SYMMETRY OF LINKING NUMBERS

BY DR. BRIAN STEERE

LET S^q, S^p BE DIFFERENTIABLY, DISJOINTLY IMBEDDED IN S^{n+1} VIA i_q, i_p RESPECTIVELY. THEN $S^{n+1} - i_p(S^p)$ HAS THE HOMOTOPY TYPE OF S^{n-p}. LET h_p BE A HOMOTOPY EQUIVALENCE.

$$h_p : S^{n+1} - i_p(S^p) \to S^{n-p}$$

SO $h_p i_q : S^q \to S^{n-p}$

LET $\alpha = [h_p i_q] \in \pi_q(S^{n-p})$

WE CALL α A <u>LINKING ELEMENT OF S^q WITH S^p</u>. LET $\beta = [h_q i_p]$ IN $\pi_p(S^{n-q})$ AND E DENOTE THE SUSPENSION HOMOMORPHISM $E : \pi_m(S^k) \to \pi_{m+1}(S^{k+1})$

WE THEN HAVE THE FOLLOWING SYMMETRY RELATION BETWEEN THE LINKING ELEMENTS α, β:

$$E^{n-q}\alpha = \pm E^{n-p}\beta$$

WE WOULD LIKE TO FIND SOME SUCH SYMMETRY RELATIONS FOR THE CASE OF THREE SPHERES S^p, S^q, S^r DIFFERENTIABLY DISJOINTLY IMBEDDED IN S^{n+1} VIA i_p, i_q, i_r RESPECTIVELY.

BY POINCARÉ DUALITY IT CAN BE SHOWN THAT $S^{n+1} - i_p(S^p) \vee i_q(S^q)$ IS OF THE SAME $n-1$ HOMOTOPY TYPE AS $S^{n-p} \vee S^{n-q}$ HENCE THE IMBEDDING OF S^r IN $S^{n+1} - i_p(S^p) \vee i_q(S^q)$ REPRESENTS AN ELEMENT β_r IN $\pi_r(S^{n-p} \vee S^{n-q})$

HILTON (JOURNAL LONDON MATH. SOCIETY 1956) HAS GIVEN A DECOMPOSITION OF $\pi_r(S^{n-p} \vee S^{n-q})$ AS:

$$\pi_r(S^{n-p} \vee S^{n-q}) = \pi_r(S^{n-p}) \oplus \pi_r(S^{n-q})$$

$$\oplus\ \pi_r(S^{2n-p-q-1}) \oplus \text{OTHERS}$$

FOR $\beta_r \in \pi_r(S^{n-p} \vee S^{n-q})$ THE PROJECTION OF β_r IN $\pi_r(S^{n-p})$ AND IS CALLED <u>A LINKING ELEMENT</u>

OF S^r WITH S^p (REFERENCE: KERVAIRE VOL. 1 ANNALS OF MATH 1959)

LET $h(\beta_r)$ BE THE PROJECTION OF β_r IN $\pi_r(S^{2n-p-q-1})$. WE ALSO HAVE $h(\beta_q)$ IN $\pi_q(S^{2n-p-r-1})$ WE SHALL PROVE THE FOLLOWING SYMMETRY RELATION: $\underline{E^{n-q+2}h(\beta_q) = (-1)^{r+q}E^{n-r+2}h(\beta r)}$

FIRST LET M, N BE TWO DISJOINT FRAMED MANIFOLDS IN S^{n+1}. THAT IS, FOR EACH MANIFOLD WE CAN CHOOSE A CONTINUOUS BASIS FOR ITS NORMAL BUNDLE.

IF M IS OF DIMENSION p THEN APPLYING THE THOM CONSTRUCTION ONE GETS AN ELEMENT OF $\pi_{n+1}(S^{n+1-p})$

FOR ONE CONSTRUCTS A TUBULAR NBD. ABOUT M^p. THE NORMAL BUNDLE HAS FIBRE DIMENSION $n+1-p$.

WE HAVE $M^p \times D^{n+1-p} \xrightarrow{\text{PROJ.}} D^{n+1-p} \xrightarrow{\text{QUOTIENT MAP}} S^{n+1-p}$

AND EXTEND MAP TO THE REST OF S^{n+1}.

THIS GIVES US AN ELEMENT

$\gamma(M) \in \pi_{n+1}(S^{n+1-p})$.

FOR $\alpha \in \pi_r(S^n)$ CHOOSE A DIFFERENTIABLE REPRESENTATIVE f. LET $*$ BE A BASE POINT OF S^n. FOR x A REGULAR VALUE OF f IN $S^n - a$ $f^{-1}(x) \equiv M$ IS A DIFFERENTIABLE FRAMED MANIFOLD IN S^r OF DIMENSION $p = r-n$. WE CLAIM THE THOM CONSTRUCTION YIELDS

$\gamma(M) = \alpha \in \pi_r(S^{r-(r-n)})$.

NOW CONSIDER M^p, N^q TWO FRAMED MANIFOLDS DISJOINT IN S^{n+1}. APPLYING THE THOM CONSTRUCTION WE GET $\gamma(M) \in \pi_{n+1}(S^{n+1-p})$ $\gamma(N) \in \pi_{n+1}(S^{n+1-q})$

USING THE HILTON DECOMPOSITION WE GET AN ELEMENT OF $\pi_{n+1}(S^{n+1-p} \vee S^{n+1-q})$

FOR M^p, N^q FRAMED DISJOINT MANIFOLDS IN S^{n+1} THERE IS A CONSTRUCTION WHICH YIELDS $\gamma(M^p, N^q) \in \pi_{n+2}(S^{2n-p-q+2})$

LET $\mathcal{U}, V$ BE TWO FRAMED MANIFOLDS IN $S^{n+1} \times I \subseteq S^{n+2}$ SUCH THAT

1) $\partial \mathcal{U} = M \times 0 \cup M' \times 1$

2) $\partial V = N \times 0 \cup N' \times 1$

3) $\mathcal{U}, V$ INTERSECT $S^{n+1} \times 0$, $S^{n+1} \times 1$ TRANSVERSALLY

4) M', N' ARE SEPARATED BY AN EQUATOR.

LET $W = \mathcal{U} \cap V$ A CLOSED FRAMED MANIFOLD SET $\Upsilon(M, N)$ TO BE ELEMENT OBTAINED FROM W BY THE THOM CONSTRUCTION.

<u>LEMMA I</u> $\Upsilon(M, N)$ INDEPENDENT OF $\mathcal{U}, V$. |

DEFINE $h' : \pi_r(S^i \vee S^j) \to \pi_{r+1}(S^{i+j})$ AS FOLLOWS: LET $\alpha \in \pi_r(S^i \vee S^j)$ AND f A DIFFERENTIABLE (EXCEPT AT BASE PT.) REPRESENTATIVE OF α.

LET $x \in S^i - *$, $y \in S^j - *$ BE REGULAR VALUES. THEN WE GET $f^{-1}(x) = M^{r-i}$ $f^{-1}(y) = N^{r-j}$ DISJOINT FRAMED MANIFOLDS IN S^r

WE THEN HAVE

$$\gamma(M,N) \in \pi_{r+1}(S^{2r-(r-i)-(r-j)}) = \pi_{r+1}(S^{i+j})$$

$$h'(\alpha) \equiv \cdot\, \gamma(M,N)$$

<u>LEMMA 2</u> RECALL FOR $\alpha \in \pi_r(S^i \vee S^j)$ $h(\alpha)$ IS PROJECTION OF α IN $\pi_r(S^{i+j-1})$ WE CLAIM $h'(\alpha) = (-1)^j E\, h(\alpha)$. |

SUPPOSE S^p, S^q, S^r ARE DIFFERENTIABLY IMBEDDED IN S^{n+1} SUCH THAT EACH PAIR IS J-EQUIVALENT TO 0. THEN

<u>LEMMA 3</u> THERE EXIST DISJOINT MANIFOLDS V_q^{p+1}, V_p^{q+1} IN S^{n+1} SUCH THAT

$$\partial V_q^{p+1} = S^p \qquad \partial V_p^{q+1} = S^q.$$ |

CONSIDER (i,j,k) A PERMUTATION OF (p,q,r) DEFINE $M_k^i = S^i \cap V_j^{k+1}$

LET W^i BE A FRAMED MANIFOLD IN $S^{n+1} \times I$ SUCH THAT $\partial W = V_{j+1}^i \times 0 \cup V_{k+1}^i \times 1 \cup S^i \times I$ THIS FOLLOWS FROM:

<u>LEMMA 4</u>

V_q^{p+1}, V_r^{p+1} ARE FRAMED COBORDANT MANIFOLDS.

(REF. HAEFLIGER ANNALS 1962)

LEMMA 5 $\gamma(M^i_k, M^i_j) = \pm E h(\beta_i)$

LET $T = V_k^{i+1} \times I \cap W^j \cap \overline{W}^k$

WHERE $\overline{W}^k$ IS THE IMAGE OF W^k UNDER THE MAP $\kappa : S^{n+1} \times I \to S^{n+1} \times I$ BY

$\kappa(x, t) = (x, 1-t)$

$\partial T = A \cup B$ WHERE $A = (V_k^{i+1} \times I) \cap (S^j \times I) \cap \overline{W}^k$

$B = S^i \times I \cap W^j \cap \overline{W}^k$

A A REPRESENTATIVE WHICH UNDER THOM CONSTRUCTION WILL GIVE $\gamma(M^i_j, M^i_k)$ AND B WILL GIVE $\gamma(M^j_i, M^j_k)$.

THIS YIELDS

THEOREM 6 $E^{n-k+2} h(\beta_k) = (-1)^{k+1} E^{n-i+2} h(\beta_i)$

MANIFOLDS WITH CONTRACTIBLE COMPLEMENT

BY DR. G. M. KELLY

WE SHALL CONSIDER CLOSED CONNECTED MANIFOLDS.

CONSIDER SUCH A MANIFOLD M^n, N^{n-k} A MANIFOLD, $N^{n-k} \subset M^n$ (N^{n-k} NOT NECESSARILY A SUBMANIFOLD) AND $M^n - N^{n-k}$ CONTRACTIBLE.

EXAMPLES $\quad M = P_n$ (R, C, QUATERNIONS, CAYLEY NOS.)

DIMENSIONS $\quad n, 2n, 4n, 16$

SUBMANIFOLD $\quad N = P_{n-1}$ (R, C, QUATERNIONS, CAYLEY NOS.)

DIMENSIONS $\quad n-1, 2n-2, 4n-4, 8$

COMPLEMENT OF N IS: $R^{n-1}, R^{2n-2}, R^{4n-4}, R^8$.

WE SHALL PROVE THE FOLLOWING:

THEOREM 1 $\quad k$ DIVIDES n AND $k = 1, 2, 4, 8$

THEOREM 2 $\quad k \neq 1$ IMPLIES $\pi_1(M) = 0$

THEOREM 3 $\quad$ IF $k = 1$ $\quad M \simeq P_n(R)$

IF $k = 2$ $\quad M \simeq P_{n/2}(C)$

WE ACTUALLY ONLY NEED: N HAS HOMOTOPY TYPE OF AN $n-k$ MANIFOLD.

PROOF OF THEOREM I

LET US CALCULATE BETTI NUMBERS MOD 2.
LET $A \subset M$ BE A CLOSED SUBSET.
WE HAVE THE FOLLOWING:

$$\begin{array}{ccccccccc}
\cdots \to & \check{H}^{r}(M) & \to & \check{H}^{r}(A) & \to & \check{H}^{r+1}(M,A) & \to & \check{H}^{r+1}(M) \to \cdots \\
& \| \wr & & \| \wr & & \| \wr & & \| \wr \\
\cdots \to & H_{n-r}(M) & \to & H_{n-r}(M, M-A) & \to & H_{n-r-1}(M-A) & \to & H_{n-r-1}(M) \to \cdots
\end{array}$$

$\check{H}^{*}$ CECH COHOMOLOGY, H_{*} SINGULAR HOMOLOGY.

APPLYING THIS TO $A = N$ AND USING
$M - N$ CONTRACTIBLE WE HAVE

$$H_r(M-N) = 0 \quad \text{IF } r \neq 0$$

$$\text{SO} \quad \check{H}^{r}(M,N) = 0 \quad \text{IF } r \neq n$$

HENCE:

$$0 = \check{H}^{n-1}(M,N) \to \check{H}^{n-1}(M) \to \check{H}^{n-1}(N) \xrightarrow{\searrow 0 \nearrow} \check{H}^{n}(M,N) \to \underset{\mathbb{Z}_2}{\check{H}^{n}(M)} \to \underset{0}{\check{H}^{n}(N)} \to \cdots$$

THIS GIVES $\check{H}^{r}(M) \simeq \check{H}^{r}(N)$ FOR $r \neq n$.

LET p_r = RANK OF $\check{H}^{r}(M)$

THEN $\begin{cases} p_r = p_{n-r} \\ p_r = p_{n-k-r} & r \neq n, -k \end{cases}$

WHICH YIELDS: $p_{n-r} = p_{n-r-k} \qquad r \neq n, -k$
$\qquad\qquad p_r = p_{r-k} \qquad r \neq 0, n+k$

IF $k \nmid r$ THEN
$$0 = p_r = p_{r+k} = p_{r+2k} = \cdots$$

IN PARTICULAR IF $k \nmid n$ THEN $p_n = 0$ CONTR!

SO $k \mid n$.

$p_0 = p_k = p_{2k} = \quad = p_n = 1 \qquad n = mk$

LET z_i BE GENERATOR OF $\check{H}^{ik}(M)$

WE HAVE: $1, z_1, z_2, \ldots z_m$

THEN $z_i \cdot z_{m-i} = z_m$

LET w_j BE GENERATOR OF $\check{H}^{jk}(N)$

AND WE HAVE: $1, w_1, \ldots w_{m-1}$

IF $i : N \to M \qquad i^* z_i = w_i$

HENCE $w_i \cdot w_{(m-1)-i} = w_{m-1}$

IMPLIES $z_i \cdot z_{m-1-i} = z_{m-1}$ (i^* IS AN ISOMORPHISM)

$z_1 \cdot z_i \cdot z_{m-1-i} = z_{m-1} \cdot z_1 = z_m$

THUS $z_1 \cdot z_i = z_{i+1}$

SO WE HAVE $1, z_1, z_1^2 = z_2, z_1^3 = z_3$

F. ADAMS HAS SHOWN THAT THIS IMPLIES $k = 1, 2, 4, 8$ FINISHING PROOF OF THM 1 ∎

PROOF OF THEOREM 2

FOR $k \neq 1$ ANY LOOP IN M MAY BE PUSHED OFF N, i.e. HOMOTOPIC TO A LOOP IN M − N WHICH BEING CONTRACTIBLE IMPLIES $\pi_1(M) = 0$. ∎

FOR $k=1$ $\pi_1(M) = Z_2$ BECAUSE IF WE LOOK AT THE TWO FOLD COVERING, WE GET $\tilde{M} - \tilde{N}$ TO BE TWO CONTRACTIBLE PIECES AND A SIMILAR ARGUMENT AS ABOVE SHOWS $\tilde{M}$ IS SIMPLY CONNECTED.

WE NOW PROVE:

THEOREM 3 IF $k=1$ $M \simeq P_n(R)$
IF $k=2$ $M \simeq P_{n/2}(C)$

PROOF

LET $k=1$. TAKE UNIVERSAL COVERING SPACE P OF M. IT IS A DOUBLE COVERING SINCE $\pi_1(M) = Z_2$. LET Q COVER N. Q IS CONNECTED AND $P - Q$ = 2 COPIES OF $M - N$. ONE ESTABLISHES FOR P THAT, $p_0 = 1 = p_n$, $p_1 = p_2 = \ldots = p_{n-1} = 0$ AND THAT $P \simeq S^n$. USING THIS ONE PROVES $M \simeq P_n(R)$.

NOTE: Q IS NOT NECESSARILY SIMPLY CONNECTED.

LET $k=2$ $\pi_r(P_n(C)) = \begin{cases} Z & r=2 \\ 0 & 2<r\leq 2n \end{cases}$

BY OBSTRUCTION THEORY ONE CAN CONSTRUCT A MAP $f: M \to P_{n/2}(C)$ WITH $f^*(w_1) = z_1$ SO $f^*(w_1^i) = z_1^i = z^i$ HENCE f^* IS AN ISO. SO BY J. H. C WHITEHEAD'S THM. f IS A HTY. EQUIV.

INVOLUTIONS AND BORDISMS

BY P.E.CONNER

DEFN. LET X BE A TOPOLOGICAL SPACE
M^n A CLOSED DIFFERENTIABLE MANIFOLD
f A CONTINUOUS MAP $f: M^n \to X$
GIVEN TWO SUCH PAIRS (M^n, f) (N^n, g)
WE SAY THEY ARE <u>DIFFEOMORPHIC</u> IF WE CAN COMPLETE THE FOLLOWING TO A COMMUTATIVE DIAGRAM WHERE h IS A DIFFEOMORPHISM.

$$\begin{tikzcd} M^n \arrow[dr, "f"] \arrow[d, dashed, "h"'] & \\ N^n \arrow[r, "g"'] & X \end{tikzcd}$$

WE SAY (M^n, f) <u>BORDS IN X</u> $\Leftrightarrow$ THERE EXISTS B^{n+1} SUCH THAT $\partial B^{n+1} = M^n$ AND THERE EXISTS $F: B^{n+1} \to X$ SUCH THAT $F|_{\partial B^{n+1}} = f$

WE SAY (M_1^n, f_1) IS <u>BORDANT</u> TO (M_2^n, f_2) WRITTEN $(M_1^n, f_1) \sim (M_2^n, f_2)$ IFF $(M_1^n \cup M_2^n, f_1 \cup f_2)$ BORDS IN X.

(WHERE WE TAKE THE DISJOINT UNION)

THIS IS AN EQUIVALENCE RELATION. THE CLASS OF (M^n, f) WILL BE DENOTED $[M^n, f]$. WHEN WE CONSIDER UNORIENTED

MANIFOLDS WE WRITE $[M^n, f]_2$.

THESE UNORIENTED BORDISM CLASSES FORM AN ABELIAN GROUP WITH

$$[M_1^n, f_1] + [M_2^n, f_2] \equiv [M_1^n \cup M_2^n, f_1 \cup f_2]_2$$

THIS GROUP IS DENOTED $\mathcal{N}_n(X)$. EACH ELEMENT IS ITS OWN INVERSE. IF X IS A POINT P THEN $\mathcal{N}_n(P) = \mathcal{N}_n$ THE UNORIENTED BORDISM GROUP OF THOM.

LET $\mathcal{N}_*(X) = \sum_0^\infty \mathcal{N}_n(X)$ THIS IS A RIGHT $\mathcal{N}$ MODULE OVER THE UNORIENTED THOM COBORDISM RING $\mathcal{N}$.

LET $f : M^n \to X$, V^m A CLOSED DIFFERENTIABLE MANIFOLD.

WE DEFINE $f' : M^n \times V^m \to X$ BY

$$f'(x,y) \equiv f(x)$$

THEN WE HAVE A RIGHT OPERATION OF $\mathcal{N}$ ON $\mathcal{N}_*(X)$ BY:

$$[M^n, f]_2 [V^m]_2 \equiv [M^n \times V^m, f']_2$$

WHICH MAKES $\mathcal{N}_*(X)$ INTO A GRADED RIGHT MODULE.

IF X IS A FINITE COMPLEX $\mathcal{N}_*(X)$ IS A FREE $\mathcal{N}$ MODULE

ISOMORPHIC TO $H_*(X;\mathbb{Z}_2)\otimes \eta$. ONE CAN SHOW GIVEN $c_k \in H_k(X;\mathbb{Z}_2)$ THERE EXISTS $f: M^k \to X$ SUCH THAT FOR σ_k GENERATOR OF $H^k(M^k, \mathbb{Z}_2)$ $f_*(\sigma_k) = c_k$. THEN CHOOSING AN ADDITIVE BASE $\{c_k\}$ OF $H_*(X;\mathbb{Z}_2)$ IN EACH DIMENSION THE ISOMORPHISM IS OBTAINED BY $c_k \otimes 1 \to [M^k, f]$

NOW CONSIDER A DIFFEOMORPHISM $T: M^n \to M^n$ SUCH THAT T^2 = IDENTITY AND T HAS NO FIXED POINTS. WE CALL T A *FIXED POINT FREE INVOLUTION*. WE SAY (T_1, M_1^n) IS *EQUIVARIANTLY DIFFEOMORPHIC* TO (T_2, M_2^n) IF THERE IS A DIFFEOMORPHISM $h: M_1^n \to M_2^n$ WHICH MAKES THE FOLLOWING DIAGRAM COMMUTATIVE:

$$\begin{tikzcd} M_1^n \arrow[r, "h"] \arrow[d, "T_1"'] & M_2^n \arrow[d, "T_2"] \\ M_1^n \arrow[r, "h"] & M_2^n \end{tikzcd}$$

WE SAY (T, M^n) *BORDS* IF THERE EXISTS (Υ, B^{n+1}) WITH NO

FIXED POINTS SUCH THAT $(\Upsilon|_{\partial B^{n+1}}, \partial B^{n+1})$ IS EQUIVARIANTLY DIFFEOMORPHIC TO (T, M^n).

GIVEN (T_1, M_1^n) AND (T_2, M_2^n) WE FORM $(T, M_1^n \cup M_2^n)$ (DISJOINT UNION) WHERE $T|_{M_i^n} = T_i$.

THEN (T_1, M_1^n) IS <u>EQUIVARIANTLY BORDANT</u> TO (T_2, M_2^n) WRITTEN $(T_1, M_1^n) \sim (T_2, M_2^n) \Leftrightarrow (T, M_1^n \cup M_2^n)$ EQUIVARIANTLY BORDS. THIS IS AN EQUIVALENCE RELATION.

FOR UNORIENTED MANIFOLDS WE FORM EQUIVALENCE CLASSES $[T, M^n]_2$ WHICH BECOME AN ABELIAN GROUP UNDER:

$$[T_1, M_1^n]_2 + [T_2, M_2^n]_2 \equiv [T, M_1^n \cup M_2^n]_2$$

THIS GROUP WE WRITE AS $\eta_n(\mathbb{Z}_2)$

ONE HAS $\eta_n(\mathbb{Z}_2) \simeq \eta_n(K(\mathbb{Z}_2, 1))$

FORMING $\eta_*(\mathbb{Z}_2) \equiv \sum_{n=0}^{\infty} \eta_n(\mathbb{Z}_2)$

WE HAVE $\eta_*(\mathbb{Z}_2)$ IS A FREE η MODULE WITH GENERATORS $[A, S^n]_2$ WHERE A IS THE ANTIPODAL MAP ON THE n-SPHERE.

EXAMPLE

LET $CP(2)$ DENOTE THE COMPLEX PROJECTIVE PLANE.

$$T : (x, [z_1, z_2, z_3]) \to (-x, [\bar{z}_1, \bar{z}_2, \bar{z}_3])$$

IS A FIXED POINT FREE INVOLUTION ON $S^1 \times CP(2)$ AND

$$[T, S^1 \times CP(2)]_2 = [A, S^0]_2 [W^5]_2 + [A, S^1]_2 [CP(2)]_2$$

WHERE W^5 IS THE WU-DOLD MANIFOLD.

NOTE $[T, M^n][V^m] \equiv [T', M^n \times V^m]$

$$T' : (x, y) = (T(x), y)$$

WE NOW WISH TO CONSIDER INVOLUTIONS WHICH ARE NOT FIXED POINT FREE. CONSIDER (T, M^n) T NOT NECESSARILY FIXED POINT FREE. FOR $0 \leq m \leq n$ LET F^m DENOTE THE m-DIMENSIONAL FIXED POINT SET. (F^m IS A UNION OF m-DIML. SUBMANIFOLDS OF M^n) LET US CHOOSE A RIEMANNIAN METRIC ON THE TANGENT BUNDLE OF M^n SUCH THAT THE INVOLUTION IS AN ISOMETRY. LET N_m DENOTE THE NORMAL BUNDLE OF F^m.

N_m IS AN $n-m$ PLANE BUNDLE.
WE HAVE AN INVOLUTION Υ ON N_m
WHICH IS AN ANTIPODAL MAP ON FIBRES.
WE CLAIM THAT THE EXPONENTIAL
MAP : $(\Upsilon, N_m) \to (T, M^n)$ IS AN
EQUIVARIANT DIFFEOMORPHISM AROUND THE
FIXED POINT SET.

NOW REPLACE THE NORMAL BUNDLE
N_m BY THE NORMAL SPHERE BUNDLE
B_m (WHICH HAS MANIFOLD DIMENSION $n-1$)
THEN ONE CAN SEE THAT

$$\Sigma_0^n [\Upsilon, B_m]_2 = 0 \in \mathcal{N}_{n-1}(\mathbb{Z}_2)$$

(Υ IS FIXED PT. FREE ON B_m)

WE CLAIM $N_m \to F^m$ DETERMINES $[M^n]_2$

WE HAVE (T, M^n) T NOT NECESSARILY
FIXED POINT FREE. WE DEFINE
TWO INVOLUTIONS ON $M^n \times I$

$$T_1 : (x, t) \to (T(x), 1-t)$$

$$T_2 : (x, t) \to (x, 1-t)$$

WE NOW IDENTIFY TWO COPIES OF $M^n \times I$
ALONG THEIR BOUNDARY (M^n CLOSED)

$$(x, 0) = (x, 0)$$

$$(x, 1) = (T(x), 1).$$

LET V^{n+1} BE THE RESULTING CLOSED MANIFOLD. T_1 AND T_2 THEN YIELD AN INVOLUTION ON V^{n+1} FOR $T_1(x,0) = (T(x),1)$
$T_2(x,0) = (x,1)$

AND $T_1(Tx,1) = (x,0)$
$T_2(x,1) = (x,0)$

NOW THE FIXED POINT SET OF T_2 IS $M^n \times \frac{1}{2}$ AND THE FIXED POINT SET OF T_1 IS $F^m \times \frac{1}{2}$. $0 \leq m \leq n$. SO CONSIDERING F^m AS SITTING IN V^{n+1} ITS NORMAL BUNDLE IS NOW $N_m \oplus R^1$ (R^1 TRIVIAL LINE BUNDLE) PASSING TO THE SPHERE BUNDLE B_m' WE HAVE BY A PREVIOUS FORMULA

$$[A, S^0]_2 \, [M^n]_2 + \sum_0^n [T, B_m']_2 = 0 \in \mathcal{N}_n(\mathbb{Z}_2)$$

PASSING TO QUOTIENT SPACES WE HAVE $[M^n]_2 = \sum_{m=0}^{n} [B_m'/T]_2$

PROVING

<u>THEOREM</u> IF M^n HAS A FIXED POINT FREE INVOLUTION THEN M^n COBORDS. |

WE NOTE THAT $[CP(n)]_2 = [P(n) \times P(n)]_2$

THERE EXIST THEOREMS STATING $[M^n \times M^n]_2$ IS COBORDANT TO AN ORIENTABLE MANIFOLD.

WE PROVE THIS FOR n ODD.

CONSIDER THE INVOLUTION ON $M^n \times M^n$

$$T : (x, y) \to (y, x)$$

SO $F^n = \{(x,x)\} = M^n$

THE NORMAL BUNDLE TO F^n IS JUST THE TANGENT BUNDLE OF M^n

PROCEEDING AS BEFORE WE HAVE

$$[M^n \times M^n]_2 = [B_n'/T]_2$$

B_n'/T IS ORIENTABLE, BECAUSE $W_1 = 0$ IF n IS ODD.

NOW CONSIDER M^n CLOSED, UNORIENTED DIFFERENTIABLE MANIFOLD AND T AN INVOLUTION ON M^n WHICH MAY HAVE FIXED POINTS. AS BEFORE WE FORM EQUIVALENCE CLASSES $[T, M^n]_2$ AND CONSTRUCT FROM THEM AN ABELIAN GROUP WHICH WE DENOTE BY I_n EVERY ELEMENT OF WHICH HAS ORDER 2.

LET $I_* = \sum_0^\infty I_n$. ONE CAN GAIN INFORMATION ABOUT I_* AS FOLLOWS:

LET $B(O(n-r))$ BE THE CLASSIFYING SPACE OF $n-r$ PLANE BUNDLES OVER CLOSED DIFFERENTIABLE MANIFOLDS.

SO ξ / M^n IS DETERMINED BY A HOMOTOPY CLASS $[f]$ $f : M^n \to B(O(n-r))$

WE LET $\mathfrak{M}_n \equiv \sum_0^n \eta_r(B(O(n-r)))$

WE HAVE A NATURAL HOMOMORPHISM

$$\mathfrak{M}_n \to \eta_{n-1}(Z_2)$$

FOR, GIVEN $[M^r, f]_2$. $f : M^r \to B(O(n-r))$ YIELDS AN $n-r$ PLANE BUNDLE OVER M^r. PUTTING A RIEMANNIAN METRIC ON THE BUNDLE AND TAKING THE SPHERE BUNDLE B_{n-1} WE GET AN $n-1$ DIMENSIONAL MANIFOLD AND A FIXED PT. FREE INVOLUTION

$$T : (x, y) \to (x, \text{ANTIPODE OF } y)$$

WE MAP $[M^r, f]_2 \to [T, B_{n-1}]_2$.

ALSO THERE EXISTS A NATURAL HOMOMORPHISM $I_n \to \mathfrak{M}_n$.

CONSIDER $[T, M^n]_2$ FOR $0 \leq r \leq n$

LET F^r BE THE FIXED PT. SET AND AS BEFORE CONSTRUCT AN $n-r$ NORMAL SPHERE BUNDLE B_r' OVER F^r WHICH IS THEREFORE INDUCED BY A MAP $f_r : F^r \to B(O(n-r))$.

SO MAP : $[T, M^n]_2 \to \sum_{r=0}^{n} [F^r, f_r]_2$

THESE HOMOMORPHISMS GIVE RISE TO A SPLIT EXACT SEQUENCE :

$$0 \to I_n \to \mathfrak{M}_n \to \eta_{n-1}(\mathbb{Z}_2) \to 0$$

SINCE BOTH $\mathfrak{M}_n$ AND $\eta_{n-1}(\mathbb{Z}_2)$ ARE KNOWN ONE KNOWS IN A SENSE I_n.

<u>THEOREM</u>

LET G BE A CONNECTED COMPACT LIE GROUP. $H \subseteq G$ A CLOSED SUBGROUP. THEN IF $[G/H]_2 \neq 0$ H HAS MAXIMAL 2-RANK. ∎

RECALL THAT THE RANK OF A LIE GROUP IS THE DIMENSION OF THE MAXIMAL TORUS IT CONTAINS. BOREL DEFINED THE NOTION OF p-RANK AS FOLLOWS. CONSIDER AN ELEMENTARY p-GROUP $(\mathbb{Z}_p)^k$ THE <u>p-RANK</u> OF G IS THE LARGEST INTEGER k SUCH THAT $(\mathbb{Z}_p)^k \subseteq G$

FOR A SUBGROUP H OF G WE SAY H HAS <u>MAXIMAL p-RANK</u> $\Leftrightarrow$ p RANK OF H $=$ p RANK OF G.

AS AN ILLUSTRATION OF THE ABOVE THEOREM CONSIDER T A MAXIMAL TORUS IN $SO(3)$

THEN $SO(3)/T = S^2$ WHICH COBORDS.

TAKING THE NORMALIZER OF T, $N(T)$

$SO(3)/N(T) = P^2$ WHICH DOES NOT COBORD

HENCE $N(T)$ HAS MAXIMAL 2 - RANK.

FOR $p > 2$ WE REQUIRE G TO BE ORIENTABLE AND WE CONSIDER ORIENTABLE BORDISM CLASSES.

LET $k \geq 1$ AND LET $(Z_2)^k$ ACT AS A GROUP OF TRANSFORMATIONS ON M^n

A <u>STATIONARY POINT</u> IS A POINT LEFT FIXED BY EVERY ELEMENT OF $(Z_2)^k$

WE HAVE :

<u>LEMMA</u> IF $(Z_2)^k$ HAS NO STATIONARY POINTS THEN $[M^n]_2 = 0$

<u>NOTE</u> IF $k=1$ THIS IS JUST OUR PREVIOUS THEOREM.

CONSIDER THE RING η_* OF UNORIENTED BORDISM CLASSES. $(\eta_* = \eta_*(P)$ P A POINT$)$

WE DEFINE A SUBRING $R((\mathbb{Z}_2)^k)$.

$[\ ]_2 \in R((\mathbb{Z}_2)^k)$ IF THERE EXISTS A REPRESENTATIVE M^n SUCH THAT $(\mathbb{Z}_2)^k$ ACTS WITH ONLY A FINITE NUMBER OF STATIONARY POINTS.

FOR $k=1$ $R((\mathbb{Z}_2)^k)$ IS $\{0,1\} = \mathbb{Z}_2$

FOR $k=2$ $R(\mathbb{Z}_2+\mathbb{Z}_2)$ IS GENERATED BY $[P^2]$

WE DESCRIBE AN ACTION OF $\mathbb{Z}_2+\mathbb{Z}_2$ ON P^2

$$(0,1):\quad [x_1, x_2, x_3] \to [-x_1, x_2, x_3]$$
$$(1,0):\quad [x_1, x_2, x_3] \to [x_1, -x_2, x_3]$$
$$(1,1):\quad [x_1, x_2, x_3] \to [-x_1, -x_2, x_3]$$

THE STATIONARY POINTS ARE $[1,0,0]$ $[0,1,0]$ AND $[0,0,1]$

FOR $k>2$ THE STRUCTURE OF $R((\mathbb{Z}_2)^k)$ IS NOT KNOWN

CONSIDER (T, M^n) SUPPOSE THE FIXED POINT SET IS DIFFEOMORPHIC TO $P(2r)$ WE CAN SHOW THAT THIS IMPLIES $n = 4r$

CONJECTURE: M^n IS COBORDANT TO $P(2r) \times P(2r)$.

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RELATIONS AMONG STIEFEL WHITNEY CLASSES

BY F. P. PETERSON

WE WILL WORK WITH $\mathbb{Z}_2$ COEFFICIENTS.

DEFN. LET A BE A HOPF ALGEBRA OVER $\mathbb{Z}_2$ AND LET H BE AN A-MODULE WHICH IS AN ALGEBRA OVER $\mathbb{Z}_2$. THEN IF $m: H \otimes H \to H$ IS AN A MAPPING WE CALL H AN ALGEBRA OVER THE HOPF ALGEBRA A.

FOR SUCH AN H AND $\psi: A \to A \otimes A$

$$\psi(a) = \sum_{i=1}^{k} a_i' \otimes a_i'' \quad \text{THEN IF} \quad m(h \otimes h') \equiv h \cdot h'$$

WE HAVE $\quad a \cdot (h \cdot h') \equiv \psi(a)(h \otimes h') = \sum (a_i' h) \cdot (a_i'' h')$

FROM NOW ON LET A DENOTE THE STEENROD ALGEBRA GENERATED BY Sq^i. FOR Sq^i WE HAVE:

$$(0) \quad Sq^0 = \text{IDENTITY}$$

$$(1) \quad Sq^i(h_i) = h_i^2$$

$$(2) \quad Sq^i(h_j) = 0 \quad \text{IF } i > j$$

$$(3) \quad Sq^r(h \cdot h') = \sum_{i=0}^{r} (Sq^i h)(Sq^{r-i} h')$$

DEFN H IS A P-ALGEBRA OVER A IF H IS AN ALGEBRA OVER THE HOPF ALGEBRA A AND H SATISFIES POINCARÉ DUALITY.

THEN $h_i \in H^i$ IS DETERMINED BY THE VALUES $h_i \cdot h'_{n-i}$ FOR ALL $h'_{n-i} \in H^{n-i}$.

EXAMPLE $H = H^*(M^n)$ M^n CONNECTED, CLOSED DIFFERENTIABLE MANIFOLD.

LET US ASSUME H IS A P-ALGEBRA OVER A. WE DEFINE A RIGHT OPERATION OF A ON H BY:

$$(h_i)a_j \cdot h'_{n-i-j} \equiv h_i \cdot a_j(h'_{n-i-j})$$

LET $(1) Sq^i \equiv V_i \in H^i$

(1 IS IDENTITY OF H)

THEN $(1) Sq^i \cdot h'_{n-i} = 1 \cdot Sq^i(h'_{n-i}) = Sq^i(h'_{n-i})$

DEFINE $W_i = \sum_{j=0}^{i} Sq^j(V_{i-j})$

PROPOSITION IF H IS A P-ALGEBRA OVER A

THEN $(*)$: $(h)\chi(Sq^i) = \sum_{j=0}^{i} u_j \cdot Sq^{i-j}(h)$

WHERE χ IS THE CANONICAL ANTI-AUTOMORPHISM OF A AND $u_j = (1)\chi(Sq^j)$

WE HAVE AN INDUCTIVE FORMULA FOR $\chi(Sq^i)$ DEFINED BY

$$\sum_{i=0}^{r} \chi(Sq^i)\, Sq^{r-i} = \begin{cases} 0 & r > 0 \\ 1 & r = 0 \end{cases}$$

$\chi^2 = $ IDENTITY

PROOF INDUCTION ON i : $r+s+t=n$

$$(h_r)\chi(Sq^i)\cdot h_s = h_r\cdot\chi(Sq^i)h_s$$

$$\underset{\substack{\uparrow \\ \text{CARTAN} \\ \text{FORMULA}}}{=} \chi(Sq^i)(h_r\cdot h_s) + \sum_{j=0}^{i-1}\chi(Sq^{i-j})(h_r)\cdot\chi(Sq^j)(h_s)$$

$$= u_i\cdot h_r\cdot h_s + \sum_{j=0}^{i-1}\chi(Sq^{i-j})(h_r)\chi(Sq^j)\cdot h_s$$

CANCELLING h_s AND USING INDUCTION HYPOTHESIS WE HAVE

$$(h_r)\chi(Sq^i) = u_i\cdot h_r + \sum_{j=0}^{i-1}\sum_{k=0}^{j} u_k\cdot Sq^{j-k}\left(\chi(Sq^{i-j})\right)(h_r)$$

$$= u_i\cdot h_r + \sum_{k=0}^{i-1} u_k\cdot Sq^{i-k}(h_r) = \sum_{k=0}^{i} u_k\cdot Sq^{i-k}(h_r)$$

DEFN H IS A LEFT-RIGHT ALGEBRA OVER A

IF 1) H IS A LEFT ALGEBRA OVER THE HOPF ALGEBRA A

2) H IS A RT. ALGEBRA OVER A

3) $(*)$ HOLDS

DEFN LET $V_i \equiv (1)Sq^i$

$$W_i = \sum_{r=0}^{i} Sq^r(V_{i-r}) \qquad \text{AND}$$

$\overline{W_i}$ IS DEFINED BY

$$\sum_{j=0}^{i}\overline{W_j}\cdot W_{i-j} = 0 \qquad (i>0)$$

PROPOSITION IF H IS A LEFT-RIGHT ALGEBRA OVER A THEN $\overline{W_i} = u_i$ WHERE

$$u_i \equiv (1)\,\chi(Sq^i)$$

PROOF
$$\sum_{j=0}^{i} u_j \cdot \sum_{r=0}^{i-j} Sq^r (V_{i-j-r})$$

$$= \sum_{j=0}^{i} \sum_{r=0}^{i-j} u_j \cdot Sq^r (V_{i-j-r})$$

$$= \sum_{l=0}^{i} (V_{i-l})\,\chi(Sq^l) = \sum_{l=0}^{i} (1)\, Sq^{i-l}\chi(Sq^l) = 0$$

THEOREM LET H BE A LEFT ALGEBRA OVER THE HOPF-ALGEBRA A. GIVEN u_i $i \geq 0$ $u_0 = 1$ THEN $(*)$ MAKES H INTO A LEFT-RIGHT ALGEBRA OVER A $\Leftrightarrow$ u_i SATISFY THE W_u FORMULA

$$Sq^r u_i = \sum_{t=0}^{r} \binom{i-r+t-1}{t} u_{r-t} \cdot u_{i+t}$$

PROOF $\Leftarrow$: WE HAVE $H^*(BO) = Z_2(W_1, \ldots)$

CASE 1 $\quad H = H^*(BO \times K(Z_2, k))$

$$u_i = \overline{W_i} \otimes 1$$

LET R BE A RELATION

DIMENSION $(h) R = N$

$$K(Z_2, k)^{N+1} \subseteq R^{2N+3}$$

LET M = DOUBLED NBD. OF $K(\mathbb{Z}_2, h)^{N+1} \equiv K$

$K \subset M$ IS A HOMOTOPY EQUIVALENCE UP TO DIM $N+1$. LET M' = SOME PRODUCT OF REAL PROJECTIVE SPACES S.T. FOR $\tau_{M'} : M' \to BO$ $\tau_{M'}^*$ IS A MONOMORPHISM UP TO DIM $N+1$. THERE EXISTS $f : M \to K(\mathbb{Z}_2, h)$ REPRESENTING THE FIRST NON-VANISHING COHOMOLOGY CLASS.

(M, $h-1$ CONNECTED)

THEN FOR $\tau_{M'} \times f : M' \times M \to BO \times K(\mathbb{Z}_2, h)$

$$g = (\tau_{M'} \times f)^* : H^*(BO \times K(\mathbb{Z}_2, h)) \to H^*(M' \times M)$$

IS A MONOMORPHISM UP TO DIM. $N+1$.

AND $u_i = \overline{W}_i \otimes 1 \Rightarrow g(u_i) = \overline{W}_i(M' \times M)$

$g((h)R) = (g(h))R = 0$ BUT g 1-1

$\Rightarrow (h)R = 0$.

<u>CASE 2</u> H ARBITRARY DIM $h = k$

$$\begin{array}{c} H^*(BO \times K(\mathbb{Z}_2, k)) = H^*(BO) \otimes H^*(\mathbb{Z}_2, k) \\ \big\downarrow e \\ H \end{array}$$

$e(\overline{W}_i \otimes 1) = u_i \qquad e(1 \otimes \iota) = h$

CLAIM e IS A RING HOMOMORPHISM PRESERVING THE LEFT ACTION OF $\mathcal{A}$ (BY WU FORMULAS).

$$(h)R = e((1 \otimes i)R) = e(0) = 0.$$

WE OMIT $\Rightarrow$, WHICH USES THE ADEM RELNS.

<u>COROLLARY</u> GIVEN A LEFT-RT. ALGEBRA H OVER $\mathcal{A}$ THERE EXISTS A UNIQUE $\gamma^* : H^*(BO) \to H$ WHICH IS A LEFT-RIGHT HOMOMORPHISM.

WE NOW CONSIDER RELATIONS AMONG STIEFEL WHITNEY CLASSES.

WE HAVE $\Upsilon^* : H^*(BO) \to H^*(M)$

$$\bigcap_{M^n} \text{KERNEL } \Upsilon^k_{M^n} \equiv I_n(\phi, \text{GEOMETRY})$$

FOR $S \subset H^*(BO)$ THEN WE DEFINE:

$$I_n(S, \text{GEOMETRY}) \equiv \bigcap_{M^n} \text{KER } \Upsilon^*_{M^n}$$

FOR ALL M^n (DIFFERENTIABLE MANIFOLD) SUCH THAT $\gamma^*_{M^n}(S) = 0$.

$$I_n(S, \text{ALGEBRA}) \equiv \bigcap_{H-P \text{ ALGEBRA}} \text{KER } \Upsilon^*_H$$

FOR ALL n-DIML. P-ALGEBRAS SUCH THAT $\Upsilon^*_H(S) = 0$.

I_n ARE IDEALS IN $H^*(BO)$

$I_n(S,\text{ ALGEBRA}) \subset I_n(S,\text{ GEOMETRY})$

E. BROWN HAS SHOWN

$I_n(\phi,\text{ GEOMETRY})^k = 0 \qquad k \leq n/2.$

(1) $Sq^i \equiv \cdot\, v_i \in I_n(\phi,\text{ ALGEBRA}) \qquad i > n/2$

$T_H^*(v_i)\cdot h_{n-i} = T^*(1)\cdot Sq^i(h_{n-i}) = 0$

ALL h_{n-i}

LET $F_n = Z_2$ MODULE GENERATED

BY $\{x \mid x \in (H^j(BO))Sq^i\}$ FOR $i > n-i-j$

THEN $T_H^*(x)\cdot h_{n-i-j} = T^*(h_j)\cdot Sq^i(h_{n-i-j})$

$= 0$ ALL h_{n-i-j}

THUS $T_H^*(x) = 0$ PROVING

~~PROPOSITION~~ $F_n \subset I_n(\phi,\text{ ALGEBRA})$.

THEOREM $F_n = I_n(\phi,\text{ GEOMETRY})$

COROLLARY $I_n(\phi,\text{ ALGEBRA}) = I_n(\phi,\text{ GEOMETRY})$

COROLLARY $I_n(\phi,\text{ GEOMETRY})^{[n/2]+1} = V_{[n/2]+1}$

FOR ARBITRARY S WE DON'T HAVE

$I_n(S,\text{ GEOMETRY}) = I_n(S,\text{ ALGEBRA})$

e.g. $n=32 \qquad S = \{w_1, w_2, w_4, w_8, w_{24}\}$

ON STIEFEL-WHITNEY CLASSES

BY F. P. PETERSON

WE RECALL SOME DEFINITIONS AND THEOREMS FROM OUR PREVIOUS TALK. WE SHALL WORK WITH $\mathbb{Z}_2$ COEFFICIENTS.

DEFN. LET A BE A HOPF ALGEBRA OVER $\mathbb{Z}_2$ AND LET H BE AN ALGEBRA OVER $\mathbb{Z}_2$, + A-MODULE THEN IF $m: H \otimes H \to H$ IS AN A MAPPING WE CALL H AN <u>ALGEBRA OVER THE HOPF ALGEBRA A</u>

FOR SUCH AN H AND $\psi: A \to A \otimes A$

$$\psi(a) = \sum_{i=1}^{k} a_i' \otimes a_i'' \quad \text{THEN IF } m(h \otimes h') \equiv h \cdot h'$$

WE HAVE $a(h \cdot h') \equiv \psi(a)(h \otimes h') = \sum (a_i' h) \cdot (a_i'' h')$

FROM NOW ON LET A DENOTE THE STEENROD ALGEBRA GENERATED BY Sq^i, FOR Sq^i

WE HAVE:

(0) Sq^0 = IDENTITY

(1) $Sq^i(h_i) = h_i^2$

(2) $Sq^i(h_j) = 0$ IF $i > j$

(3) $Sq^r(h \cdot h') = \sum_{i=0}^{r} (Sq^i h) \cdot (Sq^{r-i} h')$

DEFN H IS A <u>POINCARÉ ALGEBRA OVER A</u> IF H IS AN ALGEBRA OVER THE HOPF ALGEBRA A AND H SATISFIES POINCARÉ DUALITY.

THEN $h_i \in H^i$ IS DETERMINED BY THE VALUES $h_i \cdot h'_{n-i}$ FOR ALL $h'_{n-i} \in H^{n-i}$.

<u>EXAMPLE</u> $H = H^*(M^n)$ M^n CONNECTED, CLOSED DIFFERENTIAL MANIFOLD.

ASSUME H IS A POINCARÉ ALGEBRA OVER A. WE DEFINE A RIGHT OPERATION OF A ON H BY:

$$(h_i)a_j \cdot h'_{n-i-j} \equiv h_i \cdot a_j(h'_{n-i-j})$$

<u>PROPOSITION</u> IF H IS A POINCARÉ ALGEBRA OVER A THEN (*)

$$(h)\chi(Sq^i) = \sum_{j=0}^{i} u_j \cdot Sq^{i-j}(h)$$

WHERE χ IS THE CANONICAL ANTI-AUTOMORPHISM OF A AND $u_j \equiv (1)\chi(Sq^j)$

WE NOW GENERALIZE OUR NOTION OF POINCARÉ ALGEBRA TO:

<u>DEFN</u> H IS A <u>LEFT-RIGHT ALGEBRA OVER A</u> IF

1) H IS A LEFT ALGEBRA OVER THE HOPF ALGEBRA A.
2) H IS A RIGHT ALGEBRA OVER A
3) (*) HOLDS.

<u>DEFN</u> LET $V_i \equiv (1)Sq^i$

$$W_i \equiv \sum_{r=0}^{i} Sq^r(V_{i-r}) \quad \text{AND}$$

$\overline{W_i}$ DEFINED BY $\sum_{j=0}^{i} \overline{W_j}\, W_{i-j} = 0 \quad (i>0)$.

PROPOSITION IF H IS A LEFT-RIGHT ALGEBRA OVER $\mathcal{A}$ THEN $\overline{W_i} = u_i$ WHERE $u_i \equiv (1)\gamma^*(Sq^i)$. ∎

THEOREM LET H BE A LEFT ALGEBRA OVER THE HOPF ALGEBRA $\mathcal{A}$. GIVEN u_i, $i \geq 0$, $u_0 = 1$ THEN $(*)$ MAKES H INTO A LEFT-RIGHT ALGEBRA OVER $\mathcal{A}$ $\Leftrightarrow$ THE u_i SATISFY THE WU FORMULA

$$Sq^r u_i = \sum_{t=0}^{r} \binom{i-r+t-1}{t} u_{r-t} \cdot u_{i+t}. \quad ∎$$

COROLLARY GIVEN A LEFT-RIGHT ALGEBRA H OVER $\mathcal{A}$ THERE EXISTS A UNIQUE $\gamma_H^*: H^*(BO) \to H$. WHICH IS A LEFT-RIGHT HOMOMORPHISM. ∎

CONSIDER: M A C^∞ MANIFOLD. AND $\gamma_{M^n}^*: H^*(BO) \to H^*(M)$

FOR $S \subseteq H^*(BO)$ (S AN ARBITRARY SET)

THEN $I_n(S, \text{GEOMETRY}) \equiv \cap \{\ker \gamma_{M^n}^* \mid \gamma_{M^n}^*(S) = 0\}$

$I_n(S, \text{ALGEBRA}) \equiv \cap \{\ker \gamma_H^* \mid H$ n-DIML. POINCARÉ ALG. S.T. $\gamma_H^*(S) = 0\}$

OBVIOUSLY $I_n(S, \text{ALGEBRA}) \subseteq I_n(S, \text{GEOMETRY})$.

LET $F_n \equiv \mathbb{Z}_2$ MODULE GENERATED BY $\{x \mid x \in H^j(BO)Sq^i$ FOR $i > n-i-j\}$

FOR SUCH AN X $\quad \gamma_H^*(x) \cdot h_{n-i-j} = \gamma_H^*(h_j) \cdot S_q^i(h_{n-i-j}) = 0$

PROVING

<u>PROPOSITION</u> $\quad F_n \subseteq I_n$ (ϕ, ALGEBRA) $|$

AND SO $\quad F_n \subseteq I_n$ (S, ALGEBRA) ANY S. $|$

<u>LEMMA</u> $\quad F_n = \mathbb{Z}_2$ MODULE GENERATED BY

$\{ x \in H^j(BO)\, a \,/\,$ s.t. $a(\iota) = 0 \quad \iota$ CANONICAL GENERATOR OF $H^{n-k}(\mathbb{Z}_2, n-k)$

$k = \dim a + j$

<u>PROOF</u> OMITTED. $|$

TODAY WE SHALL PROVE THE FOLLOWING:

<u>THEOREM</u> $\quad F_n = I_n$ (ϕ, GEOMETRY)

<u>IMMEDIATE COROLLARY</u> $\quad I_n$ (ϕ, ALGEBRA) $= I_n$ (ϕ, GEOMETRY)

<u>PROOF (OF THEOREM)</u>

FOR V A VECTOR SPACE OVER $\mathbb{Z}_2$

$V^* \equiv \mathrm{HOM}(V, \mathbb{Z}_2)$ FOR X A TOPOLOGICAL SPACE $\eta_*(X)$ IS THE GROUP OF UNORIENTED BORDISM CLASSES

$[M^n, f] \qquad f : M^n \to X$ A CONTINUOUS MAP.

CONSIDER: $\quad H^*(BO) \xrightarrow{\otimes \iota} H^*(BO \times K(\mathbb{Z}_2, n-k)) \xrightarrow{\theta_2}$

$\xrightarrow{\theta_2} \eta(K(\mathbb{Z}_2, n-k))^*$.

Θ_2 IS DIMENSION PRESERVING DEFINED AS FOLLOWS: CONSIDER $[M^n, f]$

$$f : M^n \to K(\mathbb{Z}_2, n-k)$$

$$\gamma_{M^n} \times f : M^n \to BO \times K(\mathbb{Z}_2, n-k)$$

$$(\gamma_{M^n} \times f)^* : H^*(BO \times K(\mathbb{Z}_2, n-k)) \to H^n(M^n) \simeq \mathbb{Z}_2$$

SO $\Theta_2(a)[M^n, f] \equiv (\gamma_M \times f)^*(a)$.

<u>LEMMA</u> $\ker(\Theta_2(\otimes i))^k = I_n(\phi, \text{GEOMETRY})^k$

<u>PROOF</u> $\Theta_2(R \otimes i)[M^n, f] \equiv (\gamma_M \times f)^*(R \otimes i)$

$= \gamma_M^*(R) \cdot f^*(i)$ GIVES THE EQUALITY. ∎

LET $\eta_* = \eta_*(\text{pt.})$

WE CONSIDER THE FOLLOWING DIAGRAM:

$$\begin{tikzcd}
\eta^* \otimes A \arrow[r, "\otimes i"] \arrow[d, "\psi_1 \cong"'] & \eta^* \otimes A \otimes H^*(\mathbb{Z}_2, n-k) \arrow[r, "\Theta_1"] \arrow[d, "\psi_2 \cong"] & \eta^* \otimes H^*(\mathbb{Z}_2, n-k) \arrow[d, "\psi_3 \cong"] \\
H^*(BO) \arrow[r, "\otimes i"] & H^*(BO \times K(\mathbb{Z}_2, n-k)) \arrow[r, "\Theta_2"] & \eta(K(\mathbb{Z}_2, n-k))^*
\end{tikzcd}$$

OUR OBJECT BEING BY THE PREVIOUS LEMMA TO EXAMINE $\ker(\Theta_2(\otimes i))$ WHICH IS EQUIVALENT TO EXAMINING $\ker(\Theta_1(\otimes i))$.

FIRST LET US DEFINE THE VARIOUS MAPS INVOLVED.

$$\Theta_1(\eta \otimes a \otimes u) = \eta \otimes \chi(a)(u),$$

BY THE THOM ISOMORPHISM $\eta_* \cong \pi_*(MO)$ (MO THE THOM SPACE OF THE STANDARD CLASSIFYING BUNDLE OVER BO.) LET

$h: \pi_*(MO) \longrightarrow H_*(BO)$ WHERE h IS THE HUREWICZ MAP FOLLOWED BY REDUCING MOD 2.

WE CAN CHOOSE $J: H_*(MO) \to \pi_*(MO)$ SUCH THAT $Jh = 1$, HENCE h^* IS ONTO AND J^* IS 1-1.

$$\eta^* \underset{J^*}{\overset{h^*}{\rightleftarrows}} H^*(MO) \xleftarrow{\varphi} H^*(BO) \xrightarrow{D} H^*(BO)$$

WHERE φ IS THE THOM-GYSIN ISOMORPHISM AND D IS AN ISOMORPHISM GIVEN BY $D(w_i) = \overline{w}_i$. LET ψ_0 DENOTE THE COMPOSITE MAP $\psi_0 = D\varphi^{-1}J^*$

SO ψ_0 IS 1-1.

WE NOW DEFINE ψ_1 AS THE FOLLOWING COMPOSITE MAP

$$\eta^* \otimes \overline{W} \xrightarrow{J^*} H^*(MO) \xrightarrow{\varphi^{-1}} H^*(BO) \xrightarrow{D} H^*(BO)$$

WHERE $\bar{\jmath}^*(\eta \otimes a) = a(\jmath^*(\eta))$.

$\bar{\jmath}^*$ IS AN ISOMORPHISM, HENCE SO IS $\psi_1 = D\varphi^{-1}\bar{\jmath}^*$

<u>LEMMA</u> $\psi_1(\eta \otimes a) = \psi_0(\eta) \cdot \chi(a)$

<u>PROOF</u> WE CONSIDER THE SPECIAL CASE $a = S_q^i$

THEN $\psi_1(\eta \otimes a) = D\varphi^{-1}\bar{\jmath}^*(\eta \otimes a) = D\varphi^{-1}(a(\jmath^*(\eta)))$

NOW $\jmath^*(\eta) = \varphi(x) = x \cup \bar{U}$ ($\cup$ = CUP PRODUCT)

SO $\psi_1(\eta \otimes S_q^i) = D\varphi^{-1}(S_q^i(x \cup \bar{U}))$

$= D\varphi^{-1}\left(\sum_{j=0}^{i}(S_q^{i-j}x) \cup (S_q^j \bar{U})\right) = D\sum_{j=0}^{i} S_q^{i-j}x \cdot u_j$

NOW $\psi_0(\eta)\chi(a) = (D\varphi^{-1}\jmath^*(\eta))\chi(S_q^i)$

AND $D\varphi^{-1}\jmath^*(\eta) = Dx$

SO $\psi_0(\eta)\chi(S_q^i) = (Dx)\,\chi(S_q^i)$

$\underset{\text{BY } (*)}{=} \sum_{j=0}^{i} u_j \cdot S_q^{i-j}(Dx) = \sum_{j=0}^{i} S_q^{i-j}(Dx) \cdot \bar{W}_j$

($u_j = \bar{W}_j$ BY PREV. PROP.) $= D\sum_{j=0}^{i} S_q^{i-j}x \cdot u_j$

WE DEFINE ψ_2 BY

$\psi_2(\eta \otimes a \otimes u) \equiv \psi_1(\eta \otimes a) \otimes u$. ψ_2 IS AN ISOMORPHISM.

WE DEFINE ψ_3 BY:

$$\psi_3(\eta \otimes u)[M^n, f] \equiv (\gamma_{M^n} \times f)^*(\psi_0(\eta) \otimes u)$$

<u>LEMMA</u> ψ_3 IS AN ISOMORPHISM FOR ARBITRARY X

I.E. $\psi_3 : \eta^* \otimes H^*(X) \to \eta(X)^*$ IS AN ISOMORPHISM.

<u>PROOF</u> OMITTED. BY STANDARD ARGUMENTS IT IS SUFFICIENT TO PROVE ψ_3 IS AN ISOMORPHISM FOR $X = PT$.

$$\eta^* \otimes Z_2 \to \eta^*$$

WE NOW EXAMINE THE KERNEL OF $\Theta_1 \otimes i$

$$\sum_j \eta_j \otimes \chi(a_j) \xrightarrow{\otimes i} \sum_j \eta_j \otimes \chi(a) \otimes i$$

$$\xrightarrow{\Theta_1} \sum_j \eta_j \otimes a_j(i) = 0 \qquad (\chi^2 = 1)$$

WHERE η_j RUN OVER GENERATORS, SO $a_j(i) = 0$.

THUS FOR $x = \sum \eta_j \otimes \chi(a_j)$

$$x \in \ker(\Theta_1 \otimes i)^k \Leftrightarrow a_j(i) = 0$$

$$\psi_1(x) = \psi_1\left(\sum_j \eta_j \otimes \chi(a_j)\right) = \sum \psi_0(\eta_j) \cdot a_j$$

THE $\psi_1(x) \in F_n$ BY A PREVIOUS LEMMA (Pg. 4)

SO $F_n^k \supseteq \ker(\Theta_2(\otimes i))^k$

SO BY LEMMA Pg. 5

$$F_n{}^{k} \supseteq I_n(\phi, \text{GEOMETRY})^{k}$$

SO $I_n(\phi, \text{GEOMETRY}) \supseteq I_n(\phi, \text{ALGEBRA})$

$$\supseteq F_n \supseteq I_n(\phi, \text{GEOMETRY})$$

(Pg. 4)

Q. E. D.

COROLLARY $I_n(\phi, \text{GEOMETRY})^{k} = 0$ IF $k \leq n/2$

COROLLARY $I_n(\phi, \text{GEOMETRY})^{n} = \{ (Sq^i + (1)\, Sq^i v) \cdot H^{n-i}(BO) \}_{i=1,\dots,n}$

COROLLARY $I_n(\phi, \text{GEOMETRY})^{[n/2]+1} = \{ v_{[n/2]+1} \mid v_i \equiv (1) Sq^i \}$

$$I_n(\phi, \text{GEOMETRY})^{[n/2]+2} = \{ v_{[n/2]+2},\ Sq^1(v_{[n/2]+1}),\ w_1 \cdot v_{[n/2]+1} \}$$

HOMOTOPY GROUPS OF SYMMETRIC SPACES

BY BRUNO HARRIS

REFERENCE ANNALS OF MATH SEPT. 1962.

LEMMA 1 CONSIDER THE FIBRATION

$$\begin{array}{ccc} SO_{2n-1} & \xrightarrow{i} & SO_{2n} \\ & & \downarrow p \\ & & S^{2n-1} \end{array}$$

THEN FROM THE HOMOTOPY SEQUENCE WE HAVE THE FOLLOWING EXACT SEQUENCE MODULO C_2 (THE CLASS OF ALL FINITE GROUPS WHOSE ORDERS ARE NOT DIVISIBLE BY 2) WHICH SPLITS.

$$0 \to \pi_j(SO_{2n-1}) \xrightarrow{i_*} \pi_j(SO_{2n}) \xrightarrow{p_*} \pi_j(S^{2n-1}) \to 0$$

OR EQUIVALENTLY THE SEQUENCE IS EXACT AND SPLITS WHEN TENSORED WITH Q_2 $\left(Q_2 = \left\{ \frac{m}{2^n} \,/\, m \in \mathbb{Z} \right\} \right)$

LEMMA 2 CONSIDER THE FIBRATION

$$\begin{array}{ccc} S^{2n-1} & \to & V_{2n+1,2} \\ & & \downarrow \\ & & S^{2n} \end{array}$$

$V_{2n+1,2}$ THE MANIFOLD OF UNIT TANGENT VECTORS ON S^{2n}.

WE GET THE FOLLOWING EXACT SEQUENCE WHICH SPLITS:

$$0 \to \pi_j(V_{2n+1,2}) \otimes Q_2 \to \pi_j(S^{2n}) \otimes Q_2 \overset{\partial}{\underset{E}{\rightleftarrows}} \pi_{j-1}(S^{2n-1}) \otimes Q_2 \to 0$$

THE SPLITTING IS GIVEN BY

$$E : \pi_{j-1}(S^{2n-1}) \to \pi_j(S^{2n})$$ THE

SUSPENSION HOMOMORPHISM.

WE SHALL GET THE ABOVE TWO RESULTS FROM MORE GENERAL THEOREMS.

LET G BE A COMPACT CONNECTED LIE GROUP AND σ AN INVOLUTION ON G.

LET K BE IDENTITY COMPONENT OF THE FIXED POINT SET OF σ AND FORM THE SYMMETRIC SPACE G/K

CONSIDER THE FIBRATION

$$K \xrightarrow{i} G \xrightarrow{p} G/K$$

WE HAVE:

<u>THEOREM 1</u> SUPPOSE

$$0 \to H_*(K, \mathbb{Q}) \xrightarrow{i_*} H_*(G, \mathbb{Q})$$ IS

EXACT WHERE $\mathbb{Q}$ = RATIONALS THEN

$$0 \to \pi_*(K) \otimes \mathbb{Q}_2 \to \pi_*(G) \otimes \mathbb{Q}_2 \to \pi_*(G/K) \otimes \mathbb{Q}_2 \to 0$$

IS EXACT AND SPLITS.

NOW σ ACTS ON $\pi_j(G)$ BY

$$\sigma[f] = [\sigma f]$$ CLAIM

σ IS -1 ON $\pi_*(G/K) \otimes \mathbb{Q}_2$

σ IS $+1$ ON $\pi_*(K) \otimes \mathbb{Q}_2$

SO $\pi_*(G) \otimes \mathbb{Q}_2 \approx i_* \pi_*(K) \otimes \mathbb{Q}_2 \oplus (\approx \pi_*(G/K) \otimes \mathbb{Q}_2)$

TELLS US HOW σ ACTS ON $\pi_*(G) \otimes \mathbb{Q}_2$.

<u>COROLLARY</u> LEMMA 1 WITH: $G = SO_{2n}$, $K = SO_{2n-1}$

$G/K = S^{2n-1}$.

<u>PROOF OF THEOREM 1</u>

FIRST WE PROVE A WEAKER THEOREM.

LET $\mathcal{P} = \left\{ p \mid p \text{ PRIME}, p = 2 \text{ OR } G \text{ HAS TORSION FOR } p \right\}$

LET $\mathbb{Q}_{\mathcal{P}} = \left\{ \frac{m}{p_1^{r_1} \cdots p_n^{r_n}} \mid p_i \in \mathcal{P}, m \in \mathbb{Z} \right\}$.

<u>THEOREM 1.5</u> IF $0 \to H_*(K; \mathbb{Q}_{\mathcal{P}}) \xrightarrow{i_*} H_*(G; \mathbb{Q}_{\mathcal{P}})$

IS EXACT THEN $0 \to \pi_*(K) \otimes \mathbb{Q}_{\mathcal{P}} \to \pi_*(G) \otimes \mathbb{Q}_{\mathcal{P}} \to$

$\to \pi_*(G/K) \otimes \mathbb{Q}_{\mathcal{P}} \to 0$ IS EXACT AND SPLITS.

<u>PROOF</u> IN COHOMOLOGY WE HAVE

$H^*(G; \mathbb{Q}_{\mathcal{P}}) = \Lambda(x_1, \ )$ THE EXTERIOR

ALGEBRA ON ODD DIMENSIONAL GENERATORS

WHICH ARE PRIMITIVE AND $H^*(G; \mathbb{Q}_{\mathcal{P}})$

$\cong \Lambda' \otimes \Lambda''$ BOTH SUBHOPF ALGEBRAS

GENERATED BY ODD DIMENSIONAL PRIMITIVE ELEMENTS. WITH $i^*: \Lambda' \xrightarrow{\approx} H^*(K; \mathbb{Q}_p)$

$$p^*: H^*(G/K; \mathbb{Q}_p) \xrightarrow{\approx} \Lambda''.$$

NOW $\sigma = 1$ ON Λ' FOR σ LEAVES K FIXED.

AND CLAIM $\sigma(x) = -x$ IF x ~~*~~ IS A PRIMITIVE ELEMENT OF Λ''.

TO SHOW THIS IT IS SUFFICIENT SINCE p COMMUTES WITH INVOLUTION $(p: G \to G/K)$ TO SHOW $\sigma y = -y$ FOR $y \in H^*(G/K; \mathbb{Q}_p)$

IN FACT WE CAN ASSUME y IS IN $H^*(G/K; R)$ R = REALS. AND BY de RHAM'S THEOREM THAT y IS A DIFFERENTIAL FORM ON G/K AND IS INVARIANT. AND FOR ANY DIFFERENTIAL FORM $\sigma(w) = (-1)^{\deg w} w$.

NOW CONSIDER $q: G/K \to G$

DEFINED BY $q(gK) = g\,\sigma(g^{-1})$

LET $f: K \times G/K \to G$ BE THE COMPOSITION OF THE FOLLOWING MAPS

$$K \times G/K \xrightarrow{i \times q} G \times G \xrightarrow{m} G$$

CLAIM f INDUCES AN ISOMORPHISM f^* IN COHOMOLOGY WITH COEFFS. $\mathbb{Q}_p$ AND AN APPLICATION OF A THEOREM OF J. H. C. WHITEHEAD WILL YIELD

$$0 \to \pi_*(K) \otimes \mathbb{Q}_p \to \pi_*(G) \otimes \mathbb{Q}_p \to \pi_*(G/K) \otimes \mathbb{Q}_p \to 0,$$

A SPLIT EXACT SEQUENCE.

WE PROVE f^* IS AN ISOMORPHISM IN COHOMOLOGY WITH COEFFS. IN $\mathbb{Q}_p$.

WE HAVE $G \underset{q}{\overset{p}{\rightleftarrows}} G/K$

$qp(g) = g\,\alpha(g^{-1})$ IF z IS A PRIMITIVE ELEMENT IN $H^*(G;\mathbb{Q}_p)$ THE $p^*q^*(z) = z - \alpha(z)$ SUPPOSE $z \in \Lambda''$ THEN $\alpha(z) = -z$

HENCE $p^*q^*(z) = 2z$ NOW $p^*q^* : \Lambda'' \to \Lambda''$ AND IS MULTIPLICATION BY 2. HENCE NO TWO TORSION IMPLIES p^*q^* IS AN ISOMORPHISM.

THUS $q^* : \Lambda'' \xrightarrow{\cong} H^*(G/K;\mathbb{Q}_p)$.

ONE CAN SHOW $q^* : \Lambda' \to 0$.

NOW $i : K \to G$ AND

$$i^* : \Lambda' \xrightarrow{\cong} H^*(K; Q_p)$$

$$i^* : \Lambda'' \to 0$$

THIS THEN PROVES φ^* IS AN ISOMORPHISM, (PROVING THM. 1.5)

NOW CONSIDER THE EXACT SEQUENCE IN THEOREM 1.

$$0 \to H_*(K, Q) \xrightarrow{i_*} H_*(G, Q)$$

THE ONLY POSSIBLE (G, K)'s SATISFYING THIS CONDITION ARE $(\mathrm{SPIN}\,2n, \mathrm{SPIN}\,2n-1)$ (SU_{2n+1}, SO_{2n+1}), (SU_{2n}, Sp_n) WHICH HAVE ONLY 2-TORSION AND A SPECIAL CASE WHICH HAS 2,3 TORSION THEN THE WEAKER THEOREM 1.5 GIVES US THEOREM 1.

WE NOW PROVE A THEOREM WHICH WILL YIELD LEMMA 2 AS A COROLLARY. CONSIDER (G, K) WITH σ AN AUTOMORPHISM BUT NOT NECESSARILY OF PERIOD 2.

SUPPOSE THERE EXISTS A CONNECTED LIE GROUP, L, $L \supseteq G$ WITH A 1-PARAMETER SUBGROUP $u(t) \subseteq L$ SUCH THAT

1) $u(1)^{-1} g\, u(1) = \sigma(g)$ FOR ALL $g \in G$

2) $u(t)^{-1} k\, u(t) = k$ FOR $k \in K$, ALL t

(K FIXED POINT SET OF σ)

CONSIDER:

$$\begin{tikzcd}
G \arrow[r, "p"] & G/K \arrow[r, hook, "i"] & L/K \arrow[d, "l"] \\
G/K \arrow[u, "q"] \arrow[ur, dashed, "p \circ q"'] & & L/G
\end{tikzcd}$$

THEN $p \circ q$ EXTENDS TO A MAP f OF THE UNREDUCED CONE OVER G/K

$f : C(G/K) \to L/K$ DEFINED AS:

$$\left(C(G/K) = \frac{G/K \times [0,1]}{G/K \times 1} \right)$$

$$f(gK, t) = g\, u(t)\, \sigma(g^{-1}) K$$

FOR $t = 1$

$$f(gK, 1) = g\, u(1)\, \sigma(g^{-1}) K$$

$$= g\, u(1)\, u(1)^{-1} g^{-1} u(1) K = u(1) K$$

WHICH IS INDEPENDENT OF g.

HENCE f IS A MAP ON THE CONE.

FOR $t = 0$

$$f(gk, 0) = g\,u(0)\,\sigma(g^{-1})\,K = g\,\sigma(g^{-1})\,K$$

$= pq(gK)$ HENCE f IS AN

EXTENSION OF $p \circ q$.

LET $S(G/K)$ DENOTE THE UNREDUCED SUSPENSION WE THEN HAVE THE COMMUTATIVE DIAGRAM:

$$\begin{tikzcd}
C(G/K) \arrow[r, "f"] \arrow[d] & L/K \arrow[d, "l"] \\
S(G/K) \arrow[r, "F"] & L/G
\end{tikzcd}$$

$$F(gk, t) = g\,u(t)\,G$$

FOR $t = 0$ $\quad F(gk, 0) = gG =$ IDENTITY OF L/G

FOR $t = 1$ $\quad F(gK, 1) = g\,u(1)\,G = u(1)\,\sigma(g)\,G$

$= u(1)\,G$ INDEPENDENT OF g.

COMMUTATIVE DIAGRAM SINCE:

$$\begin{tikzcd}
(gK, t) \arrow[r, "f"] \arrow[d] & g\,u(t)\,\sigma(g^{-1})\,k \arrow[d, "l"] \\
(gk, t) \arrow[r, "F"] & g\,u(t)\,G = g\,u(t)\,\sigma(g^{-1})\,G
\end{tikzcd}$$

(F A BOTT MAP)

THEOREM 2 LET (G, K, σ) SATISFY THE CONDITION OF THEOREM 1 AND $L \supseteq G$ BE SUCH THAT CONDITIONS 1) AND 2) HOLD.

THEN THE FIBRATION

$$G/K \hookrightarrow L/K \xrightarrow{f} L/G$$

YIELDS THE SPLIT EXACT SEQUENCE

$$0 \to \pi_j(L/K)\otimes \mathbb{Q}_2 \to \pi_j(L/G)\otimes\mathbb{Q}_2 \underset{E}{\overset{\partial}{\rightleftarrows}} \pi_{j-1}(G/K)\otimes\mathbb{Q}_2 \to 0$$

THE SPLITTING GIVEN BY E THE COMPOSITION OF: $\pi_{j-1}(G/K) \xrightarrow[\text{SUSPENSION}]{} \pi_j(S(G/K)) \xrightarrow{F_*} \pi_j(L/G)$

PROOF $\partial \circ E = p_* q_*$ ON $\pi_{j-1}(G/K)$

AND $p_* q_*$ WE SAW IN PROOF OF THEOREM 1 WAS AN ISOMORPHISM IN $\otimes\mathbb{Q}_2$

Q.E.D.

COROLLARY LEMMA 2.

PROOF LET $G = SO_{2n}$, $K = SO_{2n-1}$ LET $A = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix}$ σ ACTS ON G BY CONJUGATION WITH A. $G/K = S^{2n-1}$ $L = SO_{2n+1}$

$L/G = S^{2n}$, $L/K = V_{2n+1,2}$

NOW E IS SUSPENSION SINCE

$F: L/G = S^{2n} \to S(G/K) = S(S^{2n-1})$ IS IDENTITY MAP. |

THE METHOD OF INFINITE REPETITION IN "PURE" TOPOLOGY

BY B. MAZUR

LET M_1^n, M_2^n BE DIFFERENTIAL MANIFOLDS AND $f: M_1^n \to M_2^n$ A CONTINUOUS MAP. WE SAY f IS A STABLE EQUIVALENCE IF WE CAN FIND AN INTEGER k AND A DIFFEOMORPHISM F SUCH THAT THE FOLLOWING DIAGRAM IS HOMOTOPY COMMUTATIVE:

$$\begin{array}{ccc} M_1^n & \xrightarrow{f} & M_2^n \\ \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \pi} \\ M_1 \times R^k & \xrightarrow{F} & M_2^n \times R^k \end{array}$$

(π IS THE NATURAL PROJECTION MAP)

STABLE HOMEOMORPHISM THEOREM

f IS A STABLE EQUIVALENCE $\Longleftrightarrow$

(1) f IS A HOMOTOPY EQUIVALENCE

(2) f IS TANGENTIAL, i.e. IF Υ_i DENOTES THE STABLE CLASS OF THE TANGENT BUNDLE OVER M_i THE $f^* \Upsilon_2 = \Upsilon_1$

($f^* \Upsilon_2$ IS THE STABLE CLASS OF THE PULL BACK BUNDLE)

WE WOULD LIKE TO PROVE THIS THEOREM

FOR $\left\{\begin{array}{l}\text{DIFFERENTIAL}\\ \text{COMBINATORIAL}\\ \text{TOPOLOGICAL}\end{array}\right.$ MANIFOLDS

(IN THE LATTER TWO CASES THE TANGENT BUNDLE IS REPLACED BY THE TANGENT MICROBUNDLE)

OUTLINE OF PROOF FOR THE DIFFERENTIAL CASE

LET E BE ANY DIFFERENTIAL MANIFOLD AND $f: E \to E$ AN <u>OPEN DIFFERENTIAL IMBEDDING</u> (I.E. (1) f IS AN IMBEDDING (2) $f(\text{INT } E)$ IS OPEN IN E (3) $f(E) \subseteq \text{INT } E$)

WE HAVE $E \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} \dots$

LET THE INJECTIVE LIMIT BE DENOTED $E(f)$ AN <u>INFINITE REPETITION SPACE</u>

IT IS A DIFFERENTIAL MANIFOLD OF DIM. OF E.

<u>LEMMA</u>

IF $E = M^n \times \text{INTERIOR } D^k$ (D^k THE k DISK) AND f HOMOTOPIC TO 1 THEN $E(f) \approx E$ (DIFFEOMORPHIC)

<u>PROOF</u> ONE CAN SHOW f ISOTOPIC TO $\frac{1}{2}$

$\frac{1}{2}: M^n \times \text{INT } D^k \to M \times \text{INT } D^k$

VIA $(m, d) \to (m, d/2)$ AND

$E(f) \approx E(\tfrac{1}{2}) \approx M \times R^k \approx E.$ |

PROOF OF THM $\Leftarrow$: LET $E_1 = M_1 \times R^k$ $E_2 = M_2 \times R^k$ WE CAN CONSTRUCT OPEN DIFFERENTIAL IMBEDDINGS $G: E_1 \to E_2$ $F: E_2 \to E_1$ WITH G, F HOMOTOPY INVERSES OF EACH OTHER.

WE HAVE $E_1 \xrightarrow{G} E_2 \xrightarrow{F} E_1 \xrightarrow{G} E_2 \xrightarrow{F} \cdots$

WE GET AN INFINITE REPETITION SPACE WHICH CONSIDERING $E_1 \xrightarrow{FG} E_1 \xrightarrow{FG} E_1 \xrightarrow{FG} \cdots$ IS $E_1(FG)$ AND CONSIDERING

$$E_2 \xrightarrow{GF} E_2 \xrightarrow{GF} E_2 \xrightarrow{GF} E_2 \to \cdots$$

IS $E_2(GF)$.

HENCE $E_1(FG) \approx E_2(GF)$

APPLYING THE LEMMA WE HAVE $E_1 \approx E_1(FG)$ $E_2 \approx E_2(GF)$

HENCE $E_1 \approx E_2$. THE DIFFEOMORPHISM SATISFIES THE CONDITIONS OF THE THEOREM.

WE OMIT $\Rightarrow$: |

CONSIDER THE PAIR (Y, X) OF TOPOLOGICAL SPACES $X \subseteq Y$. WE WISH TO CONSTRUCT A "CANONICAL" NEIGHBORHOOD OF X IN Y.

THOUGH THE THEORY WE CONSTRUCT IS QUITE GENERAL THE MAIN APPLICATION WE HAVE IN MIND IS THE CASE WHERE Y IS A MANIFOLD.

SUCH A THEORY IS TO HAVE THE FOLLOWING PROPERTIES:

0) CONSIDER $A \subseteq B \subseteq C$ $\quad B$ OPEN IN C
A CANONICAL NBD. OF A IN B IS ALSO A CANONICAL NBD. OF B IN C.

1) IT IS TO BE A "FORMAL" THEORY.

2) IF $\mathcal{U}_1, \mathcal{U}_2$ ARE TWO CANONICAL NBDS. OF X IN Y THEN THERE IS AN ISOMORPHISM $\mathcal{U}_1 \xrightarrow{\varphi} \mathcal{U}_2$, WHICH IS 1 ON X.

3) a) IF (Y, X) IS A DIFFERENTIAL MANIFOLD PAIR THEN AN OPEN TUBULAR NBD. OF X IN Y IS CANONICAL.

b) IF Y IS A COMBINATORIAL MANIFOLD X A "NICE" SUBCOMPLEX THEN AN OPEN WHITEHEAD REGULAR NBD. OF X IN Y IS CANONICAL.

c) LET A BE PARACOMPACT AND $f : A \to X$. CONSIDER THE OPEN MAPPING CYLINDER $M_f = A \times [0, 1) \cup_f X$. THEN FOR $Y = M_f$, Y ITSELF IS A CANONICAL NBD. OF X.

4) DEFINE $\underline{\Pi}_*(X) \equiv$ PROJECTIVE LIMIT$_{Y \supseteq \mathcal{O}_{\text{open}} \supseteq X}$ $\pi_*(\mathcal{O})$

IF $\mathcal{U}$ IS A CANONICAL NBD. OF X IN Y THEN $\eta : \underline{\Pi}_*(X) \to \pi_*(\mathcal{U})$ IS AN ISOMORPHISM.

5) LOCAL CRITERIA FOR EXISTENCE OF CANONICAL NBDS.

<u>CONSEQUENCES OF THE THEORY:</u>

A) TOPOLOGICAL INVARIANCE OF THE WHITEHEAD REGULAR NBD. CONSIDER TWO TRIANGULATIONS (K_1, L_1) (K_2, L_2) OF THE PAIR (Y, X) THE TWO WHITEHEAD REGULAR NBDS. ASSOCIATED WITH X ARE CANONICAL BY 3 b) AND HENCE HOMEOMORPHIC BY 2).

B) THEOREM OF THE CONE.

IF Z IS PARACOMPACT AND

THE OPEN CONE CZ IS A TOPOLOGICAL n-MANIFOLD $\left(CZ = \underbrace{Z \times (0,1]}_{Z \times 1} \right)$

THEN $CZ \approx R^n$

PROOF LET $Y = CZ$ AND $X = \{v\}$ THE VERTEX OF THE CONE. BY 3c) (CONSIDERING THE MAP $f : Z \to \{v\} = X$ THEN $M_f = CZ$) Y IS A CANONICAL NBD. OF X. NOW BY HYPOTHESIS Y IS LOCALLY EUCLIDEAN SO THERE EXISTS A NBD. $\mathcal{O}$ ABOUT v, $\mathcal{O} \approx R^n$. $\mathcal{O}$ IS A CANONICAL NBD. BEING A MAPPING CYLINDER OVER A PT. BY UNIQUENESS THM. $\mathcal{O} \approx Y$ SO $Y \approx R^n$. ∎

c) FORMALITY OF THE THEORY ALLOWS US TO DEFINE LOCAL KNOT GROUPS FOR TAME IMBEDDINGS.

CONSTRUCTION OF THE CANONICAL NBDS.

LET X BE AN ARBITRARY TOPOLOGICAL SPACE AND CONSIDER THE TRIPLE

$$\mathcal{E} = (E, X, i) \qquad i : E \to E \qquad i|_X = 1$$

e.g. $E = X \times [0,1)$ $\quad i : (x,t) \to (x, t/2)$.

WE SAY $K \subseteq E$ IS <u>BOUNDED</u> IF K IS CLOSED AND $i(K)$ IS CLOSED

(IN EXAMPLE $E = X \times [0,1)$ BOUNDED SETS DON'T RUN OUT TO $X \times 1$)

CONSIDER $f : E \to E$ SUCH THAT $f|_X = 1$

f IS <u>BOUNDED</u> $\Leftrightarrow$ $f(K)$ IS BOUNDED FOR ALL BOUNDED K.

f IS AN <u>AUTOMORPHISM</u> IF $f = 1$ ON $E - K$ FOR SOME BOUNDED K.

A TRIPLE $\mathcal{E} = (E, X, i)$ IS A <u>DILATION SPACE</u> IF 1) THERE EXISTS AN AUTOMORPHISM h SUCH THAT $hi^2 = i$ ON $i(E)$.

2) THERE EXIST STRETCHING MAPS, I.e. FOR K BOUNDED THERE EXISTS A BOUNDED f WITH $f = 1$ ON $i^2(E)$ AND $f\,i(E) \supseteq K$.

3) THERE EXIST COMPRESSING MAPS, i.e. LET $X \subseteq \mathcal{U}$ OPEN THEN THERE EXISTS $f: E \to E$ $f(E) \subseteq \mathcal{U}$ AND $f = 1$ ON A NBD. OF X.

<u>EX.</u> E VECTOR BUNDLE OVER A PARACOMPACT SPACE X.

LET $E(i)$ DENOTE THE INFINITE REPETITION SPACE

<u>DEFN.</u> A <u>CANONICAL NBD.</u> OF X IN Y IS $\mathcal{U}$ OPEN $\supseteq X$ SUCH THAT $\mathcal{U} \approx E(i)$ THE ISOMORPHISM BEING THE IDENTITY ON X.

THIS SATISFIES 0) – 4). |

ALGEBRAIC BORDISM

BY F. P. PETERSON

TODAY WE WILL DEFINE AN ALGEBRAIC ANALOG OF COBORDISM THEORY AND SHOW THAT MANY OF THE THEOREM CARRY OVER. THIS IS MOTIVATED BY THE DESIRE TO UNDERSTAND THE ALGEBRAIC PART OF THE "RELATIONS THEOREM". PROVED IN THE LAST LECTURE. AGAIN THIS IS JOINT WORK WITH E. H. BROWN.

WE USE THE NOTATION OF OUR PREVIOUS TALK. WE WILL WORK WITH $\mathbb{Z}_2$ COEFFICIENTS.

LET H BE A POINCARÉ ALGEBRA OF FINITE TYPE, OF DIMENSION n.

<u>DEFN</u> $H = \partial(H', H'')$ MEANS WE HAVE:

$$\begin{tikzcd} H' \arrow[rr, "i"] & & H \arrow[dl, "\delta"] \\ & H'' \arrow[ul, "j"] & \end{tikzcd}$$

WHICH IS EXACT.

WHERE: H' IS A CONNECTED LEFT-RIGHT ALGEBRA OVER THE STEENROD ALGEBRA A, i IS A LEFT-RIGHT HOMOMORPHISM, H'' IS A LEFT MODULE OVER A AND OVER H', j IS A LEFT A-MAPPING.

FURTHERMORE THE PAIRING $H' \otimes H'' \to H''$ PRESERVES STEENROD OPERATIONS. ALSO $\delta: H^n \to (H'')^{n+1}$ IS AN ISOMORPHISM. LEFSCHETZ DUALITY ALSO HOLDS, THAT IS WE HAVE A DUAL PAIRING $(H')^i \otimes (H'')^{n+1-i} \to (H'')^{n+1}$.

GIVEN AN n-MANIFOLD M WHICH IS THE BOUNDARY OF AN $n+1$ MANIFOLD W THEN
$$H^*(M) = \partial(H^*(W), H^*(W,M))$$
WITH HOMOMORPHISMS:
$$\to H^q(W,M) \xrightarrow{j} H^q(W) \xrightarrow{i} H^q(M) \xrightarrow{\delta} H^{q+1}(W,M) \to$$
WHICH IS THE MOTIVATION OF OUR DEFINITION.

IN A PREVIOUS TALK WE PROVED THAT FOR A LEFT RIGHT ALGEBRA H' THERE EXISTS A UNIQUE LEFT-RIGHT HOMOMORPHISM $\gamma_{H'}^*: H^*(BO) \to H'$

<u>THEOREM</u> $H = \partial(H', H'') \Leftrightarrow \gamma_H^n = 0$

<u>PROOF</u> $\Rightarrow$. EASY

$\Leftarrow$ LET $\{Y\}$ DENOTE THE SUBALGEBRA GENERATED BY $Y \subseteq H$ AND CLOSED UNDER STEENROD OPERATIONS.

BY HYPOTHESIS $\{\mathrm{IM}\ \tau_H\}^n = 0$ LET H' BE A MAXIMAL SUBALGEBRA OF H SUCH THAT $H' \supseteq \{\mathrm{IM}\ \tau_H\}$ AND $(H')^n = 0$

I.E. IF $h \notin H'$ THEN $\{h, H'\}^n \neq 0$

LET i DENOTE INCLUSION OF H' IN H.

DEFINE $H'' = H/H'$ AND REINDEX BY ADDING $+1$ TO DIMENSION. THIS QUOTIENT DEFINES δ. $j = 0$

$$h' \cdot [h] \equiv [h' \cdot h]$$

. THIS MAKES H'' INTO A LEFT MODULE OVER H'.

LEFSHETZ DUALITY:

$$h'[h] = 0 \qquad \text{ALL } [h]$$

THEN $(H')^n = 0 \Rightarrow h' \cdot h = 0$ FOR ALL $h \in H^{n-i}$ THUS $h' = 0$ BY POINCARÉ DUALITY IN H.

$$h' \cdot [h] = 0 \qquad \text{ALL } h' \in (H')^{n-i}$$

FIXED $[h] \in (H'')^{i+1}$

THEN $h' \cdot h = 0$ ALL $h' \in (H')^{n-i}$ FIXED $h \in H^i$

TO PROVE $h \in H'$. SUPPOSE NOT, THEN $\{h, H'\}^n \neq 0$. TO SHOW

$$Sq^I_2(h) \cdot h' = 0 \quad \text{IN DIM. } n$$

WE USE INDUCTION

$$Sq^I(h)\cdot h' = Sq^I(h\cdot h') + \Sigma \underbrace{Sq^I(h)\cdot Sq^{I_2}(h')}_{0}$$

$$= v_I \cdot h \cdot h' = h(h'\cdot v_I) = 0$$

BECAUSE $v_I \in \operatorname{Im} \tilde{i}_H \subset H'$

WHERE $v_I \equiv (1)\, Sq^I$.

BY ASSUMPTION $\Theta(h)\cdot h' \neq 0$ WHERE THE ~~=~~ LOWEST DIMENSIONAL Θ WHICH IS NOT A STEENROD OPERATION IS

$h \cdot Sq^i(h)$ THUS $\dim.\ \Theta(h) > 2i$

SO $i < \frac{n}{2}$. LET $g \in H^{n-i}$, $g\cdot h \neq 0$

AND $g(H')^i = 0$. NOW $g \notin H'$ BECAUSE

$g\cdot h \neq 0$ THUS $\{g, H^*\}^n \neq 0$. SO

WE GET $n - i < \frac{n}{2}$

HENCE $(n-i+i) < n$ CONTRADICTION!

<u>DEFN.</u> H IS <u>COBORDANT TO 0</u>, WRITTEN

$H \sim 0$ IF $H = \partial(H', H'')$.

WE DEFINE $H_1 \# H_2$ BY

$(H_1 \# H_2)^i = H_1^i \oplus H_2^i$ IF $i \neq 0, n$

$(H_1 \# H_2)^j = \mathbb{Z}_2$ FOR $j = 0, n$.

THIS IS THE CONNECTED SUM OF H_1 AND H_2

WE SAY H_1 IS COBORDANT TO H_2

WRITTEN $H_1 \sim H_2$ IFF $H_1 \# H_2 \sim 0$

PROPOSITION $\gamma_{H_1 \# H_2} = \gamma_{H_1} \oplus \gamma_{H_2}$ ∎

BY OUR PREVIOUS THEOREM AND PROPOSITION THIS YIELDS

THEOREM $H_1 \sim H_2 \Leftrightarrow \gamma_{H_1}{}^n = \gamma_{H_2}{}^n$

DEFN. WE CALL THE SET ${}^a\eta_*$ OF EQUIVALENCE CLASSES OF POINCARÉ ALGEBRAS OF FINITE TYPE THE UNORIENTED ALGEBRAIC COBORDISM RING (TENSORING GIVES THE MULTIPLICATION $\oplus$* THE ADDITION)

DEFINE $\varphi : \eta_* \to {}^a\eta_*$ BY

$$\varphi : [M] \to [H^*(M)]$$

φ IS A MONOMORPHISM BY THOM'S THEOREM.

(* $\oplus$ DENOTES DIRECT SUM)

WE NOW DEFINE ALGEBRAIC BORDISM GROUP.

LET X BE A FIXED LEFT ALGEBRA OVER THE STEENROD ALGEBRA. X HAVING A UNIT AND OF FINITE TYPE.

LET $f : X \to H$ BE A LEFT HOMOMORPHISM. WE SAY $(H, f) \sim 0$ IF $H = \partial(H', H'')$ AND THERE EXISTS $\bar{f} : X \to H'$ SUCH THAT $i\bar{f} = f$, (H, f) IS SAID TO <u>BORD IN X</u>.

(H_1, f_1) IS <u>BORDANT</u> TO (H_2, f_2) WRITTEN $(H_1, f_1) \sim (H_2, f_2)$ IF $(H_1 \# H_2, f_1 \# f_2)$ BORDS IN X.

THE EQUIVALENCE CLASSES OF POINCARÉ ALGEBRAS OF DIMENSION n BECOME UNDER DIRECT SUM THE <u>ALGEBRAIC BORDISM GROUP</u> ${}^{a}\eta_n(X)$. FORMING THE DIRECT SUM AND USING TENSOR MULTIPLICATION WE GET THE <u>ALGEBRAIC BORDISM RING</u> ${}^{a}\eta_*(X)$.

FOR K A SPACE WE HAVE

$$\varphi : \eta_*(K) \longrightarrow {}^a\eta_*(H^*(K))$$

BY $\varphi : [M, g] \longrightarrow [H^*(M), g^*]$

$g : M \to K$ A CONT. MAP.

<u>THEOREM</u> $\varphi(K)$ IS AN ISOMORPHISM.

THIS INCLUDES THE PREVIOUS CASE WHEN K = POINT.

<u>PROOF</u> OMITTED.

WE DEFINE ${}^a\Theta : H^*(BO) \otimes_{Z_2} X \to {}^a\eta_*(X)^*$

BY ${}^a\Theta(u \otimes x)([M, f]) = \gamma_M(u) \cdot f(x) \in Z_2$

<u>THEOREM</u> ${}^a\eta_*(X)$ IS NATURALLY EQUIVALENT TO $\eta_* \otimes X^*$ WHERE X IS A LEFT ALGEBRA OVER THE STEENROD ALGEBRA A.

<u>PROOF</u> OMITTED

<u>THEOREM</u> ${}^a\Theta$ IS AN EPIMORPHISM WITH KERNEL $F(X)$ WHERE

$$F(X) = IM\left(H^*(BO) \otimes X \otimes \bar{A} \to H^*(BO) \otimes X\right)$$

WHERE $\bar{A}$ = POSITIVE DIML. ELEMENTS OF A

AND $H^*(BO) \otimes X$ IS A LEFT-RIGHT ALGEBRA OVER $\mathcal{A}$.

PROOF $F(X) \subseteq$ KERNEL IS EASY.

TO SHOW $F(X) \supseteq$ KERNEL.

LET $u \in (H^*(BO) \otimes X)^n$, $u \notin F(X)^n$

TO PROVE ${}^a\theta(u) \neq 0$, i.e. TO FIND H, f SUCH THAT

${}^a\theta(u)[H,f] \neq 0$. LET

$z : (H^*(BO) \otimes X)^n \to \mathbb{Z}_2$ SUCH THAT

$z(u) \neq 0$ AND $z(F(X)^n) = 0$.

DEFINE $J \subseteq H^*(BO) \otimes X$ BY

$$J^i = \{X \mid X \cdot (H^*(BO) \otimes X)^{n-i} \subseteq \ker z\}$$

J IS AN IDEAL CLOSED UNDER LEFT $\mathcal{A}$ OPERATIONS. FOR LET $x \in J^i$, $y \in (H^*(BO) \otimes X)^j$ THEN

$$(x \cdot y)(H^*(BO) \otimes X)^{n-i-j} \subseteq \ker z.$$

SO J IS AN IDEAL.

NOW LET $x \in J^i$ BY INDUCTION

ONE SHOWS $Sq^j(x) \cdot h \equiv Sq^j(x \cdot h)$ MOD KER Z USING THE CARTAN FORMULA.

ONE CAN SHOW THAT

$$Sq^j(x \cdot h) + v_j(x \cdot h) \in F^n(X) \subset \ker Z .$$

THUS $Sq^j(x) \cdot h \equiv x \cdot (h \cdot v_j)$ MOD KER Z,

BUT $x \cdot (h \cdot v_j) \in \ker Z$ $\quad (v_j \equiv \chi(Sq^j))$.

HENCE $\overline{J}$ IS CLOSED UNDER LEFT $\mathcal{A}$-OPERATIONS.

$$H \equiv \frac{H^*(BO) \otimes X}{J}$$ IS A LEFT ALGEBRA OVER $\mathcal{A}$. LET $\eta : H^*(BO) \otimes X \to H$ BE THE QUOTIENT MAP. TO SHOW H SATISFIES POINCARÉ DUALITY:

$[x] \cdot [v] = [0] \in H^n$ MEANS $x \cdot v \in \ker Z$ (SINCE $J^n = \ker Z$) FOR ALL v HENCE $x \in J^i$ HENCE $[x] = [0]$.

DEFINE $f : X \to H$ BY $f(x) = \eta(1 \otimes x)$. ONE CHECKS EASILY THAT $\theta(u)([H, f]) = \eta(u) \neq 0$.

COHOMOLOGY OF FIBRE SPACES

BY WILLIAM MASSEY

THIS TALK REPRESENTS JOINT WORK WITH F. PETERSON

I. LET $\mathfrak{A}$ DENOTE THE STEENROD ALGEBRA OVER $\mathbb{Z}_2$.

WE HAVE A DIAGONAL MAP

$$\psi : \mathfrak{A} \to \mathfrak{A} \otimes_{\mathbb{Z}_2} \mathfrak{A} .$$

GIVEN M, N MODULES OVER $\mathfrak{A}$ AND $\mathbb{Z}_2$ THEN THE MAP ψ MAKES $M \otimes_{\mathbb{Z}_2} N$ INTO AN $\mathfrak{A}$ MODULE BY $\alpha(m \otimes n) \equiv \sum_i (\alpha_i' m) \otimes (\alpha_i'' n)$ WHERE $\psi(\alpha) = \sum_i \alpha_i' \otimes \alpha_i''$.

ALSO $\mathrm{HOM}_{\mathbb{Z}_2}[M, N]$ CARRIES AN $\mathfrak{A}$ MODULE STRUCTURE.

LET A BE AN $\mathfrak{A}$-MODULE WHICH IS ALSO AN ALGEBRA OVER $\mathbb{Z}_2$. THEN, IF $\mu : A \otimes A \to A$ IS AN $\mathfrak{A}$ MAPPING A IS AN <u>ALGEBRA OVER THE STEENROD ALGEBRA $\mathfrak{A}$</u>.

LET X BE A TOPOLOGICAL SPACE FOR $u \in H^i(X; \mathbb{Z}_2)$ WE HAVE

$$Sq^j(u) = \begin{cases} 0 & \text{IF } j > i \\ u^2 & \text{IF } j = i \end{cases}$$

THE FIRST CONDITION GIVES US A DECREASING SEQUENCE OF LEFT IDEALS

$\mathcal{O}\!\iota = B(0) \supset B(1) \supset B(2) \supset \dots$

WHERE $B(i)$ = LEFT IDEAL GENERATED BY Sq^j $j > i$.

IN GENERAL ANY $\mathcal{O}\!\iota$ MODULE M s.t. $B(n) \cdot M^n = 0$ IS CALLED <u>UNSTABLE</u>

LET M BE A GRADED UNSTABLE MODULE OVER $\mathcal{O}\!\iota$ THEN $U(M)$ DENOTES THE FREE GRADED ALGEBRA OVER $\mathcal{O}\!\iota$ GENERATED BY M $\left(U(M) = T(M) / \{ Sq^i u_i - u_i \otimes u_i \} \right)$

GIVEN THE PAIR OF SPACES (X, Y) WE FORM $H^*(X, Y; Z_2)$ WHICH IS A MODULE OVER $H^*(X; Z_2)$ AND ALSO A MODULE OVER $\mathcal{O}\!\iota$. THESE TWO MODULE STRUCTURES ARE RELATED BY: IF $v \in H^*(X; Z_2)$ AND $u \in H^*(X, Y; Z_2)$ AND $\alpha \in \mathcal{O}\!\iota$ THEN

$$(*) \qquad \alpha(v \cdot u) = \sum_i (\alpha_i' v)(\alpha_i'' u)$$

WHERE $\psi(\alpha) = \sum_i \alpha_i' \otimes \alpha_i''$

ABSTRACTING THIS NOTION WE CONSIDER R AN ALGEBRA OVER $\mathcal{O}\!\iota$ AND M A MODULE OVER $\mathcal{O}\!\iota$ AND R THEN M IS AN <u>$\mathcal{O}\!\iota - R$ MODULE</u> $\Leftrightarrow$ $(*)$ HOLDS FOR $\alpha \in \mathcal{O}\!\iota$, $v \in R$ AND $u \in M$.

WE SHALL CONSTRUCT A RING R' WHICH HAS THE PROPERTY THAT THERE EXISTS A 1-1 CORRESPONDENCE BETWEEN $\mathfrak{A}-R$ MODULES AND R' MODULES.

WE LET $R' = R \otimes_{\mathbb{Z}_2} \mathfrak{A}$ AND DEFINE $(r \otimes \alpha) m = r(\alpha m)$ AND $(r \otimes \alpha)((s \otimes \beta) m) = [(r \otimes \alpha)(s \otimes \beta)] m$ WHERE $(r \otimes \alpha)(s \otimes \beta) = \sum_i [r(\alpha_i' s)] \otimes [\alpha_i'' \beta]$

THE MULTIPLICATION GIVES R' A RING STRUCTURE.

R' WITH THIS RING STRUCTURE WE CALL THE <u>SEMI-TENSOR PRODUCT OF R AND $\mathfrak{A}$</u> AND DENOTE IT BY $R \odot \mathfrak{A}$

II. WE SHALL WORK WITH $\mathbb{Z}_2$ COEFFICIENTS.

CONSIDER A FIBRE BUNDLE ξ, $p: E \to B$ WITH BUNDLE GROUP $O(n)$ AND FIBRE $V_{n,k}$ THE STIEFEL MANIFOLD OF k-FRAMES IN n-SPACE.

LET E_T DENOTE THE MAPPING CYLINDER (E_T IS OBTAINED FROM $E \times I \cup B$ BY IDENTIFYING $(e, 0)$ WITH $p(e)$)

THE PAIR (E_T, E) GIVES US

$$\begin{tikzcd} \xrightarrow{\delta} H^q(E_T, E) \arrow[r] \arrow[dr, "i"] & H^q(E_T) \arrow[r] \arrow[d, phantom, "\wr"] & H^q(E) \\ & H^q(B) \arrow[ur, "p^*"] & \end{tikzcd}$$

WHERE i IS INDUCED BY THE ISOMORPHISM $H^q(E_T) \to H^q(B)$.

CONSIDER THE UNIVERSAL BUNDLE γ:

$$\begin{array}{c} BO(n-k) \\ \downarrow V_{n,k} \\ BO(n) \end{array}$$

THEN WE HAVE:

$$0 \to H^*(BO(n-k)_T, BO(n-k)) \xrightarrow{i} H^*(BO(n)) \xrightarrow{p^*} H^*(BO(n-k)) \to 0$$

WHERE $H^*(BO(n)) = Z_2[w_1, .. w_n]$. THE w_i ARE THE STIEFEL WHITNEY CLASSES.

THEN $\ker p^* = \{ w_{n-k+1}, . w_n \}$

THERE EXISTS A UNIQUE $\mathcal{U}_i \in H^i(BO(n-k)_T, BO(n-k))$ SUCH THAT $i(\mathcal{U}_i) = w_i$ FOR $n-k < i \leq n$.

WE PULL BACK THIS INFORMATION IN THE FOLLOWING MANNER:

THERE EXISTS $f : B \to BO(n)$ SUCH THAT $\xi = f^*(\gamma)$. LET $\hat{f} : E \to BO(n-k)$ BE INDUCED BY f WHICH ALSO INDUCES

$\hat{f} : E_T \to BO(n-k)_T$ AND HENCE

$\hat{f} : (E_T, E) \to (BO(n-k)_T, BO(n-k))$

THEN DEFINE $\mathcal{U}_i(\xi) = \hat{f}^*(\mathcal{U}_i)$.

THUS THERE EXISTS A UNIQUE $\mathcal{U}_i(\xi)$ SUCH THAT $i(\mathcal{U}_i(\xi)) = w_i(\xi)$.

USING THE WU FORMULA

$$Sq^i w_j = \sum_{s=0}^{i} \binom{j-i+s-1}{s} w_{j+s} w_{i-s}$$

ONE CAN SHOW $H^*(E_T, E)$ IS AN $H^*(B) \odot \mathfrak{A}$ MODULE.

THE ABOVE FORMULA PULLS BACK IN ξ TO:

$$Sq^i\left(U_j(\xi)\right) = \sum_{s=0}^{i} \binom{j-i+s-1}{s} w_{i-s}(\xi)\, U_{j+s}(\xi)$$

LET $M(\xi)$ BE THE SUBMODULE OF $H^*(E_T, E)$ OVER $H^*(B)$ GENERATED BY $U_{h-k+1}(\xi), \ldots, U_h(\xi)$. THE FORMULA ABOVE SHOWS $M(\xi)$ IS A MODULE OVER $\mathfrak{A}$ AND HENCE A MODULE OVER $H^*(B) \odot \mathfrak{A}$.

NOW ASSUME $V_{h,k}$ IS TOTALLY NON HOMOLOGOUS TO ZERO IN E, i.e. IF j IS THE INJECTION MAP $j : V_{h,k} \to E$ THEN

$$j_* : H_*(V_{h,k}) \to H_*(E) \text{ IS } 1-1$$

OR EQUIVALENTLY $j^* : H^*(E) \to H^*(V_{h,k})$ IS ONTO AND HENCE BY STANDARD ARGUMENTS IN SPECTRAL SEQUENCES

$$p^* : H^*(B) \to H^*(E) \text{ IS } 1-1.$$

HENCE OUR EXACT SEQUENCE BECOMES

$$0 \to H^*(B) \xrightarrow{p^*} H^*(E) \xrightarrow{\delta} H^*(E_T, E) \to 0$$

LET $N(\xi) = \delta^{-1}(M(\xi))$

WE THEN HAVE THE FOLLOWING EXACT SEQUENCE OF $H^*(B) \odot \mathfrak{A}$ MODULES:

$$0 \to H^*(B) \xrightarrow{p^*} N(\xi) \xrightarrow{\delta} M(\xi) \to 0 \text{ ,}$$

ANALOGOUS TO $\mathfrak{U}(N(\xi))$ WE MAY DEFINE $\mathfrak{U}_{H^*(B)}(N(\xi))$.

THEN WE HAVE THE FOLLOWING THEOREM WHICH WE STATE WITHOUT PROOF:

THEOREM

$$H^*(E) \simeq \mathfrak{U}_{H^*(B)}(N(\xi)) \quad \text{WHERE}$$

THE ISOMORPHISM IS AN ISOMORPHISM OF ALGEBRAS OVER $\mathfrak{A}$. |

TOPOLOGICAL CLASSIFICATIONS OF IMMERSIONS BY SMALE, THOM AND HIRSCH

J. LEVINE

REFERENCES:

S. SMALE	ANNALS OF MATH	VOL. 69 (1958)
R. THOM	SEM. BOURBAKI	1957-1958
M. HIRSCH	TRANS. AMS	VOL. 93 (1959)

LET M^p BE A COMPACT MANIFOLD AND $V^n \subseteq M^p$ A COMPACT SUBMANIFOLD. LET X^r BE AN UNBOUNDED MANIFOLD $r > p \geq n$.

GIVEN A MANIFOLD A, $T(A)$ WILL DENOTE THE TANGENT BUNDLE WITH RIEMANNIAN METRIC AND $T_k(A)$ THE ASSOCIATED BUNDLE OF ORTHONORMAL k-FRAMES (THE FIBRE IS $V_{a,k}$ THE STIEFEL MANIFOLD OF ORTHONORMAL k-FRAMES IN $a = \dim A$ SPACE)

LET $f : M \to X$ BE C^∞ THEN WE HAVE $df : T(M) \to T(X)$. f IS AN IMMERSION IF df IS 1-1 AND AN IMBEDDING IF f IS 1-1 ALSO (M COMPACT)

LET $f : M \times I \to X$ BE A DIFFERENTIABLE HOMOTOPY IT IS A REGULAR HOMOTOPY IF EACH f_t IS AN IMMERSION AND AN ISOTOPY IF EACH f_t IS AN IMBEDDING.

LET U_i BE A NBD. OF V IN M $i = 1, 2$. TWO IMMERSIONS $f_i : U_i \to X$ ARE EQUIVALENT MODULO V DENOTED $f_1 \underset{V}{\sim} f_2 \Leftrightarrow f_1|V = f_2|V$ AND $df_1|V = df_2|V$. AN EQUIVALENCE CLASS OF SUCH IMMERSIONS WILL BE CALLED AN M-IMMERSION OF V IN X. THE SET OF SUCH WILL BE DENOTED $I_M(V, X)$. THE SET OF IMMERSIONS OF V RESP. M IN X WILL BE DENOTED $I(V, X)$ RESP. $I(M, X)$.

THESE SETS ARE TOPOLOGIZED BY THE C^1 TOPOLOGY THAT IS f IS NEAR g IF $f(x)$ IS NEAR $g(x)$ FOR ALL x AND $df(t)$ IS NEAR $dg(t)$ FOR ALL TANGENT VECTORS t.

WE THEN HAVE THE FOLLOWING SEQUENCE:

$$I(M,X) \xrightarrow{p} I_M(V,X) \xrightarrow{p'} I(V,X)$$

WHERE GIVEN ANY IMMERSION $f: M \to X$, $p(f)$ DENOTES ITS EQUIVALENCE CLASS IN $I_M(V,X)$.

FOR ANY M-IMMERSION, α, OF V IN X LET g BE A REPRESENTATIVE, THEN $p'(\alpha) \equiv g/V$ IMMERSING V IN X.

<u>THEOREM</u> p AND $p' \circ p$ ARE (SERRE) FIBRE MAPS. (HENCE SO IS p'). |

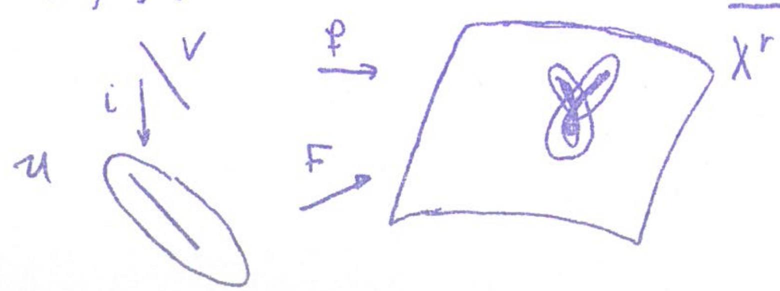
WE SHALL SKETCH A PROOF OF THIS THEOREM FOR $\overline{p} = p' \circ p$ THE PROOF FOR p BEING SIMILAR.

A <u>TUBULAR NBD.</u> OF V^n IN M^p IS A COMPACT SUBMANIFOLD $\mathcal{U}^p \subseteq M^p$ WITH $V \subseteq \mathcal{U} \subseteq M$ SUCH THAT $\mathcal{U} \cap (\overline{M - \mathcal{U}}) \cap V = \emptyset$ AND THE DISTANCE FROM V TO $\overline{M - \mathcal{U}}$ IS SMALL.

LET $\partial \mathcal{U}$ DENOTE $\mathcal{U} \cap \overline{M - \mathcal{U}}$.

GIVEN AN IMMERSION $f: V^n \to X^r$. LET $\mathcal{U}^r$ BE A COMPACT MANIFOLD $i: V \to \text{INT}\, \mathcal{U}$ AN IMBEDDING AND $F: \mathcal{U} \to X$ AN IMMERSION SUCH THAT $F \circ i = f$ THEN $(\mathcal{U}, F, i)$ IS SAID TO BE A <u>TUBULAR NBD. OF f</u>.

i.e.



NOW TO PROVE $\bar{p} = p' \circ p$ IS A FIBRE MAP IT IS SUFFICIENT TO PROVE IT IS A FIBRE MAP OF $\bar{p}^{-1}(W)$ WHERE W RUNS OVER AN OPEN COVERING OF $I(V, X)$.

CONSIDER THE DIAGRAM:

$$\begin{array}{ccc} & \overset{f_0}{\nearrow} & \bar{p}^{-1}(W) \subseteq I(M, X) \\ & & \bar{p}\downarrow \\ I \times I^n & \xrightarrow{\tilde{G}} & W \subseteq I(V, X) \end{array}$$

$\tilde{G}(t, u) \equiv g_{t,u} \in W$. $f_{0,u} : M \to X$ SUCH THAT $f_{0,u}|V = g_{0,u}$. TO CONSTRUCT IMMERSIONS $f_{t,u} : M \to X$ SUCH THAT $f_{t,u}|V = g_{t,u}$.

FOR SIMPLICITY WE CONSIDER THE SPECIAL CASE $n=0$. GIVEN $g \in I(V, X)$ WE SHALL CONSTRUCT NBDS. W_g WITH RESPECT TO WHICH WE SHALL PROVE THE THEOREM.

THUS WE HAVE g_t A REGULAR HOMOTOPY IN W, $f_0|V = g_0$ TO CONSTRUCT A REGULAR HOMOTOPY f_t IN $\bar{p}^{-1}(W)$ SUCH THAT $f_t|V = g_t$.

OUR METHOD IS TO DEFORM $f_0 : M \to X$ INTO A "WELL POSITIONED" IMMERSION (THE DEFORMATION FIXED ON V) ~~———~~ AND PROVE THE THEOREM FOR THAT CASE.

WE OMIT THE DEFORMATION STAGE.

LET $f : M \to X$ BE AN IMMERSION $f|_V \equiv g$

LET $(\mathcal{U}, G, i)$ BE A TUBULAR NBD. OF g

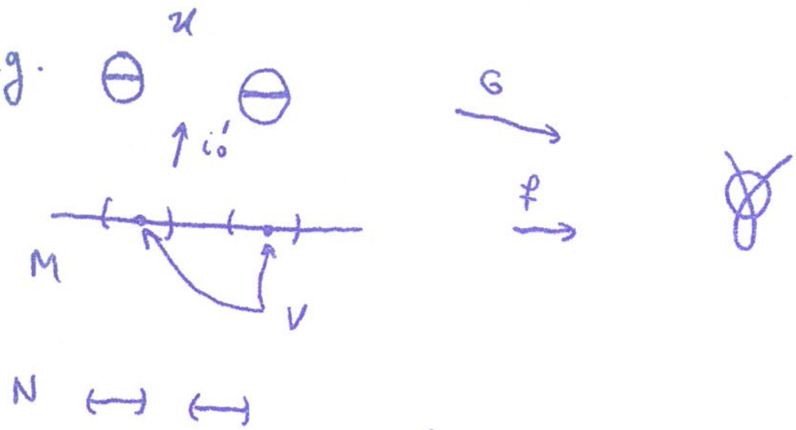
WE SAY f IS <u>WELL-POSITIONED</u> WITH RESPECT TO $(\mathcal{U}, G, i)$ IF THERE EXISTS A TUBULAR NBD. N OF V IN M WHICH CAN BE LIFTED INTO $\mathcal{U}$ NICELY, THAT IS

THERE EXISTS AN EXTENSION i_0' OF i

$$i_0' : (N, \bar{\partial} N) \to (\mathcal{U}, \partial \mathcal{U}) \quad \text{SUCH THAT}$$

$$G \circ i_0' = f_{|N}$$

e.g.



WE PROVE THE THEOREM FOR SUCH AN f.

WE NOW DEFINE W_g TO CONSIST OF ALL IMMERSIONS $g' : V \to X$ SUCH THAT THERE EXISTS AN IMBEDDING $i' : V \to \text{INT}\,\mathcal{U}$ WITH $G \circ i' = g'$.

<u>LEMMA</u> LET $\mathcal{U}$, M BE COMPACT MANIFOLDS AND $f_t : M \to \text{INT}\,\mathcal{U}$ IMBEDDINGS. THEN THERE EXISTS A FAMILY H_t OF DIFFEOMORPHISMS OF $\mathcal{U}$ SATISFYING

(i) $H_0 = 1$

(ii) $H_t = 1$ IN A NBD. OF $\partial \mathcal{U}$

(iii) $H_t \circ f_0 = f_t$

WE OMIT THE PROOF. |

CONSIDERING OUR $g_t : V \to X$ g_t IN W_g ($g_0 = g$, $f_0 = f$) WE HAVE IMBEDDINGS $i_t : V \to \text{INT}\,\mathcal{U}$ WITH $G \circ i_t = g_t$. APPLYING THE LEMMA TO THE i_t WE HAVE A FAMILY H_t OF DIFFEOMORPHISMS OF $\mathcal{U}$ WHICH ARE THE IDENTITY NEAR $\partial \mathcal{U}$ AND $H_t \circ i_0 = i_t$.

WE HAVE $i_0' : (N, \bar{\partial} N) \to (\mathcal{U}, \partial \mathcal{U})$

DEFINE $f_t|N$ BY $f_t = G \circ H_t \circ i_0'$

WHERE $\bar{\partial} N = N \cap (\overline{M-N})$

NOW $f_t|V = G \circ H_t \circ i_0'|V = G \circ H_t \circ i_0 = G \circ i_t = g_t$

AND NEAR $\bar{\partial} N$

$$f_t = G \circ i_0' = f_0$$

SO WE CAN DEFINE $f_t|\overline{M-N}$ TO BE $f_0|\overline{M-N}$

THEN f_t PIECES TOGETHER INTO AN IMMERSION PROVING THE THEOREM. |

<u>COROLLARY</u> LET $M \supseteq N \supseteq V$ COMPACT MANIFOLDS X UNBOUNDED. CONSIDER $I(M,X) \xrightarrow{\psi} I_M(N,X) \xrightarrow{\varphi} I_M(V,X)$

BY THE THEOREM ψ AND $\varphi\psi$ ARE FIBRE MAPS

HENCE

$\varphi : I_M(N,X) \to I_M(V,X)$ IS A FIBRE MAP. |

LET f IMMERSE V IN X LET x_0 BE A BASE POINT OF V y_0 A BASE POINT OF X f IS A <u>BASED IMMERSION</u> IF

(i) $f(x_0) = y_0$

(ii) $df(V_{x_0})$ IS PRESCRIBED.

CALL THE EQUIVALENCE CLASSES OF BASED IMMERSIONS $\tilde{I}_M(V,X)$.

<u>LEMMA</u> $\tilde{I}_{S^k}(D^i, X^r)$ $r \geq k \geq i$ IS CONTRACTIBLE.

<u>PROOF</u> BY WELL KNOWN TECHNIQUES ANY $f \in \tilde{I}_{S^k}(D^i, X^r)$ IS REGULARLY HOMOTOPIC TO A LINEAR MAP INTO A COORDINATE NBD. OF y_0 WHICH CORRESPONDS TO THE DIFFERENTIAL OF f AT x_0. |

NOW WE HAVE FIBRE SPACES

$$\tilde{I}_{S\ell}(D^i, X) \to \tilde{I}_{S\ell}(S^{i-1}, X)$$
$$\tilde{I}_{S\ell}(S^i, X) \to \tilde{I}_{S\ell}(D^i, X).$$

THE FIBRES IN BOTH CASES ARE EQUAL CALL THEM $F_{i,\ell}$. BY THE HOMOTOPY SEQUENCE OF A FIBRE SPACE WE HAVE:

$$\pi_j(F_{i,\ell}) \simeq \pi_j(\tilde{I}_{S\ell}(S^i, X)) \simeq \pi_{j+1}(\tilde{I}_{S\ell}(S^{i-1}, X)) \quad \text{SO}$$

$$\pi_0(F_{i,\ell}) \simeq \pi_0(\tilde{I}_{S\ell}(S^i, X)) \simeq \pi_i(\tilde{I}_{S\ell}(S^0, X)).$$

NOW $\tilde{I}_{S\ell}(S^0, X) = I_{S\ell}(PT., X) = T_\ell(X)$

SO $\pi_i(T_\ell(X)) \simeq \pi_0(\tilde{I}_{S\ell}(S^i, X))$.

IN PARTICULAR

$$\pi_\ell(T_\ell(X)) \simeq \pi_0(I(S^\ell, X))$$

LET $X = R^n$ $n > \ell$

THEN $\boxed{\pi_0(I(S^\ell, R^n)) \simeq \pi_\ell(V_{n,\ell})}$ ▮

<u>REMARK</u>

THE ISOMORPHISM

$$\pi_0(F_{i,\ell}) \to \pi_\ell(T_\ell(X))$$

IS GOTTEN AS FOLLOWS:

LET $f, g \in F_{i,\ell}$ CHOOSE A FIXED ℓ-FRAME $v_1, \dots v_\ell$ ON D^i

$\bar{f} : D^i \to T_\ell(X)$

BY $\bar{f}(x) = (df(v_1), \dots df(v_\ell))$ $\bar{f}, \bar{g}$ AGREE ON S^{i-1}

SO PATCH TOGETHER TO GET $\Omega(f, g) : S^i \to T_\ell(X)$

FIX $f_0 \in F_{i,\ell}$ FOR ANY f

WE MAP $f \to \Omega(f_0, f)$. ▮

HOMOTOPY OF SPACES WITH AN INVOLUTION

J. LEVINE

THIS TALK REPRESENTS JOINT WORK WITH P.E. CONNER.

LET X BE A SPACE WITH INVOLUTION T AND AT LEAST ONE FIXED POINT x_0.
LET F BE THE FIXED POINT SET OF T.

GIVEN INTEGERS $p, q \geq 0$ WE DEFINE AN INVOLUTION T_p ON S^{p+q} BY $T_p(x_0, \ldots, x_{p+q}) = (x_0, \ldots x_p, -x_{p+1}, \ldots -x_{p+q})$
FIXED PT. SET OF T_p IS S^p.

CONSIDER $f : (S^{p+q}, D^p_-) \to (X, x_0)$ SUCH THAT
$f \circ T_p = T \circ f$ WHERE $D^p_- = \{x \in S^{p+q} / \ x_p \leq 0\}$

DENOTE BY $A_{p,q}(X, x_0, T)$ THE SET OF HOMOTOPY CLASSES OF SUCH EQUIVARIANT MAPS f.
IF $p \geq 1$ $A_{p,q}$ IS A GROUP (ADD ON ANY COORDINATE $x_0, \ldots x_{p-1}$) AND IF $p \geq 2$ $A_{p,q}$ IS ABELIAN. $A_{p,0} \simeq \pi_p(F, x_0)$.

WE HAVE THE FOLLOWING EXACT SEQUENCE:

$$\to \pi_{p+q}(X) \xrightarrow{\varphi} A_{p,q} \xrightarrow{\psi} A_{p,q-1} \xrightarrow{\rho} \pi_{p+q-1}(X) \xrightarrow{\varphi} A_{p-1,q} \to \cdots$$
$$\cdots \to A_{1,q} \to A_{1,q-1} \to \pi_q(X).$$

WHERE ρ DENOTES PASSAGE TO ORDINARY HOMOTOPY
ψ = RESTRICTION TO EQUATOR $(x_{p+q} = 0)$.
φ IS DEFINED AS FOLLOWS:
CONSIDER $f : (S^{p+q}, D^p_-) \to (X, x_0)$
DEFINE A NEW MAP $g : S^{p+q} \to X$
BY $g|D^p_+ = f|D^p_+$

AND SINCE g IS TO BE EQUIVARIANT

WE ARE FORCED TO DEFINE $g|_{D_-^p} = T \circ f \circ T_p|_{D_-^p}$

THUS $g \circ T_p = T \circ g$.

AND WE DEFINE $\varphi[f] = [g]$.

WE HAVE THEN THE EXACT COUPLE:

$$\begin{tikzcd} A \arrow[rr, "\psi"] & & A \arrow[dl, "\rho"] \\ & E \arrow[ul, "\varphi"] & \end{tikzcd}$$

$A = \{A_{p,q}\}$ IS A BIGRADED GROUP WHERE $A_{p,q} \equiv 0$ FOR $p < 2$ OR $q < 0$.

$$E = \{E_{p,q} \mid E_{p,q} = \pi_{p+q}(X)\}$$

CONSIDERING THE RESULTING SPECTRAL SEQUENCE

$$E^2_{p,q} = \frac{\ker(\rho \circ \varphi)}{\mathrm{Im}(\rho \circ \varphi)}$$

$\rho \circ \varphi = 1 \pm T_*$ THE SIGN DEPENDS ON q.

IT TURNS OUT THAT $E^2_{p,q} \simeq H^q(\mathbb{Z}_2; \pi_{p+q}(X))$

WHERE $\mathbb{Z}_2$ ACTS ON $\pi_{p+q}(X)$ THROUGH T_*.

$$\pi_p(F) = A_{p,0} \supset A^2_{p,0} \supset A^3_{p,0} \supset \dots \supset A^r_{p,0} \supset \dots$$

$$A^r_{p,0} / A^{r+1}_{p,0} \simeq \ker \varphi_r \subseteq E_{p,r-1}$$

WHERE φ_r IS HOMOMORPHISM IN THE r-th DERIVED COUPLE.

IN $\mathbb{Z}_2$ HOMOLOGY WE HAVE THE SMITH SEQUENCE

$$H_n(X) \to H_n(X/T, F) \xrightarrow{\Delta} \begin{array}{c} H_{n-1}(X/T, F) \\ \oplus \\ H_{n-1}(F) \end{array} \to H_{n-1}(X)$$

(X/T ORBIT SPACE OF X UNDER THE ACTION OF T)

WE THEN HAVE THE FOLLOWING COMMUTATIVE DIAGRAM:

$$\begin{array}{ccccccc}
\pi_{p+q}(X) & \to & A_{p,q} & \xrightarrow{\psi} & A_{p,q-1} & \to & \pi_{p+q-1}(X) \to \\
\downarrow h & & \downarrow h_1 & & \downarrow h_2 & & \downarrow h \\
H_{p+q}(X) & \to & H_{p+q}(X/T, F) & \xrightarrow{\Delta} & H_{p+q-1}(X/T, F) & \to & H_{p+q-1}(X) \to \\
 & & & & \oplus & & \\
 & & & & H_{p+q-1}(F) & &
\end{array}$$

WHERE h IS THE HUREWICZ MAP.

WE DEFINE $\bar{h} : A_{p,q} \to H_{p+q}(X/T, F)$ $q \geq 1$ AS FOLLOWS: LET $f : S^{p+q} \to X$ REPRESENT $\alpha \in A_{p,q}$

NOW $f_* : H_{p+q}(S^{p+q}/T_p, S^p) \to H_{p+q}(X/T, F)$

(BECAUSE $f \circ T_p \simeq T \circ f$). SINCE $H_{p+q}(S^{p+q}/T_p, S^p) \simeq Z_2$

LET σ BE THE GENERATOR THEN

$\bar{h}(\alpha) = f_*(\sigma)$.

THEN WE DEFINE $h_1 = \bar{h}$

$$h_2(\alpha) = \begin{cases} \bar{h}(\alpha) \oplus 0 & \text{IF } q \geq 2 \\ 0 \oplus h(\alpha) & \text{IF } q = 1 \end{cases}$$

WE NOW CONSIDER THE SPHERICAL HOMOLOGY OF F.

$S_p(F) = h(\pi_p(F)) \subseteq H_p(F)$.

WE HAVE $h : A_{p,0} \to H_p(F)$

$A_{p,0} \supseteq A^2_{p,0} \supseteq A^3_{p,0} \supseteq \dots \supseteq A^r_{p,0} \supseteq \dots$

$A^r_{p,0} / A^{r+1}_{p,0} \simeq$ SUBGROUP OF $E^r_{p,r-1}$

IF X IS AN n-COMPLEX AND T SIMPLICIAL WE CLAIM $h(A^{n-p+2}_{p,0}) = 0$, THIS IS SEEN FROM THE FOLLOWING COMMUTATIVE DIAGRAM:

$$\begin{tikzcd} A_{p,n-p+1} \arrow[r, "\psi"] \arrow[d, "h_1"] & A_{p,n-p} \arrow[r, "\psi"] \arrow[d, "h_1"] & \cdots & A_{p,1} \arrow[r, "\psi"] \arrow[d, "h_1"] & A_{p,0} \arrow[d, "h"] \\ 0 = H_{n+1}(X/T, F) \arrow[r, "\tilde{\Delta}"] & H_n(X/T, F) \arrow[r, "\tilde{\Delta}"] & \cdots & H_{p+1}(X/T, F) \arrow[r, "\tilde{\Delta}"] & H_p(F) \end{tikzcd}$$

WHERE $\tilde{\Delta}$ IS Δ FOLLOWED BY PROJECTION

$$H_n(X/T, F) \oplus H_n(F) \to H_n(X/T, F).$$

THEN SINCE $A_{p,0}^{n-p+2} = \text{IMAGE } \psi^{n-p+2}$ AND DIAGRAM IS COMMUTATIVE WE HAVE $h(A_{p,0}^{n-p+2}) = 0$. ∎

SO $S_p(F) \simeq$ QUOTIENT GROUP OF $A_{p,0} / A_{p,0}^{n-p+2}$

THUS $S_p(F)$ HAS A FILTRATION THE QUOTIENTS OF WHOSE SUCCESSIVE TERMS ARE QUOTIENT GROUPS OF SUBGROUPS OF $E^r_{p,r-1}$ $1 \le r \le n-p+1$. $(p \ge 2)$.

WE THEN HAVE THE FOLLOWING THEOREM

__THEOREM__ IF $\pi_p(X) = 0$ AND $H^i(\mathbb{Z}_2; \pi_{p+i}(X)) = 0$ $1 \le i \le n-p$ THEN $S_p(F)$.

__PROOF__ USE THE FILTRATION OF $S_p(F)$, THE FACT THAT $E^2_{p,q} \simeq H^q(\mathbb{Z}_2, \pi_{p+q}(X))$ AND $\pi_p(X) = 0$. ∎

__COROLLARY__ IF $\pi_i(X) = 0$ $p \le i \le n$ THEN $S_p(F) = 0$. ∎

__EXAMPLE__ LET $P^n(C)$ DENOTE COMPLEX PROJECTIVE SPACE THEN THE FIXED POINT SET OF ANY INVOLUTION OF $P^n(C)$ HAS NO HOMOLOGY ABOVE DIMENSION 2. ∎

STABLE STRUCTURES ON MANIFOLDS

BY HERMAN GLUCK

REFERENCE: M. BROWN AND H. GLUCK, STABLE STRUCTURES ON MANIFOLDS. BULLETIN OF THE A.M.S. 69 (1963) (Pg. 51–58)

THE MATERIAL HERE GIVES A PROOF OF THE SCHOENFLIES PROBLEM FOR $S^{n-1} \times S^1$, NAMELY:

THEOREM LET f_1, f_2 BE LOCALLY FLAT EMBEDDINGS OF S^{n-1} INTO $S^{n-1} \times S^1$ WHOSE IMAGES EITHER BOTH SEPARATE OR BOTH DO NOT SEPARATE $S^{n-1} \times S^1$ THEN THERE IS A HOMEOMORPHISM h OF $S^{n-1} \times S^1$ ONTO ITSELF SUCH THAT $hf_1 = f_2$.

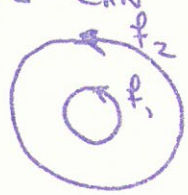
(AN EMBEDDING $f: S^{n-1} \to M^n$ A TOPOLOGICAL MANIFOLD IS LOCALLY FLAT IF THERE EXISTS FOR EACH POINT $x \in f(S^{n-1})$ A COORD. NBD. $(\mathcal{U}, k)$ ON M^n ABOUT x SUCH THAT $k(\mathcal{U} \cap f(S^{n-1}))$ IS OPEN IN R^{n-1})

NOTATION LET M^n BE AN n-MANIFOLD $\mathrm{HOM}(S^{n-1}, M^n)$ THE SET OF ALL LOCALLY FLAT EMBEDDINGS OF S^{n-1} INTO M^n AND $H(S^n)$ THE SET OF ALL HOMEOMORPHISMS OF S^n.

LET $f_0, f_1 \in \mathrm{HOM}(S^{n-1}, S^n)$ THEN IF f_0 AND f_1 ARE HOMOTOPIC VIA AN EMBEDDING $F: S^{n-1} \times I \to S^n$ WE SAY f_0 AND f_1 ARE STRICTLY ANNULARLY EQUIVALENT WRITTEN $f_0 \underset{A}{\sim} f_1$.

f_0 AND f' ARE <u>ANNULARLY EQUIVALENT</u>
DENOTE $f \underset{a}{\sim} f'$ IF THERE EXISTS A SEQUENCE OF ELEMENTS OF $\mathrm{HOM}(S^{n-1}, S^n)$ $f = f_0, f_1, \ldots, f_\ell = f'$ SUCH THAT $f_i \underset{A}{\sim} f_{i+1}$.

<u>DEFN</u> LET $f_1, f_2 \in \mathrm{HOM}(S^{n-1}, S^n)$ WITH DISJOINT IMAGES. f_1 AND f_2 ARE <u>SIMILARLY ORIENTED</u> IF WE CAN ORIENT THE REGION BETWEEN THEM.

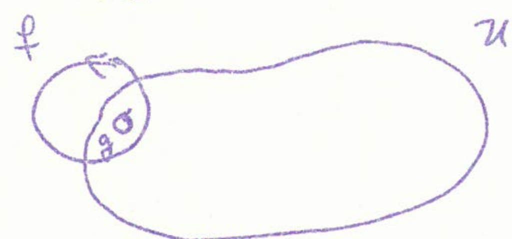


<u>LEMMA</u> IF f_1 AND f_2 ARE SIMILARLY ORIENTED AND $f_1 \underset{a}{\sim} f_2$ THEN $f_1 \underset{A}{\sim} f_2$.

THE PROOF OMITTED HERE USES ELEMENTARY TECHNIQUES AND THE FOLLOWING LEMMA

<u>LEMMA</u> GIVEN $f \in \mathrm{HOM}(S^{n-1}, S^n)$ $\mathcal{U}$ OPEN IN S^n THEN THERE EXISTS $g \in \mathrm{HOM}(S^{n-1}, S^n)$ $g(S^{n-1}) \subseteq \mathcal{U}$ AND $g \underset{A}{\sim} f$.

<u>PROOF</u> WE CAN ALWAYS ASSUME WE HAVE THE FOLLOWING PICTURE:



THEN SHRINK $f(S^{n-1})$ INSIDE $\mathcal{U}$ TO GET g.

STABLE HOMEOMORPHISMS

LET $h \in H(S^n)$ BE SUCH THAT THERE EXISTS $\mathcal{U}$ OPEN IN S^n WITH $h|\mathcal{U} = 1$. h IS SAID TO BE SOMEWHERE THE IDENTITY.

$SH(S^n)$ THE GROUP OF STABLE HOMEOMORPHISMS OF S^n WILL BE THE SET OF PRODUCTS OF HOMEOMORPHISMS EACH OF WHICH IS SOMEWHERE THE IDENTITY.

ANDERSON AND FISHER HAVE SHOWN $SH(S^n)$ IS A NORMAL SUBGROUP OF $H(S^n)$ AND IS A SIMPLE GROUP.

NOTE: IF $g \in H(S^n)$ AND AGREES ON SOME OPEN SET WITH $h \in SH(S^n)$ THEN SINCE $g = h(h^{-1}g)$ $g \in SH(S^n)$.

IT CAN BE SHOWN THAT ORIENTATION PRESERVING DIFFERENTIABLE HOMEOMORPHISMS AND PIECEWISE LINEAR HOMEOMORPHISMS OF S^n ARE STABLE WHICH LEADS US TO THE CONJECTURE THAT ALL ORIENTATION PRESERVING HOMEOMORPHISMS ARE STABLE.

DEFN. LET $f, f' \in \text{HOM}(S^{n-1}, S^n)$. IF THERE IS AN ELEMENT $h \in SH(S^n)$ SUCH THAT $hf = f'$ WE SAY f AND f' ARE STABLY EQUIVALENT AND WRITE $f \underset{s}{\sim} f'$.

NOTE $H(S^n)$ ACTS ON STABLE EQUIVALENCE CLASSES AS FOLLOWS $h[f] \equiv [hf]$

IF $f_1 \underset{s}{\sim} f_2$ THEN THERE EXISTS $h_0 \in SH(S^n)$ WITH $h_0 f_1 = f_2$ SO $(h h_0 h^{-1}) h f_1 = h f_2$ AND SINCE $SH(S^n)$ IS A NORMAL SUBGROUP OF $H(S^n)$ WE HAVE $h h_0 h^{-1} \in SH(S^n)$ AND HENCE $h f_1 \underset{s}{\sim} h f_2$. ∎

IF $H^+(S^n)$ DENOTES THE SET OF ALL ORIENTATION PRESERVING HOMEOMORPHISMS OF S^n ONE CAN SHOW $SH(S^n)$ IS A NORMAL SUBGROUP OF $H^+(S^n)$ AND $H^+(S^n) / SH(S^n)$ ACTS ON $HOM_s(S^{n-1}, S^n)$ (STABLE EQUIVALENCE CLASSES) EFFECTIVELY AND TRANSITIVELY SO THERE IS A 1-1 ONTO CORRESPONDENCE BETWEEN
$H^+(S^n) / SH(S^n)$ AND $HOM_s(S^{n-1}, S^n)$. ∎

IT TURNS OUT THAT STABLE AND ANNULAR EQUIVALENCE ARE THE SAME IN $HOM(S^{n-1}, S^n)$. THIS RESULT IS USED TO PROVE:

<u>THEOREM</u> LET $h \in SH(S^n)$, E_1, E_2 CLOSED n-CELLS IN S^n WITH LOCALLY FLAT BOUNDARY IF $E_1 \cup h E_1$ IS DISJOINT FROM E_2 THEN THERE IS A STABLE HOMEOMORPHISM h' OF S^n WHICH AGREES WITH h ON E_1 AND $h' | E_2 = 1$. ∎

<u>COROLLARY</u> CONSIDERING $h = h'(h'^{-1} h)$ WE SEE ANY STABLE HOMEOMORPHISM h CAN BE WRITTEN AS THE PRODUCT OF TWO HOMEOMORPHISMS EACH OF WHICH IS SOMEWHERE THE IDENTITY. ∎

INSTEAD OF S^n LET US NOW CONSIDER M^n A CONNECTED n-MANIFOLD. AN EMBEDDING $f: D^n$ (n-DISK) $\to M^n$ IS <u>LOCALLY FLAT</u> IF $f|S^{n-1}$ IS LOCALLY FLAT. AGAIN WE DEFINE AS BEFORE THE FOLLOWING SETS: $\mathrm{HOM}(D^n, M^n)$, $H(M^n)$, $SH(M^n)$.

LET $h \in H(M^n)$ (ALL HOMEOMORPHISMS ARE TO BE ONTO). WE SAY h IS <u>ALMOST EVERYWHERE THE IDENTITY</u> IF THERE EXISTS A CLOSED n CELL IN M^n WITH LOCALLY FLAT BOUNDARY SUCH THAT $h|M^n - E = 1$. $SH_0(M^n)$ WILL BE THE GROUP OF ALL SUCH PRODUCTS. FISHER HAS SHOWN $SH_0(M^n)$ IS THE INTERSECTION OF ALL NON-TRIVIAL NORMAL SUBGROUPS OF $H(M^n)$ AND IS ALGEBRAICALLY SIMPLE.

WE HAVE $SH_0(S^n) = SH(S^n)$.

AGAIN WE DEFINE STRICT ANNULAR, ANNULAR AND STABLE EQUIVALENCE IN $\mathrm{HOM}(D^n, M^n)$ AND CAN ESTABLISH ANNULAR EQUIVALENCE = STABLE EQUIVALENCE.

WE HAVE

<u>LEMMA</u> LET $\mathcal{U}$ BE OPEN IN M^n, $f \in \mathrm{HOM}(D^n, M^n)$ THEN THERE EXISTS $f' \in \mathrm{HOM}(D^n, M^n)$ WITH $f' \underset{a}{\sim} f$ AND $f'(D^n) \subseteq \mathcal{U}$.

A RELATION BETWEEN STRICT ANNULAR AND ANNULAR EQUIVALENCE IS OBTAINED FROM THE FOLLOWING:

THEOREM IF $f \underset{n}{\sim} f'$ AND $f(D^n) \cap f'(D^n) = \phi$ THEN THERE EXISTS A g SUCH THAT

$$f \underset{A}{\sim} g \underset{A}{\sim} f'._{1}$$

NOTE $SH(M^n)$ IS "COMPLETE". THAT IS, IF $g = h$ ON U $h \in SH(M^n$ THEN $g \in SH(M^n)$ SINCE $g = h(h^{-1}g)$.

LET $HOM_S(D^n, M^n)$ DENOTE STABLE EQUIVALENCE CLASSES. THERE EXISTS A 1-1 CORRESPONDENCE

$$H(M^n)/SH(M^n) \rightarrowtail HOM_S(D^n, M^n).$$

GIVEN U OPEN AND CONNECTED $\subseteq M^n$ THE INJECTION $i : U \to M^n$ INDUCES

$$i_* : HOM_S(D^n, U) \to HOM_S(D^n, M^n)$$

WHICH IS ONTO BY LEMMA Pg 5. QUESTION IS i_* 1-1?

IF E IS AN n-CELL WITH LOCALLY FLAT BOUNDARY THEN $i_* : HOM_S(D^n, M^n - E) \to HOM_S(D^n, M^n)$ IS 1-1.

THE THEOREM AND COROLLARY Pg.4 TRANSLATE WORD FOR WORD WITH S^n REPLACED BY M^n.

WE HAVE THEOREM LET $h \in SH(M^n)$ E_1 CLOSED n-CELL IN M^n WITH LOCALLY FLAT BOUNDARY SUCH THAT $E_1 \cap h(E_1) = \phi$. THEN THERE EXISTS A CLOSED n-CELL E_2 WITH LOCALLY FLAT BOUNDARY SUCH THAT

$E_1 \cup h(E_1) \subseteq$ INTERIOR E_2 AND THERE EXISTS $h' \in SH(M^n)$ SUCH THAT $h'|_{E_1} = h|_{E_1}$ AND $h'|_{M^n - E_2} = 1$

STABLE STRUCTURES

DEFN A HOMEOMORPHISM $f : \mathcal{U}$ OPEN IN R^n ONTO V OPEN IN R^n IS STABLE AT $x \in \mathcal{U}$ IFF THERE EXISTS A NBD $\mathcal{U}_x$ OF x IN $\mathcal{U}$ AND $h \in SH(R^n)$ SUCH THAT $f|_{\mathcal{U}_x} = h|_{\mathcal{U}_x}$. IF f IS STABLE AT EVERY PT. OF $\mathcal{U}$ WE SAY f IS STABLE ON $\mathcal{U}$ AND f IS CALLED A STABLE COORDINATE TRANSFORMATION. THUS WE HAVE THE NOTION OF A STABLE STRUCTURE ON A TOPOLOGICAL MANIFOLD.

ONE CAN SHOW 1. SIMPLY CONNECTED 2. ORIENTABLE DIFFERENTIABLE 3. ORIENTABLE PIECEWISE LINEAR 4. ORIENTABLE TRIANGULABLE MANIFOLDS HAVE STABLE STRUCTURES.

FINALLY WE HAVE:

THEOREM LET M^n BE A CONNECTED TOPOLOGICAL MANIFOLD THE FOLLOWING FOUR CONDITIONS ARE EQUIVALENT:

1. M^n HAS A STABLE STRUCTURE
2. IF $h \in SH(M^n)$, E_1, E_2 CLOSED n-CELLS WITH LOCALLY FLAT BOUNDARIES SUCH THAT $E_1 \cup hE_1 \subset \operatorname{int} E_2$ THEN THERE IS A STABLE HOMEOMORPHISM h' OF M^n SUCH THAT $h'|_{E_1} = h|_{E_1}$ AND $h'|_{M^n - E_2} = 1$.
3. $f \underset{R}{\simeq} f'$ $f(D^n) \subset \operatorname{int} f'(D^n)$ THEN $f \underset{A}{\simeq} f'$.
4. FOR $\mathcal{U}$ OPEN CONNECTED SUBSET OF M^n $i_* : \mathrm{Hom}_S(D^n, \mathcal{U}) \to \mathrm{Hom}_S(D^n, M^n)$ IS 1-1.

WHITEHEAD PRODUCTS AND COHOMOLOGY OPERATIONS

BY F. P. PETERSON

THIS IS JOINT WORK WITH E. H. BROWN.

CONSIDER S^n AND α A GENERATOR OF $\pi_n(S^n)$. FORM THE WHITEHEAD PRODUCT $[\alpha,\alpha] \in \pi_{2n-1}(S^n)$. THEN $[\alpha,\alpha] \neq 0$ IFF S^n IS NOT AN H-SPACE.

LET $Y = S^n \cup_{[\alpha,\alpha]} e^{2n}$, u A GENERATOR OF $H^n(Y)$ AND v A GENERATOR OF $H^{2n}(Y)$.
IF n IS EVEN THEN $u \cup u = 2v$.
IF n IS ODD THEN NO PRIMARY COHOMOLOGY OPERATION IS $\neq 0$.

<u>THEOREM</u> IF $n+1 \neq 2^r$ FOR ANY r THEN WE WILL CONSTRUCT A SECONDARY COHOMOLOGY OPERATION Φ SUCH THAT $\Phi(u) = v$
IF $n+1 = 2^r$, $r > 3$ THEN THERE EXISTS A TERTIARY COHOMOLOGY OPERATION Φ SUCH THAT $\Phi(u) = v$.

<u>PROOF</u> LET $u \in H^n(X, Z_2)$ SO u CORRESPONDS TO A MAP $u: X \to K(Z_2, n)$.

CONSIDER THE DIAGRAM:

$$\begin{array}{ccccc} & & E & & \\ & & \downarrow p & & \\ X & \xrightarrow{u} & K(Z_2,n) & \xrightarrow{\varphi_1} & \prod_{n<i\leq 2n} K(Z_2,i) \end{array}$$

WHERE E IS INDUCED FROM THE PATH SPACE OVER $\prod_{n<i\leq 2n} K(\mathbb{Z}_2, i)$ VIA φ_1. (φ_1 ARBITRARY SO FAR)

IF $\varphi_1(u)=0$ THEN THERE EXISTS $\bar{u}: X \to E$

DEFINE $\Phi(u) = \{ \bar{u}^*(\varphi_2) \,/\, p\bar{u} = u \}$

WHERE $\varphi_2 \in H^{2n}(E)$.

<u>LEMMA</u> ASSUME Φ IS GIVEN SUCH THAT IF $\Phi(x)$ AND $\Phi(y)$ ARE DEFINED THEN SO IS $\Phi(x+y)$ AND $\Phi(x+y) = \Phi(x) + \Phi(y) + x \cup y$

FOR SUCH Φ, $\Phi(u) = V \cdot 1$

NOW $\alpha : S^n \to S^n \subset Y$ INDUCES

α' :

$$\begin{tikzcd} S^n \vee S^n \arrow[r, "\alpha'"] \arrow[d, phantom, "\cap"] & Y \\ S^n \times S^n \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

WHICH EXTENDS TO g BECAUSE $[\alpha, \alpha] = 0$ IN Y.

THEN $g^*(\Phi(u)) = \Phi(g^*(u)) = \Phi(\alpha \otimes 1 + 1 \otimes \alpha)$

$= \Phi(\alpha) \otimes 1 + 1 \otimes \Phi(\alpha) + \alpha \otimes \alpha = \alpha \otimes \alpha \neq 0.$

SUPPOSE $n+1 \neq 2^r$ ANY r

NOW $Sq^{n+1} = \sum_{i=1}^{\ell} a_i' a_i$ $\quad$ DIM $a_i > 0$, DIM $a_i'' > 0$

(I.E. $Sq^{n+1} = Sq^j Sq^{n+1-j} + \sum_{i>0} \binom{n-j-i}{j-2i} Sq^{n+1-i} Sq^i$

$n+1 = j + 2^k \qquad 0 < j < 2^k$)

NOW $\sum_{i=1}^{l} a_i' a_i = 0$ FOR ANY COHOMOLOGY CLASS OF DIMENSION $\leq n$ AND SO IS AN UNSTABLE RELATION.

WE HAVE

$$\begin{array}{ccc}\prod_{i=1}^{l} K(\mathbb{Z}_2, n+\dim a_i - 1) & \xrightarrow{j} & E \\ & & \downarrow p \\ & & K(\mathbb{Z}_2, n) \xrightarrow{\varphi_1} \prod_{i=1}^{l} K(\mathbb{Z}_2, n+\dim a_i)\end{array}$$

LET $i_i \in H^{n+\dim a_i}(K(\mathbb{Z}_2, n+\dim a_i))$

AND ${}'i_i \in H^{n+\dim a_i - 1}(K(\mathbb{Z}_2, n+\dim a_i - 1))$

THE SUSPENSION OF i_i. THEN THERE EXISTS $\varphi_2 \in H^{2n}(E)$ SUCH THAT

$$\sum a_i'({}'i_i) = j^*(\varphi_2).$$

TO SEE THIS, COMPUTE THE TRANSGRESSION:

$$\tau\left(\sum_{i=1}^{l} a_i'({}'i_i)\right) = \sum_{i=1}^{l} a_i' a_i(i) = Sq^{n+1}(i) = 0$$

THUS WE CAN CONSTRUCT A SECONDARY COHOMOLOGY OPERATION Φ. DEFINED ON $\cap \ker a_i$ WITH VALUES IN $H^{2n}(X) / \sum_{i=1}^{l} \operatorname{im} a_i'$

ALSO THERE EXISTS A SPACE ${}^{-1}E$ SUCH THAT $\Omega({}^{-1}E) = E$ ($\Omega(X) \equiv$ LOOP SPACE OF X).

– WE HAVE:

$$\begin{tikzcd} \prod_{i=1}^{l} K(\mathbb{Z}_2, n+\dim a_i) \arrow[r, "^{-1}j"] & ^{-1}E \arrow[d] \\ & K(\mathbb{Z}_2, n+1) \arrow[r, "\varphi_1"] & \prod_{i=1}^{l} K(\mathbb{Z}_2, n+1+\dim a_i) \end{tikzcd}$$

E IS AN H-SPACE SO IN $H^*(E)$ A HOPF ALGEBRA, WE MAY TALK ABOUT PRIMITIVE ELEMENTS. THE φ_2 CONSTRUCTED IS NOT PRIMITIVE.

BY A THEOREM OF G. W. WHITEHEAD IT IS SUFFICIENT TO SHOW φ_2 IS NOT A SUSPENSION.

ASSUME THERE EXISTS $\psi \in H^{2n+1}(^{-1}E)$ SUCH THAT $'\psi = \varphi_2$.

THEN $'(^{-1}j^*(\psi)) = j^*(\varphi_2) = \sum_{i=1}^{l} a_i'('\iota_i)$

$= '\left(\sum_{i=1}^{l} a_i'(\iota_i)\right)$.

SUSPENSION IS A MONOMORPHISM IN THIS DIMENSION SO $^{-1}j^*(\psi) = \sum_{i=1}^{l} a_i'(\iota_i)$

NOW $0 = \tau\left(\sum_{i=1}^{l} a_i'(\iota_i)\right) = \sum_{i=1}^{l} a_i' a_i(\iota_{n+1})$

$= Sq^{n+1}(\iota_{n+1}) = \iota_{n+1}^2 \neq 0$ CONTR!

<u>LEMMA</u> $\underline{\Phi}$ IS QUADRATIC.

<u>PROOF</u>

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\bar{x}, \bar{y}"] \arrow[r, "x, y"'] & K(\mathbb{Z}_2, n) \end{tikzcd}$$

LET μ DENOTE THE MULTIPLICATION ON E.

$$\mu : E \times E \to E$$

$$\mu^*(\varphi_2) = \varphi_2 \otimes 1 + 1 \otimes \varphi_2 + p^*(i_n) \otimes p^*(i_n)$$

LIFTING OF $x+y$ IS $\mu(\bar{x} \times \bar{y})\Delta$ (Δ IS DIAGONAL MAP)

THEN $(\mu(\bar{x} \times \bar{y})\Delta)^*\varphi_2 = \Delta^*(\bar{x} \times \bar{y})^*(\varphi_2 \otimes 1 + 1 \otimes \varphi_2 + p^*(i_n) \otimes p^*(i_n))$

$$= \Phi(x) + \Phi(y) + x \cup y$$

THIS CONCLUDES THE PROOF OF THE THEOREM. ▮

IF n IS EVEN ONE CAN CONSTRUCT A SPACE X SUCH THAT $\pi_n(X) = \mathbb{Z}$, $\pi_{2n-1}(X) = \mathbb{Z}$, $[\alpha, \alpha] \neq 0$ (α A GENERATOR OF $\pi_n(X)$) AND $\pi_i(X) = 0$ $i \neq n, 2n-1$.

<u>QUESTION</u> IF n IS ODD, WHAT IS THE MINIMUM NUMBER OF HOMOTOPY GROUPS AN $(n-1)$ CONNECTED SPACE MUST HAVE SUCH THAT $[\alpha, \alpha] \neq 0$ FOR SOME $\alpha \in \pi_n(X)$.

<u>COROLLARY</u> (OF CONSTRUCTION) THERE EXISTS SUCH A SPACE WITH $\ell + 2$ HOMOTOPY GROUPS. ▮

EXAMPLE $\quad n \equiv 1 \mod 4$

$$Sq^{n+1} = Sq^2 Sq^{n-1} + Sq^1 (Sq^{n-1} Sq^1)$$

IMPLIES $\quad \pi_i(E') = \begin{cases} \mathbb{Z}_4 & i = n \\ \mathbb{Z}_2 & i = 2n-2 \\ \mathbb{Z}_2 & i = 2n-1 \end{cases}$

WHERE WE HAVE:

$$\begin{tikzcd}
K(\mathbb{Z}_2, 2n-1) \arrow[r] & E' \arrow[d] & \\
\prod_{i=1}^{l} K(\mathbb{Z}_2, \dim a_i + n-1) \arrow[r] & E \arrow[r, "\varphi_2"] \arrow[d] & K(\mathbb{Z}_2, 2n) \\
& K(\mathbb{Z}_2, 1) \arrow[r] & \prod_{i=1}^{l} K(\mathbb{Z}_2, \dim a_i + n)
\end{tikzcd}$$

A PRODUCT THEOREM FOR 2-MANIFOLDS

BY R. L. FINNEY

WE WILL CONSIDER LOCALLY FINITE SIMPLICIAL COMPLEXES.

HAUPTVERMUTUNG: LET $h : |K_1| \to |K_2|$ BE A HOMEOMORPHISM THEN K_1 AND K_2 ARE COMBINATORIALLY EQUIVALENT, THAT IS HAVE ISOMORPHIC SUBDIVISIONS.

REMARK

C.D. PAPAKYRIAKOPOULOS PROVED THE HAUPTVERMUTUNG FOR $\dim(K_i) = 2$ $i = 1, 2$.

MOISE AND BING PROVED IT FOR K_i 3-MANIFOLDS.

J. MILNOR HAS GIVEN A COUNTER EXAMPLE IN DIMENSION 6. THESE COMPLEXES WERE NOT MANIFOLDS AND THE HAUPTVERMUTUNG IS STILL OPEN FOR K_i MANIFOLDS.

IN TRYING TO PROVE THE HAUPTVERMUTUNG IN DIMENSION 3 I FOUND IT NECESSARY TO ESTABLISH THE FOLLOWING LEMMA:

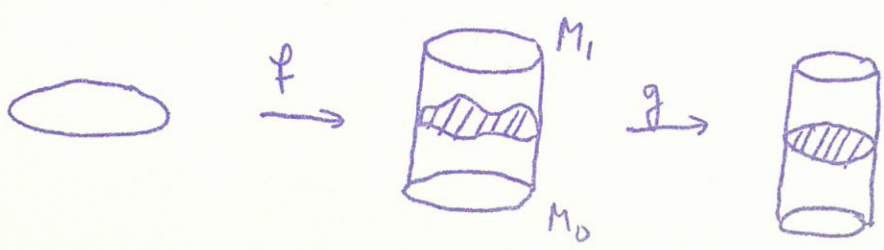
<u>LEMMA</u> LET M BE A COMPACT 2-MANIFOLD. LET $M \times I$ AND M BE TRIANGULATED (THE TRIANGULATION OF $M \times I$ NOT NECESSARILY INDUCED FROM THAT OF M). LET $f: M \to M \times I$ BE A PIECEWISE LINEAR HOMEOMORPHISM (i.e. f IS LINEAR ON EACH SIMPLEX OF SOME SUBDIVISION OF M) SUCH THAT:

1. $f(M)$ SEPARATES $M_0 \equiv M \times 0$ FROM $M_1 \equiv M \times 1$
2. IF $\partial M \neq \emptyset$ THEN $f(\partial M) = f(M) \cap (\partial M \times I)$

THEN THERE EXISTS A PIECEWISE LINEAR HOMEOMORPHISM $g: M \times I$ ONTO $M \times I$ SUCH THAT $g f(M) = M \times \frac{1}{2}$ AND $g|_{M_0 \cup M_1} = 1$

IN ADDITION IF $f(\partial M) = \partial M \times \frac{1}{2}$ THEN $g = 1$ ON $\partial M \times I$. |

FOR M = 2 DISK WE HAVE:



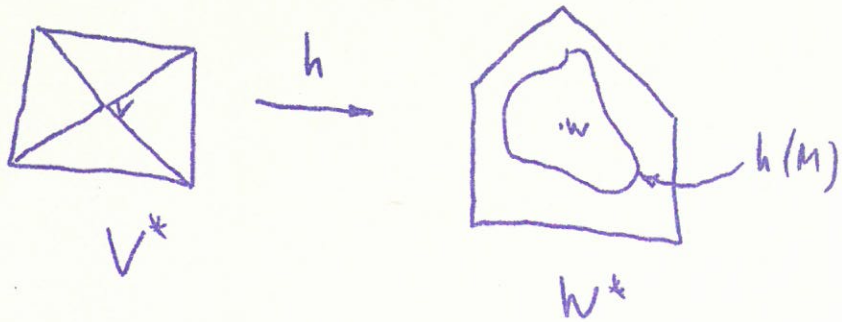
WE APPLY THIS LEMMA IN THE FOLLOWING CASE:

LET $h : |K_1| \to |K_2|$ BE A HOMEOMORPHISM. DIM $K_i = 3$ AND K_1 LOCALLY EUCLIDEAN EVERYWHERE EXCEPT AT THE VERTEX V. WE SHALL CONSTRUCT A COMBINATORIAL EQUIVALENCE. LET $h(v)$ BE THE VERTEX W AND V^* DENOTE THE CLOSED STAR OF V, W^* THE CLOSED STAR OF W. NOW $\partial(V^*)$ AND $\partial(W^*)$ ARE HOMEOMORPHIC COMPACT 2-MANIFOLDS. (1. IT CAN BE SHOWN THAT GIVEN ANY VERTEX $V \in K$ A LOCALLY FINITE SIMPLICIAL COMPLEX THE HOMOLOGY GROUPS OF $\partial(V^*)$ ARE INDEPENDENT OF THE TRIANGULATION. 2. COMPACT 2-MANIFOLDS ARE CHARACTERIZED BY THEIR HOMOLOGY GROUPS)

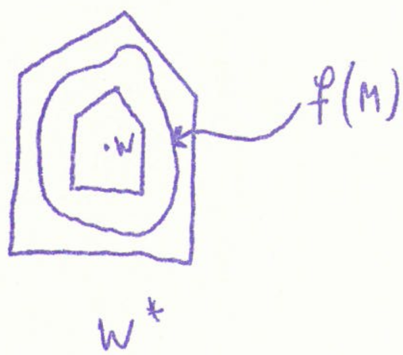
LET $M = \partial V^*$ AND E DENOTE THE SUBCOMPLEX WHICH IS THE COMPLEMENT OF THE OPEN STAR OF V. WE CAN ASSUME $h(M)$ IS INSIDE W^*

BY A THEOREM OF BING THERE IS A PIECEWISE LINEAR HOMEOMORPHISM

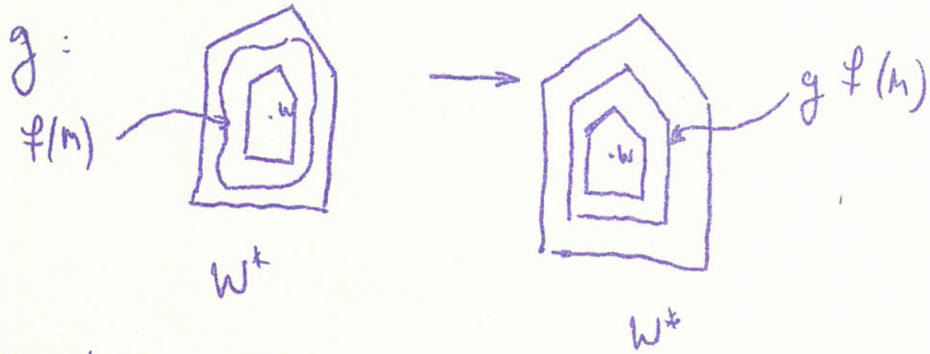
$f : E \to |K_2|$ SUCH THAT f IS POINTWISE NEAR h (WHICH INSURES $f(M)$ IS INSIDE W^*)



BY A RETRACTION OF ∂W^* WE HAVE THE FOLLOWING PICTURE OF THE IMAGE OF f.



APPLYING THE LEMMA WE HAVE A PIECEWISE LINEAR HOMEOMORPHISM



AND WE EXTEND g TO ALL OF K_2 BY LETTING IT BE THE IDENTITY OUTSIDE $\partial(W^*)$.

THUS $gf : E \to K_2$.

SINCE $\underline{|K_2| - gf(E)}$ = CONE OVER $gf(M)$

AND V^* IS THE CONE OVER M

WE HAVE AN OBVIOUS (RADIAL) EXTENSION OF gf WHICH PROVIDES US WITH THE DESIRED COMBINATORIAL EQUIVALENCE. ∎

WE NOW SKETCH A PROOF OF THE LEMMA FOR M AN ORIENTABLE COMPACT 2-MANIFOLD.

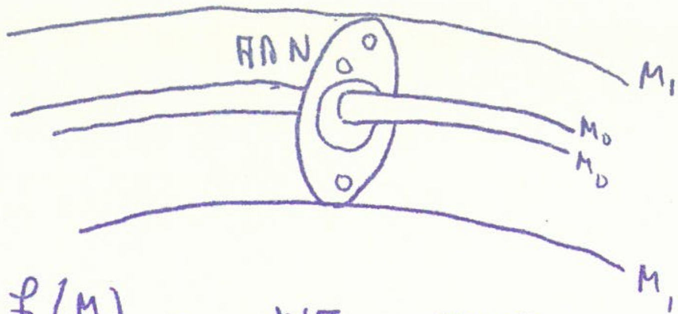
THUS M IS HOMEOMORPHIC TO A 2-SPHERE WITH h HANDLES AND l HOLES. FOR $h=0, l=0$ THE LEMMA IS KNOWN. IT IS ALSO KNOWN FOR $M \times I$ A SOLID CYLINDER I.E. THE CASE $h=0, l=1$.

SO WE ASSUME M IS NEITHER OF THE ABOVE CASES AND WE SHALL REDUCE OURSELVES TO THE CASE $h=0, l=1$.

FIRST IMBED $M \times I$ IN EUCLIDEAN 3-SPACE.

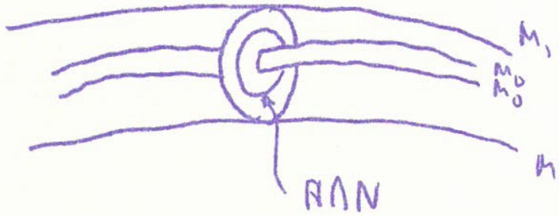
<u>REMOVING HANDLES</u>

LOCALLY LOOKING AT A "THICKENED" HANDLE WE HAVE:



LET $N = f(M)$. WE PASS A PLANE THROUGH THE HANDLE TO GET AN ANNULUS A. CHOOSING THE PLANE SUCH THAT IT MEETS NO VERTICES OF N WE HAVE $N \cap A$ IS A COLLECTION OF SIMPLE CLOSED CURVES.

THE PROCEDURE NOW IS TO MOVE N AROUND SO THAT $A \cap N$ IS JUST A SINGLE SIMPLE CLOSED CURVE (WHICH MUST THEREFORE SEPARATE $A \cap M_0$ FROM $A \cap M_1$) I.E.



WE THEN CUT THE "THICKENED" HANDLE AT THIS PLANE TO GET A MANIFOLD WITH $h-1$ HANDLES AND $\ell+2$ HOLES: I.E.

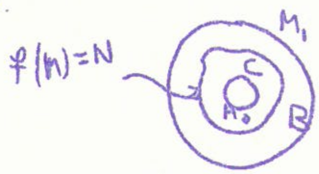
AN INDICATION OF HOW TO MOVE N AROUND IS AS FOLLOWS:

THE PLANE INTERSECTS THE HANDLE IN AN ANNULUS A. $A \cap N$ IS THE DISJOINT UNION OF SIMPLE CLOSED CURVES.

WE NOW USE A LEMMA DUE TO H. GLUCK

<u>LEMMA</u> CONSIDER M A COMPACT 2-MANIFOLD $f: M \to M \times I$ A P.L. HOMEOMORPHISM $f(M)$ SEPARATING M_0 FROM M_1 AND IF $\partial M \neq \emptyset$ THEN $f(\partial M) = f(M) \cap (\partial M \times I)$.

LET C AND B DENOTE THE TWO COMPONENTS OF $M \times I - f(M)$, i.e.



THEN WE HAVE:

$$\begin{tikzcd} & \pi_1(N) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ \pi_1(\bar{B}) \arrow[dr, "\cong"'] & & \pi_1(\bar{C}) \arrow[dl, "\cong"] \\ & \pi_1(\bar{B} \cup \bar{C}) & \end{tikzcd}$$

THE MAPS ALL INDUCED VIA INJECTION MAPS. |

I. NOW LET α BE A SIMPLE CLOSED CURVE OF $A \cap N$ WITH $\alpha \sim 0$ (HOMOTOPIC) IN A SO $\alpha \sim 0$ IN $M \times I$ THEN BY THE LEMMA $\alpha \sim 0$ ON N.

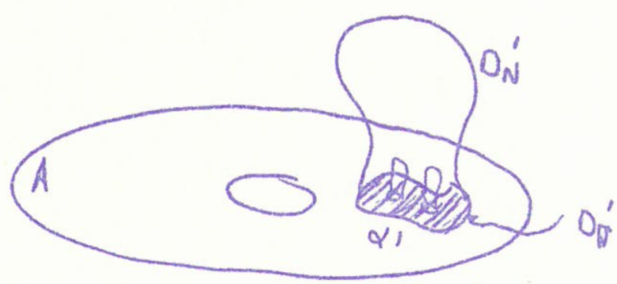
HENCE α BOUNDS A DISK D_N ON N. $D_N \cap A$ IS A DISJOINT UNION OF SIMPLE CLOSED CURVES.

GIVEN A COLLECTION OF DISJOINT SIMPLE CLOSED CURVES ON A DISK WE SAY A CURVE IN THIS COLLECTION IS <u>INNERMOST</u> IF THE DISK IT BOUNDS CONTAINS NO OTHER CURVES OF THE COLLECTION.

PICK AN INNERMOST CURVE α' ON D_N. THEN α' BOUNDS A DISK D_N' ON D_N.

NOW α' BOUNDS A DISK D_A' ON A.

$D_N' \cup D_A'$ IS S^2 SO THE UNION BOUNDS A 3 CELL IN $M \times I$. WE HAVE:



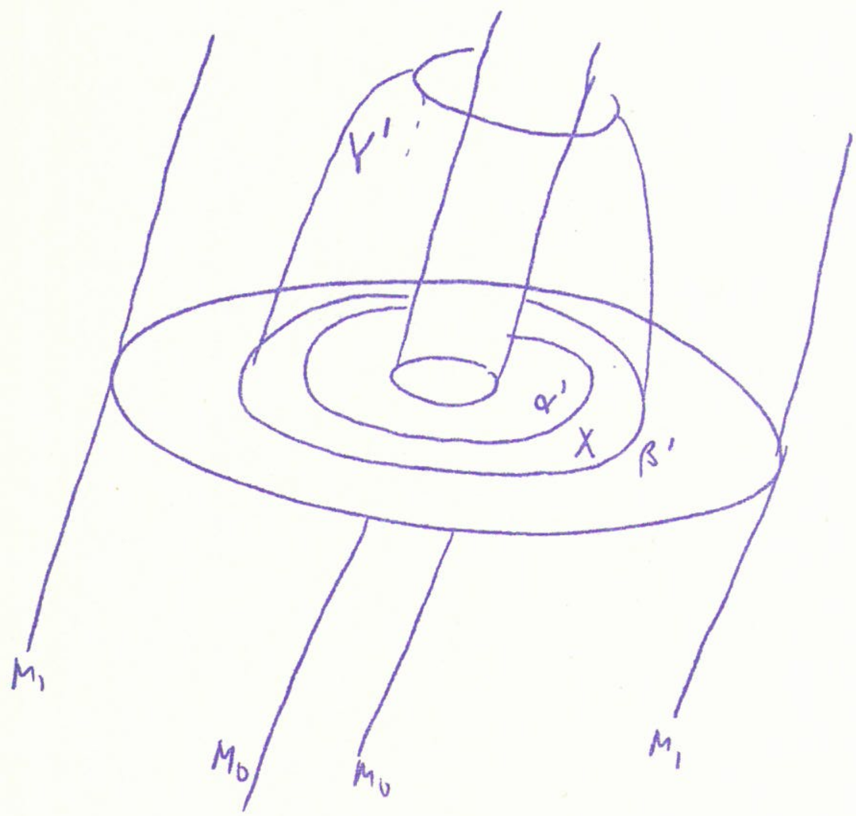
THE IDEA NOW IS TO PULL D_N' THROUGH A WHILE LEAVING $N \cap (A - D_A')$ FIXED, THUS DECREASING THE NUMBER OF INTERSECTIONS OF N WITH A.

II. NOW LET α BE A CURVE NOT NULL HOMOTOPIC IN A. THEN α IS HOMOTOPIC TO $A \cap M_0$ IN $M \times I$ HENCE HOMOTOPIC TO $A \cap M_0$ IN N. BY A THEOREM ANY TWO SUCH α, β BOUND AN ANNULUS Y ON N. $Y \cap A$ IS A COLLECTION OF DISJOINT SIMPLE CLOSED CURVES.

ON AN ANNULUS A PAIR IN THE COLLECTION OF DISJOINT SIMPLE CLOSED CURVES IS <u>INNERMOST</u> IF THE ANNULUS THEY BOUND CONTAINS NO OTHER MEMBERS OF THE COLLECTION.

PICK AN INNERMOST PAIR α', β'. THEY BOUND AN ANNULUS Y' IN N, A SUBANNULUS OF Y. THEN ALSO BOUND AN ANNULUS X IN A.

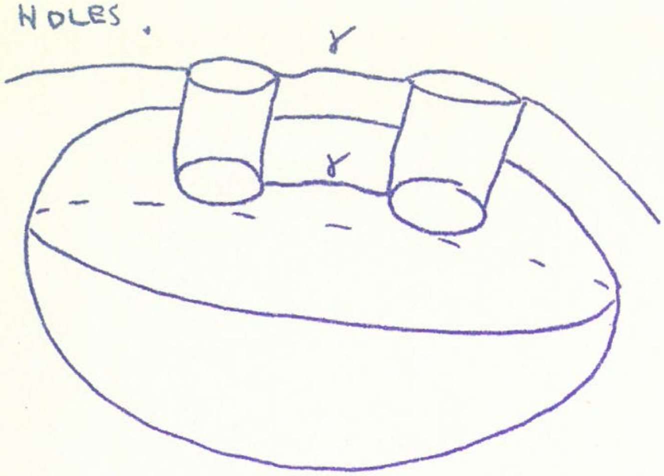
$X \cup Y'$ IS A TORUS WHICH BOUNDS A SOLID TORUS IN $M \times I$. ONE USES THE SOLID TORUS TO PULL Y' THRU A.



FINALLY SINCE N SEPARATES M_0 AND M_1 WE MUST HAVE ONE CURVE LEFT N TO $A \cap M_0$ IN $M \times I$. ONE DEFORMS SUCH THAT $N \cap A = (M \times \frac{1}{2}) \cap A$.

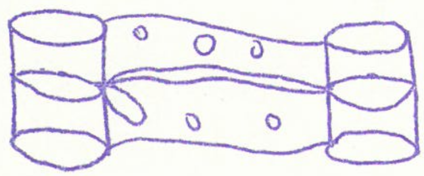
AFTER REMOVING HANDLES WE HAVE LEFT ONLY HOLES.

M×I



WE CONNECT TWO HOLES BY A CURVE γ CHOSEN SUCH THAT $(\gamma \times I)^{\circ} \cap N$ CONTAINS NO VERTEX OF N.

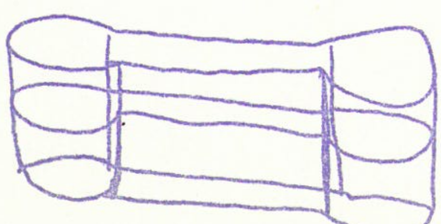
HENCE WE HAVE THE FOLLOWING "LOCAL" PICTURE



BY METHODS SIMILAR TO THOSE ALREADY APPLIED WE CAN DEFORM N SUCH THAT $N \cap (\gamma \times I)$ LOOKS LIKE

WHICH CAN BE STRAIGHTENED TO

. WE THEN SPLIT ALONG THE DISK $\gamma \times I$ TO GET ONE HOLE



WE ULTIMATELY REDUCE OURSELVES TO THE CASE M HAS NO HANDLES AND ONE HOLE. ∎