) \longrightarrow \widetilde{KO}^{n}(X \wedge CP_2)$$

Theorem $\widetilde{KC}^{n}(X \wedge S^5) \cong \widetilde{KO}^{n}(X \wedge P)$

where $P = S^2 \cup_{\eta 2} e^5$ (c.f. $CP_2 = S^2 \cup_{\eta} e^4$)

Proof

$CP_2 \longrightarrow S^2 CP_2 \longrightarrow SP$ is a cofibration and gives rise to the exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{KO}^{*}(X \wedge SP) \arrow[r] & \widetilde{KO}^{*}(X \wedge S^2 CP_2) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \widetilde{KO}^{*}(X \wedge CP_2) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \cdots \\
\cdots \arrow[r] & \widetilde{KC}^{*}(X) \arrow[r] & \widetilde{K}^{*}(X) \arrow[r] & K^{*}(X) &
\end{tikzcd}$$

The isomorphisms are given by the first theorem and the second theorem then follows by the 5-lemma.

Knowing $K^*(X)$ we can compute $KC^*(X)$.

We have cofibrations $CP_2 \longrightarrow S^2P \longrightarrow SP^1 (= S^3 \cup_{\eta_3} e^7)$

$$\text{and} \quad S^6 \xrightarrow{\eta_3} S^3 \longrightarrow SP^1$$

From the second one we see that $KO^*(S^3) \longrightarrow KO^*(S^6)$ is the zero map. Hence we have a short exact sequence

$$0 \longleftarrow KO^*(S^3) \longleftarrow KO^*(SP^1) \xleftarrow{\delta} KO^*(S^6) \longleftarrow 0$$

For instance

$$0 \longleftarrow \underset{\underset{Z}{\parallel}}{KO^7(S^3)} \longleftarrow KO^7(SP^1) \longleftarrow \underset{\underset{Z}{\parallel}}{KO^6(S^6)} \longleftarrow 0.$$

Hence $KO^*(SP^1) \cong KO^*(S^3) \oplus KO^*(S^7)$.

and $\widetilde{KO}^*(X \wedge SP^1) \cong \widetilde{KO}^*(X \wedge S^3) \oplus \widetilde{KO}^*(X \wedge S^7)$

Desuspending and using the first cofibration we have an exact sequence

$$\begin{tikzcd}
\widetilde{KO}^*(X) \oplus \widetilde{KO}^*(X \wedge S^4) \arrow[rr, "(\text{degree } 0)"] \arrow[d, equal] & & \widetilde{KC}^*(X) \arrow[ddl, "(\text{degree } +1)"] \\
\widetilde{K}Sp^*(X) & & \\
& K^*(X) \arrow[uul, "(\text{degree } 0)"'] &
\end{tikzcd}$$

Remark. The following lemma is very useful in finding cofibres.

Lemma Let A, A', B, B' be compact spaces and $\alpha : A \to B$, $\alpha' : A' \to B'$, $a : A \to A'$, $b : B \to B'$ be maps such that $b \circ \alpha = \alpha' \circ a$.

Let $C = B \cup_\alpha CA$. and $\beta : B \longrightarrow C$ be obvious map

" $C' = B' \cup_{\alpha'} CA'$ " $\beta' : B' \longrightarrow C'$ " " "

" $A'' = A' \cup_a CA$ " $a' : A' \longrightarrow A''$ " " "

" $B'' = B' \cup_b CB$ " $b' : B' \longrightarrow B''$ " " "

" $C_0'' = C' \cup_c CC$ where $c : C \longrightarrow C'$ is " "

" $C_1'' = B'' \cup_{\alpha''} CA''$ " $\alpha'' : A'' \longrightarrow B''$ " " "

Let $c' : C' \to C_0''$ and $\beta'' : B'' \to C_1''$ be obvious maps.

Then C_0'' and C_1'' are homeomorphic and the following diagram is strictly commutative.

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "a"'] & B \arrow[r, "\beta"] \arrow[d, "b"] & C \arrow[d, "c"] \\
A' \arrow[r, "\alpha'"] \arrow[d, "a'"'] & B' \arrow[r, "\beta'"] \arrow[d, "b'"] & C' \arrow[d, "c'"] \\
A'' \arrow[r, "\alpha''"] & B'' \arrow[r, "\beta''"] & C_1'' \approx C_0''
\end{tikzcd}$$

Theorem $$\begin{tikzcd}
RSp(G)^\wedge \oplus RO(G)^\wedge \arrow[r] \arrow[d, "\alpha'"] & R(G)^\wedge \arrow[d, "\alpha"] \\
KSp^0(G) \oplus KO^0(B_G) \arrow[r] & K^0(B_G)
\end{tikzcd}$$

Then α' is an isomorphism.

Note $KSp^1(B_G) = KO^1(B_G)$.

Corollaries ① $\widetilde{K}^*(X) = 0 \iff \widetilde{KC}^*(X) = 0 \iff \widetilde{KO}^*(X) = 0$

┌ Proved using Bott's exact sequence

$$\begin{tikzcd}
KO^*(X) \arrow[rr] & & KO^*(X) \arrow[dl] \\
& K^*(X) \arrow[ul] &
\end{tikzcd}$$
┘

② Suppose that $G \subset KO^*(X)$ is a torsion free subgroup.

Then $G \otimes KO^*(pt.) \longrightarrow KO^*(X)$ ($\cup$ multiplication) is an isomorphis

$\iff G \otimes KC^*(pt.) \longrightarrow KC^*(X)$ is an isomorphism

$\iff G \otimes K^*(pt.) \longrightarrow K^*(X)$ is an isomorphism.

Example $X = \mathbb{R}P^\infty$

Now $\widetilde{K}^0(\mathbb{R}P^\infty) = J_2$ = 2-adic numbers

Hence $KO^*(\mathbb{R}P^\infty) = KO^*(pt.) \otimes J_2$.

KUNNETH THEOREMS FOR BORDISM THEORIES

P. Landweber

2nd November 1964.

Although our notation is that of the complex case, except where indicated, all that we do applies also to the oriented case.

Def. $\mathcal{U}_n(X)$ = equivalence classes of pairs (M^n, f) where M^n is a closed differentiable n-manifold with a weakly almost complex structure, $f : M^n \longrightarrow X$, and $(M^n, f) \sim (N^n, g)$ if $\exists$ a weakly almost complex manifold B^{n+1} and $F : B^{n+1} \longrightarrow X$ s.t. $\partial B^{n+1} = M^n + (\underset{\|}{-N^n})$

N^n with opposite $\mathcal{U}$-structure.

and $F/_{M^n} = f$, $F/_{N^n} = g$.

Denote the equivalence class by $[M^n, f]$, and let

$$\mathcal{U}_*(X) = \sum_n \mathcal{U}_n(X).$$

Product Structure

$$\mathcal{U}_n(X) \otimes \mathcal{U}_m(Y) \longrightarrow \mathcal{U}_{m+n}(X \times Y)$$

by $[M^n, f] . [N^n, g] \longrightarrow [M^n \times N^m, f \times g]$.

$\mathcal{U}_*(X)$ is a graded module over $\mathcal{U}_*(pt) = \mathcal{U}$ and $\mathcal{U} = \mathbb{Z}[x_2, x_4, \dots]$. Hence we have a product of graded $\mathcal{U}$-modules

$$\alpha : \mathcal{U}_*(X) \otimes \mathcal{U}_*(Y) \longrightarrow \mathcal{U}_*(X \times Y).$$

Def $\mu : \mathcal{U}_n(X) \longrightarrow H_n(X; \mathbb{Z})$ by

$\mu([M^n, f]) = f_*(\sigma_n)$ where $\sigma_n \in H_n(M^n; \mathbb{Z})$ is the fundamental class.

Def. Let G be a finite group. Then $\mathcal{U}_n(G)$ = classes of weakly almost complex manifolds M^n, on which G acts freely by diffeomorphisms which preserve the $\mathcal{U}$-structure.

Then $\mathcal{U}_n(G) \cong \mathcal{U}_n(B_G)$

⌈This is easily seen since $M^n \longrightarrow M^n/G$ is a principal G-bundle, which determines an element of $\mathcal{U}_n(B_G)$ ⌋

In particular $\mathcal{U}_*(B_{\mathbb{Z}_p}) \cong \mathcal{U}_*(\mathbb{Z}_p)$

Conner and Floyd showed in the oriented case:

Theorem ① $\alpha: \Omega_*(B_{\mathbb{Z}_p}) \otimes_{\Omega} \Omega_*(Y) \longrightarrow \Omega_*(B_{\mathbb{Z}_p} \times Y)$

is a monomorphism for any C.W. complex Y.

② If X is a C.W. complex and $\Omega_*(X)$ is a free Ω-module, then

$$\alpha: \Omega_*(X) \otimes_{\Omega} \Omega_*(Y) \longrightarrow \Omega_*(X \times Y)$$

is an isomorphism for any C.W. complex Y.

(This is true for any multiplicative homology theory).

Note If $H_*(X)$ is a free abelian group then $\Omega_*(X)$ is free over Ω (Corollary to Lemma 2)

We will prove:

Theorem Let X, Y be C.W. complexes with base points and X be of finite type (in particular X^k is finite). Suppose $\mu: \mathcal{U}_*(X) \longrightarrow H_*(X; \mathbb{Z})$ is onto. Then $\exists$ a natural Kunneth formula

$$0 \longrightarrow \tilde{\mathcal{U}}_*(X) \otimes_{\mathcal{U}} \tilde{\mathcal{U}}_*(Y) \xrightarrow{\alpha} \tilde{\mathcal{U}}_*(X \wedge Y) \xrightarrow{\beta} \tilde{\mathcal{U}}_*(X) *_{\mathcal{U}} \tilde{\mathcal{U}}_*(Y) \longrightarrow 0$$

(Torsion product

Note (i) It suffices to prove the theorem for X finite.

(ii) Example $X = B_{Z_p}$. Then $H_*(X) = Z_p$ in odd dimension and in dimension $2n-1$ we have

$$S^{2n-1}/Z_p \longrightarrow B_{Z_p} = K(Z_p, 1)$$

Lemma 1 Let X be a finite C.W. complex with base pt. Let N be onto. Then there is an inclusion

$$i : A \longrightarrow S^m X$$

of a finite C.W. complex A into a suitably high suspension of X s.t. the reduced bordism exact sequence of $(S^m X, A)$ is

$$0 \longrightarrow \underset{\underset{S^m X / A}{\parallel}}{\tilde{\mathcal{U}}_*(B)} \longrightarrow \tilde{\mathcal{U}}_*(A) \longrightarrow \tilde{\mathcal{U}}_*(S^m X) \longrightarrow 0$$

and $\tilde{\mathcal{U}}_*(A)$, $\tilde{\mathcal{U}}_*(B)$ are free over $\mathcal{U}$.

Lemma 1 $\Rightarrow$ reduced Kunneth formula

We have

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\mathcal{U}}_*(S^m X) \underset{\mathcal{U}}{*} \tilde{\mathcal{U}}_*(Y) \arrow[r] & \tilde{\mathcal{U}}_*(B) \underset{\mathcal{U}}{\otimes} \tilde{\mathcal{U}}_*(Y) \arrow[r] \arrow[d, "\alpha_A"] & \tilde{\mathcal{U}}_*(A) \underset{\mathcal{U}}{\otimes} \tilde{\mathcal{U}}_*(Y) \arrow[r] \arrow[d, "\alpha_B"] & \tilde{\mathcal{U}}_*(S^m X) \underset{\mathcal{U}}{\otimes} \tilde{\mathcal{U}}_*(Y) \arrow[r] \arrow[d, "\alpha"] & 0 \\
{} \arrow[r] & \tilde{\mathcal{U}}_*(S^m X \wedge Y) \arrow[u, dashed, "\beta"] \arrow[r] & \tilde{\mathcal{U}}_*(B \wedge Y \arrow[r] & \tilde{\mathcal{U}}_*(A \wedge Y) \arrow[r] & \tilde{\mathcal{U}}_*(S^m X \wedge Y) \arrow[r] & {}
\end{tikzcd}$$

The second result of Conner and Floyd is true for complex bordism and so α_A and α_B are isomorphisms. This defines β and proves exactness.

Lemma 2 Let X be as in the theorem. Let $\{c_i\}$ be generators of $H_n(X; Z)$. Pick $\{\gamma_i\}$ so that $N(\gamma_i) = c_i$.

Then ⓐ $\{\gamma_i\}$ generate $\tilde{\mathcal{U}}_*(X)$ as a $\mathcal{U}$-module.

ⓑ If $\tilde{H}_*(X; Z)$ is free and c_i are free generators, then the γ_i's are free generators for $\tilde{\mathcal{U}}_*(X)$ over $\mathcal{U}$.

Corollary $\tilde{H}_*(X)$ free $\Rightarrow$ $\tilde{\mathcal{U}}_*(X)$ is free over $\mathcal{U}$.

Proof (of Lemma 2)

We consider the bordism spectral sequence. There is a filtration

$$0 \subset J_{0,n} \subset \cdots \subset J_{n-1,1} \subset J_{n,0} = \tilde{\mathcal{U}}_n(X)$$

where $J_{p,\,n-p} = \mathrm{Im}\,\{\tilde{\mathcal{U}}_n(X_p) \longrightarrow \tilde{\mathcal{U}}_n(X)\}$.

Then $$\frac{J_{p,\,n-p}}{J_{p-1,\,n-p+1}} \cong E^{\infty}_{p,\,n-p}.$$

$$E^2_{p,q} \cong \tilde{H}_p(X;\mathcal{U}_q) \cong \tilde{H}_p(X)\otimes\mathcal{U}_q$$

Now $\mu : \tilde{\mathcal{U}}_n(X) \longrightarrow \tilde{H}_n(X;\mathbb{Z})$ is edge homomorphism.

$$\left(\tilde{\mathcal{U}}_n(X) = J_{n,0} \xrightarrow{epi} E^{\infty}_{n,0} \xrightarrow{mono} E^2_{n,0} = \tilde{H}_n(X:\mathbb{Z})\right)$$

Further, μ is onto $\Leftrightarrow$ the spectral sequence collapses.

⌈ $\mathrm{Im}\,\mu$ = permanent cycles in $E^2_{n,0}$

$\therefore$ $\alpha\otimes 1 \in E^2_{p,0} = \tilde{H}_p(X)\otimes\mathcal{U}_0$ is killed.

But $\mathcal{U}$ acts on the whole spectral sequence and the differentials are homomorphisms over $\mathcal{U}$ ⌋

$\therefore$ $E^{\infty}_{p,\,n-p} = E^2_{p,\,n-p}$ and assertions of Lemma 2 follow.

Remark If $\tilde{H}_*(X)$ is free then μ is onto and so we don't need to assume that μ is onto in the corollary.

Lemma 3 Same as Lemma 1 with homology replacing complex bordism.

Lemma 2 and Lemma 3 $\Rightarrow$ Lemma 1

We have

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\mathcal{U}}_*(B) \arrow[r] \arrow[d, "\mu"] & \tilde{\mathcal{U}}_*(A) \arrow[r] \arrow[d, "\mu"] & \tilde{\mathcal{U}}_*(S^m X) \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \tilde{H}_*(B;\mathbb{Z}) \arrow[r] & \tilde{H}_*(A;\mathbb{Z}) \arrow[r] & \tilde{H}_*(S^m X;\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

Since $\tilde{H}_*(B:\mathbb{Z})$ and $\tilde{H}_*(A;\mathbb{Z})$ are free, so are $\tilde{\mathcal{U}}_*(B)$ and $\tilde{\mathcal{U}}_*(A)$. Exactness of the top line comes by diagram chasing.

There remains:

<u>Proof of Lemma 3</u>

$$\begin{tikzcd} \tilde{\mathcal{U}}_*(X) \arrow[r, "\nu"] \arrow[d, "\cong"'] & \tilde{H}_*(X;\mathbb{Z}) \arrow[d, "\cong"] \\ \tilde{\mathcal{U}}^*(D_n(X)) \arrow[r, "\nu"] & \tilde{H}^*(D_nX) \end{tikzcd}$$

$\nu : \tilde{\mathcal{U}}_*(X) \longrightarrow \tilde{H}_*(X;\mathbb{Z})$ is onto

(Alexander - Spanier Duality isomorphisms)

$\therefore$ The lower ν is onto.

Let $\{c_i\}$ be generators for $\tilde{H}^*(D_nX)$ and pull back to $\{\gamma_i\} \subset \tilde{\mathcal{U}}^*(D_n(X))$. Let γ_i have dimension n_i.

$$\gamma_i \in \tilde{\mathcal{U}}^{n_i}(D_nX) \cong [S^k(D_nX), M_{k+n_i}]$$

where M_n is the n^{th} object of the Thom spectrum

.... $MU(n)$, $SMU(n)$, ...

γ_i is represented by $[f_i] \in [S^k(D_nX), M_i]$

Let $M = \prod M_i$.

$f = \prod f_i \ : \ S^k(D_nX) \longrightarrow M$

Now $H^*(M_i)$ is free and so $H^*(M)$ is free.

Let
$$A = DM \xrightarrow{\quad i \quad} D(S^k(D_nX))$$
$$\| $$
$$S^mX.$$

THE HOMOTOPY GROUPS OF A WEDGE OF SUSPENSIONS

Dr. G. J. Porter 9th November 1964

These notes are intended to serve as an appendix to the mimeographed notes 'A generalization of the Hilton - Milnor Theorem' by Gerald J. Porter and notation is in accordance with this.

1. Motivation

A primary homotopy operation of type $(n_1, \dots, n_k ; m)$ is a function $\theta : \sum_{i=1}^{k} \pi_{n_i}(X) \longrightarrow \pi_m(X)$ defined for every X. The primary homotopy operations are in (1-1) correspondence with the elements of $\pi_m(S^{n_1} \vee \dots \vee S^{n_k})$, the correspondence given as follows:

Let $[\theta] \in \pi_m(S^{n_1} \vee \dots \vee S^{n_k})$, $[f_i] \in \pi_{n_i}(X)$.
Then the f_i's define $f_1 | f_2 | \dots | f_k : S^{n_1} \vee \dots \vee S^{n_k} \longrightarrow X$ and we define the operation by composition of θ with $f_1 | \dots | f_k$.

$$S^m \xrightarrow{\theta} S^{n_1} \vee \dots \vee S^{n_k} \xrightarrow{f_1 | \dots | f_k} X.$$

Similarly the groups $\pi_m(T_{k-i}(S^{n_1}; \dots S^{n_k}))$ define i-ary homotopy operations. We have

$$\begin{tikzcd}
S^m \arrow[r, "\theta"] & S^{n_1}\vee\cdots\vee S^{n_k} \arrow[r, "f_1|\cdots|f_k"] \arrow[dd] & X \\
T_{k-1} \arrow[ur, phantom, "="] & & \\
S^m \arrow[r, "\theta_2"] & \bigcup\limits_{1\le i<j\le k} S^{n_i}\times S^{n_k} \arrow[uur, dashed, "\phi_2"] \arrow[d] & \\
T_{k-2} \arrow[ur, phantom, "="] & \vdots \arrow[d] & \\
S^m \arrow[r, "\theta_i"] & T_{k-i}(S^{n_1}, \dots S^{n_k}) \arrow[uuuur, dashed, "\phi_i"'] &
\end{tikzcd}$$

and the i-ary homotopy operation is defined whenever there exists a map ϕ_i extending $f_1|\dots|f_k$.

More generally we can replace spheres by suspensions of spaces and homotopy groups by homotopy classes of maps so that a homotopy operation is a function

$$: \sum_{i=1}^{k} \pi(\Sigma X_i, Y) \longrightarrow \pi(\Sigma Z, Y)$$

and the i-ary operations come from the diagram

$$\begin{tikzcd}
\Sigma Z_1 \arrow[r, "\theta_1"] & \Sigma X_1\vee\cdots\vee\Sigma X_k \arrow[r, "f_1|\cdots|f_k"] \arrow[d] & Y \\
\Sigma Z_2 \arrow[r, "\theta_2"] & T_{k-2}(\Sigma X_1, \dots, \Sigma X_k) \arrow[ur, dashed, "\phi_2"] \arrow[d] & \\
 & \vdots \arrow[d] & \\
\Sigma Z_i \arrow[r, "\theta_i"] & T_{k-i}(\Sigma X_1, \dots, \Sigma X_k) \arrow[uuur, dashed, "\phi_i"'] &
\end{tikzcd}$$

2. Proof of Theorem 2

$F_i \subset PX_1 \times \dots \times PX_n$ is the subspace with at least i coords loops. In general suppose $Y_j \subset K_j$,

$j = 1, \dots n$, and K_j contractible. Let

$$J_i(Y_1, \dots Y_n) = \{(k_1, \dots, k_n) \mid \text{at least } i \text{ of the } k_j \in Y_j\}$$

It suffices to prove

Lemma $$J_i(Y_1, \dots, Y_n) \sim \bigvee_{j=n-i+1}^{n} \bigvee^{\binom{j-1}{n-i}} \bigvee_{1 \le i_1 < \dots < i_j \le n} \Sigma^{n-i} \Lambda(Y_{i_1}, \dots Y_{i_j})$$

Proof

Let $J_i^n = J_i(Y_1, \dots, Y_n)$. Then $J_i^n \sim J_{i-1}^{n-1} \times Y_n \cup J_i^{n-1} \times K_n$ and J_i^{n-1} is contractible in J_i^n and Y_n is contractible in K_n.

In general, let A be contractible in X and B contractible in Y. Then

$$X \times B \underset{A \times B}{\cup} A \times Y \sim \frac{X \times B}{B} \vee \Sigma \Lambda(A, B) \vee \frac{A \times Y}{A}$$

⌈ Up to homotopy equivalence we have $A \subset CA \subset X$.

$$X \times B \cup A \times Y \sim X \times B \underset{CA \times B}{\cup} (CA \times B \cup A \times CB) \underset{A \times CB}{\cup} A \times Y.$$

Then factor out $CA \vee CB$ ⌋

We may assume $X = \Sigma Z$. Then

$$\frac{\Sigma Z \times B}{B} \sim \Sigma \Lambda(Z, B) \vee \Sigma Z$$

⌈ Suppose we have a cofibration

$$\begin{tikzcd} Z \arrow[r, "i"] & \Sigma \Lambda(Z, B) \arrow[r] & \dfrac{\Sigma \Lambda(Z, B)}{Z} \arrow[r, shift left, "\partial"] & \Sigma Z \arrow[l, shift left, "j"] \arrow[r] & \dots \end{tikzcd}$$

s.t. $i \sim 0$ and $\exists\, j$ s.t. $\partial j \sim 1$.

Then $$\frac{\Sigma \Lambda(Z, B)}{Z} \sim \Sigma \Lambda(Z, B) \vee \Sigma Z$$

But $$\frac{\Sigma \Lambda(Z, B)}{Z} \sim \frac{CZ \times B \cup Z \times CB}{Z \times CB} \sim \frac{CZ \times B}{Z \times B}$$

$$\cdots \sim \frac{C_+ Z \times B \cup C_- Z \times B}{C_- Z \times B} \sim \frac{\Sigma Z \times B}{B} \quad \rule{1em}{0.4pt}\!\!\rfloor$$

To start off the induction we require the cases J_0^r and J_r^r. $J_0^r = K_1 \times \dots \times K_{rr}$ is contractible. For J_r^r we use

$$\Sigma(Y_1 \times \dots \times Y_r) \sim \bigvee_{j=1}^{r} \bigvee_{1 \le i_1 < \dots < i_j \le r} \wedge(Y_{i_1}, \dots Y_{i_j}, S^0).$$

3. Examples (i) $F_1 = \Sigma^{n-1} \wedge (\Omega X_1, \dots, \Omega X_n)$

Let $X_1 = \dots = X_n = CP^\infty$. Then $\Omega CP^\infty = S^1$.

$\therefore \Omega T_1((CP^\infty)^n) \sim S^1 \times \dots \times S^1 \times \Omega S^{2n-1}$

(ii) If $r < 11n - 6$

$$\pi_r(T_5{}^{10}(S^n)) \sim \sum_{j=1}^{10} \pi_r(S^n) \oplus \sum_{k=6}^{10} \sum_{j=1}^{i_k} \pi_r(S^{k(n-1)+5})$$

where $\sum_{j=1}^{n} G = \underbrace{G \oplus \dots \oplus G}_{n \text{ times.}}$

$i_6 = 210$

$i_7 = 1980$

$i_8 = 10,425$

$i_9 = 40,040$

$i_{10} = 126,156$

HONEST STABLE HOMOTOPY AND ABELIAN CATEGORIES

Prof. Peter Freyd 16th November 1964

$\mathcal{T}$ will denote any of 3 topological categories.

$\mathcal{T}_{sc}$ = category of simplicial complexes
$\mathcal{T}_{fcw}$ = " " finite C.W. "
$\mathcal{T}_{acw}$ = " " all " ".

For $X, Y \in \mathcal{T}$ let (X, Y) denote homotopy classes of maps from X to Y, and let S be the suspension functor. Then we have

$$(X, Y) \xrightarrow{S} \underset{\text{group}}{(SX, SY)} \xrightarrow{S} \underset{\text{abelian group.}}{(S^2X, S^2Y)} \longrightarrow \cdots$$

__Def__. The stable homotopy group from X to Y is

$$(X, Y)_{\mathcal{S}} = \lim_{j\to\infty} (S^jX, S^jY)$$

__Def__ Stable Category $\mathcal{S}_0(\mathcal{T})$ is the category whose objects are spaces in $\mathcal{T}$ and whose maps are $(X, Y)_{\mathcal{S}} = \lim\limits_{j\to\infty} (S^jX, S^j$

We compose two maps $S^jX \longrightarrow S^jY$, $S^kY \longrightarrow S^kZ$ by suspending the appropriate one.

We want to enlarge $\mathcal{S}_0$ so that S is an automorphism.

Def $\mathcal{S}$ is the category whose objects are pairs $\langle X, n\rangle$ $X \in \mathcal{T}$, $n \in \mathbb{Z}$ and whose maps are

$$(\langle X, n\rangle, \langle Y, m\rangle) = \lim_{j \to \infty} (S^{j+n} X, S^{j+m} Y)$$

Then $\langle X, n\rangle \longrightarrow \langle SX, n\rangle$ and $\langle X, n\rangle \longrightarrow \langle X, n+1\rangle$ are naturally equivalent and S is an automorphism with $S^{-1}\langle X, n\rangle = \langle X, n-1\rangle$.

Of particular importance are the stable homotopy groups of the sphere $G_n = (S^n, S^0)_{\mathcal{S}}$. $\sum_{n=0}^{\infty} G_n$ is a skew-commutati ring.

The main result is:

Theorem $\exists$ an abelian category $\mathcal{A}$ with enough projectives and injectives s.t. $\mathcal{S} \subset \mathcal{A}$, $\mathcal{S}$ is a full subcategory (i.e. $(X, Y)_{\mathcal{S}} = (X, Y)_{\mathcal{A}}$) and

1) $A \in \mathcal{S} \iff A$ is projective

2) $A \in \mathcal{S} \iff A$ is injective

3) $\forall A \in \mathcal{A}$, $\exists X \longrightarrow Y \in \mathcal{S}$ s.t. $A = \mathrm{Im}(X \longrightarrow Y)$.

4) Let $f : X \longrightarrow Y \in \mathcal{T}$ and let C_f = mapping cone. Then $X \longrightarrow Y \longrightarrow C_f$ represents an exact sequence in $\mathcal{A}$.

Proof

By examining the assertions of the theorem we are able to work backwards and find $\mathcal{A}$.

4) $\Rightarrow$ $(Z, X)_{\mathcal{S}} \longrightarrow (Z, Y)_{\mathcal{S}} \longrightarrow (Z, C_f)_{\mathcal{S}}$
$(C_f, Z)_{\mathcal{H}} \longrightarrow (Y, Z)_{\mathcal{H}} \longrightarrow (X, Z)_{\mathcal{H}}$ are exact.

. From 3) Suppose $f : A \to B \in \mathcal{A}$. We can represent A and B by

$$\begin{tikzcd}
X \arrow[r, two heads] \arrow[d, dashed] & A \arrow[r, tail] \arrow[d, "f"] & Y \arrow[d, dashed] \\
X' \arrow[r, two heads] & B \arrow[r, tail] & Y'
\end{tikzcd}$$

f induces maps (not unique) : $X \to X'$, $Y \to Y'$.

Let $G \subset (X, X')_{\mathcal{S}} \times (Y, Y')_{\mathcal{S}}$ be the maps induced by all maps : $A \to B$.

Let $K \subset (X, X')_{\mathcal{S}} \times (Y, Y')_{\mathcal{S}}$ be the maps induced by the zero map : $A \to B$.

Then $(A, B)_{\mathcal{A}} = G/_K$.

__Def.__ Given $\mathcal{S}$ let $\mathcal{S}^{\to}$ be the category whose objects are maps : $X \to Y$ and whose maps are commutative squares

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X' \arrow[r] & Y'
\end{tikzcd}.$$

Then $\mathcal{S}$ is embedded in $\mathcal{S}^{\to}$ by $X \to (: X \xrightarrow{id} X)$

Define
$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X' \arrow[r] & Y'
\end{tikzcd} \equiv 0$$
iff the diagonal $X \to Y' = 0$ (ie. is null homotopic).

Define $\mathcal{A} = \mathcal{S}^{\to}/_{\equiv}$

Then $\mathcal{S} \to \mathcal{S}^{\to} \to \mathcal{S}^{\to}/_{\equiv}$ i.e. $\mathcal{S} \subset \mathcal{A}$.

__Alternative description of $\mathcal{A}$__

For every $X \to Y \in \mathcal{S}$ $\exists$ an embedding $X \to \bar{Y}$ s.t. $(X \to Y) \simeq (X \to \bar{Y}) \in \mathcal{S}$.

Let $\mathcal{S}^{\to}$ have as objects pairs $X \subset Y$ and
$$\begin{tikzcd}
X \arrow[r, phantom, "\subset"] \arrow[d] & Y \arrow[d] \\
X' \arrow[r, phantom, "\subset"] & Y'
\end{tikzcd}$$
as maps.

An element of $(X \subset Y, X' \subset Y')$ is represented by a $\mathcal{J}$-map $f : S^j Y \longrightarrow S^j Y'$ for some j s.t. $f(S^j(X)) \subset S^j(X')$.

Let $g : S^{j'} Y \longrightarrow S^{j'} Y'$ represent another element.

Define $f \equiv g$ iff $\exists\ k, l$ s.t. $S^{k+j} X \longrightarrow S^{k+j} Y \xrightarrow{S^k f} S^{k+j}$

is homotopic to $S^{l+j'} X \longrightarrow S^{l+j'} Y \xrightarrow{S^l g} S^{l-j'}$

Then $\mathcal{A} = \dfrac{\mathcal{S}^{\rightarrow}}{\equiv}$.

Further Results

5) $\text{Ext}^{n+3}(A, B) \cong \text{Ext}^n(SA, SB)$ for $n > 0$.

6) $\mathcal{A}(\mathcal{J}_{fcw})$ is self dual.

Hypothesis Let $f : X \longrightarrow Y \in \mathcal{S}_{f.c.w}$

$$\text{If},\ \forall n,\ (S^n, X) \xrightarrow{(S^n f)} (S^n, Y) = 0$$

$$\text{then } X \xrightarrow{f} Y = 0.$$

If the hypothesis is true then

1. $\exists$ elements in $\mathcal{M} = \sum_{n=0}^{\infty} G_n$ of arbitrary order
2. $\mathcal{M}$ is not nilpotent.
3. $\alpha \neq 0 \implies \alpha \mathcal{M} \neq 0$.

Note We can find a counter-example to the hypothesis for the case $\mathcal{S}_{acw}$.

SPACES WITH MULTIPLICATION AND COMULTIPLICATION

R. Holzsager

23rd November 1964

Space = 0-connected space with base-pt. with the homotopy type of a C.W. complex.

Def H-space X is an H-space if ∃ an extension (up to homotopy) $: X \times X \to X$ of the folding map f.

$$\begin{tikzcd} X \vee X \arrow[r, phantom, "\subset"] \arrow[dr, "f"'] & X \times X \arrow[d, dashed] \\ & X \end{tikzcd}$$

H'-space X is an H'-space if ∃ a map making the diagram homotopy commutative

$$\begin{tikzcd} X \vee X \arrow[r, phantom, "\subset"] & X \times X \\ & X \arrow[ul, dashed] \arrow[u, "\Delta = \text{diagonal}"'] \end{tikzcd}$$

Remark In an H'-space all cup products are trivial.

Lemma If n is odd then $H_i(K(Q; n); \mathbb{Z}) = 0$ for $i > n$ ($\mathbb{Q}$ = rationals)

Proof

By induction on n. It is true for $n=1$. Asume it for n (odd). We the consider the homology spectral sequences of the fibrations

$$\begin{tikzcd} K(Q,n) \arrow[r] & E \arrow[d] & \text{and} & K(Q,n+1) \arrow[r] & E' \arrow[d] \\ & K(Q,n+1) & & & K(Q,n+2) \end{tikzcd}$$

to prove it for $n+2$.

Remark Adams has shown that S^1, S^3, S^7 are the only spheres which admit a multiplication. (all admit comultiplication) Browder has shown that if M is a finite complex having an H and H' structure then M has Poincaré Duality and

$H_*(M) = H_*(S^n)$.

<u>Theorem</u> a) If $G \subset \mathbb{Q}$ then the Moore space $M(G, 2k+1)$ $k>0$ is an H-space $\Longleftrightarrow$ either $\frac{G}{2G} = 0$ or $k = 1$ or 3.

b) If $\mathbb{Z}_{p^\infty}$ = p-primary component of $\mathbb{Q}/\mathbb{Z}$ (p a prime) then $M(\mathbb{Z}_{p^\infty}, 2k)$ $k>0$ is an H space $\Longleftrightarrow$ either $p \neq 2$ or $k = 1, 3$.

c) Call X, Y <u>mutually prime</u> if $H_i(X) \otimes H_j(Y) = 0$ and $\mathrm{Tor}(H_i(X), H_j(Y)) = 0$ for any $i, j > 0$. Then a space is both an H-space and an H' space $\Longleftrightarrow$ it is homotopy equivalent to either S^1 or to the wedge of a pairwise mutually prime family of spaces occurring in (a) and (b).

<u>Proof</u>

(a) Let $M = M(G, 2k+1)$ $k>0$. The only obstructions to making M an H-space lie in

$$H^{4k+2}(M \times M, M \vee M; \pi_{4k+1}(M)) = \mathrm{Hom}(G \otimes G, \pi_{4k+1}(M))$$

and $$H^{4k+3}(M \times M, M \vee M; \pi_{4k+2}(M)) = \mathrm{Ext}(G \otimes G, \pi_{4k+2}(M)).$$

We show first that the second of these groups vanishes. For all primes p s.t. $\frac{G}{pG} = 0$, the map $M \longrightarrow K(\mathbb{Q}, 2k+1)$ corresponding to $G \longrightarrow \mathbb{Q}$ induces homology isomorphims modulo the class of torsion groups with no p-primary component. Using our result on $K(\mathbb{Q}, n)$ for n odd and Serre's work we see that $\pi_r(M)$ has no p-component for $r > 2k+1$ and p s.t. $\frac{G}{pG} = 0$. For primes p s.t. $\frac{G}{pG} \neq 0$ $\exists$ a monomorphism $\mathbb{Z} \longrightarrow G$ s.t. the corresponding map $S^{2k+1} \longrightarrow M$ induces homology isomorphims modulo the torsion groups with no p-component. Hence $\exists$ an element of $G \otimes G$ which is not divisible by any prime dividing the order of $\pi_{4k+2}(M)$. Let it generate a subgroup $\mathbb{Z}$ and apply $\text{Ext}(\ \ , \pi_{4k+2}(M))$ to $0 \to \mathbb{Z} \to G\otimes G \to \frac{G\otimes G}{\mathbb{Z}} \to 0$ to get the result.

Now $\text{Hom}(G\otimes G; \pi_{4k+1}(M))$ is given by the Whitehead products

$$G \otimes G = \pi_{2k+1}(M) \otimes \pi_{2k+1}(M) \longrightarrow \pi_{4k+1}(M)$$

and we can show that these are zero if $k = 1$ or 3 or $\frac{G}{2G} = 0$.

Conversely if $k \neq 1$ or 3 and $\frac{G}{2G} \neq 0$ consider the projective plane $P(M)$. Then $H^*(P(M); \mathbb{Z}_2) = \frac{\mathbb{Z}_2[x]}{(x^3)}$ which is contradicted by the results of Adem and Adams.

b) $\Rightarrow$ By the same argument as above on the projective plane

$\Leftarrow$ Let G be the kernel of some epimorphism $\mathbb{Q} \to \mathbb{Z}_{p^\infty}$

e.g.
$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r] & \mathbb{Q} \arrow[r] \arrow[dr] & \mathbb{Z}_{p^\infty} \arrow[r] & 0 \\ & & & \mathbb{Q}/\mathbb{Z} \arrow[u] & \end{tikzcd}$$

Consider the fibration
$$\begin{tikzcd} F \arrow[r] & M(G, 2k+1) = M \arrow[d, "(\text{corr. to } G \subset \mathbb{Q})"] \\ & K(\mathbb{Q}, 2k+1) \end{tikzcd}$$

The a consideration of the homology spectral sequence of this fibration shows that $F = M(\mathbb{Z}_{p^\infty}, 2k)$.

Consider the commutative diagram
$$\begin{tikzcd} F \vee F \arrow[r, "f"] \arrow[d] & F \arrow[d, "\text{inclusion}"] \\ F \times F \arrow[r, "m"] \arrow[dr, "\text{inclusion}"'] & M \\ & M \times M \arrow[u, "m"'] \end{tikzcd}$$

where f is the folding map and m is the multiplication on M which exists by a). The obstructions to making F into an H-space lie in $H^i(F\times F, F\vee F; \pi_i(M,F)) = H^i(F\times F, F\vee F; \pi_i(K(\mathbb{Q}, 2k+1)))$ $= 0$ by Kunneth and Universal Coefficient Theorem.

c). $\Leftarrow$ Moore spaces have the homotopy type of suspensions. $\therefore$ so does a wedge of them and is thus an H'-space. For a mutually prime family of spaces occurring in (a) and (b) the wedge has the same homotopy type as the cartesian product and $\therefore$ is an H-space.

$\Rightarrow$ Let X be both an H-space and an H'-space. Then for F a field, $\dim \tilde{H}^*(X;F) \leq 1$.

$\ulcorner$ Let x, y be linearly independent and $m: X \times X \longrightarrow X$ be a multiplication. Then

$$m^*(x) = x \otimes 1 + 1 \otimes x + \bar{x}$$
$$m^*(y) = y \otimes 1 + 1 \otimes y + \bar{y}$$
$$\therefore \quad 0 = m^*(xy) = y \otimes x \pm x \otimes y \neq 0 \quad \lrcorner$$

Now X an H'-space $\Rightarrow$ $\pi_1(X)$ is free (Eilenberg and Ganea)

X an H-space $\Rightarrow$ $\pi_1(X)$ is abelian.

$\therefore \quad \pi_1(X) = \mathbb{Z}$ or 0.

(i) $\underline{\pi_1(X) = \mathbb{Z}}$

By the remark above $H_*(X) \cong H_*(S^1)$. Find a map $f: X \longrightarrow S^1$ realising this isomorphism and we will show that f is a homotopy equivalence. Let Φ be the fibre of the fibration arising from f. Then $\pi_1(S^1)$ acts trivially on $H_*(\Phi)$.

$\ulcorner$ ℓ a loop in S^1. Cover it by $\ell' \in X$.

Define $h_t(\phi) = m(\phi, \ell'(t))$ a covering homotopy.

Then $\ell(x) = h_{1*}(x) = x$ $\lrcorner$

A consideration of the homology spectral sequence of the fibration shows that $H_*(\Phi)$ is trivial and since Φ is simply connected, Φ is contractible and $X \sim S^1$

(iii) $H_1(X) = 0$

$H_*(X)$ has torsion except in at most one dimension. Let $\exists$ some p torsion. The we have a monomorphism $\mathbb{Z}_p \longrightarrow H_i(X)$ for some i and $H^{i+1}(X; \mathbb{Z}_p) = \operatorname{Ext}(H_i(X), \mathbb{Z}_p) \neq 0$. Since X has no other mod. p cohomology we see that $H_j(X)$ is divisible by p for every j (otherwise $\operatorname{Hom}(H_j(X), \mathbb{Z}_p) \neq 0$) Also the p-component of $H_*(X)$ is $\mathbb{Z}_{p^\infty}$.

Suppose for a field F $\exists$ $0 \neq x \in H^i(X; F)$ for i even. Then $0 = m^*(x^2) = (x \otimes 1 + 1 \otimes x)^2 = 2(x \otimes x) \neq 0$ unless char. $F = 2$. Consideration of $H^i(X; \mathbb{Z}_4) \longrightarrow H^i(X; \mathbb{Z}_2) \longrightarrow 0$ takes care of the case $F = \mathbb{Z}_2$. $\therefore$ The non-trivial cohomology over $\mathbb{Q}$ or $\mathbb{Z}_p$ occurs in odd dimension and so in homology the torsion occurs in even dimensions and the torsion free part in an odd dimension.

If we can show that $\exists$ maps : $M(G; j) \longrightarrow X$

$$M(\mathbb{Z}_{p^\infty}; i) \longrightarrow X$$

realising the homology then the induced map on the wedge will be a homotopy equivalence by J. H. C. Whitehead's theorem. The pairwise mutual primeness follows from the above description of $H_*(X)$.

If the torsion-free part is zero then Serre's mod $\mathcal{C}$ Theorem implies the realisability of the $\mathbb{Z}_{p^\infty}$ part. If $H^n(X; \mathbb{Q}) = \mathbb{Q}$ for some n take a map $f: X \longrightarrow K(\mathbb{Q}, n)$ representing a generator and consider the fibre F of the associated fibration. Examination of the spectral sequence shows F to be of the form already dealt with.

It remains to realise the torsion free part, i.e. we must find a lifting

$$\begin{tikzcd}
F \arrow[r] & X \arrow[d, "f"] \\
M(G,n) \arrow[r] \arrow[ur, dashed] & K(Q,n)
\end{tikzcd}$$

The obstructions to this lie in $H^i(M(G,n); \pi_{i-1}(F))$

$= 0$ if $i \neq n, n+1$ and

$$H^n(M(G,n); \pi_{n-1}(F)) = Hom(G, \pi_{n-1}(F))$$

$$H^{n+1}(M(G,n); \pi_n(F) = Ext(G, \pi_n(F)).$$

Examination of $\pi_{n-1}(F)$ and $\pi_n(F)$ shows that both there groups vanish.

CHERN CHARACTERS AND EMBEDDING OF COMPLEXES

Dr. S. Gitler

30th November 1964

For X a finite C.W. complex we prove (Corollary to Theorem A) a weakened form of a theorem of Atiyah and Hirzebruch (Theorem B).

Recall

Theorem (Adams) If X is a finite C.W. complex with $H^*(X;\mathbb{Z})$ free and $x \in H^{2q}(X;\mathbb{Z})$, then $\exists\, N \in \widetilde{KU}(X)$ with $ch(N) = x + \sum x_{2q+2k}$, $x_{2q+2k} \in H^{2q+2k}(X;\mathbb{Q})$ where $2^k x_{2q+2k}$ is integral mod 2. (i.e no 2's in denom.)

Also $c(Sq^{2k})\, x = \rho(2^k x_{2q+2k})$ where ρ = reduction mod 2 and $c : \mathcal{A} \longrightarrow \mathcal{A}$ is the canonical anti-automorphism of the Steenrod algebra over $\mathbb{Z}_2$.

Theorem (Thom) If $X \subset S^n$ then $c(Sq^k) : H^q(X) \longrightarrow H^{q+k}(X)$ is trivial if $n-q < 2k+1$

Proof

Let Y be an n dual of X. Then

$Sq^k : H^{n-q-k-1}(Y) \longrightarrow H^{n-q-1}(Y)$ corresponds under Alexander-Spanier-Whitehead Duality to $c(Sq^k)$ of above.

Combining the two theorems we get:

Corollary If $X \subset S^n$, $H^*(X;\mathbb{Z})$ torsion free, then for any bundle $N \in \widetilde{KU}(X)$ with $ch\,N = x + \sum x_{2q+2k}$, $2^{k-1} x_{2q+2k}$ is integral mod 2 if $n - 2q < 4k+1$

Theorem A Let X be as in Adam's Theorem. If $X \subset S^M$ and $ch(N) = x + \sum x_{2q+2k} = x + \sum ch_{q+k}(N)$, then $2^{k-N} ch_{q+k}(N)$ is integral mod 2 where N is the largest integer s.t. $4N \leq 4k + 2q - M + 4$

Suppose X is $2k$ dimensional. Then if $N \in \widetilde{KU}(X)$, $ch_k(N) = \frac{s_k(N)}{k!}$ where $s_k(N)$ is an integer.

Corollary If X is $2k$ dimensional and $\exists\, N \in \widetilde{KU}(X)$ with $s_k(N)$ odd, then $X \not\subset \mathbb{R}^{4k - 4\alpha(k)}$ where $\alpha(k)$ = number of terms in the dyadic expansion of k.

Compare this with:

Theorem B (Atiyah and Hirzebruch) Under the conditions of the above Corollary with X a manifold, we have $X \not\subset \mathbb{R}^{4k - 2\alpha(k)}$

Remark ① Universal examples of torsion free spaces are $K(\mathbb{Z}, q)$ and $BU(2q, \dots \infty)$. Stong showed

$$H^*(BU(2q, \dots \infty); \mathbb{Z}_2) = \frac{H^*(\mathbb{Z}, 2q; \mathbb{Z}_2)}{\mathcal{J}(Sq^3 i_{2q})} \otimes P(\text{a polynomial algebra})$$

and this approach yields Atiyah and Hirzebruch's result for $\alpha(k) \leq 6$.

② The proper setting for all this is that of S-embeddings

Def. A complex S-embeds in S^n if $\exists$ a subcomplex Y of S^n s.t. $S^t X$ has the same homotopy type as $S^t Y$.

Problem Is a π-manifold of dim. n S-emboddable in $(n+1)$ space?

The results hold for S-embeddings instead of merely embedding

Proof of Theorem A

We make use of Maunder's axiomatic treatment of higher order stable operations.

Consider $Sq^1 Sq^{2r} + Sq^{01} Sq^{2r-2} + Sq^{2r} Sq^1 = 0$ where $Sq^{01} = [Sq^1, Sq^2]$ and let $\overline{\Phi}^{(2)}_{2r}$ be the operation associated with this relation.

Then $Sq^1 \overline{\Phi}^{(2)}_{2r} + Sq^{01} \overline{\Phi}^{(2)}_{2r-2} + Sq^{2r} \overline{\Phi}^{(2)}_{0} = 0$ with 0 indeterminacy. ($\overline{\Phi}^{(2)}_{0}$ = 2nd order Bockstein).

Similarly using an Nth order relation of degree $2r+1$ we can define $\Phi^{(N}_{2}$

Theorem For any integral class x of dim. $\leq q = 2r - 4N + 5$ and any choice of the operations $\overline{\Phi}^{(N)}_{2r}$, $\overline{\Phi}^{N}_{2r}(x)$ is defined and if dim $x < q$ then $\overline{\Phi}^{(N)}_{2r}(x)$ is a stable primary operation (modulo indeterminacy)

Maunder's Theorem If in Adams theorem $2^{k-N-1} x_{2q+2k}$ is integral then $\exists$ an Nth order operation $\widetilde{\Phi}^{(N)}_{2k}$ s.t. $\rho(2^{k-N-1} x_{2q+2k}$ lies in $\widetilde{\Phi}^{(N)}_{2k}\, \rho(x)$ where $\widetilde{\Phi}^{(N)}_{2k}$ is "dual" to $\overline{\Phi}^{(N)}_{2k}$.

We can strengthen this to read "if $\overline{\Phi}^{(N)}_{2k}$ vanishes then $2^{k-N-2} x_{2q+2k}$ is integral."

These two theorems then imply Theorem A.

SPACES SATISFYING POINCARÉ DUALITY

Dr. R. Spivak 4th December 1964

X satisfies Poincaré Duality if $\exists\ \mu \in H_n(X)$ s.t.

$$\cap \mu : H^*(X) \xrightarrow{\cong} H_*(X).$$

A complex satisfying Poincaré Duality is called a *P-space*

A pair (X, Y) satisfies Poincaré Duality if $\exists\ \mu \in H_n(X, Y)$

s.t. 1) $\cap \mu : H^*(X) \xrightarrow{\cong} H_*(X, Y)$

2) $\cap \mu : H^*(X, Y) \xrightarrow{\cong} H_*(X)$

and. 3) $\cap \partial \mu : H^*(Y) \xrightarrow{\cong} H_*(Y)$.

A pair of complexes satisfying Poincaré Duality is called a *P-pair*.

Problem When does a P-space have the homotopy type of a C^∞ manifold?

Answer If n is odd, then $X \sim C^\infty$ manifold if $\exists$ a vector bundle over X s.t. T(vector bundle) is reducible.

Def. If $\pi : E \to X$ is a fibre space (throughout this will mean Hurewicz fibre space.) then $T(\pi) = CE \cup_\pi X$ is the *Thom space*.

For vector bundles this coincides (up to homotopy equivalence) with the usual definition.

Def. (X, x_0) is *reducible* (S reducible) if $\exists\ n$ and a map (S map) $f : (S^n, a) \longrightarrow (X, x_0)$ s.t. f_* is an isomorphism in H_p for $p \geq n$.

(X, x_0) is *coreducible* (S-coreducible) if $\exists\ n$ and a map (S map)

$f : (X, x_0) \longrightarrow (S^n . a)$ s.t. f^* is an isomorphism in H^p for $p \leq n$.

Def $\pi : E \longrightarrow X$ is *reducible* if $T(\pi)$ is.

Def $\pi : E \longrightarrow X$ is *spherical of fibre dimension k* if $\pi^{-1}(x) \sim S^{k-1}$.

Analagous to the Whitney sum for vector bundles we can form $\pi_1 \oplus \pi_2$ for fibre spaces. Let n_X = trivial bundle $X \times S^{n-1}$.

Def π_1 and π_2 are *stably fibre homotopy equivalent* (written $\pi_1 \underset{S}{\simeq} \pi_2$) if $\exists$ n, m s.t. $\pi_1 \oplus n_X \sim \pi_2 \oplus m_X$.

Main Theorem If X is a $\mathcal{P}$ space then $\exists$ one and (up to $\underset{S}{\simeq}$) only one reducible spherical fibre space over X.

Proof

Existence – Embed $X \subset R^{n+k}$. Let $(N, \partial N)$ be a regular neighbourhood. Then $\mathcal{P}(N ; \partial N , X) \xrightarrow{w} X$ is a fibre space, and $T(w) \sim \frac{N}{\partial N}$ and so is reducible.

To prove that it is spherical consider the fibration

$\partial N \sim \mathcal{P}(N; \partial N, N) \longrightarrow N$ which has the same fibre F. Then $H^*(\partial N) \cong H^*(N) \oplus H^{*+k-1}(N)$ and a spectral sequence argument shows that the fibre has the cohomology of a $k-1$ sphere. Since $\pi_1(F) = \pi_2(N, \partial N) = 0$ we have the result.

By embedding in half-space we can prove a similar result for $\mathcal{P}$ pairs.

Before proving uniqueness we remark that if X is a manifold and N is tubular neighbourhood from the normal bundle $\nu : E \longrightarrow X$, then $(\mathcal{P}(N; \partial N, X) \longrightarrow X) \underset{S}{\simeq} \nu$.

This follows from:

Prop. Let $\pi : E \to X$ be a fibre space. C_π = mapping cylinder.

Then $(\gamma(C_\pi, E, X) \longrightarrow X) \sim \pi$.

Uniqueness We recall Atiyah's method to prove that for X a C^∞ manifold and $\pi : E \to X$ a reducible vector bundle, then $\pi \underset{S}{\sim}$ normal bundle of X. We have

1) A vector bundle $\pi : E \to X$ is $\underset{S}{\sim}$ trivial $\Leftrightarrow T(\pi)$ is S-coreducible.

2) If $(M, \partial M)$ is a C^∞ manifold with boundary then $M/\partial M$ is an S dual of T(normal bundle of M).

3) If $\pi : A \to X$ is a disc bundle then $(A, \dot{A})$ is a C^∞ manifold with boundary.

4) $\tau_A \sim \pi^*(\tau_X) \oplus \pi^*(A)$ for $\pi : A \to X$ a disc bundle.

Let $\omega : A \to X$ be reducible disc bundle associated with π. Then $-\omega^*(A) \oplus \omega^*(-\tau_X) = -\tau_A$ and so we require that $T(-\tau_A)$ is S coreducible. i.e. that an S dual of $T(-\tau_A)$ is S reducible. i.e. that $A/\dot{A}$ is S reducible.

For X a P space we replace 'vector bundle' by 'fibre space' then 1) remains the same.

2) becomes: If (X, Y) is a P pair then X/Y is an S dual of $T(\nu_X)$ where ν_X denotes the reducible spherical fibre space over X defined in the first part of the proof

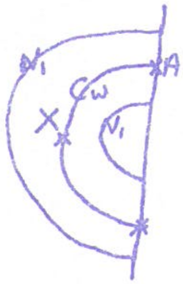
i.e. ν_X is $\gamma(N; \partial N, X) \longrightarrow X$

3) becomes: If $\pi : E \to X$ is a spherical fibre space then $\exists$ an equivalent fibre space $\omega : A \to X$ s.t. (C_ω, A)

is a P pair and $(C_\omega, A) \sim (C_\pi, E)$.

4) Let $\pi : E \longrightarrow X$ and $\omega : A \longrightarrow X$ be as in 3).
Let $i : X \longrightarrow C_\omega$. Then $-\pi \oplus \nu_X \sim i^*(\nu_{C_\omega})$

⌈This is proved by embedding C_ω in half space and showing that



$$\begin{tikzcd}
\nu(N; \partial N, X) \arrow[d] & \sim & \nu(N; N_1, X) \arrow[d] & \oplus & \nu(C_\omega; A, X) \arrow[d] \\
X & & X & & X
\end{tikzcd}$$

⌋

The analogue of Atiyah's proof then goes through.

———— · ————

The unique class of reducible spherical fibre spaces over X determines a unique homotopy class of maps $\alpha : X \longrightarrow BSH$. We may ask if α factors through BSO.

$$\begin{tikzcd}
 & BSO \arrow[dr] & \\
X \arrow[ur, dashed] \arrow[rr, "\alpha"] & & BSH
\end{tikzcd}$$

We have exact sequences

$$\begin{tikzcd}
\pi_n(BSO) \arrow[r] & \pi_n(BSH) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \pi_{n-1}(F) \arrow[r] & \pi_{n-1}(BSO) \\
\pi_{n-1}(SO) \arrow[r, "J_{n-1}"] & \Pi_{n-1} \arrow[r] & \pi_{n-1}(F) \arrow[r] & \pi_{n-2}(SO)
\end{tikzcd}$$

stable group of sphere

and $\pi_{n-1}(F) = \dfrac{\Pi_{n-1}}{\mathrm{Im}\, J_{n-1}} \left(\oplus\, Z\right)$

Milnor and Kervaire have defined a function

$$\psi : H^k(K, L : Z) \longrightarrow H^n(K, L; \pi_{n-1}(S^k))$$

provided that $H^p(K, L : G) = 0$ for $1 < p < n$ and all coefficient groups G. This has the property that it is

- an homomorphism when stabilised by

$$H^n(K, L; \pi_{n-1}(S^k)) \longrightarrow H^n(K, L; \pi_{n-1})$$

Let X be an $(n-1)$ connected P space of dim N.

$$\psi : H^{N-n}(X; \mathbb{Z}) \longrightarrow H^N(X; \pi_{n-1})$$

Then $\psi = \smile \psi^n(x)$ where $\psi^n(x) \in H^n(X; \pi_{n-1})$.

<u>Theorem</u> $\exists$ an exact sequence

$$\pi_{n-1}(SO) \longrightarrow \pi_{n-1} \xrightarrow{\phi} \pi_{n-1}(F)$$

so that under the map $H^n(X, \pi_{n-1}) \longrightarrow H^n(X, \pi_{n-1}(F))$ induced by ϕ, $\psi^n(x) \longrightarrow \theta^n(x)$, the primary obstruction to lifting α.

<u>Proof</u>

By means of the universal example $BH[n, \infty)$.

THE KERVAIRE INVARIANT FOR 8k+2 MANIFOLDS

Prof. E. H. Brown Jr. 14th December 1964

This talk represents joint work with F. P. Peterson.

Let T_n = tangent cell bundle of S^n, $p : T_n \longrightarrow S^n$

L_n = 2 copies of T_n plumbed together as follows:

Take a disc $D^n \subset S^n$. Then $p^{-1}(D^n) \sim D^n \times D^n$. Paste the 2 copies of T_n together on $D^n \times D^n$ by identifying (x, y) in one with (y, x) in the other.

<u>Theorem</u> If n is odd, ∂L_n is homeomorphic to S^{2n-1}.

<u>Theorem 1</u> If $n \equiv 1 \bmod 4$, $n \neq 1$, ∂L_n is not diffeomorphic to S^{2n-1} (Proved by Kervaire for $n = 5, 9$)

Let $K_n = L_n \cup D^{2n}$ (paste the disc onto ∂L_n).

<u>Theorem 2</u> If $n \equiv 1 \bmod 4$, $n \neq 1$, K_n does not admit a differentiable structure.

<u>Note</u> 1. Theorem 2 $\Longrightarrow$ Theorem 1.

2. If $n \equiv 1 \bmod 4$ and K_n has a differentiable structure then K_n is stably $\|$sible (since all obstructions are zero).

Hereafter $n \equiv 1 \bmod 4$, $n \neq 1$, and all coeffs. are $\mathbb{Z}_2$.

<u>Proof of Theorem 2</u>

<u>Outline</u> We define a secondary cohomology operation ϕ from $H^n(X)$ to $H^{2n}(X)$ s.t.

(i) $\phi(u+v) = \phi(u) + \phi(v) + uv$ mod. indeterminacy.

(ii) If M is a closed compact C^∞ 2n manifold with $\pi_1(M)=0$, $W_2(M) = 0$, then $\phi : H^n(M) \longrightarrow H^{2n}(M)$ without indeterminacy.

(iii) Define $A(M) = \sum_{i=1}^{r} \phi(\lambda_i)(M)\, \phi(\mu_i)(M) \in \mathbb{Z}_2$ where $\{\lambda_i, \mu_i\}_{i=1 \dots r}$ is a symplectic basis for $H^n(M)$.

i.e. $\lambda_i \lambda_j = \mu_i \mu_j = 0 \quad \lambda_i \mu_j = \delta_{ij} \quad \forall\, i, j.$

Then $A(M)$ is independent of the basis.

(iv) $A(K_n) = 1$ (Use (i))

(v) If M is stably ‖-sible, $A(M) = 0$.

<u>The operation ϕ</u>

$$Sq^{n+1} = Sq^2 Sq^{n-1} + Sq^1(Sq^2 Sq^{n-2})$$
$$= 0 \quad \text{on } H^n(X).$$

Hence we get an operation

$$\phi : H^n(X) \cap \operatorname{Ker} Sq^{n-1} \cap \operatorname{Ker} Sq^2 Sq^{n-2} \longrightarrow \frac{H^{2n}(X)}{Sq^2 H^{2n-2}(X) + Sq^1 H^{2n-1}(X)}$$

(i) is proved by means of the universal example for ϕ.

If M is a closed compact C^∞ manifold of dim 2n, then

$$Sq^1 H^{2n-1}(M) = W_1(M) . H^{2n-1}(M)$$
$$Sq^2 H^{2n-2}(M) = \left(W_2(M) + W_1(M)^2\right) H^{2n-2}(M)$$

$\therefore$ If $W_1(M) = W_2(M) = 0$, $\phi : H^n(M) \cap \operatorname{Ker} Sq^{n-1} \longrightarrow H^{2n}(M)$

If $\pi_1(M) = 0$ then $H^{2n-1}(M) = 0$ by Poincaré Duality and so $Sq^{n-1} H^n(M) = 0$.

To deal with the non-simply connected case we have to modify (iii) by working with SU cobordism. Hereafter all manifolds have an SU structure on their normal bundle.

We define $\phi : \Omega_{2n}(K(Z_2, n), SU) \longrightarrow Z_2$ as follows: Let $f : M^{2n} \longrightarrow K(Z_2, n)$ and ι_n be fundamental class in $H^n(K(Z_2, n))$.

If $Sq^{n-1} f^* \iota_n = 0$ define $\phi\{M^{2n}, f\} = \phi(f^* \iota_n)(M)$.

If $Sq^{n-1} f^* \iota_n \neq 0$ we can perform surgery on M^{2n} to make this so.

Define $A : \Omega_{2n}(SU) \longrightarrow Z_2$ by

$$A\{M\} = \sum_{i=1}^{r} \phi\{M, \lambda_i\} \phi\{M, \mu_i\}$$

where $\{\lambda_i, \mu_i\}$ is a symplectic basis for $H^n(M)$.

Note 1. Both these definitions make sense.

2. For an SU manifold $W_2 = 0$.

Lemma (Corollary to Theorem of Lashof, Rothenberg, Floyd and Conner) If M^{2n} is stably ||-sible then $\{M^{2n}\} = \{S^1 \times S^1 \times N\}$ in $\Omega_{2n}(SU)$.

It remains, therefore, to compute $A(S^1 \times S^1 \times N)$.

Let $v_{n-1} \in H^{n-1}(BSU)$ be s.t. $Sq^{n-1} : H^{n-1}(N) \longrightarrow H^{2n-2}(N)$ is $\cup\ \nu_N^* v_{n-1}$ where $\nu_N : N \longrightarrow BSU$ is classifying map of $\nu(N)$. Let $\nu_N^* v_{n-1} = v_{n-1}(N)$ so that $u^2 = v_{n-1}(N) . u$. for $u \in H^{n-1}(N)$

Case I $v_{n-1}^2 = 0$ (Case II $v_{n-1}^2 \neq 0$ is not done here)

Let $U = \{u \mid u \in H^{n-1}(N), u v_{n-1}(N) = u^2 = 0\} \subset H^{n-1}(N)$.

Let $\{r_i, s_i\}$ be a symplectic basis for U. Then a basis for $H^n(S^1 \times S^1 \times N)$ is

λ	μ
$x \otimes x \otimes u_i$	$1 \otimes 1 \otimes z_i$
$x \otimes 1 \otimes r_i$	$1 \otimes x \otimes s_i$
$1 \otimes x \otimes r_i$	$x \otimes 1 \otimes s_i$
$x \otimes 1 \otimes v_{n-1}$	$1 \otimes x \otimes v_{n-1}$

where x is a generator of $H^1(S^1)$ and $u_i z_j = \delta_{ij}$

Then $\phi = 0$ on either the λ or μ term for all pairs of basis elements except the last.

$\ulcorner$ $Sq^{n-1}(1 \otimes 1 \otimes z_i) = 0$ for dimensional reasons

$\therefore \phi(1 \otimes 1 \otimes z_i) = 1 \otimes 1 \otimes \phi(z_i) = 0$ for dim. reasons

$Sq^{n-1}(x \otimes 1 \otimes r_i) = x \otimes 1 \otimes Sq^{n-1} r_i = x \otimes 1 \otimes v_{n-1}(N)\, r_i = 0$

and $\phi(1 \otimes x \otimes r_i) = 1 \otimes \phi(x \otimes r_i) = 0$ for dim reasons $\lrcorner$

By symmetry we get

$$A(S^1 \times S^1 \times N) = \phi(S^1 \times S^1 \times N,\ x \otimes 1 \otimes v_{n-1}(N))$$

$$= \phi(S^1 \times Q,\ x \otimes v_{n-1}(Q)) \text{ where } Q \overset{SU}{\sim} S^1 \times N$$

and $\pi_1(Q) = 0$.

$$= \phi(x \otimes v_{n-1}(Q))(S^1 \times Q)$$

$$= \phi'(v_{n-1}(Q))(Q) \qquad \text{(suspended operation)}$$

$$= Sq^2_{\nu_Q}(v_{n-1}^2)(Q) \qquad (\nu_Q : Q \longrightarrow BSU)$$

$$= Sq^2_{T(\nu_Q)}(v_{n-1}^2\, U_k) \text{ (top class)} \quad \text{where } U_k$$

is the Thom class and

$$\begin{tikzcd} T(\nu(Q)) \arrow[r, "T(\nu_Q)"] & MSU_k \\ S^{2n-1+k} \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}$$

$$= Sq^2_f(v_{n-1}^2\, U_k)(S^{2n-1+k})$$

$$= Sq^2_{g\eta}(v_{n-1}^2\, U_k)(S^{2n-1+k}) \quad \text{since } f = g\eta \text{ where } \eta$$

is a suspension of the Hopf map.

$$= Sq^2_\eta(g^*(v_{n-1}^2\, U_k))(S^{2n-1+k})$$

$$= Sq^2_\eta\, Sq^2_\eta(g^*(v_{n-1}^2\, U_k))(S^{2n+k}) \quad \text{since } Sq^2_\eta \text{ is}$$

an isomorphism.

$A(S^1 \times S^1 \times N) = \bar{\Psi}_{\eta\eta\eta}(g^*(V^2_{n-1}\ U_k))(S^{2n+k})$ (Adem)

$= \bar{\Psi}_{g\eta\eta}\ (V^2_{n-1}\ U_k)\,(S^{2n+k})$

$= \bar{\Psi}_{\vartheta_M}\ (V^2_{n-1})\,(M)$ where $M \overset{SU}{\sim} S^1 \times S^1 \times N$

and M is 3 connected.

If M is stably //sible, $\vartheta_M \sim 0$

$\therefore A(S^1 \times S^1 \times N) = 0.$

ON SU COBORDISM, KO CHARACTERISTIC CLASSES AND THE ARF INVARIANT

Prof. F. P. Peterson 4th January 1965

This talk represents joint work with D. W. Anderson and E. H. Brown Jr.

We recall

(i) If $\mathfrak{N}_*$ = unoriented cobordism ring and $[M] \in \mathfrak{N}_*$, then

$[M] = 0 \Longleftrightarrow$ all Stiefel-Whitney numbers vanish.

(ii) If Ω_* = oriented cobordism ring and $[M] \in \Omega_*$, then

$[M] = 0 \Longleftrightarrow$ all Stiefel-Whitney numbers and all Pontryagin numbers vanish.

(iii) If Ω_*^U = unitary cobordism ring and $[M] \in \Omega_*^U$, then

$[M] = 0 \Longleftrightarrow$ all Chern numbers vanish.

Our main result (Theorem 1) is a similar statement for Ω_*^{SU}.

<u>Note</u> It is known that the vanishing of the Stiefel-Whitney, Pontryagin, and Chern numbers is not sufficient.

M^n a manifold. Our first aim is to define a fundamental class $\{M\} \in KO_n(M^n)$.

Let k be dimension of normal bundle ν, $\gamma_k = (ESU(\frac{k}{2}), p, BSU(\frac{k}{2}))$ be univeral bundle for $SU(\frac{k}{2})$. By abuse of notation we have $\nu : M^n \longrightarrow BSU(\frac{k}{2})$ which induces

$$\tilde{\nu} : T(\nu) \longrightarrow T(\gamma_k)$$

The Thom isomophism $\phi : KO^a(BSU(\frac{k}{2})) \cong \widetilde{KO}^{a+k}(T(\gamma_k))$ is given by $\phi(x) = p^*(x) \cup U$ for some $U \in \widetilde{KO}^k(T(\gamma_k))$. Now $M^n \cup pt.$ is an $n+k$ dual of $T(\nu)$ and so by Alexander Duality $\widetilde{KO}^k(T(\nu)) \cong \widetilde{KO}_n(M^n \cup pt.) = KO_n(M^n)$.

Define $\{M\}$ to be the class corresponding to $\tilde{\nu}^*(U)$.

<u>Def.</u> Given $x \in KO^a(BSU(\frac{k}{2}))$ define

$$x(M) = \langle \nu^*(x), \{M\} \rangle \in KO^{a-n}(pt.) \text{ (Kronecker Index).}$$

<u>Theorem 1</u> If $[M] \in \Omega_*^{SU}$ then $[M] = 0 \Longleftrightarrow$ all Chern numbers and all $x(M)$ vanish.

We now define $[M_k] \in \Omega_{8k}^{SU}$ as follows:

$[M_1] \in \Omega_8^{SU} = Z$ is a generator.

Take M_k s.t. $\langle \nu^* s_{(k,k)}(\text{Pontryagin classes}), \{M\} \rangle \not\equiv 0 \bmod 2$

if k is not a power of 2.

and s.t. $\langle \nu^* s_{(2^r,2^r,2^r,2^r)}(\text{Pontryagin classes}), \{M\} \rangle \not\equiv 0 \bmod 2$

if k is a power of 2.

Take $\alpha \in \Omega_1^{SU} = Z_2$.

<u>Theorem 2</u> (Conner and Floyd and Lashof and Rothenberg)

$$0 \longrightarrow \mathrm{Tor}\,\Omega_*^{SU} \longrightarrow \Omega_*^{SU} \longrightarrow \Omega_*^{U} \text{ is exact.}$$

Further, a basis for $\mathrm{Tor}\,\Omega_*^{SU}$ is given by (monomials in $[M_r]$) $\times \alpha$ and (monomials in $[M_r]$) $\times \alpha^2$.

Theorem 3 The elements in Ω_*^{SU} which have framed manifolds as representatives are $[M_1]^r \times \alpha \quad r \geqslant 0$

and $[M_1]^r \times \alpha^2 \quad r \geqslant 0.$

Corollary If $\underline{\Phi} : \Omega^{SU}_{8r+2} \longrightarrow Z_2$ is Kervaire Invariant (c.f. E. H. Brown's talk on December 14th 1964), then $\underline{\Phi}$ (framed manifold) $= 0$ for $r \geqslant 1$.

Proof

$$
\begin{aligned}
\underline{\bar{\Phi}}\,([M_1]^r \times \alpha^2) &= \underline{\bar{\Phi}}\,([M_1]^r \times \alpha \times \alpha) \\
&= \underline{\bar{\Phi}}\,(N \times \alpha) \\
&= \underline{\bar{\Phi}}\,(\tilde{S}^{8r+1} \times S^1) \\
&= \underline{\bar{\Phi}}\,(\tilde{S}^{8r+2}) = 0 .
\end{aligned}
$$

Theorem 4 Using characteristic numbers one can write down a basis for $\mathrm{Hom}\,(\Omega^{SU}_{8r+2} \otimes Z_2,\ Z_2)$

Corollary $\underline{\bar{\Phi}} = K_r$ = some linear combination of Chern numbers and KO numbers.

Let X be a finite C.W. complex, $\pi_1(X) = 0$, and ξ a k-plane bundle over X. We may ask: when is X of the same homotopy type as a C^∞ manifold s.t. ξ goes into the normal bundle? Necessary conditions are that $\exists\, g \in H_n(X; Z)$ s.t. $\cap g : H^i(X : Z) \cong H_{n-i}(X; Z)$ and $\phi(g) \in H_{n+k}(T(\xi); Z)$ is spherical.

A theorem of Browder and Novikov states that if n is odd these conditions are sufficient.

If $n = 4s$ and the Hirzebruch index formula holds then we have the same conclusion.

We have the similar result:

Theorem 5 If $n = 8r+2$ and ξ is an SU-bundle s.t. $Sq^2 : H^{n-2}(X;Z_2) \longrightarrow H^n(X;Z_2)$ is zero and $\Phi(X) = K_r(\xi;\beta)$ for some $\beta \in h^{-1}(\phi(g))$ (h is Hurewicz map), then $\exists$ an SU-manifold M and $f: M \longrightarrow X$ s.t. $f^*(\xi) = \nu$ and f is a homotopy equivalence.

Note Although K_r was only defined on manifolds, it can be defined on bundles in the presence of such a β.

Proofs

We merely indicate some of the techniques.

The E_2 term of the Adams spectral sequence for $\pi_*(T(\gamma_k)) = \pi_*(MSU(\frac{k}{2}))$ can be computed (Peterson's seminar last summer). By use of a result of Conner and Floyd of the characteristic number type we can calculate d_2 and hence E_3. We then prove the

Theorem $E_3 = E_\infty$.

This gives Theorem 2. Further monomials in M_r $r > 1$ can be detected by Pontryagin numbers mod 2.

For sufficiently large k $\exists$ an isomorphim

$$\psi : \pi_{n+k}(MSU(\tfrac{k}{2})) \xrightarrow{\cong} \Omega_n^{SU}.$$

$MSU(\frac{k}{2})$ is $k-1$ connected and let $g: S^k \longrightarrow MSU(\frac{k}{2})$

be the generator in dimension k. We define a sequence of elements $\mu_r \in \pi_{8r+1+k}(S^k)$ as follows:

$\mu_0 = \eta$, the Hopf map.

$\mu_r = \{8\sigma, 2\iota, \mu_{r-1}\}$ where $\sigma \in \pi_{7+s}(S^s)$
$16\sigma = 0.$

<u>Lemma</u> $\psi(g_\#(\mu_r)) = [M_1]^r \times \alpha$

<u>Lemma</u> (J. F. Adams) If $k \equiv 0 \mod 8$ then

$$\begin{tikzcd}
\widetilde{KO}^0(S^k) \arrow[r, "(\mu_r)^*"] \arrow[d, phantom, "\parallel"] & \widetilde{KO}^0(S^{8r+1+k}) \arrow[d, phantom, "\parallel"] \\
\mathbb{Z} & \mathbb{Z}_2
\end{tikzcd}$$

is non-zero

In his thesis Anderson defined $\pi^i \in KO^0(BSU)$.
If γ is a polynomial then

$$\langle \nu^* \gamma(P_i), \{M\} \rangle = \pm \langle \nu^* \gamma(\pi^i), \{M\} \rangle$$

for M of right degree. Now monomials in M_r $r>1$ are detected by polynomials in P's and ∴ by same polynomials in π's.

EMBEDDINGS IN THE TRIVIAL RANGE

Prof. H. Gluck 11th and 18th January 1965

Ref. "Embeddings in the trivial range" H. Gluck.
Bull. Am. Math. Soc. 69 1963

Basic Problem: Classify the embeddings of a k-dim. object P^k into an n-manifold M^n when k is small compared to n, (where the underlined terms have still to be interpreted.)

Small means $2k+2 \leq n$.

Classify means classify under equivalence by ambient isotopy. (2 embeddings f_0, f_1 are ambient isotopic $\Longleftrightarrow \exists\ h: M^n \rightarrow M^n$ s.t. h is isotopic to identity and $hf_0 = f_1$)

General Embedding Problem - Given an object P^k and an n-manifold M^n, classify the embeddings of P^k into M^n under equivalence by ambient isotopy.

This projects naturally into the

Homotopy Problem - Given an object P^k and an n-manifold M^n, classify the continuous maps of P^k into M^n under equivalence by homotopy.

Claim In the trivial range, $2k+2 \leq n$, this natural projection is bijective.

Note If we interpret the above problem piecewise linearly, the claim has been proved by Gugenheim; if we interpret it differentiably then the claim has been proved by Whitney. We will give an interpretation of "embedding" in the topological case for which the claim is true. It is palpably untrue if we consider all possible embeddings.

We take object = compact k-polyhedron

manifold = manifold which can support a combinatorial structure.

embedding = locally tame embedding.

Def. Let $f : \underset{\text{compact k-polyhedron}}{P^k} \longrightarrow \underset{\text{comb. manifold}}{M^n}$ be an embedding.

Then f is *locally tame* if there is a triangulation of P^k s.t. for each point $x \in P^k$ there is an open nghd. U of $f(x)$ in M^n and a triangulation of U as a combinatorial manifold in terms of which f is piecewise linear on some nghd. of x.

It is a corollary to Theorem 1 that the definition is independent of the triangulation of P^k.

Remark Whenever we have a metric on M^n we will require that M^n be complete in the metric and that sets of sufficiently small diameter be enclosable in an n cell lying in M^n.

Def. M a manifold with a metric. A a subset of M. $\varepsilon \geq 0$.

An *ε-push* of (M, A) is a homeomorphism h of M s.t.

1) h is an ε-homeomorphism

2) $h|_{M - U_\varepsilon(A)} = 1$

3) h is isotopic to 1 s.t. each h_t of the isotopy satisfies 1) and 2)

The main theorems are:

Theorem 1 Let f be a locally tame embedding of the polyhedron P^k into the combinatorial manifold M^n. If $2k+2 \leq n$, then for each $\varepsilon > 0$, $\exists$ an ε-push h of $(M^n, f(P^k))$ s.t. $hf : P^k \longrightarrow M^n$ is p.w.l. w.r.t. arbitrary, preassigned triangulations of P^k and M^n.

Theorem 2 If $2k+2 \leq n$, then for each $\varepsilon > 0$, $\exists$ $\delta > 0$ s.t. if f, f' are locally tame embeddings of P^k into M^n with $d(f, f') < \delta$, then $\exists$ an ε-push h of $(M^n, f(P^k))$ s.t. $hf = f'$.

Theorem 3 Let f, f' be locally tame embeddings of P^k into M^n. If $2k+2 \leq n$ and f is homotopic to f', then $\exists$ a homeomorphism $h : M^n \longrightarrow M^n$ which is isotopic to the identity, s.t. $hf = f'$.

A Formal Section

X = metric space. $A \subset X$ with $\bar{A}$ compact.

M = topological manifold with a complete metric d.

Hom $(X, A; M)$ will denote any family of embeddings of X into M, all of which agree on $X - A$.

Then $$d(f, f') = \underset{x \in X}{\text{l.u.b.}}\, d(f(x), f'(x)) = \underset{x \in \bar{A}}{\text{l.u.b.}}\, d(f(x), f'(x))$$ makes Hom $(X, A; M)$ into a metric space.

Def. $F \subset$ Hom is dense if it is dense in the usual sense.

<u>Def</u> $F \subset Hom$ is <u>solvable</u> if for any $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. if $f, f' \in F$ and $d(f, f') < \delta$, then $\exists$ an ε-push h of $(M, f(A))$ s.t. $hf = f'$!

We then have.

<u>Theorem A</u> Let $F \subset F' \subset Hom(X, A; M)$. Suppose that for each $f \in F'$ and each $\varepsilon > 0$, $\exists$ an ε push h of $(M, f'(A))$ s.t. $hf' = f \in F$. Then if F is solvable, so is F' (easily proved)

<u>Theorem B</u> The union of 2 dense, solvable subsets of Hom is dense and solvable.

As an application of Theorem B we prove

<u>Theorem 4</u> Suppose

1) M^n is a possibly non-compact combinatorial n-manifold.
2) $\tilde{M}^n$ .. .. , topologically embedded in M^n.
3) $\tilde{P}^k$ is a possibly infinite simplicial complex, p.w.l. embedded as a closed subset of $\tilde{M}^n$.
4) $\tilde{L}$ a subcomplex of $\tilde{P}^k$ s.t. closure of $\tilde{P}^k - \tilde{L}$ is p.w.l. embedded in M^n as well as in $\tilde{M}^n$.

If $2k+2 \leq n$, then for any $\varepsilon > 0$ $\exists$ an ε push h of $(M^n, \tilde{P}^k - \tilde{L})$ s.t. $h/_{\tilde{P}^k} : \tilde{P}^k \longrightarrow M^n$ is p.w.l. and $h/_{\tilde{L}} = 1$

<u>Proof</u>

Let $Hom(\tilde{P}^k, \tilde{P}^k - \tilde{L} ; \tilde{M}^n)$ = all topological embeddings which restrict to inclusion on $\tilde{L}$.

Let $F \subset \text{Hom}$ denote the p.w.l. embeddings of $\tilde{P}^k \longrightarrow \tilde{M}^n$

" $F' \subset \text{Hom}$ " " " " " $\tilde{P}^k \longrightarrow M^n$

Then F and F' are dense and solvable in $\text{Hom}(\tilde{P}^k, \tilde{P}^k - \tilde{L}; \tilde{M}^n)$

$\therefore$ $F \cup F'$ is " " " " "

by Theorem B.

Using the denseness of F' find $f' \in F'$ within $\delta_{F \cup F'}(\varepsilon)$ of the inclusion $i : \tilde{P}^k \subset \tilde{M}^n$ which is in F. Since $F \cup F'$ is solvable, find an ε push h of $(\tilde{M}^n, \tilde{P}^k - \tilde{L})$ s.t. $hi = f'$. Then extend h via the identity to an ε push of $(M^n, \tilde{P}^k - \tilde{L})$.

Proofs

Proof (almost) of Theorem 1 (we prove Theorem 1 first in the case where P^k has the triangulation in which f is locally tame.)

Def A set $A \subset P^k$ is small if $\exists$ nghds U of A in P^k and V of $f(A)$ in M^n s.t.

① $f(U) = V \cap f(P^k)$

② V has a combinatorial triangulation s.t. $f|_U : U \longrightarrow V$ is p.w.l.

Note The finite union of disjoint small sets is small.

By taking a subdivision if necessary we can assume that the star of every simplex of P^k is small. Let P^i = i-skeleton of P^k. N_i = closed regular nghd. of P^i in $(P^k)''$.

Then N_0 is small (by Note above)

and closure $(N_{r+1} - N_r)$ is small $r = 0, \ldots k-1$.

Apply Theorem 4 to get an $\frac{\varepsilon}{k+1}$ push h_0 which straightens out $f|_{N_0}$.

Apply Theorem 4 to get an $\frac{\varepsilon}{k+1}$ push h_i which straightens out

$$h_{i-1} \cdots h_0 f|_{N_k} \qquad i=1, \dots k.$$

Take $h = h_k h_{k-1} \cdots h_0$.

Proofs of Theorems 1 and 2

We use Theorems A and B to remove the restriction imposed above. Let $\mathrm{Hom}(P^k, M^n)$ = set of all locally tame embeddings of $P^k \to M^n$ and $2k+2 \leq n$. Let M^n have an arbitrary combinatorial triangulation. Let $(P^k)_1$, $(P^k)_2$ be 2 different triangulations of P^k. Let F_1 = p.w.l. emboddings w.r.t. $(P^k)_1$

F_2 = " " " $(P^k)_2$

F_1' = locally tame " " $(P^k)_1$

F_2' = " " " " $(P^k)_2$

Then F_1 and F_2 are each dense and solvable in $\mathrm{Hom}(P^k, M^n)$ (by p.w.l. theory). The "almost" proof and Theorem A $\Rightarrow$ F_1' and F_2' are dense and solvable in $\mathrm{Hom}(P^k, M^n)$. Theorem B $\Rightarrow$ $F_1' \cup F_2$ is dense and solvable.

Take $f_1' \in F_1'$ and choose $f_2 \in F_2$ s.t. $d(f_1', f_2) < \delta_{F \cup F'}(\varepsilon)$ (denseness of F_2). Then $\exists$ ε-push h of $(M^n, f_1'(P^k))$ s.t. $hf_1' = f_2$ (solvability) and Theorem 1 follows since f_2 is p.w.l. w.r.t. $(P^k)_2$.

Also, $\mathrm{Hom}(P^k, M^n) = F_1'$ = solvable and Theorem 2 follows.

Proof of Theorem 3

Hom (P^k, M^n) is solvable and for this we find $\delta(1)$. Let f, f^* be 2 homotopic locally tame embeddings. Then $\exists$ $g_1 (= f), g_2, \cdots g_{r-1}, g_r (= f^*) : P^k \longrightarrow M^n$ s.t. $d(g_i, g_{i+1}) < \delta(1)$. Also Hom (P^k, M^n) is dense in the mapping space $(M^n)^{P^k}$ (since it contains the p.w.l. ones and they are dense). Hence $\exists$ $f_1 (= f), f_2, \cdots f_{r-1}, f_r (= f^*) \in$ Hom (P^k, M^n) s.t. $d(f_i, f_{i+1}) < \delta(1)$. Solvability $\Longrightarrow$ $\exists$ 1 pushes h_i s.t. $h_i f_{i-1} = f_i$ $i = 2, \ldots r$. Take $h = h_r \cdots h_1$.

Applications

Def. $f : M^k \subset M^n$ is <u>locally flat</u> if for each $x \in M^k$, $\exists$ a nghd. U of $f(x)$ in M^n s.t $(U, U \cap f(M^k))$ is homeomorphic to (R^n, R^k).

If f is a locally flat embedding of a combinatorial manifold M^k into the combinatorial manifold M^n, then f is locally tame.

Thus the embedding theorem is trivial for locally flat embeddings of combinatorial manifolds in R^n in the range $2k+2 \leq n$.

Sketch of Proof of Theorem B (by Homma's technique used in "On the embedding of polyhedra in manifolds" Yokohama Math. Journal. 10 (1962) 5-10)

Note (i) If h is an ε push of $(M. A)$ then h^{-1} is a 2ε-push of $(M. h(A))$.

(ii) If h_1 is an ε_1 push of (M, A)

h_2 " " ε_2 " " $(M, h_1(A))$

$\vdots$

h_r " " ε_r " " $(M, h_{r-1} \cdots h_1(A))$

then $h_r \dots h_1$ is a $(\varepsilon_1 + \dots \varepsilon_r)$ push of (M, A).

(iii) Suppose $\varepsilon_1 + \dots + \varepsilon_r + \dots = \Sigma$ with h_i an ε_i-push of $(M, h_{i-1} \dots h_1(A))$ $i = 1, 2, 3, \dots$. Let $g = \dots h_3 h_2 h_1$. Then g is not necessarily a Σ push of (M, A), but it is if g is (1-1).

Let F, F' be dense and solvable in Hom. Write $\delta_F(\varepsilon) = \delta(\varepsilon)$, $\delta_{F'}(\varepsilon) = \delta'(\varepsilon)$.

Claim $\delta_{F \cup F'}(\varepsilon) = \min\left(\delta\left(\frac{\varepsilon}{6}\right), \delta'\left(\frac{\varepsilon}{6}\right)\right)$ makes $F \cup F'$ solvable.

Take $f \in F$, $f' \in F'$ s.t. $d(f, f') < \delta_{F \cup F}(\varepsilon)$. We construct sequences $\{f_i\} \in F$ and $\{f_i'\} \in F'$ and ε_i (ε_i') pushes h_i (h_i') for $i = 1, 2, \dots$ s.t. $h_i f_{i-1} = f_i$ $h_i' f_{i-1}' = f_i'$, satisfying the following conditions:

For each n, C_n is a covering of $\bar{A}$ by sets of diameter $< \frac{1}{n}$. We require

ⓐ $2\varepsilon_{i+1} < \varepsilon_i$.

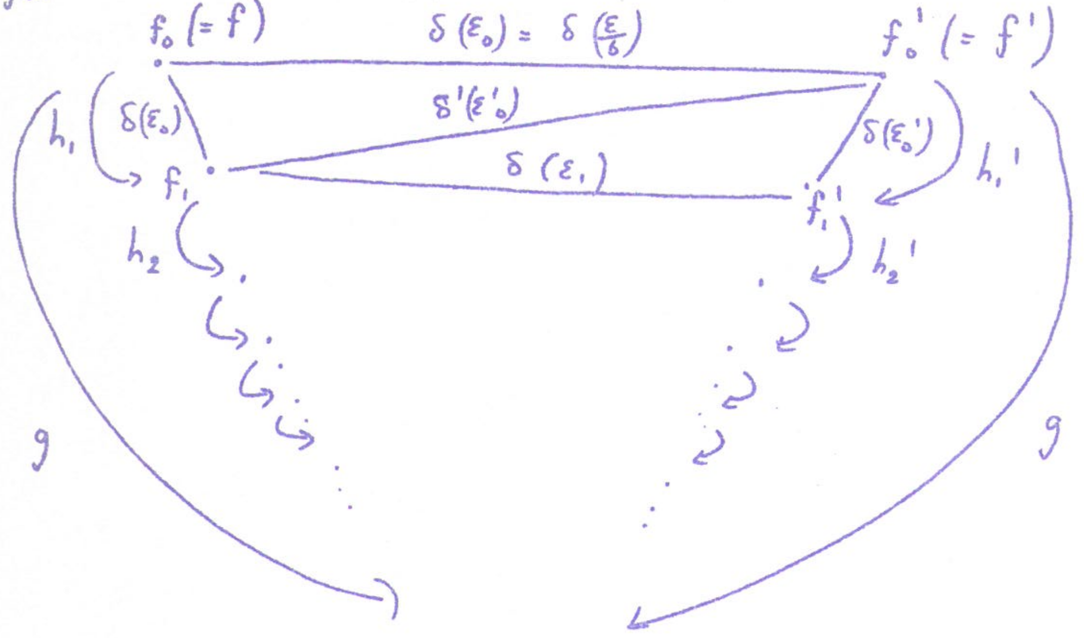
ⓑ $4\varepsilon_{i+1} < d(g_i(U), g_i(V))$ $\forall$ disjoint $U, V \in C_i$ where $g_i = h_i h_{i-1} \dots h_1$.

ⓒ $U_{\varepsilon_i}(g_i(\bar{A})) \subset g_i(U_{\frac{1}{i}}(\bar{A}))$

and similar primed conditions.

"By" "taking $\varepsilon_0 = \varepsilon/6$, finding h_1 which pushes f_0 very close to f_0',
h_1' " " f_0' " " " f_1,
etc.

we get



Then $g = \dots\, h_i h_{i-1} \cdots h_2 h_1$

$g' = \dots\, h_i' h_{i-1}' \cdots h_2' h_1'$

are homeomorphisms and hence $\varepsilon/3$ pushes.

$\therefore$ $(g')^{-1} g$ is a $2\frac{\varepsilon}{3} + \varepsilon/3 = \varepsilon$ push.

The Semi-Simplicial Free Lie Ring

James W. Schlesinger

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1. Introduction

Recent developments in semi-simplicial topology give us a purely algebraic method for finitely computing the homotopy groups of a connected simplicial complex, or more generally a reduced semi-simplicial complex K. Briefly the method consists of computing the homotopy spectral sequence which is associated with the filtration of the free group complex GK by its lower central series Γ_r GK.

The functor G was defined by D. M. Kan [6.] who proved that GK serves as a loop space for K. The lower central series of GK was studied by E. Curtis [2,3] who proved that the associated spectral sequence converges to the homotopy groups of GK. Furthermore he proved that the homotopy groups of the factor groups $\Gamma_r GK/\Gamma_{r+1} GK$ depend only on the homotopy groups of the first factor group $GK/\Gamma_2 GK$, that is on the homology groups of K.

In this paper we shall compute a few of these groups in the case when K is an n+1 sphere and thus $GK/\Gamma_2 GK$ is a $K(Z,n)$ space [6.]. Curtis has shown that $\sum_{r=1}^{\infty} \Gamma_r GK/\Gamma_{r+1} GK$ is the semi-simplicial free Lie ring $LK(Z,n) = \sum_{r=1}^{\infty} L^r K(Z,n)$ which is obtained by applying the functor L (or the functors L^r) to each dimension of $K(Z,n)$, and to the face and degeneracy operations. Since LK is a direct sum of the complexes $L^r K$, the module of products of weight r or of r factors, it is sufficient to compute the homotopy groups $\pi L^r K(Z,n)$. The results we obtain are for $r = p$ a prime.

$$\pi_{n+k} L^pK(Z,n) \approx \begin{cases} Z \text{ if } p=2, n \text{ is odd, and } k=n \\ Z_p \quad \text{if } k \quad -1 \pmod{2(p-1)} \text{ and } 0<k<n(p-1) \\ 0 \text{ otherwise.} \end{cases}$$

2. The Free Lie Ring

In [2.], Curtis defined a certain finite ordered set $B(r)$ of basic bracketting types t, and a natural filtration F^tL^rM, whose factor groups G^tL^rM may be naturally decomposed according to the class of t.

Proposition 1. (Curtis [2.]) If M is free, then G^tL^rM is free and there are natural isomorphisms:

(a) $G^tL^rM \approx G^{t_o} L^{r_o} M \otimes SP^h G^{t_1} L^{r_1} M$ where t is of class I, $t = t_o\, t_1 \dots t_h$, $t_o > t_1 = \dots = t_h$, $r = r_o + hr_1$, and $r_1 > 1$.

(b) $G^t L^rM \approx J^sG^{t_o} L^{r_o} M$ where t is of class II, $t = t_o^s$, $r = sr_o$, and $r_o > 1$.

(c) $G^t L^r M \approx J^rM$ where t is of class III.

Proposition 2. If M is free, $r > 1$, then there is a natural short exact sequence connecting the functor J^r with the symmetric tensor product SP^r. It is:

$$0 \to J'M \xrightarrow{i} M \otimes SP^{r-1} M \xrightarrow{i} SP^rM \to 0,$$

where $i([\dots[m_1,m_2],\dots,m_r])=m_1 \otimes (m_2,\dots,m_r)-m_2 \otimes (m_1,m_3,\dots,m_r)$ and j is the multiplication map in the symmetric ring.

This result allows us to simplify Curtis' computation of $\pi_* J^rK(Z,1)$ since $\pi_* SP^rK(Z,1)$ is known [4.].

Proposition 3. (Dold. Puppe [4.]

$$\pi_k\, SP^rK(Z,1) \approx \begin{cases} Z & \text{for } r = k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 4. (Curtis).

$$\pi_k \; J^r K(Z,1) \approx \begin{cases} Z & \text{for } r = k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5. (Dold Puppe)

$$\pi_k \; SP^r K(Z,2) \approx \begin{cases} Z & \text{if } k = 2r \\ 0 & \text{otherwise.} \end{cases}$$

The homomorphism:

$\pi_{2r}(K(Z,2) \otimes SP^{r-1}\, K(Z,2)) \to \pi_{2r}\; SP^r K(Z,2)$ is multiplication by r.

Proposition 6.

$$\pi_k J^r K(Z,2) \approx \begin{cases} Z_r & \text{if } k = 2r^{-1} \\ 0 & \text{otherwise} \end{cases}$$

3. The Computation of $\pi\; L^p K(Z,1)$ and $\pi\; L^p K(Z,2)$

Proposition 7.

$$\pi_* L^r K(Z,1) \approx \begin{cases} 0 & \text{if } r \text{ is odd, } r > 1 \\ \pi_* L^{r/2}\, K(Z,2) & \text{if } r \text{ is even.} \end{cases}$$

Proof: Since $G^t L^r K(Z,1)$ is contractable unless t is a basic type in c^2 the only member of $B(2)$, we can conclude that $L^{r/2} L^2 K(Z,1)$ is a deformation retract of $L^r K(Z,1)$. Furthermore $L^2 K(Z,1) = J^2 K(Z,1) = K(Z,2)$.

To compute $\pi_* L^p K(Z,2)$ we recall an additional result of Dold and Puppe on the homotopy groups of symmetric products.

Proposition 8. If M is a free semi-simplicial module all of whose homotopy groups are finite with no p-primacy component, then $SP^r M$ has only finite homotopy groups with no p-primacy component.

Proposition 9. If $r > 1$ then $\pi_k \, G^t L^r K(Z,2)$ is finite and has no p-primacy component unless p/r.

Proof: Let $\mathcal{M}_p$ be the category of semi-simplicial complexes that have only finite homotopy groups with no p-primacy component. Proposition 2 and 8 combine to imply that $J^s M$ is in $\mathcal{M}_p$ provided M in in $\mathcal{M}_p$. Furthermore the Kunneth formula implies that $M \otimes N$ is in $\mathcal{M}_p$ provided either M or N is in $\mathcal{M}_p$. Curtis' decomposition and a straight forward induction on r yield the result.

Proposition 10. If p is a prime number, then

$$\pi_k \, L^p K(Z,2) \approx \pi_k \, J^p K(Z,2) \approx \begin{cases} Z_p & \text{if } k = 2p - 1 \\ 0 & \text{otherwise} \end{cases} .$$

Proposition 11. If $r > 2$ then $\pi_* L^r K(Z,1)$ is finite and has no p-primacy component unless 2p divides r.

If $r > 1$ then $\pi_* L^r K(Z,2)$ is finite and has no p-primacy component unless p divides r.

4. An Operation in L

In this section we introduce an operation in a semi-simplicial Lie ring L which gives it the structure of a graded differential Lie ring. This operation will be used to show that the suspension process is in some sense repetitive when r is a prime.

Definition 4.1 Let (α, β) be a shuffle permutation of type (p,q) which is denoted by the ordered sets $\alpha = \{\alpha(i)\}_{i=0}^{p-1}, \beta = \{\beta(i)\}_{i=0}^{q-1}$. The degeneracy operation S^α (or S^β) is obtained from the word:

$$S_{p+q-1} \cdots S_1 \, S_0,$$

by deleting those symbols S_j whose subscripts j are in α (or β);

$$S^\alpha = S_{p+q-1} \cdots \widehat{S}_{\alpha(p-1)} \cdots \widehat{S}_{\alpha(0)} \cdots S_0 .$$

Definition 4.2 Let x and y be two simplexes of dimensions p and q in L. We define the double bracket:

$[\![c, y]\!] = \sum (-1)^{(\alpha,\beta)} [S^{\alpha} x, S^{\beta} y]$ where the sum is to be over all shuffles of type (p,q) and $(-1)^{(\alpha,\beta)}$ denotes the sign of the permutation (α,β).

Proposition 12. The double bracket has the following properties:

(a) $[\![x,y]\!] + (-1)^{pq} [\![y,x]\!] = 0$ where $\dim x = p$ and $\dim y = q$.

(b) $(-1)^{pr}[\![x,y,z]\!] + (-1)^{rq}[\![z,x,y]\!] + (-1)^{qp}[\![y,z,x]\!] = 0$ where $\dim z = r$ and $[\![x,y,z]\!] = [\![[\![x,y]\!], z]\!]$.

(c) $\partial[\![x,y]\!] = [\![\partial x, y]\!] + (-1)^{p} [\![x, \partial y]\!]$.

Remark: If L is a free module then:

(a) implies $[\![x,x]\!] = 0$ if the dimension of x is even and

(b) implies $[\![x,x,x]\!] = 0$ for all x.

5. The Cross-effects of the Functor L

The cross effects of the functor L and the functors L^r may be obtained from their two-fold cross-effects. These in turn may be obtained from our description of the ideal I which is the kernel of the homomorphism $L(A \oplus B) \to L(B)$. The ideal I is the ideal generated by A and it must be the direct sum of $L(A)$ and the two-fold cross-effects $L(A,B)$.

Proposition 13. For any two free Abelian groups A and B there is a natural short exact sequence:

$$0 \to L(A \oplus A \otimes \sum_{r=1}^{\infty} B^{r}) \to L(A \oplus B) \to L(B) \to 0,$$

where $\sum_{r=1}^{\infty} B^{r}$ is the tensor ring, without unit, generated by B.

Proof: Every element of I may be written in terms of products of elements in either A or B; the subgroup Σ generated by products containing exactly one factor in A is isomorphic to $I/[I,I]$. In general $L^{r}\Sigma$ is the subgroup generated by products with exactly r

factors in A, hence $I = L\Sigma$.

Since multiplication in I is anti-commutative, we may write ~~with~~ generators of Σ in the form $[\ldots[a,w],\ldots,w_s]$ where w_i is in $L(B) \subset \sum_{r=1}^{\infty} B^r$, see [1]. The Jacobi identity allows us to identify $[\ldots[a,w_1],\ldots,w_s]$ with $a \otimes w_1 \otimes \ldots \otimes w_s$ in $A \oplus A \otimes \sum_{r=1}^{\infty} B^r$.

To describe the free Lie ring $L(A_1 \oplus \ldots \oplus A_s)$, we consider the free Lie ring generated by the symbols $\{1,\ldots,s\}$. To each basic product b in the Hall basis [5] of this Lie ring, we assign a group A_b which is a suitable tensor product of the groups $A_1,\ldots,A_s$.

Proposition 14. If $A_1,\ldots,A_s$ are free Abelian groups then

$$L(A_1 \oplus \ldots \oplus A_s) \approx \sum_b L(A_b).$$

$$L^r(A_1 \oplus \ldots \oplus A_s) \approx \sum_{d|r} \sum_b L^{r/d}(A_b),$$

where b is a basic product of weight d.

6. The Computation of $\pi L^pK(Z,n)$

The complex $WK(Z,n-1)$ is a twisted direct sum of the complexes $K(Z,n-1)$ and $K(Z,n)$. The kernel I of the homomorphism $LWK(Z,n-1) \to LK(Z,n)$ serves as a loop space for $LK(Z,n)$ since $LWK(Z,n-1)$ is contractable. Let I^2 denote $[I,I]$ and I^{k+1} denote $[I^k,I]$. Let $I(r)$ denote $I \cap L^rWK(Z,n-1)$ and $I^k(r)$ denote $I^k \cap L^rWK(Z,n-1)$. $I(r)$ serves as a loop space for $L^rK(Z,n)$ and is filtered by its powers:

$$I^r(r) \subset I^{r-1}(r) \subset \ldots \subset I^k(r) \subset \ldots \subset I(r).$$

Let b be a basic product in the symbols $\{n,n-1\}$ and let the index of b be the number of factors of n-1. Let K_b be a suitable tensor product of $K(Z,n)$ and $K(Z,n-1)$ so that K_b is homotopically equivalent to a K(Z,n(wt.b)-index b). By proposition 14 the factor groups may be described by:

$$I^k(r)/I^{k+1}(r) \approx \sum_{d|(r,k)} \sum_{b} L^d(K_b),$$

where b is a basic product of weight r/d and index k/d.

When d = 1, the generator of the non-trivial homotopy group $\pi_{rn-k}L'K_b$ may be covered in $I^k(r)$ by a suitable double braketted product of the generators x and y of WK(Z,n-1). This leads us to consider a certain chain complex $\mathcal{L}(n)$; generated by x,y and the operation [[,]]. $\mathcal{L}(n)$ is a subcomplex of LWK(Z,n-1), viewed as a chain complex, but it is completely determined by the dimensions of x and y, the properties of the operation [[,]] and the fact that LWK(Z,n-1) is free. Let $\mathcal{L}^r(n)$ denote $\mathcal{L}(n) \cap L^r WK(Z,1)$. When r is an odd prime then one can show by the "five lemma" or a spectral sequence argument that;

$$\pi_k\ I(r) \approx \pi_k L'K(Z,n-1) \oplus H_k(\mathcal{L}^r(2)).$$

However the homology groups $H_*(\mathcal{L}^r(n))$ are determined, up to the dimensions in which they occur, by the parity of n. This together with the results of section 3 gives us:

Proposition 15. If p is a prime then:

$$\pi_{n+k}L^pK(Z,n) \approx \begin{cases} Z \text{ if } p = 2, n \text{ is odd and } k = n \\ Z_p \text{if } k -1(\text{modulo } 2(p-1) \text{ and } 0<k<n(p-1) \\ 0 \quad \text{otherwise.} \end{cases}$$

(This result for p = 2 may be computed directly).

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Homeomorphisms of the plane with no fixed points.

Dr. S. Andrea   3-1-65

Ref: Brouwer ; Math Ann. 1912 v.72

Given a homeo. $T: \mathbb{R}^2 \to \mathbb{R}^2$ we wish to see what conditions we must require so that T be "similar" to the translation $L: (x,y) \to (x+1,y)$. Since L preserves orientation, we must require T also to preserve orientation.

Prop: T orientation preserving fixed point free $\Leftrightarrow$

for any compact connected $C \subset \mathbb{R}^2$, $C \cap TC = \phi \Rightarrow$

$C \cap T^n C = \phi \quad \forall n \in \mathbb{Z} \ n \neq 0.$

This follows from

Proof of Prop. $\Rightarrow$ Suppose assertion false, then we'll have q + $T^n q$ on same curve C. Around every point on C take a nbhd. $U \ni: U \cap TU \neq \phi$ Let $\mathcal{U}$ be this nbhd of C, it is arc-wise connected. Let p be "first" common boundary pt of U and TU, $T^{-1}p$ is a bdry pt. of U + take a curve going from $T^{-1}p \to q \to T^n q \to p$ disjoint from its image but not from its n^{th} iterated image. This is a contradiction by following lemma.

Lemma: (Brouwer 1912) T orientation preserving + fixed pt. free Given p, consider a curve C connecting p + $T(p)$, if $D = C - T(p)$ and $D \cap TC = \phi \Rightarrow D \cap T^nC = 0 \quad \forall n \in \mathbb{Z} \quad n \neq 0$.

Def: T is a free mapping of $\mathbb{R}^2$ $\Leftrightarrow$ iff above thm. holds

Cor: (Brouwer) For any pt. $p \in \mathbb{R}^2$; T free $\Rightarrow T^n p \to \infty \quad n \to \pm\infty$

Pf: Consider a large compact set A around p + $T(p)$, $\exists\, U(p)$ and $U(T(p))$ with $U(p) \cap U(T(p)) = \phi$. By compactness cover by a finite # of such open sets. $\therefore$ A has a finite # of iterates.

Cor: 2. Montgomery: Consider $F : D' \to D'$ a homeo. of open unit disc preserving orientation and area, then $\exists$ a fixed pt.

Pf: Clear:

Def: A free map T is conjugate to a translation $\Leftrightarrow T = ULU^{-1}$ U some homeo. $\mathbb{R}^2 \to \mathbb{R}^2$.

This does not classify all free maps as is seen by the following example

$= T$. sends every pt. one unit along arrow.

Although this is not conjugate to a translation, restricting to each of these three areas is.

This suggests the following def.

Def: If T is some free map, + p, q points in the plane, then $p \sim q$ (p codivergent to q) $\Leftrightarrow$ there is a curve Λ joining p and q such that $T^n(\Lambda) \to \infty$ as $n \to \pm\infty$

The equivalence classes are called fundamental regions.

Some known results on

Thm (A): The following are equivalent

(a) T has one fundamental region

(b) T conjugate to a Translation.

This will follow from the following:

Thm (B) (Sperner (1933)) The following are equivalent

① For any compact A, $T^n A \to \infty$ $n \to \pm\infty$

② T conjugate to a translation.

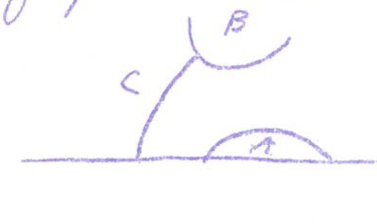
[$\mathbb{R}^2/T$ Hausdorff <=> ①]

Thm :(C) Let T be any free mapping if $Q = \bigcup_{-\infty < n < \infty} T^n \binom{\text{any fixed}}{\text{compact set}}$ then Q does not meet some non bounded connected B

[This thm. also holds for the closed half plane]

Proof of B from C

Take p & Tp & a curve C between them. On C select a minimal subsegment S s.t. the end pts. correspond under T. This line goes to ∞ in both directions & divides the plane into two half planes. We will show compact A in either goes to ∞.



The iterates of B never hit A & iterates of C go to ∞ & thus B ∪ C sweeps A to infinity.

Now B and C ⇒ A by properties of $\mathbb{R}^2/T$.

$\mathbb{R}^2/T$ has following properties

1) locally homeomorphic to $\mathbb{R}^2$ 2) orientable

3) $T^n A \to \infty \Rightarrow$ Hausdorff 4) connected

4) $\mathbb{R}^2$ is universal covering manifold

5) $\pi_1(\mathbb{R}^2/T)$ = group of covering transformations.
$\overset{\cong}{\mathbb{Z}}$

All of this implies that $\mathbb{R}^2/T$ is homeomorphic to a cy

& $\mathbb{R}^2$ is the usual covering of the cylinder

[Note: There is a T such that T has n-fundamental regions

$\forall n > 0,\ n \neq 2$].

Concordance is equivalent to smoothing

James Munkres

Let h: $K \to M$ and k: $K \to N$ be smooth (C^∞) triangulations of the non-bounded manifolds M and N . M. Hirsch defines the differentiable structures on K induced by h and k to be <u>concordant</u> if there is a differentiable structure on $K \times I$, compatible with its piecewise-linear structure, whose restrictions to $K \times 0$ and $K \times 1$ equal those induced by h and k , respectively; and he has constructed a theory of obstructions to the existence of a concordance [1].

The definition may be reformulated as follows: concordance is equivalent to the existence of a <u>combinatorial deformation</u> between the combinatorial equivalence $f = kh^{-1}: M \to N$ and a diffeomorphism. This means that there exist piecewise-smooth triangulations

$$\tilde{h}: K \times I \longrightarrow M \times I \quad \text{and} \quad \tilde{k}: K \times I \longrightarrow N \times I$$

whose restrictions to $K \times 1$ equal h and k , respectively, such that $\tilde{k}\tilde{h}^{-1}|(M \times 0)$ is a diffeomorphism [3].

On the other hand, we have defined a notion of when f is <u>smoothable</u> to a diffeomorphism; this notion is weaker than the requirement that f be combinatorially deformable to a diffeomorphism. We have constructed a theory of first-order and higher-order obstructions to the existence of a smoothing of f [2,3]. Each of the two obstruction theories seems to have insights and applications denied to the other, so that settling the differences between them has become an issue.

Our purpose here is to prove that if f is smoothable to a diffeomorphism, then f is combinatorially deformable to a diffeomorphism. This implies that all obstructions vanish in one theory if and only if they vanish in the other.

The method of proof is the following: We define a smooth cell decomposition of M to be a regular cell complex whose topological space is M such that the closure of each m-cell is smooth, in the sense that it lies in a smooth open submanifold of M of dimension m. If $h: K \to M$ is a smooth triangulation and σ is a simplex of K, then $h(\overline{\sigma})$ is a smooth cell in M. However, if τ is a cell of the usual dual cell decomposition of K, then $h(\overline{\tau})$ is not in general a smooth cell. We prove that it is possible to modify h by a smooth isotopy so as to obtain a smooth triangulation $h': K \to M$ for which the sets $h'(\tau)$ do give a smooth cell decomposition of M. To insure that the sets $h'(\overline{\tau})$ are smooth, we choose h' so that they intersect each set $h(\sigma)$ orthogonally, under suitably chosen coordinate systems. The proof requires considerable care; it seems to us of some independent interest.

The advantage of h' for smoothing purposes is the following: In general, if we take the combinatorial equivalence $f: kh^{-1}: M \to N$ and start to smooth it, the smoothed map f_m will not preserve the simplicial structures of M and N. In fact, the sets $f_m h(\overline{\sigma})$ will not even be smooth cells in N. However, if h' and k' are the special triangulations mentioned above, and we start to smooth

$f' = k'(h')^{-1}$, then we find that at each stage the smoothed map f'_m may be chosen so as to carry the smooth cell $h'(\tau)$ onto the smooth cell $k'(\tau)$, and thus preserve the (dual) cell structures.

The main theorem follows: If f may be smoothed to a diffeomorphism, then so may the map f' ; let this diffeomorphism be g . The composite $g^{-1}f'$ is a combinatorial equivalence of M with itself, carrying each smooth cell $h'(\tau)$ onto itself. Proving that $g^{-1}f'$ is combinatorially deformable to the identity follows from techniques of J. H. C. Whitehead, along with the Alexander construction for deforming a piecewise-linear homeomorphism of a ball to the identity.

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Homotopy Theory of Banach Manifolds

Prof. R. Palais 3-8-65

I Types of Banach Manifolds

1) $Gl(V)$ V - a Banach space

$F(V,W)$ Fredholm maps etc.

2) $S(V)$ unit sphere in a B. space V etc.

3) Manifolds of smooth maps i.e. M compact C^∞ manifold V paracompact finite dimensional manifold

$C^0(M,V) \dots C^k(M,V) \dots$ are B.-manifolds C^∞ is not.

$H^k(M,V)$ Soboleff maps $k > \frac{\dim M}{2}$; $W^k_p(M,V)$ $k > \frac{\dim M}{p}$

These are the natural objects to study in many problem of non-linear analysis.

II

Given $J: \mathcal{M}(M,V)$ [one of the mapping spaces] $\to R$.

consider $J(F) = \int_M F(j_k(f))$ $j_k(f) = k^{th}$ jet.

e.g. the length function.

The problem in calculus of variations is to look for critical points + to relate these to the topology of $\mathcal{M}(M,V)$.

In general, J will be bounded below + we will impose certain "ellipticity" conditions

(C) If $S \subseteq \mathcal{M}(M,V)$ on which J is bounded, but $\|dJ\|$ is not bounded away from 0 $\Rightarrow \exists$ a critical pt. in $\bar{S}$.

Since $\mathcal{M}(M,V) \subseteq \mathcal{M}(M,\mathbb{R}^n) \Rightarrow \exists$ a norm on the tangent space

Thm: Let $\mathcal{M}$ be a complete C^2 Finsler manifold + given $J: \mathcal{M} \to R$ C^2, bounded below, + satisfies (C) then

(1) there are at least cat($\mathcal{M}$) critical pts. of J.

(2) If M is Riemannian + critical pts. are non-degenerate then we have Morse inequalities.

III Properties of ANR's.

1) (Smirnov) X^0 paracompact + locally metrizable $\Rightarrow X$ metrizable if X is locally complete $\Rightarrow X$ can be given a complete metric.

2) (Hanner) If X is metrizable + a countable union of open ANR's $\Rightarrow X$ is an ANR. A topological union of ANR's is an ANR.

3) A B. space is an AR

These imply: If X is paracompact + locally ANR, then X is ANR. Moreover if X admits a complete metric locally, then it is even a closed nbhd. retract in B.space.

4) Every paracompact C°, B-manifold is an ANR & in fact a closed nbhd. retract of a B. space.

IV Domination by a simplicial complex

Def: $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ $\quad g\circ f \sim id_X \iff Y$ dominates X.

This is a transitive relation.

It will be enough for us to show that every open set in a B-manifold is dominated by a simplicial complex.

Lemma: Let X be paracompact + $\{W_\alpha\}$ an open covering $\Rightarrow \exists$ a locally finite refinement $\{\mathcal{O}_\beta\}_{\beta\in B}$ of $\{W_\alpha\}_{\alpha\in A}$ $\ni$: $\bigcap_{i=1}^{N}\mathcal{O}_{\beta_i} \neq 0 \Rightarrow \bigcup_{i=1}^{N}\mathcal{O}_{\beta_i} \subset W_\alpha$ some α.

Thm 1. Every paracompact $\mathcal{O}$ in a locally convex top. v.s. X is dominated by a simplicial complex

Idea of Proof: Cover $\mathcal{O}$ by convex open sets W_α + can get locally finite open cover of $\mathcal{O}$, $\{\mathcal{O}_\beta\}$ as above then $\bigcup \mathcal{O}_{\beta_i}$ has its convex hull in $\mathcal{O}$.

Let $\{\varphi_\beta\}$ be a partition of unity with supp $\varphi_\beta \subset \mathcal{O}_\beta$ define $f: \mathcal{O} \to |N|$

N nerve of $\{\mathcal{O}_\beta\}$ $\quad f(x) = \sum \varphi_\beta(x)\,\beta$ + define $g: |N| \to X$ $g(\sum t_i \beta_i) = \sum t_i X(\beta_i)$

where $X(p_c) \in \mathcal{O}_{p_c}$

Theorem now follows from facts about convex hulls + fact that line segment from $gf(x)$ to $x \subset \mathcal{O}$.

Thm II: Every paracompact C^0 manifold is dominated by a simplicial complex + hence has the homotopy type of a simplicial complex.

Thm (J.H.C. Whitehead) Given X, Y dominated by simp. cxs. K, L then a weak homotopy equivalence between X, Y is a strong homotopy equivalence. $\therefore$ if X dominated by a C-W complex it is of the same homotopy type.

Cor: If X is a paracompact C° B-manifold + if $\pi_n(X) = 0 \Rightarrow X$ is contractible + hence an AR.

Thm (Kuiper) If H is Hilbert space, then $\pi_n(Gl(H)) = 0$ $\quad Gl(H)$: automorphisms of H in norm top.

Cor: (Dold) $Gl(H)$ is an AR.

Cor: Every Hilbert bundle is trivial

V Principle of Irrelevance of Topology.

Thm 1. Let X be a metrizable locally convex topological vector space, E any dense s/space in the finite top. Given $\mathcal{O}$, open in X, let $\tilde{\mathcal{O}} = \mathcal{O} \cap E$ as a s/space of E, then $\tilde{\mathcal{O}} \to \mathcal{O}$ is a w.h.e. If E is of countable dimension [then it is a top. v.s. in finite top.], then $\tilde{\mathcal{O}} \to \mathcal{O}$ is a h.e.

Cor: Let $X_1 X_2$ be be metrizable l.c. t.v.s. + let X_1 be a dense linear s/space of X_2 with a (possibly) stronger top. Let$_2$ be open in X_2 + let $\mathcal{O}_1 = \mathcal{O}_2 \cap X_1$ as a s/sp. of $X \Rightarrow \mathcal{O}_1 \hookrightarrow \mathcal{O}_2$ is a h.e.

Applications 1) $\mathcal{U}$ open in $\mathbb{R}^n$, M a compact manifold $X_1 = C^k(M, \mathbb{R}^n)$ $X_2 = C^\circ(M, \mathbb{R}^n)$. Let $\mathcal{O}_2 = C^\circ(M, \mathcal{U})$, $\mathcal{O}_1 = C^k(M, \mathcal{U}) \Rightarrow \mathcal{O}_1 \hookrightarrow \mathcal{O}_2$ is a h.e.

2) Let V be a σ-compact f.d. manifold, then embed V as a closed s/space of $\mathbb{R}^n$ + let $\mathcal{U}$ be a tubular nbhd. + $\pi : \mathcal{U} \to V$ the projection + $i : V \hookrightarrow \mathcal{U}$.

then
$$\begin{tikzcd}
C^k(M, U) \arrow[r, phantom, "\simeq"] \arrow[d, shift left, "c^+"] & C^0(M, U) \arrow[d, shift left, "c^+"] \\
C^k(M, V) \arrow[u, shift left, "f_*"] \arrow[r, Rightarrow, "f"'] & C^0(M, V) \arrow[u, shift left, "f_*"]
\end{tikzcd}$$
$\Rightarrow f \simeq$

Same argument works for C^∞ although it is not a B-manifold + will also work for other mapping spaces.

Cor: Let X be a B-sp. + $\{E_n\}$ an increasing sequence of f.d. s/spaces $\ni$: $\cup\{E_n\}$ is dense in X. Given O open in X, $O_n = O \cap E_n$ $O_\infty = \varinjlim O_n$, then $O_\infty \to O$ is a h.e.

Pf: Take $E = \cup E_n$ then $O_\infty = \tilde{O}$.

Poincaré Duality Prof. E. Curtis 4-4-65

This form + method of proof is due to A. Dold.

Let M be a n manifold . K a compact $\subset M$, where M is either orientable or we use $\mathbb{Z}_2$- coefficients.

Thm: $\check{H}^i(K) \cong H_{n-i}(M, M-K)$ Cohomology is Čech cohomology, Homology is singular.

Cor: M compact . $M = K$ then $\check{H}^i(M) \cong H_{n-i}(M)$

Each pt. p of M lies in a nbhd. V homeo. to $\mathbb{R}^n$; M is Hausdorff.

$H_n(M, M-p) \cong H_n(V, V-p) \cong \mathbb{Z}$. M orientable means can choose $\varphi(p)$ a generator of $H_n(M, M-p)$ continuously with p.

Thm: (A) For any compact $K \subset M$, $H_i(M, M-K) = 0$ $i > n$.

(B) Let $a, b \in H_n(M, M-K)$ s.t. $H_n(M, M-K) \xrightarrow[i_*]{} H_n(M, M-p)$.

$i_*(a) = i_*(b)$ $\forall p \in K \Rightarrow a = b$.

(C) $\exists\, \varphi(K) \in H_n(M, M-K) \ni:$ $i_* \varphi(K) = \varphi(p) =$ gen'r $\in H_n(M, M-p)$.
$\forall p \in K$.

(D) $\check{H}^i(K) \xrightarrow[\cap\varphi(K)]{} H_{n-i}(M, M-K)$ is an iso.

This cap- product is defined as follows:

We have a map $H^i(X) \otimes H_n(X \times Y) \longrightarrow H_{n-i}(Y)$
$u \otimes z \rightarrow u \backslash z$

letting $Y = X$ $z = d_* w$ $u \cap W = u \backslash d_* w$

In the relative case we get $H^i(X) \otimes H_n(X, A) \to H_{n-i}(X, A)$

This is natural. and $1 \cap z = z$ $\forall z \in H_n(X, A)$.

We need $\check{H}^i(X) \otimes H_n(X, A) \to H_{n-i}(X, A)$ but for this merely use map $\check{H}^i(X) \to H^i(X)$

Let $V \supset K$, then have pairing $\check{H}^i(V) \otimes H_n(V, V-K) \to H_{n-i}(V, V-K) \cong H_{n-i}(M, M-K)$

we have map $\cap\, \varphi(K) : \check{H}^i(V) \longrightarrow H_{n-i}(M, M-K)$ + take limit over $V \supset K$

+ limit homo $\cap\, \varphi(K) : \check{H}^i(K) \longrightarrow H_{n-i}(M, M-K)$. It is this which we claim is the iso. Note we need Čech cohomology because of use of limits.

Case 1. $M = \mathbb{R}^n$ $K = pt.$ A, B, C, D clear.

(2) $M = \mathbb{R}^n$, K = simplex p. deformation retract of K.
Also clear.

Inductive step: Suppose A, B, C, D true for M, K; M, L; $M, K \cap L$; then also true for $M, K \cup L$. where M is an n-manifold + K, L compact subsets.

Use Mayer-Vietoris (A) is clear

(B)

$$\begin{tikzcd}
H_{n+1}(M, M-K\cap L) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_n(M, M-K\cup L) \arrow[r] \arrow[ddd] & H_n(M, M-K) \\
0 & & \oplus \\
& & H_n(M, M-L) \arrow[dl] \arrow[d] \\
& H_n(M, M-p) & H_n(M, M-K\cap L) \arrow[l] \\
& & \iota_* \varphi(K) - j_* \varphi(L)
\end{tikzcd}$$

clear by exactness.

(C) same diagram.

$\iota_* \varphi(K) - j_* \varphi(L)$ is 0 in $H_n(M, M-p)$ $\forall p \Rightarrow \exists\, \varphi(K \cup L) \in H_n(M, M-K\cup L)$

$\varphi(K \cup L)$ satisfies (C).

(D) follows from commutativity of diagram linking cohomology exact sequence with homology by cap products + by 5-lemma.

(3) A, B, C, D true for $M = \mathbb{R}^n$, K = finite simplicial complex.

(4) K arbitrary compact $M = \mathbb{R}^n$. Δate $\mathbb{R}^n$ + let $A = \cup$ all simplices which intersect K.

+ take limits over subdivisions

(5) M arbitrary, K compact $K \subset B \subset V$ V is homeo to $\mathbb{R}^n$ B homeo. to unit ball.

$$\begin{tikzcd}
H^i(K) \cap H_n(M, M-K) \arrow[r] & H_{n-i}(M, M-K) \\
H^i(K) \cap H_n(V, V-K) \arrow[r] \arrow[u, phantom, "\cong" rotate=90] & H_{n-i}(V, V-K) \arrow[u, phantom, "\cong" rotate=90]
\end{tikzcd}$$

(6) M arbitrary, K compact arbitrary. Cover M by $\{V_\lambda\}$ homeo to $\mathbb{R}^n$. let $V_1 \dots V_m$ be subcovering of $\{V\}$ for which $B_1 \dots B_m$ cover K. + let $K_i = K \cap B_i$ + thm true for M, K_i. Thm. true for $M, K_1 \cup \dots \cup K_l$; M, K_{l+1}; $M, (K_1 \cup \dots \cup K_l) \cap K_{l+1}$

$\Rightarrow$ thm. true for $M, K_1 \cup \dots \cup K_{l+1}$

$\Rightarrow$ thm. true for M, K.

Spectral Operations

Dr. T. C. Kuo 4-12-65

Spectral operations are natural transformation of functors $E_n^{p,q}(\ ;G) \rightarrow E_t^{a,b}(\ ;G')$

e.g. in category of Serre fibres where objects are fibre maps & morphisms are commutative squares

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

Consider three categories

$\mathcal{K}$ Kan's c.s.s. fibrations. Objects are simplicial fibrations

$$\begin{tikzcd} X \arrow[r, "\mu"'] \arrow[d, "P"'] \arrow[dr, phantom, "\circlearrowright"] & X' \arrow[d, "P'"] \\ B \arrow[r] & B' \end{tikzcd}$$

are the morphisms.

$\mathcal{E}$ onto c.s.s. maps are objects &

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & X' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

are morphisms, all maps c.s.s.

$\mathcal{F}$ increasingly filtered c.s.s. complexes & filtration preserving maps.

$F_0X \subset F_1X \subset \ldots \quad X = \cup F_n(X)$

$I : \mathcal{K} \rightsquigarrow \mathcal{E}$ is the inclusion functor

To define $\psi : \mathcal{E} \rightsquigarrow \mathcal{F}$, given $X \xrightarrow[f]{} Y \in Ob(\mathcal{E})$, $F_n(X) = f^{-1}(Y^{(n)})$ where $Y^{(n)}$ is the n-skeleton

For $X = \overset{\infty}{\underset{n=0}{\cup}} F_nX$ of $\mathcal{F}$, we define a decreasing filtration on $C^*(X;G)$

$F^nC^* =$ all $\gamma \ni ;\ \gamma \mid F_{n-1}X = 0$. this decreasing filtration gives rise to a spectral sequence $E_*^{*,*}(X;G)$ & each term $E_n^{p,q}(\ ;G)$ is a contravariant functor on $\mathcal{F}$ & thus we have defined $\mathcal{F} \overset{E_n^{p,q}}{\rightsquigarrow} G$

Thus we have $\quad$ Nat Trans $(E_n^{p,q} \circ \psi \circ I,\ E_t^{a,b} \circ \psi \circ I)$ operations on $\mathcal{K}$

$\xleftarrow{I^*}$ Nat. Trans $(E_n^{p,q} \circ \psi,\ E_t^{a,b} \circ \psi)$ operations on $\mathcal{E}$

$\xleftarrow[\psi^*]{}$ Nat. Trans $(E_n^{p,q},\ E_t^{a,b})$ operations on $\mathcal{F}$.

Remarks: (1) known (i) ψ^* is neither onto nor one-one

(ii) all operations on $\mathcal{F}$ are known

(ii) not known, whether I^* is onto or one-one.

(iii) all spectral operations developed by Araki, Vasquez & Kristensen are operations on $\mathcal{E}$.

We shall determine all operations (single-valued or many-valued) on $\mathcal{E}$ in case $n \geq q+2$ & coefficients in $\mathbb{Z}_2$ (or $\mathbb{Z}_p$, p a prime)

Thm:1. The functor $E_n^{p,q}$ on $\mathcal{E}$ (i.e. $E_n^{p,q} \circ \psi$) admits a universal example

More precisely $\exists$ an object $P_n^{p,q} : K_n^{p,q} \longrightarrow B_n^{p,q}$ & a fundamental class $\{\xi\} \in E_n^{p,q}(P_n^{p,q})$ with the property that

$\forall\ f: X \to Y \in \mathcal{E}$ & any class $\{\gamma\} \in E_n^{p,q}(f)$ $\exists$ a morphism $(\bar\gamma, \tilde\gamma): f \to P_n^{p,q}$

$$\begin{tikzcd} X \arrow[r, "\bar\gamma"'] \arrow[d, "f"'] & K_n^{p,q} \arrow[d, "P_n^{p,q}"] \\ Y \arrow[r, "\tilde\gamma"'] & B_n^{p,q} \end{tikzcd}$$

is a commutative diagram

$(\bar\gamma, \tilde\gamma) : \underline{E}(P_n^{p,q}) \longrightarrow E(f) \ni: (\bar\gamma, \tilde\gamma)^*\{\xi\} = \{\gamma\}$

If $n \geq q+2 \Rightarrow d_n$ vanishes

Proof: In ordinary case Δ_q's are the models, we define $Z^n(\Delta_q; \pi)$ etc. In this case let the surjective maps $\begin{matrix}\Delta_n\\ \downarrow f\\ \Delta_t\end{matrix}$ be the models & we can then consider $Z_n^{p,q}\left(\begin{matrix}\Delta_n\\ \downarrow\\ \Delta_t\end{matrix}; G\right)$ as in the ordinary case. Then let $(K_n^{p,q})_n = \bigcup\limits_{\substack{\text{disjoint union}\\ \text{all surjective}\\ \alpha \ni: \mathrm{dom}(\alpha)=n\\ t\text{ - arbitrary}}} Z_n^{p,q}\left(\begin{matrix}\Delta_n\\ \downarrow \alpha\\ \Delta_t\end{matrix}\right)$.

with universal example constructed well-known argument shows that any single-valued or many-valued operation $T: E_n^{p,q} \to E_t^{a,t}$ is completely determined by operation on fundamental class.

$$\begin{tikzcd} \{\xi\} \arrow[d, mapsto] & E_n^{p,q}(P_n^{p,q}) \arrow[r] \arrow[d] & E_t^{a,b}(P_n^{p,q}) \arrow[d] \\ \{\gamma\} & E_n^{p,q}(f) \arrow[r] & E_t^{a,b}(f) \end{tikzcd}$$

Thus if $\tilde\gamma^*\{\xi\} = \{\gamma\}$

$T\{\gamma\} = \tilde\gamma^* T\{\xi\}$

but since $(\bar\gamma, \tilde\gamma)$ not necessarily unique may get a many valued operation by considering all $(\bar\gamma, \tilde\gamma)$.

Thus to determine all $E_n^{p,q} \to E_*^{**}$ $\equiv$ to compute $E_*^{**}(P_n^{p,q})$.

Thm:2. For $n \geq q+2$ $t = 2, 3, \dots, \infty$ there is an iso. $E_t^{**}(P_n^{p,q}) \cong H^*(\mathbb{Z}_2, p+q; \mathbb{Z}_2)$ the latter furnished with a bigrading to be defined below.

Def: If A is a graded commutative algebra $H^*(A, p; \mathbb{Z}_2) = H^*(\bar{B}^p A; \mathbb{Z}_2)$

Lemma: $H^*(H_*(\mathbb{Z}_2, q; \mathbb{Z}_2), p; \mathbb{Z}_2) \cong H^*(\mathbb{Z}_2, p+q; \mathbb{Z}_2)$ [only reason need $\mathbb{Z}_p$ where $\mathbb{Z}_p$ a prime so can use Cartan's result.]

pf. By Cartan $\exists$ an acyclic construction where

$$\begin{tikzcd}
H_*(Z_2,q;Z_2) = H_*(q) \arrow[rr] \arrow[dd] & & CH_*(q) \arrow[d] & & \\
& & H_*(q+1) \arrow[r] \arrow[dl, dotted, "f_1"'] \arrow[ddl, dotted, "\bar f_1"] & CH_*(q+1) \arrow[d] & \\
H_*(q) \arrow[r] & BH_*(q) \arrow[d] & & H_*(q+2) \arrow[dl, dotted, "f_2"'] \arrow[ddl, dotted, "\bar f_2"] & \} \; p\text{-times} \\
& \bar B H_*(q) \arrow[r] & B\bar B H_*(q) \arrow[d] & & \\
& & \bar B^2 H_*(q) & &
\end{tikzcd}$$

where the diagram is completable by algebraic homotopies. f are cohomology equivalences

$\Rightarrow H_*(p+q)$ and $\bar B^p(H_*(q))$ are homotopically equivalent.

$H^*(\bar B^p(H_*(q)); Z_2)$ is bigraded + this bigrading is the desired one.

Typology

April 9, 1965
Dr. Guggenheim

ef: Genre - type. (The term genre was first used by Mazur + will be used for the remainder of the seminar).

Given two categories B, $\mathcal{a}$ + bigger category E such that ob(E) are those of B and $\mathcal{a}$ + Map(E) are Map(B), Map($\mathcal{a}$) + some maps $B \to \mathcal{a}$ from an object in B to one in $\mathcal{a}$.

Thus we can compose maps $B' \xrightarrow{g} B \xrightarrow{E} A \xrightarrow{f} A'$ + define fE, Eg + all laws hold. The new morphisms are the morphisms of the genre.

Examples (i) B = category of polyhedra

$\mathcal{a}$ = category of smooth manifolds.

$B \xrightarrow[E]{} A$ is a piecewise differentiable map e.g. triangulations + theory of smoothings is about this genre.

(ii) extensions Ext(B, A) E is an extension + then maps on A, B induce extensions.

(iii) principal fibre spaces.

B = top. spaces $\mathcal{a}$ = top. gps. Maps on spaces + gps. induce new fibre spaces + E is taken to be the space.

(in this + many other contexts it is often easier to replace maps in category by homotopy classes of maps.)

Def: Given $B \xrightarrow{E} A$, if for every $B \xrightarrow{E'} A'$ we have an f such that

$$\begin{tikzcd} B \arrow[r, "E"] \arrow[dr, "E'"'] & A \arrow[d, dotted, "f"] \\ & A' \end{tikzcd}$$

then E is called left-universal + dually rt. universal if there is always a g ∋:

$$\begin{tikzcd} B' \arrow[d, dotted] \arrow[dr, "E'"] & \\ B \arrow[r, "E"'] & A \end{tikzcd}$$

A unimorphism is one which is both left + rt. universal. We can have unimorphisms without bijectivity.

In (i) unimorphisms hardly ever occur + one probably has to change the maps in $\mathcal{a}$ + in general unimorphisms occur very rarely except in the case of (iii) where for instance using Kan universal bundle + appropriate maps we get a unimorphism.

e.g. $\overline{W}\Gamma \to \Gamma$ is a unimorphism if on rt. hand side we use group homotopy classes.

In a category, left inverse ⇒ mono, in a genre this is not but we have the following lemma.

Lemma: If E is left universal & $\exists$ any monic $E' : B \to A' \Rightarrow E$ is monic.

Because $E' = fE$ $\qquad Eg_1 = Eg_2 \Rightarrow E'g_1 = E'g_2 \Rightarrow g_1 = g_2$.

Dual lemma: E is rt. universal, & $\exists$ any epic $E' : B \to A \Rightarrow E$ is epic.

Given $0 \to A \to C \to B \to 0$

$$0 \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(A, G) \to \mathrm{Ext}(B, G)$$

$$f \dashrightarrow Ef$$

If G is projective $\Rightarrow f \to Ef$ is onto i.e. E rt. universal

If G is injective $\Rightarrow E$ left universal.

— — — — — — — —

In an Abelian category $\mathrm{Hom}(B, A)$ is an Abelian gp. (Freyd regards this as a theorem). Very often the genres $B \to A$ form an Abelian gp.

(i) e.g. $\mathrm{Ext}(B, A)$ where B is a co-commutative H-alg. & A a comm H. alg. & a genre is a central extension. [see Guggenheim ; Amer. J. Math 349-382 (84)]

(ii) also classically $\mathrm{Ext}(B, A)$ is an Abelian gp.

(iii) Again in theory of vector bundles (also micro-bundles) after stabilization i.e. in $\tilde{K}$ and $\widetilde{KO}$ we get inverses.

It is interesting to note that in all of these the proof of inverse is essentially different.

We would want ideally to have have

$B_1 \xrightarrow{E_1} A_1$ & B to have sums A products & get

$B_2 \xrightarrow[E_2]{} A_2$ $\qquad B_1 + B_2 \xrightarrow{(E_1, E_2)} A_1 \times A_2$.

However in (i) B's have product & in (iii) vector plo. are products for B & sum for A which gives Whitney sum.

N-spheres in a 2·N Manifold

Prof. M.A. Kervaire 4-26-65

Problem: Given M^{2n} a connected differentiable manifold $n > 2$ $\alpha \in \pi_n(M^{2n})$, does there exist $f : S^n \longrightarrow M^{2n}$ a differentiable embedding representing α.

Answer: Yes if M is simply connected

$\exists$ always an immersion $g : S^n \rightarrow M^{2n}$ representing α.

Whitney lemma:

Given $\varphi : D^p \longrightarrow M^n$ $\quad p + q = n$

$\psi : D^q \longrightarrow M^n$ $\quad p, q > 2$

$\varphi(D^p) \cap \psi(D^q) = 2$ pts. with opposite orientation coefficients.

(P, Q) (P', Q') are the two pts.

Let u be path on D^p $P \rightarrow P'$

v be path on D^q $Q \rightarrow Q'$

and $\varphi(u) \cdot \psi(v^{-1}) \sim$ constant in M. Then

1) φ is diffeotopic to φ_0

2) $\varphi_0(D^p) \cap \psi(D^q) = \phi$

3) φ_0 and φ agree in nbhd. of ∂D^p

Thm: Let $a \in H_n(\bar{M}^{2n})$ be a lifting of α

α is representable $\Longleftrightarrow$ $a \cdot Ta = 0$ $\forall\, T \neq id$, T covering transformation

Reidemeister scalar product.

$M^m \rightarrow \bar{M}^n$ be a universal covering. + M a regular cell complex

Let ξ_i^n be cells in M, with η_i^{m-n} complementary cells

The ξ'n are lifted to X_i^n so that η_i^{m-n} are lifted to y_i^{m-n}

then $X_i^m \cdot Y_j^{m-n} = \delta_{ij}$

$a \in H_n(\bar{M})$, $b \in H_{m-n}(\bar{M})$ $\quad a = \sum \alpha_i X_i^n$ $\quad b = \sum \beta_i y_i^{m-n}$ $\quad \alpha, \beta \in \pi$

$\langle a, b \rangle = \sum \alpha_i \bar{\beta}_i$ $\qquad \bar{\ } : Z[\pi] \rightarrow Z[\pi]$

$\sigma \longrightarrow \sigma^{-1}$

$H_n(\bar{M}) \otimes H_{m-n}(\bar{M}) \longrightarrow Z[\pi]$ $\qquad \langle a, b \rangle = \sum_{T \in \pi} (a \cdot Tb) T$

Let $M^{2n} = S^n \times S^n \vee S^{2n+1} \times S^1$ $\pi_1(M) = J$ with t as a generator

$a = x + (t + t^{-1})y$
$\downarrow$
α is representable if n is odd but not if n is even.

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Thm: $\exists$ M^{2n+1} closed differentiable combinatorial manifold

$\ni$: $\pi_i(M) = \pi_i(S^1)$ $i < n$

$$\pi_n(M) = \mathbb{Z}[J] \Big/ (t^{-1} - 1 + t)$$

which is differential

$\iff$ $\exists$ V^{2n} closed, diff., almost stably parallelizable manifold
$H_*(V) = H_*(S^n \times S^n)$ & simply connected $\epsilon(V) = 1$

There exists combinatorial manifold $\underline{M}^{2n+1}$ with specifications, but no such V^{2n} can exist $\therefore$ $\underline{M}^{2n+1}$ can not be given a differential structure.

Outline of construction of $\underline{M}^{2n+1}$:

Take two bundles of tangent disc bundle of S^n + construct W by plumbing W can be embedded in S^{2n+1}. One gets $\Sigma^{2n-1} \subset S^{2n+1}$ by $\Sigma^{2n-1} = \partial W$ then $\pi_i(S^{2n+1} - \Sigma)$ are the right homotopy groups.

A. Wasserman Equivariant Differential Topology

Let G be a compact Lie gp. acting on a manifold i.e. we are given $\varphi : G \times M \to M$

e.g. V = orthogonal representation

We can consider $S(V)$. unit ball

$G_k(V)$ k-planes in V + also the universal bundle

$$\begin{tikzcd} M_k(V) \arrow[d, "\pi"] \\ G_k(V) \end{tikzcd}$$

We wish to prove the important theorems of differential topology for the equivariant case.

Let $W \subset V$ be an invariant subspace of dim. k. $W \in G_k(V)$.

then if $G_k(V)_G$ is the fixed point set

$G_k(V)_G \supset G_W(V)$ are the invariant k-planes equivariantly isomorphic under the action

[Palais + Mostow: Any compact G manifold can be embedded in a v. space].

Consider $\mathcal{Y}(V)$. $M \in \mathcal{Y}(V) \Leftrightarrow \forall x \in M$ $\exists$ $U_x \to V^t - 0$

Whitney Th: $f : M^n \to V^t$, $M \in \mathcal{Y}(V) \Rightarrow \exists \bar{f}$ approx. to f (C^∞ + uniform,

$\ni$: $\bar{F}$ is a (1-1) immersion if $t \geq 2n$ $(2n+1)$.

Proof:

Lemma 1) $M^n \subset V^t$ for some power t.

Lemma 2) $f : M^n \to V^t$ is an immersion, W an irred representation

W occurs s times in V^t then if $s \geq 2n$ $\Rightarrow$ P (an equivariant projection)

$: V^t \to V^t$ $\ni$: $\ker P \simeq W$ + $P \circ f$ is an immersion.

$$\begin{tikzcd} T(M) \arrow[d] & \supset ST(M) \arrow[r, "d\tilde{f}"'] & S(V^t) & S(\mu(V^t)) \arrow[l, "\iota"] \arrow[d] \\ M & & & G_W(V^t) \end{tikzcd}$$

$\widetilde{df} : (v) = \dfrac{df(v)}{\| df(v) \|}$ $\iota(P, v) = v$ then since W is irreducible

ι is an injection

+ rest of argument follows by dimension count.

* Thus we can always consider $V_x \to V^{2n+1}$ + get $M \to V^{2n+1}$.

Coborderism Theory. Take $M_1^n, M_2^n \in \mathcal{G}(V)$

$M_1^n \sim M_2^n$ means $\exists$ $Q^{n+1} \in \mathcal{G}(V)$ $\ni: \partial Q = M_1 \cup M_2$

This is an equivalence relation + we get $\eta_n(V) = \mathcal{G}(V)/\sim$

Given $M \subset V^{2n+1}$ take a tubular nbhd. $\gamma(M) \to \mu_k(V^{2n+1})$ + constru

thus $\eta_n(V) \to [S(V^{2n+1} \oplus R)$, Thom space of $\mu_k(V^{2n+1})]_G$

$S(V \oplus R^n) = S^n(V)$ + define $\pi_n^V(X, x) = [S^n(V), *, X\ x]_G$

$G\,x = x$ $\quad x =$ north pole or south pole.

π_n^V is a group $n \geq 1$, abelian $n > 1$ + we get

$$\theta : \eta_n(V) \to \pi_1^{V^{n+h}}(T_k(V^{n+h} \oplus R), \infty) \qquad h \geq n+3$$

If G is finite or G is Abelian then θ is an isomorphism.

In all of these arguments, transverse regularity must be replaced by canonical + transverse regularity.

Def: is c.t.r. $\iff$ take $x \in V_G^{n+h}$ $\quad f(x) \to G_k(V^{n+h} \oplus h)$

look at $(S_x)_G^{\perp} \xrightarrow{(df)_G^{\perp}} (\pi^{-1}(f(x))_G^{\perp}$

$$(df)_G^{\perp} = 1 - f(x)$$

Bundle Theory.

$E \downarrow B$ is a G-v.b. $\iff \forall b \in B$ $\quad \pi^{-1}(b) \to V'|G_1$

is G_b equivariant linear mono.

Given f, $f^*(\mu_k)$ is defined + all such bundles are derived this way

$$\begin{tikzcd}
f^*(\mu_k) \arrow[r] \arrow[d] & \mu_k(V^t) \arrow[d] \\
M \arrow[r, "f"'] & G_k(V^t)
\end{tikzcd}$$

Twist-spinning spheres in spheres

Dr. B. J. Sanderson (Joint work with W. C. Hsiang)

The idea is due to Artin (1926)

Given a knotted $S^1 \subset \mathbb{R}^3$, we can produce a knotted $S^2 \subset \mathbb{R}^4$ this is called spinning around a circle

Consider $R_3^+ \subset R_4$

+ take $\dfrac{R_3^+ \times S^1}{\sim \text{ equiv}}$

$$p : \partial R_3^+ \times S^1 \longrightarrow \partial R_3^+$$

$$\{x \sim y \mid px = py\}$$

+ this spinning produces a knotted 2-sphere in R^4

Zeeman twisted this procedure

We will go around S^l + use (S^m, Σ^n) where Σ^n is a submanifold of S^m + is a homotopy n-sphere which is homeomorphic $n \geq 4$ but may have an exotic differentiable structure. Our result will be (S^{m+l}, Σ^{n+l}).

The set of all of these can be made into a gp. take the h-cobordism classes of pairs + form a gp. $\Theta^{m,n}$

$$(S_1^m, \Sigma_1^n) \sim (S_2^m, \Sigma_2^n) \text{ if } \partial(W_1^{m+1}, N^{n+1}) = (S_1^m, \Sigma^n) - (S_2^m, \Sigma_2^n)$$

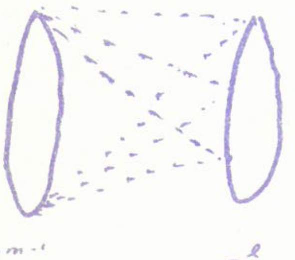
and $\Sigma_i^n \subset N$ $S_i^m \subset W$ are homotopy equivalences.

Thm: $n \geq 4$, $m - n \geq 3$ (Smale) $(S_1^m, \Sigma_1^n) \overset{h}{\sim} (S_2^m, \Sigma_2^n) \Longleftrightarrow (S_1^m, \Sigma_1^n) \approx (S_2^m, \Sigma_2^n)$

diffeo. orientation preserving

Thm: (Haefliger) $\Theta^{m,n} = \Theta^n$ $m > 3/2n$

$$S^{m+l} = S^m \circ S^l$$



S^m splits into two halves D_+^m, D_-^m + assume $\Sigma^n \subset D_+^m$ + for every point on S^l we get D_+^m + as the pt. varies on S^l, we get spinning.

$$S^{m+l} = S^{m-1} \circ S^l$$
$$= S^{m-1} \times D^{l+1} \cup D^m \times S^l$$

the intersection is

$S^{m-1} \times D^{\ell+1} \cap D^m \times S^\ell = S^{m-1} \times S^\ell$ & we want to glue it back so as to preserve $S^{n-1} \times S^\ell \subset S^{m-1} \times S^\ell$

This suggests to twist by a diffeo. $(S^{m-1}, S^{n-1}) \times S^\ell$

Thm (Smale) $m-n \geq 3$ $n \geq 4$

i) $S^m - \Sigma^n \approx S^{m-n-1} \times R^{n+1}$

ii) $(S^m, \Sigma^n) \approx (D_+^m, D_+^n) \cup_h (D_-^m, D_-^n)$

i) $h \times 1 : (S^{m-1}, S^{n-1}) \times S^\ell \circlearrowleft$

ii) $F : (S^{m-1}, S^{n-1}) \times S^\ell \circlearrowleft$

$(x, y) \rightarrow (\alpha_y x, y)$ $\alpha : S^\ell \rightarrow SO(n) \times SO(m-n) \subset SO(m)$

$\Phi : \Theta^{m,n} \times \pi_\ell(SO(n) \times SO(m-n)) \rightarrow \Theta^{m+\ell, n+\ell}$

A representative for $\Phi([S^n, \Sigma^m], \alpha)$ is given by

$((\partial D_+)^{m-1}, (\partial D_+)^{n-1}) \times D^{\ell+1} \underset{\substack{\text{glue by} \\ F \circ (h \times 1)}}{\cup} (D_-^m, D_-^n) \times S^\ell$

without F called spinning with F called twist spinning.

[what Zeeman did was construct a map

$\vartheta_s^{3,1} \times \underset{\overset{\shortparallel}{\mathbb{Z}}}{\pi_1(SO(2))} \rightarrow \vartheta_s^{4,2}$]

$$\begin{tikzcd}
(\partial D_+)^{m-1} \times D^{\ell+1} \arrow[d, "id"] & \cup_{F \circ (h \times 1)} & D^m \times S^\ell \arrow[d, "F \circ \tilde{h} \times 1"] \\
S^{m+\ell} \approx \partial D_+^m \times D^{\ell+1} & \cup_{id} & D^m \times S^\ell
\end{tikzcd}$$

We want to extend $F \circ (h \times 1)$ over $D^m \times S^\ell$ but can do for $F \circ h$

look at s/manifold & replace m by n in above $h | \partial D^n$ extends to a homeo.

1) Is Φ non-trivial

2) Is it bilinear? Yes over first factor, only bilinear over second if element in $SO(n)$ comes from $SO(n-1)$.

* We will answer 1) by a relationship of φ with the following (relative version) of a pairing of Munkres-Milnor-Novikov.

$$\begin{tikzcd} \psi : \pi_0(\mathrm{Diff}_c(R^{m-1}, R^{n-1})) \times \pi_l(SO(n-1) \times SO(m-n)) \arrow[d] \\ \pi_0(\mathrm{Diff}_c(R^{m+l-1}, R^{n+l-1}) \end{tikzcd}$$

Gives non-relative version by elementary ...

$$(h \times 1)^{-1} \circ F \circ (h \times 1) \circ F^{-1} : R^{m+l-1}, R^{n+l-1} \circlearrowleft$$

$$\overset{\shortparallel}{(R^{m-1}, R^{n-1}) \times D^l} \qquad (x, y) \longrightarrow (\alpha_y x, y)$$

$$\alpha : R^l \to SO(n-1) \times SO(m-n)$$

this gives rise to $\hat{\psi} : \Theta^{m,n} \times \pi_l(\;) \to \Theta^{m+l,n+l}$

Thm: $\hat{\psi}$ coincides with φ_{n-1}

Milnor showed ψ non-trivial ∴ so is φ_n

$$\begin{tikzcd} \varphi : \Theta^{m,n} \times \pi_l \arrow[r] \arrow[d, "\gamma \times p"] & \Theta^{m+l,n+l} \arrow[d, "\nu"] \\ \pi_{n-1}(SO(m-n)) \times \pi_l(SO(m-n)) \arrow[r, "\text{Samuelson}"] & \pi_{n+l-1}(SO(m-n)) \end{tikzcd}$$

Levine reduced calculation of $\Theta^{m,n}$ to a calculation of

$\pi_n(G_k, SO(k))$ $\quad k = m-n \quad G_k = \deg 1$ maps $S^{k-1} \to S^{k-1}$

mod AnS invariant, up to extension.