UNIVERSAL RELATIONS BETWEEN
STIEFEL-WHITNEY CLASSES

BY E. BROWN

Let \( \gamma = (E, B, p) \) denote a real
\( n \)-plane bundle, that is, a fibre bundle
with real \( n \)-dimensional vector space
as fibre and \( GL(n, \mathbb{R}) \) as structural
group.

\[
\begin{array}{ccc}
\gamma & \xrightarrow{p} & V^n \\
\downarrow & & \downarrow GL(n, \mathbb{R}) \\
B & & B
\end{array}
\]

Let \( W_i(\gamma) \in H^i(B; \mathbb{Z}_2) \) be a
Stiefel Whitney class of \( \gamma \).

Origin of \( W_i(\gamma) \):

To \( \gamma \) is associated a bundle
of \( k \) frames \( E^k \).

\[
E^k = \left\{ (v_1, \ldots, v_k) \mid \text{v_1, \ldots, v_k linear indep. vectors of } \mathbb{R}^{n+k} \right\}
\]

\[
\begin{array}{ccc}
E^k & \xrightarrow{v_{m,k}} & V_{m,k} \\
\downarrow & & \downarrow B
\end{array}
\]

Any fibre is \( V_{m,k} \) the Stiefel
manifold of all \( k \)-frames in
\( n \)-space. We note that \( V_{m,k} \)
is \( n-(k+1) \) connected.

The primary obstruction to a
cross section of \( E^k \) is \( W_{n-k+1}(\gamma) \).
$W_{n-k+1} \in H^{n-k+1} (B, \Pi_{n-k} (V, k))$

where $\Pi_{n-k} (V, k)$ is $Z_2$, $Z$, or $Z$ twisted.

If we reduce this group to $Z_2$, we get rid of twisting and obtain a class $\overline{W_{n-k+1}} \in H^{n-k+1} (B, Z_2)$.

Define: $W_l (\eta) = \overline{W^{l-k+1}}$

The Whitney class of the bundle $\eta$.

We look at the universal bundle $\gamma^h \{ E \rightarrow V^h \}$ where $G_h$ is the infinite Grassman manifold of $h$-planes in $R^q$ and $E$ consisting of all pairs $(H, x)$ with $H$ an $h$-plane and $x$ a vector in $H$.

Then

$$H^* (G_h, Z_2) = Z_2 (W_1 (\gamma^h), \ldots, W_h (\gamma^h))$$

where $Z_2 (W_1, \ldots, W_h)$ is the polynomial algebra over $Z_2$ generated by the Whitney classes.

Suppose $M^h$ a differentiable manifold. Look at the tangent bundle $T(M^h)$ and let $\gamma (M^h)$ be the $h$-plane bundle $\{ T(M^h) \rightarrow \gamma (M) \}$.
WE DEFINE: \( W_c(M) = W_c(\gamma(M)) \)

THE PROBLEM IS: WHAT UNIVERSAL RELATIONS HOLD FOR STIEFEL-WHITNEY CLASSES?

LOOK AT THE KERNEL \( I_M \)
OF \( \mathbb{Z}^2 \left( W_c(M^n), W_c(M^n) \right) \to H^*(M, \mathbb{Z}^2) \)
DEFINED BY \( W_c \to W_c(M) \)

Let \( \delta_n = \bigwedge M^I M \). IF \( M \) IS A \( C^0 \) COMPACT MANIFOLD WITH OR WITHOUT BOUNDARY IT IS KNOWN THAT \( \delta_n \neq 0 \).

\( \delta_n \) BECOMES A GRADED POLYNOMIAL ALGEBRA BY DEFINING \( \text{dim. } W_c = c \)

DOLO SHOWED THAT \( (\delta_n)_n \neq 0 \), WHICH COMES FROM THE WU FORMULAS.

WE SHALL SHOW \( (\delta_n)_2 = 0 \) FOR \( 2 < n \), AND SHALL USE THE FOLLOWING THEOREM. LET \( K \) BE A FINITE CW COMPLEX OF \( \text{dim. } = n \), \( \gamma \) A REAL \( m \)-FRAME PLANE BUNDLE OVER \( K \).

**THEOREM** IF \( 2n \leq m \) THERE EXISTS A COMPACT \( C^0 \) MANIFOLD \( M^m \), A CW COMPLEX \( K' \) CONTAINED IN \( M \), AND A HOMOTOPY EQUIVALENCE \( g: K' \to K \) SUCH THAT \( g^*(\gamma) \) IS EQUIVALENT TO \( c^*(\gamma(M^n)) \).
IDEA OF THE PROOF: GIVEN $\gamma$ A REAL $m$-FRAME BUNDLE OVER $K$, TO PUT A MANIFOLD $M^m$ AROUND $K$, $i: K \to M$, SUCH THAT THE INDUCED BUNDLE $i^*(\gamma(M^m))$ IS EQUIVALENT TO $\gamma$. WE CAN DO THIS IF WE ARE WILLING TO CHANGE $K$ UP TO HOMOTOPY TYPE.

**Corollary** $(d_m)_n = 0$ IF $2h \leq m$

LET US LOOK AT $G_m$, THE CLASSIFYING SPACE FOR $GL(m, \mathbb{R})$. LOOK AT THE ASSOCIATED $m$-PLANE BUNDLE

$$3 \left( \begin{array}{c} E \\ G_m \end{array} \right) \to V^m$$

LET $G_n^m$ DENOTE THE $n$-SKELETON OF $G_m$. IT IS THE COMPLEX $K$ TO WHICH WE APPLY THE THEOREM. HENCE WE ASSUME $2h \leq m$. WE GET THE INDUCED BUNDLE $3'$ OVER $G_n^m$

$$\gamma(M) \left( \begin{array}{c} T(M) \\ M \end{array} \right) \to 3' \left( \begin{array}{c} E' \\ G_n^m \end{array} \right) \to V^m$$

AND WE CAN FIND $M$ SUCH THAT WE HAVE A BUNDLE MAP $\lambda$. 
Now \( W_i(M) \cong W_i(Y(M)) \) and Stiefel-Whitney classes behave functorially so
\[
\ell^*(W_i(M)) = W_i(3') \quad \ell^*(W_i(3)) = W_i(3')
\]
Any relations among \( W_i(M), \ldots, W_n(M) \) means we have the same relation in \( W_i(3'), \ldots, W_n(3') \) and hence the same relation holds in \( W_i(3), \ldots, W_n(3) \) for \( \ell^* \) is a monomorphism in dimensions \( \leq n \) (for \( G^k \) is the \( k \)-skeleton of \( G^n \)).

**Defn.** Let \( \ell: X \to Y \subset Z \) the mapping cylinder is the quotient space obtained from \( X \times I \cup Y \) by identifying:
\[
(x, 1) \sim \ell(x) \quad (x, 0) \sim x
\]
Thus \( C_\ell \cong X \sqcup Y \). \( \ell \) is the attaching map.

Let us suppose \( K \subset M \) and \( N = \partial M \neq \emptyset \), \( M \) a compact \( C^\infty \) manifold.

**Defn.** \( M \) is a tubular nbhd. of \( K \) if there exists a map \( F: N \to K \) such that \( CF \) is the underlying space of \( M \) and \( \rho: CF \to R^1 \) by \( \rho(x, t) = t \), \( \rho(y) = 1 \) defines a \( C^\infty \) map on \( M - K \), i.e. \( M - K \subset M = CF \to R^1 \) is \( C^\infty \).
**Examples:**

1. \( K = \text{pt.} \)
   \[ M = C_{k^{+1}} = E, \]
   \[ N = \partial M = S^{k}, \]
   \[ F = \text{constr. map} \]

2. \( K = \text{closed interval} \)

Now let \( K \) be the given finite CW complex of \( \text{dim} = n \). A real \( m \)-plane bundle over \( K \).

To construct \( M^{m} \) which is a tubular nbhd. of \( K \) such that for \( i \): \( K \to M \)

\( i^{*}(\gamma(\gamma_{1})) \sim \gamma \quad z_{n} < m \)

**Method:** Adding handles to \( M \)

(i) Take vertices of \( K \) and around each vertex place a disk of dimension 1.

(ii) Around each 1-cell of \( K \) place the \( M \) of example 2 above.

We have:

The choice in choosing above tubular nbds. allows us to shift the tangent bundle nicely.
WE GIVE THE INDUCTION STEP TO THE PROCESS OF CONSTRUCTING $M^m$. LET $K = L \cup e^{h+1}$. $L$ A CW COMPLEX, $\dim L \leq m$, $\dim (h+1) \leq m$.

Suppose $L \subseteq M^m$, such that $M^m$ is an $m$-DIMENSIONAL TUBULAR NBHD. OF $L$. WE SHALL ENLARGE $M^m$ TO ENCLOSE A CELL.

Adding A HANDLE

$N = \partial M$

$M^m$

$\varphi(S^h)$

$L$

$F : N \to L$ ALONG LINE.

$C_F = M$

Suppose $\varphi : S^h \to L$ IS THE MAP BY WHICH THE CELL $e^{h+1}$ IS ATTACHED TO $L$. WE KNOW $\dim L + h < m$ THEREFORE WE CAN JIGGLE $\varphi$ SO THAT $L$ IS MISSED, i.e. $\varphi \sim \varphi'$ IN $M$ SUCH THAT $\varphi'(S^h) \cap L = \emptyset$.

HENCE THERE EXISTS $g : S^h \to N$ A $C^\infty$ IMBEDDING SO THAT $g \circ \varphi \in M = C_F$. $F_g : S^h \to L$

Certainly $F_g \circ \varphi$ IN $M$ AND $.1. IN L$

WE WANTED TO INCLOSE $K = L \cup e^{h+1}$ VIA $\varphi$. IN FACT WE INCLOSE $K' = L \cup e^{h+1}$ VIA $F_g$. THIS WILL CLEARLY SUFFICE.
WE NOW DRAW PICTURES OF HOW TO PASTE ON A HANDLE.

LET \( g : S^n \to N \) BE OUR IMBEDDING.

SUPPOSE \( g(S^n) \) HAS A TRIVIAL NORMAL BUNDLE IN \( N \), FOR SIMPLICITY.

IN THE EUCLIDEAN PLANE:

[Diagram showing a handle to be attached]

SUPPOSE WE COULD DEFINE A \( C^\infty \) NON-DEGENERATE FCN. ON \( \mathbb{R}^2 \) WITH LEVEL CURVES. (\( C^\infty \) EXCEPT ON \( h = 0 \))

IF \( Q_0 \) DENOTES SET ON WHICH \( h = 0 \)

AND IF \( Q_1 \) " " " " \( h = 1 \)

WE MAP \( Q_0 \to Q_1 \) ALONG THE FLOW LINES. NOW TAKE THE HANDLE AND PASTE IT ON M:
UNIVERSAL RELATIONS BETWEEN
STIEFEL-WHITNEY CLASSES CONT'0.

BY E. BROWN

Today we shall prove:

**Theorem** Let $K$ be a finite CW complex of dim. = $n$, $\gamma$ a real $m$-plane bundle over $K$, then if $2n \leq m$ there exists a compact $C^\infty$ manifold $M$ of dim. = $m$, a CW complex $K'$ contained in $M$, and a homotopy equivalence $\varphi: K' \to K$ such that $\varphi^*(\gamma)$ and $\varphi^*(\gamma(M))$, (where $\iota$ is inclusion map $K' \to M$) are equivalent.

**Proof:** The proof is by induction on the cells of $K$.

We give the inductive step. Let $K = L \cup e^{p+1}$, $\varphi: S^p \to L$ is the attaching map.

$\gamma$ an $m$-plane bundle, $2\dim K \leq m$ and suppose $M$ is a tubular nbhd. of $L$ $N = \partial M \neq \emptyset$, and that we know $\partial L \sim \gamma(M)\vert L$.

**Notation:** If $3$ is a bundle over $X$ $\forall x \in X$, $3\mid_x$ is the induced bundle.
Our object is to obtain $M'$, a tubular nbhd. of $K$ such that $\gamma \sim \gamma(M') \mid K$
(more or less).

Last time we observed that by changing $K$ up to homotopy type we construct
$g : S^p \to \partial M$ which is a
$C^\infty$ embedding s.t. $\gamma = Fg$.
(where $F : \partial M = N \to L$ and $M = C_F$)

(Note: If $K$ is a simplicial complex,
we do not have to change $K$, however
the argument then is more elaborate
than the one we give here.)

The process of going from $M$ to $M'$
is by adding a handle to $M$. Put
a riemannian metric on $N = \partial M \neq \emptyset$
so we can talk about $J = \text{normal}
$bundle of $g(S^p)$ in $\partial M$, and
assume $J$ is trivial. (Actually
this will be shown in the proof)

There are two steps: 1. How to
attach a handle 2. How to
attach a handle so that the
tangent bundle condition is satisfied.
STEP 1. LET \( V_1, V_2, \ldots, V_{m-p-1} \) BE LINK INDEP. CROSS SECTIONS OF \( J \).

LET \( \phi : S^p \times E^{m-p-1} \to \partial M \)

BY \( \phi(x, y_1, \ldots, y_{m-p-1}) = \exp g(x) \sum_{i=1}^{m-p-1} y_i (v_i, g(x)) \)

WHERE \( E^{m-p-1} = \{ y = (y_1, \ldots, y_{m-p-1}) \mid |y| \leq 1 \} \)

WHERE \( |y| = \sqrt{\sum y_i^2} \). WE MAY ASSUME \( \phi \) IS AN EMBEDDING.

WE ATTACH A HANDLE:

THE USUAL WAY TO ATTACH \( E^{p+1} \times E^{m-p-1} \) TO \( M \) IS TO USE \( \phi : S^p \times E^{m-p-1} \) AS AN ATTACHING MAP.

HOWEVER WE MUST ATTACH A HANDLE SO THAT THE RESULT IS A TUBULAR NBD. OF \( K = L \cup E^{p+1} \).

WE USE THE FOLLOWING FCT. \( h \) ON \( E' \times E' \to R' \) AS DESCRIBED LAST TIME.

\[ h(s, r) \]

AND \( h \) IS \( C^\infty \).
Define \( H : E^{p+1} \times E^{m-p-1} \to R \)
by \( H(x, y) = h \left( \frac{1}{x_1}, \frac{1}{y_1} \right) \).

The higher dimensional analogue of the above diagram is:
\[ U = \{ (x, y) \in E^{p+1} \times E^{m-p-1} \mid 0 \leq H(x, y) \leq 1 \} \]

That is, if \( p \geq 0 \) and \( m = 3 \), \( U \) is obtained by revolving the above diagram about the \( z \) axis.

Let \( V = \{ (x, y) \in U \mid 1/x_1 \geq \frac{1}{2} \} \).

We map \( V \) into \( M \) by \( \psi : V \to M \):
\[ \psi(x, y) = \left( \phi \left( \frac{x}{1/x_1}, y \right), z/|x_1|-1 \right) \]
\[ \leq M \times [0, 1] \]
(In \( V \), \( 1/x_1 \) runs from \( \frac{1}{2} \) to 1.)

Now we define \( M' = M' (M, U, V, \ldots, V_{m-p-1}) \)
\[ = M \cup U \text{ with } (x, y) \in U \text{ identified with } \psi(x, y) \in M. \]

\[ \partial M \times [0, 1] \]
\[ M = C_F \]
\[ F : N \to L \]

The handle attached
STEP 2.

LET \( \overline{E}^{p+1} = \{ x \in E^{p+1} \mid |x| \leq \frac{1}{2} \} \)

\( \overline{s}^p = \overline{E}^{p+1} \)

LET \( K' = K - \overline{E}^{p+1} = L \cup \overline{E}^{p+1} - \overline{E}^{p+1} \)

WE IMBED \( K' \) IN \( M \). \( L \) IS ALREADY IN \( M \), WE PUT IN \( \overline{E}^{p+1} - \overline{E}^{p+1} \) BY IDENTIFYING \( x \in \overline{E}^{p+1} - \overline{E}^{p+1} \), i.e. \( |x| \geq \frac{1}{2} \)

WITH \( \left( g \left( \frac{x}{|x|} \right), 2|x|-1 \right) \in \partial M \times (0, 1] \).

NOW \( \gamma(M) |_{\overline{s}^p} = \emptyset + \gamma^p(\overline{s}^p) + \overline{s}' \)

WHERE \( \overline{s}' \) IS A LINE BUNDLE.

SO \( \gamma |_{K'} \sim \gamma(M) |_{K'} \) BECAUSE

THIS HOLDS FOR \( L \) AND \( L \) IS A DEFORMATION RETRACT OF \( K' \).

LET \( \Lambda \) BE THE BUNDLE EQUIVALENCE

\( \Lambda : \gamma |_{K'} \sim \gamma(M) |_{K'} \)
We know $\gamma^p(S^6) + 3^1 \cong 3^2$ is a trivial bundle, being the sum of a tangent bundle of a sphere and a trivial bundle.

Now via $\mu$ we have $\gamma^1 S^p = 3^1 + 3^2$. We seek to extend this splitting to $\gamma^1 E^{p+1}$. We obtain a set of cross sections for $3^2$.

We paste $\mu$ on $M$, $\mu \in E^{p+1} E^{m-p-1}$.

$(x, y) \in \{ (x, y) \mid |x| \leq 1, |y| \leq 1 \}$ and tangential vectors $\frac{\partial}{\partial y_i}$ go into cross sections of $3^2$.

(rotate: about vertical axis. looking down:)}
WE THUS OBTAIN CROSS SECTIONS FOR $\Sigma_2$ AND SO $p+1$ CROSS SECTIONS OF $\gamma |_{\Sigma^p}$, I.E. OF $\gamma(M) |_{\Sigma^p}$ AND EXTEND TO CROSS SECTIONS OF $\gamma |_{\Sigma^{p+1}}$. THE OBSTRUCTION TO THIS EXTENSION LIES IN $\pi_p(V_{m,p+1})$ WHICH IS ZERO SINCE $2(p+1) \leq m$.

LABEL THESE CROSS SECTIONS $w_1, \ldots, w_{p+1}$.

$\gamma(M) |_{\Sigma^p} = 1 + 3z_2$

$\gamma |_{\Sigma^1} = 3_1 + 3z_2$

$\gamma |_{\Sigma^{p+1}} = 3_1 + 3z_2$

(BUNDLE OUGH A DISK.)

LET $u_1, \ldots, u_{m-p-1}$ BE CROSS SECTIONS OF $\Sigma_1 |_{\Sigma^{p+1}}$.

LET $u_i = \lambda_i^{-1} u_i |_{\Sigma^p}$. THE $u_i$'S SERVE AS CROSS SECTIONS OF $\lambda$ SO $\lambda$ IS A TRIVIAL BUNDLE. NOW $\overline{\lambda} : \gamma \to \gamma(n) |_{\Sigma} / k$ IS GIVEN BY:

$\overline{\lambda}(u) = \lambda(u) / |_{\Sigma} : E \pi / k$,

$\overline{\lambda} (\Sigma a_i v_i + \Sigma b_i v_i)$

$= \Sigma v_i / 2 \dot{x_i} + \Sigma b_i / \dot{\gamma_i}, \forall \Sigma E; e |_{\Sigma^{p+1}}$.

WE CAN CHECK THAT THESE PIECE TOGETHER NICELY.
Homotopy Commutativity and Rotation Groups

By Ioan James

Defn.

Let $G$ be a topological group and $G', G''$ subspaces then $G'$ homotopy commutes, $\text{iff}$ the map $C : G' \times G'' \to G$

defined by $C(x, y) = xyx^{-1}y^{-1}$ is null homotopic. If $G' + G''$ are subgroups then $C : (G', e) \to C$

$C : (e, G'') \to C$ so $C$ defines a map $\tilde{C}$ on the smash product

$\tilde{C} : G' \wedge G'' \to G$

$(G' \wedge G'' = \frac{G' \times G''}{(g', e)\cup(e, g'')})$

Then $C$ is null homotopic iff $\tilde{C}$ is null homotopic.

We now define the group $O_m$.

For the reals $O_m$ is the real orthogonal group

" " Complexes " " Unitary group

" " Quaternions " " Symplectic group

We can embed $O_m$ in $O_{m+1}$ by considering $O_m$ as transformations on $m+1$ dimensional vector space over the corresponding field $F$ leaving the last coordinate fixed.

We regard $O_m, O_n \subseteq \mathbb{C}^m, m/n \leq \epsilon$.
We want to find \( m, h, t \) such that \( O_m \) and \( O_h \) homotopy commute in \( O_t \). Does there exist such a \( t \)?

Yes. Let \( t \geq m + h \).

Embed \( O_m \) in \( O_t \) as \( O_t \) (i.e., imbedding matrices).

Now by conjugating with respect to a fixed element in the path component of \( e \) which is homotopically trivial, we shift \( O_h \) to \( O_h' \).

Thus \( O_m, O_h' \) commute in \( O_t \) so \( O_m + O_h' \) homotopy-commutes in \( O_t \).

**Defn.** Let \( \ell(m, h) \) be the smallest integer \( t \) such that \( O_m, O_h \) homotopy-commute in \( O_t \). In general, \( \ell(m, h) \) is different for the reals, complexes, quaternions.

**Theorem.** \( \ell(m, h) = m + h \) except for \( m = n = 1 \) in complex case when \( F \) is complexes or quaternions.

Proof. Let \[ m + n = k + t, \quad t \geq m, n \]
we have:
\[ c : O_m \times O_n \rightarrow O_t \]
for what \( t \) is \( c \) null homotopic?
we know if \( k = 0 \) \( c \) is null homotopic
so assume \( k \neq 1 \).
Again we shift \( O_n \) to \( O_n' \)
by a homotopically trivial mapping.

\[
\begin{array}{|c|c|}
\hline
O_m & O_{m-k} \\
\hline
O_n' & O'_{n-k} \\
\hline
\end{array}
\]

Now \( O_m \times O_{m-k} \rightarrow O_t \)
\( O_n \times O'_{n-k} \rightarrow O_t \)
are both homotopically trivially.
Define \( O_{m,k} = O_m / O_{m-k} \).

Then \( c : O_n \times O_n \rightarrow O_t \)
induces
\( c' : O_{m,k} \times O'_{n,k} \rightarrow O_t = O_{m+n-k} \).

We show \( c' \) indep. of coset representatives.

Let \( x, y \in O_m \) \( xny \) iff \( x = ya \)
\( a \in O_{m-k} \). Let \( z \in O_n' \).

Then \( c : (x,z) \rightarrow xz x^{-1} z^{-1} = ya z a^{-1} y^{-1} z^{-1} \)
but \( a \in O_{m-k} \), \( z \in O_n' \) \( a z a^{-1} = z \)
\( \therefore c(x,z) = c(y,z) \)
Similarly \( c' \) is indep. of coset repr. chosen in \( O_{m,k} \).
Notation: \((X, Y) = \text{homotopy classes of maps } X \to Y\).

Let \(E\) be a fiber space, with base space \(B\) and fiber \(F\), and let \(K\) be a complex, \(\text{sk} K\) its suspension. Then \(F \to E \to B\) induces the exact sequence:

\[
\begin{align*}
\text{sk} (E) \xrightarrow{\pi} \text{sk} (B) \xrightarrow{\Delta} (K, F) \xrightarrow{I} (K, E)
\end{align*}
\]

\(\Delta\) is the transgression operator.

Suppose we let \(K = \bigotimes_{m} \bigotimes_{k} \mathbb{R}^{m+k}\), then \(\text{sk} K = \bigotimes_{m} \bigotimes_{k} \mathbb{R}^{m+k}\)

and \(\mathbb{R}^{m+k+h} \to \mathbb{R}^{m+k} \to \mathbb{R}^{m+k, h}\)

be the fiber space \((F \to E \to B)\)

\[
\bigotimes_{m} \bigotimes_{k} \mathbb{R}^{m+k} \xrightarrow{c'} \bigotimes_{m+k+h} \mathbb{R}^{m+k, h}
\]

\(\{c'\} = \gamma \in (K, F)\)

Assume \(r \geq 1\), claim \(I(\gamma) = 0\).

By exactness therefore there exists \(B \in (\text{sk}, B)\), \(\exists \gamma : \Delta(B) = \gamma\).

If \(c'\) is null homotopic, i.e. \(\gamma = 0\) then by exactness \(\exists \phi \in (SK, E)\)

\(\phi : \Delta(\gamma) = B\) and conversely if such an \(\alpha\) exists then \(c'\) is null homotopic.
\textbf{DEFN.} \textit{The Join} of $A_0$ and $A_1$

Denoted $A_0 \ast A_1$ is $A_0 \times I \times A_1$

with $(x_0, t, x_1) \sim x_t$ for $t = i$

We define a map called the \textit{Intrinsic Join} $S$ $k_0 = O_m^k, k_0 \ast O_h^k \rightarrow O_{m+h}^k, k = B$

by

\[ k \begin{bmatrix} R \cos \theta \\ B \sin \theta \end{bmatrix} \]

$0 \leq \theta \leq \frac{\pi}{2}$

\textbf{Conjecture} we can choose $B$ above such that $B$ contains

the \textit{Intrinsic Join} $I$.

R. Bott proves this for $k = 1$.

When $k = 1$ \textit{Intrinsic Join} is a \textbf{Homeomorphism}.

We now examine the case $k = 1$. We have

\[ O_h \ast O_m \rightarrow O_{m+h-1} \]

\[ \begin{array}{c}
\downarrow P \\
O_{h,1} \times O_{m,1}
\end{array} \]

\[ \text{A comm. diagram} \]

\textbf{Question} when is $C$ null homotopic?

($P$ is smash of proj. maps $O_h \rightarrow O_m/O_{m-1}$

$O_h \rightarrow O_h/O_{k-1}$)

We now choose $k = O_m \ast O_h$
AND THE SAME FIBER SPACE

LET \( L = \Omega_m,1 \times \Omega_n,1 \)

WE HAVE THE FOLLOWING DIAGRAM DEFINING THE MAP \( \tilde{h} \cdot \tilde{h}' = h' \cdot P \)

\[
\begin{align*}
SK &= \Omega_m,1 \times \Omega_n,1 \quad \longrightarrow \quad \Omega_{m+n},1 \\
SP &\quad \downarrow \\
SL &= \Omega_m,1 \times \Omega_n,1
\end{align*}
\]

WHERE \( \tilde{h}' \) IS IN \( (SL,E) \xrightarrow{\Pi'} (SL,B) \xrightarrow{D} (L,F) \rightarrow (L,E) \)

\[
\{\tilde{h}'\} = \{c'\}
\]

WE HAVE THE INDUCED COMM. DIAGRAM

\[
\begin{align*}
(SK,B) &\xrightarrow{\Delta} (K,F) \\
(SP)^* &\quad \downarrow \quad \uparrow P^* \\
(SL,B) &\xrightarrow{D} (L,F) \\
\{\tilde{h}'\} &\xrightarrow{\Delta} \{c'\}
\end{align*}
\]

UNDER WHICH:

\[
\{h \circ P\} = \{h\} \quad \longrightarrow \quad \{c' \circ P\} = \{c\}
\]

SO WE HAVE ESTABLISHED:

**CRITERION** \( \Omega_m,1, \Omega_n,1 \) COMMUTE IN \( \Omega_{m+n-1} \)

IFF \( h \) CAN BE LIFTED TO A MAP

\( g: SK \rightarrow E \), i.e. A MAP \( \Omega_m \times \Omega_n \rightarrow \Omega_{m+n} \)
To prove the thm. of Bott that $l(m, n) = m + n$ it is suff. to prove $O_m, O_n$ do not commute in $O_{m+n-1}$ and this is $\iff$ to showing $l$ does not have a lift.

We prove this for the complex numbers. We consider the projection map $O_2 \to O_{q-1, 1} = \text{sphere of dim } 2^{q-1}$. Suppose we have a map back $O_{q-1, 1} \to O_2$, the composition therefore is a map $S^{2^{q-1}} \to S^{2^{q-1}}$ and we can talk of its degree. It is known there exist maps of degree $l \iff (q-1)!/l$. Thus the smallest possible degree $O_{q-1} \to O_{q-1}$ is $(q-1)!$. Now $O_{m+n-1} \equiv \text{intrinsic join}$ $O_m, O_n, 1$

Suppose $O_m, O_n$ HTY = c. in $O_{m+n-1}$ then there exists $g : O_m \star O_n \to O_{m+n}$ the composite map $O_{m+n-1} \equiv O_m, O_n, 1 \to O_{m+n} \to O_{m+n-1}$ then has degree $(m-1)! \cdot (n-1)! / A$ proper divisor of $(m+n-1)!$ if $m \neq n$. 

7
Thus $O_m, O_n$ don't homotopy commute in $O_{m+n-1}$ except when $m = n = 1$.

For the real orthogonal groups using sphere bundles and Stiefel Whitney classes we obtain the following:

Theorem: Suppose $m + h \neq 4$ or $8$

Then $O_m, O_n$ don't homotopy commute in $O_{m+n-1}$ if $m$ or $n$ is even or $d(m) = d(h)$.

($d(q)$ is the greatest power of $2$ which divides $q^{-1}$.)

Cor. $O_n$ commutes with itself in $O_{2n-1}$ if $2n = 4$ or $8$.

If $m + h = 12$ for pairs $(3, 9), (5, 7)$ we don't know whether $O_m, O_n$ homotopic in $O_{m+n-1}$.

Note: $O_m, O_n$ never homotopic in $O_{m+n-2}$, for apply this to one of $(m, n), (m-1, n), (m, n-1)$. 
DEFN. Let $X$ be a top. space, $G$ a compact Lie group. If $\mathcal{X}$ is the set of homeomorphisms of $X$ with compact open topology an action of $G$ on $X$ is a homomorphism of $G \to \mathcal{X}$ (continuous).

If $X$ is a differentiable manifold and $\mathcal{X}$ is the set of diffeomorphisms of $X$ with the $C^1$ topology then a differentiable action of $G$ on $X$ is a cont. homomorphism of $G \to \mathcal{X}$.

Two actions are equivalent if they differ by an inner automorphism of $\mathcal{X}$.

PROBLEMS

a) If $X$ is a manifold and $G$ acts on $X$ can a diff. structure for $X$ be chosen making the action differentiable?

b) If $X$ is a diff. manifold and $G$ acts on $X$ is the action equiv. to a diff. action?

c) If $X$ is diffeomorphic to $E^n$ or $S^n$ is a diff. action of $G$ on $X$ equiv. to a linr. action?
a) IS INCORRECT IF G = IDENTITY GROUP
   BY KERVAIRE'S WORK. IT IS ALSO
   INCORRECT FOR NON-TRIVIAL G.

b) IS INCORRECT. WE HAVE BOCHNER'S
   THM. IF G ACTS. DIFF. ON X
   AND P IS A STATIONARY POINT THEN
   DIFF. COOKOS. CAN BE CHOOSEN AROUND P
   IN WHICH ACTION IS LINEAR.
   BY AN EXAMPLE OF BING Z_2 ACTS
   ON S^3 \phi. FIXED AT SET IS
   ALEXANDER'S HORNED SPHERE AND
   BY BOCHNER'S THM. THIS ACTION
   IS NOT EQUIV. TO A DIFF. ACTION.

WE SHALL DISCUSS TWO PAPERS:
1. EXAMPLES OF DIFF. GROUP ACTIONS ON
   SPHERES. BY MONTGOMERY AND SAMELSON
2. SOME EXAMPLES OF CONTRACTIBLE OPEN
   3-MANIFOLDS. BY D.R. Mc MILLAN

IN THE PAPER OF MONTGOMERY AND SAMELSON
WE HAVE THE FOLLOWING:

THEOREM LET G BE A NON-TRIVIAL
COMPACT LIE GROUP, CHOOSE H > 0
S.T. G HAS A REAL LINR. REPR.
OF DEG. H LEAVING ONLY THE ORIGIN
FIXED. THEN \exists A SEQ. OF DIFF.
ACTIONS ON S^{n+2} WHICH HAVE NON-
HOMEOMORPHIC STATIONARY SETS. (EACH
SET WILL BE A HOMOLOGY SPHERE BUT
NOT A SPHERE)
Proof: Let $K$ be a finite acyclic 2-complex with $\pi_1(K)$ the doubled icosahedral group. It is a perfect group (i.e., it is isomorphic to its commutator group) given by 2 generators and 2 relations.

Aside: If $\Gamma$ is a perfect group with $k$ generators and $l$ relations, then the 2-complex $K$, induced by $\Gamma$, is acyclic. We have for each gen. a copy of $S^1$ and for each reln. a 2-cell. Then $\pi_1(K) = \Gamma$ so $H_1(K) = \mathbb{Z}^k$.

(Because $\Gamma$ is a perfect group) Now $H_2(K) = \mathbb{Z}_2$ the group of 2-cycles ($B_2 = 0$ because $K$ is 2-complex).

So $H_2(K)$ is free. Thus suppose to show its rank $b_2$ is 0.

Now the Euler characteristic of $K$ is $1 - k + k$ so $1 - k + k = 1 - b_1 + b_2$.

But $H_1(K) = 0$ so $b_1 = 0$ thus $b_2 = 0$.

Thus $K$ is acyclic. We see that our particular $K$ above is acyclic.

Let $K = K_1 \cup \cdots \cup K_l$. 

\[ l +imes \]
Then $K_i$ is an acyclic 2-complex 
$\tilde{\Pi}_i(K_i)$ has minimal set of 2c generators. 
\[ \therefore \tilde{\Pi}_i(K_i) \neq \tilde{\Pi}_i(K_j) \]
For $i \neq j$.

Now imbed $K_i$ in $S^5$, (two complex in 5 space)
and let $T_i$ be a reg. nbhd.
(in the sense of Whitehead) of $K_i$.
$\partial T_i = M_i$, a manifold.

Let $Y_i = S^5 - T_i^0$ ($T_i^0 = \text{interior of } T_i$)

Claim: (a) $Y_i$ is a 5-manifold
with $\partial Y_i = M_i$, clearly.

(b) $\tilde{\Pi}_i(M_i) = \tilde{\Pi}_i(K_i)$

Proof $T_i \cong_k K_i$ (homotopy equivalent)

Because $K_i$ is a deformation retract of $T_i$. By taking a 2-complex out of $T_i$ we do not change $\tilde{\Pi}_i(T_i)$ because we are in 5-space.

So $\tilde{\Pi}_i(T_i) = \tilde{\Pi}_i(T_i - K_i)$

But $M_i$ is a deformation retract of $T_i - K_i$.

Thus $\tilde{\Pi}_i(K_i) = \tilde{\Pi}_i(T_i) = \tilde{\Pi}_i(T_i - K_i) = \tilde{\Pi}_i(M_i)$. 1
(c) \( Y_i \) is simply connected. Given a loop in \( Y_i \) it bounds a disk in \( S^5 \) and we can push the disk off \( K_i \).

We use retraction of \( T_i - K_i \) onto \( M_i \) to push disk into \( Y_i \).

(d) \( Y_i \) is acyclic, \( T_i \sim K_i \),

\[ \Rightarrow \quad T_i \text{ acyclic}, \quad \text{since} \quad Y_i - M_i = S^5 - T_i \]

Then by Alexander duality,

\[ Y_i - M_i \text{ is acyclic} \quad \text{and} \quad Y_i - M_i \sim Y_i \Rightarrow Y_i \text{ acyclic}. \]

So \( Y_i \) is a contractible 5-dimensional manifold with \( D = M_i \).

\( Y_i \) admits a smooth structure.

This is proven directly in the paper of Montgomery and Samelson or can be proven using Munkres' obstruction theory.

CLAIM: \( Y_i \times D^h \cong S^{n+5} \) \( n \geq 10 \)

By a well known fact we need only prove combinatorially equivalent.

Now \( Y_i \times D^h \) is a regular nbd.

of \( Y_i \) in \( R^{n+5} \) and by a result of J.H.C. Whitehead which states
That 2-simply connected $\pi$-complexes
of same homotopy type have isomorphic
regular nbdos. in $R^{n+s}$, we have $Y_i \times D^h \cong D^{n+s}$.

The seq. of boundaries are non-
homeomorphic homology spheres.

$\partial (Y_i \times D^h) = S^{n+h} = M_i \times D^h \cup Y_i \times S^{n-1}$

We define an action on $S^{n+h}$.

Consider a linear action on $D^h$
leaving only the origin fixed
and trivial on $Y_i$. This induces
a diff. action on $\partial (Y_i \times D^h) = S^{n+h}$
for the fixes of $Y_i$. We have:

$F(Y_i \times D^h) = Y_i \times 0 \quad \text{Thus}$

$F(S^{n+h}) = \partial Y_i \times 0 = M_i \times 0$.

And since the $M_i$ are not
homeomorphic (having as we have
shown different fundamental groups)
the thm. is proven.

Turning now to the paper of Dr. McMillan
we have:

Theorem. Let $G$ be a non-trivial compact
lie group choose $h > 2$ s.t. $G$
has a repr. of $O(n)$ leaving only
the origin fixed. Then $f$.
(a) Uncountably many diff. actions of \( G \) on \( \mathbb{R}^{h+3} \) with non-homeomorphic fixed pt. sets.

(b) Uncountably many cont. actions of \( G \) on \( S^{h+3} \) with non-homeo. fixed pt. sets.

**Proof:** McMillan constructs uncountably many non-homeo. open contractible 3-manifolds \( \{ M_x \} \) and proves:

**Lemma:** Each \( M_x \) has large compact sets with connected complement.

We define: \( X \) is simply connected at \( M \) \( \iff \) for each compact \( K \subset X \), \( \exists \) compact \( L \) with \( K \subset L \subset X \) s.t. the induced map \( \pi_1(X - L) \to \pi_1(X - K) \) is trivial.

There is the following:

**Stallings Theorem:** If \( X \) is a contractible open combinatorial \( h \)-manifold simply connected at \( \emptyset \) \( \& \) if \( h \geq 5 \) then \( X \simeq \mathbb{R}^h \) comb. diffeo.

We shall show \( M_x \times \mathbb{R}^h \) simply connected at \( \emptyset \) \& hence by Stallings' thm. \( M_x \times \mathbb{R}^h \simeq \mathbb{R}^{h+3} \) diffeo. for \( h \geq 2 \).
We show $M_x \times R^1$ simply connected at $\infty$. The proof generalizes to $M_x \times R^n$.

By McMillan's lemma, we choose $L' \subseteq M_x$ and $m \in L'L= L' \times [-m, m] = L$ and $M_x - L'$ is connected.

Now $K_2$ is simply connected. For project any loop onto $M_x$ (dashed line) and then contract via contraction of $M_x$.

(M, recall is an open contractible 3 manifold)

Similarly $K_1$ is simply connected and $K_1 \cap K_2 = M_x - L'$ is connected.

$\Rightarrow K_1 \cup K_2$ is simply connected. Thus $M_x \times R^1 - L$ is simply connected.
Now having established:
\[ M \times \mathbb{R}^n \cong \mathbb{R}^{n+3} \]

We define an action by \( G \) on \( \mathbb{R}^n \) leaving origin fixed, and on \( M \) by leaving every pt. fixed. This is a different action of \( G \) on \( \mathbb{R}^{n+3} \) with fixed set \( M_0 \times 0 \), proving a).

To prove b) we note the action above induces a cont. action on \( S^{n+3} \) with fixed pt. set \( M_0 \) (the 1 pt. compactification of \( M_0 \)). Then \( M_0 \neq M_0 \) for \( x \neq b \). For suppose not, then we may assume the homeo. map \( \alpha : \alpha \to \alpha \) and \( M_0 = M_0 \) for \( x \neq b \), contr!

Thus b) is proven.

Remark: The following theorem is true:

Theorem: To within equivalence \( \exists \) at most countably many diff. actions of a compact group on a compact diff. manifold. \( \Box \)
DEFN. Let $A, B$ be left $R$ modules, and $F$ a submodule of $A \times B$. Then the triple $\varphi = (A, B, F)$ is a generalized homomorphism or correspondence, if $F$ has the property that for $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in F$, then $\varphi$ is an ordinary homomorphism.

$D\varphi = \{ a / a \in A \} \exists b \in B \exists (a, b) \in F \}$ called the domain of $\varphi$.

$J\varphi = \{ b / b \in B, \exists (a, b) \in F \}$ is called the indeterminacy of $\varphi$.

$\varphi = (A, B, F)$ induces a homomorphism $A \cong D\varphi \to B / J\varphi$. The relation is 1-1 between correspondences and induced homomorphisms.

Obviously $D\varphi = A$, $J\varphi = 0 \Rightarrow \varphi$ is a homo.

Examples of correspondences

1) Secondary cohomology operations

2) Consider a short exact seq. of chain complexes

\[ 0 \rightarrow C_1 \xrightarrow{i} C_2 \xrightarrow{p} C_3 \rightarrow 0 \]
WHERE $C_\ell$ IS A CHAIN COMPLEX (WE DISREGARD THE GRADING), WE OBTAIN THE EXACT TRIANGLE:

$$
\begin{array}{c}
H_1 \\
\downarrow_c \\
H_2 \quad \cdots \quad \downarrow_c \quad \cdots \\
\end{array}
\xrightarrow{\ell^\chi} H_3
$$

(OR LONG EXACT SEQUENCE). $E_\chi$ THE CONNECTING HOMOMORPHISM. LET $h_\chi : C_\ell \to H_\chi$.

$h_\chi$ IS A CORRESPONDENCE. THEN $E_\chi = h_\chi^{-1} P^{-1} d_2 c^{-1} h_1$ IS A HOMOMORPHISM, ALTHOUGH COMPOSED OF THREE CORRESPONDENCES, $p^{-1}$, $h_\chi^{-1}$, $h_1$.

NOTATION IN $E_\chi$ ABOVE, $h_\chi^{-1}$ ACTS FIRST, ETC.

NOTE: IF $F \subseteq A \times B$, LET $F^\#$ BE THE CORRESPONDING SUBMODULE OF $B \times A$. SO $P^{-1}$ ABOVE IS: $(C_3, C_2, (\text{GRAPH OF } p)^\#)$

AND IN GENERAL IF $\varphi$ IS A CORRESPONDENCE $\varphi = (A, B, F)$ $\varphi^\# = (B, A, F^\#)$

WE SHALL RESERVE THE NOTATION $\varphi^{-1}$ FROM NOW ON TO GENUINE INVERSES.

3) SPECTRAL SEQUENCES:

WE HAVE: $E_1, d_1 r s.t. E_{r+1} \cong H(E_r)$

SUPPOSE $a \in E_1$ IS S.T. $d_1, d_2, d_3, \ldots$ VANISH ON $a$, THEN $d_1$ IS A CORRESPONDENCE ON $E_1$. 
WE WISH NOW TO: 1) GET AN AXIOMATIC SET UP FOR THESE CORRESPONDENCES.
2) ESTABLISH THE FUNDAMENTAL PROPS. OF CORRS.
3) COMPARE OUR SET UP WITH THE NOTION OFABELIAN CATEGORIES.

DEFN. AN INVOLUTION OR I-CATEGORY \( \mathcal{K} \) IS:

I: A CATEGORY WITH OBJECTS \( A, B \) AND CORRESPONDENCES OR MORPHISMS \( \varphi \in \mathcal{K}(A, B) \)

\[ \text{Hom}_\mathcal{K}(A, B) \]

II: IF \( \varphi \in \mathcal{K}(A, B) \) THEN \( \exists \varphi^\# \in \mathcal{K}(B, A) \)
CALLED THE CONVERSE OF \( \varphi \).

III: \( \exists \) A PARTIAL ORDERING "\( \leq \)" AMONG THE CORRESPONDENCES.

Moreover the following BASIC AXIOMS HOLD:

\[(\varphi \circ \gamma)^\# = \gamma^\# \circ \varphi^\#\]

\[\varphi^\# \circ \varphi^\# = \varphi\] (HENCE THE NAME INVOLUTION)

\[\varphi_1 \leq \varphi_2 \Rightarrow \varphi_1 \circ g \leq \varphi_2 \circ g\] FOR ALL \( g \).

\[\varphi_1 \leq \varphi_2 \Rightarrow \varphi_1^\# \leq \varphi_2^\#\]

Let \( S \) DENOTE THE CATEGORY OF SETS

"B" "G" "M" "K" "K" WITH BASE PT.

(NON-ABELIAN) GROUPS (NON-ABELIAN) MODULES
**Additional Axioms:**

\[(k-1)\]: \(\exists \text{ an object } O \text{ s.t. } K(0,0) = \{1\}\)

(i.e. has only one morphism). \(K(0, A)\) contains a smallest element \(\omega_A\) and a largest element \(\Omega_A\).

**Defn.** Let \(\varphi \in K(A, B)\), the in determinacy of \(\varphi\)
denoted \(I \varphi\) is:

\[I \varphi = \omega_B \cdot \varphi \in K(0, B)\]

(Notation: \(A \xrightarrow{\varphi} B \xrightarrow{\psi} C\) then \(A \xrightarrow{\varphi \circ \psi} C\))

**Examples of categories satisfying above requirements are:**

i) \(S\), \(O = \emptyset\) empty set, \(K(\emptyset, A) = \emptyset\)

ii) \(B\), sets with base pt. \(\ast\) \(O = \{\ast\}\)

\(O \times A \cong A\), \(\ast\) base pt. maps to base pt.

\(\Rightarrow K(0, A) = \{\ast\}\) so \(\ast = \omega_A = \text{base pt. of } A\).

And \(I \varphi = \text{image of base pt. of } A\).

iii) \(G\), iv) \(M\).

**Defn.** \(B \varphi = \Omega_B \cdot \varphi \in K(0, B)\) called

the image (bild) of \(\varphi\)

\[K \varphi = \omega_B \cdot \varphi \# \in K(0, A)\]

\[D \varphi = \Omega_B \cdot \varphi \# \in K(0, A)\]

Called the kernel \& domain of \(\varphi\), respectively.
DEFN. \( \phi \) is a **proper morphism** of the category iff \( \mathcal{I}\phi = \omega_B \), \( \mathcal{D}\phi = \pi_B \).

\( \mathcal{C}(K) \) is the subcategory of proper morphisms. Then if \( \phi \in K(A,B) \), \( \gamma \in K(B,C) \) proper morphisms, then \( \gamma \circ \phi \in K(A,C) \) is a proper morphism since
\[
\mathcal{I}(\gamma \circ \phi) = \mathcal{W}_A (\gamma \circ \phi) = (\mathcal{W}_A \phi) \gamma = (\mathcal{I}\phi) \gamma = \mathcal{I} \gamma = \omega_C.
\]

**Question** When is \( \mathcal{I}\phi \circ \mathcal{I}\phi = 1_B \)?

If \( g \mathcal{I}\phi \circ \gamma \) then \( \mathcal{I}\phi \circ \mathcal{W}_C \gamma \mathcal{I}\phi \circ \gamma \)

**Axiom (K-2)**

(a) \( \mathcal{I}\phi \circ \mathcal{I}\gamma \Rightarrow g \mathcal{I}\phi \circ \gamma \)

(True for \( G, M \) not for \( S, B \))

(b) \( \mathcal{B}\phi = \mathcal{B} \gamma \Rightarrow g \mathcal{I}\phi \circ \gamma \)

(True for \( B, S, M \) not for \( S \))

DEFN. A morphism \( \phi \) is called **epi**

\( \mathcal{B}\phi = \pi_B \phi \)

It is called **mono** \( \mathcal{K}\phi = \mathcal{W}_A \phi \)

and **iso** \( \mathcal{E}\phi + \mathcal{M}\phi \).

**Proposition** \( \phi \) is an isomorphism \( \Leftrightarrow \phi \) is an equivalence (i.e. \( \phi \) is \( \phi^{-1} \)).
Axiom (K-3)

If $u \in K(0,A)$ then

(a) There exists a monomorphism $m: U \rightarrow H$ s.t. $bm = u$
(b) There exists an epimorphism $e: A \rightarrow Q$ s.t. $ke = u$.

(a) is true for $(S, B, S, N)$
(b) is true for $(S, B, M)$

**Defn.** $C \in K$ is a **chain object** if there exists $d \in K(C, C)$, an object $H$, and $h: C \rightarrow H$ s.t. $h = wh$ and $Bh = d$.

$Dh = Kd$ and $Kh = Bd$. $H$ is called a **homology object**.

**Note.** Given $C, d \in K(C, C)$, then $Bd$ of $Kd$ is a homology object exists.

Given a short exact seq. of chain objects then the connecting homomorphism is:

$\varepsilon_x = (#_3 \# p \# d_2 \# h_1 \#)$, $\varepsilon_x$ is a proper morphism.

**Defn.** A **spectral sequence** is an object $C$ with morphisms $d_1, d_2, d_3, \ldots$ s.t.

$Bd_r \subset Kd_r$

and $Dd_r = Kd_{r-1}$

$s_r = B d_{r-1}$
Now \( \text{Bd}_r \subset \text{Kd}_r \Rightarrow \text{Homology object} \ (H_r, h_r) \) of \( (C, d_r) \) exists.

We have:

\[
\begin{array}{ccc}
C & \xrightarrow{d_r} & C \\
\downarrow{h_{r-1}} & & \downarrow{h_{r-1}} \\
H_{r-1} & \xrightarrow{d_r} & H_{r-1}
\end{array}
\]

Where \( d_r = h_{r-1} \cdot d_r \cdot h_{r-1}^{-1} \).

We define \( \overline{h_r} : H_{r-1} \to H_r \) as \( \overline{h_r} = h_{r-1} \cdot h_r \).

**Proposition**

(a) \( d_r \) is a proper morphism,

(b) \( (H_r, \overline{h_r}) \) is a homology object of \( (H_{r-1}, d_r) \) \( (= E_r) \).

**Note** (K-3)(a) is only used to prove:

\( \text{Bd} \subset \text{Kd} \Rightarrow \text{a homology object exists.} \)

Thus in the category \( G \) of (non-abelian) groups we have spectral sequences.

Also in a category where (K-3) does not hold but \( C, d_1, d_2, \ldots \) exists with \( \text{Id}_r = \text{Bd}_{r-1} \) and \( \text{Dd}_r = \text{Kd}_{r-1} \) and if a homology object exists then \( (C, d_1, d_2, \ldots) \) is a spectral sequence.

**Defn.** \( (A, C) \) is an exact couple if

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & A \\
\downarrow{\eta} & & \downarrow{\gamma} \\
C & \xrightarrow{g, h} & C
\end{array}
\]

is an exact triangle with \( g, h \) proper morphisms.
DEFINING \( d_i = b(F^r)^{-1} g \) THEN
\( C, d_1, d_2, \ldots, d_r, \ldots \) is a SPECTRAL SEQUENCE.

For:
\[
Bd_r = \sum_c b(F^r) g_c \in \Omega_c g = \Omega \cdot h^\# \\
< Wg^\# \cdot f^{r-1} h^\# = Kd_r \\
\text{so } Bd_r = Kd_r.
\]

\[
Dd_r \leq \sum g^\# \cdot f^{r-1} h^\# = L(\cdot f^{r-1} h^\# \\
= \begin{cases} \\
Wg^\# \cdot f^{r-2} h^\# = Kd_{r-1} & r \geq 1 \\
\end{cases}
\]

THEOREM 1 GIVEM AN ABELIAN CATEGORY \( \Omega \)
THERE EXISTS AN I-CATEGORY \( K \) SATISFYING
\( (K-1) - (K-3) \) SUCH THAT \( \varepsilon(K) \equiv \Omega \).

THEOREM 2 IF \( K, K' \) ARE I-CATEGORIES
SATISFYING \( (K-1) - (K-3) \) AND S.T.
\( \varepsilon(K) \equiv \varepsilon(K') \) THEN \( K \equiv K' \).

THEOREM 3 IF \( K \) IS AN I-CATEGORY SATISFYING
\( (K-1) - (K-3) \) THEN \( \varepsilon(K) \) IS A QUASI-EXACT
ABELIAN CATEGORY WHERE:

DEFN. A CATEGORY \( C \) IS QUASI-EXACT IF:
1) \( \exists \text{ Ob}_A, c(A, B) \)
2) Kernels, Cokernels, Coimages, Images Exist.
IF \( \phi \in c(A, B) \) THE KERNEL OF \( \phi \),
\( \ker \phi \) IS A PAIR \( (U, u) \)
\[ U \rightarrow A \text{ s.t.} \]

1) \( u \) is injective, i.e. \( ku = k'u \Rightarrow x = x' \)

2) \( u \neq = O_{ub} \)

3) Given \( (u', u') \in E \), a unique \( f \) s.t.

Following diagram is commutative:

\[
\begin{array}{ccc}
U & \xrightarrow{u} & A \\
& \downarrow{f} & \downarrow{g} \\
& B & \xleftarrow{O_{ub}} \\
\end{array}
\]

Thus \( \text{ker} f \) is soln. to a universal problem. Similarly one defines image of \( f \), cokernel of \( f \), coimage of \( f \).

3) \( C(R, B) \) has addition, satisfying distributive laws.

**Additional Axioms**

\((k-4)\) \( K(R, B) \) is a lattice; i.e.

\[ g_1, g_2 \in K(R, B) \text{ then } g_1, g_2 \in K(R, C) \]

and \( f \in K(R, B) \Rightarrow f(g_1, g_2) \in f g_1, f g_2 \).

Now \( I \neq \in K_{g_0} \) \( (v = 1, 2) \Rightarrow f(g_1, g_2) \geq f g_1, f g_2 \)

so we have:

\((k-5)\) \( I \neq \in K_{g_1} \)
(K-6) Given $A_1, A_2 \in K \exists g_v : P \to A_v$

$(v=1,2)$ s.t. $g_v$ is a proper morphism

and $g_1 \# g_2$ is the largest el. in $K(A_1, A_2)$

call it $\Omega_{A_1, A_2}$ (we know $\Omega_{A_1, A_2}$

exists, K-6 tells us that it can be

written as $g_1 \# g_2$).

With these three additional axioms holding

for K we can show $E(K)$ is abelian.

**Theorem** $K \to E(K)$ and $\alpha \to K(\alpha)$

are isomorphisms under isomorphisms of categories.

(by thm. 1 given $\Omega \in K$ s.t. $E(K) \equiv \Omega$

and if $K'$ is any category s.t. $E(K') \equiv \Omega$

then by thm. 2 $K \equiv K'$ so the map

$\Omega \to K$ is 1-1 so we write this $K$

as $K(\Omega)$, this is our map $\alpha \to K(\Omega)$

in thm. above.)

Applications to Secondary Cohomology Operations

**Defn.** Suppose $(\alpha, \beta)$ are correspondences s.t. for any top. space

we have:

$$H^*(X) \xrightarrow{\alpha} H^*(X) \xrightarrow{\beta} H^*(X)$$

\(\alpha \beta = 0\)
Then a correspondence $\phi: H^n(X) \rightarrow H^{n+p}(X)$ is a secondary cohomology operation associated with $(\kappa, \beta)$ iff

1) If $\ell: X \rightarrow Y$ is cont.

Then in

$$H^n(X) \xrightarrow{\phi} H^{n+p}(X)$$

$$\ell^* \uparrow \quad \uparrow \ell^*$$

$$H^n(Y) \xrightarrow{\phi} H^{n+p}(Y)$$

we have: $\phi \ell^* \subset \ell^* \phi$

2) If $0 \rightarrow X \xrightarrow{i} Y \xrightarrow{\pi} Y/X = Z$

and $Z$ satisfies homotopy ext. property

Then in

$$H^*(X) \xleftarrow{i^*} H^*(Y) \xrightarrow{\pi^*} H^*(Z) \xleftarrow{\Delta^*} H^*(X)$$

we have: $\ell \phi \Delta = \kappa \pi \beta$
STRUCTURAL STABILITY OF DIFFERENTIAL MAPPINGS

R. Thom

THEOREM

Let \( f : \mathbb{R} \to \mathbb{R} \) be of class \( C^m \)
and s.t. \( f^{(k)}(t) \neq 0 \) on \([a, b]\).
Then there exists \( \psi : u \to t \)

A local diffeo. \( \psi \neq 0 \)
on \([a, b]\) with inverse \( \psi : t \to u \)
s.t. \( \psi(t) = P_s^L(f(t)) \),
where \( P_s^L \) is a polyn. of deg. \( s \).
We may choose \( \psi \)
s.t. \( \psi \) is of class \( C^{m-1} \),
and \( \psi = \text{identity} \)
outside \([a', b']\), \( a' \leq a, b \leq b' \).

To prove this theorem we need:

LEMMA

Let \( f, g \) be real valued fcts. of class \( C^m \)
\( f \) min at 0, \( g \) min. at \( b \) and \( f + g \) have

Same order of contact:

Then there exists a change of variables

Which is a local diffeomorphism \( \psi : u \to t \)
s.t.

Locally \( \phi^{-1}(\psi(u)) = g(u) \).

Proof

It is suff. to prove this for

\( \phi = t^{\frac{1}{m-1}} \).

Now \( g(u) = \int_0^u + L(u) u, L(u) \) class \( C^{m-1} \),

Assume A > 0. Then let \( \psi^{(m-1)} = \left( A^\frac{1}{m-1} u \right)^{\frac{m-1}{m-1}} \left[ 1 + \frac{L(u)}{A} u \right] \)
\[ \gamma = A \left[ u \left( 1 + \frac{f'(u) \cdot u}{1} \right)^{\frac{1}{n-1}} \right] \]

Then \( \psi : u \rightarrow \gamma \) is a local diffeomorphism.

\[ \psi(f(u)) = (\psi(u))^{k-1} = \gamma^{k-1} = \lambda \cdot u^{\frac{k}{k-1}} + f'(u) \cdot u = g(u) \]

Graphically:

\[ f(t_1) = y_1, \quad g(u_1) = y_1 \]

Given \( u_1 \) s.t. \( g(u) = y_1 \), find \( t_1 \) s.t. \( f(t_1) = y_1 \).

Then \( u_1 \rightarrow t_1 \) is the local diffeo.

More generally we do the same with

\[ f \quad g \]

where \( f, g \) have max of same order of contact.

Further if \( f, g \) have max, min in same orientation with same order of contact we always can find a local diffeo. Change of variables.

Now to prove the thm. consider \( k = 1 \)

Then \( f \) has at most 3 critical values.
WE WISH TO FIND A POLYNOMIAL (DASHED LINE) 
\[ P_+ \] WITH MAX. MIN. IN SAME ORIENTATION 
AND SAME ORDER OF CONTACT.

THEN BY LEMMA 1 WE GET A DIFFEOMORPHIC 
CHANGE OF VARIABLES \[ \Psi : u \to t \]
S.T. \[ \Psi(\Psi(u)) = P_+(u) \].

THUS IT IS SUFF. TO EXHIBIT SUCH POLYNOMIALS.
THIS IS DONE IN AMER. MATH. MONTHLY NOV. 57

WE GIVE A TOPOLOGICAL CONSTRUCTION.

LEMMA 2. THERE EXISTS A POLYNOMIAL OF
DEG. \[ \Psi \] WITH THE SAME CRITICAL VALUES AND
SAME ORDER OF CONTACT AS \[ \Psi \].

PROOF. WE CONSIDER THE PROBLEM IN THE
COMPLEX PLANE: \[ \Psi = P_+(x) \].

MAP RIEMANN SPHERES: \( S(x) \to S(\Psi) \)
IS A RAMIFIED COVERING, CRITICAL VALUES OF
\( P_+ \) AND RAMIFICATION PTS.

\[ \xymatrix{ 0 \\
X \ar[r] & Y \ar[r] & \circ \ar[l] ^{\times} & 0
} \]

\( S(\Psi) \) \( (\Psi^{-1}) \) PTS. CYCLIC OF ORG. 2
\( P \) SHEETS COVERING.

BY GLOBAL ISOTOPY CAN SHIFT CRITICAL PTS.
TO A FIXED SET LEAVING \( \Psi \) FIXED.
WE GET $S(x) \rightarrow S(y)$ MUST BE A POLYN.
OF DEG. $k - 1$

FROM AN ALGEBRAIC PT. OF VIEW $k-1$

CRITICAL VALUES OF A POLYNOMIAL OF DEG. $k$
DO NOT UNIQUELY DETERMINE IT, SINCE WE HAVE
TRANSITION BY $x \rightarrow ax + b$. WE NORMALIZE
THE POLYNOMIAL $p_k \cdot x \cdot \prod_{i \neq k} \prod_{\text{roots} = 0}
\Psi(x) = 1$

$p_k = x^k + \cdots + \Psi_k$.

WE NEED TO KNOW IF CRITICAL VALUES ARE
REAL THE CORR. COMPLEX POLYN. HAS REAL
COEFFS. WANT $p_k(x) = \Psi_k(x)$.

WE WANT TO SHOW SYMMETRY CAN BE LIFTED

CASE $k=3$ SO AT MOST $k-1 = 2$ CRITICAL PTS.

COVERING BY 3-SHEETS
(Recall if \( w = z^2 \) we have a 2-fold covering and in \( w \)-plane:

\[
\begin{array}{cc}
\text{Image of} & \text{Image of real } z\text{-axis} \\
\text{imaginary } z\text{-axis} & \\
\end{array}
\]

Counter image of real \( w \)-axis is:

Correspondingly, we have:

\[
\begin{array}{c}
c_1 \\
c_2 \\
c_3 \\
\end{array}
\]

Counter image:

Case \( p = q \):

\[
\begin{array}{c}
(12) \\
(23) \\
(34) \\
\end{array}
\]

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\]

\[
\begin{array}{c}
C_1 \\
C_2 \\
C_3 \\
\end{array}
\]

\[
\begin{array}{c}
\Omega_1 \\
\Omega_2 \\
\Omega_3 \\
\end{array}
\]

\[
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\end{array}
\]

We have:

Counter image:
Claim the corresponding polynomial cannot have real coefficients. For we cannot get an axis of symmetry with respect to one of its lines.

So a polynomial of degree 4 all of whose critical values are real, but not having real coefficients.

However, let us reorder the sheets.

With counter image

So corresponding polynomial has real coefficients.

If critical values involve distinct leaves then we may permute without affecting the topological structure of the covering.
Given now a polynomial of order $k$

Assume all $k-1$ zeros of the derivative are real.

We can always reduce the topological covering to that of a polynomial like:

A Tschebysheff poly. gives this transformation.
By a transformation using a Tschebyscheff polynomial, by permuting only distinct critical values you can get any other topological type of any other fct. of deg. \( f \).

**Defn.** \( f, g : X \to Y \) are of same topological type if \( \exists \) homomorphisms \( h, h' \)

S.t. we have the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{h'} \\
X & \xrightarrow{g} & Y
\end{array}
\]

Consider now \( f, g \) to be our polynomials \( P_k \) of degree \( k \) and assume \( \frac{k-1}{k} \) roots of derivative are all real.

We examine the topological type of \( P_k \)

Consider the object \((1, \ldots, k)\) and a permutation of it \( w \). Then \( w \) may be broken up into \( \frac{k-1}{k-1} \) transpositions \( w = \nu_1 \ldots \nu_{k-1} \)
CALL THIS A PRESENTATION OF W.

TO PRESENTATIONS W, W' ARE EQUIVALENT

IF \( W = \omega_1 \cdot \omega_i \omega_j \cdot \omega_{k-1} \)

\( W' = \omega_1 \cdot \omega_j \omega_i \cdot \omega_{k-1} \)

\( \omega_i, \omega_j \) DISTINCT TRANSPOSITIONS.

FOR ONE PRESENTATION OF A TSCHEGYSHEFF POLYN. ANY OTHER MEMBER OF EQUIV.

CLASS CORR. TO POLYN. OF SAME

TOPOLOGICAL TYPE.
DEFN. A homotopy \( H: \mathbb{X} \times I \to \mathbb{X} \) is a pseudo-isotopy on \( \mathbb{X} \) if
\[
H_\varepsilon \text{ is a homeomorphism onto for all } \varepsilon < 1.
\]
For example, the unit disk in the plane may be shrunk to the origin radially by a pseudo-isotopy on the plane which contracts the disk and which is the identity of a nbd. of the disk.

Given a map \( \varphi: \mathbb{X} \to \mathbb{X} \) there exist a pseudo-isotopy \( H \) on \( \mathbb{X} \) s.t. \( H_1(x) = \varphi(x) \).

Remarks: 1) Bing showed, using the work of Youngs, Floyd and Fort, that for \( \mathbb{X} = S^2 \), then \( \varphi \) nonmonotone (i.e. \( \varphi^{-1}(\text{pt.}) \) is connected) is a suff. condition.
2) This condition is not suff. for \( \mathbb{X} = S^3 \).
3) If \( \mathbb{X} = \Delta \) and \( \varphi \) simplicial then a suff. cond. is \( \varphi^{-1}(\text{pt.}) \) is cellular.

I.e. \( \varphi^{-1}(\text{pt.}) = \bigwedge \{ \text{closed 3-cells } C_e / C_{e+1} \subseteq C_e \} \)
NOTE: \( \circ \) ARE CLEARLY

CELLULAR.

\( S' \) IS NOT CELLULAR.

The topologist's sine curve

\[ \sin \] IS CELLULAR, BUT NOT A CELL.

Borsuk's example of the worm-hole cylinder is cellular but not a cell.

**Theorem** \( X = \text{compact } n\text{-manifold} \)

\( (\emptyset = \emptyset) \) \( \Phi : X \to X \), IF \( \exists \)

A pseudo-isotopy \( H \) on \( X \) s.t.

\( H_1(x) = \Phi(x) \) THEN \( \Phi^{-1}(pt.) \) IS

CELLULAR, i.e. \( \bigwedge \{ \text{closed } n\text{-cells } C_i / C_{i+1} \subseteq C_i \} \}

**Proof** LET \( A = \Phi^{-1}(q) \) TO SHOW

\( A = \bigwedge \{ \text{closed } n\text{-cells } C_i / C_{i+1} \subseteq C_i \} \}

\( \text{as } t \to 1 \) \( H_t(A) \to q \)

(\( \text{diam } H_t(A) \to 0 \))

LET \( B_1 \) BE A BALL ABOUT \( q \).

\( \text{choose } \varepsilon_1 < 1 \) s.t. \( H_{\varepsilon_1}(A) \subseteq B_1 \)

LET \( C_i = H_{\varepsilon_1}^{-1}(B_i) \) A CELL.
WE SEEK $t_2 < 1$ s.t. THE FOLLOWING PICTURE HOLDS:

\[ H^*_2(A) \quad B_1 \quad H^*_2(C_1) \]

\[ H^*_2(B_2) \]

Define $C_2 = H^*_2(B_2)$.

Given $t_i$, $B_i$, $C_i$ to find $t_i$, $B_i$, $C_i$ s.t.

\[ C_i \subseteq C^0, \quad A \subseteq C^0, \quad t_i \to 1 \]

Diam. $B_i \leq 2$ Diam. $H^*_2(A)$.

A SIMILAR PROBLEM: $K$ COMPACT $C$ INTERIOR OF A CLOSED $h$-CELL $Z$. FIND AN $h$-CELL $Z'$ s.t. $Z' \subseteq Z^0$ AND s.t.

\[ K \subseteq Z'^0 \quad \text{AND} \quad \text{Diam.} \quad Z' \leq 2 \text{ Diam.} \quad K \]

EXAMPLES SHOW THAT NO SUCH $Z'$ MAY EXIST, BUT IF ONE REQUIRES ALSO THAT $d(K, Z) \geq 4$ Diam. $K$ AND THAT $Z \subseteq E^h$, $Z \subseteq E^h$

THEN ONE MAY TAKE

\[ Z' = \{ P \mid d(P, \overline{P}) \leq 2 \text{ Diam.} \quad K \} \]
WHERE \( \bar{p} \) IS A FIXED PT. OF \( K \) AND \( d \) IS THE USUAL CARTESIAN METRIC. 1

BACK TO OUR PROOF. LET \( g : B_1 \rightarrow B \), B CLOSED UNIT BALL IN \( \mathbb{E}^n \)

\[
g(H_{t_2}(A)) \quad g_{H_{t_2}(C_1)}
\]

To construct \( B_2 \)

Find \( t_2 \) LARGE ENOUGH SO THAT

\[
H_{t_2}(A) \subset B_1
\]

\[
T \equiv g \left( H_{t_2}(C_1) \cap B_1 \right)
\]

Pick \( t_2 \) s.t.

1) \( H_{t_2}(A) \subset B_1 \),

2) \( d \left( g(H_{t_2}(A)), \text{BDRT} \right) \geq 4 \text{Diam} g_{H_{t_2}(A)} \)

3) \( 1 > t_2 > \frac{t_1 + 1}{2} \)

THE FACT THAT \( p \left( H_{t_2}(A), H_{t_2}(C_1) \right) > \varepsilon > 0 \) (NOTE: DIAM \( H_{t_2}(A) \to 0 \)) ALLOWS US TO

(2). 1) AND 3) GIVE NO TROUBLE. THE GENERAL INDUCTION STEP GOES ALONG THE SAME LINES.
Examples:

1) We construct a map \( \varphi : S^3 \rightarrow S^3 \) which is monotone, but for which there exists a pt. \( q \) s.t. \( \varphi^{-1}(q) \) is a 1-sphere (in particular, \( \varphi^{-1}(q) \) is not cellular).

Consider the two disks \( \Delta_1, \Delta_2 \) in \( S^3 \), intersecting in a line-segment which is a radius of each disk. Let \( D \) be the following decomposition of \( S^3 \):

\[
\begin{array}{c}
\Delta_1 \\
\text{D}
\end{array} \quad \text{d \in D if}
\]

1) \( d \) is a pt. of \( S^3 \) \( = (\Delta_1 \cup \Delta_2) \)
2) \( d = \text{bdy} (\Delta_1) \)
3) \( d = \text{bdy} (\Delta_2) \)
4) \( d \) is one of the uncountably many figure-eights lying in \( (\Delta_1 \cup \Delta_2) \) (see fig.)

This decomposition is due to R.H. Bing.

We show \( D^* \cong S^3 \) where \( D^* \) is the hyperspace of the decomposition \( D \).

Considering the horizontal disk as a deflated balloon we get the following figure.
When it is inflated.

In this way the decomposition \( D \) induces a decomposition \( D' \) of the 3-cell \( C^3 = S^3 - \text{int. (balloon)} \).

The hyperspace \( D'^* \) is homeomorphic to \( D^* \).

We now collapse the balloon against its "vertical axis" to get a decomposition \( D \) of \( S^3 \) whose non-degenerate els. are the uncountably many figure \( D \) shaped arcs suggested in the figure below.

The hyperspace \( D'^* \) is homeomorphic to \( D^* \).

But the els. of the decomposition \( D \) may be collapsed to give the trivial decomposition \( D \) of \( S^3 \).
AND \(D^* \times D^* \simeq S^3\)

THE HOMEOMORPHISMS FIT TOGETHER TO GIVE \(D^* \simeq S^3\).

LET \(\Phi\) BE THE MAP OF \(S^3\) INTO \(D^*\) THAT SENDS EACH PT. OF \(S^3\) TO THE EL. OF \(D\) IN WHICH IT LIES. \(\Phi\) IS A MAP OF \(S^3\) INTO \(S^3\), AND \(\partial \Phi(\Delta_1) = \Phi^{-1}(q)\) FOR SOME \(q \in S^3\).

(2) AN EXAMPLE OF A PSEUDO-ISOTOPY \(H\) ON \(E^2\) FOR WHICH \(H^{-1}(\text{origin})\) IS THE NON-NEGATIVE PORTION OF THE \(y\)-AXIS. IN PARTICULAR, \(H^{-1}(0)\) IS NOT CELLULAR.

\[H = \begin{cases} 
(x, y) / y \geq |x| \\
(x, y) / 0 \leq y \leq |x| \\
(x, y) / y < 0
\end{cases}\]

LET \(A = (x, y) / y \geq |x|\)
\(B = (x, y) / 0 \leq y \leq |x|\)
\(C = (x, y) / y < 0\)

DEFINE \(H_+(x, y) = \begin{cases} 
(1-t)(x, y) + t(0, x) \quad \text{on} \quad A \\
(x-ty, y) \quad \text{on} \quad B \\
(x, y) \quad \text{on} \quad C
\end{cases}\)
An Fomulae of Thom and Wu.

$M$ compact diff $n$-manifold, $\mu \in H^\ast(M; \mathbb{Z}_2)$,

$V_i \in H^\ast(M; \mathbb{Z}_2)$ uniquely defined by $\langle Sq^i x, \mu \rangle = \langle \cup V_i x, \mu \rangle$.

$w_k = \sum_{i+j=k} Sq^i V_j$.

This can be defined in more general situation — do they satisfy same relations?

$H = \frac{\tilde{H}}{\varphi \sim 0}$, $Sq$ 'operators'.

1) $\Delta \varphi = \sum_{i+j=k} Sq^i h \cdot Sq^j k$ or $\Delta a = \sum a' \varphi a''$.

2) $Sq^i h = 0$ if $h \in H^i$; $i < j$.

3) $\exists \mu \in H^u \ast \ast$ $\langle h, \mu \rangle$ is dual pairing of $H^\ast H^\ast \ast \mathbb{Z}_2$.

Define $A \ast$ operate on right by

$\langle h \ast a, \mu \rangle = \langle h, a \mu \rangle$, $h \in H^u \ast$, $a \in A$.

E.g. $e_H$ the unit in $H$, $e_H Sq^i = V_i$ above.

Define words

i) $\varphi$

ii) closed under left + rt. $A$ operation

iii) "c" cup product

iv) "l" linear combination.

If $H$ an algebra above, define $\theta_H : \text{words} \to H$ by

$\theta_H(\varphi) = e_H$, $\theta_H(aW) = a \theta_H(W)$, $\theta_H(Wa) = \theta_H(W) a$.
\[ W = W \Leftrightarrow \Theta_H(W) = \Theta_H(W') \quad \text{all } H, \text{equivalence classes form } V, \text{ which has } +, \text{ A operations. This is object of study.} \]

Th. 1. Use polynomial alg. on \( W, \ldots \), where \( u_i \): defined by \[ u_i = e_i(\zeta(S_q)) \text{, } i = 1, \ldots \]

Th. 2. Given \( N \), integer. Let \( m \) run over monomials \( u_i \) \text{deg } m \leq N. \text{Then } \exists \text{ compact diffeo. manifold } D \text{ values of monomials } u \text{ in } H^*(D; \mathbb{Z}_2) \text{ are linear.} \]

Cor. 3. \( W \in U, \Theta_H(W) = 0 \) if \( H = H^*(D; \mathbb{Z}_2) \), all diffeo. manifolds \( D \Rightarrow W = 0 \).

Prove Th. 2: \( x \in H^N(p^\infty; \mathbb{Z}_2), \alpha \in A^i \), \( i + j + k = 2^{s_0} - 1 \),

then \( \forall \alpha \in A^i \), \( x^\alpha = x^\alpha(\zeta^\alpha) \).

Then \( \forall \alpha \in A^i \), \( x^\alpha \cdot x^\delta = x^\alpha \cdot (\zeta^\alpha \cdot x^\delta) \).

If \( i + j + k = N \), \( \alpha \in A^i \), \( a(x^\alpha \cdot x^\delta) = 0 \).

\[ a(x^\alpha \cdot x^\delta) = 0 = a(x^\alpha \cdot x^\delta + \sum a^n \cdot x^n \cdot (\zeta^\alpha \cdot x^\delta) \quad \text{by induction on } i \]

Thus, \( a(x^\alpha \cdot x^\delta) = 0 \).
Lemma 5. \( n = 2^i - 2 \) \((i \geq 2)\), \( M = \mathbb{P}^n \times \chi H^i(M; \mathbb{Z}_2) \).

\( \varepsilon (\chi (S^n_1)) = 1, \varepsilon i (\chi (S^n_2)) = 0, i > 1. \)

Proof: \( \langle \varepsilon (\chi (S^n_1)) \rangle, \chi^{-n+i} \rangle > 0 \chi (S^n_2) \chi^{-n+i}, \mu > \)

\( \varepsilon (\chi (S^n_1)) \chi^{-n+i} = \chi (S^n_2) \chi^{-n+i} \chi = 0 \) \( i > 0 \) \( \mu H^i (\mathbb{P}^n, \mathbb{Z}_2) \)

Then \( \chi (\chi (S^n_1) \chi^{-n+i} = x^{-n+i} S^n_1 x = 0 \) \( i > 0 \) \( \chi H^i (\mathbb{P}^n, \mathbb{Z}_2) \)

Lemma 6. \( \varepsilon a = \sum \varepsilon M' \varepsilon a' \otimes \varepsilon M'' a'' \)

Proof: \( \varepsilon a \chi^{-i} \otimes \chi^b = \sum \varepsilon M' \varepsilon a' \chi^{-i} \otimes \varepsilon M'' \varepsilon a'' \chi^b \)

\( \varepsilon M' \varepsilon a' \chi^{-i} \otimes \varepsilon M'' \varepsilon a'' \chi^b = \sum \varepsilon M' \varepsilon a' \chi^{-i} \otimes \varepsilon M'' \varepsilon a'' \chi^b \)

Lemma 7. \( D = \frac{N}{n} \prod \mathbb{P}^n (i) \quad n = 2^i - 2 \geq N \)

\( \varepsilon (\chi (S^n_1)) = \varepsilon (S^n_1) \text{ elementary symmetric function } \varepsilon X_1 \cdots X_N. \)

Proof: \( \varepsilon S^n_1 = \sum \varepsilon S^n_j \otimes S^n_j \)

\( \Delta \chi (S^n_1) = \sum \chi S^n_j \otimes \chi S^n_j, \quad \Delta \chi (S^n_1) = \sum \chi S^n_j \otimes \chi S^n_j \)

\( \varepsilon \chi (S^n_1) = \sum \varepsilon (S^n_j \otimes \chi S^n_j \otimes \chi S^n_j \cdots \chi S^n_j) \)

\( \prod \varepsilon (S^n_j \otimes \chi S^n_j \cdots \chi S^n_j) \)

\( \varepsilon \chi (S^n_1) = \sum \varepsilon (X_1 \cdots X_i) \)

\( \therefore \) \( H_2 \) as monomials in elementary sym. functions are independent in \( H^* (D) \).
To prove Th 1, enough to show \( U \) is gen. (mult.) by \( v \).

**Lemma 8.** \( d > 0, h \in H(ueU, u = Eb) \) then
\[
E_a h = ah + \sum_{\alpha > 0} (a' h) a'' + ha
\]

**Proof:** \( h \in H, h \in A \), \( h \in H \)

\[
E_a h - h = a(h^2h) = ah - h + \sum a'h a'' h + h' ah
\]

\[
= ah - h + \sum (a'h) a'' h + <ha, h> dl h.
\]

**Lemma 9.** \( U \) is mult. gen. by \( E_a, a \in A \).

**Proof:** To prove \( a \in A \), \( W = \text{poly} \in Eb \), the \( aW + Wa \)-axes.

Induction: \( d > 0, 0 \in \mathbb{N} \).

Start with: \( a(Eb) = E_a \cdot Eb + \sum (a' Eb) a'' + (Eb) a \) (Lemma 8)

But \( (Eb) a = E(ba) \). By induction \( a(Eb) \in \text{poly} \in eb' s \).

If \( W \) is, then \( aW \) is in terms of \( c(Eb) \) with \( d > c = h \)

\( Wa = (Ea) \cdot W - aW - \sum (a' W) a'' \) by Lemma 8.

\( 0 \in \mathbb{N} \).

Let \( I(U) = \sum u^d, D(U) = \text{decomposable alts., i.e.} \)

\[
u = \sum u^d, u, u'' \in I(U)
\]

**Lemma 10.** \( u \in D(U), a \in A \Rightarrow u a \in D(U)\)

**Proof by induction** Lemma 8

\[
E_a u = a u + \sum (a' u) a'' + u a
\]

\( \text{by induction} \)
Lemma 11. If $d \equiv b \pmod{D(0)}$

$$a(\varepsilon b) = \varepsilon(b \chi(a)) \pmod{D(0)}$$

Proof by induction:

$$a(\varepsilon b) = \varepsilon a \cdot \varepsilon b - \varepsilon(ba) - \sum_{\varepsilon D} (a \cdot \varepsilon b) a''$$ by lemma 8.

$$\sum_{\varepsilon D} (b \chi(a')a'') \pmod{D} \text{ by 10}$$

$$\therefore a(\varepsilon b) = -\varepsilon(ba) - \sum \varepsilon(b \chi(a')a'') \pmod{D}$$

$$\therefore a(\varepsilon b) = \varepsilon(b \chi(a))$$

Lemma 12. $I(0)/D(0)$ is spanned by $\varepsilon_i$, $i \geq 0$.

Proof. $I(0)/D(0)$ is spanned by $\varepsilon_0, a$ spanning set for $I(A)$.

$$S = \sum_0 \varepsilon_i s_i, \ldots s_i^{i+1}, i \geq 2, i \neq 2i, 2i+1 \geq 0.$$ spans $I(A)$.

$X(S)$ spans $I(A)$ also. $X(S)$ contains $\chi(S_i)$. Every other element of $X(S)$ is a $\chi(S_i)$ with $d \leq j$. But $\varepsilon d \cdot (\varepsilon d) = 0$.

$e \cdot \chi(S_i) \in D(0)$ by lemma 11, $\therefore I(0)/D(0)$ spanned by $\varepsilon_i$.

$\therefore \text{the } 1.$
Morton Brown's Proof of the Generalized
Schoenflies Theorem

By A. Wasserman

1960, p. 74.

We shall prove:

Generalized Schoenflies Theorem:

Let \( h \) be a homeomorphic imbedding
of \( S^{n-1} \times I \) into \( S^n \). Then the
closure of either complementary
domain of \( h(S^{n-1} \times \frac{1}{2}) \) is an \( n \)-cell.

We first give some defns. and prove
some preliminary theorems.

Defns.

1. If \( Q \) is an \( n \)-cell \( Q \), denotes
its boundary and \( Q^0 \) denotes its interior.

2. \( I \) is unit interval \([0,1]\).

3. If \( f : X \rightarrow Y \) (cont.) then an
inverse set of \( f \) is a set \( M \subset X \)
which contains more than one pt.

And such that

For some \( y \in f(X) \) \( M = f^{-1}(y) \).

(\( so \ Y \) is \( T_1 \), \( \Rightarrow \) \( M \) is closed)

4. A set \( M \) is cellular in an
n-DIMENSIONAL COMPACT METRIC SPACE $S^n$

IF THERE EXIST $n$-CELLS $Q_1, Q_2, \ldots$ IN $S$

SUCH THAT $Q_{i+1} \subset Q_i$ AND $\bigcap Q_i = M$

PRELIMINARY THEOREMS

THEOREM 0 LET $Q$ BE AN $n$-CELL AND LET

$f : Q \to S^n$ AND SUPPOSE INVERSE SETS

ARE IN $Q^0$ THEN $f(Q)$ IS THE UNION OF $f(Q)$

AND ONE OF ITS COMPLEMENTARY DOMAINS.

PROOF USE METHODS SUCH AS THOSE IN

HUREWICZ AND WALLMAN'S BOOK (P. 97).

THEOREM 1 LET $Q$ BE AN $n$-CELL. SUPPOSE

$M$ IS A CELLULAR SUBSET OF $Q^0$ THEN

THERE IS A MAP $f : Q \to Q$ SUCH THAT

$f | Q = I$ AND $M$ IS THE

ONLY INVERSE SET UNDER $f$.

PROOF

WE WANT A MAP $h$ CONTRACTING $M$ TO A PT.

AND $I$ ON $Q$. WE DO THIS

BY CONTRACTIONS ON THE $Q_i$ (SINCE $M$ IS

CELLULAR). $h_i$ HOMEOM. OF $Q_i$ I.D. ON

BOUNDARY AND SUCH THAT $O(h_i) \leq 1$
WE DEFINE $h_{\partial Q}$ to be $h_{i}$ on $Q - Q$, 
and such that $\text{diam } h_{\partial Q} (Q_{i}) < \frac{1}{i^2}$. 

LET $f = \lim h_{i}$.

**THEOREM 2** LET $S$ BE A TOPOLOGICAL $h$-1 
SPHERE IN $S^n$ AND LET $D$ BE ONE OF ITS 
COMPLEMENTARY DOMAINS. SUPPOSE $f : D \to E^n$ 
AN $h$-CELL SUCH THAT THE ONLY INVERSE 
SET OF $f$ IS A CELLULAR SUBSET $M$ 
OF $D$. THEN $D$ IS AN $h$-CELL.

**PROOF**

LET $\bar{D}$ BE IDENTIFICATION MAP $\bar{D} \to \bar{D}/M$ 
THEN $\bar{D}$ INDUCES A CONT. MAP 
$f' : \bar{D}/M \to E^n$, $f'$ IS 1-1, CONT. 
$\bar{D}/M$ IS COMPACT SO $f'$ IS A HOMEOMORPHISM.

THUS SUFF. TO SHOW $D$ NORM. WRT $\bar{D}/M$ 

LET $Q$ BE AN 
$h$-CELL IN $D$ SUCH 
THAT $M \subset Q$.

THEN $M$ IS CELLULAR IN $Q$. BY THM. 1

THERE EXISTS $g : Q \to Q$, $g|_{Q} = \text{IDENTITY}$ 
AND THE ONLY INVERSE SET UNDER $g$ IS $M$.

WE DEFINE $g' : \bar{D} \to \bar{D}$ BY:
\begin{align*}
g' \circ q = g \circ q & \quad \text{identity} \\
\text{Then } (pg') & : \overline{D} \to \overline{D}/\kappa \\
\text{is a homeomorphism.}
\end{align*}

\textbf{Theorem 3} \quad \text{Let } Q \text{ be an } h\text{-cell}

\text{and suppose } f \text{ maps } Q \text{ into } \mathbb{S}^n.

\text{Suppose also that } M \subset Q^0 \text{ is the only inverse set under } f, \text{ then } M

\text{is cellular in } Q.

\textbf{Proof} \quad \text{Let } V = \{ x \mid x \in Q, d(x, M) < \frac{1}{\epsilon} \}.

\text{To construct sets: } Q_1 V_1, Q_{l+1} V_{l+1} \cap Q_l.

\text{However it is sufficient to construct}

\text{for any open set } U (V_{l+1} \cap Q_l)

\text{containing } M, U \text{ in interior of an } h\text{-cell}

(Q_l) \text{ an } h\text{-cell } Q' (Q_{l+1}) \text{ such that } M \subset Q' \subset (Q_{l+1}).

\text{Now by Thm. 0 } f(Q) = f(Q) \cup D

\text{where } D \text{ is a complementary domain of } f(Q).

\text{Let } U \text{ be an open set in } Q^0 \cup U \cap M.
The idea here is to construct a homeomorphism \( g : Q \to Q \) which pulls \( Q \) inside \( U \) and is the identity on an open set \( V \supset M \). We may do this on \( Q \) since it can be done in \( S^h \). More precisely, \( M \subset U \subset Q \), \( U \) open implies \( f(U) \) open in \( D \). Let \( V' \) be a nbhd of \( f(M) \), \( V' \subset f(U) \).

Let \( h \) be a homeomorphism \( S^h \to S^h \) which pulls \( D \) inside \( f(U) \) and is the identity on \( V' \). Let \( V = f^{-1}(V') \).

Now \( g(x) = \begin{cases} x & x \in M \\ f^{-1}h^{-1}f(x) & x \notin M \end{cases} \)

This is a homeomorphism which is the desired one. \( g(Q) = Q' \) is an \( h \)-cell and \( M \subset V = g(V) \subset Q'^0 \).

**Theorem 4** Let \( f : S^h \to S^h \) with precisely two inverse sets \( A \) and \( B \). Then \( A \) and \( B \) are cellular in \( S^h \).
Proof: Let \( d \) be an \( n-1 \) sphere in \( S^n - (A \cup B) \) each of whose complementary domains has an \( n \)-cell for its closure. (If \( d \) separates \( \mathbb{R}^n \) and \( B \) Thm. 4 is immediate from Thm. 3.)

Let \( Q \) be the \( n \)-cell whose boundary is \( d \) and which contains \( A \cup B \) in its interior.

\[
S^n \quad \xrightarrow{\phi} \quad S^n
\]

Let \( \phi(A) = a, \, \phi(B) = b. \)

By Thm. 0 \( \phi(Q) = \phi(Q) \cup D, \quad D = \phi(a, b) \).

Let \( U \) be an open subset of \( D \setminus a, \) and \( U \not
\phi. \) Then let \( h \) be a homeomorphism \( S^n \to S^n \) which pulls \( \phi(Q) \) inside \( U \) and is the identity on a small neighborhood \( V \) of \( a \).

Let \( g : Q \to \mathbb{QCS}^n \) be defined as:

\[
g(x) = \begin{cases} x \in A & \text{if } \phi(x) \\ \phi^{-1} \phi(x) \in A & \text{if } x \in A. \end{cases}
\]

Thus, the only inverse set of \( g \) is \( B \) so by Thm. 3 \( B \) is cellular in \( Q \) so cellular in \( S^n. \)
Similarly $A$ is cellular in $S^n$.

We now prove:

**Generalized Schoenflies THM.**

Let $h$ be a homeomorphic imbedding of $S^{n-1} \times I$ into $S^n$. Then the closure of either complementary domain of $h(S^{n-1} \times \frac{1}{2})$ is an $n$-cell.

**Proof.**

$S^n = h(S^{n-1} \times 1) = \mathcal{N} \cup \mathcal{A}'$

Let $h(S^{n-1} \times 0) \in \mathcal{A}'$

$S^n = h(S^{n-1} \times 0) = \mathcal{B} \cup \mathcal{B}'$

Let $h(S^{n-1} \times 1) \in \mathcal{B}'$

Thus $S^n = h(S^{n-1} \times I) \cup \mathcal{N} \cup \mathcal{B}$.

Let $\varphi : S^n \to S^n$, $\varphi : \overline{N} \to \overline{N.P.}$, $\varphi : \overline{B} \to \overline{S.P.}$

And $\varphi : h(S^{n-1} \times \frac{1}{2})$ onto equator with only $\overline{H}$ and $\overline{B}$ as involutive sets.

Let $D_A, D_B$ be complementary domains.
OF $h(S^{n-1} \times \frac{1}{2})$ WHICH CONTAIN A, B RESPECTIVELY, BY THM. 4
A, B ARE CELLULAR IN $S^n$
HENCE CELLULAR IN $D_A, D_B$
RESPECTIVELY. THEN BY THM. 2
$\bar{D}_A, \bar{D}_B$ ARE $n$-CELLS.
Seminar of Professor G. Whitehead

The Generalized Bar Construction and The Lower Central Series for Free Group Complexes.
by E. Curtis

1. Semi-simplicial object

Definition. A semi-simplicial (s.s.) object over the category \( \mathcal{G} \) shall be a contravariant functor \( X \) from the category \( \mathcal{G} \) of finite ordered sets (and monotone maps) to \( \mathcal{G} \).

The objects in \( \mathcal{G} \) are the sets \( \{ m_0, m_1, \ldots, m_k \} \). A monotone map \( \alpha : \{ m_0, m_1, \ldots, m_k \} \rightarrow \{ m_0, m_1, \ldots, m_j \} \) must satisfy

\[ \alpha(i) \leq \alpha(j) \quad \text{if} \quad i \leq j. \]

\( X[\mathcal{G}] \) is usually denoted by \( X_{\mathcal{G}} \) and is called the (set of) m-simplices of \( X \).

S.s. sets, s.s. monoids, s.s. groups are examples of s.s. objects; this last is the term used when the category \( \mathcal{G} \) is not specified.

For example, if \( A \) is a topological space, \( S(A) \), the Eilenberg total singular complex of \( A \) is a s.s. set. \( C_S(A) \) is a s.s. abelian group, formerly referred to as an FD-module.
Certain maps \( \varepsilon_i : [q] \to [q+i] \) namely

\[ \varepsilon_i(y) = y, \quad y < i \]
\[ \varepsilon_i(y) = y+1, \quad y \geq i \]

and \( \eta_i : [q+i] \to [q] \) where \( \eta_i(y) = y, \quad y \leq i \)
\[ \eta_i(y) = y-i, \quad y > i \]

serve as generators for the maps in \( \partial \).

\( X \varepsilon_i \) is called \( \partial_i \); the \( i^\text{th} \) face operator.
\( X \eta_i \) is called \( s_i \); the \( i^\text{th} \) degeneracy.

The usual semi-simplicial relations are all consequences of the relations among the \( \varepsilon_i, \eta_i \).

**Def.** A s.s. object with base point is a pair \((X, b)\), where \( X \) is an s.s. object, and \( b = b_0, s_0 b_0, \ldots, s_{n-1} \ldots s_1 s_0 b_0, \ldots \) is the sub-s.s. set of a point \( b \in X_0 \). When the base point is unspecified, the notation \((X, *)\) may be used.

**Def.** A s.s. object \( X \) is said to satisfy the extension condition if, given \( x_0, \ldots, x_k, \ldots x_{q+1} \in X_q \), such that \( \partial_j x_j = \partial_{j-1} x_{j+1} \) for \( 1 < j < q+1 \), then there is an \( x \in X_{q+1} \) with \( \partial_i x = x_i \), all \( i \neq k \).
For example, $S(A)$ satisfies the extension condition.

S.s. objects satisfying the extension condition allow a definition of homotopy, see [5], and the $\pi_n(X,x)$ are defined: for $n \geq 1$, they are groups, and for $n \geq 2$, abelian groups.

2. S.s. groups

Lemma (Moore [6]) If $G$ is a s.s. group, then it satisfies the extension condition.

Define $$(N_6)_q = \bigcap \{(\ker d)_i : G_i \to G_{i-1}\}
\quad \text{for } q = 0
$$

and differential operator on $N_6$ given by

$$d = j(0) = j_0 : (N_6)_q \to (N_6)_{q-1}
$$

Then image $j(0)$ is a normal subgroup of $\ker j(0)$ and the quotients $\frac{\ker d(0)}{\im j(0)}$ are defined to be the homology of $N_6$, $H_q(N_6)$. Then it turns out that $H_q(N_6) \cong \overline{Hq}(G,e)$ ($e = \text{identity of } G$ in every dimension, always taken as the base point of a c.s. group).
Def. A map $p : X \to B$ (i.e. $p : X_q \to B_q$ commuting with the face and degeneracy operators) is called a fibre map if given $y \in B_{q+1}$ and $x_0, \ldots, x_k, x_{q+1} \in X_q$ with $p(x) = y$ and $p(x_i) = \eta_{q+1}^{-1} x_i$, $(q, s + k + 1)$, then there is an $x \in X_{q+1}$ such that $p(x) = y$, and $p(x) = x$. The fibre $F_{q+1}$ is given by

$$F_n = p^{-1}(b_n),$$

$b$ a base point for $B$.

Then there is the usual exact sequence:

$$\to \pi_n(B_{q+1}) \to \pi_n(F_{q+1}) \to \pi_n(X_{q+1}) \to \pi_{n-1}(B_{q+1}) \to \cdots$$

If $p : B \to B$ is a surjection of s.s. groups, then $p$ is a fibre map.

3. Kan's Construction $6X$

For each s.s. set with base point $(X,b)$, Kan [55] has defined a s.s. group $6X = 6(X,b)$ which is free in every dimension, and which serves as a (s.s. version of) loop space for $(X,b)$. 
Let $\beta_2 G = [G, G]$ the commutator subgroup.

Then as proven in [5],

$$\pi_{n-1}(G \times \{e\}, e) \cong H_n(X)$$

and the projection $p : G \times \{e\} \rightarrow G \times \{e\}$ induces the Hurewicz homomorphism $h$:

$$\pi_n(X, e) \cong \pi_{n-1}(G, e) \quad \xrightarrow{h} \quad \pi_{n-1}(G \times \{e\}, e)$$

The Hurewicz theorem as given in [5] involves showing that if $G$ is a s.s. free group (e.g. $G = G \times G$), then $\beta_2 G$ has connectivity one higher than $G$.

4. The Lower Central Series.

For any group $G$, let $F_1 G = G$

and $F_r G$ defined inductively by $F_r G = [F_{r-1} G, G]$, then

$$G > \beta_2 G > \cdots > \beta_r G > \cdots$$

is called the lower central series for $G$. 
$\Gamma_r$ will be considered as a functor from $G$ (the category of groups) to itself. If $\mathcal{G}$ is a s.s. group (i.e., $\mathcal{G}$ is a contravariant functor from $\Omega$ to $G$) then $\Gamma_r \mathcal{G}$ is the s.s. group which is the composite functor $\Gamma_r \mathcal{G}$ is thus obtained by applying $\Gamma_r$ to each dimension $\mathcal{G}_g$, and to the face and degeneracy operators. The same remarks apply to the functor $L^r$ from $G$ to $A$ (the category of abelian groups), defined by

$$L^r \mathcal{G} = \frac{\Gamma_r \mathcal{G}}{\Gamma_r \mathcal{G}^+}$$

The process of composing the s.s. object $X$ which is a functor $X : \Omega_{\geq 0} \rightarrow \mathcal{E}$ with a functor $T : \Omega_{\geq 0} \rightarrow \mathcal{E}$ to obtain a new s.s. object $TX : \Omega_{\geq 0} \rightarrow \mathcal{E}$ is called "prolongation" by Dold [17].

A s.s. object $X$ is called $n$-connected if $\pi_c(X, x) = 0$ for all $c \leq n$.

For each real number $a$, let $\lfloor a \rfloor$ denote $a$ if $a$ is an integer, otherwise $\lfloor a \rfloor$ is to be the next integer larger than $a$. 
Theorem 5. If $G$ is a c.s. free group which is $n$-connected ($n \geq 0$), then $\text{Tr} \frac{G}{\text{Tr}+1} G$ is $n + \frac{\log_2 r^2}{2}$-connected.

The next sections outline the techniques used in the proof of Thm 1. It would be interesting to know the connectivity of the c.s. groups $\text{Tr} G$ themselves.

5. Witt's Theorem

A theorem of Witt [8], related to the better known Poincaré-Birkhoff-Witt Theorem, states that if $G$ is a free group, the maps $\text{Tr} \frac{G}{\text{Tr}+1} G$ depend (even as functors) only on $G/\text{Tr} G$. That is, for any group $G$,

let $L(G) = \sum_{r \geq 1} L_r(G)$ be the Lie ring of the group $G$ (cf. [7]). For any abelian group $M$, let $L(M) = \sum_{r \geq 1} L_r(M)$ be the free Lie ring over $M$ (cf. [4a]). Then there is a natural transformation $\Theta$ from the composite functor $\text{Tr} \circ \text{Tr}$ to the functor $L$ (considered as functors from $G$ to $A$). The Witt Theorem...
Then states that if $g$ is free,

$$\Theta : L^1(L^1g) \to L^1(g)$$

is an isomorphism.

6. A Theorem of A. Dold \cite{13}

**Theorem** If $T$ is a covariant functor from $A$ to itself, then if $M$ is a s.s. free abelian group, $\pi_*(TM,e)$ depends only on $\pi_*(M,e)$.

**Remark** Under our notation, $\pi_*(M,e)$ (base point understood to be the identity) is sometimes referred to as the "homology" of $M$. Let $T$ be the functor from s.s. abelian groups to s.s. abelian groups given by $T M = M$

and differential operator on $M$ given by $D = \Sigma (-1)^i$.

Then the normalization theorem of Eilenberg-MacLane (see for example \cite{33}) shows that

$$H_*(NM) \cong H_*(T^iM),$$

and then both of these are $\pi_*(M,e)$.
Putting Witt's and Dold's theorem together, we see that \( \pi_\ast (L^H S) \) depends only on \( \pi_\ast (L^{H_2} \mathbb{K}) \) if \( S \) is a free s.s. group, and we are reduced to studying the functor \( \star \) on s.s. free abelian groups. This is most conveniently done by means of the generalized bar construction of Dold–Puppe.


Let \( M \) be a s.s. abelian group. Then the cone \( cM \) and the suspension \( EM \) are defined by taking for \( cM \) any s.s. abelian group which is contractible and contains \( M \) as a direct summand; \( EM \) is defined to be the quotient:

\[
0 \rightarrow M \rightarrow cM \rightarrow EM \rightarrow 0
\]

Then the boundary homomorphism \( \Delta \) is an isomorphism:

\[
\Delta : \pi_{n+1} (EM) \xrightarrow{\cong} \pi_n (M)
\]

\[
H_{n+1} (EM) \rightarrow H_n (NM)
\]
Let $T$ be a functor from $a$ to $a$ (with $T(0) = 0$), and apply (the prolongation of) $T$

$$
\begin{array}{cccc}
& T_i & \longrightarrow & T_p \\
TM & \longrightarrow & TCM & \longrightarrow & TEM \\
\end{array}
$$

$T_i$ is injective, $T_p$ is surjective.

Then

$$
\begin{array}{cccc}
\pi_n(TM) & \longrightarrow & \pi_n(Ker T_p) & \longrightarrow & \pi_{n+1}(TEM) \\
\Delta & \longrightarrow & \pi & \longrightarrow & 0
\end{array}
$$

The composite $D^{-1}(T_i)_* = \sigma$ is called the suspension.

The generalized bar construction uses the cross effects of $T$ to provide a construction of a double complex (double differential graded) whose homology (under the total differential operator) is that of $NTEM$.

The cross effects of $T$, denoted by $T(M_1, \ldots, M_n)$, are defined in [43], where it is shown that they satisfy:

$$
T(M_1, \ldots, M_n) = \sum_{\tau} T(M_{\tau_1}, \ldots, M_{\tau_s})
$$

where $\tau = (\tau_1, \ldots, \tau_s), 1 \leq \tau_1 < \cdots < \tau_s \leq n$

e.g. $T(A+B) = T(A) + T(B) + T(A/B)$
Let $\Sigma_k M$ denote the direct sum of the copies of $M$, and let

$\alpha_i' : \Sigma_k M \to \Sigma_{k-1} M$ for $1 \leq i \leq k-1$

by $\alpha_i' (m_1, \ldots, m_k) = (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_k)$

then

$T(\alpha_i') : T(\Sigma_k M) \to T(\Sigma_{k-1} M)$

and let $\alpha_i$ be the composite

\[ T_{\Sigma_k} (M) \xrightarrow{T} T_{\Sigma_{k-1}} (M) \]

(definition)

\[ T(M_{1} \cdots M) \xrightarrow{\alpha_i} T(\Sigma_k M) \xrightarrow{T} T(\Sigma_{k-1} M) \]

$K$ factors

Let $d = \sum (-1)^i \alpha_i$; then $d^2 = 0$

If $M$ is a s.s. abelian group, let $\hat{\gamma}M$

be the double differential group:

\[ \hat{\gamma}M \xrightarrow{d} T_{\Sigma_{p+1}} (M) \xrightarrow{T_{\Sigma p}} T_{\Sigma p} (M) \]

\[ d = \sum (-1)^i T_{\Sigma p} \]

$T_{\Sigma p} (M_{p-1})$
Theorem: \( TM \) is a double differential group under \( d \) and \( \delta \), and the homology of the associated simply differential group \( tRM \), \( (tRM)_n = \sum_{p+q=n} T^p_{\mathbb{R}^q}(M) \) under \( D = d \oplus (-1)^p d \) becomes the homology of \( NTEM \), and the injection \( c : TM \to \tilde{T}M \) induces the suspension \( \sigma : \\

\[ 
\pi_n(TM) = H_n(NTM) \xrightarrow{\sigma} H_{n+1}(NTEM) = \pi_{n+1}(TEM) \xrightarrow{\partial} H_{n+1}(tRM) 
\]

In the next few pages we shall briefly outline why this theorem is true. For complete details see [33].

Let \( \Delta(m) \) be the s.s. set of the \( m \)-simplex that is \( \Delta(m)_p = \{ \text{all monotone maps } [p] \to [m] \} \)

- \( \Delta(0)_p \) has but one (map): \( \gamma_p^0 \)
- \( \Delta(1)_p \) has \( (m+1) \) elements: \( \gamma_p^0, \gamma_p^1, \ldots, \gamma_p^m \)

where \( \gamma_k^p : (k+1) \to 0 \)

where \( (k, k+1), \ldots, p \to 1 \)
Let $D(0) \rightarrow D(1)$ by $\varepsilon^0 r^p = r^p_{p+1}$ or $\varepsilon^1 r^p = r^p_0$.

Let $Z$ denote the free abelian group functor, and define

$$0 \rightarrow Z D(0) \rightarrow Z D(1) \rightarrow \frac{Z \varepsilon^0 D(0)}{Z (\varepsilon^0 D(0) + \varepsilon^1 D(0))} \rightarrow 0$$

$A, C, S$ are s.s. free abelian groups,

$A_p$ is freely generated by $r^p$

$C_p$ is freely generated by $r^p_0 \ldots r^p$

$S_p$ is freely generated by $r^p \ldots r^p$

Given a s.s. abelian group $M$, form the double objects $A \diamond M$, $C \diamond M$, $S \diamond M$, where for example $(S \diamond M)_{p, q} = S_p \otimes M_q$.

The diagonal objects and total objects are (respectively)

for example

$$d \left( S \diamond M \right)_n = (S \diamond M)_n, u_n = S_n \otimes M_n$$

$$t \left( S \diamond M \right)_n = \sum (S \diamond M)_{p, q} = \sum S_p \otimes M_q$$

$$p+q=n$$

$$p+q=n$$
The double s.s. objects have two sets of s.s. operators, denoted $d^i$, $s_i^c$ for the first index $(p)$ and $d^i, s^c_i$ for the second index $(q)$. The diagonal object is regarded as a differential group under $d = d^1 \circ d^2 = d^2 \circ d^1$, $s = \sum s^c_i d^i$, $s^c = \sum s^c_i d^i$.

The total object is a differential group under $D = D^1 + (s^c_i) D^2$.

Let $d(C \otimes M) = CM$, and since $d(Q \otimes M) = M$,
\[d(S \otimes M) = EM\]
we have constructed the cone and suspension.

Now apply $T$ to the double objects, and obtain the double objects:
\[T(C \otimes M), T(C \otimes M), T(S \otimes M)\]

The Eilenberg-Zilber-Cartier [3] theorem shows that the diagonal objects (e.g. $d T(S \otimes M) = TEM$) and the total objects (e.g. $T T(S \otimes M)$) have naturally isomorphic homology.
Now apply $N'$ to the double objects, where

\[ N' (\text{double object}) = \bigcup \text{ (kernel $D_i'$)} \]

and $D'$ = Unusual $\alpha_i$

Then

\[ N' T(\mathfrak{H}^\theta M)_{\rho,*} = TM \quad \rho = 0 \]
\[ = 0 \quad \rho \neq 0 \]

Also, and this is the crucial observation:

\[ N' T(\mathfrak{H}^\theta M) \cong YM \]

for

\[ T(\mathfrak{H}^\theta M)_{\rho,*} = T \left( \sum_{k=1}^{p} Y_{k} \rho \times M \right) \]
\[ = \sum_{r} T \left( \frac{Y_{k}}{r} \rho \times M \right) \]

But image $s_{i-1} = \sum_{r \neq i} T \left( \frac{Y_{k}}{r} \rho \times M \right) \]

so

\[ N' T(\mathfrak{H}^\theta M)_{\rho,*} = \sum_{r \neq (i_1, \ldots, i)} T \left( \frac{Y_{k}}{r} \rho \times M \right) \]

Then

\[ N' T(\mathfrak{H}^\theta M)_{\rho,*} = \bigcup_{D'} T(\mathfrak{H}^\theta M)_{\rho,*} \]
\[ \times T \left( \frac{Y_{k}}{r} \rho \times M \right) \]
\[ \times T_{E \rho \mathbb{R}^3}(M) \]

And it turns out that the differentials correspond, etc., and $\beta_0 \neq \alpha_0$.

In the applications, it will be convenient to filter

$TM$ by $F^p TM = \bigoplus_{m \leq p} T_{m^2} M_q$

Then in the associated spectral sequence, $E^1$

with its differential becomes the sequence

$H^*(NTK) \xrightarrow{d^1} H^*(NT_{k+1} K)$

and the $E^r$ converge to $E^\infty$ which is the

graded group associated with the filtration on $H^*(NTK)$

Recall that an s.s. object $X$ is called $n$-connected

if $\pi_q(X,*) = 0$ for all $q \leq n$

Lemma Let $T$ be a functor from $A$ to $A$

with $T(0) = 0$, prolonged to a functor on s.s.

abelian groups. Suppose that for all $0$-connected

s.s. free abelian groups $K$, $TK$ is $n$-connected.

Then for all finite s.s. free abelian groups $L$ which

are $n$-connected, $TL$ is $(N+n)-$ connected.

Proof. First note that if $TK$ is $N+n$

connected for all a s.s. free abelian groups $K$, then
The same is true of all the cross effects of $T$ (using the decomposition $T(K_1, \ldots, K_n) = \sum_{i} T(K_1, \ldots, K_i)$).

To prove the lemma it suffices to prove the following inductive (on $n$) step:

If $TK$ is $(N+n-1)$-connected for all $n-1$-connected $s.s.$ free abelian groups $K$, then $TL$ is $N+n$-connected for all $n$-connected $s.s.$ free abelian groups $L$.

This is proven by taking for $K$, any $s.s.$ free abelian group such that

$$H_{q-1}(NK) \times H_q(NL) \quad \text{all } q$$

so that $H_q(NEK) \times H_q(NL) \quad \text{all } q$.

By Dold's theorem, $EK$ may be substituted for $L$ before applying $T$, i.e. $H_q(NTEK) \times H_q(NTL)$.

Now apply the generalized bar const. to $TK$ and obtain $PK$. In the (above described) associated spectral sequence, $E'_{p,q} = H_q(NT_{p+1}K)$, which is 0 for $q \leq N+n-1$ by the inductive assumption. Then $H_{p+q}(PK) = 0$, for all $p+q \leq 1+N+n-1$ (since $p \geq 1$), by standard spectral sequence arguments.
9. The Connectivity of $L^r(M)$

In this section is outlined the proof of the following, which in conjunction with the Hurewicz-Kan Theorem (Section 3) implies Theorem 1.

**Theorem 2** If $M$ is a free s.s. abelian group which is $n$-connected, $L^r(M)$ is in $n + \frac{3}{\log_2 r}$-connected.

This relies on a decomposition: let $M$ be a free abelian group.

**Theorem 3** There is a finite filtration on $L^r(M)$ say $F^t L^r(M)$ for $0 < 1 < \cdots < t < \cdots < n(r)$, where the associated quotients $q^t = F^t \to F^{t-1}$ satisfy a recursion relation (i) or (ii) (depending on $r$):

(i) $q^t L^r(M) \cong q^t \otimes q^{t-1} L^r(M) \otimes \text{Sym}^a(q^{t-1} L^r(M))$

(ii) $q^t L^r(M) \cong q^{t-1} L^r(M) \otimes (q^{t-1} L^r(M))$

The only term which does not decompose further is $q^{n(r)} L^r(M)$ which is $L^r(M)$.

$$\bigoplus [L^r(M) \otimes L^{2r-1}(M)]$$

and which will be called $M^r(M)$.
The filtration on $G_r(M)$ is defined by a lexicographic ordering of the "types" of the basic commutators [see 7a] which occur. For example
\[ \left[ [m_1, m_2, m_3], [m_4, m_5, m_6] \right] \]
and
\[ \left[ [\left[ [m_1, m_2, m_3], m_4 \right], [m_5, m_6] \right] \]
are both commutators of weight 6, and in the ordering, the "type" of the former will precede that of the latter. An element in $G_r(M)$ is of filtration $\leq t$ if it may be written as a sum of basic commutators all of types $\leq t$. The argument to prove Theorem 3 is a modification of that in M. Hall [7a].

The functor $MK(M)$ is simpler than $G_r(M)$ and satisfies:

**Lemma** If $M$ is a free $\mathbb{Q}$-connected s.s. abelian group, $MK(M)$ is $r-1$ connected.
This lemma may be proven by a technique similar to that of Dold- Puppe [3] in their treatment of $\Sigma^r(M)$.

Finally, Theorem 2 (and with it, Theorem 1) is proven by induction on $r$. It is because of the possibility of the decomposition (ii) (note a composite functor)
that the connectivity of $\Sigma^r(M)$ only increases with $\log r$. This is illustrated by the submodule $Y^2(y^2(M)) \subset \gamma^4(M)$ which has connectivity only 2 higher than that of $M$.
Also $Y^2(y^2(y^2(M))) \subset \gamma^8(M)$ has connectivity only 3 higher than that of $M$.

---

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NUMERICAL INVARIANTS OF HOMOTOPY TYPE

BY. F. PETERSON

LET $C$ BE THE CATEGORY OF SPACES WITH BASE POINT AND HOMOTOPY CLASSES OF MAPS.

DEFN. $X$ IS AN $H$-SPACE IF WE CAN COMPLETE THE FOLLOWING DIAGRAM:

\[
\begin{array}{c}
X \\
\downarrow f \\
X \times X \\
\downarrow \\
X
\end{array}
\]

WHERE $\eta$ IS THE FOLDING MAP.

DEFN. $X$ IS AN $H^*$-SPACE OR HAS CATEGORY $\leq 1$ IF WE CAN COMPLETE THE FOLLOWING DIAGRAM:

\[
\begin{array}{c}
X \\
\downarrow c \\
X \times X \\
\downarrow \\
X
\end{array}
\]

WHERE $\Delta$ IS THE DIAGONAL MAP.

LET $S, T$ BE FUNCTORS $\mathbb{C} \rightarrow \mathbb{C}$, $\gamma, \eta$ NATURAL TRANSFORMATIONS, $I$ THE IDENTITY FUNCTOR.

DEFN. $(S, T, \gamma, \eta)$ IS A STRUCTURE.
DEFN. \( X \) is structured with respect to \( t \) if there exists \( t : T(X) \to I(X) \) such that the following diagram is commutative:

\[
\begin{array}{c}
T(X) \xrightarrow{t} I(X) \\
\uparrow \gamma(X) \quad \uparrow \delta(X) \\
S(X) \xrightarrow{\eta(X)} I(X)
\end{array}
\]

(This is dualized in:

\[
\begin{array}{c}
R(X) \xrightarrow{\alpha(X)} I(X) \\
\downarrow \iota(X) \quad \downarrow \varphi(X) \\
Q(X) \xleftarrow{\Delta(X)} I(X)
\end{array}
\]

DEFN. \( FV^X_{i=1} \) denotes the fat weoge \((X_1, \ldots, X_n)\) with at least one \( X_i = x \) \((x = \text{base pt. of } X)\).

DEFN. If we can complete the following diagram:

\[
\begin{array}{c}
FV^X_{i=1} \xrightarrow{\rho} X \\
\downarrow \iota \quad \downarrow \Delta \\
X \xleftarrow{h+1} X
\end{array}
\]

then \( X \) is of category \( \leq n \).

So we have generalized the notion of category from 1 (\( H^X \)-space) to \( n \). We seek a dual notion that for
\[ h \neq 1 \text{ picks out only } H\text{-spaces.} \]

**Defn.** If \( X, Y \) are structured with respect to \( t, t' \) respectively and \( \varphi : X \to Y \) then \( \varphi \) is a homomorphism if the following diagram is commutative:

\[
\begin{array}{ccc}
T(X) & \xrightarrow{T(\varphi)} & T(Y) \\
\downarrow t & & \downarrow t' \\
I(X) & \xrightarrow{\varphi} & I(Y)
\end{array}
\]

Assume that \( X \) is structured with respect to \( t, t' : X \to Y \) and \((E_{\varphi}, X, p)\) is fibre space induced by \((P(Y), Y, p')\) where \( P(Y) \) is the path space of \( Y \).

We seek to find conditions on \( \varphi \) such that \( E_{\varphi} \) is structured and \( p \) is a homomorphism.

**Basic Assumption** (\( \star \)):

\[ \pi(T(X), \Omega) \xrightarrow{\varphi^*} \pi(S(X), \Omega) \to 0 \]

is exact for loop spaces \( \Omega \).

\((\pi(A, B) \text{ denotes the homotopy classes of maps } A \to B)\)

**Defn.** Let \( X \) be structured with respect to \( t \). Then \( \varphi : X \to Y \) is primitive if there exists \( t' : T(Y) \to Y \).
(t' not necessarily a structure)

Such that the following diagram is commutative:

\[
\begin{array}{ccc}
T(x) & \xrightarrow{T(f)} & T(y) \\
\downarrow & & \downarrow \\
I(x) & \xrightarrow{f} & I(y)
\end{array}
\]

**Theorem**

If \((S, T, f, n)\) satisfy (*), and \(f\) is primitive, then \(E_f\) is structured such that \(\rho\) is a homomorphism.

**Proof**

Use the following commutative diagram with exact rows:

\[
\begin{array}{c}
\pi(S(x), x) \\
\uparrow \quad \pi(T(x), x) \\
\pi(T(E), 'y') \quad \pi(T(E), E) \quad \pi(T(E), x) \quad \pi(T(E), y) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\pi(S(E), 'y') \quad \pi(S(E), E) \quad \pi(S(E), x)
\end{array}
\]

**Examples**

1. \(E_f\)

\[
\begin{array}{ccc}
X & \xrightarrow{f} & K(T, h) \\
\downarrow & & \\
X & \xrightarrow{\phi} & \quad \quad \quad \quad \quad \quad \\
\end{array}
\]

\(X\) an H-space \(\phi\) represents a cohomology class.
\[ \epsilon^*(\varphi) = \varphi \otimes 1 + 1 \otimes \varphi + \alpha \]
\[ \varphi \text{ is primitive } \iff \alpha = 0. \]

2. Consider the natural map \( S\Omega X \to X \) and diagram:

\[
\begin{array}{ccc}
S\Omega X \times S\Omega X & \to & S\Omega X \\
\downarrow & & \downarrow \\
S\Omega X \vee S\Omega X & \to & X
\end{array}
\]

Then for \( T(x) = S\Omega X \times S\Omega X \)
\( S(x) = S\Omega X \vee S\Omega X \)
condition \((*)\) is fulfilled.

The diagram can be filled in \( \iff S\Omega x \) is homotopy abelian
( Theorem of Stasheff)

Now assume that \( X \) is structured with respect to \( \epsilon \) and is
an \( H \)-space.

\[ \epsilon : X \times X \to X \]
\[ \epsilon^*(\varphi) = \varphi \otimes 1 + 1 \otimes \varphi + \sum \alpha \otimes \beta \]
\[ \varphi \text{ is primitive if } \sum \alpha \otimes \beta = 0 \]
For example, let \( E \mathcal{F} \)

\[
X = K(\mathcal{Z}, 2) \xrightarrow{\mathcal{F}} K(\mathcal{Z}, 8)
\]

Let \( \mathcal{F} \) represent \((C \cup)^3\)

Then \( e^*(\mathcal{F}) = \mathcal{F} \otimes 1 + 10 \mathcal{F} + i^2 \otimes 1 + 10i^2 \)

\( i \) generator of \( K(\mathcal{Z}, 2) \).

\( \mathcal{F} \) is not primitive therefore \( E \mathcal{F} \) is not an \( H \)-space.

But we can conclude \( \Omega E \mathcal{F} \) is homotopy abelian although \( E \mathcal{F} \) is not an \( H \)-space as \( i^*(c^2) = 0 \)

**Theorem**

If \( \pi_i(X) = 0 \), \( i \leq n-1 \)

Then \( \Omega X \) is homotopy abelian.

\( \iff \) \( X \) is an \( H \)-space.

Consider the group \( \mathit{H} \left[ \bigoplus_{i=1}^{n-1} \pi_i(\Omega X), \Omega X \right] \)

Let \( g_i \) represent the projection of \( \pi_i(\Omega X) \) into its \( i \)-th factor.

\[
[\ldots\left[\alpha_1, \alpha_2\right], \alpha_3, \ldots, \alpha_{n+1}]\]

In particular, we have an \( n+1 \) fold commutator of \( \left\{ \alpha_i \right\}_{i=2}^{n+1} \).
DEFN. NILPOTENCY OF $X \leq h$ IF

FOR ANY CHOICE OF $h+1$ FOLD COMMUTATORS
OF $\{ \alpha_i \}_{i=1}^{h+1}$, IT IS TRIVIAL.

CONSIDER THE GROUP $\pi \left[ S^X, \bigwedge_{i=1}^{h+1} S^X \right]$.

LET $\beta_i$ REPRESENT THE INJECTION
OF $S^X$ INTO THE $i$-TH FACTOR
OF $\bigwedge_{i=1}^{h+1} S^X$.

DEFN. CONNILPOTENCY OF $X \leq h$ IF

FOR ANY CHOICE OF $h+1$ FOLD COMMUTATORS
OF $\{ \beta_i \}_{i=1}^{h+1}$, IT IS TRIVIAL.

THEOREM $X$ HAS NILPOTENCY $\leq h$ $\iff$

WE CAN COMPLETE THE FOLLOWING DIAGRAM:

$$
\begin{array}{c}
\bigwedge_{i=1}^{h+1} S^X \\
\downarrow \\
\bigwedge_{i=1}^{h} S^X \\
\downarrow \\
V S^X \\
\downarrow \\
X
\end{array}
$$

WHERE $C(H)$ DENOTES THE COWEAK OVER $H$
$H \ast H$ DENOTES THE SMASH PRODUCT.
However we don't have a dual for this theorem giving necessary and sufficient conditions for conilpotency \( \leq n \).

However we do have: A space \( Q_n^h(X) \),

A diagram:

\[
\begin{array}{c}
X \\
\downarrow \Delta \downarrow \\
X \rightarrow \bigoplus_{i=1}^{n} X \\
\downarrow \\
\bigwedge_{i=1}^{n} X \rightarrow \bigoplus_{i=1}^{n} S_{\bigwedge X} \end{array}
\]

And theorem: conilpotency \( \leq n \iff e_j \Delta = 0 \)

Conilpotency \( \leq n \iff \Delta \) factors through \( Q_n^h(X) \).

Example: Let \( X = S^2 \cup g \cdot e^2 \)

where \( g = \gamma \circ w \gamma : S^3 \rightarrow S^2 \)

\( \gamma \) is Hopf map, \( w \) Blakers-Massey map.

Then weak category \( X = 2 \), conilpotency \( X = 1 \).
conilp, for \( n=1 \) yields:

\[
\begin{align*}
X & \rightarrow X \times X \rightarrow \omega (S X \times S X) \\
\downarrow & \\
X \wedge X & \rightarrow \omega S (X \wedge X) \\
\text{conilp, } n=1 & \iff e \not\in 0.
\end{align*}
\]

To dualize this we have a notion of **flat product** \( X \square X \)

\[
\begin{align*}
\omega (S X \vee S X) & \longrightarrow X \vee X \\
\downarrow & \\
(S \square X) \vee (S \square X) & \vee (X \square X) \rightarrow X \square X
\end{align*}
\]

_duality doesn't hold because _

\( X \square X \) appears and not \( S \square (X \square X) \)

but it is not clear how to dualize

_in case of \( n \).

we would like a structure for

coCat \( X \leq n \) such that:

coCat \( X \leq 1 \)iff \( X \) is an \( n \)-space.
SOME NON-STABLE SPHERE BUNDLES
OVER REAL PROJECTIVE SPACE

BY J. LEVINE

WE WILL STUDY THE FOLLOWING PROBLEM:
FOR INTEGERS \( k, n \) WITH \( k \leq n \) DESCRIBE THE
\( k \)-PLANE BUNDLES OVER \( \mathbb{P}^n(\mathbb{R}) \).

FIRST WE ESTABLISH A CLASSIFICATION
THEOREM BY TECHNIQUES SIMILAR TO THE
FELDBAUER CLASSIFICATION OF BUNDLES OVER
SPHERES.

LET \( S^{n-k} = (n-k) \) SPHERE, \( D^n = n \) DISK,
\( G \) A TOPOLOGICAL GROUP.

DEFN. IF \( \Phi : S^{n-k} \to G \) SUCH THAT \( \Phi(-x) = \Phi(x)^{-1} \)
WE CALL \( \Phi \) EQUIVARIANT. IF \( \Phi \) IS A
HOMOTOPY OF EQUIVARIANT MAPS, WE CALL \( \Phi \)
AN EQUIVARIANT HOMOTOPY, OR AN E-HOMOTOPY.

LET \( (E, \rho, P^h) \) BE A PRINCIPAL \( G \)-BUNDLE
WITH USUAL RIGHT TRANSLATION OF \( E \) BY \( G \).
THEN IF \( h : D^n \to P^h \) WHERE \( h \) IDENTIFIES
ANTIPODAL PTS. OR IF \( \partial D^n = S^{n-k} \) THEN \( h \) HAS
A LIFTING \( H \).

\[ \begin{array}{ccc}
D^n & \xrightarrow{h} & P^h \\
\downarrow H & & \downarrow \rho \\
E^n & \xrightarrow{p} & P^h
\end{array} \]
IF $x \in S^{n-1}$ THEN $h(x)$ AND $h(-x)$ LIE IN SOME FIBRE OF $p$. WE DEFINE $\psi : S^n \to G$ BY THE FORMULA:

$$h(x) = h(-x) \psi(x) \quad \text{for} \ x \in S^{n-1}$$

IT IS STRAIGHTFORWARD TO CHECK THAT $\psi$ IS EQUIVARIANT, $\psi$ IS CALLED A CHARACTERISTIC MAP OF $E$.

**Lemma** Any two characteristic maps $\psi_0, \psi_1$ of $E$ are $e$-homotopic up to conjugacy, i.e. there exists $a \in G$, such that $\psi_1$ and $a^{-1} \psi_0 a$ are $e$-homotopic.

**Proof** If $h_0$ and $h_1$ are liftings of $h$ corresponding to $\psi_0$ and $\psi_1$ define $g : D^n \to G$ by $h_1(x) = h_0(x) g(x)$.

WE ESTABLISH THAT $\psi_1(x) = g(-x)^{-1} h_0(x) g(x)$

NOW DEFINE $g_t : S^n \to G$ BY

$$g_t(x) = g(-x)^{-1} h_0(x) g(x)$$

THEN $g_0 = a^{-1} \psi_0 a$ where $a = g(0)$

$g_1 = \psi_1$

WE CHECK THAT $g_t$ IS EQUIVARIANT AND THE LEMMA IS PROVEN.
THEOREM  THE CORRESPONDENCE OF PRINCIPAL G-BUNDLES TO THE CONJUGATE PHOMOTOPY CLASS OF ITS CHARACTERISTIC MAPS IS A 1-1 CORRESPONDENCE.

PROOF (1) ONTO: Suppose $\varphi : S^{n-1} \to G$ is equivariant. We define a principal bundle $E_\varphi$ with $\varphi$ as a characteristic map.

Consider the product bundle $D^n \times G$

with subbundle $S^{n-1} \times G$

we define a bundle map $S^{n-1} \times G \to S^{n-1} \times G$

by $\tilde{\varphi}(x,y) = (-x, \varphi(x)y)$ clearly $\varphi$ covers the antipodal map of $S^{n-1}$.

We define $E_\varphi$ by collapsing $D^n \times G$ through $\varphi$. This covers a collapsing of $D^n$ through the antipodal map of $S^{n-1}$. Thus there is defined in a natural manner a projection $\rho: E_\varphi \to \rho^n$ such that the following diagram is commutative!
\[ D^n \times G \xrightarrow{\overline{\Phi}} E_p \]
\[ \downarrow \quad h \downarrow \quad \downarrow p \]
\[ D^n \xrightarrow{h} P^n \]

WHERE \( \overline{\Phi} \) IS THE COLLAPSING MAP.

Since \( \overline{\Phi} \) is a bundle map, we can check that \( G \) acts without fixed pt. on \( E_p \) and \( E_p/G = P^n \). \( (\overline{\Phi} \) is a bundle map by a theorem of Gleason.

We define \( h : D^n \rightarrow E_p \) covering \( h \)
by \( h(x) = \overline{\Phi}(x,e) \) and \( \Phi \) is the characteristic map. This proves onto.

(2) ONE-ONE: Let \( E_0, E_1 \) be bundles with characteristic maps \( \Phi_0, \Phi_1 \) derived from liftings \( H_0, H_1 \) of \( h \).
We prove the following:
(a) If \( \Phi_0 = a^{-1} \Phi_1 a \), then \( E_0 \) is equivalent to \( E_1 \).
(b) If \( \Phi_0 \) is \( a \)-homotopic to \( \Phi_1 \), then \( E_0 \) is equivalent to \( E_1 \).

These clearly establish one-one.

Proof of (a): We define an equivalence \( F : E_0 \rightarrow E_1 \) by \( F(H_0(x)y) = H_1(x)a_y \).
This is well-defined; if $h_0(x) y = h_0(-x) y_0(x) y$

then $F(h_0(-x) y_0(x) y) = h_1(-x) a y_0(x) y$

$= h_1(-x) y_0(x) a y = h_1(x) a y = F(h_0(x) y)$.

Now $F$ is a bundle map and

covers the identity map of $p^1$.

Therefore it is an equivalence.

Proof of (b): Let $\varphi_t : S^{n-1} \to G$ be

an $\mathcal{C}$-homotopy from $\varphi_0$ to $\varphi_1$. We

may assume $E_0 = E_{\varphi_0}$, $E_1 = E_{\varphi_1}$.

Since by (a) these are already equivalent, we use $\varphi_t$ to

construct a bundle $E$ over $I \times D^n$.

As above we take $I \times D^n \times G$

and define a bundle map

$\varphi : I \times S^{n-1} \times G \to I \times S^{n-1} \times G$ by:

$\varphi(t, x, y) = (t, -x, \varphi_t(x) y)$

by collapsing under $\varphi$, we obtain $E$.

Clearly $E|_{I \times D^n} = E_0$, $E|_{I \times D^n} = E_1$.

So $E_0 \sim E_1$.

Remark: One can prove this theorem by replacing $G$ by $\Omega B G$ and using the inverse operation of a loop space.

By studying $[p^n, B G]$. But the passage to $G$ seems to require more knowledge of the relation between

the multiplicative structure of $G$ and $\Omega B G$.!!
We wish to study $\pi_{n-1}^i(G)$ the set of $i$-homotopy classes of $S^{n-1} \to G$. It is useful to generalize the situation. We replace the topological group $G$ by a connected space $X$, $n$-simple for all $n$, with involution $T$ and fixed pt. $x_0$. We replace the antipodal map on $S^n$ by involutions $T_c$ defined by the formula $T_c(x,y) = (x,-y)$ for $x \in D^n$.

In the first $i$ coordinates fixed and reversing the rest.

$(x,y) \in S^n \implies (x,y) = (x_1, \ldots, x_i, y_{i+1}, \ldots, y_{n+1})$.

So fixed pt. set of $T_c$ is $S^{i-1}$.

We define $\pi_{n-i}^i(X, x_0; T)$ as the set of homotopy classes of mappings $f: (S^n, S^{n-1}) \to (X, x_0)$ s.t. $fT_c = Tf$ such an $f$ is $i$-equivariant when $i = 0$, $X = G$ this agrees with our previous defn. of equivariant.

We note some properties:

1. If $i > 0$, $\pi_{n-i}^i(X, x_0; T)$ is a group. Multiplication is defined as follows:

\[ f \cdot g \mapsto f \circ g \quad g \mapsto g \to x \]
The $i$-th preserves parallels along $i$-th axis so if $\phi, \psi$ are $i$-equivariant so is $\psi \circ \phi$.

(2) There is a group action of $\Pi_n$ on $\Pi_n^i$. $\phi \rightarrow \psi$ $i$-equivariant

Note that $T_i$ exchanges hemispheres.

Let $\phi, \psi : S^h \rightarrow X$, $\phi$ $i$-equivariant and let $\psi \circ \phi$ on lower hemisphere.

We define $h : S^h \rightarrow X$ by:

$h = \begin{cases} \phi \text{ on upper hemisphere} \\ T \phi T_i \text{ on lower hemisphere} \end{cases}$

Then $h$ is well defined and equivariant.

Consider a sequence of mappings:

$\Pi_{n+1} \xrightarrow{\psi_{i+1}} \Pi_n \xrightarrow{\phi} \Pi_n^i \xrightarrow{\psi_i} \Pi_{n-1} \xrightarrow{\psi_i} \Pi_{n-1}$

$\psi_{i+1}$ and $\psi_i$ are passage to ordinary homotopy class. $\psi_i$ is restriction to $n-1$ sphere along $i+1$ st. axis.

$\phi_i(x) = \alpha \cdot x_0$. When $i > 0$, these maps are homomorphisms.

We can prove several things about these maps.
**Theorem** (a) This sequence is exact, i.e.,

When $i = 0$ this is well-defined if $x_0$ is considered as base pt. of $T_h^0$.

Moreover, at $T_h^i$ we have

$$
\psi_i(\beta_1) = \psi_i(\beta_2) \iff \alpha \cdot \beta_1 = \beta_2
$$

For some $\alpha$.

(b) $\psi_i(\alpha \cdot \beta) = \psi_i(\beta) + \alpha + (-1)^{h-i+1} T_x(\alpha)$

$$
\psi_i(\phi_i(\alpha)) = \alpha + (-1)^{h-i+1} T_x(\alpha)
$$

(c) $\psi_h : T_h^i \to T_h^i$ is an isomorphism.

(d) By induction on $h - i$, we can study $T_h^i$. If $h - i = 0$ use (c).

Using above sequence, by induction we know all but $T_h^i$.

Consider the following situation:

Let $(x_j, x_0)$ be a 2-fold covering of $(x_1, x_0)$ $\tau$ a covering transformation $P : (\tilde{x}, x_0) \to (x, x_0)$ the covering map.

**Lemma** There exist two involutions of $\tilde{x}$ covering $\tau$, say $T'$ and $T''$, and $T'T'' = \tau$.

Call $T'$ that one such that $T'(\tilde{x}_0) = x_0$.

If $i > 0$, $\rho^i : T_h^i(\tilde{x}, x_0, T') \to T_h^i(x, x_0, T)$ is a homomorphism.

**Proposition** $\rho^i$ is a monomorphism. If $i > 1$, $\rho^i$ is an isomorphism.
\[ \rho^*_x : \pi^0_*(\tilde{X}, T') \cup \pi^0_*(\tilde{X}, T'') \rightarrow \pi^0_*(X, T) \]

is defined.

**Proposition** If \( n > 1 \), \( \rho^*_x \) is onto.

The inverse sets under \( \rho^*_x \) are of the form \( \{ \beta, t_x(\beta) \} \)

where \( t_x \) acts on \( \pi^0_*(\tilde{X}, T') \)

and \( \pi^0_*(\tilde{X}, T'') \).

---

**Table of \( \mathbb{P}^k \)-plane bundles over \( \mathbb{P}^n \)**

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \mathbb{P}^k )</th>
<th>( \mathbb{P}^n )</th>
<th>( \mathbb{P}^k )</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>( 2T )</td>
<td>( N + T )</td>
<td>( A_{2i} )</td>
</tr>
<tr>
<td>3</td>
<td>( 3T )</td>
<td>( N + 2T )</td>
<td>( 2N + T )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \mathbb{P}^k )</th>
<th>( \mathbb{P}^n )</th>
<th>( \mathbb{P}^k )</th>
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</thead>
<tbody>
<tr>
<td>2</td>
<td>( 2T )</td>
<td>( N + T )</td>
<td>( 2N + T )</td>
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<tr>
<td>3</td>
<td>( 3T )</td>
<td>( N + 2T )</td>
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<td>4</td>
<td>( 4T )</td>
<td>( N + 3T )</td>
<td>( B_{2i} )</td>
</tr>
<tr>
<td>5</td>
<td>( 5T )</td>
<td>( N + 4T )</td>
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(1) Columns are stable classes. The last row for each \( h \) is the beginning of the stable range

(2) \( T \) = trivial line bundle

(3) \( N \) = non-trivial line bundle

(4) \( A_i, B_i, C_i \) = sequence of bundles distinguished by Euler class \( \{ \} \) in \( H^k(\mathbb{P}^n, \mathbb{Z}) \) with twisted integers, \( k \) even.
(5) \(\oplus\) = A bundle which has no particular recognition \((x) = \text{indistinguishable by any characteristic class.}\)

**Note:** (i) All bundles over \(P^3\) are sums of line bundles.

(ii) Any bundle over \(P^n, P^3\) or \(P^4\) with trivial Stiefel Whitney class is trivial.

**Conjecture:** For any \(n, k\) there are only a finite number of bundles unless \(n = k\) is even in non-orientable case.

It may be for all \(P^n\), \(n\) even, stable triviality of \(n\)-plane bundle implies triviality.

---

**Computation of the Table**

Recall the exact sequence:

\[
\cdots \to \pi_h \phi_m \phi_i \psi_i \psi_{i-1} \psi_{i-2} \cdots \phi_0 \psi_o \phi_0 \cdots \phi_0 \psi_0 \phi_0 \phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \phi_7 \phi_8 \to \pi_{n-i} \pi_{n-i-1} \phi_0 \psi_o \phi_0 \cdots
\]

CASE PT. \(\psi_o\)

**Note** our strong exactness at \(\pi_{n-i}\) implies:

\(\pi_{n-i} = 0 \Rightarrow \psi_o\) is a monomorphism.

If \(\chi = \text{topological group } G\) and \(T = \text{inversion}\)

\(\pi_{n-1} (G; T) / \pi_{0}(G)\) classifies \(G\)-bundles over \(P^n\).

By passing to characteristic map of a bundle, let \(G = \text{O}_k = \text{Rk}\), \(k\), from defn. of characteristic map

(a) \(\pi_{n-1} (\text{Rk})\) = orientable bundles over \(P^n\).

(b) \(\pi_{n-1} (\text{rk})\) = non-orientable bundles over \(P^n\).

If \(f, \phi_o, \phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8\) characteristic maps of \(\alpha, \beta\) then

\[
\begin{pmatrix}
\phi_0 \\
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4 \\
\phi_5 \\
\phi_6 \\
\phi_7 \\
\phi_8
\end{pmatrix}
\]

is characteristic map of \(\alpha \beta\).
IF $k=1, \quad O_k = \gamma + 1, -1$ \[ \text{THENCE} \]
\[ \Pi^0_{n-1}(R_1) = 1, \quad \Pi^0_{n-1}(A_1) = -1 \]

LET $a_i$ BE MATRIX, IN ANY $O_k, \quad k \geq i$.
\[
\begin{pmatrix}
-1 & \gamma \\
-1 & 0 \\
0 & 1 \\
0 & 1
\end{pmatrix}
\]

NOW $a_i$ IS A FIXED POINT OF $T_j$ FROM (b).

IF WE CONSIDER $a_i$ AS AN ELEMENT OF $\Pi^0_{n-1}(O_k; T)$ IT REPRESENTS THE BUNDLE $\gamma(N + (k-i)T)$.

WE CALCULATE $\Pi^0_{n-1}(R_2), \Pi^0_{n-1}(H_2)$

\[
\begin{align*}
\phi_1 & \quad \phi_0 & \quad \psi_0 \\
\Pi_1 & \quad \Pi_0 & \quad \Pi_0
\end{align*}
\]

CONSIDER THIS EXACT SEQUENCE WHERE $X = R_2$ OR $H_2$ \quad $X_0 = E$ OR $A_1$.

NOW $\Pi_0^0 = \Pi_0 = X_0$ \quad Thus $\phi_0$ IS ONTO.

NOW $\phi, \phi_1(x) = x - T_x(x) = \gamma(2x) \quad \{ 0 \quad x = R_2$

\begin{align*}
\phi_1 & \quad \phi_0 \\
\Pi_1 & \quad \Pi_0
\end{align*}

NOW $\phi_1$ IS AN ISOMORPHISM SO

IMAGE $\phi_1 = \left\{ \begin{array}{ll}
2\Pi_1 & x = R_2 \\
0 & x = H_2
\end{array} \right.$

\[
\phi_0(h) = h \times x_0 = x_0 \Leftrightarrow \left\{ \begin{array}{ll}
h \text{ EVEN} & x = R_2 \\
h = 0 & x = H_2
\end{array} \right.$

HENCE $\Pi_1^0(R_2) = \left\{ e_1, 1 \cdot e_2 \right\} = \left\{ e_1, a_2 \right\}$ \quad $\Pi_1^0(H_2) = \left\{ n - a_1 \right\}$ \quad \text{SINCE}

\[
W(e) \neq W(a_2), \quad \Pi_1^0(H_2) = \left\{ n - a_1 \right\}
\]

UNDER CONJUGATION $n \cdot a_1 \rightarrow (-n) \cdot a_1$.
Stable class of these will follow from Euler class, which comes out now.

\[ n=3 \]

\[ 0 = \pi_2 \to \pi_2^0 \to \pi_1^0 \to \pi_1 \]

If \( x = R_2 \), \( \phi_0 = 0 \) since \( \pi_1^0 = \{ \text{constant} \} \)

Therefore \( \pi_2^0(R_2) \cong \pi_1^0(R_2) = \{ e, a_2 \} \) by exactness.

If \( x = R_2 \), \( \psi_0(h \cdot a_1) = h + T_x(h) = 2h \)

(Since \( \psi_0 \) is, in terms of bundles, just inducing bundle under \( S^h \to \mathbb{P}^h \))

This verifies statement about Euler class if \( h \cdot a_1 = a_h \)

Thus image \( \psi_0 = \phi_1 \)

Hence by exactness \( \pi_2^0(R_2) = a_1 \)

\[ h > 3 \]

\[ 0 = \pi_{h-1} \to \pi_{h-1}^0 \to \pi_{h-2}^0 \to \pi_{h-2} = 0 \]

Thus \( \pi_{h-1}^0 \cong \pi_{h-2}^0 \) and we get result by induction.

Now \( \pi_{h-1}^0(1^{R_k}) \), \( \pi_{h-1}^0(R_k) \) for \( k > 2 \).

\[ n=2 \]

\[ Z_2 = \pi_1 \to \pi_1^0 \to \pi_0^0 = \chi_0 \]

Thus \( \phi_0 \) is onto and \( \pi_1^0 \) contains at most two sets. These are \( \{ e, a_2 \} \) and \( \{ a_1, a_3 \} \).

\[ n=3 \]

\[ 0 = \pi_2 \to \pi_2^0 \to \pi_1^0 \to \pi_1 \]

\( \psi_0 = 0 \) since \( \pi_1^0 = \{ \text{constant} \} \) \( \therefore \psi_0 \) is an isomorphism and \( \pi_2^0 \cong \pi_1^0 \).
We now illustrate technique for $n=4$ by calculating $\Pi_3^0(R_3)$.

\[ \Pi_3^1(R_3) \rightarrow \Pi_3(R_3) \rightarrow \Pi_2^0(R_3) \rightarrow \Pi_2(R_3) \rightarrow \Pi_2^0(R_3) \rightarrow \Pi_2(Z) \cdot \{ e_0, a_2 \} \]

Thus $\Pi_3^0(R_3) = \{ h \cdot e, h \cdot a_2 \}$

Let $x_0 = e$ or $a_2$, then $(zh) \cdot x_0 = x_0$

1. $\phi_0(Zh) = 0$ because $\phi_1(h) = h - T_x(h) = 2h$ thus $zh \in \text{image } \phi_1$

Thus $\Pi_3^0(R_3) = \{ e, a_2, 1 \cdot e, 1 \cdot a_2 \}$

Claim: (a) 1. $e = e$
(b) 1. $a_2 \neq 0$

Proof of (a): consider the sequence where $x_0 = e$. It suffices to show 1 e image $\phi_1$.
But 1 is represented by $\phi : S^3 \rightarrow R_3$ defined as follows: represent $R^3$ as rotations of purely imaginary quaternions $S^3$ by unit quaternions.

$\phi(q) \cdot 2 = q \cdot \overline{2}$, note $\phi(s^0) = e$.

Note that $T_1 : S^3 \rightarrow S^3$ coincides with conjugation of quaternions:

$T_1(a_0 + a_1 + a_2 + a_3) = (a_0 - ca_1 - da_2 - ha_3)$

Therefore need to show $\phi(q)^{-1} = \phi(\overline{q})$.

$\phi(q)$ is conjugation by $q$, so $\phi(q)^{-1} = \phi(\overline{q})$ or $T \phi = \phi T_1$.

Proof of (b): we assume $a_2 = 1 \cdot a_2$.
\[ \pi_4^0(R^3) \rightarrow \pi_3^0(R^3) \rightarrow \pi_2^0(R^3) \rightarrow \pi_3(R^3) \]

\[ \mathbb{Z}_2 \xrightarrow{\phi_0} \{e, \alpha \cdot e, a_2, \alpha \cdot a_2\} \]

Thus \( \pi_3^0(R^3) = \{e, \alpha \cdot e, a_2, \alpha \cdot a_2\} \)

\[ \phi_0(\alpha \cdot x_0) = \alpha + T_\star(\alpha) = 0 \quad \text{therefore} \quad \phi_0 = 0. \]

We shall contradict this by showing \( \phi_0 \) is onto.

\[ S^4 \xrightarrow{\phi} S^3 \xrightarrow{\phi} R^3 \]

is the essential map, where \( Sh \) is suspension of Hopf map. Note \( hT_0 = h \) since \( h \) collapses great circles.

Note \( ST_0 = T_1 \),

\[ (\phi \circ Sh) \circ T_0 = \phi \circ S(hT_0) \circ T_0 = \phi \circ S h \circ ST_0 \circ T_0 \]

\[ = \phi \circ S h \circ T_1 \circ T_0 \quad \text{now} \quad T_1 \circ T_0 \text{ inverts coordinate along which we have suspended} \ h. \quad \text{therefore} \]

\( (Sh)(T_1T_0) = (T_1T_0)(Sh) \) and so

\[ (\phi \circ Sh) \circ T_0 = \phi(T_1T_0), \quad (Sh) = T_1T_0(Sh) = T_1(Sh) \]

(since \( \phi(T_0) = \phi \)). Thus \( \phi(Sh) \) is 0-equivariant, so \( \phi_0 \) is not zero.

Contradiction.
COHOMOLOGY AND ALGEBRAS OVER

HOPF ALGEBRAS

BY N.E. STEENROD

LET R BE A COMMUTATIVE RING WITH UNIT.

IF \( \mathfrak{A} = \bigoplus_{k \geq 0} \mathfrak{A}_k \) IS A GRADED \( R \)-MODULE

THEN \( \bigotimes \mathfrak{A} \) IS A GRADED \( R \)-MODULE WHERE

\[
(\bigotimes \mathfrak{A})_k = \bigoplus_{i+j=k} \mathfrak{A}_i \otimes \mathfrak{A}_j.
\]

DEFNS.

AN ALGEBRA OVER R IS A GRADED R-MODULE, \( \mathfrak{A} \), WITH A

HOMOMORPHISM \( \gamma : \bigotimes \mathfrak{A} \rightarrow \mathfrak{A} \)

OF GRADED R-MODULES.

\( \mathfrak{A} \) IS ASSOCIATIVE IF THE FOLLOWING DIAGRAM IS COMMUTATIVE:

\[
\begin{array}{ccc}
\mathfrak{A} \otimes \mathfrak{A} & \xrightarrow{\mu} & \mathfrak{A} \\
\downarrow{\mu} & & \downarrow{\gamma} \\
\mathfrak{A} \otimes \mathfrak{A} & \rightarrow & \mathfrak{A}
\end{array}
\]

A UNIT FOR \( \mathfrak{A} \) IS AN ALGEBRA HOMOMORPHISM \( \eta : R \rightarrow \mathfrak{A} \)

SUCH THAT THE FOLLOWING DIAGRAM IS COMMUTATIVE:

\[
\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{\eta} & \mathfrak{A} \\
\downarrow{\mathfrak{A}} & & \downarrow{\mathfrak{A}} \\
\mathfrak{A} \otimes \mathfrak{A} & \rightarrow & \mathfrak{A}
\end{array}
\]

A COALGEBRA OVER R IS A GRADED R-MODULE, \( \mathfrak{A} \), WITH A

HOMOMORPHISM \( \gamma : \bigotimes \mathfrak{A} \leftarrow \mathfrak{A} \)

OF GRADED R-MODULES.

\( \mathfrak{A} \) IS ASSOCIATIVE IF THE FOLLOWING DIAGRAM IS COMMUTATIVE:

\[
\begin{array}{ccc}
\mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A} & \xrightarrow{\Delta} & \mathfrak{A} \otimes \mathfrak{A} \\
\downarrow{\delta} & & \downarrow{\gamma} \\
\mathfrak{A} & \rightarrow & \mathfrak{A}
\end{array}
\]

A UNIT FOR \( \mathfrak{A} \) IS A COALGEBRA HOMOMORPHISM \( \epsilon : \mathfrak{A} \leftarrow R \)

SUCH THAT THE FOLLOWING DIAGRAM IS COMMUTATIVE:

\[
\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{\epsilon} & \mathfrak{A} \\
\downarrow{\mathfrak{A}} & & \downarrow{\mathfrak{A}} \\
\mathfrak{A} \otimes \mathfrak{A} & \rightarrow & \mathfrak{A}
\end{array}
\]

AND COMPOSITION IS IDENTITY MAP OF \( \mathfrak{A} \).
An augmentation of an algebra with unit is a homomorphism $\varphi : \mathbb{A} \to R$ of algebras with unit. $\mathbb{A}$ with $\varphi$ is an augmented algebra.

$A$ is commutative if the diagram:

\[
\begin{array}{ccc}
\mathbb{A} \otimes R & \xrightarrow{\varphi} & R \\
\downarrow T & & \downarrow \eta \\
\mathbb{A} \otimes \mathbb{A} & \xrightarrow{\varphi} & \mathbb{A}
\end{array}
\]

is commutative.

where $T(x \otimes y) = (-1)^{pq}(y \otimes x)$ for $x \in \mathbb{A}$, $y \in \mathbb{A}$.

Defn. A Hopf algebra is a graded $R$-module $\mathbb{A}$, together with a map $\eta : \mathbb{A} \to R$ which makes $\mathbb{A}$ into an algebra with unit $\eta : R \to \mathbb{A}$ and a map $\gamma : \mathbb{A} \to \mathbb{A} \otimes \mathbb{A}$ which makes $\mathbb{A}$ into a coalgebra with unit $\varphi : \mathbb{A} \to R$ such that:

1) $\eta$ is an augmentation for the coalgebra

2) $\varphi$ is an augmentation for the algebra

3) The following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{A} \otimes \mathbb{A} & \xrightarrow{\varphi} & \mathbb{A} \\
\downarrow \psi & & \downarrow \Phi \\
\mathbb{A} \otimes \mathbb{A} & \xrightarrow{\varphi} & \mathbb{A}
\end{array}
\]
3) is equivalent to either 3') $\psi$ is a map of algebras or 3'') $\psi$ is a map of coalgebras.

DEF: The Hopf algebra $\mathcal{H}$ is connected if either of the equivalent conditions holds:
1) $\eta : R \rightarrow A_0$
2) $\varepsilon : A_0 \rightarrow R$

REMARK Hopf proved that the rational cohomology ring of a group manifold $G$ has a certain form (exterior algebra on odd-dimensional generators) by algebraic techniques, essentially using the fact that $H^*(G; \mathbb{Z}_0)$ is a Hopf algebra ($\mathbb{Z}_0 = \text{rationals}$).

PROPOSITION If $\mathcal{H}$ is a Hopf algebra and $C$ is the category of $\mathcal{H}$-modules and $\mathcal{H}$-mappings then $C'$ admits an internal tensor product $\otimes_R$ satisfying the usual identities.

PROOF Let $X$ be a graded $\mathcal{A}$-module, $Y$ a graded $\mathcal{B}$-module. $X \otimes_R Y$ is an $A \otimes B$ module by defining:
$(a \otimes b)(x \otimes y) = \pm ax \otimes by$ and this is natural.
Let $Y$ be an $A$-module then $X \otimes_Y Y$ is an $A$-module, but this is not in the category $C$. However, using $\gamma : A \to A \otimes A$ $X \otimes_Y Y$ becomes an $A$-module by defining $a \cdot (x \otimes y) = \gamma(a)(x \otimes y)$.

The fact that $\gamma$ is associative proves $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$ as $A$-modules.

Also $X \otimes Y \cong Y \otimes Z$ as $A$-modules provided that coalgebra $A$ is commutative.

**Note:** $R$ is a Hopf algebra.

**Defn.** Let $A$ be a Hopf algebra over $R$ and $X$ an $A$-module which is an algebra over $R$. Then if $m : X \otimes X \to X$ is an $A$-mapping we call $X$ an algebra over the Hopf algebra $A$.

For such an $X$ and $\gamma : A \to \text{Nor}$ $\gamma(a) = \sum q_i \otimes a_i$ then

if $m(xy) = xy$ we have

$a \cdot (xy) = \sum (-1)^{(x)} (a_i)(a_i')(a_i'' x) \cdot (a_i'' y)$

($x$ in exponent is degree of $x$)
Examples

1. Our main example of a Hopf algebra is the reduced power (Steenrod) algebra $O_\ast$, defined as follows. Consider cohomology of $X$ with coefficients in $\mathbb{Z}_2$. Then we have isomorphisms $S^c_2 : H^q(X) \to H^{q+c}(X)$ for $c = 0, 1, 2, \ldots$ and $O_\ast$ is the graded operator algebra generated by $S^c_2$.

For $S^c_2$ we have:

0. $S^0_2 = \text{identity}$

1. $S^c_2 x = x^c$ if $x \in H^q(X)$, i.e., if degree $x = q$

2. $S^c_2 (x) = 0$ if $i > \text{deg} x$

3. $S^r_2 (xy) = \sum_{i=0}^{r} (S^i_2 x) (S^{r-i}_2 y)$

Milnor proved that there is a unique co-product $\gamma : O_\ast \to O_\ast \otimes O_\ast$ making $O_\ast$ into a Hopf algebra such that

$\gamma(S^c_2) = \sum_{i+j=c} S^i_2 \otimes S^{j-i}_2$, (3)

Then simply the statement that $H^*(X; \mathbb{Z}_2)$ is an algebra over $O_\ast$.
If $A$ is a graded $R$-module, let its dual $\mathcal{A}_*^r$ be the graded $R$-module $\mathcal{A}_*^r = \bigoplus_{n \geq 0} \text{Hom}_R(\mathcal{A}_n, R)$ and if $\phi : A \to B$ is a map of graded $R$-modules, $\phi_* : B_*^r \to A_*^r$ defined by $\phi_*(b_*)a = b_* \phi(a)$ is the dual map.

By considering $\mathcal{A}_*^r$, the dual of $\phi$ becomes $\phi_* : \mathcal{A}_*^r \to A_*^r \otimes \mathcal{A}_*^r$ and the dual of $\phi$ is multiplication $\phi_* : A_*^r \otimes A_*^r \to A_*^r$. $\phi$ is commutative, hence $\phi_*$ is commutative.

Using this and the fact that $A_*^r$ is a finitely generated free $R$-module, Milnor proved $\mathcal{A}_*^r$ is a polynomial algebra, and this then gave information about $\mathcal{A}_*^r$.

2. Let $X$ be an algebra over $R$ and let $G$ be the group of automorphisms of $X$ as an algebra. We then have maps: $G \to G \times G \to G \to G$.

Let $R(G)$ denote the group ring of $G$. 
WE HAVE MAPS
\[ R(\mathcal{O}) \rightarrow R(\mathcal{O}) \otimes R(\mathcal{O}) \rightarrow R(\mathcal{O}) \]

\( R(\mathcal{O}) \) acts as endomorphisms on \( \mathcal{X} \) and is a Hopf algebra over \( \mathcal{X} \) is an algebra over the Hopf algebra \( R(\mathcal{O}) \).

\( \mathcal{O} \) is always a Hopf algebra.

\[
\begin{align*}
R & \rightarrow R \otimes R \\
\otimes R & \rightarrow R \\
K & \rightarrow K
\end{align*}
\]

given an algebra over \( K \) then.

By \( 2 \) it is an algebra over a Hopf algebra. Thus the notion of an algebra over a Hopf algebra is a true generalisation of the notion of an algebra.

WHOLESALE CONSTRUCTION OF ALGEBRAS

OVER HOPF ALGEBRAS

LET \( M \) BE AN ARBITRARY GRADED \( A \)-module WHERE \( A \) IS A HOPF ALGEBRA OVER \( R \).

WE DEFINE THE TENSOR ALGEBRA \( T(A) \) OF \( M \) BY:
\[ T(n) = R + M + M \otimes M + \cdots + \text{homo} \otimes M + \cdots \]

\text{DEGREE} \quad 0 \quad 1 \quad 2 \quad \cdots \quad n \quad \cdots

\( T(M) \) is an \( R \)-module, for \( M \otimes_R M \)

(INTERNAL TENSOR PRODUCT) is an \( R \)-module

(by using diagonal map \( \psi : R \to R \otimes R \))

and \( \psi \) associative says \( M \otimes M \otimes M \)

is an \( R \)-module proving \( M \otimes \cdot \otimes M \)

is an \( R \)-module. And \( T(n) \)

being direct sum of \( R \)-modules is

therefore an \( R \)-module. \( \psi \) associative

induces multiplication \( T(M) \otimes T(M) \to T(M) \)

\[ (M \otimes M \otimes M \otimes M) \to M^{2n+2} \] so that

\( T(M) \) is an algebra over the hopf

algebra \( A \). Furthermore \( T(M) \)

satisfies a universal property. That is,

given \( M \to A \)

\[ \begin{array}{c}
\uparrow \\
\psi \\
\downarrow \\
T(M)
\end{array} \]

there exists a unique extension \( \phi \).

Assume \( \psi : R \to R \otimes R \) is commutative

the commutator of \( x \in M_0, y \in M_2 \)

is \[ x \otimes y - (-1)^{|x||y|} y \otimes x \] and
is denoted \((x, y)\). The ideal of commutators is an \(A\)-module and \(T(M)/\text{ideal of commutators}\) is an algebra over \(A\) and is commutative. It is a universal gadget for commutative \(A\)-algebras.

Now let \(M\) be an \(A\)-module where \(O\) is the reduced power (Steenrod) algebra. Suppose \(S^l_2(x) = 0\) if \(l > \text{deg} \cdot x\).

Let \(\Lambda(M) = Z_M + M + M \otimes M + \ldots\).

\[ S^l_2(x) \in M \quad x \otimes x \in M \otimes M \]

The ideal generated by \((S^l_2 x - x \otimes x)\) is an \(O\)-module. The proof of this uses the Adem relations. Then \(\Lambda(M)/\) \(\text{ideal} (S^l_2 x - x \otimes x)\) is a commutative algebra over \(O\) and is a universal gadget.
Consider the Eilenberg–Mac Lane complex $K(\mathbb{Z}_2, k)$ then Serre has given the structure of $H^*(K(\mathbb{Z}_2, k); \mathbb{Z}_2)$.

We claim: $H^*(K(\mathbb{Z}_2, k); \mathbb{Z}_2) = U(\text{free } k\text{-module on one } k\text{-diml. generator})$.

**Theorem.** Let $G$ be a group manifold such that $H^*(G; \mathbb{Z}_2)$ is primitively generated and $P$ be the set of primitive elements of $H^*(G; \mathbb{Z}_2)$.

(i.e. under $H^*(G; \mathbb{Z}_2) \rightarrow H^*(G; \mathbb{Z}_2) \otimes H^*(G; \mathbb{Z}_2)$ (induced by $m: G \times G \rightarrow G$) an element $x \in H^*(G; \mathbb{Z}_2)$ is primitive if $m^*(x) = x \otimes 1 + 1 \otimes x$)

Then $H^*(G; \mathbb{Z}_2) \cong U(P)$

**Cor.** $H^*(SO(n)) \cong U(H^*(P_0^{n-1}))$.

More generally (but this does not follow from the theorem)

$H^*(V_{n, h}^k) \cong U(H^*(P_0^{k-1}))$. 
SYMMETRIC PRODUCT OF COMPLEXES

By C. Curtis

REFERENCE: DOLD-PUPPE, ANNALS OF FOURIER INSTITUTE.

WE HAVE

THEOREM (DOLD-THOM) IF \( X \) IS A SEMI-SIMPLICIAL SET AND CONNECTED THEN

\[ \pi_i(\text{sym}^\infty X) \cong H_i(X), \quad i \geq 1. \]

DOLD AND PUPPE HAVE REFINED THIS TO

THEOREM IF \( X \) IS A SEMI-SIMPLICIAL SET AND \( r-1 \) CONNECTED, \( r-1 \geq 1 \), THEN

\[ \pi_i(\text{sym}^n X) \cong H_i(X), \quad \text{for} \quad i \leq r-2+2n, \]

NOTE. THE THEOREM IS NOT TRUE FOR

\[ i = r-2+2n, \quad \text{let} \quad r = 3, \quad n = p, \]

\( p \) A PRIME NUMBER. IF \( X = S^3 \)

THEN \( \pi_{2n+1}(\text{sym}^n S^3) \cong \mathbb{Z}_p \)

BUT \( H_{2n+1}(S^3) = 0 \)

WE SHALL PROVE

THEOREM \( \pi_i(\text{sym}^n X) \to \pi_i(\text{sym}^{n+1} X) \)

IS AN ISOMORPHISM FOR \( i \leq r+2n-2 \),

WHICH PROVES THE DOLD-PUPPE THM.
To prove this it will be sufficient to prove
\[ H_i(\text{sym}^N X) \Rightarrow H_i(\text{sym}^{N+1} X) \]
for \( i < \frac{p}{2} + 2N - 2 \) and for this sufficient to prove
\[ H_j(\text{sym}^{N+1} X, \text{sym}^N X) = 0 \]
for \( j < \frac{p}{2} + 2N - 1 \). Now let
\[ C_X(X) = P + K = L \]
where \( P \) is SS ABELIAN GROUP OF A POINT
and \( K \) is a SS FREE ABELIAN GROUP
complex which is \( i-1 \) connected
that is \( H_j(K) = 0 \) for \( j < k \).

**Formula of Steenrod**

\[ \text{sym}^L = P + \text{sym}^1 K + \ldots + \text{sym}^n K \]

Since \( \text{sym}^1 P \cong P \) and \( P \otimes A \cong A \)
(Actually \( \text{sym}^L = \sum \text{sym}^i P \otimes \text{sym}^{n-i} K \))
then\[ C_X(\text{sym}^N X) = \text{sym}^L = \text{sym}^{N+1} K \]
(look at \( C_X(\text{sym}^{N+1} X), C_X(\text{sym}^{N+2} X) \))
so we are reduced to showing that

**Lemma** IF \( K \) IS A FREE ABELIAN GROUP
complex \( i-1 \) connected \( i \geq 2 \), then
\( \text{sym}^{N+1} K \) is \( i+2N-1 \) connected \( i \)
\[ H_j(\text{sym}^{N+1} K) = 0 \] for \( j < i - 2 + 2N \) \( (n = N+1) \)
Now take $K$ to be finitely generated, i.e. $K^i$ is finitely for each $i$.

For arbitrary $K$ take direct limit.

Our next reduction is:

**Lemma**: If $K = K^1 \oplus \cdots \oplus K^r$ and the previous lemma holds for each $K^i$ then it holds for $K$.

**Proof**: $\text{sym}^n K = \sum_{i_1, \ldots, i_r} \text{sym}^{i_1} K^{i_1} \oplus \cdots \oplus \text{sym}^{i_r} K^{i_r}$

And use the Kunneth formula.

In fact the first time homology is non-zero occurs in dimension

$$\sum_{j=1}^r (k + 2(c_j - 2)) = k \cdot r + 2n - 2r$$

$$= (k - 2) \cdot r + 2n \geq k - 2 + 2n.$$  

Now to show $H_c(\text{sym}^k K) = 0$ if $k \leq k - 2 + 2n$.

It is suff. to show

$$H_c(\text{sym}^k K \otimes \mathcal{O}) = H_c(\text{sym}^k (K \otimes \mathcal{O})) = 0$$

if $k \leq k - 2 + 2n$ for each prime field $\mathcal{O}$.

But by dold $H_c(\text{sym}^k (K \otimes \mathcal{O}))$ depends only on $H_c(K \otimes \mathcal{O})$, so replace
K by a direct sum of spheres, i.e.

\[ K = \sum \mathbb{C}_x(S^k) \]

and by the above lemma it is sufficient to consider only one sphere and prove:

If \( K = \mathbb{C}_x(S^k) \) \( m > k \)

then \( H_i(\text{sym}^m K) = 0 \) \( i < 2n + k + 2 \).

For this we use the generalized bar construction.

Let \( \Omega \) be the category of semi-simplicial abelian groups and \( T \) any functor \( T : \Omega \rightarrow \Omega \).

E.g., \( T = \text{sym}^m \) and \( (TK)_h = TK_h \) etc.

E.g., \( K = \mathbb{C}_x(S^m) \)

form \( T(K_1 + \ldots + K_n) = \sum T(K_1 \mid K_n) \)

so \( T(K_1 + K_2) = T(K_1) + T(K_1 \mid K_2) + T(K_2) \).

Form \( \alpha \) \( \alpha : T(K) \rightarrow T(K') \rightarrow \ldots \)

\( T_{p-1}(\mathbb{R}) \rightarrow T_p(\mathbb{R}) \)

\( \alpha \) induced by \( K \times K \rightarrow K \) (addition)

\( s_1 = 0 \) \( s_2 = \Sigma (-1)^i \alpha_i \)

\( d_0 = \Sigma (-1)^i T d_i \)
ON DOUBLE COMPLEX \( \text{LET } D = d \pm d \)

THEN \( H_k (\text{DOUBLE COMPLEX}) = H_{kn} (\text{EK}) \)

\( \text{EK} = \text{SUSPENSION OF K} \)

\( \text{LET } Q = \text{SS GROUP OF A POINT (Z IN ALL DIMENSIONS)} \)
\( C = \text{SS GROUP OF } \Delta(1)/\text{ONE END PT} \)
\( S = \text{SS GROUP OF } \Delta(1)/\text{BOTH END PTS} \)

SO WE HAVE: \( 0 \rightarrow Q \rightarrow C \rightarrow S \rightarrow 0 \)

GENERATORS:
\( x_0, x_0, y_p, y_{1p}, \ldots y_{kp} \)

\( \delta_i x_{kp} = x_{k-1} \quad \text{IF } k < -p \)
\( \delta_i y_{kp} = y_{k+1} \quad \text{IF } k > -p \)
\( s_i x_{kp} = x_{kp+1} \quad \text{IF } k < -p \)
\( s_i y_{kp} = y_{kp} \quad \text{IF } k > -p \)

WE FORM DOUBLE COMPLEXES
\( Q \otimes K, C \otimes K, S \otimes K \)

(\text{WHERE } (A \otimes B)_{p,q} = A_p \otimes B_q \text{ ) K IS A}
\text{SS ABELIAN GROUP.})

APPLY \( T : T(S_p \otimes K_q) = T(K_q^+) \text{ ) P} \)

NOW \( H_j (TS \otimes K) = H_j (TSDK) \)

FURTHERMORE WE HAVE THE COMMUTATIVE
Diagram:

\[ H_n(TK) \xrightarrow{\iota^*_x} H_{n+n}(\text{double complex}, D) \]
\[ \downarrow \]
\[ \iota'_x \]
\[ H_{n+n}(T(EK)) \]

Where \( \iota^*_x \) is induced by the injection of \( TK \rightarrow \text{double complex} \). This is the main assertion of the generalized bar construction.

It may also be shown that if \( K \) is \( 2^q \)-connected, \( T(K|K) \) is \( 2^q \)-connected, etc.

So using only the first part of the double complex \( T(K) \)

\[ \downarrow \]
\[ T(K) \rightarrow T(K|K) \]

We find that

\[ \iota^*_x \]

\[ H_j(T(K|K)) \xrightarrow{\alpha^*_x} H_j(TK) \rightarrow H_{j+n}(T(EK) \rightarrow H_{j+n}(HK|K)) \]

\[ \iota'_x \text{ isomorphism if } j < 2^q \]

Exactness if \( j < 3^q \),

Now let \( K = EQ = \text{augmented chain complex of } S' \).

Lemma: \( \text{Sym}^h K \) is contractible

i.e. \( H_j(\text{Sym}^h K) = 0 \) for all \( j \).
**Proof**

**Note That**

\[ \text{Sym}^n(\mathbf{K}'|\cdot,|\mathbf{K}) = \sum \text{Sym}^n \mathbf{K}' \otimes \cdots \otimes \text{Sym}^n \mathbf{K} \]

\[ \sum_{i,j} = n, i_j \geq 0 \]

**e.g.** \[ \text{Sym}^n(M + L) = \text{Sym}^n M + \cdots + \text{Sym}^n M \otimes \text{Sym}^{n-1} L + \cdots + \text{Sym}^n L \]

**Let us apply this to \( Q \); each \( \mathcal{Q}_p = \mathbb{Z} \)**

\[ \text{Sym}^n \mathcal{Q} \otimes \mathcal{Q} \]

\[ T(Q) \leftarrow T(Q/Q) \leftarrow \cdots \]

\[ \sum \text{Sym}^i \mathcal{Q} \otimes \cdots \otimes \text{Sym}^i \mathcal{Q} \rightarrow \sum \text{Sym} \mathcal{Q} \otimes \cdots \otimes \text{Sym} \mathcal{Q} \]

**if** \( i_1 = 1 \) \( \Rightarrow s = 0 \)

**if** \( i_1 > 1 \) \( \Rightarrow s \) maps \( \text{Sym}^i \rightarrow \text{Sym}^{i-1} \otimes \text{Sym}^1 \)

**then** \( ds + sd = \text{idemity} \)

**This shows** \( H_j(\text{Sym}^n \mathbf{K}) = 0 \)

**Let** \( \mathbf{K} = \mathbb{C}_K (s) \)

\[ E \mathbf{K} = \mathbb{C}_K (s^2) \]

\[ \text{Sym}^n(\mathbf{K}) \leftarrow \cdots \leftarrow \text{Sym}^n(\mathbf{K}|\cdot,|\mathbf{K}) \]

**Then** \( H_j(\text{Sym}^n(E \mathbf{K})) = \mathbb{Z} \quad j = 2h \)

\[ = 0 \quad j < 2h \]
\[ \text{And}\quad H_j \left( \text{Sym}^n (\tilde{C}_k(S^3)) \right) = \mathbb{Z} \quad \text{if} \quad j = 2n+1 \]

\[ = 0 \quad \text{if} \quad j < 2n+1 \]

\[ \text{And in general}\quad H_j \left( \text{Sym}^n (\tilde{C}_k(S^d)) \right) = 0 \]

\[ \text{if} \quad j < 2n+k-2 \]