UNIVERSAL RELATIONS BETWEEN STIEFEL-WHITNEY CLASSES

BY E. BROWN

LET $\eta = (E, B, p)$ DENOTE A REAL n-PLANE BUNDLE, THAT IS, A FIBRE BUNDLE WITH REAL n-DIMENSIONAL VECTOR SPACE AS FIBRE AND $GL(n,R)$ AS STRUCTURAL GROUP.

$$\eta \left\{ \begin{tikzcd} E \arrow[d, "p"'] \arrow[d, phantom, shift left=4, "V^n"] \\ B \end{tikzcd} \right. \quad GL(n,R)$$

LET $W_i(\eta) \in H^i(B; Z_2)$ BE A STIEFEL WHITNEY CLASS OF η.

ORIGIN OF $W_i(\eta)$:

TO η IS ASSOCIATED A BUNDLE OF k FRAMES E^k.

$$E^k = \left\{ (v_1, . . v_k) \middle| \begin{array}{l} v_1, . , v_k \text{ LINR. IND. VECTORS OF } \pi^{-1}(b) \\ \text{FOR SOME } b \in B \end{array} \right\}$$

$$\begin{tikzcd} E^k \arrow[d, "V_{n,k}"] \\ B \end{tikzcd}$$

ANY FIBRE IS $V_{n,k}$ THE STIEFEL MANIFOLD OF ALL k-FRAMES IN n-SPACE. WE NOTE THAT $V_{n,k}$ IS $n-(k+1)$ CONNECTED.

THE PRIMARY OBSTRUCTION TO A CROSS SECTION OF E^k IS $W_{n-k+1}(\eta)$

$$W_{n-k+1} \in H^{n-k+1}\left(B;\ \pi_{n-k}(V_{n,k})\right)$$

WHERE $\pi_{n-k}(V_{n,k})$ IS $\mathbb{Z}_2$, $\mathbb{Z}$, OR $\mathbb{Z}$ TWISTED. IF WE REDUCE THIS GROUP TO $\mathbb{Z}_2$ WE GET RID OF TWISTING AND OBTAIN A CLASS $\overline{W}_{n-k+1} \in H^{n-k+1}(B; \mathbb{Z}_2)$

DEFINE: $W_i(\eta) = \overline{W}^{(n-i+1)}$ $\quad i = 0, 1, 2, .\ n$

THE <u>WHITNEY CLASS OF THE BUNDLE η</u>

WE LOOK AT THE UNIVERSAL BUNDLE

$$\gamma^n \left\{ \begin{tikzcd} E \arrow[d] \\ G_n \end{tikzcd} \right. V^n$$

WHERE G_n IS THE INFINITE GRASSMAN MANIFOLD OF n-PLANES IN R^∞ AND E CONSISTING OF ALL PAIRS (H, x) WITH H AN n-PLANE AND x A VECTOR IN H.

THEN

$$\boxed{H^*(G_n, \mathbb{Z}_2) = \mathbb{Z}_2(W_1(\gamma^n), .. , W_n(\gamma^n))}$$

WHERE $\mathbb{Z}_2(W_1, . , W_n)$ IS THE POLYNOMIAL ALGEBRA OVER $\mathbb{Z}_2$ GENERATED BY THE WHITNEY CLASSES.

SUPPOSE M^n A DIFFERENTIABLE MANIFOLD. LOOK AT THE TANGENT BUNDLE $T(M^n)$ AND LET $\tau(M^n)$ BE THE n-PLANE BUNDLE

$$\left. \begin{tikzcd} T(M^n) \arrow[d] \\ M^n \end{tikzcd} \right\} \tau(M)$$

WE DEFINE: $W_i(M) \equiv W_i(\Upsilon(M))$

THE PROBLEM IS: WHAT UNIVERSAL RELATIONS HOLD FOR STIEFEL-WHITNEY CLASSES?

LOOK AT THE KERNEL I_M OF $\mathbb{Z}_2(W_1(\gamma^n), \ldots, W_n(\gamma^n)) \longrightarrow H^*(M, \mathbb{Z}_2)$ DEFINED BY $W_i \to W_i(M)$

LET $\mathcal{I}_n = \bigcap_M I_M$. IF M IS A C^∞ COMPACT MANIFOLD WITH OR WITHOUT BOUNDARY IT IS KNOWN THAT $\mathcal{I}_n \neq 0$.

$\mathcal{I}_n$ BECOMES A GRADED POLYNOMIAL ALGEBRA BY DEFINING DIM. $W_i = i$

DOLD SHOWED THAT $(\mathcal{I}_n)_n \neq 0$, WHICH COMES FROM THE Wu FORMULAS.

WE SHALL SHOW $(\mathcal{I}_n)_q = 0$ FOR $2q \leq n$, AND SHALL USE THE FOLLOWING THEOREM. LET K BE A FINITE CW COMPLEX OF DIM $= n$, η A REAL m-FRAME PLANE BUNDLE OVER K.

THEOREM IF $2n \leq m$ THERE EXISTS A COMPACT C^∞ MANIFOLD M^m, A CW COMPLEX K' CONTAINED IN M, AND A HOMOTOPY EQUIVALENCE $g: K' \to K$ SUCH THAT $g^*\eta$ IS EQUIVALENT TO $\iota^*(\Upsilon(M^m))$.

$$\begin{tikzcd}
\gamma(M^m) & T(M^m) \arrow[d] & & E^m \arrow[d] & \eta \\
 & M^m & K' \arrow[l, "i"'] \arrow[r, "g"] & K &
\end{tikzcd}$$

IDEA OF THE PROOF: GIVEN η A REAL m-FRAME BUNDLE OVER K, TO PUT A MANIFOLD M^m AROUND K, $i : K \to M$, SUCH THAT THE INDUCED BUNDLE $i^*(\gamma(M^m))$ IS EQUIVALENT TO η. WE CAN DO THIS IF WE ARE WILLING TO CHANGE K UP TO HOMOTOPY TYPE.

COROLLARY $(d_m)_n = 0$ IF $2n \leq m$

LET US LOOK AT G_m THE CLASSIFYING SPACE FOR $GL(m, R)$. LOOK AT THE ASSOCIATED m-PLANE BUNDLE

$$\begin{tikzcd}
\zeta & E \arrow[d] & V^m \\
 & G_m &
\end{tikzcd}$$

LET G_m^n DENOTE THE n-SKELETON OF G_m. IT IS THE COMPLEX K TO WHICH WE APPLY THE THEOREM. HENCE WE ASSUME $2n \leq m$. WE GET THE INDUCED BUNDLE ζ' OVER G_m^n

$$\begin{tikzcd}
\gamma(M) & T(M) \arrow[d] & \zeta' & E' \arrow[d] \arrow[lll] & V^m & \zeta & E \arrow[d] & V^m \\
 & M & & G_m^n \arrow[ll, "h"', "\text{INJECTION}"] \arrow[rrr, "g"] & & & G_m &
\end{tikzcd}$$

AND WE CAN FIND M SUCH THAT WE HAVE A BUNDLE MAP h.

NOW $W_i(M) \equiv W_i(\tau(M))$ AND STIEFEL WHITNEY CLASSES BEHAVE FUNCTORIALLY SO

$$k^*(W_i(M)) = W_i(\xi')$$

$$g^*(W_i(\xi)) = W_i(\xi')$$

ANY RELATIONS AMONG $W_1(M), \ldots, W_n(M)$ MEANS WE HAVE THE SAME RELATION IN $W_1(\xi'), \ldots, W_n(\xi')$. AND HENCE THE SAME RELATION HOLDS IN $W_1(\xi), \ldots, W_n(\xi)$ FOR g^* IS A MONOMORPHISM IN DIMENSIONS $\leq n$ (FOR G_m^n IS THE n-SKELETON OF G_m) . |

DEFN. LET $f : X \to Y$ C_f THE *MAPPING CYLINDER* IS THE QUOTIENT SPACE OBTAINED FROM $X \times I \cup Y$ BY IDENTIFYING:

$$(x, 1) \sim f(x)$$
$$(x, 0) \sim x \quad .$$

THUS $C_f \supseteq X, Y$. f IS THE *ATTACHING MAP*. |

LET US SUPPOSE $K \subseteq M$ AND $N = \partial M \neq \emptyset$, M A COMPACT C^∞ MANIFOLD.

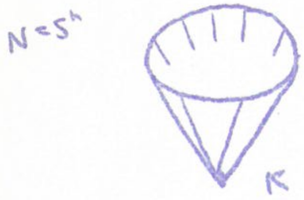
DEFN. M IS A *TUBULAR NBD. OF K* IF THERE EXISTS A MAP $F : N \to K$ SUCH THAT C_F IS THE UNDERLYING SPACE OF M AND $\rho : C_F \to R^1$ BY $\rho(x, t) = t$, $\rho(y) = 1$ DEFINES A C^∞ MAP ON $M - K$, i.e. $M - K \subseteq M = C_F \hookrightarrow R^1$ IS C^∞.

EXAMPLES: ① $K = PT.$

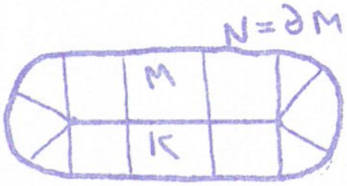
$M = C_F = E^{n+1}$

$N = \partial M = S^n$

F = CONST. MAP



② K = CLOSED INTERVAL



$F : N \to K$
ALONG FLOW LINES.

NOW LET K BE THE GIVEN FINITE CW COMPLEX OF DIM $= n$. η A REAL m-PLANE BUNDLE OVER K.

TO CONSTRUCT: M^m WHICH IS A TUBULAR NBD. OF K SUCH THAT FOR $i : K \to M$

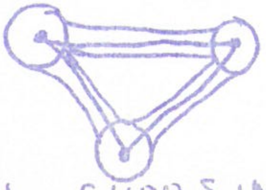
$i^*(\tau(M)) \sim \eta$ $\quad 2n \leq m$

METHOD: ADDING HANDLES TO M

(i) TAKE VERTICES OF K AND AROUND EACH VERTEX PLACE A DISK OF DIMENSION 1.

(ii) AROUND EACH 1-CELL OF K PLACE THE M OF EXAMPLE ② ABOVE.

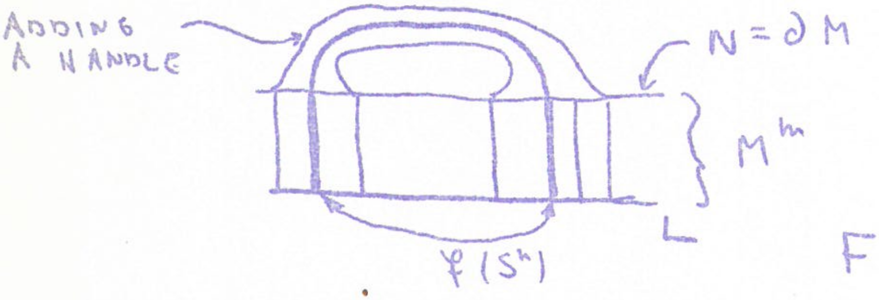
WE HAVE:



THE CHOICE IN CHOOSING ABOVE TUBULAR NBDS. ALLOWS US TO SHIFT THE TANGENT BUNDLE NICELY.

WE GIVE THE INDUCTION STEP OF THE PROCESS OF CONSTRUCTING M^m:

LET $K = L \cup e^{n+1}$. L A CW COMPLEX, $2 \cdot \dim L \leq m$, $2(n+1) \leq m$. SUPPOSE $L \subseteq M^m$, SUCH THAT M^m IS AN m-DIMENSIONAL TUBULAR NBD. OF L. WE SHALL ENLARGE M^m TO ENCLOSE A CELL.



$F: N \to L$ ALONG LINE.

$C_F = M$

SUPPOSE $f: S^n \to L$ IS THE MAP BY WHICH THE CELL e^{n+1} IS ATTACHED TO L. WE KNOW $\dim L + n < m$ THEREFORE WE CAN JIGGLE f SO THAT L IS MISSED. I.E. $f \sim f'$ IN M SUCH THAT $f'(S^n) \cap L = \phi$.

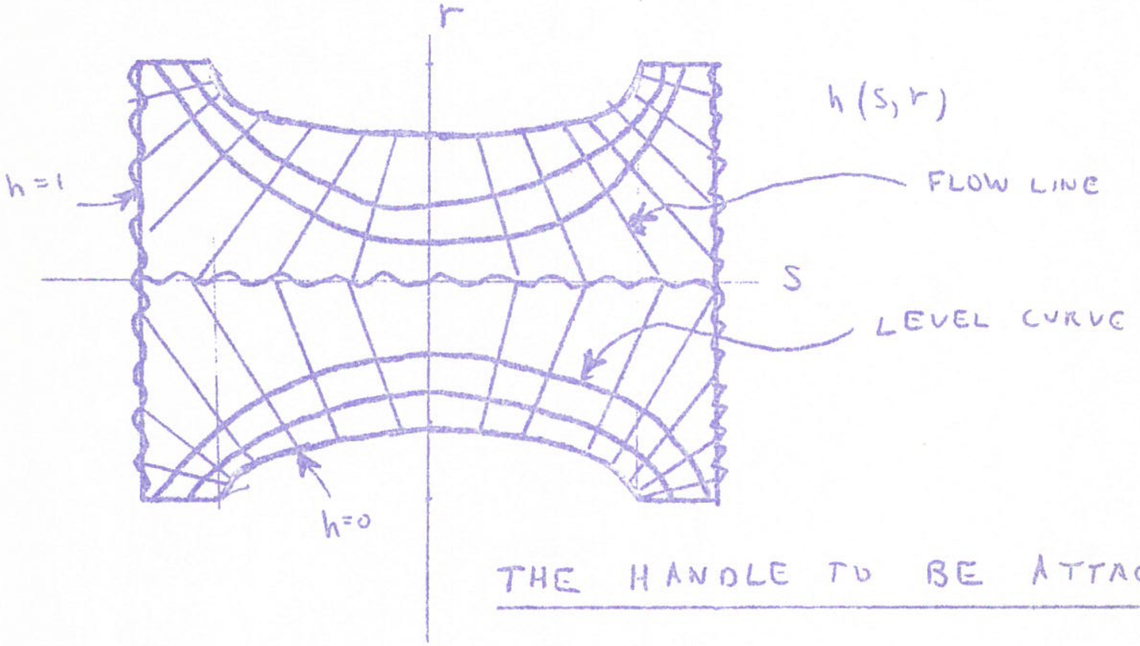
HENCE THERE EXISTS $g: S^n \to N$ A C^∞ IMBEDDING SO THAT $g \sim f$ IN $M = C_F$. $Fg: S^n \to L$ CERTAINLY $Fg \sim f$ IN M AND $\therefore$ IN L

WE WANTED TO INCLOSE $K = L \cup e^{n+1}$ VIA f. IN FACT WE INCLOSE $K' = L \cup e^{n+1}$ VIA Fg. THIS WILL CLEARLY SUFFICE.

WE NOW DRAW PICTURES OF HOW TO PASTE ON A HANDLE.

LET $g : S^n \to N$ BE OUR IMBEDDING. SUPPOSE $g(S^n)$ HAS A TRIVIAL NORMAL BUNDLE IN N, FOR SIMPLICITY.

IN THE EUCLIDEAN PLANE:

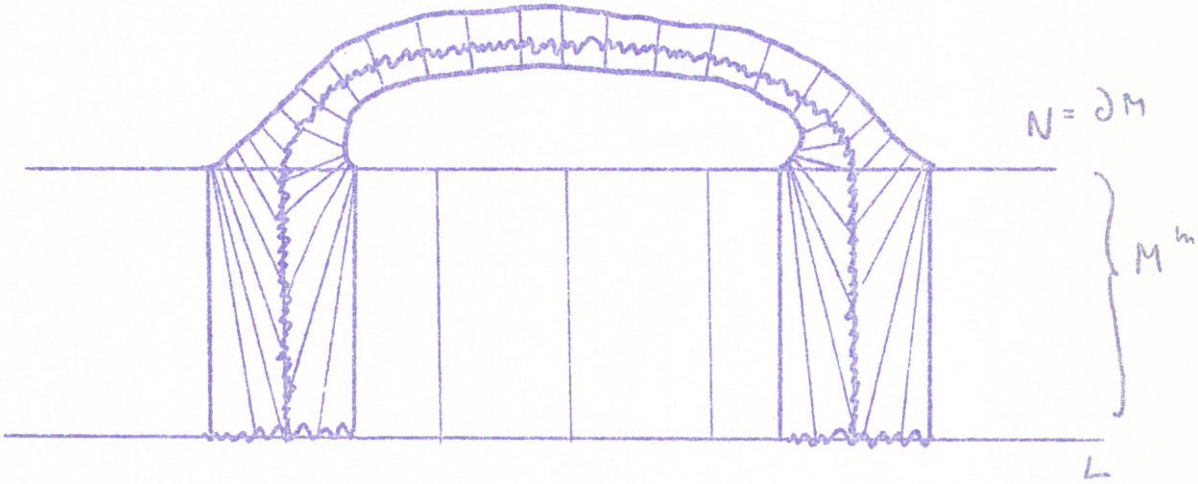


THE HANDLE TO BE ATTACHED

SUPPOSE WE COULD DEFINE A C^∞ NON-DEGENERATE FCN. ON R^2 WITH LEVEL CURVES. (C^∞ EXCEPT ON )

IF Q_0 DENOTES SET ON WHICH $h=0$
& IF Q_1 " " " " $h=1$

WE MAP $Q_0 \to Q_1$ ALONG THE FLOW LINES. NOW TAKE THE HANDLE AND PASTE IT ON M:



THE HANDLE ATTACHED

UNIVERSAL RELATIONS BETWEEN STIEFEL-WHITNEY CLASSES CONT'D.

BY E. BROWN

TODAY WE SHALL PROVE:

THEOREM LET K BE A FINITE CW COMPLEX OF DIM. $= n$, η A REAL m-PLANE BUNDLE OVER K. THEN IF $2n \leq m$ THERE EXISTS A COMPACT C^∞ MANIFOLD M OF DIM. $= m$, A CW COMPLEX K' CONTAINED IN M, AND A HOMOTOPY EQUIVALENCE $g : K' \to K$ SUCH THAT $g^* \eta$ AND $\iota^*(\tau(M))$, (WHERE i IS INJ. MAP $K' \to M$) ARE EQUIVALENT.

PROOF: THE PROOF IS BY INDUCTION ON THE CELLS OF K.

WE GIVE THE INDUCTIVE STEP.

LET $K = L \cup e^{p+1}$ $f : S^p \to L$ IS THE ATTACHING MAP.

η AN m-PLANE BUNDLE, $2 \dim K \leq m$ AND SUPPOSE M IS A TUBULAR NBD. OF L $N = \partial M \neq \emptyset$. AND THAT WE KNOW

$$\eta|L \sim \tau(M)|L.$$

NOTATION: IF ξ IS A BUNDLE OVER X + $A \subseteq X$, $\xi_{|A}$ IS THE INDUCED BUNDLE.

OUR OBJECT IS TO OBTAIN M' A TUBULAR NBD. OF K $\ni$ $\eta \sim \gamma(M')|K$ (MORE OR LESS)

LAST TIME WE OBSERVED THAT BY CHANGING K UP TO HOMOTOPY TYPE WE CONSTRUCT $g: S^p \to \partial M$ WHICH IS A C^∞ IMBEDDING s.t. $f = Fg$. (WHERE $F: \partial M = N \to L$ & $M = C_F$)

(<u>NOTE:</u> IF K IS A SIMPLICIAL COMPLEX WE DO NOT HAVE TO CHANGE K, HOWEVER THE ARGUMENT THEN IS MORE ELABORATE THAN THE ONE WE GIVE HERE.)

THE PROCESS OF GOING FROM M TO M' IS BY ADDING A HANDLE TO M. PUT A RIEMANNIAN METRIC ON $N = \partial M \neq \phi$ SO WE CAN TALK ABOUT ν = NORMAL BUNDLE OF $g(S^p)$ IN ∂M, AND ASSUME ν IS TRIVIAL. (ACTUALLY THIS WILL BE SHOWN IN THE PROOF)

THERE ARE TWO STEPS: 1. HOW TO ATTACH A HANDLE 2. HOW TO ATTACH A HANDLE SO THAT THE TANGENT BUNDLE CONDITION IS SATISFIED.

STEP 1. LET $V_1, V_2, .. V_{m-p-1}$ BE LINR. INDEP. CROSS SECTIONS OF ν.

LET $\phi : S^p \times E^{m-p-1} \to \partial M$

BY $\phi(x, y_1, .. y_{m-p-1}) = \text{EXP}_{g(x)} \sum_{i=1}^{m-p-1} y_i (V_i\, g(x))$

WHERE $E^{m-p-1} = \{ y = (y_1, .., y_{m-p-1}) / |y| \leq 1 \}$

WHERE $|y| = \sqrt{\sum y_i^2}$. WE MAY ASSUME

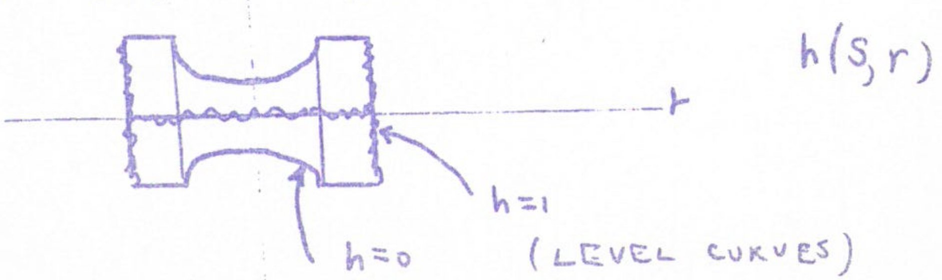
ϕ IS AN EMBEDDING.

WE ATTACH A HANDLE:

THE USUAL WAY TO ATTACH $E^{p+1} \times E^{m-p-1}$ TO M IS TO USE $\phi: S^p \times E^{m-p-1}$ AS AN ATTACHING MAP.

HOWEVER WE MUST ATTACH A HANDLE SO THAT THE RESULT IS A TUBULAR NBD. OF $K = L \cup e^{p+1}$

WE USE THE FOLLOWING FCT. h ON $E^1 \times E^1 \to R^1$ s DESCRIBED LAST TIME.



AND h IS C^∞.

DEFINE $H: E^{p+1} \times E^{m-p-1} \to R$

BY $H(x,y) = h(|x|, |y|)$

THE HIGHER DIMENSIONAL ANALOGUE OF THE ABOVE DIAGRAM IS

$$\mathcal{U} = \{ (x,y) \in E^{p+1} \times E^{m-p-1} / \ 0 \leq H(x,y) \leq 1 \}$$

THAT IS, IF $p=0$ & $m=3$ $\mathcal{U}$ IS OBTAINED BY REVOLVING THE ABOVE DIAGRAM ABOUT THE S AXIS.

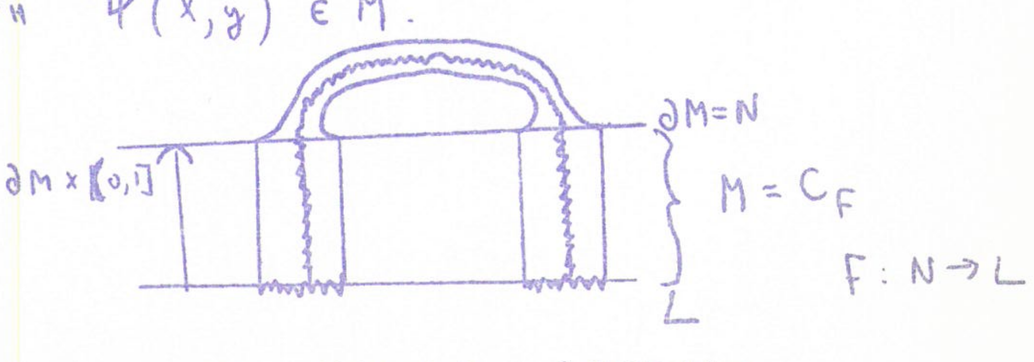
LET $V = \{ (x,y) \in \mathcal{U} \ / \ |x| \geq \frac{1}{2} \}$

WE MAP V INTO M BY $\psi : V \to M$

$$\psi(x,y) = \left(\phi\left(\frac{x}{|x|}, y \right), 2|x| - 1 \right)$$
$$\subseteq \partial M \times [0,1]$$

(IN V, $|x|$ RUNS FROM $\frac{1}{2}$ TO 1)

NOW WE DEFINE $M' = M'(M, \mathcal{U}, v_1, \ldots v_{m-p-1})$
$= M \cup \mathcal{U}$ WITH $(x,y) \in \mathcal{U}$ IDENTIFIED WITH $\psi(x,y) \in M$.



THE HANDLE ATTACHED

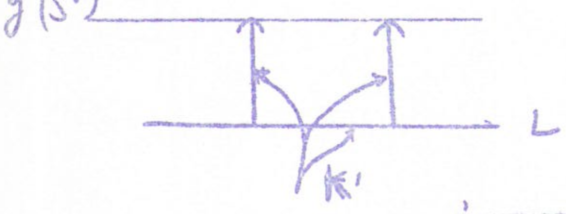
STEP 2.

LET $\overline{E}^{p+1} := \{ x \in E^{p+1} \mid |x| \leq \frac{1}{2} \}$

& $\overline{S}^p = \dot{\overline{E}}^{p+1}$

LET $K' = K - \overline{E}^{p+1} = L \cup e^{p+1} - \overline{E}^{p+1}$

WE IMBED K' IN M. L IS ALREADY IN M, WE PUT IN $e^{p+1} - \overline{E}^{p+1}$ BY IDENTIFYING $x \in e^{p+1} - \overline{E}^{p+1}$, i.e. $|x| \geq \frac{1}{2}$ WITH $\left(g\left(\frac{x}{|x|}\right), 2|x|-1 \right) \in \partial M \times (0,1]$.



NOW $\boxed{\tau(M)|_{\overline{S}^p} = \partial + \tau^p(S^p) + \xi'}$

WHERE ξ' IS A LINE BUNDLE.

SO $\eta|_{K'} \sim \tau(M)|_{K'}$ BECAUSE THIS HOLDS FOR L AND L IS A DEFORMATION RETRACT OF K'.

LET μ BE THE BUNDLE EQUIVALENCE

$$\mu : \eta|_{K'} \rightarrow \tau(M)|_{K'}$$

WE KNOW $\gamma^p(S^p) + \xi' \cdot\equiv\cdot \xi_2$ IS A TRIVIAL BUNDLE, BEING THE SUM OF A TANG. BUNDLE OF A SPHERE AND A TRIVIAL BUNDLE.

NOW VIA μ WE HAVE

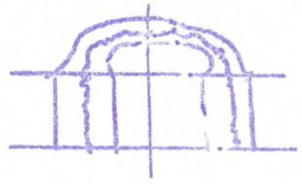
$$\eta|_{\bar{S}^p} = \xi_1 + \xi_2$$

. WE SEEK TO EXTEND THIS SPLITTING TO $\eta|\bar{E}^{p+1}$. WE OBTAIN A SET OF CROSS SECTIONS FOR ξ_2: WE PASTE $\mathcal{U}$ ON M, $\mathcal{U} \subseteq E^{p+1} \times E^{m-p-1}$

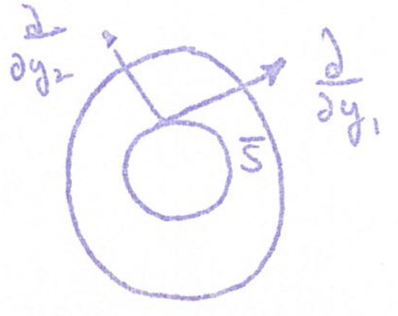
$$= \{ (x, y) \mid |x| \leq 1, |y| \leq 1 \}$$

AND TANG. VECTORS $\frac{\partial}{\partial y_i}$ GO INTO CROSS SECTIONS OF ξ_2.

(ROTATE:



ABOUT VERTICAL AXIS. LOOKING DOWN:



)

WE THUS OBTAIN CROSS SECTIONS FOR ζ_2 AND SO $p+1$ CROSS SECTIONS OF $\eta|\bar{S}^p$, i.e. OF $\tau(M)|\bar{S}^p$ AND EXTEND TO CROSS SECTIONS OF $\eta|\bar{E}^{p+1}$. THE OBSTRUCTION TO THIS EXTENSION LIES IN $\pi_p(V_{m,p+1})$ WHICH IS ZERO SINCE $2(p+1) \leq m$. LABEL THESE CROSS SECTIONS $\overline{w_1}, \dots \overline{w_{p+1}}$.

$$\begin{tikzcd}
\tau(M)|\bar{S}^p = \nu + \zeta_2 \\
\eta|\bar{S}^p = \zeta_1 + \zeta_2 \arrow[u, "\mu"]
\end{tikzcd}$$

$$\underbrace{\eta|\bar{E}^{p+1}}_{} = \bar{\zeta}_1 + \bar{\zeta}_2$$

(BUNDLE OVER A DISK.)

LET $\bar{v}_1, \dots, \bar{v}_{m-p-1}$ BE CROSS SECTIONS OF $\bar{\zeta}_1|\bar{E}_{p+1}$

LET $v_i = \mu^{-1}\bar{v}_i|\bar{S}^p$. THE v_i'S SERVE AS CROSS SECTIONS OF ν SO ν IS A TRIVIAL BUNDLE. NOW $\bar{\mu}: \eta \to \tau(M')|_K$ IS GIVEN BY:

$$\bar{\mu}(v) = \mu(v) \qquad v \in E_{\eta|K'}$$

$$\bar{\mu}\left(\sum a_i\bar{v}_i + \sum b_i\bar{w}_i\right)$$

$$= \sum v_i \frac{\partial}{\partial x_i} + \sum b_i \frac{\partial}{\partial \cdot}, \quad v \in E_{\eta|\bar{E}^{p+1}}$$

WE CAN CHECK THAT THESE PIECE TOGETHER NICELY. ∎

HOMOTOPY COMMUTATIVITY AND ROTATION GROUPS

BY IOAN JAMES

DEFN.

LET G BE A TOPOLOGICAL GROUP AND G', G'' SUBSPACES THEN G' HOMOTOPY COMMUTES, HTY-C, IFF THE MAP $c: G' \times G'' \to G$ DEFINED BY $c(x,y) = xyx^{-1}y^{-1}$ IS NULL HOMOTOPIC. IF G' + G'' ARE SUBGROUPS THEN

$$c: (G', e) \to e$$

$$c: (e, G'') \to e$$

SO c DEFINES A MAP $\tilde{c}$ ON THE SMASH PRODUCT

$$\tilde{c}: G' \divideontimes G'' \to G$$

$$\left(G' \divideontimes G'' \equiv \frac{G' \times G''}{(G', e) \cup (e, G'')} \right)$$

THEN c IS NULL HOMOTOPIC IFF $\tilde{c}$ IS NULL HOMOTOPIC.

WE NOW DEFINE THE GROUP O_m.

FOR THE REALS O_m IS THE REAL ORTHOGONAL GROUP
" " COMPLEXES " " " UNITARY GROUP
" " QUATERNIONS " " " SYMPLECTIC GROUP

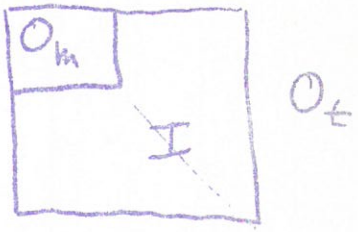
WE CAN IMBED O_m IN O_{m+1} BY CONSIDERING O_m AS TRANSFORMATIONS ON $m+1$ DIML. VECTOR SPACE OVER THE CORRESPONDING FIELD F LEAVING THE LAST COORDINATE FIXED.

WE REGARD $O_m, O_n \subseteq O_t$
$m,n \leq t$

WE WANT TO FIND m, n, t SUCH THAT O_m AND O_n HOMOTOPY COMMUTE IN O_t. DOES THERE EXIST SUCH A t?

YES. LET $t \geq m+n$.

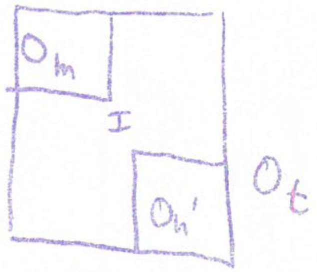
IMBED O_m IN O_t AS



(i.e. IMBEDDING MATRICES)

NOW BY CONJUGATING WITH RESPECT TO A FIXED ELEMENT IN THE PATH COMPONENT OF e WHICH IS HOMOTOPICALLY TRIVIAL WE SHIFT O_n TO O_n'.

THUS



AND O_m + O_n' COMMUTE IN O_t SO $O_m + O_n$ HTY-C IN O_t.

DEFN. LET $\ell(m,n)$ BE THE SMALLEST INTEGER t SUCH THAT O_m, O_n HTY-C. IN O_t. IN GENERAL $\ell(m,n)$ IS DIFFERENT FOR THE REALS, COMPLEXES, QUATERNIONS.

THEOREM $\ell(m,n) = m+n$, EXCEPT FOR $m=n=1$ IN COMPLEX CASE. WHEN F IS COMPLEXES OR QUATERNIONS.

REF. R. BOTT COMM. MATH. HELV. 1960

PROOF LET $m+n=k+t$, $t \geqq m,n$

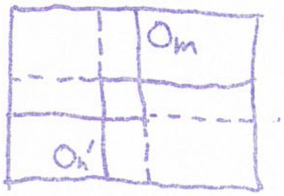
WE HAVE: $c: O_m \divideontimes O_n \longrightarrow O_t$

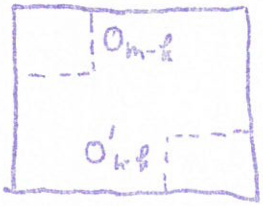
FOR WHAT t IS c NULL HOMOTOPIC?

WE KNOW IF $k=0$ c IS NULL HOMOTOPIC

SO ASSUME $k \geqq 1$.

AGAIN WE SHIFT O_n TO O_n' BY A HOMOTOPICALLY TRIVIAL MAPPING





NOW
$$O_{m-k} \divideontimes O_n \longrightarrow O_t$$
$$O_m \divideontimes O_{n-k} \longrightarrow O_t$$
ARE BOTH HOMOTOPICALLY TRIVIALLY.

DEFINE $O_{m,k} \equiv. O_m / O_{m-k}$

THEN $c: O_m \divideontimes O_n \rightarrow O_t$ INDUCES

$c': O_{m,k} \divideontimes O_{n,k} \longrightarrow O_t = O_{m+n-k}$

WE SHOW c' INDEP. OF COSET REPRESENTATIVES.

LET $x, y \in O_m$ $x \sim y$ IFF $x = ya$ $a \in O_{m-k}$. LET $z \in O_n'$

THEN $c: (x,z) \longrightarrow xzx^{-1}z^{-1} = yaza^{-1}y^{-1}z^{-1}$

BUT $a \in O_{m-k}$, $z \in O_n' \Rightarrow aza^{-1} = z$

$\therefore c(x,z) = c(y,z)$ SIMILARLY c' IS INDEP. OF COSET REPR. CHOSEN IN $O_{n,k}$.

NOTATION: (X, Y) = HOMOTOPY CLASSES OF MAPS $X \to Y$.

LET E BE A FIBER SPACE, WITH BASE SPACE B AND FIBER F. AND LET K BE A COMPLEX, SK ITS SUSPENSION. THEN $F \to E \to B$ INDUCES THE EXACT SEQUENCE:

$$(SK,E) \xrightarrow{\pi} (SK,B) \xrightarrow{\Delta} (K,F) \xrightarrow{I} (K,E)$$

Δ IS THE TRANSGRESSION OPERATOR.

SUPPOSE WE LET $K = O_{m,k} \divideontimes O_{n,k}$.

THEN $SK = O_{m,k} * O_{n,k}$

AND $O_{m+n-k} \to O_{m+n} \to O_{m+n,k}$

BE THE FIBER SPACE $(F \to E \to B)$

$$O_{m,k} \divideontimes O_{n,k} \xrightarrow{c'} O_{m+n-k}$$

$\{c'\} = \gamma \in (K,F)$

ASSUME $k \geq 1$ CLAIM $I(\gamma) = 0$.

BY EXACTNESS THEREFORE THERE EXISTS $\beta \in (SK, B) \ni \Delta(\beta) = \gamma$.

IF c' IS NULL HOMOTOPIC, i.e. $\gamma = 0$ THEN BY EXACTNESS $\exists\ \alpha \in (SK,E) \ni \pi(\alpha) = \beta$ AND CONVERSELY IF SUCH AN α EXISTS THEN c' IS NULL HOMOTOPIC.

DEFN. THE **JOIN** OF A_0 AND A_1 DENOTED $A_0 * A_1$ IS $A_0 \times I \times A_1$ WITH $(x_0, t, x_1) \sim x_i$ FOR $t = i$

WE DEFINE A MAP CALLED THE **INTRINSIC JOIN** $: SK = O_{m,k} * O_{n,k} \to O_{m+n,k} = B$

BY

$$\begin{array}{c|c|c|} & m & n \\ \hline k & A\cos\theta & B\sin\theta \\ & & \\ \hline \end{array} \qquad 0 \leq \theta \leq \pi/2$$

CONJECTURE WE CAN CHOOSE β ABOVE SUCH THAT β CONTAINS THE INTRINSIC JOIN.

R. BOTT PROVES TH'S FOR $k = 1$.

WHEN $k=1$ INTRINSIC JOIN IS A HOMEOMORPHISM.

WE NOW EXAMINE THE CASE $k=1$. WE HAVE

$$\begin{tikzcd} O_n \divideontimes O_m \arrow[rr, "c"] \arrow[dr, "P"'] & & O_{m+n-1} \\ & O_{n,1} \divideontimes O_{m,1} \arrow[ur, "c'"'] & \end{tikzcd}$$

. A COMM. DIAGRAM

QUESTION WHEN IS c NULL HOMOTOPIC?

(P IS SMASH OF PROJ. MAPS. $O_n \to O_m/O_{m-1}$
$O_n \to O_n/O_{n-1}$)

WE NOW CHOOSE $K = O_m \divideontimes O_n$

AND THE SAME FIBER SPACE

LET $L = O_{m,1} * O_{n,1}$

WE HAVE THE FOLLOWING DIAGRAM DEFINING THE MAP h. $h = h' \cdot p$

$$\begin{tikzcd}
SK = O_m * O_n \arrow[rr, "h"] \arrow[dr, "SP"'] & & O_{m+n,1} \\
& SL = O_{m,1} * O_{n,1} \arrow[ur, "h'"'] &
\end{tikzcd}$$

h' = INTRINSIC JOIN, A HOMEO.

WHERE BY R. BOTT'S RESULT $\{h'\} \in (SL, B)$ IS $\ni$ IN $(SL, E) \xrightarrow{\Pi'} (SL, B) \xrightarrow{D} (L, F) \xrightarrow{I} (L, E)$

$$D\{h'\} = \{c'\}$$

WE HAVE THE INDUCED COMM. DIAGRAM

$$\begin{tikzcd}
(SK, B) \arrow[r, "\Delta"] & (K, F) \\
(SL, B) \arrow[u, "(SP)^*"] \arrow[r, "D"] & (L, F) \arrow[u, "p^*"']
\end{tikzcd}$$

UNDER WHICH:

$$\begin{tikzcd}
\{h' \circ p\} = \{h\} \arrow[r, "\Delta"] & \{c' \circ p\} = \{c\} \\
\{h'\} \arrow[u, "(SP)^*"] \arrow[r, "D"] & \{c'\} \arrow[u]
\end{tikzcd}$$

SO WE HAVE ESTABLISHED:

<u>CRITERION</u> O_m, O_n COMMUTE IN O_{m+n-1} IFF h CAN BE LIFTED TO A MAP $g : SK \to E$, I.E. A MAP $O_m * O_n \longrightarrow O_{m+n}$

TO PROVE THE THM. OF BOTT THAT $l(m,n) = m+n$ IT IS SUFF. TO PROVE O_m, O_n DO NOT COMMUTE IN O_{m+n-1} AND THIS IS $\Leftrightarrow$ TO SHOWING h DOES NOT HAVE A LIFT.

WE PROVE THIS FOR THE COMPLEX NUMBERS. WE CONSIDER THE PROJECTION MAP $O_q \to O_{q,1}$ = SPHERE OF DIM $2q-1$. SUPPOSE WE HAVE A MAP BACK, $O_{q,1} \to O_q$ THE COMPOSITION THEREFORE IS A MAP: $S^{2q-1} \to S^{2q-1}$ AND WE CAN TALK OF ITS DEGREE. IT IS KNOWN THERE EXIST MAPS OF DEGREE l $\Leftrightarrow (q-1)!/l$. THUS THE SMALLEST POSSIBLE DEGREE $O_{q,1} \to O_{q,1}$ IS $(q-1)!$

NOW

$$\begin{tikzcd} O_{m+n,1} & \underset{\text{INTRINSIC JOIN}}{\equiv} & O_{m,1} \arrow[d, "\text{DEG }(m-1)!"'] & * & O_{n,1} \arrow[d, "\text{DEG }(n-1)!"] \\ & & O_m & * & O_n \end{tikzcd}$$

SUPPOSE O_m, O_n HTY-C. IN O_{m+n-1} THEN THERE EXISTS $g : O_m * O_n \to O_{m+n}$ THE COMPOSITE MAP

$$O_{m+n,1} \equiv O_{m,1} * O_{n,1} \longrightarrow O_m * O_n \xrightarrow{g} O_{m+n} \xrightarrow{\text{PROJ.}} O_{m+n,1}$$

THEN HAS DEGREE $(m-1)!\,(n-1)!$ A PROPER DIVISOR OF $(m+n-1)!$ IF m OR $n \neq 1$

THUS O_m, O_n DON'T HOMOTOPY COMMUTE IN O_{m+n-1} EXCEPT WHEN $m = n = 1$.

FOR THE REAL ORTHOGONAL GROUPS USING SPHERE BUNDLES AND STIEFEL WHITNEY CLASSES WE OBTAIN THE FOLLOWING:

THEOREM: SUPPOSE $m+n \neq 4$ OR 8 THEN O_m, O_n DON'T HOMOTOPY COMMUTE IN O_{m+n-1} IF m OR n IS EVEN OR $d(m) = d(n)$.
($d(q) \equiv$. GREATEST POWER OF 2 WHICH DIVIDES q^{-1}).

COR. O_n COMMUTES WITH ITSELF IN O_{2n-1} $\Longleftrightarrow$ $2n = 4$ OR 8.

IF $m+n = 12$ FOR PAIRS $(3,9)$ $(5,7)$ WE DON'T KNOW WHETHER O_m, O_n HTY-C. IN O_{m+n-1}.

NOTE: O_m, O_n NEVER HTY-C. IN O_{m+n-2}, FOR APPLY THM TO ONE OF (m,n); $(m-1,n)$; $(m,n-1)$.

CONTINUOUS, DIFFERENTIABLE, AND LINR. ACTIONS OF COMPACT LIE GROUPS

BY R. PALAIS

DEFN. LET X BE A TOP. SPACE, G A COMPACT LIE GROUP. IF Y IS THE SET OF HOMEOMORPHISMS OF X WITH COMPACT OPEN TOPOLOGY AN ACTION OF G ON X IS A HOMOMORPHISM OF $G \to Y$. (CONTINUOUS)

IF X IS A DIFFERENTIABLE MANIFOLD AND Y IS THE SET OF DIFFEOMORPHISMS OF X WITH THE C' TOPOLOGY THEN A DIFFERENTIABLE ACTION OF G ON X IS A CONT. HOMOMORPHISM OF $G \to Y$.

TWO ACTIONS ARE EQUIVALENT IF THEY DIFFER BY AN INNER AUTOMORPHISM. OF Y.

PROBLEMS

a) IF X IS A MANIFOLD AND G ACTS ON X CAN A DIFF. STRUCTURE FOR X BE CHOSEN MAKING THE ACTION DIFFERENTIABLE?

b) IF X IS A DIFF. MANIFOLD AND G ACTS ON X IS THE ACTION EQUIV. TO A DIFF. ACTION?

c) IF X IS DIFFEOMORPHIC TO E^n OR S^n IS A DIFF. ACTION OF G ON X EQUIV. TO A LINR. ACTION?

a) IS INCORRECT IF $G =$ IDENTITY GROUP BY KERVAIRE'S WORK. IT IS ALSO INCORRECT FOR NON-TRIVIAL G.

b) IS INCORRECT. WE HAVE <u>BOCHNER'S THM.</u> IF G ACTS. DIFF. ON $\underline{X}$ AND p IS A STATIONARY POINT THEN DIFF. COORDS. CAN BE CHOSEN AROUND p IN WHICH ACTION IS LINEAR.

BY AN EXAMPLE OF BING $\mathbb{Z}_2$ ACTS ON S^3 $\ni$ FIXED PT SET IS ALEXANDER'S HORNED SPHERE AND ∴ BY BOCHNER'S THM. THIS ACTION IS NOT EQUIV. TO A DIFF. ACTION.

WE SHALL DISCUSS TWO PAPERS:

1. EXAMPLES OF DIFF. GROUP ACTIONS ON SPHERES. BY MONTGOMERY AND SAMELSON
2. SOME EXAMPLES OF CONTRACTIBLE OPEN 3-MANIFOLDS. BY D. R. Mc MILLAN

IN THE PAPER OF MONTGOMERY AND SAMELSON WE HAVE THE FOLLOWING:

<u>THEOREM</u> LET G BE A NON-TRIVIAL COMPACT LIE GROUP, CHOUSE $h > 10$ S.T. G HAS A REAL LINR. REPR. OF DEG. h LEAVING ONLY THE ORIGIN FIXED. THEN $\exists$ A SEQ. OF DIFF. ACTIONS ON S^{h+4} WHICH HAVE NON-HOMEOMORPHIC STATIONARY SETS. (EACH SET WILL BE A HOMOLOGY SPHERE BUT NOT A SPHERE)

PROOF LET K BE A FINITE ACYCLIC 2-COMPLEX WITH $\pi_1(K)$ THE DOUBLED ICOSAHEDRAL GROUP. IT IS A PERFECT GROUP (i.e. = TO ITS COMMUTATOR GROUP) GIVEN BY 2 GENERATORS AND 2 RELATIONS.

ASIDE: IF Γ IS A PERFECT GROUP WITH k GENERATORS & k RELATIONS THEN THE 2-COMPLEX, K, INDUCED BY Γ IS ACYCLIC, WE HAVE FOR EACH GEN. A COPY OF S^1 AND FOR EACH RELN. A 2-CELL.

THEN $\pi_1(K) = \Gamma$ SO $H_1(K) = \Gamma / [\Gamma,\Gamma] = 0$ (BECAUSE Γ IS A PERFECT GROUP) NOW $H_2(K) = Z_2$ THE GROUP OF 2-CYCLES ($B_2 = 0$ BECAUSE K A 2-COMPLEX) SO $H_2(K)$ IS FREE. THUS SUFF. TO SHOW ITS RANK b_2 IS 0.

NOW EULER CHARACTISTIC OF K IS $1 - k + k$. SO

$$1 - k + k = 1 - b_1 + b_2$$

BUT $H_1(K) = 0$ SO $b_1 = 0$ THUS $b_2 = 0$.

THUS K IS ACYCLIC.

WE SEE ∴ THAT OUR PARTICULAR K ABOVE IS ACYCLIC.

LET $K_i = K \underbrace{\vee \cdots \vee}_{i \text{ times}} K$

THEN K_i IS AN ACYCLIC 2-COMPLEX $\pi_1(K_i)$ HAS MINIMAL SET OF $2i$ GENERATORS. $\therefore \pi_1(K_i) \not\cong \pi_1(K_j)$ FOR $i \neq j$.

NOW IMBED K_i IN S^5. (TWO COMPLEX IN 5 SPACE)

AND LET T_i BE A <u>REG. NBD.</u> (IN THE SENSE OF WHITEHEAD) OF K_i.

$\partial T_i = M_i$ A MANIFOLD.

LET $Y_i = S^5 - T_i^{\circ}$ (T_i° = INTERIOR OF T_i)

<u>CLAIM</u>: (a) Y_i IS A 5-MANIFOLD WITH $\partial Y_i = M_i$ CLEARLY. $\cdot|$

(b) $\pi_1(M_i) = \pi_1(K_i)$

<u>PROOF</u> $T_i \underset{h.t.}{\sim} K_i$ (HOMOTOPY EQUIVALENT) BECAUSE K_i IS A DEFORMATION RETRACT OF T_i. BY TAKING A 2-COMPLEX OUT OF T_i WE DO NOT CHANGE $\pi_1(T_i)$ BECAUSE WE ARE IN 5-SPACE.

SO $\pi_1(T_i) = \pi_1(T_i - K_i)$

BUT M_i IS A DEFORM. RETRACT OF $T_i - K_i$ THUS

$$\pi_1(K_i) = \pi_1(T_i) = \pi_1(T_i - K_i) = \pi_1(M_i) \cdot|$$

(c) Y_i IS SIMPLY CONNECTED. GIVEN A LOOP IN Y_i IT BOUNDS A DISK IN S^5 AND WE CAN PUSH THE DISK OFF K_i & USE RETRACTION OF $T_i - K_i$ ONTO M_i TO PUSH DISK INTO Y_i.

(d) Y_i IS ACYCLIC. $T_i \underset{h.t.}{\sim} K_i$ $\Rightarrow$ T_i ACYCLIC, SINCE $Y_i - M_i = S^5 - T_i$ THEN BY ALEXANDER DUALITY

$Y_i - M_i$ IS ACYCLIC AND

$Y_i - M_i \underset{h.t.}{\sim} Y_i \Rightarrow Y_i$ ACYCLIC. |

SO Y_i IS A CONTRACTIBLE 5-MANIFOLD WITH $\partial = M_i$.

Y_i ADMITS A DIFF. STRUCTURE. THIS IS PROVEN DIRECTLY IN THE PAPER OF MONTGOMERY AND SAMELSON OR CAN BE PROVEN USING MUNKRES' OBSTRUCTION THEORY.

<u>CLAIM</u>: $Y_i \times \underset{\substack{\uparrow \\ n\text{-DISK.}}}{D^n} \underset{\text{DIFFEO.}}{\approx} D^{n+5}$ $\quad n \geq 10$

BY A WELL KNOWN FACT WE NEED ONLY PROVE COMBINATORIALLY EQUIVALENT.

NOW $Y_i \times D^n$ IS A REGULAR NBD. OF Y_i IN R^{n+5} AND BY A RESULT OF J. H. C. WHITEHEAD WHICH STATES

THAT 2-SIMPLY CONNECTED k-COMPLEXES OF SAME HOMOTOPY TYPE HAVE ISOMORPHIC REGULAR NBDS. IN R^{k+5}, WE HAVE

$$Y_i \times D^n \underset{\text{COMB.}}{\approx} D^{n+5} .$$

THE SEQ. OF BOUNDARIES ARE NON-HOMEOMORPHIC HOMOLOGY SPHERES.

$$\partial | Y_i \times D^n) = S^{n+4} = M_i \times D^n \cup Y_i \times S^{n-1}$$

WE DEFINE AN ACTION ON S^{n+4}:

CONSIDER A LINR. ACTION ON D^n LEAVING ONLY THE ORIGIN FIXED AND TRIVIAL ON Y_i. THIS INDUCES A DIFF. ACTION ON $\partial (Y_i \times D^n) = S^{n+4}$. FOR THE FIXED PT. SET WE HAVE:

$$F(Y_i \times D^n) = Y_i \times 0$$

THUS

$$F(S^{n+4}) = \partial Y_i \times 0 = M_i \times 0 .$$

AND SINCE THE M_i ARE NOT HOMEOMORPHIC (HAVING AS WE HAVE SHOWN DIFFERENT FUND. GROUPS) THE THM. IS PROVEN. |

TURNING NOW TO THE PAPER OF D.R. McMILLAN WE HAVE:

<u>THEOREM</u> LET G BE A NON-TRIVIAL COMPACT LIE GROUP CHOOSE $n \geq 2$ S.T. G HAS A REPR. OF DEG. n LEAVING ONLY THE ORIGIN FIXED. THEN $\exists$:

(a) UNCOUNTABLY MANY DIFF. ACTIONS OF G ON R^{n+3} WITH NON-HOMEOMORPHIC FIXED PT. SETS.

(b) UNCOUNTABLY MANY CONT. ACTIONS OF G ON S^{n+3} WITH NON-HOMEO. FIXED PT. SETS.

PROOF: McMILLAN CONSTRUCTS UNCOUNTABLY MANY NON-HOMEO. OPEN CONTRACTIBLE 3-MANIFOLDS $\{M_\alpha\}$ AND PROVES:

LEMMA EACH M_α HAS LARGE COMPACT SETS WITH CONNECTED COMPLEMENT.

WE DEFINE: X IS SIMPLY CONNECTED AT ∞ $\Leftrightarrow$ FOR EACH COMPACT $K \subseteq X$, $\exists$ COMPACT L WITH $K \subseteq L \subseteq X$ s.t. THE INDUCED MAP

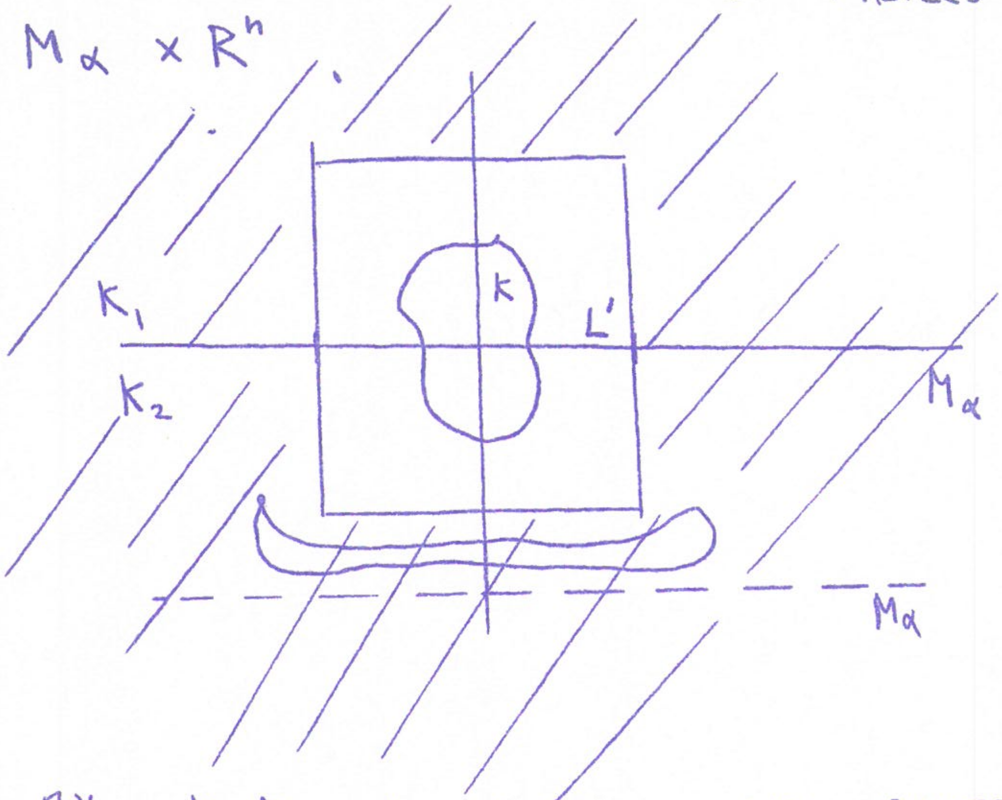
$$\pi_1(X - L) \to \pi_1(X - K) \text{ IS TRIVIAL.}$$

THERE IS THE FOLLOWING:

STALLINGS THEOREM IF X IS A CONTRACTIBLE OPEN COMBINATORIAL n-MANIFOLD SIMPLY CONNECTED AT ∞ + IF $n \geq 5$ THEN $X \underset{\substack{\text{COMB.}\\ \text{DIFFEO.}}}{\approx} R^n$

WE SHALL SHOW $M_\alpha \times R^n$ SIMPLY CONNECTED AT ∞ + HENCE BY STALLING'S THM. $M_\alpha \times R^n \underset{\text{DIFFEO.}}{\approx} R^{n+3}$ FOR $n \geq 2$.

WE SHOW $M_\alpha \times R^1$ SIMPLY CONNECTED AT ∞. THE PROOF GENERALIZES TO $M_\alpha \times R^n$.



BY McMILLAN'S LEMMA WE CHOOSE $L' \subseteq M_\alpha$ AND $m \ni L = L' \times [-m, m] \supseteq K$ AND $M_\alpha - L'$ IS CONNECTED.

NOW K_2 IS SIMPLY CONNECTED, FOR PROJECT ANY LOOP ONTO M_α (DASHED LINE) AND THEN CONTRACT VIA CONTRACTION OF M_α. (M_α RECALL IS AN OPEN CONTRACTIBLE 3 MANIFOLD) SIMILARLY K_1 IS SIMPLY CONNECTED AND $K_1 \cap K_2 \simeq M_\alpha - L'$ IS CONNECTED $\Rightarrow$ $K_1 \cup K_2$ IS SIMPLY CONNECTED. THUS $M_\alpha \times R^1 - L$ IS SIMPLY CONNECTED. |

NOW HAVING ESTABLISHED:

$$M_\alpha \times R^n \underset{\text{DIFFEO}}{\approx} R^{n+3} \qquad n \geq 2$$

WE DEFINE AN ACTION BY G ON R^n LEAVING ORIGIN FIXED, AND ON M_α BY LEAVING EVERY PT. FIXED. THIS IS ∴ A DIFF ACTION OF G ON R^{n+3} WITH FIXED SET $M_\alpha \times 0$ PROVING a).

TO PROVE b) WE NOTE THE ACTION ABOVE INDUCES A CONT. ACTION ON S^{n+3} WITH FIXED PT. SET $\hat{M}_\alpha$ (THE 1 PT. COMPACTIFICATION OF M_α). THEN $\hat{M}_\alpha \not\equiv \hat{M}_\beta$ FOR $\alpha \neq \beta$. FOR SUPPOSE NOT, THEN WE MAY ASSUME THE HOMEO. MAP $\infty \to \infty$ AND ∴ $M_\alpha \equiv M_\beta$ FOR $\alpha \neq \beta$. CONTR! THUS b) IS PROVEN. |

REMARK THE FOLLOWING THEOREM IS TRUE:

THEOREM TO WITHIN EQUIVALENCE $\exists$ AT MOST COUNTABLY MANY DIFF. ACTIONS OF A COMPACT GROUP ON A COMPACT DIFF. MANIFOLD. |

CORRESPONDENCES IN ABELIAN CATEGORIES

BY. D. PUPPE

DEFN. LET A, B BE LEFT R MODULES, AND F A SUBMODULE OF $A \times B$. THEN THE TRIPLE $f = (A, B, F)$ IS A GENERALIZED HOMOMORPHISM OR CORRESPONDENCE. IF F HAS THE PROPERTY THAT FOR $a \in A$ $\exists$ A UNIQUE b s.t. $(a, b) \in F$, THEN f IS AN ORDINARY HOMOMORPHISM.

$Df \mathrel{.=.} \{ a / a \in A + \exists\ b \in B \ni (a, b) \in F \}$ CALLED THE DOMAIN OF f.

$If \mathrel{=.} \{ b / b \in B, \text{ s.t. } (0, b) \in F \}$ IS CALLED THE INDETERMINACY OF f.

$f = (A, B, F)$ INDUCES A HOMOMORPHISM

$$A \supseteq Df \to B/If$$

. THE RELATION IS 1-1 BETWEEN CORRESPONDENCES AND INDUCED HOMOMORPHISMS.

OBVIOUSLY $Df = A$, $If = 0 \Rightarrow f$ IS A HOMO.

EXAMPLES OF CORRESPONDENCES

1) SECONDARY COHOMOLOGY OPERATIONS

2) CONSIDER A SHORT EXACT SEQ. OF CHAIN COMPLEXES,

$$0 \to C_1 \xrightarrow{i} C_2 \xrightarrow{p} C_3 \to 0$$

WHERE C_i IS A CHAIN COMPLEX (WE DISREGARD THE GRADING). WE OBTAIN THE EXACT TRIANGLE:

$$\begin{tikzcd}
H_1 \arrow[dr, "i_*"'] & & H_3 \arrow[ll, "\varepsilon_*"'] \\
& H_2 \arrow[ur, "p_*"'] &
\end{tikzcd}$$

(OR LONG EXACT SEQUENCE). ε_* THE CONNECTING HOMOMORPHISM. LET $h_V : C_V \to H_V$, h_V IS A CORRESPONDENCE. THEN $\varepsilon_* = h_3^{-1} p^{-1} d_2 i^{-1} h_1$ IS A HOMOMORPHISM, ALTHOUGH COMPOSED OF THREE CORRESPONDENCES, p^{-1}, h_3^{-1}, h_1.

NOTATION IN ε_* ABOVE, h_3^{-1} ACTS FIRST, ETC.

NOTE: IF $F \subseteq A \times B$, LET $F^{\#}$ BE THE CORRESPONDING SUBMODULE OF $B \times A$. SO p^{-1} ABOVE IS: $(C_3, C_2, (\text{GRAPH OF } p)^{\#})$ AND IN GENERAL IF f IS A CORRESPONDENCE $f = (A, B, F)$ $f^{\#} \equiv (B, A, F^{\#})$ WE SHALL RESERVE THE NOTATION f^{-1} FROM NOW ON TO GENUINE INVERSES.

3) SPECTRAL SEQUENCES.

WE HAVE: E_γ, d_γ s.t. $E_{r+1} \cong H(E_r)$ SUPPOSE $a \in E_1$ IS s.t. $d_1, d_2, d_3, \ldots$ VANISH ON a, THEN d_j IS A CORRESPONDENCE ON E_1.

WE WISH NOW TO: 1) GET AN AXIOMATIC SET UP FOR THESE CORRESPONDENCES.

2) ESTABLISH THE FUNDAMENTAL PROPS. OF CORRS.

3) COMPARE OUR SET UP WITH THE NOTION OF ABELIAN CATEGORIES.

<u>DEFN.</u> AN <u>INVOLUTION</u> OR <u>I-CATEGORY</u> $\mathcal{K}$ IS:

I: A CATEGORY WITH <u>OBJECTS</u> A, B AND <u>CORRESPONDENCES</u> OR <u>MORPHISMS</u> $f \in \mathcal{K}(A,B) = \mathrm{HOM}_{\mathcal{K}}(A,B)$.

II: IF $f \in \mathcal{K}(A,B)$ THEN $\exists\ f^{\#} \in \mathcal{K}(B,A)$ CALLED THE <u>CONVERSE</u> OF f.

III: $\exists$ A PARTIAL ORDERING "$\subset$" AMONG THE CORRESPONDENCES.

MOREOVER THE FOLLOWING <u>BASIC AXIOMS</u> HOLD:

$$(fg)^{\#} = g^{\#} f^{\#}$$

$$f^{\#\#} = f \qquad \text{(HENCE THE NAME INVOLUTION)}$$

$$f_1 \subset f_2 \Rightarrow f_1 g \subset f_2 g \qquad \text{FOR ALL } g.$$

$$f_1 \subset f_2 \Rightarrow f_1^{\#} \subset f_2^{\#}$$

LET S DENOTE THE CATEGORY OF SETS

" B " " " " " WITH BASE PT.

" G " " " " (NON. ABEL.) GROUPS

" M " " " " MODULES

ADDITIONAL AXIOMS:

(K-1) : $\exists$ AN OBJECT 0 s.t. $K(0,0)=\{1\}$ (i.e. HAS ONLY ONE MORPHISM). $\mathcal{K}(0,A)$ CONTAINS A SMALLEST ELEMENT ω_A AND A LARGEST ELEMENT Ω_A.

DEFN. LET $f \in \mathcal{K}(A,B)$, THE INDETERMINACY OF f DENOTED If IS:

$$If \equiv . \ \omega_A \cdot f \in \mathcal{K}(0,B)$$

(NOTATION: $A \xrightarrow{f} B \quad B \xrightarrow{g} C$ THEN $A \xrightarrow{fg} C$)

EXAMPLES OF CATEGORIES SATISFYING ABOVE REQUIREMENTS ARE:

i) S, $0 = \phi$ EMPTY SET, $\mathcal{K}(\phi, A) = \phi$

ii) B, SETS WITH BASE PT. $*$ $\quad 0 = \{*\}$

$0 \times A \cong A$, + BASE PT. MAPS TO BASE PT.

$\Rightarrow \mathcal{K}(0,A) = \{*\}$ SO $\Omega_A = \omega_A =$ BASE PT. OF A. AND $If =$ IMAGE OF BASE PT. OF A.

iii) G. iv) M.

DEFN. $Bf \equiv . \ \Omega_A \cdot f \in \mathcal{K}(0,B)$ CALLED THE IMAGE (BILD) OF f)

$$\left.\begin{aligned} Kf &\equiv . \ \omega_B f^{\#} \\ Df &\equiv . \ \Omega_B f^{\#} \end{aligned}\right\} \in \mathcal{K}(0,A)$$

CALLED THE KERNEL + DOMAIN OF f, RESPECTIVELY.

DEFN. f IS A **PROPER MORPHISM** OF THE CATEGORY IFF $If = w_B$, $Df = \Omega_A$

$\mathcal{E}(K)$ IS THE SUBCATEGORY OF PROPER MORPHISMS.

($f \in K(A,B)$, $g \in K(B,C)$ PROPER MORPHISMS , THEN $fg \in K(A,C)$ IS A PROPER MORPHISM SINCE
$I(fg) \equiv w_A(fg) = (w_A \cdot f)g = (If)\cdot g$
$= w_B \cdot g = Ig = w_C \cdot$)

QUESTION FOR $f : A \to B$, WHEN IS $f^{\#} f = 1_B$?

IF $g f^{\#} f \subset g$ THEN $wf \subset wg f^{\#} f \subset wg$

AXIOM (K-2) (a) $If \subset Ig \Rightarrow g f^{\#} f \subset g$
(TRUE FOR G, M NOT FOR S, B)
(b) $Bf \supset Bg \Rightarrow g f^{\#} f \supset g$
(TRUE FOR B, G, M NOT FOR S)

DEFN. A MORPHISM f IS CALLED **EPI**
$\Leftrightarrow Bf = \Omega_B f$

IT IS CALLED **MONO** $\Leftrightarrow Kf = w_A f^{\#}$

AND **ISO** $\Leftrightarrow$ EPI + MONO .

PROPOSITION f IS AN ISOMORPHISM $\Leftrightarrow$ f IS AN EQUIVALENCE . (+ IF SO $f^{\#} = f^{-1}$).

AXIOM (K-3)

IF $u \in K(0,A)$ THEN

(a) $\exists$ A [redacted] MONOMORPHISM $m: U \to A$ s.t. $Bm = u$

(b) $\exists$ AN EPIMORPHISM $e: A \to Q$ s.t. $Ke = u$.

(a) IS TRUE FOR (S, B, G, M)

(b) IS TRUE FOR (S, B, M)

DEFN. $C \in K$ IS A CHAIN OBJECT IF $\exists$ $d \in K(C,C)$, AN OBJECT H, AND $h: C \to H$ s.t. $Ih = \omega_H$ AND $Bh = \Omega_H$

$Dh = Kd$ AND $Kh = Bd$. H IS CALLED A HOMOLOGY OBJECT.

NOTE GIVEN C, $d \in K(C,C)$ THEN $Bd \subset Kd$ $\Rightarrow$ A HOMOLOGY OBJECT EXISTS.

GIVEN A SHORT EXACT SEQ. OF CHAIN OBJECTS THEN THE CONNECTING HOMOMORPHISM IS:

$$\varepsilon_* = h_3^{\#} p^{\#} d_2 i^{\#} h_1^{\#}$$

, ε_* IS A PROPER MORPHISM.

DEFN. A SPECTRAL SEQUENCE IS AN OBJECT, C WITH MORPHISMS $d_1, d_2, d_3, \ldots$ s.t.

$$Bd_r \subset Kd_r$$

AND

$$Dd_r = Kd_{r-1}$$

$$Id_r = Bd_{r-1}$$

NOW $Bd_r \subset Kd_r \Rightarrow$ HOMOLOGY OBJECT (H_r, h_r) OF (C, d_r) EXISTS.

WE HAVE:

$$\begin{tikzcd} C \arrow[r, "d_r"] \arrow[d, "h_{r-1}"'] & C \arrow[d, "h_{r-1}"] \\ H_{r-1} \arrow[r, "\overline{d_r}"] & H_{r-1} \end{tikzcd}$$

WHERE $\overline{d_r} \equiv h_{r-1}^{\#} \, d_r \, h_{r-1}$.

WE DEFINE $\overline{h_r} : H_{r-1} \rightarrow H_r$ AS $\overline{h_r} = h_{r-1}^{\#} \, h_r$

PROPOSITION (a) $\overline{d_r}$ IS A PROPER MORPHISM, (b) $(H_r, \overline{h_r})$ IS A HOMOLOGY OBJECT OF $(H_{r-1}, \overline{d_r})$ $(= E_r)$.

NOTE (K-3)(a) IS ONLY USED TO PROVE: $Bd \subset Kd \Rightarrow$ A HOMOLOGY OBJECT EXISTS.

THUS IN THE CATEGORY $\mathcal{G}$ OF (NON-ABEL.) GROUPS WE HAVE SPECTRAL SEQUENCES.

ALSO IN A CATEGORY WHERE (K-3) DOES NOT HOLD BUT $C, d_1, d_2, \dots$ EXISTS WITH $Id_r = Bd_{r-1}$ AND $Dd_r = Kd_{r-1}$ AND IF A HOMOLOGY OBJECT EXISTS THEN $(C, d_1, d_2, \dots)$ IS A SPECTRAL SEQUENCE.

DEFN. (A, C) IS AN *EXACT COUPLE* IF $\exists$

$$\begin{tikzcd} A \arrow[rr, "f"] & & A \arrow[dl, "g"] \\ & C \arrow[ul, "h"] & \end{tikzcd}$$

AN EXACT TRIANGLE WITH g, h PROPER MORPHISMS.

DEFINING $d_r = h\,(f^\#)^{r-1}\,g$ THEN $C, d_1, d_2, \ldots, d_r, \ldots$ IS A SPECTRAL SEQUENCE.

FOR: $Bd_r = \Omega_c\, h(f^\#)^{r-1} g \subset \Omega_c g \underset{\substack{\uparrow \\ \text{EXACTNESS}}}{=} w h^\#$

$\subset w g^\# f^{r-1} h^\# = K d_r$

SO $Bd_r = Kd_r$.

$$Dd_r \subset \Omega g^\# f^{r-1} h^\# = \Omega f^{r-1} h^\# = \begin{cases} \Omega h^\# = \Omega & r \leq 1 \\ \Omega f f^{r-2} h^\# \end{cases}$$

$= w g^\# f^{r-2} h^\# = K d_{r-1} \qquad r > 1.$

THEOREM 1 GIVEN AN ABELIAN CATEGORY $\mathfrak{A}$ THERE EXISTS AN I-CATEGORY K SATISFYING (K-1) — (K-3) SUCH THAT $\mathcal{E}(K) \cong \mathfrak{A}$.

THEOREM 2 IF K, K' ARE I-CATEGORIES SATISFYING (K-1) - (K-3) AND S.T. $\mathcal{E}(K) \cong \mathcal{E}(K')$ THEN $K \cong K'$.

THEOREM 3 IF K IS AN I-CATEGORY SATISFYING (K-1) — (K-3) THEN $\mathcal{E}(K)$ IS A *QUASI-EXACT ABELIAN* CATEGORY WHERE:

DEFN. A CATEGORY C IS *QUASI-EXACT* IF:

1) $\exists\ 0_{AB} \in C(A, B)$

2) KERNELS, COKERNALS, COIMAGES, IMAGES EXIST.

IF $f \in C(A, B)$ THE *KERNEL OF f*, *$\operatorname{KER} f$* IS A PAIR (U, u)

$\mathcal{U} \xrightarrow{u} A$ s.t.

i) u IS INJECTIVE, i.e. $xu = x'u \Rightarrow x = x'$

ii) $uf = 0_{\mathcal{U}B}$

iii) GIVEN $(\mathcal{U}', u')$ $\exists$ A UNIQUE g s.t. FOLLOWING DIAGRAM IS COMMUTATIVE:

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "u"] & A \arrow[r, "f"] & B \\
 & \mathcal{U}' \arrow[ul, dashed, "g"] \arrow[u, "u'"'] \arrow[ur, "0_{\mathcal{U}'B}"'] &
\end{tikzcd}$$

THUS KER f IS SOLN. TO A UNIVERSAL PROBLEM. SIMILARLY ONE DEFINES IMAGE OF f, COKERNEL OF f, COIMAGE OF f.

3) $C(A,B)$ HAS ADDITION, SATISFYING DISTRIBUTIVE LAWS.

ADDITIONAL AXIOMS

(K-4) $K(A,B)$ IS A LATTICE; i.e. $g_1, g_2 \in K(B,C)$ THEN $g_1 \cap g_2 \in K(B,C)$ AND $f \in K(A,B) \Rightarrow f(g_1 \cap g_2) \subset fg_1 \cap fg_2$ ..

NOW $If \subset Kg_\nu$ $(\nu = 1,2) \Rightarrow f(g_1 \cap g_2) \supset fg_1 \cap fg_2$

SO WE HAVE:

(K-5) $If \subset Kg_1$

(K-6) GIVEN $A_1, A_2 \in \mathcal{K}$ $\exists\, g_\nu : P \to A_\nu$ $(\nu = 1, 2)$ s.t. g_ν IS A PROPER MORPHISM AND $g_1^\# g_2$ IS THE LARGEST EL. IN $K(A_1, A_2)$ CALL IT $\Omega_{A_1 A_2}$ (WE KNOW $\Omega_{A_1 A_2}$ EXISTS, K-6 TELLS US THAT IT CAN BE WRITTEN AS $g_1^\# g_2$).

WITH THESE THREE ADDITIONAL AXIOMS HOLDING FOR K WE CAN SHOW $\mathcal{E}(K)$ IS ABELIAN.

THEOREM $\mathcal{K} \to \mathcal{E}(K)$ AND $\mathfrak{A} \to K(\mathfrak{A})$ ARE ISOMORPHISMS UNDER ISOMORPHISMS OF CATEGORIES.

(BY THM. 1 GIVEN $\mathfrak{A}$ $\exists\, \mathcal{K}$ s.t. $\mathcal{E}(K) \cong \mathfrak{A}$ AND IF $\mathcal{K}'$ IS ANY CATEGORY s.t. $\mathcal{E}(K') \cong \mathfrak{A}$ THEN BY THM. 2 $\mathcal{K} \cong \mathcal{K}'$ SO THE MAP $\mathfrak{A} \to \mathcal{K}$ IS 1-1 SO WE WRITE THIS $\mathcal{K}$ AS $K(\mathfrak{A})$. THIS IS OUR MAP $\mathfrak{A} \to K(\mathfrak{A})$ IN THM. ABOVE.)

APPLICATIONS TO SECONDARY COHOMOLOGY OPERATIONS

DEFN. SUPPOSE (α, β) ARE CORRESPONDENCES s.t. FOR ANY TOP. SPACE WE HAVE:

$$H^*(X) \xrightarrow{\alpha} H^*(X) \xrightarrow{\beta} H^*(X)$$

$\alpha\beta = 0$

THEN A CORRESPONDENCE $\phi : H^n(X) \to H^{n+p}(X)$ IS A *SECONDARY COHOMOLOGY OPERATION ASSOCIATED WITH (α, β)* IFF

1) IF $f : X \to Y$ IS CONT.

THEN IN

$$\begin{tikzcd}
H^n(X) \arrow[r, "\phi"] & H^{n+p}(X) \\
H^n(Y) \arrow[u, "f^*"] \arrow[r, "\phi"] & H^{n+p}(Y) \arrow[u, "f^*"']
\end{tikzcd}$$

WE HAVE: $\phi f^* \subset f^* \phi$

2) IF $0 \to X \xrightarrow{i} Y \xrightarrow{j} Y/X = Z$

AND Z SATISFIES HOMOTOPY EXT. PROPERTY

THEN IN $H^*(X) \xleftarrow{i} H^*(Y) \xleftarrow{j} H^*(Z) \xleftarrow{h} H^*(X)$

WE HAVE: $i \phi h = \alpha j^{\#} \beta$.

STRUCTURAL STABILITY OF DIFFERENTIAL MAPPINGS

R. THOM

THEOREM LET $f : R \to R$ BE OF CLASS C^m AND S.T. $f^{(k)}(t) \neq 0$ ON $[a, b]$ THEN THERE EXISTS $\psi : u \to t$ A LOCAL DIFFEO. $\psi' \neq 0$ ON $[a, b]$ WITH INVERSE $\varphi : t \to u$ S.T. $f(t) = P_k(\varphi(t))$. WHERE P_k IS A POLYN. OF DEG. k. WE MAY CHOOSE ψ S.T. ψ IS OF CLASS C^{m-1}, AND ψ = IDENTITY OUTSIDE $[a', b']$ $a' \leq a$, $b \leq b'$.

TO PROVE THIS THEOREM WE NEED:

LEMMA 1

LET f, g BE REAL VALUED FCTS. OF CLASS C^m f MIN AT O, g MIN. AT β AND $f + g$ HAVE SAME ORDER OF CONTACT.

THEN THERE EXISTS A CHANGE OF VARIABLES WHICH IS A LOCAL DIFFEOMORPHISM, $\psi : u \to t$ S.T. LOCALLY $f(\psi(u)) = g(u)$.

PROOF IT IS SUFF. TO PROVE THIS FOR $f = t^{k-1}$

NOW $g(u) = Au^{k-1} + h(u)\, u^k$, $h(u)$ CLASS C^{m-k}

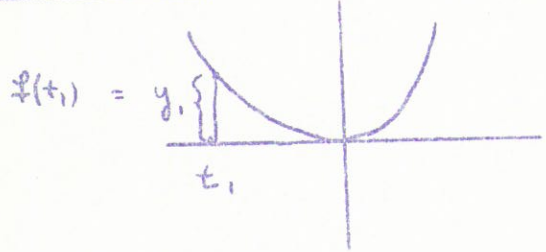
ASSUME $A > 0$. THEN LET $\tau^{k-1} = \left(A^{\frac{1}{k-1}} u\right)^{k-1} \left[1 + \frac{h(u)}{A} u\right]$

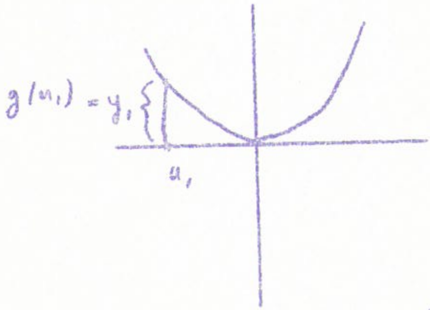
AND $\tau = A^{\frac{1}{k-1}} u \left[1 + \frac{h(u)\cdot u}{A} \right]^{\frac{1}{k-1}}$

THEN $\psi : u \to \tau$ IS A LOCAL DIFFEOMORPHISM.

& $f(\psi(u_1)) = (\psi(u_1))^{k-1} = \tau^{k-1} = Au^{k-1} + h(u)u^k = g(u)$. ▮

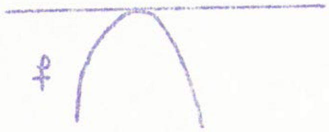
GRAPHICALLY:

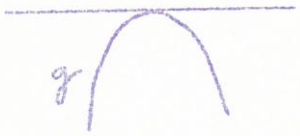




GIVEN u_1 s.t. $g(u_1) = y_1$ FIND t_1 s.t. $f(t_1) = y_1$.
THEN $u_1 \to t_1$ IS THE LOCAL DIFFEO.

MORE GENERALLY WE DO THE SAME WITH





WHERE f, g HAVE MAX OF SAME ORDER OF CONTACT.

FURTHER IF f, g HAVE MAX, MIN IN SAME ORIENTATION WITH SAME ORDER OF CONTACT WE ALWAYS CAN FIND A LOCAL DIFFEO. CHANGE OF VARIABLES.

NOW TO PROVE THE THM. CONSIDER $k = 4$
THEN f HAS AT MOST 3 CRITICAL VALUES

WE WISH TO FIND A POLYNOMIAL (DASHED LINE) P_f WITH MAX., MIN. IN SAME ORIENTATION AND SAME ORDER OF CONTACT.

THEN BY ~~THE~~ LEMMA I WE GET A DIFFEOMORPHIC CHANGE OF VARIABLES $\psi : u \to t$ s.t. $f(\psi(u)) = P_f(u)$.

THUS IT IS SUFF. TO EXHIBIT SUCH POLYNOMIALS. THIS IS DONE IN AMER. MATH. MONTHLY NOV. 57

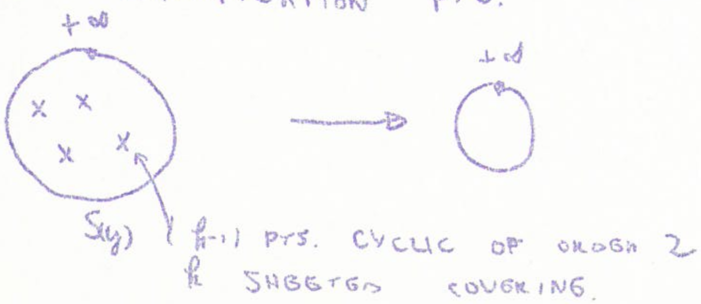
WE GIVE A TOPOLOGICAL CONSTRUCTION.

__LEMMA 2__ THERE EXISTS A POLYNOMIAL OF DEG. k WITH THE SAME CRITICAL VALUES AND SAME ORDER OF CONTACT AS f.

__PROOF__ WE CONSIDER THE PROBLEM IN THE COMPLEX PLANE. $y = P_k(x)$.

MAP RIEMANN SPHERES: $S(x) \xrightarrow{P_k} S(y)$

IS A RAMIFIED COVERING, CRITICAL VALUES OF P_k ARE RAMIFICATION PTS.



BY GLOBAL ISOTOPY CAN SHIFT CRITICAL PTS. TO A FIXED SET LEAVING $+\infty$ FIXED.

WE GET $S_{(x)} \to S_{(y)}$ MUST BE A POLYN. OF DEG. k.

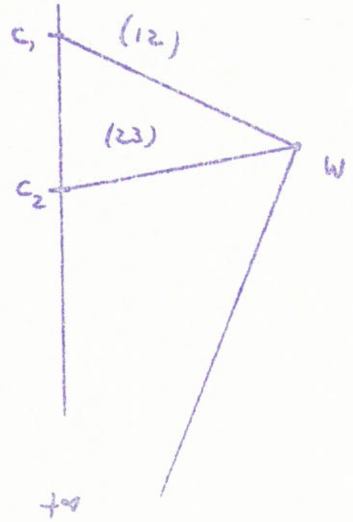
FROM AN ALGEBRAIC PT. OF VIEW $k-1$ CRITICAL VALUES OF A POLYNOMIAL OF DEG. k DO NOT UNIQUELY DETERMINE IT, SINCE WE HAVE TRANSLATION BY $x \to ax+b$. WE NORMALIZE THE POLYNOMIAL P_k $\ni$. A_1 = SUM OF ROOTS = 0 & s.t. $A_0 = 1$

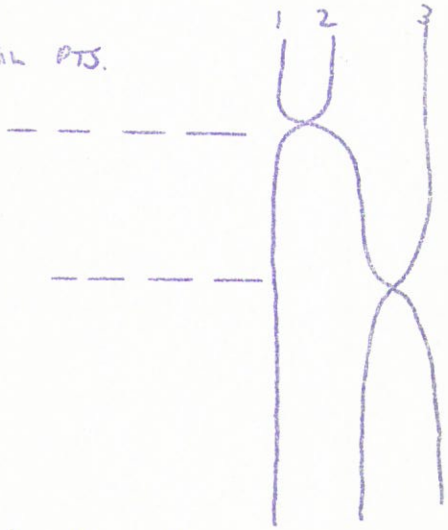
$$P_k = x^k + A_2 x^{k-2} + \dots + A_k \, .$$

WE NEED TO KNOW IF CRITICAL VALUES ARE REAL THE CORR. COMPLEX POLYN. HAS REAL COEFFS. WANT $P_k(\bar{x}) = \overline{P_k(x)}$.

WE WANT TO SHOW SYMMETRY CAN BE LIFTED.

<u>CASE $k=3$</u> SO AT MOST $k-1=2$ CRITICAL PTS.

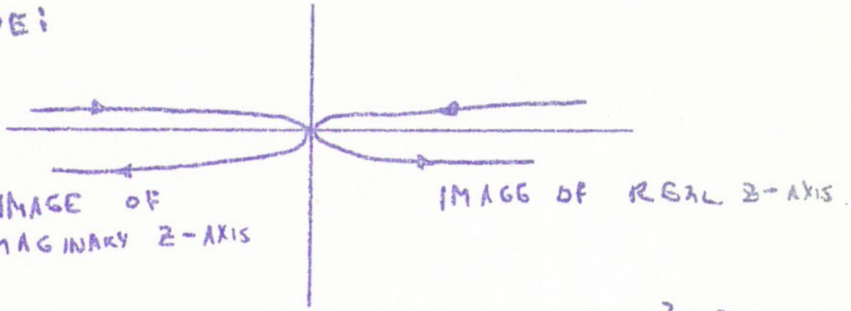




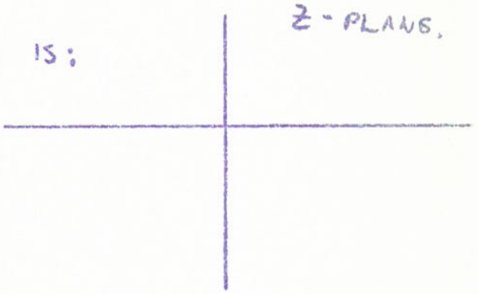
COVERING BY 3-SHEETS

(RECALL IF $W = Z^2$ WE HAVE A 2-FOLD COVERING

AND IN W-PLANE:

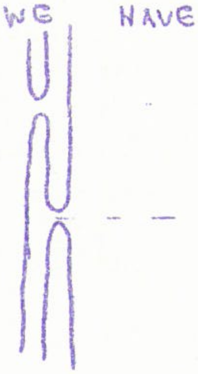


COUNTER IMAGE OF REAL W-AXIS IS:



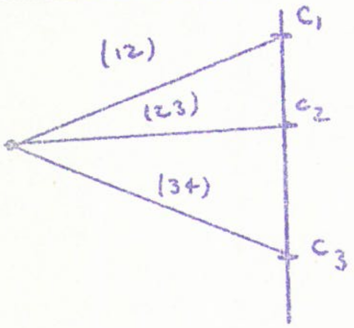
)

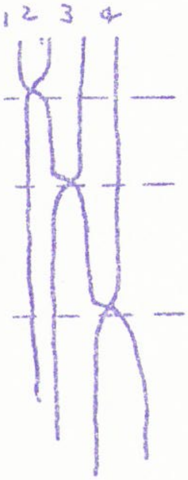
CORRESPONDINGLY WE HAVE:



COUNTER IMAGE

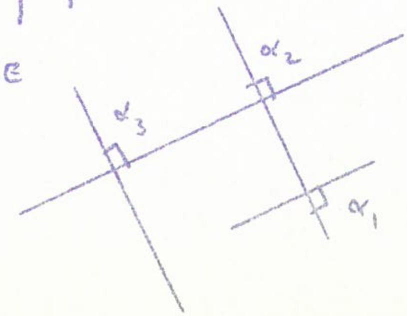
CASE $k=4$.





∴ WE HAVE

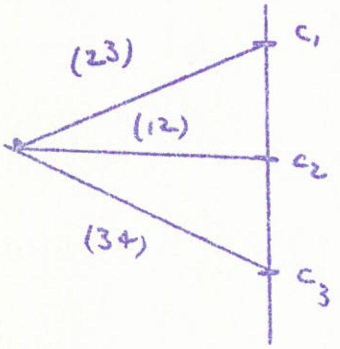
COUNTER IMAGE

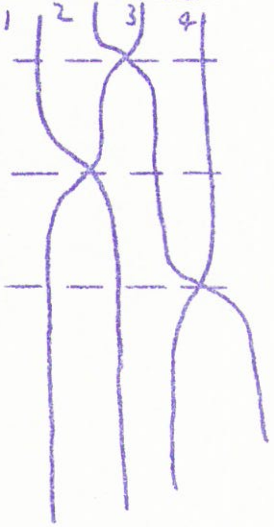


CLAIM THE CORRESPONDING POLYN. CANNOT HAVE REAL COEFFS. FOR WE CANNOT GET AN AXIS OF SYMMETRY WITH RESPECT TO ONE OF ITS LINES.

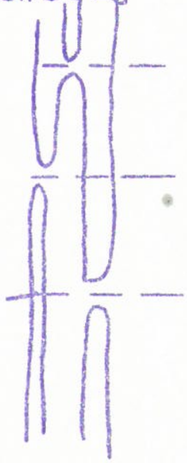
SO ∃ A COMPLEX POLYN. OF DEG. 4 ALL OF WHOSE CRITICAL VALUES ARE REAL BUT NOT HAVING REAL COEFFS.

HOWEVER LET US REORDER THE SHEETS

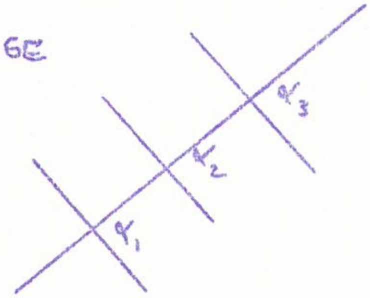




∴ HAVE



WITH COUNTER IMAGE

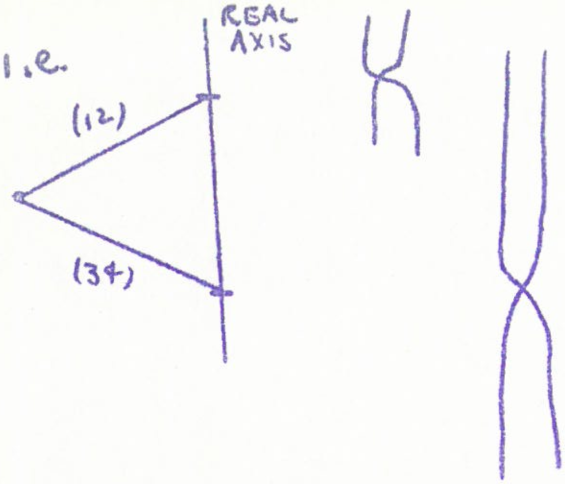


(SYMMETRIC)

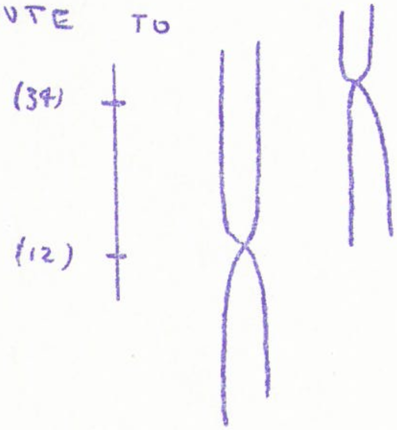
SO CORRESPONDING POLYN. HAS REAL COEFFS.

IF CRITICAL VALUES INVOLVE DISTINCT LEAVES THEN WE MAY PERMUTE WITHOUT AFFECTING THE TOPOLOGICAL STRUCTURE OF THE COVERING.

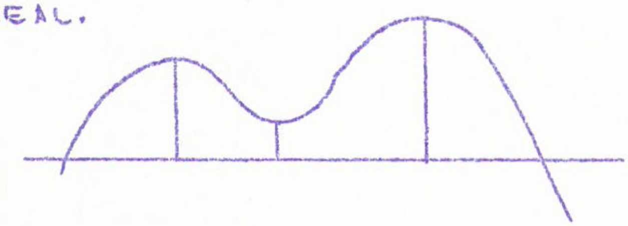
i.e.



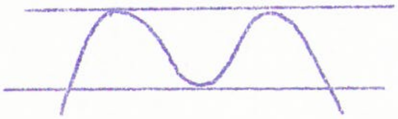
PERMUTE TO



GIVEN NOW A POLYNOMIAL OF ORDER k
ASSUME ALL $k-1$ ZEROS OF THE DERIVATIVE
ARE REAL.

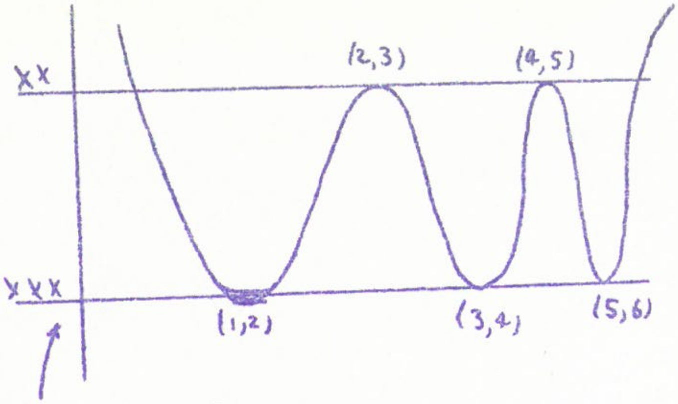


WE CAN ALWAYS REDUCE THE TOPOLOGICAL COVERING TO THAT OF A POLYNOMIAL LIKE:



A TSCHEBYSHEFF POLYN. GIVES THIS TRANSFORMATION.

$k = 6$



PERMUTATIONS OF SHEETS INVOLVING DISTINCT LEAVES

BY A TRANSFORMATION USING A TSCHEBYSCHEFF POLYNOMIAL, BY PERMUTING ONLY DISTINCT CRITICAL VALUES YOU CAN GET ANY OTHER TOPOLOGICAL TYPE OF ANY OTHER FCT. OF DEG. k.

DEFN. $f, g : X \to Y$ ARE OF SAME TOPOLOGICAL TYPE IF $\exists$ HOMEOMORPHISMS h, h' S.T. WE HAVE THE FOLLOWING COMMUTATIVE DIAGRAM:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "h'"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

CONSIDER NOW f, g TO BE OUR POLYNOMIALS P_k OF DEGREE k AND ASSUME $(k-1)$ ROOTS OF DERIVATIVE ARE ALL REAL.

WE EXAMINE THE TOPOLOGICAL TYPE OF P_k

CONSIDER THE OBJECT $(1, .., k)$ AND A PERMUTATION OF IT, W. THEN W MAY BE BROKEN UP INTO $k-1$ TRANSPOSITIONS $W = \alpha_1 .. \alpha_{k-1}$

CALL THIS A PRESENTATION OF W.

TO PRESENTATIONS W, W' ARE EQUIVALENT

IF $W = \alpha_1 \cdot \; \alpha_i \alpha_j \cdot\cdot \; \alpha_{k-1}$

$W' = \alpha_1 \cdot \; \alpha_j \alpha_i \cdot\cdot \; \alpha_{k-1}$

α_i, α_j DISTINCT TRANSPOSITIONS.

FOR ONE PRESENTATION OF A TSCHEBYSHEFF POLYN. ANY OTHER MEMBER OF EQUIV. CLASS CORR. TO POLYN. OF SAME TOPOLOGICAL TYPE. ∎

PSEUDO-ISOTOPIES ON COMPACT MANIFOLDS

BY R. FINNEY

DEFN. A HOMOTOPY $H: \underline{X} \times \underline{I} \xrightarrow{\text{ONTO}} \underline{X}$ IS A PSEUDO-ISOTOPY ON $\underline{X}$ IF H_t IS A HOMEOMORPHISM ONTO FOR ALL $t<1$.

FOR EXAMPLE, THE UNIT DISK IN THE PLANE MAY BE SHRUNK TO THE ORIGIN BY A PSEUDO-ISOTOPY ON THE PLANE WHICH CONTRACTS THE DISK RADIALLY AND WHICH IS THE IDENTITY OUTSIDE OF A NBD. OF THE DISK.

GIVEN A MAP $f: \underline{X} \xrightarrow{\text{ONTO}} \underline{X}$ UNDER WHAT CONDITIONS DOES THERE EXIST A PSEUDO-ISOTOPY H ON $\underline{X}$ S.T. $H_1(x) = f(x)$.

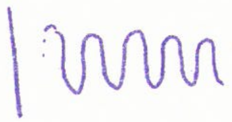
REMARKS: 1) BING SHOWED, USING THE WORK OF YOUNGS, FLOYD AND FORT, THAT FOR $\underline{X} = S^2$, THEN f MONOTONE (I.E. $f^{-1}(\text{PT.})$ IS CONNECTED) IS A SUFF. CONDITION.

2) THIS CONDITION IS NOT SUFF. FOR $\underline{X} = S^3$

3) IF $\underline{X} = \Delta'd\ S^3$ AND f SIMPLICIAL THEN A SUFF. COND. IS $f^{-1}(\text{PT.})$ IS CELLULAR I.E. $f^{-1}(\text{PT.}) = \cap\{\text{CLOSED 3-CELLS } C_i \mid C_{i+1} \subseteq C_i^{\circ}\}$

<u>NOTE:</u>  +  ARE CLEARLY CELLULAR.

 S^1 IS NOT CELLULAR.

THE TOPOLOGIST'S SINE CURVE  IS CELLULAR, BUT NOT A CELL.

BORSUK'S EXAMPLE OF THE WORM-HOLE CYLINDER IS CELLULAR BUT NOT A CELL.

<u>THEOREM</u> X = COMPACT n-MANIFOLD $(\partial = \phi)$ $f : X \xrightarrow{\text{ONTO}} X$. IF $\exists$ A PSEUDO-ISOTOPY H ON X s.t. $H_1(x) = f(x)$ THEN $f^{-1}(\text{PT.})$ IS CELLULAR, i.e. $= \bigcap \{ \text{CLOSED } n \text{ CELLS } C_i / C_{i+1} \subseteq C_i^{\circ} \}$

<u>PROOF</u> LET $A = f^{-1}(q)$ TO SHOW

$$A = \bigcap \{ \text{CLOSED } n\text{-CELLS } C_i / C_{i+1} \subseteq C_i^{\circ} \}$$

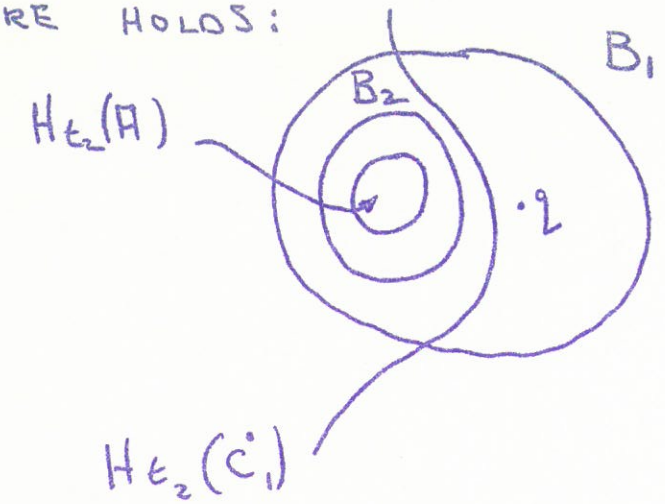
+ AS $t \to 1$ $H_t(A) \to q$ $(\text{DIAM } H_t(A) \to 0)$

LET B_1 BE A BALL ABOUT q.

CHOOSE $t_1 < 1$ s.t. $H_{t_1}(A) \subset B_1^{\circ}$

LET $C_1 = H_{t_1}^{-1}(B_1)$ A <u>CELL</u>.

WE SEEK $t_2 < 1$ s.t. THE FOLLOWING PICTURE HOLDS:



DEFINE $C_2 = H_{t_2}^{-1}(B_2)$.

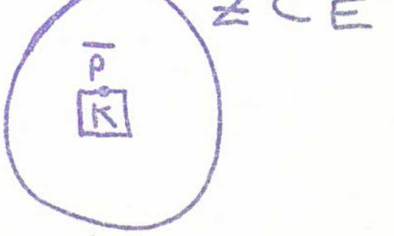
GIVEN t_{i-1}, B_{i-1}, C_{i-1} TO FIND t_i, B_i, C_i s.t. $C_i \leq C_i^{\circ}$, $A \leq C_i^{\circ}$ $t_i \to 1$

DIAM. $B_i \leq 2$ DIAM. $H_{t_i}(A)$.

<u>A SIMILAR PROBLEM:</u> K COMPACT $\subset$ INTERIOR OF A CLOSED n-CELL, Z. FIND AN n-CELL Z' s.t. $Z' \subset Z^{\circ}$ AND s.t. $K \subset Z'$. AND DIAM. $Z' \leq 2$ DIAM. K

EXAMPLES SHOW THAT NO SUCH Z' MAY EXIST, BUT IF ONE REQUIRES ALSO THAT $d(K, \dot{Z}) \geq 4$ DIAM. K

AND THAT $Z \subset E^n$

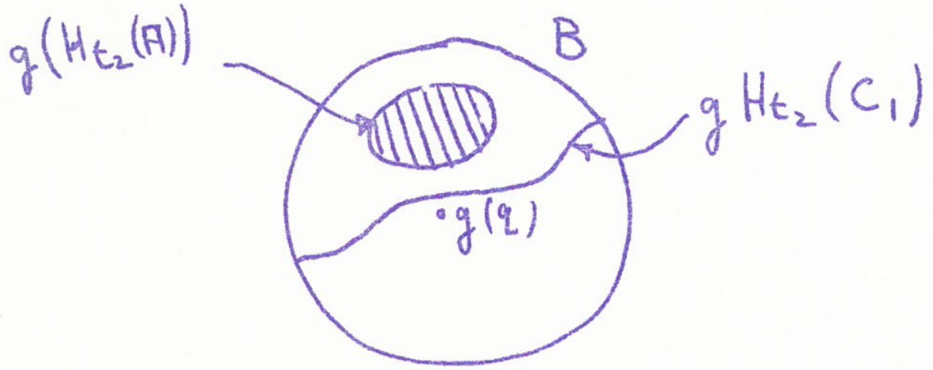


THEN ONE MAY TAKE

$Z' = \{ p \mid d(p, \bar{p}) \leq 2 \text{ DIAM. } K \}$

WHERE $\bar{p}$ IS A FIXED PT. OF K AND d IS THE USUAL CARTESIAN METRIC.

BACK TO OUR PROOF. LET $g : B_1 \to B$, B CLOSED UNIT BALL IN E^n



TO CONSTRUCT B_2

FIND t_2 LARGE ENOUGH SO THAT

$$H_{t_2}(A) \subset B_1$$

$$T \equiv g(H_{t_2}(C_1) \cap B_1)$$

PICK t_2 s.t.

1) $H_{t_2}(A) \subseteq \overset{\circ}{B_1}$
2) $d(g(H_{t_2}(A)); \text{BDRY } T) \geq 4 \text{ DIAM } g H_{t_2}(A)$
3) $1 > t_2 > \frac{t_1 + 1}{2}$

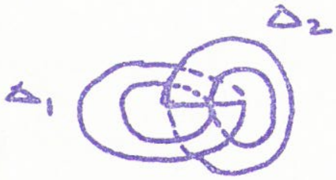
THE FACT THAT $\rho(H_t(A), H_t(\dot{C}_1)) > \varepsilon > 0$ (NOTE: DIAM $H_t(A) \to 0$) ALLOWS US TO DO 2). 1) AND 3) GIVE NO TROUBLE. THE GENERAL INDUCTION STEP GOES ALONG THE SAME LINES.

EXAMPLES:

① WE CONSTRUCT A MAP $f: S^3 \to S^3$ WHICH IS MONOTONE, BUT FOR WHICH $\exists$ A PT. q s.t. $f^{-1}(q)$ IS A 1-SPHERE (IN PARTICULAR, $f^{-1}(q)$ IS NOT CELLULAR)

CONSIDER THE TWO DISKS Δ_1, Δ_2 IN S^3 INTERSECTING IN A LINE-SEGMENT WHICH IS A RADIUS OF EACH DISK.

LET D BE THE FOLLOWING DECOMPOSITION OF S^3:



D

$d \in D$ IF

1) d IS A PT. OF $S^3 - (\Delta_1 \cup \Delta_2)$
2) $d = \text{BDY}(\Delta_1)$
3) $d = \text{BDY}(\Delta_2)$
4) d IS ONE OF THE UNCOUNTABLY MANY FIGURE-EIGHTS LYING IN $(\Delta_1 \cup \Delta_2)$ (SEE FIG.)

THIS DECOMPOSITION IS DUE TO R.H. BING.

WE SHOW $D^* \approx S^3$ WHERE D^* IS THE HYPERSPACE OF THE DECOMPOSITION D.

CONSIDERING THE HORIZONTAL DISK AS A DEFLATED BALLOON WE GET THE FOLLOWING FIGURE:

WHEN IT IS INFLATED.

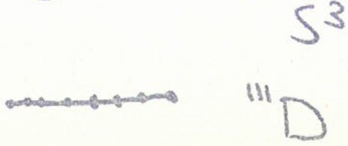
IN THIS WAY, THE DECOMPOSITION D INDUCES A DECOMPOSITION $'D$ OF THE 3-CELL $C^3 = S^3 - \text{INT.}\ (\text{BALLOON})$

THE HYPERSPACE $'D^*$ IS HOMEOMORPHIC TO D^*

WE NOW COLLAPSE THE BALLOON AGAINST ITS "VERTICAL AXIS" TO GET A DECOMPOSITION $''D$ OF S^3 WHOSE NON-DEGENERATE ELS. ARE THE UNCOUNTABLY MANY FIGURE ⊃ SHAPED ARCS SUGGESTED IN THE FIGURE BELOW

THE HYPERSPACE $''D^*$ IS HOMEOMORPHIC TO $'D^*$.

BUT THE ELS. OF THE DECOMPOSITION $''D$ MAY BE COLLAPSED TO GIVE THE TRIVIAL DECOMP. $'''D$ OF S^3.

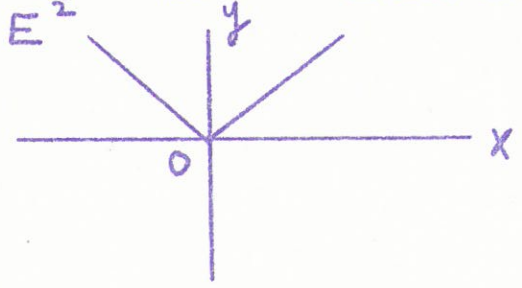


AND $''D^* \approx '''D^* \approx S^3$

THE HOMEOMORPHISMS FIT TOGETHER TO GIVE $D^* \approx S^3$.

LET f BE THE MAP OF S^3 ONTO D^* THAT SENDS EACH PT. OF S^3 TO THE EL. OF D IN WHICH IT LIES. f IS A MAP OF S^3 ONTO S^3, AND $\text{BDY}(\Delta_1) = f^{-1}(q)$ FOR SOME $q \in S^3$. |

② AN EXAMPLE OF A PSEUDO-ISOTOPY H ON E^2 FOR WHICH H_1^{-1}(ORIGIN) IS THE NON-NEGATIVE PORTION OF THE y-AXIS. IN PARTICULAR, $H_1^{-1}(0)$ IS NOT CELLULAR



LET $A = \{(x,y) / y \geq |x|\}$

$B = \{(x,y) / 0 \leq y \leq |x|\}$

$C = \{(x,y) / y < 0\}$

DEFINE
$$H_t(x,y) = \begin{cases} (1-t)(x,y) + t(0,x) & \text{ON } A \\ (x-ty,\, y) & \text{ON } B \\ (x,y) & \text{ON } C \end{cases}$$

On Formulae of Thom & Wu.

FPPeterson

M compact diff n-manifold, $\mu \in H_n(M; Z_2)$.

$v_i \in H^i(M; Z_2)$ uniquely defined by $< Sq^i x, \mu > = < v_i x, \mu >$

∇ $W_k = \sum_{i+j=k} Sq^j v_i$.

Thus can be defined in more general situation — do they satisfy same relations?

$H = \sum_{i=0}^{n} H^i$ over Z_2, Sq^i operators $\ni$

1) $Sq^i(hk) = \sum_{r+s=i} Sq^r h \cdot Sq^s k$ or $\Delta a = \sum a' \otimes a''$
$a(hk) = \sum (a'h)(a''k)$

2) $Sq^j h = 0$ if $h \in H^i$, $i < j$.

3) $\exists \mu \in H^{n*} \ni < hk, \mu >$ is dual pairing of H^i & H^{n-i} to Z_2

Define A to operate on right by

$$< ha . k, \mu > = < h . ak, \mu >, \quad k \in H^{n-i-j}, \; a \in A^j, \; h \in H^i.$$

E.g. ε_H the unit in H, $\varepsilon_H Sq^i = v_i$ above.

Define words
- i) ε
- ii) closed under left & rt. A operation
- iii) " " cupproduct
- iv) " " linear combination.

If H an algebra above, define θ_H : words $\to$ H by

$\theta_H(\varepsilon) = \varepsilon_H$, $\theta_H(aW) = a\,\theta_H(W)$, $\theta_H(Wa) = \theta_H(W)\,a$

preserve +, ·.

$W \equiv W \Leftrightarrow \theta_H(W) = \theta_H(W')$ all H, equivalence classes form U, which has $+, \cdot$, A operations. This is object of study.

Th. 1. U is a polyn. alg. on $u_1, \cdots$, where u_i defined by

$$u_i := \mathcal{E}(\chi(Sq^i)), \quad i = 1, \ldots$$

Th. 2. Given N, integer. Let m run over monomials in u_i $\deg m \leq N$. Then $\exists$ compact diff. manifold $D \ni$ values of monomials m in $H^*(D; Z_2)$ are lin. ind.

Cor. 3. $W \in \mathcal{U}$, $\theta_H(W) = 0$ if $H = H^*(D; Z_2)$, all diff. manifolds D $\Rightarrow W = 0$.

Z.B. $W = Sq^i W_k - Q(w_1, \cdots, w_{i+k})$, ~~proving~~ $w_k = \sum Sq^j(\epsilon Sq^k)$, proving Wu formula in general.

Prove Th 2 first. Lemma 4. $x \in H^1(P^\infty; Z_2)$, $a \in A^i$, $i + j + k = 2^s - 1$,

then $\quad x^j \cdot a x^k = x^k \cdot (\chi a) x^j$

$N = 2^s - 1$, then $Sq^m : H^{N-m}(P^\infty; Z_2) \to H^N(P^\infty; Z_2)$ is 0 because

$$Sq^m(x^{N-m}) = \binom{N-m}{m} x^N = 0$$ by binomial coeff. argument

$\therefore$ if $i + j + k = N$, $a \in A^i$, $a(x^k \cdot x^j) = 0 \therefore$

$a x^k \cdot x^j + \sum a' x^k \cdot a'' x^j = ($, $\dim a'' > 0)$

$a(x^k \cdot x^j) = 0 = a x^k \cdot x^j + \sum x^k \cdot \chi(a') \cdot a'' x^j$ by induction on i

$= a x^k \cdot x^j + x^k \cdot \chi(a) x^j$

Lemma 5. $n = 2^s - 2$ $(s \geq 2)$, $M = P^n$, $x \in H^1(M; Z_2)$.

$\varepsilon_H(\chi(Sq^1)) = x$, $\varepsilon_H(\chi(Sq^i)) = 0$, $i > 1$.

Proof: $\langle \varepsilon_H(\chi(Sq^i)) \cdot x^{n-i}, \mu \rangle = \langle \chi(Sq^i) x^{n-i}, \mu \rangle$

~~$= x^{n-i}, Sq^i$~~ ~~$\chi(Sq^i) x^{n-i} = x^{n}$ [illegible] x^{n+1}~~

Then $x \cdot \chi(Sq^i) x^{n-i} = x^{n-i} Sq^i x = \begin{cases} 0 & i > 0 \\ 1 & i = 1 \end{cases}$ in $H^*(P^\infty; Z_2)$

∴ lemma

Lemma 6. $\varepsilon_{M' \times M''} a = \sum \varepsilon_{M'} a' \otimes \varepsilon_{M''} a''$

Proof: $\varepsilon_{M' \times M''} a \cdot x^j \otimes x^k =$ $\quad i + j + k = n' + n''$

$\varepsilon_{M' \times M''} \cdot \sum a' x^j \otimes a'' x^k = \sum \varepsilon_{M'} a' \cdot x^j \otimes \varepsilon_{M''} a'' \cdot x^k$

$= \sum \varepsilon_{M'} a' \otimes \varepsilon_{M''} a'' \cdot x^j \otimes x^k$

Lemma 7. $D = \prod_{j=1}^{N} P^n(j)$ $\quad n = 2^s - 2 \geq N$

$\varepsilon_D(\chi(Sq^i)) = i^{\text{th}}$ elementary symmetric function in $x_1, \dots, x_N$.

Proof: $\Delta Sq^i = \sum_{r+s=i} Sq^r \otimes Sq^s$

$\therefore \Delta \chi(Sq^i) = \sum_{r+s=i} \chi Sq^r \otimes \chi Sq^s$, $\therefore \Delta^N \chi(Sq^i) = \sum_{r_1 + \dots + r_N = i} \chi Sq^{r_1} \otimes \dots \chi Sq^{r_N}$

$\therefore \varepsilon_D \chi(Sq^i) = \sum_{r_1 + \dots + r_N = i} \varepsilon_{P(1)} \chi Sq^{r_1} \otimes \dots \otimes \varepsilon_{P(N)} \chi(Sq^{r_N})$

$= \sum x_1 \cdots x_i$

∴ 14.2. as monomials in elementary sym. functions are independent in $H^*(D)$.

To prove Th 1, enough to show U is gen. (mult.) by $u_1, \dots$

Lemma 8. dim $a > 0$, $h \in H$ ($u \in U$, $u = \varepsilon b$) then

$$\underset{H}{\varepsilon} a \cdot h = ah + \sum_{\dim a', a'' > 0} (a'h) a'' + ha$$

Proof: $k \in H^{n-(i+j)}$ $a \in A^i$, $h \in H^j$

$$(\underset{H}{\varepsilon} a) \cdot h \cdot k = a(hk) = ah \cdot k + \sum a'h \cdot a''k + h \cdot ak$$

$$= ah \cdot k + \sum (a'h) a'' \cdot k + \langle ha, k \rangle \quad \text{all } k.$$

Lemma 9. U is mult. gen. by εa, $a \in A$.

Proof: To prove $a \in A$, W = polyn in εb, then aW + Wa are also.

Induction: dim $a = 0$, O.K.

start with: $a(\varepsilon b)$ $\quad a(\varepsilon b) = \varepsilon a \cdot \varepsilon b + \sum (a' \varepsilon b) a'' + (\varepsilon b) a$ (lemma 8)

but $(\varepsilon b) a = \varepsilon(ba)$, $\therefore$ by induction $a(\varepsilon b)$ is polyn. in εb's.

If W is, then aW is in terms of $c(\varepsilon b)$ with dim $c \le k$

$\therefore$ by above all O.K.

$$Wa = (\varepsilon a) \cdot W - aW - \sum (a'W) a'' \text{ by lemma 8.}$$

$\therefore$ O.K.

Let $I(U) = \sum_{j>0} U^j$, $D(U)$ = decomposable elts, i.e.

$u = \sum u' u''$, $u', u'' \in I(U)$

Lemma 10 $\quad u \in D(U)$, $a \in A \Rightarrow ua \in D(U)$

Proof by induction lemma 8

$$\underset{\in D}{\varepsilon a \cdot u} = \underset{\substack{\in D \\ \text{by Cartan}}}{au} + \underset{\in D \text{ by induction}}{\sum (a'u) a''} + ua$$

Lemma 11. If dim $b > 0$ then

$$a(\varepsilon b) = \varepsilon(b\,\chi(a)) \mod D(U)$$

Proof by induction

$a(\varepsilon b) = \underset{\in D}{\varepsilon a \cdot \varepsilon b} - \varepsilon(ba) - \sum(a' \varepsilon b)a''$ by lemma 8.

$$\parallel$$

$$\sum \varepsilon(b\chi(a')a''$$

mod D by 10

$\therefore\ a(\varepsilon b) = -\varepsilon(ba) - \sum \varepsilon(b\chi(a')a'')$ mod D

$\therefore\ a(\varepsilon b) = \varepsilon(b\chi a)$

Lemma 12 $I(U)/D(U)$ is spanned by u_i, $i > 0$.

Proof: $I(U)/D(U)$ is spanned by εa, $a \in$ spanning set for $I(A)$

$S = \{Sq^{i_1} \dots Sq^{i_t}\}$, $i_j \geq 2i_{j+1}$, $2i_t > 0$. S spans $I(A)$ (Adams)

$\chi(S)$ spans $I(A)$ also. $\chi(S)$ contains $\chi(Sq^i)$. Every other elt. of $\chi(S)$

$= d\,\chi(Sq^j)$ with dim $d < j$. But $Sq^j(\varepsilon d) = 0$ $\therefore$

$\varepsilon d\chi(Sq^j) \in D(U)$ by lemma 11, $\therefore$ $I(U)/D(U)$ spanned by u_i

$\therefore$ thm 1.

MORTON BROWN'S PROOF OF THE GENERALIZED SCHOENFLIES THEOREM

BY A. WASSERMAN

REFERENCE: AMERICAN MATH. SOCIETY BULLETIN 1960, Pg. 74.

WE SHALL PROVE:

GENERALIZED SCHOENFLIES THEOREM:

LET h BE A HOMEOMORPHIC IMBEDDING OF $S^{h-1} \times I$ INTO S^h. THEN THEN THE CLOSURE OF EITHER COMPLEMENTARY DOMAIN OF $h(S^{h-1} \times \frac{1}{2})$ IS AN h-CELL.

WE FIRST GIVE SOME DEFNS. AND PROVE SOME PRELIMINARY THEOREMS.

DEFNS.

(1) IF Q IS AN h-CELL $\dot{Q}$ DENOTES ITS BOUNDARY AND Q^0 DENOTES ITS INTERIOR.

(2) I IS UNIT INTERVAL $[0\ 1]$

(3) IF $f : X \to Y$ (CONT.) THEN AN INVERSE SET OF f IS A SET $M \subset X$ WHICH CONTAINS MORE THAN ONE PT. AND SUCH THAT FOR SOME $y \in f(X)$ $M = f^{-1}(y)$.
(SO Y IS $T_1 \Rightarrow M$ IS CLOSED)

(4) A SET M IS CELLULAR IN AN

n-DIMENSIONAL COMPACT METRIC SPACE S, IF THERE EXIST n-CELLS $Q_1, Q_2, \ldots$ IN S SUCH THAT $Q_{i+1} \subset Q_i^0$ AND $\bigcap_1^\infty Q_i = M$

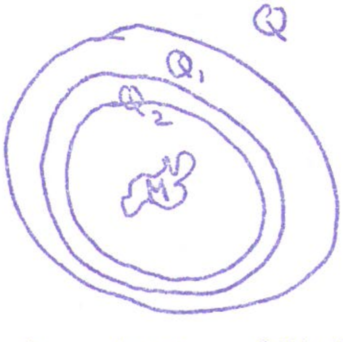
PRELIMINARY THEOREMS

THEOREM 0 LET Q BE AN n-CELL AND LET $f : Q \to S^n$ AND SUPPOSE INVERSE SETS ARE IN Q^0 THEN $f(Q)$ IS THE UNION OF $f(\dot{Q})$ AND ONE OF ITS COMPLEMENTARY DOMAINS.

PROOF USE METHODS SUCH AS THOSE IN HUREWICZ + WALLMAN's book (Pg. 97).

THEOREM 1 LET Q BE AN n-CELL. SUPPOSE M IS A CELLULAR SUBSET OF Q^0 THEN THERE IS A MAP $f : Q \xrightarrow{\text{ONTO}} Q$ SUCH THAT $f|\dot{Q}$ = IDENTITY AND M IS THE ONLY INVERSE SET UNDER f.

PROOF



WE WANT A MAP CONTRACTING M TO A PT. AND IDENTITY ON $\dot{Q}$. WE DO THIS BY CONTRACTIONS ON THE Q_i (SINCE M IS CELLULAR). h_1 HOMEO. OF Q, IDENTITY ON BOUNDARY AND SUCH THAT DIAM. $h_1(Q_1) < 1$

WE DEFINE h_{i+1} TO BE h_i ON $Q - Q_i$ AND SUCH THAT DIAM $h_{i+1}(Q_{i+1}) < \frac{1}{i+1}$

LET $f = \lim h_i$.

<u>THEOREM 2</u> LET S BE A TOPOLOGICAL $n-1$ SPHERE IN S^n AND LET D BE ONE OF ITS COMPLEMENTARY DOMAINS. SUPPOSE $f : \overline{D} \xrightarrow{\text{ONTO}} E^n$ AN n-CELL SUCH THAT THE ONLY INVERSE SET OF f IS A CELLULAR SUBSET M OF D. THEN $\overline{D}$ IS AN n-CELL.

<u>PROOF</u>

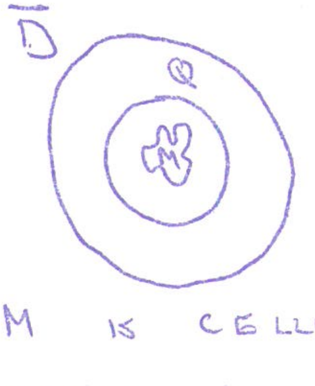
LET ρ BE IDENTIFICATION MAP $\overline{D} \to \overline{D}/M$

THEN f INDUCES A CONT. MAP

$f' : \overline{D}/_M \xrightarrow{\text{ONTO}} E^n$, f' IS 1-1, CONT.

& $\overline{D}$ IS COMPACT SO f' IS A HOMEOMORPHISM.

THUS SUFF. TO SHOW $\overline{D}$ HOMEO. WITH $\overline{D}/M$



LET Q BE AN n-CELL IN D SUCH THAT $M \subset Q^0$.

THEN M IS CELLULAR IN Q. BY THM. 1 THERE EXISTS $g : Q \to Q$ $g|_{\dot{Q}} =$ IDENTITY AND THE ONLY INVERSE SET UNDER g IS M.

WE DEFINE $g' : \overline{D} \to \overline{D}$ BY:

$$g'|Q = g|Q \qquad g'|\bar{D}-Q = \text{IDENTITY.}$$

THEN $(pg'^{-1}) : \bar{D} \to \bar{D}/M$ IS A HOMEOMORPHISM. |

THEOREM 3 LET Q BE AN n-CELL AND SUPPOSE f MAPS Q INTO S^n SUPPOSE ALSO THAT $M \subset Q^0$ IS THE ONLY INVERSE SET UNDER f. THEN M IS CELLULAR IN Q.

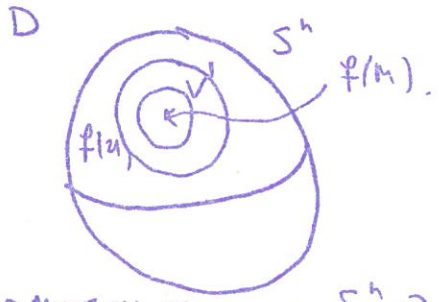
PROOF LET $V_i = \{ x \,/\, x \in Q, d(x,M) < \frac{1}{i} \}$

TO CONSTRUCT SETS: $Q_1 \subset V_1$, $Q_{i+1} \subset V_{i+1} \cap Q_i^0$.

HOWEVER IT IS SUFFICIENT TO CONSTRUCT FOR ANY OPEN SET $\mathcal{U}$ $(V_{i+1} \cap Q_i^0)$ CONTAINING M, $\mathcal{U}$ IN INTERIOR OF AN n-CELL (Q_i) AN n-CELL Q' (Q_{i+1}) SUCH THAT $M \subset Q'^0$ (Q_{i+1}^0).

NOW BY THM. 0 $f(Q) = f(\dot{Q}) \cup D$ WHERE D IS A COMPLEMENTARY DOMAIN OF $f(\dot{Q})$. LET $\mathcal{U}$ BE AN OPEN SET IN Q^0_*, $\mathcal{U} \supset M$.

THE IDEA HERE IS TO CONSTRUCT A HOMEOMORPHISM $g : Q \to Q$ WHICH PULLS Q INSIDE $\mathcal{U}$ AND IS THE IDENTITY ON AN OPEN SET $V \supset M$. WE MAY DO THIS ON Q SINCE IT CAN BE DONE IN S^n. MORE PRECISELY; $M \subset \mathcal{U} \subset Q^0$ $\mathcal{U}$ OPEN IMPLIES $f(\mathcal{U})$ OPEN IN D. LET V' BE A NBD. OF $f(M)$, $V' \subset f(\mathcal{U})$.



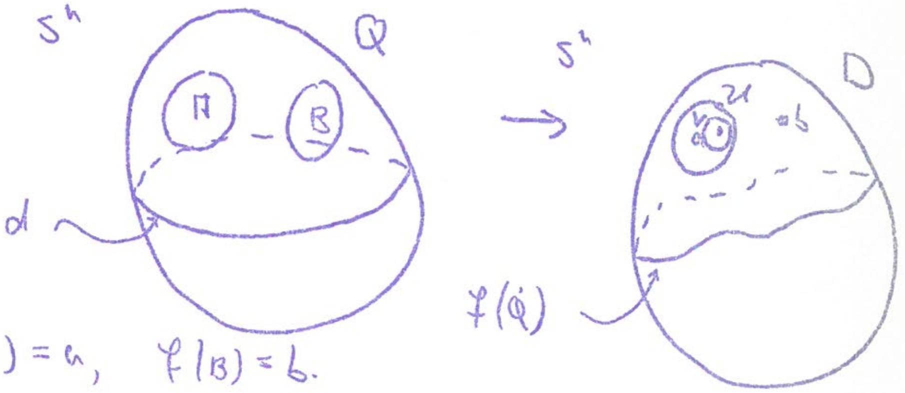
LET h BE A HOMEOMORPHISM : $S^n \to S^n$ WHICH PULLS $\overline{D}$ INSIDE $f(\mathcal{U})$ AND IS THE IDENTITY ON V'. LET $V = f^{-1}(V')$.

NOW $$g(x) = \begin{cases} x & x \in M \\ f^{-1} h^{-1} f(x) & x \notin M \end{cases}$$

IS A HOMEOMORPHISM WHICH IS THE DESIRED ONE. $g(Q) = Q'$ IS AN n-CELL AND $M \subset V = g(V) \subset Q'^{0}$. ∎

<u>THEOREM 4</u> LET $f : S^n \xrightarrow{\text{onto}} S^n$ WITH PRECISELY TWO INVERSE SETS A AND B. THEN A AND B ARE CELLULAR IN S^n.

PROOF LET d BE AN $n-1$ SPHERE IN $S^n - (A \cup B)$ EACH OF WHOSE COMPLEMENTARY DOMAINS HAS AN n-CELL FOR ITS CLOSURE. (IF d SEPARATES A AND B THM. 4 IS IMMEDIATE FROM THM. 3). LET Q BE THE n-CELL WHOSE BOUNDARY IS d AND WHICH CONTAINS $A \cup B$ IN ITS INTERIOR.



LET $f(A) = a$, $f(B) = b$.

BY THM. 0 $f(Q) = f(\dot{Q}) \cup D$, $D \supset \{a, b\}$

LET $\mathcal{U}$ BE AN OPEN SUBSET OF D $\mathcal{U} \ni a$, AND $\mathcal{U} \not\ni b$. THEN LET h BE A HOMEOMORPHISM $S^n \to S^n$ WHICH PULLS $f(Q)$ INSIDE $\mathcal{U}$ AND IS THE IDENTITY ON A SMALL NBD. V OF a.

LET $g: Q \to Q \subset S^n$ BE DEFINED AS:

$$g(x) = \begin{cases} x & x \in A \\ f^{-1} h f(x) & x \notin A. \end{cases}$$

THUS THE ONLY INVERSE SET OF g IS B SO BY THM. 3 B IS CELLULAR IN Q SO CELLULAR IN S^n.

SIMILARLY $\bar{A}$ IS CELLULAR IN S^n.

WE NOW PROVE:

GENERALIZED SCHOENFLIES THM.:

LET h BE A HOMEOMORPHIC IMBEDDING OF $S^{n-1} \times I$ INTO S^n. THEN THE CLOSURE OF EITHER COMPLEMENTARY DOMAIN OF $h(S^{n-1} \times \frac{1}{2})$ IS AN n-CELL.

PROOF.

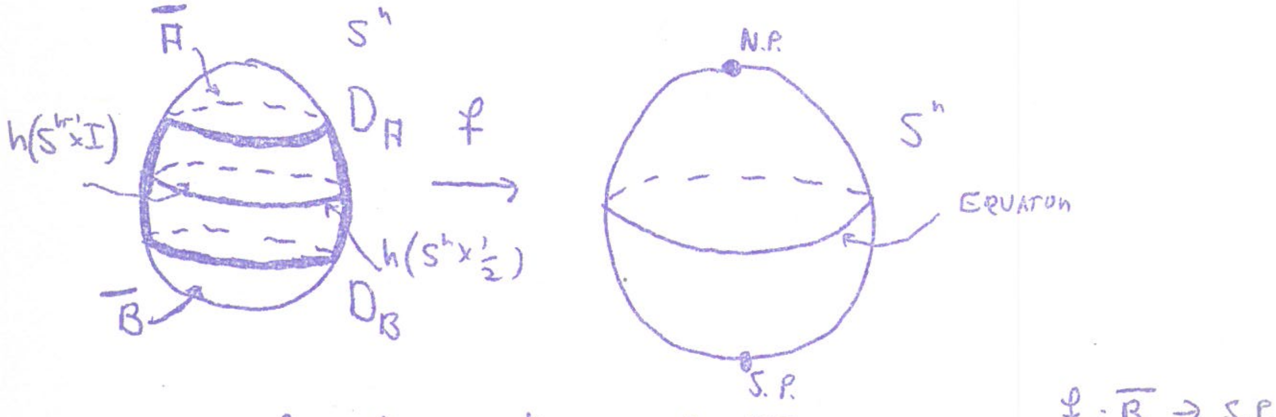
$$S^n - h(S^{n-1} \times 1) = A \cup A'$$

$$\text{LET} \quad h(S^{n-1} \times 0) \in A'$$

$$S^n - h(S^{n-1} \times 0) = B \cup B'$$

$$\text{LET} \quad h(S^{n-1} \times 1) \in B'$$

THUS $S^n = h(S^{n-1} \times I) \cup A \cup B$.



LET $f: S^n \to S^n$ $\quad f: \bar{A} \to$ N.P., $\quad f: \bar{B} \to$ S.P.

AND $f: h(S^{n-1} \times \frac{1}{2})$ ONTO EQUATOR WITH ONLY $\bar{A}$ AND $\bar{B}$ AS INVERSE SETS

LET D_A, D_B BE COMPLEMENTARY DOMAINS

OF $h(S^{n-1} \times \frac{1}{2})$ WHICH CONTAIN A, B RESPECTIVELY, BY THM. 4 A, B ARE CELLULAR IN S^n HENCE CELLULAR IN D_A, D_B RESPECTIVELY. THEN BY THM. 2 $\overline{D_A}$, $\overline{D_B}$ ARE n-CELLS. |

Seminar of Professor G. Whitehead

The Generalized Bar Construction and the Lower Central Series for Free Group Complexes.

by E. Curtis

1. Semi-simplicial objects

Definition A semi-simplicial (s.s.) object over the category $\mathcal{C}$ shall be a contravariant functor X from the category $\mathcal{O}$ of finite ordered sets (and monotone maps) to $\mathcal{C}$.

The objects in $\mathcal{O}$ are the sets $[m] = \{0, 1, \cdots, m\}$ A monotone map $\alpha : [m] \longrightarrow [n]$ must satisfy $\alpha(i) \leq \alpha(j)$ if $i \leq j$.

$X[m]$ is usually denoted by X_m and is called the (set of) m-simplices of X.

S.s. sets, s.s. monoids, s.s. groups etc are examples of s.s. objects; this last is the term used when the category $\mathcal{C}$ is not specified.

For example, if A is a topological space, $S(A)$, the Eilenberg total singular complex of A is a s.s. set. $C_* S(A)$ is a s.s. abelian group, formerly referred to as an FD-module.

Certain maps in $\mathcal{O}$, namely

$$\epsilon^i : [q] \longrightarrow [q+1] \quad \text{where} \quad \begin{array}{l} \epsilon^i(j) = j, \ j < i \\ \epsilon^i(j) = j+1, \ j \geq i \end{array}$$

and $$\eta^i : [q+1] \longrightarrow [q] \quad \text{where} \quad \begin{array}{l} \eta^i(j) = j, \ j \leq i \\ = j-1, \ j > i \end{array}$$

serve as generators for the maps in $\mathcal{O}$.

X_{ϵ^i} is called ∂_i the i^{th} face operator

X_{η^i} is called s_i the i^{th} degeneracy ".

The usual semi-simplicial relations are all consequences of the relations among the ϵ^i, η^i.

Def. A s.s. object with base point is a pair (X, b), where X is an s.s. object, and $b = b_0, s_0 b_0, \ldots, s_{n-1} \cdots s_1 s_0 b_0, \ldots$ is the sub-s.s. set of a point $b_0 \in X_0$. When the base point is unspecified, the notation $(X, *)$ may be used.

Def. A s.s. object X is said to satisfy the extension condition if given $x_0, \ldots, \hat{x}_k, \ldots x_{q+1} \in X_q$ such that $\partial_i x_j = \partial_{j-1} x_i$ for $i < j$, $i \neq k \neq j$, then there is an $x \in X_{q+1}$ with $\partial_i x = x_i$, all $i \neq k$

For example, $S(A)$ satisfies the extension condition.

S.s. objects satisfying the extension condition allow a definition of homotopy; see [5], and the $\pi_n(X, *)$ are defined: for $n \geq 1$, they are groups, and for $n \geq 2$, abelian groups.

2. S.s. groups

Lemma (Moore [6]) If G is a s.s. group, then it satisfies the extension condition.

Def. Let (NG) be the differential group given by

$$(NG)_q = \bigcap_{i \neq 0} (\ker \partial_i : G_q \longrightarrow G_{q-1})$$

and differential operator on NG given by

$$\partial = \partial^{(q)} = \partial_0 : (NG)_q \longrightarrow (NG)_{q-1}$$

Then image $\partial^{(q+1)}$ is a normal subgroup of $\ker \partial^{(q)}$ and the quotients $\frac{\ker \partial^{(q)}}{\operatorname{im} \partial^{(q+1)}}$ are defined to be the homology of NG, $H_q(NG)$. Then it turns out that $H_q(NG) \approx \pi_q(G, e)$ (e = identity of G in every dimension, always taken as the base point of a s.s. group).

Def. A map $p: X \to B$ (i.e $p: X_q \to B_q$ commuting with the face and degeneracy operators) is called a fibre map if given $y \in B_{q+1}$ and $x_0, \dots \hat{x}_k, \dots x_{q+1} \in X_q$ with $p_i x = \partial_i y$ and $\partial_i x_j = \partial_{j-1} x_i$, $i<j$, $i \neq k \neq j$, then there is an $x \in X_{q+1}$ such that $px = y$, and $\partial_i x = x_i$. The fibre F is given by

$$F_n = p^{-1}(b_n) \quad , \quad b \text{ a base point for } B.$$

Then there is the usual exact sequence:

$$\to \pi_{n+1}(B, *) \to \pi_n(F, *) \to \pi_n(X, *) \to \pi_n(B, *) \to \cdots$$

If $p: G \to B$ is a surjection of s.s. groups, then p is a fibre map.

3. Kan's Construction GX

For each s.s. set with base point (X, b), Kan [5] has defined a s.s. group $GX = G(X, b)$ which is free in every dimension, and which serves as a (s.s. version of) loop space for (X, b)

Let $\Gamma_2 G = [G, G]$ the commutator subgroup.

Then as proven in [5],

$$\pi_{n-1}(GX/\Gamma_2 GX, e) \approx H_n(X)$$

and the projection $p: GX \longrightarrow GX/\Gamma_2 GX$ induces the Hurewicz homomorphism h:

$$\begin{tikzcd}
\pi_n(X, b) \arrow[r, phantom, "\approx"] \arrow[d, "h"] & \pi_{n-1}(GX, e) \arrow[d, "p*"] \\
H_n(X) \arrow[r, phantom, "\approx"] & \pi_{n-1}(GX/\Gamma_2 GX, e)
\end{tikzcd}$$

The Hurewicz Theorem as given in [5] involves showing that if G is a s.s. free group (e.g. $G = GX$), then $\Gamma_2 G$ has connectivity one higher than G.

4. The Lower Central Series.

For any group G, let $\Gamma_1 G = G$ and $\Gamma_r G$ defined inductively by $\Gamma_r G = [\Gamma_{r-1} G, G]$, $r \geq 2$

Then

$$G \supset \Gamma_2 G \supset \dots \supset \Gamma_r G \supset \dots$$

is called the lower central series for G

Γ_r will be considered as a functor from $\mathcal{G}$ (the category of groups) to itself. If G is a s.s. group (i.e., G is a contravariant functor from $\mathcal{O}$ to $\mathcal{G}$) then $\Gamma_r G$ is the s.s. group which is the composite functor. $\Gamma_r G$ is thus obtained by applying Γ_r to each dimension G_q, and to the face and degeneracy operators. The same remarks apply to the functor L^r from $\mathcal{G}$ to $\mathcal{A}$ (the category of abelian groups), defined by

$$L^r G = \Gamma_r G / \Gamma_{r+1} G$$

The process of composing the s.s. object X (which is a functor $X : \mathcal{O} \rightsquigarrow \mathcal{C}$) with a functor $T : \mathcal{C} \rightsquigarrow \mathcal{D}$ to obtain a new s.s. object $TX : \mathcal{O} \rightsquigarrow \mathcal{D}$ is called "prolongation" by Dold [1].

A s.s. object X is called n-connected if $\pi_i(X, *) = 0$ for all $i \leq n$.

For each real number a, let $\{a\}$ denote a if a is an integer, otherwise $\{a\}$ is to be the next integer larger than a.

Theorem If G is a s.s. free group which is n-connected $(n \geq 0)$, then $\Gamma_r G / \Gamma_{r+1} G$ is $n + \{\log_2 r\}$-connected.

The next sections outline the techniques used in the proof of Thm 1. It would be interesting to know the connectivity of the s.s. groups $\Gamma_r G$ themselves.

5. Witt's Theorem

A theorem of Witt [8], related to the better known Poincaré-Birkhoff-Witt theorem, states that if G is a free group, the $\Gamma_r G / \Gamma_{r+1} G$ depend (even as functors) only on $G / \Gamma_2 G$. That is, for any group G, let $L(G) = \sum_{r \geq 1} L^r(G)$ be the Lie ring of the group G ([7]). For any abelian group M, let $\mathcal{L}(M) = \sum \mathcal{L}^r(M)$ be the free Lie ring over M ([4a]). Then there is a natural transformation θ from the composite functor $\mathcal{L}^r \circ L^1$ to the functor L^r (considered as functors from $\mathcal{G}$ to $\mathcal{A}$). The Witt theorem

Thm. states that if G is free,

$$\theta : \mathcal{L}^r(L'G) \longrightarrow L^r(G)$$

is an isomorphism.

6. A Theorem of A. Dold [1]

<u>Theorem</u> If T is a covariant functor from $\mathcal{A}$ to itself, then if M is a s.s. free abelian group, $\pi_*(TM,e)$ depends only on $\pi_*(M,e)$

<u>Remark</u> under our notation, $\pi_*(M,e)$ (base point understood to be the identity) is sometimes referred to as the "homology" of M. Let $\mathfrak{K}$ be the functor from s.s. abelian groups to s.s. abelian groups given by

$$\mathfrak{K}M = M$$

and differential operator on M given by $\partial = \Sigma(-1)^i \partial_i$

Then the normalization theorem of Eilenberg-Maclane (see for example [3]) shows that

$$H_*(NM) \approx H_*(\mathfrak{K}M)$$, and then

both of these are $\approx \pi_*(M,e)$.

Putting Witt's and Dold's theorem together, we see that $\pi_*(L^r G, e)$ depends only on $\pi_*(G/\Gamma_2, e)$ if G is a free s.s. group, and we are reduced to studying the functor $\mathcal{L}^r$ on s.s. free abelian groups. This is most conveniently done by means of the generalized bar construction of Dold-Puppe.

7. The Generalized Bar Construction [3]

Let M be a s.s. abelian group. Then the cone CM and the suspension EM are defined by taking for CM any s.s. abelian group which is contractible and contains M as a direct summand; EM is defined to be the quotient:

$$0 \longrightarrow M \xrightarrow{i} CM \xrightarrow{p} EM \longrightarrow 0$$

Then the boundary homomorphism Δ is an isomorphism

$$\begin{tikzcd} \Delta: & \pi_{n+1}(EM) \arrow[r, "\approx"] \arrow[d, phantom, "\parallel"] & \pi_n(M) \arrow[d, phantom, "\parallel"] \\ & H_{n+1}(NEM) \arrow[r] & H_n(NM) \end{tikzcd}$$

Let T be a functor from $\mathcal{A}$ to $\mathcal{A}$ (with $T(0) = 0$), and apply (the prolongation of) T

$$TM \xrightarrow{Ti} TCM \xrightarrow{Tp} TEM$$

Ti is injective, Tp is surjective.

Then

$$\begin{tikzcd} \pi_n(TM) \arrow[r, "(Ti)_*"] \arrow[rr, dashed, bend right, "\sigma"'] & \pi_n(\mathrm{Ker}\, Tp) \arrow[r, "\Delta^{-1}", "\approx"'] & \pi_{n+1}(TEM) \end{tikzcd}$$

The composite $\Delta^{-1} \circ (Ti)_* = \sigma$ is called the suspension.

The generalized bar construction uses the cross effects of T to provide a construction of a double complex (double differential group) whose homology (under the total differential operator) is that of $NTEM$.

The cross effects of T denoted by $T(M_1 | \cdots | M_n)$, are defined in [4], where it is shown that they satify:

$$T(M_1 + \cdots + M_n) = \sum_{\tau} T(M_{\tau_1} | \cdots | M_{\tau_s})$$

$$\tau = (\tau_1, \ldots, \tau_s), \quad 1 \leq \tau_1 < \cdots < \tau_s \leq n$$

e.g. $T(A+B) = T(A) + T(B) + T(A|B)$

Let $\Sigma_k M$ denote the direct sum of k copies of M, and let

$$\alpha_i' : \Sigma_k M \longrightarrow \Sigma_{k-1} M \quad \text{for } 1 \le i \le k-1$$

by $\alpha_i'(m_1, \cdots, m_k) = (m_1, \cdots, m_i + m_{i+1}, \cdots, m_k)$

Then

$$T(\alpha_i') : T(\Sigma_k M) \longrightarrow T(\Sigma_{k-1} M)$$

and let α_i be the composite

$$\begin{tikzcd}
T_{[k]}(M) \arrow[rrr, "\alpha_i"] \arrow[d, equal, "\text{(definition)}"'] & & & T_{[k-1]}(M) \arrow[d, equal] \\
T(\underbrace{M|M|\cdots|M}_{k \text{ factors}}) \arrow[r] & T_*(\Sigma_k M) \arrow[r] & T(\Sigma_{k-1} M) \arrow[r] & T(\underbrace{M|\cdots|M}_{k-1})
\end{tikzcd}$$

Let $d = \Sigma(-1)^i \alpha_i$; then $d^2 = 0$

If M is a s.s. abelian group, let $\tilde{T}M$ be the double differential group:

$$\begin{tikzcd}
 & & & {} \arrow[d] & \\
\tilde{T}M : & {} & T_{[p-1]}(M_q) \arrow[l] & T_{[p]}(M_q) \arrow[l, "d"'] \arrow[d, "\partial = \Sigma(-1)^i T_{[p]} \partial_i"] & {} \arrow[l] \\
 & & & T_{[p]}(M_{q-1}) &
\end{tikzcd}$$

<u>Theorem</u> ΥM is a double differential group under d and ∂, and the homology of the associated singly differential group $t\,\Upsilon M$, $(t\,\Upsilon M)_n = \sum_{p+q=n} T_{[p]}(M_q)$ under $D = d + (-1)^p \partial$ becomes the homology of $NTEM$, and the injection $\iota : TM \longrightarrow \Upsilon M$ induces the suspension σ :

$$\begin{tikzcd}
\pi_n(TM) \arrow[r, phantom, "="] & H_n(NTM) \arrow[r, "\sigma"] \arrow[dr, "\iota_*"'] & H_{n+1}(NTEM) \arrow[r, phantom, "="] \arrow[d, phantom, "\wr\wr"] & \pi_{n+1}(TEM) \\
 & & H_{n+1}(t\,\Upsilon M) &
\end{tikzcd}$$

In the next few pages we shall briefly outline why this theorem is true. For complete details see [3].

Let $\Delta(m)$ be the s.s. set of the m-simplex that is $\Delta(m)_p = \{$ all monotone maps $[p] \longrightarrow [m] \}$

$\Delta(0)_p$ has but one $\overset{\text{element}}{\text{(map)}}$: γ^p

$\Delta(1)_p$ has $\overset{\text{elements}}{\text{(maps)}}$: γ^p_K , $K = 0, 1, \dots p+1$

where γ^p_K : $0, 1, \dots K-1 \longrightarrow 0$

$K, K+1, \dots p \longrightarrow 1$

let $\Delta(0) \xrightarrow{\epsilon_0,\ \epsilon_1} \Delta(1)$ $\left\{ \begin{array}{l} \text{by} \quad \epsilon^0 \gamma^p = \gamma^p_{p+1} \\ \text{or} \quad \epsilon^1 \gamma^p = \gamma^p_0 \end{array} \right.$

let Z denote the free abelian group functor, and define

$$\begin{tikzcd}
0 \arrow[r] & Z\,\Delta(0) \arrow[r, "\epsilon^1"] \arrow[d, equal] & Z\,\Delta(1) / Z\,\epsilon^0 \Delta(0) \arrow[r] \arrow[d, equal] & Z\Delta(1) / Z(\epsilon^0 \Delta(0) + \epsilon^1 \Delta(0)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Q \arrow[r] & C \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

Q, C, S are s.s. free abelian groups,

Q_p is freely generated by γ^p

C_p " $\gamma^p_0 \cdots \gamma^p_p$

S_p " $\gamma^p_1 \cdots \gamma^p_p$

Given a s.s. abelian group M, form the double objects $Q \hat{\otimes} M$, $C \hat{\otimes} M$, $S \hat{\otimes} M$, where for example $(S \hat{\otimes} M)_{p,q} = S_p \otimes M_q$.

The diagonal objects and total objects are (respectively) for example

$$d(S \hat{\otimes} M)_n = (S \hat{\otimes} M)_{n,n} = S_n \otimes M_n$$

$$t(S \hat{\otimes} M)_n = \sum_{p+q=n} (S \hat{\otimes} M)_{p,q} = \sum_{p+q=n} S_p \otimes M_q$$

The double s.s. objects have two sets of s.s. operators, denoted ∂'_i, s'_i for the first index (p) and ∂''_i, s''_i for the second index (q). The diagonal object is regarded as a differential group under $\partial = \partial' \circ \partial'' = \partial'' \circ \partial'$; $\partial' = \Sigma (-1)^i \partial'_i$
$\partial'' = \Sigma (-1)^i \partial''_i$

The total object is a differential group under $D = \partial' + (-1)^p \partial''$.

Let $d(C \hat{\otimes} M) = CM$
$d(S \hat{\otimes} M) = \Sigma M$, and since $d(Q \hat{\otimes} M) = M$

we have constructed the cone and suspension.

Now apply T to the double objects, and obtain the double objects:

$$T(Q \hat{\otimes} M) \qquad T(C \hat{\otimes} M) \qquad T(S \hat{\otimes} M)$$

The Eilenberg-Zilber-Cartier [3] theorem shows that the diagonal objects (eg $d\, T(S \hat{\otimes} M) = T \Sigma M$) and the total objects (eg $t\, T(S \hat{\otimes} M)$) have naturally isomorphic homology

Now apply N' to the double objects, where

$$N'(\text{double object}) = \bigcap_{i \neq 0} (\text{kernel } \partial_i')$$

$$\approx \text{double object}/D'$$

and $D' = \bigcup \text{image } s_i'$

Then $N' \, T(Q \hat{\otimes} M)_{p,*} = TM \quad p = 0$

$= 0 \quad p \neq 0$

Also, and this is the crucial observation:

$$N' \, T(S \hat{\otimes} M) \approx \gamma M$$

for $T(S \hat{\otimes} M)_{p,*} = T\left(\sum_{K=1}^{p} \gamma_K^p \times M \right)$

$$= \sum_{\tau} T(\gamma_{\tau_1}^p \times M | \cdots | \gamma_{\tau_r}^p \times M)$$

but image $s_{i-1}' = \sum_{\tau \not\ni i} T(\gamma_{\tau_1}^p \times M | \cdots | \gamma_{\tau_r}^p \times M)$

so $D' \, T(S \hat{\otimes} M)_{p,*} = \sum_{\tau \neq (1, \cdots p)} T(\gamma_{\tau_1}^p \times M | \cdots | \gamma_{\tau_r}^p \times M)$

Then $N' \, T(S \hat{\otimes} M)_{p,*} \approx T(S \hat{\otimes} M)_{p,*} / D' \, T(S \hat{\otimes} M)_{p,*}$

$$\approx T(\gamma_1^p \times M | \cdots | \gamma_p^p \times M)$$

$$\approx T_{[p]}(M)$$

And it turns out that the differentials correspond, etc, and that $\iota_* = \sigma$.

8 Applications of the gen. Bar Const.

In the applications, it will be convenient to filter

TM by $$F^p TM = \sum_{\substack{m \le p \\ \text{all } q}} T_{[m]}(M_q)$$

Then in the associated spectral sequence, E^1 with its differential becomes the sequence

$$H_*(NTK) \xleftarrow{d_*} H_*(NT_{[2]}K) \leftarrow \cdots$$

and the E^r converge to E^∞ which is the graded group associated with the filtration on $H_*(NTEK)$

Recall that an s.s. object X is called n-connected if $\pi_q(X, *) = 0$ for all $q \le n$

<u>Lemma</u> Let T be a functor from $\mathcal{a}$ to $\mathcal{a}$ with $T(0) = 0$, prolonged to a functor on s.s. abelian groups. Suppose that for all 0-connected s.s. free abelian groups K, TK is N-connected. Then for all ~~free~~ s.s. free abelian groups L which are n-connected, TL is $(N+n)$-connected.

<u>Proof.</u> First note that if TK is $N+m$ connected for all m-connected s.s. free abelian groups K, then

The same is true of all the cross effects of T (using the decomposition $T(K_1 + \cdots + K_n) = \sum_{\tau} T(K_{\tau_1} | \cdot\cdot | K_{\tau_r})$

To prove the lemma it suffices to prove the following inductive (on n) step:

If TK is $(N+n-1)$-connected for all $n-1$ connected ss free abelian groups K, then TL is $N+n$-connected for all ~~free~~ n-connected s.s. free abelian groups L.

This is proven by taking for K, any s.s. free abelian group such that

$$H_{q-1}(NK) \approx H_q(NL) \quad \text{all } q$$

so that $H_q(NEK) \approx H_q(NL)$ all q.

By Dold's theorem, EK may be substituted for L before applying T, ie $H_q(NTEK) \approx H_q(NTL)$

Now apply the generalized bar const. to TK and obtain $\mathcal{T}K$. In the (above described) associated spectral sequence, $E^1_{p,q} = H_q(N\, T_{[p]} K)$, which is 0 for $q \leq N+n-1$ by the inductive assumption. Then $H_{p+q}(\mathcal{T}K) = 0$, for all $p+q \leq 1+N+n-1$ (since $p \geq 1$), by standard spectral sequence arguments

QED

9. The Connectivity of $\mathcal{L}^r(M)$

In this section is outlined the proof of the following, which in conjunction with the Hurewicz-Kan theorem (section 3) implies Theorem 1

Theorem 2 If M is a free s.s. abelian group which is n-connected, $\mathcal{L}^r(M)$ is $n + \{\log_2 r\}$-connected

This relies on a decomposition theorem: let M be a free abelian group.

Theorem 3 There is a finite filtration on $\mathcal{L}^r(M)$ say $F^t\mathcal{L}^r(M)$ for $0 \leq 1 \leq \cdot\cdot \leq t \leq \cdot\cdot \leq N(r)$, where the associated quotients $g^t = F^t/F^{t-1}$ satisfy for $t < N(r)$ a recursion relation (i) or (ii) (depending on t)

$$\text{(i)} \quad g^t\mathcal{L}^r(M) \approx g^{t'}\mathcal{L}^{r'}(M) \otimes \mathrm{Sym}^a(g^{t''}\mathcal{L}^{r''}(M))$$

(ii) $g^t\mathcal{L}^r(M) \approx g^{t'''}\mathcal{L}^{r'''}$ [illegible] $(g^{t^{iv}}\mathcal{L}^{r^{iv}}(M))$

The only term which does not decompose further is $g^{N(r)}\mathcal{L}^r(M)$ which is

$$\frac{\mathcal{L}^r(M)}{\sum\limits_{2 \leq i \leq r-2}[\mathcal{L}^i(M), \mathcal{L}^{r-i}(M)]}$$

and which will be called $\mathfrak{M}^r(M)$

The filtration on $\mathcal{L}^r(M)$ is defined by a lexicographic ordering of the "types" of the basic commutators [see 7a] which occur. For example

$$[[[m_1, m_2], m_3], [[m_4, m_5], m_6]]$$

and

$$[[[[m_1, m_2], m_3], m_4], [m_5, m_6]]$$

are both commutators of weight 6, and in the ordering, the "type" of the former will precede that of the latter. An element in $\mathcal{L}^r(M)$ is of filtration $\leq t$ if it may be written as a sum of basic commutators all of types $\leq$ the type corresponding to t. The argument to prove Theorem 3 is a modification of that in M. Hall [7a].

The functor $\mathcal{M}^r(M)$ is simpler than $\mathcal{L}^r(M)$ and satisfies:

<u>Lemma</u> If M is a free 0-connected s.s. abelian group, $\mathcal{M}^r(M)$ is $r-1$ connected.

This lemma may be proven by a technique similar to that of Dold-Puppe [3] in their treatment of $Sym^r(M)$.

Finally, Theorem 2 (and with it, Theorem 1) is proven by induction on r. It is because of the possibility of the decomposition (ii) (Note a composite functor) that the connectivity of $\mathcal{L}^r(M)$ only increases with log r. This is illustrated by the submodule $\mathcal{L}^2(\mathcal{L}^2(M)) \subset \mathcal{L}^4(M)$ which has connectivity only 2 higher than that of M. Also $\mathcal{L}^2(\mathcal{L}^2(\mathcal{L}^2(M))) \subset \mathcal{L}^8(M)$ has connectivity only 3 higher than that of M.

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NUMERICAL INVARIANTS OF HOMOTOPY TYPE

BY. F. PETERSON

LET $\mathcal{C}$ BE THE CATEGORY OF SPACES WITH BASE POINT AND HOMOTOPY CLASSES OF MAPS.

DEFN. X IS AN H-SPACE IF WE CAN COMPLETE THE FOLLOWING DIAGRAM:

$$\begin{tikzcd}
X \times X \arrow[dr, dashed] & \\
X \vee X \arrow[u, "j"] \arrow[r, "\eta"'] & X
\end{tikzcd}$$

WHERE η IS THE FOLDING MAP

DEFN. X IS AN H^*-SPACE OR HAS CATEGORY ≤ 1 IF WE CAN COMPLETE THE FOLLOWING DIAGRAM:

$$\begin{tikzcd}
X \vee X \arrow[d, "i"] & \\
X \times X & X \arrow[l, "\Delta"] \arrow[ul, dashed]
\end{tikzcd}$$

WHERE Δ IS THE DIAGONAL MAP

LET S, T BE FUNCTORS : $\mathcal{C} \to \mathcal{C}$

j, η NATURAL TRANSFORMATIONS, I THE IDENTITY FUNCTOR

DEFN (S, T, j, η) IS A STRUCTURE

DEFN. X IS *STRUCTURED WITH RESPECT TO* *t* IF THERE EXISTS $t : T(X) \to I(X)$ SUCH THAT THE FOLLOWING DIAGRAM IS COMMUTATIVE:

$$\begin{tikzcd} T(X) \arrow[dr, "t"] & \\ S(X) \arrow[u, "j(X)"] \arrow[r, "h(X)"'] & I(X) \end{tikzcd}$$

(THIS IS DUALIZED IN:

$$\begin{tikzcd} R(X) \arrow[d, "i(X)"] & \\ Q(X) & I(X) \arrow[ul] \arrow[l, "\Delta(X)"] \end{tikzcd}$$

)

DEFN. $F \overset{n}{\underset{i=1}{\vee}} X$ DENOTES THE *FAT WEDGE* $(x_1, .., x_n)$ WITH AT LEAST ONE $x_i = *$ ($* =$ BASE PT. OF X)

DEFN. IF WE CAN COMPLETE THE FOLLOWING DIAGRAM;

$$\begin{tikzcd} F \overset{n+1}{\underset{i=1}{\vee}} X \arrow[d, "i"] & \\ \underbrace{X \times \cdots \times X}_{n+1} & X \arrow[l, "\Delta"] \arrow[ul, dashed] \end{tikzcd}$$

THEN X IS OF *CATEGORY $\leq n$*

SO WE HAVE GENERALIZED THE NOTION OF CATEGORY FROM 1 (H^*-SPACE) TO n. WE SEEK A DUAL NOTION THAT FOR

$n \leq 1$ PICKS OUT ONLY H-SPACES.

DEFN. IF X, Y ARE STRUCTURED WITH RESPECT TO t, t' RESPECTIVELY AND $f : X \to Y$ THEN f IS A HOMOMORPHISM IF THE FOLLOWING DIAGRAM IS COMMUTATIVE:

$$\begin{tikzcd}
T(X) \arrow[r, "T(f)"] \arrow[d, "t"'] & T(Y) \arrow[d, "t'"] \\
I(X) \arrow[r, "f"] & I(Y)
\end{tikzcd}$$

ASSUME THAT X IS STRUCTURED WITH RESPECT TO t, $f : X \to Y$ AND (E_f, X, p) IS FIBRE SPACE INDUCED BY $(P(Y), Y, p')$ WHERE $P(Y)$ IS THE PATH SPACE OF Y.

WE SEEK TO FIND CONDITIONS ON f SUCH THAT E_f IS STRUCTURED AND p IS A HOMOMORPHISM.

BASIC ASSUMPTION (*) : $\pi(T(X), \Omega) \xrightarrow{j(X)^{\#}} \pi(S(X), \Omega) \to 0$ IS EXACT FOR LOOP SPACES Ω.

($\pi(A, B)$ DENOTES THE HOMOTOPY CLASSES OF MAPS $A \to B$)

DEFN LET X BE STRUCTURED WITH RESPECT TO t. THEN $f : X \to Y$ IS PRIMITIVE IF THERE EXISTS $t' : T(Y) \to Y$

(t' NOT NECESSARILY A STRUCTURE) SUCH THAT THE FOLLOWING DIAGRAM IS COMMUTATIVE:

$$\begin{tikzcd}
T(X) \arrow[r, "T(f)"] \arrow[d, "t"'] & T(Y) \arrow[d, "t'"] \\
I(X) \arrow[r, "f"] & I(Y)
\end{tikzcd}$$

THEOREM IF (S, T, j, η) SATISFY $(*)$ AND f IS PRIMITIVE THEN E_f IS STRUCTURED SUCH THAT p IS A HOMOMORPHISM.

PROOF USE THE FOLLOWING COMMUTATIVE DIAGRAM WITH EXACT ROWS:

$$\begin{tikzcd}
 & & \pi(S(X),X) \arrow[dd, bend left=60, "S(p)^{\#}"] & \pi(T(Y),Y) \arrow[d, "T(f)^{\#}"] \\
 & & \pi(T(X),X) \arrow[u, "j(X)^{\#}"] \arrow[r] \arrow[d, "T(p)^{\#}"'] & \pi(T(X),Y) \arrow[d, "T(p)^{\#}"] \\
\pi(T(E), {}'Y) \arrow[r] \arrow[d, "j(E)^{\#}"'] & \pi(T(E),E) \arrow[r] \arrow[d, "j(E)^{\#}"] & \pi(T(E),X) \arrow[r, "f_{\#}"] \arrow[d, "j(E)^{\#}"] & \pi(T(E),Y) \\
\pi(S(E), {}'Y) \arrow[r] & \pi(S(E),E) \arrow[r] & \pi(S(E),X) &
\end{tikzcd}$$

EXAMPLES 1.

$$\begin{tikzcd}
E_f \arrow[d] & \\
X \arrow[r, "f"] & K(\Pi, n)
\end{tikzcd}$$

X AN H-SPACE f REPRESPENTS A COHOMOLOGY CLASS

$$t^*(f) = f \otimes 1 + 1 \otimes f + a$$

f IS PRIMITIVE $\Leftrightarrow a = 0$.

2\. CONSIDER THE NATURAL MAP $S\Omega X \to X$ AND DIAGRAM:

$$\begin{tikzcd}
S\Omega X \times S\Omega X \arrow[rd, dashed] & \\
S\Omega X \vee S\Omega X \arrow[u] \arrow[r] & X
\end{tikzcd}$$

THEN FOR $T(X) = S\Omega X \times S\Omega X$

$S(X) = S\Omega X \vee S\Omega X$ CONDITION

$(*)$ IS FULFILLED.

THE DIAGRAM CAN BE FILLED IN

$\Leftrightarrow$ ΩX IS HOMOTOPY ABELIAN

(THEOREM OF STASHEFF)

NOW ASSUME THAT X IS STRUCTURED WITH RESPECT TO t AND IS AN H-SPACE.

$$t : X \times X \to X$$

$$t^*(f) = f \otimes 1 + 1 \otimes f + \sum \alpha \otimes \beta$$

f IS PRIMITIVE IF $\sum {}'\alpha \otimes {}'\beta = 0$

FOR EXAMPLE

$$\begin{tikzcd} E_f \arrow[d] & \\ X = K(\mathbb{Z},2) \arrow[r, "f"] & K(\mathbb{Z},8) \end{tikzcd}$$

LET f REPRESENT $(\text{CUP})^3$

THEN $t^*(f) = f \otimes 1 + 1 \otimes f + i^2 \otimes 1 + 1 \otimes i^2$

(i GENERATOR OF $K(\mathbb{Z},2)$.)

f IS NOT PRIMITIVE THEREFORE E_f IS NOT AN H-SPACE.

BUT WE CAN CONCLUDE ΩE_f IS HOMOTOPY ABELIAN ALTHOUGH E_f IS NOT AN H-SPACE AS $'(c^2) = 0$

<u>THEOREM</u> IF $\pi_i(X) = 0$, $i \leq n-1$ THEN ΩX IS HOMOTOPY ABELIAN $\Leftrightarrow$ X IS AN H-SPACE.

CONSIDER THE GROUP $\pi\left[\overset{n+1}{\underset{i=1}{X}}(\Omega X), \Omega X\right]$

LET α_i REPRESENT THE PROJECTION OF $\overset{n+1}{\underset{i=1}{X}}(\Omega X)$ INTO ITS i-th FACTOR.

LET $[\alpha, \beta] \equiv \alpha\beta\alpha^{-1}\beta^{-1}$.

IN PARTICULAR WE HAVE AN $n+1$ FOLD COMMUTATOR OF $\{\alpha_i\}_{i=1}^{n+1}$

$$[\ldots[[\alpha_1, \alpha_2], \alpha_3], \ldots \alpha_{n+1}]]\ldots]$$

DEFN. NILPOTENCY OF $X \leq n$ IF

FOR ANY CHOICE OF $n+1$ - FOLD COMMUTATORS OF $\{\alpha_i\}_{i=1}^{n+1}$ IT IS TRIVIAL. |

CONSIDER THE GROUP $\pi[SX, \bigvee_{i=1}^{n+1} SX]$

LET β_i REPRESENT THE INJECTION OF SX INTO THE i-th FACTOR OF $\bigvee_{i=1}^{n+1} SX$.

DEFN CONILPOTENCY OF $X \leq n$ IF

FOR ANY CHOICE OF $n+1$ FOLD COMMUTATORS OF $\{\beta_i\}_{i=1}^{n+1}$ IT IS TRIVIAL. |

THEOREM X HAS NILPOTENCY $\leq n \Leftrightarrow$ WE CAN COMPLETE THE FOLLOWING DIAGRAM:

$$\begin{tikzcd}
 & (\underbrace{S\Omega X \vee \cdots \vee S\Omega X}_{n+1}) \vee C(\bigwedge_{i=1}^{n+1} \Omega X) \arrow[dr, dashed] & \\
(\bigvee_{i=1}^{n+1} S\Omega X) \arrow[ur, "i"] \arrow[rr, "\text{FOLDING MAP}"'] & & X
\end{tikzcd}$$

WHERE $C(A)$ DENOTES THE CONE OVER A

$A \wedge B$ DENOTES THE SMASH PRODUCT. |

HOWEVER WE DON'T HAVE A DUAL FOR THIS THEOREM GIVING NECESSARY AND SUFFICIENT CONDITIONS FOR CONILPOTENCY $\leq n$.

HOWEVER WE DO HAVE: A SPACE $Q_n(X)$,

A DIAGRAM:

$$\begin{tikzcd}
 & Q_n(X) \arrow[d] & \\
X \arrow[r, "\Delta"] & \overset{n+1}{\underset{i=1}{\times}} X \arrow[r] \arrow[d, "j"] & \Omega\left(\bigvee_{i=1}^{n+1} SX\right) \\
 & \bigwedge_{i=1}^{n+1} X \arrow[r, "e"] & \Omega S\left(\bigwedge_{i=1}^{n+1} X\right)
\end{tikzcd}$$

AND <u>THEOREM</u> CONILPOTENCY $\leq n \Leftrightarrow ej\Delta = 0$

CONILPOTENCY $\leq n \Leftrightarrow \Delta$ FACTORS THROUGH $Q_n(X)$.

<u>EXAMPLE</u> LET $X = S^2 \cup_g e^8$

WHERE $g = \eta w \eta : S^7 \to S^2$

η IS HOPF MAP, w BLAKERS-MASSEY MAP

THEN WEAK CATEGORY $X = 2$,

CONILPOTENCY $X = 1$.

CONILP. FOR $n=1$ YIELDS:

$$\begin{tikzcd}
X \arrow[r, "\Delta"] & X \times X \arrow[r] \arrow[d, "j"] & \Omega(SX \times SX) \\
 & X \wedge X \arrow[r, "e"] & \Omega S(X \wedge X)
\end{tikzcd}$$

CONILP. $\leq 1 \iff ej\Delta = 0$.

TO DUALIZE THIS WE HAVE A NOTION OF <u>FLAT PRODUCT</u> $X \flat X$

$$\begin{tikzcd}
S(\Omega X \vee \Omega X) \arrow[rr] \arrow[d, equal] & & X \vee X \arrow[r] & X \\
(S\Omega X) \vee (S\Omega X) \vee (X \flat X) \arrow[rr, shift left, leftarrow] \arrow[rr, shift right] & & X \flat X \arrow[u] &
\end{tikzcd}$$

DUALITY DOESN'T HOLD BECAUSE $X \flat X$ APPEARS AND NOT $S\Omega(X \flat X)$

BUT IT IS NOT CLEAR HOW TO DUALIZE IN CASE OF n.

WE WOULD LIKE A STRUCTURE FOR COCAT $X \leq n$ SUCH THAT:

COCAT $X \leq 1 \iff X$ IS AN H-SPACE.

SOME NON-STABLE SPHERE BUNDLES OVER REAL PROJECTIVE SPACE

BY J. LEVINE

WE WILL STUDY THE FOLLOWING PROBLEM: FOR INTEGERS k, n $k \leq n$ DESCRIBE THE k-PLANE BUNDLES OVER $P^n(R)$.

FIRST WE ESTABLISH A CLASSIFICATION THEOREM BY TECHNIQUES SIMILAR TO THE FELDBAU CLASSIFICATION OF BUNDLES OVER SPHERES.

LET S^{n-1} = $(n-1)$ SPHERE, D^n = n DISK, G A TOPOLOGICAL GROUP.

DEFN. IF $f: S^{n-1} \to G$ SUCH THAT $f(-x) = f(x)^{-1}$ WE CALL f *EQUIVARIANT*. IF f_t IS A HOMOTOPY OF EQUIVARIANT MAPS, WE CALL f_t AN *EQUIVARIANT HOMOTOPY*, OR AN *e-HOMOTOPY*.

LET (E, p, P^n) BE A PRINCIPAL G-BUNDLE WITH USUAL RIGHT TRANSLATION OF E BY G.

THEN IF $h: D^n \to P^n$ WHERE h IDENTIFIES ANTIPODAL PTS. OF $\partial D^n = S^{n-1}$ THEN h HAS A LIFTING H.

$$\begin{tikzcd}
 & E^n \arrow[d, "p"] \\
D^n \arrow[ur, "H"] \arrow[r, "h"'] & P^n
\end{tikzcd}$$

IF $x \in S^{n-1}$ THEN $H(x)$ AND $H(-x)$ LIE IN SAME FIBRE OF p. WE DEFINE $f : S^{n-1} \to G$ BY THE FORMULA:

$$H(x) = H(-x)\, f(x) \quad \text{FOR } x \in S^{n-1}$$

IT IS STRAIGHTFORWARD TO CHECK THAT f IS EQUIVARIANT. f IS CALLED A <u>CHARACTERISTIC MAP</u> OF E.

<u>LEMMA</u> ANY TWO CHARACTERISTIC MAPS f_0, f_1 OF E ARE e-HOMOTOPIC UP TO CONJUGACY, I.E. THERE EXISTS $a \in G$ SUCH THAT f_1 AND $a^{-1} f_0 a$ ARE e-HOMOTOPIC.

<u>PROOF</u> IF H_0 AND H_1 ARE LIFTINGS OF h CORRESPONDING TO f_0 AND f_1 DEFINE $g : D^n \to G$ BY $H_1(x) = H_0(x)\, g(x)$

WE ESTABLISH THAT $f_1(x) = g(-x)^{-1} f_0(x) g(x)$

NOW DEFINE $g_t : S^{n-1} \to G$ BY

$$g_t(x) = g(-tx)^{-1} f_0(x)\, g(tx)$$

THEN $g_0 = a^{-1} f_0 a$ WHERE $a = g(0)$

$g_1 = f_1$

WE CHECK THAT g_t IS EQUIVARIANT AND THE LEMMA IS PROVEN. |

THEOREM THE CORRESPONDENCE OF PRINCIPAL G-BUNDLES TO THE CONJUGATE e-HOMOTOPY CLASS OF ITS CHARACTERISTIC MAPS IS A 1-1 CORRESPONDENCE.

PROOF (1) ONTO: SUPPOSE $f: S^{n-1} \to G$ IS EQUIVARIANT. WE DEFINE A PRINCIPAL BUNDLE E_f WITH f AS A CHARACTERISTIC MAP.

CONSIDER THE PRODUCT BUNDLE

$$\begin{tikzcd} D^n \times G \arrow[d] \\ D^n \end{tikzcd}$$

WITH SUBBUNDLE

$$\begin{tikzcd} S^{n-1} \times G \arrow[d] \\ S^{n-1} \end{tikzcd}$$

WE DEFINE A BUNDLE MAP $S^{n-1} \times G \xrightarrow{\varphi} S^{n-1} \times G$ BY $\varphi(x, y) = (-x, f(x)y)$ CLEARLY φ COVERS THE ANTIPODAL MAP OF S^{n-1}.

WE DEFINE E_f BY COLLAPSING $D^n \times G$ THROUGH φ. THIS COVERS A COLLAPSING OF D^n THROUGH THE ANTIPODAL MAP OF S^{n-1}. THUS THERE IS DEFINED IN A NATURAL MANNER A PROJECTION $p: E_f \to P^n$ SUCH THAT THE FOLLOWING DIAGRAM IS COMMUTATIVE :

$$\begin{tikzcd} D^n \times G \arrow[r, "\Phi"] \arrow[d] & E_f \arrow[d, "p"] \\ D^n \arrow[r, "h"] & P^n \end{tikzcd}$$

WHERE Φ IS THE COLLAPSING MAP.
SINCE Φ IS A BUNDLE MAP
WE CAN CHECK THAT G ACTS
WITHOUT FIXED PT. ON E_f AND
$E_f/G = P^n$. ~~ALSO~~ (Φ IS A BUNDLE
MAP BY A THEOREM OF GLEASON)
WE DEFINE $H: D^n \rightarrow E_f$ COVERING h
BY $H(x) = \Phi(x, e)$ AND f IS THE
CHARACTERISTIC MAP. THIS PROVES ONTO.

(2) <u>ONE-ONE</u>: LET E_0, E_1 BE BUNDLES
WITH CHARACTERISTIC MAPS f_0, f_1
DERIVED FROM LIFTINGS H_0, H_1 OF h
WE PROVE THE FOLLOWING:

(a) IF $f_0 = a^{-1} f_1 a$, THEN E_0 IS EQUIVALENT TO E_1.

(b) IF f_0 IS e-HOMOTOPIC TO f_1, THEN E_0 IS EQUIVALENT TO E_1.

THESE CLEARLY ESTABLISH ONE-ONE.

<u>PROOF OF (a)</u>: WE DEFINE AN EQUIVALENCE
$F: E_0 \rightarrow E_1$ BY $F(H_0(x)y) = H_1(x)ay$

THIS IS WELL-DEFINED ; IF $H_0(x)\,y = H_0(-x)\,f_0(x)\,y$ THEN $F(\,H_0(-x)\,f_0(x)y\,) = H_1(-x)\,a\,f_0(x)y$

$$= H_1(-x)\,f_1(x)\,ay = H_1(\,x)\,ay = F\,(H_0(x)y)\,.$$

NOW F IS A BUNDLE MAP AND COVERS THE IDENTITY MAP OF P^n THEREFORE IT IS AN EQUIVALENCE. |

PROOF OF (b): LET $f_t : S^{n-1} \longrightarrow G$ BE AN e-HOMOTOPY FROM f_0 TO f_1. WE MAY ASSUME $E_0 = E f_0$, $E_1 = E f_1$ SINCE BY (a) THESE ARE ALREADY EQUIVALENT. WE USE f_t TO CONSTRUCT A BUNDLE E OVER $I \times D^n$ AS ABOVE WE TAKE

$$\begin{tikzcd} I \times D^n \times G \arrow[d] \\ I \times D^n \end{tikzcd}$$

AND DEFINE A BUNDLE MAP

$$\varphi : I \times S^{n-1} \times G \to I \times S^{n-1} \times G \quad \text{BY:}$$

$$\varphi(\,t, x, y) = (t, -x,\, f_t(x)y\,)$$

BY COLLAPSING UNDER φ, WE OBTAIN E. CLEARLY $E|_{0\times P^n} = E_0$, $E|_{1\times P^n} = E_1$ SO $E_0 \simeq E_1$ |

REMARK ONE CAN PROVE THIS THEOREM, REPLACING G BY ΩB_G AND USING THE INVERSE OPERATION OF A LOOP SPACE, BY STUDYING $[P^n, B_G]$. BUT THE PASSAGE TO G SEEMS TO REQUIRE MORE KNOWLEDGE OF THE RELATION BETWEEN THE MULTIPLICATIVE STRUCTURE OF G AND ΩB_G. |

WE WISH TO STUDY $\pi^{0}_{n-1}(G)$ THE SET OF e-HOMOTOPY CLASSES OF $S^{n-1} \to G$.

IT IS USEFUL TO GENERALIZE THE SITUATION. WE REPLACE THE TOPOLOGICAL GROUP G BY A CONNECTED SPACE X, n-SIMPLE FOR ALL n, WITH INVOLUTION T AND FIXED PT. x_0. WE REPLACE THE ANTIPODAL MAP ON S^n BY INVOLUTIONS T_i DEFINED BY THE FORMULA

$$T_i(x,y) = (x,-y) \quad \text{FOR } x \in D^i.$$

i.e. LEAVING THE FIRST i-COORDINATES FIXED AND REVERSING THE REST.

$$(x,y) \in S^n \qquad (x,y) = (x_1, \dots, x_i, y_{i+1}, \dots y_{n+1})$$

SO FIXED PT. SET OF T_i IS S^{i-1}.

WE DEFINE $\pi_n^i(X, x_0; T)$ AS THE SET OF HOMOTOPY CLASSES OF MAPPINGS

$$f : (S^n, S^{n-1}) \to (X, x_0) \quad \text{s.t.}$$

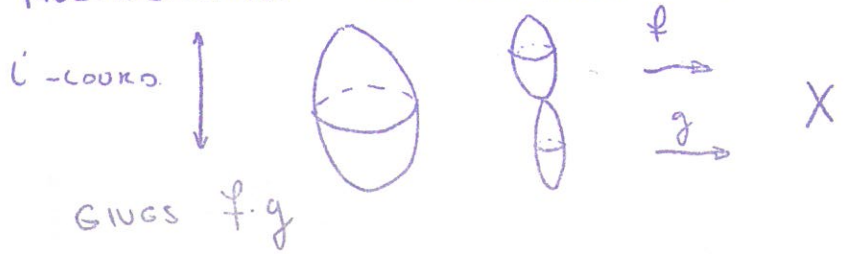
$fT_i = Tf$ SUCH AN f IS *i-EQUIVARIANT*

WHEN $i=0$, $X=G$ THIS AGREES WITH OUR PREVIOUS DEFN. OF EQUIVARIANT.

WE NOTE SOME PROPERTIES:

(1) IF $i>0$, $\pi_n^i(X, x_0; T)$ IS A GROUP

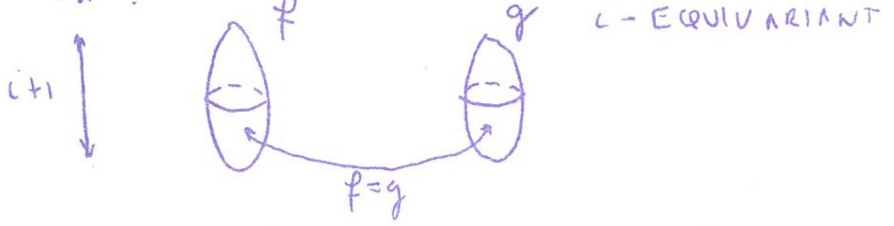
MULTIPLICATION IS DEFINED AS FOLLOWS:



GIVES $f \cdot g$

T_i PRESERVES PARALLELS ALONG i-th AXIS SO IF f, g ARE i-EQUIVARIANT SO IS $f \cdot g$

(2) THERE IS A GROUP ACTION OF π_h ON π_h^i.



NOTE THAT T_i EXCHANGES HEMI-SPHERES

LET $f, g : S^h \to X$, g i-EQUIVARIANT AND LET $f = g$ ON LOWER HEMISPHERE.

WE DEFINE $h : S^h \to X$ BY:

$$h = \begin{cases} f & \text{ON UPPER HEMISPHERE} \\ T f T_i & \text{ON LOWER HEMISPHERE} \end{cases}$$

THEN h IS WELL DEFINED AND EQUIVARIANT.

CONSIDER A SEQUENCE OF MAPPINGS:

$$\pi_h^{i+1} \xrightarrow{\varphi_{i+1}} \pi_h \xrightarrow{\phi_i} \pi_h^i \xrightarrow{\psi_i} \pi_{h-1}^i \xrightarrow{\varphi_i} \pi_{h-1}$$

φ_{i+1}, AND φ_i ARE PASSAGE TO ORDINARY HOMOTOPY CLASS. ψ_i IS RESTRICTION TO $h-1$ SPHERE ALONG $i+1$ ST. AXIS.

$\phi_i(\alpha) = \alpha \cdot X_0$. WHEN $i > 0$, THESE MAPS ARE HOMOMORPHISMS.

WE CAN PROVE SEVERAL THINGS ABOUT THESE MAPS.

THEOREM (a) THIS SEQUENCE IS EXACT. (WHEN $i=0$ THIS IS WELL-DEFINED IF x_0 IS CONSIDERED AS BASE PT. OF π_n^0) MOREOVER AT π_n^i WE HAVE

$$\psi_i(\beta_1)=\psi_i(\beta_2) \Longleftrightarrow \alpha\cdot\beta_1=\beta_2$$ FOR SOME α.

(b) $\varphi_i(\alpha\cdot\beta)=\varphi_i(\beta)+\alpha+(-1)^{n-i+1}T_*(\alpha)$

$\varphi_i(\phi_i(\alpha))=\alpha+(-1)^{n-i+1}T_*(\alpha)$

(c) $\phi_n : \pi_n \longrightarrow \pi_n^n$ IS AN ISOMORPHISM.

(d) $T_*\varphi_i=(-1)^{n-i+1}\varphi_i$

BY INDUCTION ON $n-i$, WE CAN STUDY π_n^i. IF $n-i=0$ USE (c) USING ABOVE SEQUENCE, BY INDUCTION WE KNOW ALL BUT π_n^i.

CONSIDER THE FOLLOWING SITUATION:

LET $(\tilde{X},\tilde{x}_0)$ BE A 2-FOLD COVERING OF (X,x_0) t A COVERING TRANSFORMATION

$p:(\tilde{X},\tilde{x}_0)\longrightarrow(X,x_0)$ THE COVERING MAP

LEMMA THERE EXIST TWO INVOLUTIONS OF $\tilde{X}$ COVERING T, SAY T' AND T'', AND $T'T''=t$.

CALL T' THAT ONE SUCH THAT $T'(\tilde{x}_0)=\tilde{x}_0$

IF $i>0$ $p_*:\pi_n^i(\tilde{X},\tilde{x}_0;T')\longrightarrow\pi_n^i(X,x_0;T)$ IS A HOMOMORPHISM.

PROPOSITION p_* IS A MONOMORPHISM. IF $i>1$ p_* IS AN ISOMORPHISM.

$$\rho_* : \pi_h^0(\tilde{X}, T') \cup \pi_h^0(\tilde{X}, T'') \longrightarrow \pi_h^0(X, T)$$

IS DEFINED.

<u>PROPOSITION</u> IF $h > 1$, ρ_* IS ONTO.

THE INVERSE SETS UNDER ρ_* ARE OF THE FORM $\{\beta, t_*(\beta)\}$ WHERE t_* ACTS ON $\pi_h^0(\tilde{X}, T')$ AND $\pi_h^0(\tilde{X}, T'')$.

TABLE OF k-PLANE BUNDLES OVER P^h

<u>$h=2$</u>

$k=2$	$2T$	$N+T$, A_{2i}	$2N$	A_{2i+1}
$k=3$	$3T$	$N+2T$	$2N+T$	$3N$

<u>$h=3$</u>

$k=2$	$2T$	$N+T$	$2N$	
$k=3$	$3T$	$N+2T$	$2N+T$	$3N$

<u>$h=4$</u>

$k=2$	$2T$	$N+T$	$2N$					
$k=3$	$3T$	$N+2T$, $\oplus^{(4)}$	$2N+T$, $\oplus^{(4)}$	$3N$				
$k=4$	$4T$	$N+3T$, B_{2i}	$2N+2T$, $\oplus^{(4)}$	$3N+T$, C_{2i}	$4N$	B_{2i-1}	$\oplus$	C_{2i-1}
$k=5$	$5T$	$N+4T$	$2N+3T$	$3N+2T$	$4N+T$	$5N$	↓	$\oplus$
							↓	↓
							$6N$	$7N$

<u>$h > 2$</u>

$k=2$	$2T$	$N+T$	$2N$

(1) COLUMNS ARE STABLE CLASSES. THE LAST ROW FOR EACH h IS THE BEGINNING OF THE STABLE RANGE

(2) T = TRIVIAL LINE BUNDLE

(3) N = NON-TRIVIAL LINE BUNDLE

(4) A_i, B_i, C_i = SEQUENCE OF BUNDLES, DISTINGUISHED BY EULER CLASS "i" IN $H^k(P^k, \mathbb{Z})$ WITH TWISTED INTEGERS, k EVEN.

(5) $\oplus$ = A BUNDLE WHICH HAS NO PARTICULAR RECOGNITION $(*)$ = INDISTINGUISHABLE BY ANY CHARACTERISTIC CLASS.

<u>NOTE</u>: (i) ALL BUNDLES OVER P^3 ARE SUMS OF LINE BUNDLES.

(ii) ANY BUNDLE OVER P^2, P^3 OR P^4 WITH TRIVIAL STIEFEL WHITNEY CLASS IS TRIVIAL.

<u>CONJECTURE</u>: FOR ANY n, k THERE ARE ONLY A FINITE NUMBER OF BUNDLES UNLESS $n = k$ IS EVEN IN NON-ORIENTABLE CASE.

IT MAY BE FOR ALL P^n, n EVEN, STABLE TRIVIALITY OF n-PLANE BUNDLE IMPLIES TRIVIALITY.

COMPUTATION OF THE TABLE

RECALL THE EXACT SEQUENCE:

$$\to \pi_n^{i+1} \xrightarrow{\varphi_{i+1}} \pi_n \xrightarrow{\phi_i} \pi_n^{i} \xrightarrow{\psi_i} \pi_{n-1}^{i} \xrightarrow{\varphi_i} \pi_{n-1} \to \dots \to \pi_{n-i} \xrightarrow{\phi_0} \pi_{n-i}^{0} \xrightarrow{\psi_0} \pi_{n-i-1}^{0} \xrightarrow{\varphi_0} \pi_{n-i-1}$$

BASE PT. x_0

<u>NOTE</u> OUR STRONG EXACTNESS AT π^0_{n-i} IMPLIES:

$\pi_{n-i} = 0 \Rightarrow \psi_0$ IS A MONOMORPHISM.

IF X = TOPOLOGICAL GROUP G AND T = INVERSION $\pi^0_{n-1}(G; T)/_{\pi_0(G)}$ CLASSIFIES G-BUNDLES OVER P^n BY PASSING TO CHARACTERISTIC MAP OF A BUNDLE.

LET $G = O_k = R_k \cup A_k$. FROM DEFN. OF CHARACTERISTIC MAP

(a) $\pi^0_{n-1}(R_k)$ = ORIENTABLE BUNDLES OVER P^n,
$\pi^0_{n-1}(A_k)$ = NON-ORIENTABLE BUNDLES OVER P^n.

(b) IF f, g CHARACTERISTIC MAPS OF α, β THEN $\left(\begin{array}{c|c} f & 0 \\ \hline 0 & g \end{array}\right)$ IS CHARACTERISTIC MAP OF $\alpha \oplus \beta$.

IF $k=1$, $O_k = \{+1, -1\}$ THEREFORE

$\pi_{n-1}^0(R_1) = 1$ $\quad \pi_{n-1}^0(\bar{A}_1) = -1$

LET a_i BE MATRIX, IN ANY O_k $k \geq i$,

$$\begin{pmatrix} \overbrace{\begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix}}^{i} & 0 \\ 0 & \begin{matrix}+1 & & \\ & \ddots & \\ & & +1\end{matrix}\end{pmatrix}$$

NOW a_i IS A FIXED POINT OF T; FROM (b), IF WE CONSIDER a_i AS AN ELEMENT OF $\pi_{n-1}^0(O_k; T)$ IT REPRESENTS THE BUNDLE:

$$iN + (k-i)T$$

WE CALCULATE $\pi_{n-1}^0(R_2)$, $\pi_{n-1}^0(\bar{A}_2)$

<u>$n=2$</u>

$$\begin{tikzcd} \pi_1' \arrow[r, "\varphi_1"] & \pi_1 \arrow[r, "\phi_0"] & \pi_1^0 \arrow[r, "\psi_0"] & \pi_0^0 \\ \pi_1 \arrow[u, "\phi_1"] & & & \end{tikzcd}$$

CONSIDER THIS EXACT SEQUENCE WHERE $X = R_2$ OR $\bar{A}_2$, $X_0 = e$ OR a_1.

NOW $\pi_0^0 \approx \pi_0 = X_0$ THUS ϕ_0 IS ONTO.

NOW $\varphi_1 \phi_1(\alpha) = \alpha - T_*(\alpha) = \begin{cases} 2\alpha & X = R_2 \\ 0 & X = \bar{A}_2 \end{cases}$

NOW ϕ_1 IS AN ISOMORPHISM SO

IMAGE $\varphi_1 = \begin{cases} 2\pi_1 & X = R_2 \\ 0 & X = \bar{A}_2 \end{cases}$

THUS $\phi_0(n) = n \cdot X_0 = X_0 \Leftrightarrow \begin{cases} n \text{ EVEN} & X = R_2 \\ n = 0 & X = \bar{A}_2 \end{cases}$

HENCE $\pi_1^0(R_2) = \{e, 1 \cdot e\} = \{e, a_2\}$ SINCE $w(e) \neq w(a_2)$. $\pi_1^0(\bar{A}_2) = \{n \cdot a_1\}$

UNDER CONJUGATION $n \cdot a_1 \longleftrightarrow (-n) \cdot a_1$

STABLE CLASS OF THESE WILL FOLLOW FROM EULER CLASS, WHICH COMES OUT NOW.

<u>$n=3$</u>

$$0 = \pi_2 \to \pi_2^0 \xrightarrow{\psi_0} \pi_1^0 \xrightarrow{\varphi_0} \pi_1$$

IF $X = R_2$, $\varphi_0 = 0$ SINCE $\pi_1^0 = \{\text{CONSTANT}\}$

THEREFORE $\pi_2^0(R_2) \approx \pi_1^0(R_2) = \{e, a_2\}$ BY EXACTNESS.

IF $X = \mathbb{A}_2$, $\varphi_0(n \cdot a_1) = n + T_*(n) = 2n$

(SINCE φ_0 IS, IN TERMS OF BUNDLES, JUST INDUCING BUNDLE UNDER $S^n \to P^n$ THIS VERIFIES STATEMENT ABOUT EULER CLASS IF $n \cdot a_1 = \mathbb{A}_n$)

THUS IMAGE $\psi_0 = a_1$

HENCE BY EXACTNESS $\pi_2^0(\mathbb{A}_2) = a_1$

<u>$n>3$</u> $\quad 0 = \pi_{n-1} \to \pi_{n-1}^0 \xrightarrow{\psi_0} \pi_{n-2}^0 \to \pi_{n-2} = 0$

THUS $\pi_{n-1}^0 \simeq \pi_{n-2}^0$ AND WE GET RESULT BY INDUCTION.

NOW $\pi_{n-1}^0(\mathbb{A}k)$, $\pi_{n-1}^0(R_k)$ FOR $k > 2$.

<u>$n=2$</u> $\quad Z_2 = \pi_1 \xrightarrow{\phi_0} \pi_1^0 \to \pi_0^0 = K_0$

THUS ϕ_0 IS ONTO AND π_1^0 CONTAINS AT MOST TWO SETS THESE ARE $\{e, a_2\}$ AND $\{a_1, a_3\}$.

<u>$n=3$</u> $\quad 0 = \pi_2 \to \pi_2^0 \xrightarrow{\psi_0} \pi_1^0 \xrightarrow{\varphi_0} \pi_1$

$\varphi_0 = 0$ SINCE $\pi_1^0 = \text{CONSTANT}$, $\therefore \psi_0$ IS AN ISOMORPHISM AND $\pi_2^0 \approx \pi_1^0$.

WE NOW ILLUSTRATE TECHNIQUE FOR $n=4$

BY CALCULATING $\pi_3^0(R_3)$.

$$\begin{tikzcd}
\pi_3^1(R_3) \arrow[r, "\varphi_1"] & \pi_3(R_3) \arrow[r, "\phi_0"] \arrow[d, equal] & \pi_3^0(R_3) \arrow[r, "\varphi_0"] & \pi_2^0(R_3) \arrow[r] \arrow[d, equal] & \pi_2(R_3) \\
 & \mathbb{Z} & & \{e, a_2\} & \\
\pi_3(R_3) \arrow[uu, "\phi_1"] & & & &
\end{tikzcd}$$

THUS $\pi_3^0(R_3) = \{n \cdot e, n \cdot a_2\}$

LET $X_0 = e$ OR a_2 THEN $(2n) \cdot X_0 = X_0$

I.E $\phi_0(2n) = 0$ BECAUSE:

$\varphi_1 \phi_1(n) = n - T_*(n) = 2n$ THUS $2n \in$ IMAGE φ_1

THUS $\pi_3^0(R_3) = \{e, a_2, 1 \cdot e, 1 \cdot a_2\}$

<u>CLAIM</u>: (a) $1 \cdot e = e$
(b) $1 \cdot a_2 \neq a_2$

<u>PROOF OF (a)</u>: CONSIDER THE SEQUENCE WHERE $X_0 = e$, IT SUFFICES TO SHOW $1 \in$ IMAGE φ_1

BUT 1 IS REPRESENTED BY $f : S^3 \to R_3$ DEFINED AS FOLLOWS: REPRESENT R^3 AS ROTATIONS OF PURELY IMAGINARY QUATERNIONS. S^3 BY UNIT QUATERNIONS.

$f(q) \cdot q' = q q' \bar{q}$ NOTE $f(S^0) = e$.

NOTE THAT $T_1 : S^3 \to S^3$ COINCIDES WITH CONJUGATION OF QUATERNIONS:

$$T_1(a_0 + i a_1 + j a_2 + k a_3) \to (a_0 - i a_1 - j a_2 - k a_3)$$

THEREFORE NEED TO SHOW $f(q)^{-1} = f(\bar{q})$.

$f(\bar{q})$ IS CONJUGATION BY q. SO $f(q)^{-1} = f(\bar{q})$.

OR $Tf = fT_1$.

<u>PROOF OF (b)</u>: WE ASSUME $a_2 = 1 \cdot a_2$

$$\begin{tikzcd}
\pi_4(R_3) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \pi_4^0(R_3) \arrow[r] \arrow[d, "\varphi_0"] & \pi_3^0(R_3) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \pi_3(R_3) \\
Z_2 & \pi_4(R_3) & \{e, a_2\} &
\end{tikzcd}$$

GENERATED
BY α

THUS $\pi_4^0(R_3) = \{e, \alpha \cdot e, a_2, \alpha \cdot a_2\}$

$\varphi_0(\alpha \cdot X_0) = \alpha \pm T_*(\alpha) = 0$ THEREFORE $\varphi_0 = 0$.

WE SHALL CONTRADICT THIS BY SHOWING φ_0 IS ONTO. $S^4 \xrightarrow[Sh]{} S^3 \xrightarrow{f} R_3$ IS THE ESSENTIAL MAP, WHERE Sh IS SUSPENSION OF HOPF MAP. NOTE $hT_0 = h$, SINCE h COLLAPSES GREAT CIRCLES.

NOTE $ST_0 = T_1$.

$(f \circ Sh) \circ T_0 = f \circ S(hT_0) \circ T_0 = f \circ Sh \circ ST_0 \circ T_0$

$= f \circ Sh \circ T_1 \circ T_0$ NOW $T_1 \circ T_0$ INVERTS COORDINATE ALONG WHICH WE HAVE SUSPENDED h. THEREFORE

$(Sh)(T_1 T_0) = (T_1 T_0)(Sh)$ AND SO

$(f \circ Sh) \circ T_0 = f(T_1 \circ T_0) \circ (Sh) = TfT_0(Sh) = Tf(Sh)$

(SINCE $fT_0 = f$). THUS $f(Sh)$ IS 0-EQUIVARIANT SO φ_0 IS NOT ZERO. CONTRADICTION. |

Peterson

COHOMOLOGY AND ALGEBRAS OVER HOPF ALGEBRAS

BY N. E. STEENROD

LET R BE A COMMUTATIVE RING WITH UNIT. IF $A = \sum_{n \geq 0} A_n$, A IS A GRADED R-MODULE THEN $A \otimes A$ IS A GRADED R-MODULE WHERE $(A \otimes A)_n = \sum_{i+j=n} A_i \otimes A_j$.

DEFNS.

AN ALGEBRA OVER R IS A GRADED R-MODULE, A, WITH A HOMOMORPHISM $\varphi : A \otimes A \to A$ OF GRADED R-MODULES

A IS ASSOCIATIVE IF THE FOLLOWING DIAGRAM IS COMM.:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "1 \otimes \varphi"] \arrow[d, "\varphi \otimes 1"] & A \otimes A \arrow[d, "\varphi"] \\
A \otimes A \arrow[r, "\varphi"] & A
\end{tikzcd}$$

A UNIT FOR A IS AN ALGEBRA HOMOMORPHISM $\eta : R \to A$ SUCH THAT THE FOLLOWING DIAGRAM IS COMMUTATIVE:

$$\begin{tikzcd}
& R \otimes A \arrow[r, "\eta \otimes 1"] & A \otimes A \arrow[dr, "\varphi"] & \\
A \arrow[ur, "\approx"] \arrow[dr, "\approx"'] & & & A \\
& A \otimes R \arrow[r, "1 \otimes \eta"] & A \otimes A \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

AND COMPOSITION IS IDENTITY MAP OF A

DEFNS.

A COALGEBRA OVER R IS A GRADED R-MODULE, A, WITH A HOMOMORPHISM $\psi : A \otimes A \leftarrow A$ OF GRADED R-MODULES

A IS ASSOCIATIVE IF THE FOLLOWING DIAGRAM IS COMMUTATIVE:

$$\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "1 \otimes \psi"'] \\
A \otimes A \arrow[u, "\psi \otimes 1"] & A \arrow[l, "\psi"] \arrow[u, "\psi"']
\end{tikzcd}$$

A UNIT FOR A IS A COALGEBRA HOMOMORPHISM $\varepsilon : R \leftarrow A$ SUCH THAT THE FOLLOWING DIAGRAM IS COMMUTATIVE:

$$\begin{tikzcd}
& R \otimes A \arrow[dl] & A \otimes A \arrow[l, "\varepsilon \otimes 1"'] & \\
A & & & A \arrow[ul] \arrow[dl] \\
& A \otimes R \arrow[ul] & A \otimes A \arrow[l, "1 \otimes \varepsilon"] &
\end{tikzcd}$$

AND COMPOSITION IS IDENTITI MAP OF A

AN *AUGMENTATION* OF AN ALGEBRA WITH UNIT IS A HOMOMORPHISM $\varepsilon : A \to R$ OF ALGEBRAS WITH UNIT. A WITH ε IS AN *AUGMENTED ALGEBRA*.

A IS *COMMUTATIVE* IF THE DIAGRAM:

$$\begin{tikzcd}
A \otimes A \arrow[dd, "T"] \arrow[dr, "\varphi"] & \\
 & A \\
A \otimes A \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

IS COMMUTATIVE.

AN *AUGMENTATION* OF A COALGEBRA WITH UNIT IS A HOMOMORPHISM $\eta : A \leftarrow R$ OF COALGEBRAS WITH UNIT. A WITH η IS AN *AUGMENTED COALGEBRA*

A IS *COMMUTATIVE* IF THE DIAGRAM:

$$\begin{tikzcd}
A \otimes A & \\
 & A \arrow[ul, "\psi"'] \arrow[dl, "\psi"] \\
A \otimes A \arrow[uu, "T"] &
\end{tikzcd}$$

IS COMMUTATIVE.

WHERE $T(x \otimes y) = (-1)^{pq}(y \otimes x)$ FOR $x \in A_p$, $y \in A_q$

DEFN. A *HOPF ALGEBRA* IS A GRADED R-MODULE, A, TOGETHER WITH A MAP $\varphi : A \otimes A \to A$ WHICH MAKES A INTO AN ALGEBRA WITH UNIT $\eta : R \to A$ AND A MAP $\psi : A \to A \otimes A$ WHICH MAKES A INTO A COALGEBRA WITH UNIT $\varepsilon : A \to R$ SUCH THAT:

1) η IS AN AUGMENTATION FOR THE COALGEBRA
2) ε IS AN AUGMENTATION FOR THE ALGEBRA
3) THE FOLLOWING DIAGRAM IS COMMUTATIVE:

$$\begin{tikzcd}
A \otimes A \arrow[r, "\varphi"] \arrow[d, "\psi \otimes \psi"] & A \arrow[r, "\psi"] & A \otimes A \\
A \otimes A \otimes A \otimes A \arrow[rr, "1 \otimes T \otimes 1"] & & A \otimes A \otimes A \otimes A \arrow[u, "\varphi \otimes \varphi"']
\end{tikzcd}$$

REMARK

3) IS EQUIVALENT TO EITHER 3') ψ IS A MAP OF ALGEBRAS OR 3'') φ IS A MAP OF COALGEBRAS.

DEFN. THE HOPF ALGEBRA A IS <u>CONNECTED</u> IF EITHER OF THE EQUIVALENT CONDITIONS HOLDS:

1) $\eta : R \xrightarrow{\cong} A_0$

2) $\varepsilon : A_0 \xrightarrow{\cong} R$

REMARK HOPF PROVED THAT THE RATIONAL COHOMOLOGY RING OF A GROUP MANIFOLD G HAS A CERTAIN FORM (EXTERIOR ALGEBRA ON ODD-DIMENSIONAL GENERATORS) BY ALGEBRAIC TECHNIQUES, ESSENTIALLY USING THE FACT THAT $H^*(G; Z_0)$ IS A HOPF ALGEBRA ($Z_0 \equiv$ RATIONALS).

PROPOSITION IF A IS A HOPF ALGEBRA AND $\mathcal{C}$ IS THE CATEGORY OF A-MODULES AND A-MAPPINGS THEN $\mathcal{C}$ ADMITS AN INTERNAL TENSOR PRODUCT $\otimes_R$ SATISFYING THE USUAL IDENTITIES.

PROOF

LET X BE A GRADED A-MODULE, Y A GRADED B-MODULE $X \otimes_R Y$ IS AN $A \otimes B$ MODULE BY DEFINING:

$(a \otimes b)(x \otimes y) = \pm\, ax \otimes by$ AND THIS IS NATURAL.

LET Y BE AN A-MODULE THEN $X \otimes_R Y$ IS AN $A \otimes A$ MODULE, BUT THIS IS NOT IN THE CATEGORY $\mathcal{C}$. HOWEVER USING $\psi : A \to A \otimes A$ $X \otimes_R Y$ BECOMES AN A MODULE BY DEFINING

$$a \cdot (x \otimes y) = \psi(a)(x \otimes y) .$$

THE FACT THAT ψ IS ASSOCIATIVE PROVES $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$ AS A-MODULES. ALSO $X \otimes Y \cong Y \otimes Z$ AS A-MODULES PROVIDED THAT COALGEBRA A IS COMMUTATIVE.

<u>NOTE</u> R IS A HOPF ALGEBRA.

<u>DEFN</u> LET A BE A HOPF ALGEBRA OVER R AND X AN A-MODULE WHICH IS AN ALGEBRA OVER R. THEN IF $m : X \otimes X \to X$ IS AN A-MAPPING WE CALL X AN <u>ALGEBRA OVER THE HOPF ALGEBRA A</u>.

FOR SUCH AN X AND $\psi : A \to A \otimes A$

$$\psi(a) = \sum_{i=1}^{k} a_i' \otimes a_i''$$ THEN

IF $m(x \otimes y) = xy$ WE HAVE

$$a \cdot (xy) = \sum (-1)^{(x)(a_i'')} (a_i' x) \cdot (a_i'' y)$$

((x) IN EXPONENT IS DEGREE OF X)

EXAMPLES

① OUR MAIN EXAMPLE OF A HOPF ALGEBRA IS THE REDUCED POWER (STEENROD) ALGEBRA $\mathfrak{A}$, DEFINED AS FOLLOWS. CONSIDER COHOMOLOGY OF X WITH COEFFICIENTS IN $\mathbb{Z}_2$. THEN WE HAVE HOMOMORPHISMS $Sq^i : H^q(X) \to H^{q+i}(X)$ $i = 0, 1, 2, \ldots$ AND $\mathfrak{A}$ IS THE GRADED OPERATOR ALGEBRA GENERATED BY Sq^i. FOR Sq^i WE HAVE:

(0) Sq^0 = IDENTITY

(1) $Sq^k x = x^2$ IF $x \in H^k(X)$, i.e. IF DEGREE $x = k$

(2) $Sq^i(x) = 0$ IF $i >$ DEG x

(3) $Sq^r(xy) = \sum_{i=0}^{r} (Sq^i x)(Sq^{r-i} y)$

MILNOR PROVED THAT THERE IS A UNIQUE CO PRODUCT $\psi : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ MAKING $\mathfrak{A}$ INTO A HOPF ALGEBRA SUCH THAT $\psi(Sq^i) = \sum_{j=0}^{i} Sq^j \otimes Sq^{i-j}$. (3) IS THEN SIMPLY THE STATEMENT THAT $H^*(X; \mathbb{Z}_2)$ IS AN ALGEBRA OVER $\mathfrak{A}$.

IF A IS A GRADED R-MODULE, LET ITS DUAL A^* BE THE GRADED R-MODULE

$$A^* = \sum_{n \geq 0} \mathrm{Hom}(A_n, R)$$

AND IF $f: A \to B$ IS A MAP OF GRADED R-MODULES, $f^*: B^* \to A^*$ DEFINED BY $f^*(b^*)a = b^* f(a)$ IS THE DUAL MAP.

BY CONSIDERING $\mathfrak{A}^*$ THE DUAL OF φ BECOMES $\varphi^* : \mathfrak{A}^* \to \mathfrak{A}^* \otimes \mathfrak{A}^*$ AND DUAL OF ψ IS MULTIPLICATION $\psi^*: \mathfrak{A}^* \otimes \mathfrak{A}^* \to \mathfrak{A}^*$. ψ IS COMMUTATIVE HENCE ψ^* IS COMMUTATIVE USING THIS AND FACT THAT $\mathfrak{A}_n$ IS A FINITELY GENERATED FREE R-MODULE MILNOR PROVED $\mathfrak{A}^*$ IS A POLYNOMIAL ALGEBRA AND THIS THEN GAVE INFORMATION ABOUT $\mathfrak{A}$.

② LET X BE AN ALGEBRA OVER R AND LET G BE THE GROUP OF AUTOMORPHISMS OF X AS AN ALGEBRA. WE THEN HAVE MAPS: $G \xrightarrow{d} G \times G \xrightarrow{m} G$

LET $R(G)$ DENOTE THE GROUP RING OF G.

WE HAVE MAPS

$$R(G) \rightarrow R(G) \otimes R(G) \rightarrow R(G)$$

$R(G)$ ACTS AS ENDOMORPHISMS ON X AND IS A HOPF ALGEBRA OVER R. X IS AN ALGEBRA OVER THE HOPF ALGEBRA $R(G)$.

③ R IS ALWAYS A HOPF ALGEBRA

$$\left(R \xrightarrow{\cong} R \otimes_R R \xrightarrow{\cong} R\right),$$

GIVEN AN ALGEBRA OVER K THEN BY ② IT IS AN ALGEBRA OVER A HOPF ALGEBRA. THUS THE NOTION OF AN ALGEBRA OVER A HOPF ALGEBRA IS A TRUE GENERALISATION OF THE NOTION OF AN ALGEBRA.

WHOLESALE CONSTRUCTION OF ALGEBRAS OVER HOPF ALGEBRAS

LET M BE AN ARBITRARY GRADED A-MODULE WHERE A IS A HOPF-ALGEBRA OVER R.

WE DEFINE THE TENSOR ALGEBRA $T(M)$ OF M BY:

$$T(M) = R + M + M\otimes M + \dots + \underbrace{M\otimes M\otimes \cdots \otimes M}_{k} + \dots$$

DEGREE $\quad 0 \quad 1 \quad 2 \quad \dots \quad k$

$T(M)$ IS AN $\mathcal{A}$-MODULE. FOR $M\otimes_R M$ (INTERNAL TENSOR PRODUCT) IS AN $\mathcal{A}$-MODULE (BY USING DIAGONAL MAP $\psi : \mathcal{A} \rightarrow \mathcal{A}\otimes\mathcal{A}$) AND ψ ASSOCIATIVE SAYS $M\otimes M\otimes M$ IS AN $\mathcal{A}$-MODULE PROVING $\underbrace{M\otimes \cdots \otimes M}_{k}$ IS AN $\mathcal{A}$-MODULE. AND $T(M)$ BEING DIRECT SUM OF $\mathcal{A}$-MODULES IS THEREFORE AN $\mathcal{A}$-MODULE. ψ ASSOCIATIVE INDUCES MULTIPLICATION $T(M)\otimes T(M) \rightarrow T(M)$ $\left(M^p \otimes M^q \rightarrow M^{p+q} \right)$ SO THAT $T(M)$ IS AN ALGEBRA OVER THE HOPF ALGEBRA $\mathcal{A}$. FURTHERMORE $T(M)$ SATISFIES A UNIVERSAL PROPERTY. THAT IS, GIVEN

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i"'] & \mathcal{A} \\ T(M) \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

f AN $\mathcal{A}$-HOMOMORPHISM

THERE EXISTS A UNIQUE EXTENSION g.

ASSUME $\psi : \mathcal{A} \rightarrow \mathcal{A}\otimes\mathcal{A}$ IS COMMUTATIVE

THE <u>COMMUTATOR</u> OF $x \in M_p$, $y \in M_q$ IS : $\quad x\otimes y - (-1)^{pq}\, y\otimes x$ AND

IS DENOTED (x, y). THE IDEAL OF COMMUTATORS IS AN A-MODULE AND $T(M) \big/ \text{IDEAL OF COMMUTATORS}$ IS AN ALGEBRA OVER A AND IS COMMUTATIVE. IT IS A UNIVERSAL GADGET FOR COMMUTATIVE A-ALGEBRAS.

NOW LET M BE AN $\mathfrak{A}$-MODULE WHERE $\mathfrak{A}$ IS THE REDUCED POWER (STEENROD) ALGEBRA. SUPPOSE $Sq^i(x) = 0$ IF $i > \text{DEG } x$

LET $\Lambda(M) = \mathbb{Z}_2 + M + M \otimes M + \ldots +$.

$Sq^k(x) \in M$ $\quad x \otimes x \in M \otimes M$

THE IDEAL GENERATED BY $(Sq^k x - x \otimes x)$ IS AN $\mathfrak{A}$-MODULE. THE PROOF OF THIS USES THE ADEM RELATIONS. THEN

$$\Lambda(M) \Big/ \text{IDEAL}(Sq^k x - x \otimes x) \equiv U(M)$$

IS A COMMUTATIVE ALGEBRA OVER $\mathfrak{A}$ AND IS A UNIVERSAL GADGET.

CONSIDER THE EILENBERG MAC LANE COMPLEX $K(\mathbb{Z}_2, n)$ THEN SERRE HAS GIVEN THE STRUCTURE OF $H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$

WE CLAIM:
$$H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2) = U\left(\begin{array}{c}\text{FREE } \mathfrak{A}\text{-MODULE}\\ \text{ON ONE } n\text{-DIML.}\\ \text{GENERATOR}\end{array}\right)$$

<u>THEOREM</u> LET G BE A GROUP MANIFOLD SUCH THAT $H^*(G; \mathbb{Z}_2)$ IS PRIMITIVELY GENERATED AND P BE THE SET OF PRIMITIVE ELEMENTS OF $H^*(G; \mathbb{Z}_2)$

(I.E. UNDER $H^*(G; \mathbb{Z}_2) \xrightarrow{m^*} H^*(G; \mathbb{Z}_2) \otimes H^*(G; \mathbb{Z}_2)$

(m^* INDUCED BY $m: G \times G \to G$) AN ELEMENT $x \in H^*(G; \mathbb{Z}_2)$ IS <u>PRIMITIVE</u>

$\Leftrightarrow$ $m^*(x) = x \otimes 1 + 1 \otimes x$)

THEN $H^*(G; \mathbb{Z}_2) \cong U(P)$

<u>COR</u> $H^*(SO(n)) \cong U(H^*(P^{n-1}))$,

MORE GENERALLY (BUT THIS DOES NOT FOLLOW FROM THE THEOREM)

$$H^*(V_{n,k}) \cong U\left(H^*\left(P^{n-1}/P^{n-k-1}\right)\right)$$,

Peterson

SYMMETRIC PRODUCT OF COMPLEXES

BY E. CURTIS

REFERENCE: DOLD - PUPPE, ANNALS OF FOURIER INSTITUTE.

WE HAVE

THEOREM (DOLD-THOM) IF X IS A SEMI-SIMPLICIAL SET AND CONNECTED THEN

$$\pi_i(\text{SYM}^{\infty} X) \simeq H_i(X) \qquad i \geq 1.$$

DOLD AND PUPPE HAVE REFINED THIS TO

THEOREM IF X IS A SEMI-SIMPLICIAL SET AND $k-1$ CONNECTED, $k-1 \geq 1$, THEN

$$\pi_i(\text{SYM}^N X) \approx H_i(X) \quad \text{FOR } i < k-2+2N.$$

NOTE. THE THEOREM IS NOT TRUE FOR $i = k-2+2N$. LET $k=3$ $N = p^{\nu}$ p A PRIME NUMBER. IF $X = S^3$ THEN $\pi_{2N+1}(\text{SYM}^N S^3) \approx Z_p$

BUT $H_{2N+1}(S^3) = 0$.

WE SHALL PROVE

THEOREM $\pi_i(\text{SYM}^N X) \longrightarrow \pi_i(\text{SYM}^{N+1} X)$ IS AN ISOMORPHISM FOR $i < k+2N-2$, WHICH PROVES THE DOLD-PUPPE THM.

TO PROVE THIS IT WILL BE SUFFICIENT TO PROVE $H_i(\mathrm{SYM}^N X) \xrightarrow{\approx} H_i(\mathrm{SYM}^{N+1} X)$ FOR $i < k + 2N - 2$ AND FOR THIS SUFFICIENT TO PROVE $H_j(\mathrm{SYM}^{N+1} X, \mathrm{SYM}^N X) = 0$ FOR $j < k + 2N - 1$. NOW LET $C_*(X) = P + K = L$ WHERE P IS SS ABELIAN GROUP OF A POINT AND K IS A SS FREE ABELIAN GROUP COMPLEX WHICH IS $k-1$ CONNECTED THAT IS $H_j(K) = 0$ FOR $j < k$.

FORMULA OF STEENROD

$$\mathrm{SYM}^N L = P + \mathrm{SYM}^1 K + \dots + \mathrm{SYM}^N K$$

SINCE $\mathrm{SYM}^i P \approx P$ AND $P \otimes A \approx A$

(ACTUALLY $\mathrm{SYM}^N L = \sum_{i=0}^{N} \mathrm{SYM}^i P \otimes \mathrm{SYM}^{n-i} K$)

THEN $C_*(\mathrm{SYM}^N X) = \mathrm{SYM}^N L = \mathrm{SYM}^{N+1} K$

(LOOK AT $C_*(\mathrm{SYM}^{N+1} X, C_*(\mathrm{SYM}^N X))$)

SO WE ARE REDUCED TO SHOWING THAT

LEMMA IF K IS A FREE ABELIAN GROUP COMOLEX $k-1$ CONNECTED $k \geq 2$, THEN $\mathrm{SYM}^{N+1} K$ IS $k+2N-1$ CONNECTED IE $H_j(\mathrm{SYM}^n K) = 0$ $j < k - 2 + 2n$ $(n = N+1)$

NOW TAKE K TO BE FINITELY GENERATED i.e. K_i IS FINITELY FOR EACH i.

FOR ARBITRARY K TAKE DIRECT LIMIT.

OUR NEXT REDUCTION IS:

LEMMA IF $K = K^1 \oplus \cdot \oplus K^r$ AND THE PREVIOUS LEMMA HOLDS FOR EACH K^i THEN IT HOLDS FOR K.

PROOF $$SYM^n K = \sum_{\sum_{j=1}^{r} i_j = n} SYM^{i_1} K^1 \otimes \cdot\cdot \otimes SYM^{i_r} K^r$$

AND USE THE KUNNETH FORMULA.

IN FACT THE FIRST TIME HOMOLOGY IS NON-ZERO OCCURS IN DIMENSION

$$\sum_{j=1}^{r} (k + 2 i_j - 2) = k \cdot r + 2n - 2r$$

$$= (k-2)r + 2n \geq k - 2 + 2n.$$

NOW TO SHOW $H_i(SYM^n K) = 0 \quad i < k-2+2n$

IT IS SUFF. TO SHOW

$$H_i(SYM^n K \otimes \mathfrak{f}) = H_i(SYM^n(K \otimes \mathfrak{f})) = 0$$

$i < k-2+2n$ FOR EACH PRIME FIELD $\mathfrak{f}$.

BUT BY DOLD $H_i(SYM^n(K \otimes \mathfrak{f}))$ DEPENDS ONLY ON $H_i(K \otimes \mathfrak{f})$. SO REPLACE

K BY A DIRECT SUM OF SPHERES, i.e.

$$K = \sum \tilde{C}_*(S^m)$$

AND BY THE ABOVE LEMMA IT IS SUFFICIENT TO CONSIDER ONLY ONE SPHERE AND PROVE:

IF $K = \tilde{C}_*(S^m)$ $m \geq k$

THEN $H_i(\mathrm{SYM}^n K) = 0$ $i < 2n + k - 2$.

FOR THIS WE USE THE GENERALIZED BAR CONSTRUCTION.

LET $\mathfrak{A}$ BE THE CATEGORY OF SEMI-SIMPLICIAL ABELIAN GROUPS AND

T ANY FUNCTOR $T : \mathfrak{A} \to \mathfrak{A}$.

e.g. $T = \mathrm{SYM}^n$ AND $(TK)_n = TK_n$ ETC.

e.g. $K = \tilde{C}_*(S^m)$

FORM $T(A_1 + . + A_n) = \sum_{\tilde{r}} T(A_{r_1} | . | A_{r_s})$

$(\text{SO } T(A_1 + A_2) = T(A_1) + T(A_1 | A_2) + T(A_2))$.

FORM $T(K) \xleftarrow{\alpha} T(K|K) \leftarrow \ldots$ $T_p(A)$

$$T_p(A) \xrightarrow{\alpha_i} T_{p-1}(A)$$

α INDUCED BY $K + K \to K$ (ADDITION)

$$d = \sum (-1)^i \alpha_i$$

$$\partial = \sum (-1)^i T \partial_i$$

ON DOUBLE COMPLEX LET $D = d \pm \partial$

THEN $H_n($ DOUBLE COMPLEX$) \approx H_{n+1}(EK)$

$EK \equiv$ SUSPENSION OF K

LET $Q =$ SS GROUP OF A POINT ($\mathbb{Z}$ IN ALL DIMENSIONS)
$C =$ SS GROUP OF $\Delta(1)$/ONE END PT
$S =$ SS GROUP OF $\Delta(1)$/BOTH END PTS

SO WE HAVE: $0 \to Q_p \to C_p \to S_p \to 0$

GENERATORS: γ_0 γ_0, γ_p $\gamma_1, .. \gamma_p$

$$\partial_i \gamma_k = \gamma_{k-1} \qquad i < k$$
$$\partial_i \gamma_k = \gamma_k \qquad i \geq k$$
$$s_i \gamma_k = \gamma_{k+1} \qquad i < k$$
$$s_i \gamma_k = \gamma_k \qquad i \geq k$$

WE FORM DOUBLE COMPLEXES

$Q \hat{\otimes} K$, $C \hat{\otimes} K$, $S \hat{\otimes} K$

(WHERE $(A \hat{\otimes} B)_{p,q} \equiv A_p \otimes B_q$) K IS A SS ABELIAN GROUP.

APPLY T: $T(S_p \otimes K_q) = T(\underbrace{K_q + \cdot + K_q}_{p})$

NOW $H_j(TS \hat{\otimes} K) \approx H_j(TSK)$

FURTHERMORE WE HAVE THE COMMUTATIVE

DIAGRAM:

$$\begin{tikzcd}
H_n(TK) \arrow[r, "i_*"] \arrow[dr, "\sigma"'] & H_{n+1}(\text{DOUBLEX COMPLEX}, D) \arrow[d, phantom, "\wr"] \\
& H_{n+1}(T(EK))
\end{tikzcd}$$

WHERE i_* IS INDUCED BY THE INJECTION OF $TK \to$ DOUBLE COMPLEX. THIS IS THE MAIN ASSERTION OF THE GENERALIZED BAR CONSTRUCTION.

IT MAY ALSO BE SHOWN THAT IF K IS q-CONNECTED $T(K/K)$ IS $2q$ CONNECTED, ETC.

SO USING ONLY THE FIRST PART OF THE DOUBLE COMPLEX

$$\begin{tikzcd}
& T(K) \arrow[d] & \\
& T(K) & T(K|K) \arrow[l]
\end{tikzcd}$$

WE FIND THAT

$$\to H_j(T(K|K)) \xrightarrow{\alpha_*} H_j(TK) \xrightarrow{\sigma_*} H_{j+1}(T(EK)) \to H_{j-1}(T(K|K))$$

σ_* ISOMORPHISM IF $j < 2q$

EXACTNESS IF $j < 3q$

NOW LET $K = EQ$ = AUGMENTED CHAIN COMPLEX OF S'.

LEMMA $SYM^n K$ IS CONTRACTIBLE

i.e. $H_j(SYM^n K) = 0$ $n \geq 1$ ALL j.

<u>PROOF</u> NOTE THAT

$$\mathrm{SYM}^n(K^1 | \cdots | K^r) = \sum_{\sum i_\ell = n \;\; i_\ell \geq 0} \mathrm{SYM}^{i_1} K^1 \otimes \cdot \otimes \mathrm{SYM}^{i_r} K^r$$

e.g. $\mathrm{SYM}^n(M+L) = \mathrm{SYM}^n M + \ldots + \mathrm{SYM}^i M \otimes \mathrm{SYM}^{n-i} L + \ldots + \mathrm{SYM}^n L$

LET US APPLY THIS TO Q ; EACH $Q_p = \mathbb{Z}$

$\mathrm{SYM}^n Q \simeq Q$

$$T(Q) \leftarrow T(Q|Q) \xleftarrow{d} \ldots$$

$$\sum \mathrm{SYM}^{i_1} Q \otimes \cdot \otimes \mathrm{SYM}^{i_r} Q \xrightarrow{S} \sum_{\sum j_r = n} \mathrm{SYM}^{j_1} Q \otimes \cdot \otimes \mathrm{SYM}^{j_{r+1}} P$$

IF $i_1 = 1$ $S = 0$

IF $i_1 > 1$ S MAPS $\mathrm{SYM}^{i_1} \to \mathrm{SYM}^1 \, \mathrm{SYM}^{i_1 - 1} \cdot \mathrm{SYM}^{i_r}$

THEN $ds + sd =$ IDENTITY

THIS SHOWS $H_j(\mathrm{SYM}^n K) = 0$,

LET $K = \tilde{C}_*(S)$

$EK = \tilde{C}_*(S^2)$

$$\mathrm{SYM}^n(K) \leftarrow \cdots \leftarrow \mathrm{SYM}^n(\underbrace{K | \cdots | K}_{n})$$

THEN $H_j(\mathrm{SYM}^n(EK)) = \mathbb{Z}$ $\quad j = 2n$

$= 0$ $\quad j < 2n$

AND $H_j(\mathrm{SYM}^n(\tilde{C}_*(S^3)) = \mathbb{Z} \qquad j = 2n+1$

$= 0 \qquad j < 2n+1$

AND IN GENERAL

$H_j(\mathrm{SYM}^n \tilde{C}_* S^k) = 0$

IF $j < 2n+k-2$,