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Unstable 3-primary Homotopy Groups of Spheres

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The Association of Econoinformatics, Himeji Dokkyo University

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Hirosi Toda

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March, 2003

Abstract

The $(n+k)$-dimensional homotopy group of n-sphere is denoted by $\pi_{n+k}(S^n)$. The determination of the group $\pi_{n+k}(S^n)$ is one of fundamental problems in algebraic topology, and the group is investigated by decomposing into the p-primary components ${}_p\pi_{n+k}(S^n)$ for each prime p. In the present work we consider the case that p is an odd prime, in particular the case $p = 3$.

For a fixed integer k, the groups $\pi_{n+k}(S^n)$ are isomorphic to each other in the range $n > k+1$, and as the limit the k-stem stable homotopy gruop $\pi_k^S = \lim_n \pi_{n+k}(S^n)$ is obtained. The stable groups π_k^S are computed for quite large values of k. For example, its 3-primary component ${}_3\pi_k^S$ is determined for k over one hundred.

On the other hand, unstable groups $\pi_{n+k}(S^n)$, $(n \leq k+1)$ are not so good established. The 3-primary components ${}_3\pi_{n+k}(S^n)$ were determined for $k < 46$ in 1967. Since then we have less information abount them. In the present paper we shall give tables of ${}_3\pi_{n+k}(S^n)$ for $k < 80$.

By virtue of Serre decomposition ${}_p\pi_{i+1}(S^{2m}) \cong {}_p\pi_i(S^{2m-1}) \oplus {}_p\pi_{i+1}(S^{4m-1})$, it is sufficient to compute ${}_p\pi_{n+k}(S^n)$ only for odd $n = 2m-1$, and our main tool is the double suspension homomorphism E^2 contained in the double EHP-sequence

$$\cdots \xrightarrow{H} \pi_i(Q_2^{2m-1}) \xrightarrow{P} \pi_i(S^{2m-1}) \xrightarrow{E^2}_p \pi_{i+2}(S^{2m+1}) \xrightarrow{H} \pi_{i-1}(Q_2^{2m-1}) \xrightarrow{P} \cdots .$$

Here, Q_2^{2m-1} is a homotopy fibre of the double suspension map $S^{2m-1} \longrightarrow \Omega^2 S^{2m+1}$, and its homotopy groups are verified by use of the following IJΔ-sequence:

$$\cdots \xrightarrow{\Delta} \pi_{i+2}(S^{2pm-1}) \xrightarrow{I} \pi_i(Q_2^{2m-1}) \xrightarrow{J} \pi_{i+3}(S^{2pm+1}) \xrightarrow{\Delta} \pi_{i+1}(S^{2pm-1}) \xrightarrow{I} \cdots .$$

The computation of the k-stem groups ${}_3\pi_{n+k}(S^n)$ is done by induction on k, from the results of the stable k-stem groups. Several old methods are renewed by the results of many researchers including Oka, Dyer-Lashof, Nishida, Selic, Cohen-Moore-Neisendorfer, Gray and Harper.

Contents

1 Introduction

Denote by $\pi_i(S^n)$ the i-th homotopy group of n-sphere S^n. The purpose of the present paper is to determine the p-primary component ${}_p\pi_i(S^n)$ of $\pi_i(S^n)$ for odd prime p, in prticuler $p = 3$, from several results on the stable groups $\pi_k^S = \lim \pi_{n+k}(S^n)$.

By Serre [28],[29], $\pi_i(S^n)$ is finite except $\pi_n(S^n) \cong \mathbf{Z}$ and $\pi_{4m-1}(S^{2m}) \cong \mathbf{Z} \oplus (\text{finite})$, and also we have Serre decomposition

$$ {}_p\pi_{i+1}(S^{2m}) \cong {}_p\pi_i(S^{2m-1}) \oplus {}_p\pi_{i+1}(S^{4m-1}) . $$

So, it is sufficient to determine the group ${}_p\pi_i(S^n)$ only for odd $n = 2m+1$. Usually, the determination is done for each sequence of the k-stem groups

$$ (1.1)_k \quad {}_p\pi_{3+k}(S^3) \xrightarrow{E^2} {}_p\pi_{5+k}(S^5) \xrightarrow{E^2} {}_p\pi_{7+k}(S^7) \xrightarrow{E^2} \cdots\cdots\cdots\cdots \xrightarrow{E^\infty} {}_p\pi_k^S $$

connected by the double suspensions E^2, from the knowledge of the lower stems and the p-primary stable group ${}_p\pi_k^S$.

Our main tool to determine $\pi_i(S^{2m+1})$ is the following double EHP-sequence :

$$ (1.2)_m \quad \cdots \xrightarrow{H} \pi_i(Q_2^{2m-1}) \xrightarrow{P} \pi_i(S^{2m-1}) \xrightarrow{E^2} \pi_{i+2}(S^{2m+1}) \xrightarrow{H} \pi_{i-1}(Q_2^{2m-1}) \xrightarrow{P} \cdots , $$

where Q_2^{2m-1} is a homotopy fiber of the double suspension map $i^2 : S^{2m-1} \to \Omega^2 S^{2m+1}$, H and P are double version of Hopf invariant and Whitehead product respectively. We call here the elements of $\pi_i(Q_2^{2m-1})$ as the *invariants* for convenience.

Homotopy groups of Q_2^{2m-1} may be computed by the following IJΔ-sequence which is exact after loclized at p :

$$ (1.3)_m \quad \cdots \xrightarrow{J} \pi_{i+4}(S^{2pm+1}) \xrightarrow{\Delta} \pi_{i+2}(S^{2pm-1}) \xrightarrow{I} \pi_i(Q_2^{2m-1}) $$
$$ \xrightarrow{J} \pi_{i+3}(S^{2pm+1}) \xrightarrow{\Delta} \pi_{i+1}(S^{2pm-1}) \xrightarrow{I} \cdots , $$

where $\Delta \circ E$ and $E \circ \Delta$ are p times, and $J \circ H = H_p : \pi_{i+3}(S^{2m+1}) \to \pi_{i+3}(S^{2pm+1})$ is the James-Hopf invariant.

Several lemmas for iterated suspensions in Toda[38] are reformed removing dimensional restrictions by use of Moore-type representations, exponential theorems Selic[26], Neisendorfer[19], and results of Gray[7], Harper[10] and Oka[21].

We shall give tables of ${}_p\pi_{n+k}(S^n)$ for odd n by a symbolic way.

For example, the table of 10- and 11-stem groups for $p = 3$ are presented as follows.

$$
\begin{array}{lccccc}
n = & 3 & & 5 & & 7 \\
k = 10 & \bullet & \longrightarrow & \circ & \longrightarrow & \bullet \; = < \beta_1 > \\
k = 11 & \bullet & \longrightarrow & \circ & = & \circ \; = < \alpha'_3 >
\end{array}
$$

where $\bullet$ and $\circ$ indicate cyclic groups $\mathbf{Z}/3$ and $\mathbf{Z}/9$ respectively, $=$ indicates isomorphic double suspension E^2, $\longrightarrow$ does a non-trivial E^2 which is not isomorphic. The last term $\bullet \; = < \beta_1 >$ means that ${}_3\pi_{7+10}(S^7)$ is in stable range and isomorphic to the stable group ${}_3\pi^S_{10}$ generated by the element β_1. Similar for $\circ \; = < \alpha'_3 >$. If ${}_3\pi_{n+k}(S^n)$ is a direct sum of many number of cyclic groups then the group in the table is represented by corresponding symbols $\bullet$, $\circ$ or $\triangleright \cong \mathbf{Z}/27$ stacked together vertically.

The computation is presented by using the following collection of exact sequences $(1.2)_m$ which is called *computation diagram*.

$$
\begin{array}{ccccccccc}
\pi_{-1+k}(Q^1_2) & & \pi_{1+k}(Q^3_2) & & \pi_{3+k}(Q^5_2) & & & & \\
& \nwarrow H & & E^2 \; \nwarrow H & & E^2 \; \nwarrow H & & & \\
& & \pi_{2+k}(S^3) & \longrightarrow & \pi_{4+k}(S^5) & \longrightarrow & \pi_{6+k}(S^7) & \xrightarrow{E^2} & \cdots \\
& & \nwarrow P & & \nwarrow P & & & & \\
\pi_k(Q^1_2) & & \pi_{2+k}(Q^3_2) & & \pi_{4+k}(Q^5_2) & & & & \\
& \nwarrow H & & E^2 \; \nwarrow H & & E^2 \; \nwarrow H & & & \\
& & \pi_{3+k}(S^3) & \longrightarrow & \pi_{5+k}(S^5) & \longrightarrow & \pi_{7+k}(S^7) & \xrightarrow{E^2} & \cdots .
\end{array}
$$

Computation diagrams are also symbolized. For example, the above cases $k = 10, 11$ are computed as follows.

$$
\begin{array}{lccccccc}
n = & & 3 & & 5 & & 7 & \\
& a_2 & & a_1 & & & & \\
& \nwarrow & & \nwarrow & & & & \\
k = 10 & & \bullet & \longrightarrow & \circ & \longrightarrow & \bullet & = < \beta_1 > \\
& & & & & \nwarrow & & \\
& A_2 & & A_1 & & i & & \\
& \nwarrow & & \nwarrow & & & & \\
k = 11 & & \bullet & \longrightarrow & \circ & = & \circ & = < \alpha'_3 >
\end{array}
$$

Here, roman symbols indicate appropriately defined invariants, the homomorphism P or H is erased if it is trivial and remains only the arrow if it is non-trivial.

An unstable version of the stable α-series $\{\alpha_r \in {}_p\pi^S_{rq-1}; r \geq 1\}$, $q = 2(p-1)$, is a typical example. This is, for each r, a collection of elements in $(rq-1)$- and $(rq-2)$-stem groups and corresponding elements in ${}_p\pi_i(Q^{2m-1}_2)$ such that they generates direct summands, with an exception ${}_p\pi_{2p^2-3}(S^{2p-1}) \cong \mathbf{Z}/p^2$, which are closed under EHP-sequences, as is shown in Gray[7]. The collections are chosen such that they are compatible with IJΔ-sequences. This means that in the computation of ${}_p\pi_i(S^{2m+1})$ we can exclude these collections, by remainning few residue invariants. The collections are called *unstable alpha families.*

Besides of the unstable alpha families there are some collections which can exclude before the computation. An element $\xi \in {}_p\pi_i(S^{2m+1})$ is called *simple* if $E^2\xi = 0$ and $\xi \notin \operatorname{Im} E^2$. When $m \not\equiv -1, 0 \pmod p$, this ξ is independent of the IJΔ-sequence. So, we can remove ξ together with the invariants $H(\xi)$ and $x \in \pi_i(Q_2^{2m-1})$ satisfying $P(x) = \xi$. We call these three elemnts as a *simple removable collection.*

There are another kind of removable collection including a pair $\{\xi,\ \Delta\xi \neq 0\}$ which is called as *short range removable collection.*

The results of 3-primary k-stem groups ${}_3\pi_{2m-1+k}(S^{2m+1})$ are given by symbolized tables as above, divided into two parts. One is old table for $k \leq 55$ and the other is new table for $56 \leq k < 80$. The first table revives [38], [39] for $k \leq 45$ and an anouncement of K.Maruyama and M.Mimura in 1998 for $45 < k \leq 55$.

The first cruicial point lies in 56-stem groups which contain unstable version of the relation $\beta_1^3\beta_2 = 0$. This relation holds in S^{15}. Also the 57-stem groups contain the relation $\beta_1^2\epsilon' = 0$ which holds in S^9.

$n =$	3		5		7		9		11		13		15
$k = 56$ {	●	=	●	=	●	=	●	=	●	=	●		
			●	→	○	→	●						
$k = 57$			●	→	○	→	●						

The second crucial points concentrate about the 60-stem, 76-stem and further stem groups in *deeply unstable* range. These cases there are many invariants of not stable type, and we have few effective methods which controle them. Our table has some ambiguities in the 76- and 77-stem groups of deep range.

2 Double Suspension and Lemmas

2.1 EHP sequence

Let $Q_1^n = \Omega(\Omega S^{n+1}, S^n)$ be a homotopy fiber of the canonical inclusion (suspension map) $i : S^n \to \Omega S^{n+1}$ which induces the suspension $E : \pi_i(S^n) \to \pi_{i+1}(S^{n+1})$. Then we have the following exact sequence (single EHP sequence).

(2.1.1) $$\cdots \xrightarrow{E} \pi_{i+2}(S^{n+1}) \xrightarrow{H'} \pi_i(Q_1^n) \xrightarrow{P'} \pi_i(S^n) \xrightarrow{E} \pi_{i+1}(S^{n+1}) \xrightarrow{H'} \cdots$$

The Whitehead product $w_n = [\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$ of the identity class $\iota_n \in \pi_n(S^n)$ is an image under P', and if n is even then $H'(w_n) = 2\iota_{2n-1}$. This implies the following Serre's decomposition [29] for odd prime p.

(2.1.2) $$E + w_{2m*} : \pi_i(S^{2m-1}) \oplus \pi_{i+1}(S^{4m-1}) \cong_p \pi_{i+1}(S^{2m}),$$

where $\cong_p$ indicates a mod p isomorphism, that is, its kernel and cokernel are finite and p-torsion free. So, E and w_{2m*} are mod p injective.

Let $\Omega^2 S^{2m+1} = \Omega(\Omega S^{2m+1})$ be the double loop space of S^{2m+1}. We use the notation

$$Q_2^{2m-1} = \Omega(\Omega^2 S^{2m+1}, S^{2m-1}).$$

Q_2^{2m-1} is a homotopy fibre of the canonical inclusion (double suspension map) $i^2 : S^{2m-1} \longrightarrow \Omega^2 S^{2m+1}$ which induces the double suspension homomorphism

$$E^2 = E \circ E \ : \ \pi_i(S^{2m-1}) \longrightarrow \pi_{i+2}(S^{2m+1}).$$

Then we have the following exact sequence, called *(double) EHP sequence.*

(2.1.3) $$\cdots \xrightarrow{E^2} \pi_{i+3}(S^{2m+1}) \xrightarrow{H} \pi_i(Q_2^{2m-1}) \xrightarrow{P} \pi_i(S^{2m-1}) \xrightarrow{E^2} \pi_{i+2}(S^{2m+1}) \xrightarrow{H} \cdots$$

The homomorphisms H and P are double version of Hopf invariant H' and Whitehead product P', and we refere $\pi_i(Q_2^{2m-1})$ as the group of *invariants.*

The following naturalities of H and P hold.

(2.1.4) $$H(\alpha) \circ \xi = H(\alpha \circ E^3\xi) \quad \textit{for} \quad \alpha \in \pi_{i+3}(S^{2m+1}),\ \xi \in \pi_j(S^i)$$

(2.1.5) $$P(\alpha) \circ \xi = P(\alpha \circ \xi) \quad \textit{for} \quad \alpha \in \pi_i(Q_2^{2m-1}),\ \xi \in \pi_j(S^i)$$

As is well known the homology ring of ΩS^{2m+1} is a polynomial algebra $H_*(\Omega S^{2m+1}) = \mathbf{Z}[u]$, $u \in H_{2m}$. By the spectral sequence associated to a fibering

$$\Omega^2 S^{2m+1} \longrightarrow E \longrightarrow \Omega S^{2m+1}$$

of contractible total space E, we have

(2.1.6) $\quad H_i(\Omega^2 S^{2m+1}, S^{2m-1}) \cong_p 0$ *for* $i < 2pm-2$ *and* $\cong_p \mathbf{Z}/p$ *for* $i = 2mp-2$.

The same is true for the homotopy groups, and we have the following mod p *stasbility* of homotopy groups of spheres due to Serre [29].

Theorem 2.1 (Serre [29]) *The double suspension*

$$E^2 : \pi_i(S^{2m-1}) \longrightarrow \pi_{i+2}(S^{2m+1})$$

is an isomorphism of p-primary components for $i < 2pm-3$ *and epimorphism of them for* $i = 2mp-3$.

Let $\pi_k^S = \lim_n \pi_{n+k}(S^k)$ be the stable k-stem group, then the following group is said to be in *stable range.*

(2.1.7) $$\pi_{2m-1+k}(S^{2m-1}) \cong_p \pi_k^S \quad for \quad k < 2(p-1)m-2\,.$$

Let $S^n_\infty = S^n \cup e^{2n} \cup \cdots \cup e^{kn} \cup \cdots$ be James' reduced product space [12] of n-sphere, which is homotopy equivalent to the loop space $\Omega(S^{n+1})$ by the natural H-map $i : S^n_\infty \to \Omega(S^{n+1})$, and we may identify them.

Let $S^n_k = S^n \cup e^{2n} \cup \cdots \cup e^{kn}$ be the kn-skeleton of S^n_∞ and

$$h_p : \Omega(S^{2m+1}) = S^{2m}_\infty \to S^{2pm}_\infty = \Omega(S^{2pm+1})$$

be James's extension[12] of shrinking map $S^{2m}_p \to S^{2m}_p / S^{2m}_{p-1} = S^{2pm}$. In [34], we see that the homotopy fibre of h_p is mod p equivalent to S^{2m}_{p-1} and that there exists a map $T_p : \Omega(S^{2m}_{p-1}) \to \Omega(S^{2pm-1})$ the homotopy fibre of which is mod p equivalent to S^{2m-1}. Then by localized at p, we have the following two fiberings :

(2.1.8) $\quad S^{2m}_{p-1} \to \Omega(S^{2m+1}) \xrightarrow{h_p} \Omega(S^{2pm+1}) \quad and \quad S^{2m-1} \to \Omega(S^{2m}_{p-1}) \xrightarrow{T_p} \Omega(S^{2pm-1})\,,$

and combining these, the following mod p fibering :

(2.1.9) $$\Omega^2(S^{2pm-1}) \xrightarrow{i} Q^{2m-1}_2 \xrightarrow{j} \Omega^3(S^{2pm+1}) \xrightarrow{\partial_p} \Omega(S^{2pm-1})\,.$$

Theorem 2.2 *The mod p fibering (2.1.9) induces the following IJΔ-sequence which is mod p exact.*

$$\cdots \xrightarrow{\Delta} \pi_{i+2}(S^{2pm-1}) \xrightarrow{I} \pi_i(Q_2^{2m-1}) \xrightarrow{J} \pi_{i+3}(S^{2pm+1}) \xrightarrow{\Delta} \pi_{i+1}(S^{2pm-1}) \xrightarrow{I} \cdots$$

Note that the map h_p induces James-Hopf invariant H_p satisfying

(2.1.10) $$H_p = J \circ H : \pi_{i+3}(S^{2m+1}) \longrightarrow \pi_i(Q_2^{2m-1}) \longrightarrow \pi_{i+3}(S^{2pm+1}).$$

The map T_p was re-constructed by Gray [8] such that the map

$$\partial_p : \Omega^3 S^{2pm+1} \longrightarrow \Omega S^{2pm-1}$$

of (2.1.9) satisfies a homotopy

$$\partial_p \circ i^2 \simeq \Omega f_p : \Omega S^{2pm-1} \longrightarrow \Omega^3 S^{2pm+1} \longrightarrow \Omega S^{2pm-1},$$

for a map $f_p : S^{2pm-1} \to S^{2pm-1}$ of degree p. Moreover Harper showed [10]

$$i^2 \circ \partial_p \simeq \Omega(S^2 f_p) : \Omega^3 S^{2pm+1} \longrightarrow \Omega S^{2pm-1} \longrightarrow \Omega^3 S^{2pm+1}.$$

Then we have

Lemma 2.1 *The following two compositions are both p times maps:*

$$\Delta \circ E^2 : {}_p\pi_i(S^{2pm-1}) \longrightarrow {}_p\pi_{i+2}(S^{2pm+1}) \longrightarrow {}_p\pi_i(S^{2pm-1}),$$

$$E^2 \circ \Delta : {}_p\pi_{i+2}(S^{2pm+1}) \longrightarrow {}_p\pi_i(S^{2pm-1}) \longrightarrow {}_p\pi_{i+2}(S^{2pm+1}).$$

Note that the following naturalities hold.

(2.1.11)
$$\begin{aligned}
I(\alpha \circ E^2\xi) &= I(\alpha) \circ \xi && \textit{for } \alpha \in \pi_{i+2}(S^{2pm-1}),\ \xi \in \pi_j(S^i) \\
J(\alpha \circ \xi) &= J(\alpha) \circ E^3\xi && \textit{for } \alpha \in \pi_i(Q_2^{2m-1}),\ \xi \in \pi_j(S^i) \\
\Delta(\alpha \circ E^3\xi) &= \Delta(\alpha) \circ E\xi && \textit{for } \alpha \in \pi_{i+3}(S^{2pm+1}),\ \xi \in \pi_j(S^i) \\
H_p(\alpha \circ E\xi) &= H_p(\alpha) \circ E\xi && \textit{for } \alpha \in \pi_{i+1}(S^{2m+1}),\ \xi \in \pi_j(S^i)
\end{aligned}$$

The following exponent theorem was established by Cohen-Moore-Neisendorfer [3], Neisendorfer [19] for $p = 3$, and Selic [26] for $m = 1$.

Theorem 2.3 $$p^m({}_p\pi_i(S^{2m+1})) = 0$$

This theorem is based on the fact that for a suitable H-structure on $\Omega^2 S^{2m+1}$ the p-th power $\mu^p : \Omega^2 S^{2m+1} \to \Omega^2 S^{2m+1}$ of the product μ is deformed to a map r_m : $\Omega^2 S^{2m+1} \to S^{2m-1}$, that is,

(2.1.12) $$\mu^p \simeq i^2 \circ r_m : (\Omega^2 S^{2m+1}, S^{2m-1}) \longrightarrow (\Omega^2 S^{2m+1}, S^{2m-1})$$

It follows

Lemma 2.2 ${}_p\pi_i(Q_2^{2m-1})$ *is elementary, i. e.,* $p({}_p\pi_i(Q_2^{2m-1})) = 0$.

By Theorem 2.1, $E^2 : {}_p\pi_i(S^{2pm-1}) \to {}_p\pi_{i+2}(S^{2pm+1})$ is bijective for $i < 2p^2m - 3$. From the exactness of IJΔ-sequence it follows

Proposition 2.1 *For* $i < 2p^2m - 3$, *we have the following split exact sequence.*

$$0 \longrightarrow \pi^S_{i-2pm+3} \otimes \mathbf{Z}/p \longrightarrow {}_p\pi_i(Q_2^{2m-1}) \longrightarrow \mathrm{Tor}(\pi^S_{i-2pm+2}, \mathbf{Z}/p) \longrightarrow 0\,.$$

We refere the above case as *meta-stable* range.

2.2 Homotopy groups of S^3

Consider the mod p fibering (2.1.8) of the case $m = 1$:

(2.2.1) $$S^2_{p-1} \longrightarrow S^2_\infty \xrightarrow{h_p} S^{2p}_\infty\,.$$

Let $f_\infty : S^2_\infty \to \mathbf{CP}_\infty = S^2 \cup e^4 \cup \cdots \cup e^{2k} \cup \cdots$ be an extension of the identity of S^2. Since $\mathbf{CP}_\infty$ is an Eilenberg-MacLane space of type $(\mathbf{Z}, 2)$, a homotopy fibre $\widetilde{S^2_\infty}$ of f_∞ is a 2-connective fibering over S^2_∞. $\widetilde{S^2_\infty} = \Omega\widetilde{S^3}$ for a 3-connective fibre $\widetilde{S^3}$ of S^3. The restriction f_k over S^2_k of f_∞ is a 2-connective fibering :

$$\widetilde{S^2_k} \xrightarrow{p_k} S^2_k \xrightarrow{f_k} \mathbf{CP}_\infty\,.$$

Since $S^{2p}_\infty = \Omega S^{2p+1}$ is 2-connected, the fibering (2.2.1) is lifted to the following mod p fibering :

(2.2.2) $$\widetilde{S^2_{p-1}} \xrightarrow{i} \widetilde{S^2_\infty} \xrightarrow{h_p} S^{2p}_\infty\,.$$

Since $H^*(S^2_\infty)$ is a divided polynomial algebra and $H^*(\mathbf{CP}_\infty)$ is a polynomial algebra, the relations $f^*_\infty(e^{2k}) = k! \cdot e^{2k}$ hold in cohomology classes.

In particular, the map of $2(p-1)$-skeleton $f_{p-1} : S^2_{p-1} \to \mathbf{CP}_{p-1} = S^{2p-1}/S^1$ becomes a p-equiavalence. f_{p-1} induces S^1-bundle map

$$\widetilde{f_{p-1}} : \widetilde{S^2_{p-1}} \longrightarrow S^{2p-1}$$

which is also a p-equivalence. Then we have a p-equivalence

$$\text{(2.2.3)} \qquad \widetilde{g} : S^{2p-1} \longrightarrow \widetilde{S^2_{p-1}} \quad \textit{such that} \quad \deg(\widetilde{f_{p-1}} \circ \widetilde{g}) \equiv 1 \pmod p.$$

From the fibering (2.2.2) we have the following mod p fibering

$$\text{(2.2.4)} \qquad S^{2p-1} \xrightarrow{\widetilde{g}} \Omega\widetilde{S^3} \xrightarrow{\widetilde{h_p}} \Omega S^{2p+1}.$$

Proposition 2.2 *The mod p fibering* (2.2.4) *induces the following mod p exact sequence.*

$$\cdots \xrightarrow{\Delta} \pi_i(S^{2p-1}) \xrightarrow{\widetilde{G}} \pi_{i+1}(\widetilde{S^3}) \xrightarrow{\widetilde{H_p}} \pi_{i+1}(S^{2p+1}) \xrightarrow{\Delta} \pi_{i-1}(S^{2p-1}) \xrightarrow{\widetilde{G}} \cdots$$

This exact sequence is equivalent to the IJΔ sequence of Theorem 2.2 of the case $m = 1$. In fact, Q^2_1 is homotopy equivalent to $\Omega^3\widetilde{S^3}$, but the relations with the composition are a llitle better than (2.1.11).

$$\text{(2.2.5)} \qquad \begin{aligned} \widetilde{G}(\alpha \circ \xi) &= \widetilde{G}(\alpha) \circ E\xi && \textit{for } \alpha \in \pi_i(S^{2p-1}),\ \xi \in \pi_j(S^i) \\ \widetilde{H_p}(\alpha \circ E\xi) &= \widetilde{H_p}(\alpha) \circ E\xi && \textit{for } \alpha \in \pi_{i+1}(\widetilde{S^3}),\ \ \xi \in \pi_j(S^i) \\ \Delta(\alpha \circ E^2\xi) &= \Delta(\alpha) \circ \xi && \textit{for } \alpha \in \pi_{i+1}(S^{2p+1}),\ \xi \in \pi_j(S^{i-1}) \end{aligned}$$

Let $\partial_p : \Omega^2 S^{2p+1} \longrightarrow S^{2p-1}$ be the inclusion of the homotopy fibre of the map $\widetilde{g} : S^{2p-1} \to \Omega\widetilde{S^3}$. In [27] Selic showed that ∂_p is homotopic to the map r_1 of (2.1.12). Then the Lemma 2.1 holds for the Δ of Proposition 2.2.

$$\text{(2.2.6)} \qquad \Delta(E^2\xi) = p \cdot \xi \quad \textit{and} \quad E^2(\Delta\eta) = p \cdot \eta$$

Let $K' = S^2_{p-1} \cup_g e^{2p}$ be the mapping cone of the composition

$$g = \rho_{p-1} \circ \widetilde{g} : S^{2p-1} \longrightarrow \widetilde{S^2_{p-1}} \longrightarrow S^2_{p-1}.$$

Since $\mathbf{CP}_p = \mathbf{CP}_{p-1} \cup e^{2p}$ is a mapping cone of S^1-bundle map $S^{2p-1} \to \mathbf{CP}_{p-1}$, the map $f_{p-1} : S^2_{p-1} \to \mathbf{CP}_{p-1}$ is extended to a map

$$\overline{f} : K' = S^2_{p-1} \cup_g e^{2p} \longrightarrow \mathbf{CP}_p = \mathbf{CP}_{p-1} \cup e^{2p}$$

such that the top cells are carried by deg $\equiv 1 \pmod p$. Since $x^p = \wp^1(x)$ for $x \in H^2(X; \mathbf{Z}/p)$, it holds

$$(2.2.7) \qquad \wp^1(e^2) = (e^2)^p = e^{2p}$$

for the orientations e^2 and e^{2p} of the bottom and top cells of $X = \mathbf{CP}_p$, respectively. By the naturality of $\wp^1$ for $\overline{f}$, the above equation holds also for $X = K' = S^2_{p-1} \cup_g e^{2p}$.

Let

$$(2.2.8) \qquad G : S^{2p} \longrightarrow S^3$$

be the adjoint of $i \circ g : S^{2p-1} \to S^2_{p-1} \to \Omega S^3$, and let $K = S^3 \cup_G e^{2p+1}$ be the mapping cone of G. Then we have a map $SK' \to K$ of degree 1 on the bottom cells e^3 and top cells e^{2p+1}. By the naturality of $\wp^1$ it follows

$$(2.2.9) \qquad \wp^1(e^3) = e^{2p+1} \quad in \quad H^*(K; \mathbf{Z}/p), \quad K = S^3 \cup_G e^{2p+1}.$$

The homotopy class $\{G\}$ of G belongs to $\pi_{2p}(S^3) \cong \pi_{2p}(\widetilde{S^3}) \cong_p \mathbf{Z}/p$. Theorem 2.2 shows that the order of $a\{G\}$ is p for some integer $a \equiv 1 \pmod p$. Now, we change the map $\overline{g}$ in (2.2.3) by its a times, then the homotopy classes $\{G\}$ and $\{i \circ g\}$ are of order p preserving the properties (2.2.3) to (2.2.9). We denote the class by

$$(2.2.10) \qquad \alpha_1(3) = \{G\} \in \pi_{2p}(S^3) \cong_p \mathbf{Z}/p.$$

Since the class $\{i \circ g\} \in \pi_{2p-1}(S^2_p) \cong \pi_{2p-1}(S^2_\infty)$ is order p, the map $i \circ g : S^{2p-1} \to S^2_p$ can be extended to a map

$$\overline{g} : (Y^{2p}, S^{2p-1}) \longrightarrow (S^2_p, S^2_{p-1}).$$

Lemma 2.3 *The degree of $\overline{g}$ on top cells is -1 modulo p.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc}
\pi_{2p-1}(S^{2p-1}) & \xrightarrow{\tilde{g}_*} & \pi_{2p-1}(\widetilde{S^2_{p-1}}) & \xrightarrow{\widetilde{f_{p-1}}_*} & \pi_{2p-1}(S^{2p-1}) \\
\downarrow{=} & & \rho_{p-1*}\downarrow\cong & & \downarrow\cong \\
\pi_{2p-1}(S^{2p-1}) & \xrightarrow{g_*} & \pi_{2p-1}(S^2_{p-1}) & \xrightarrow{f_{p-1*}} & \pi_{2p-1}(\mathbf{CP}_{p-1}) \\
\partial\uparrow\times p & & \partial\uparrow & & \partial\uparrow\cong \\
\pi_{2p}(Y^{2p}, S^{2p-1}) & \underset{\times x}{\xrightarrow{\bar{g}_*}} & \pi_{2p}(S^2_p, S^2_{p-1}) & \underset{\times p!}{\xrightarrow{f_{p*}}} & \pi_{2p}(\mathbf{CP}_p, \mathbf{CP}_{p-1})
\end{array}$$

Let a be the degree of $\widetilde{f_{p-1}}_* \circ \tilde{g}_*$, then $a \equiv 1 \pmod p$ by (2.2.3). Let x be the degree of $\bar{g}_*$. The commutativity of the above diagram implies $p \times a = x \times p!$. Thus $x = a/(p-1)! \equiv -1 \pmod p$. □

In the mod p exact sequence of Proposition 2.2, we can replace $\pi_{i+1}(\widetilde{S^3})$ by isomorphic $\pi_{i+1}(S^3)$ for $i+1 \neq 3$. Then the homomorpisms $\widetilde{G}$ and $\widetilde{H_p}$ are replaced by G and H_p respectively, given by the following maps :

$$S^{2p-1} \xrightarrow{g} S^2_\infty \xrightarrow{h_p} S^{2p}_\infty .$$

The shrinking map $\pi : Y^{2p} \to S^{2p} = Y^{2p}/S^{2p-1}$ induces an epimorphism

$$\pi_{2p}(S^{2p}) \longrightarrow [Y^{2p}, S^{2p}] \cong \mathbf{Z}/p .$$

Then $\pi^*(a\iota_{2p}) = [\pi] = \pi^*(\iota_{2p})$ if and on If $a \equiv 1 \pmod p$. Lemma 2.3 shows $\bar{g} \simeq -\pi : Y^{2p} \to S^{2p}$, and we have the following homotopy commutative diagram :

$$(2.2.11) \qquad \begin{array}{ccccc}
S^{2p-1} & \xrightarrow{i} & Y^{2p} & \xrightarrow{\pi} & S^{2p} \\
\downarrow{=} & & \bar{g}\downarrow & & -i\downarrow \\
S^{2p-1} & \xrightarrow{g} & S^2_\infty & \xrightarrow{h_p} & S^{2p}_\infty .
\end{array}$$

Theorem 2.4 *The following sequence is* mod p *exact for* $i+1 \neq 3$ *:*

$$\cdots \xrightarrow{H_p} \pi_{i+2}(S^{2p+1}) \xrightarrow{\Delta} \pi_i(S^{2p-1}) \xrightarrow{G} \pi_{i+1}(S^3) \xrightarrow{H_p} \pi_{i+1}(S^{2p+1}) \xrightarrow{\Delta} \cdots ,$$

where G is given by $\quad G(\xi) = \alpha_1(3) \circ E(\xi) \quad$ *for* $\quad \xi \in {}_p\pi_i(S^{2p-1})$
and the Hopf invariant H_p satisfies

$$H_p\{\alpha_1(3), p\iota_{2p}, E(\xi)\}_1 = -E^2(\xi) \quad \textit{for} \quad \xi \in \pi_i(S^{2p-1}) \textit{ with } p \cdot \xi = 0 .$$

Proof. The adjoint of g represents $\alpha_1(3)$, the first relation follows. For a coextension $\widetilde{\xi} \in \pi_{i+1}(Y^{2p})$ of ξ, the adjoint of $\overline{g}_*(\widetilde{\xi})$ belongs to $\{\alpha_1(3), p\iota, E(\xi)\}_1$ and the adjoint of $\pi_*(\widetilde{\xi})$ is $E^2(\xi)$. Then the last relation follows from the commutativity of (2.2.11). □

The following lemma will be applied to investigate the homomorphism Δ in Theorem 2.4.

Lemma 2.4 *If $E^2(\xi) = 0$ for $\xi \in {}_p\pi_i(S^{2p-1})$ then ξ is an image of Δ.*

Proof. Since $E(G(\xi)) = E(\alpha_1(3)) \circ E^2(\xi) = 0$, it follows from (2.1.2) that $G(\xi) = 0$. By the exactness of the sequence of Theorem 2.4 the lemma follows. □

2.3 Representation of invariants

In [38], we used the following representations of invariants in $\pi_i(Q_2^{2m-1})$.

We denote an invariant $x \in {}_p\pi_i(Q_2^{2m-1})$ by the symbol

$$x = Q^m(\xi) \tag{2.3.1}$$

if there exists an element $\xi' \in \pi_{i+2}(S^{2pm-1})$ such that $x = I(\xi')$ and the stable class $\xi = E^\infty(\xi')$ is not divisible by p.

Also, we denote an invariant $y \in {}_p\pi_i(Q_2^{2m-1})$ by the symbol

$$y = \overline{Q}^m(\eta) \tag{2.3.2}$$

if the stable class $\eta = E^\infty(J(y))$ is non-trivial.

Both invaiants $Q^m(\xi)$ and $\overline{Q}^m(\eta)$ are said to be of *stable type*. Remark that these invariants are not fixed only by the stable classes ξ or η. $Q^m(\xi)$ depends on the choice of the elements ξ'. $\overline{Q}^m(\eta)$ has the indeterminacy $\mathrm{Ker}(E^\infty \circ J)$.

On the other hand we use a reperesentation of invariants given by Moore[17].

It follows from (2.1.6) that we can choose a map

(2.3.3) $$g_m : Y^{2pm-2} \longrightarrow Q_2^{2m-1}$$

such that following diagram is homotopy commutative up to non-zero mod p coefficients of i^2 and i^3 .

(2.3.4) $$\begin{array}{ccccc} S^{2pm-3} & \xrightarrow{i} & Y^{2pm-2} & \xrightarrow{\pi} & S^{2pm-2} \\ \downarrow{\scriptstyle i^2} & & \downarrow{\scriptstyle g_m} & & \downarrow{\scriptstyle i^3} \\ \Omega^2 S^{2pm-1} & \longrightarrow & Q_2^{2m-1} & \longrightarrow & \Omega^3 S^{2pm+1} \end{array}$$

By considering the corresponding diagram of homotopy groups we have

Lemma 2.5 *If* $E^2 : {}_p\pi_i(S^{2pm-3}) \to {}_p\pi_{i+2}(S^{2pm-1})$ *and* $E^4 : {}_p\pi_{i-1}(S^{2pm-3}) \to {}_p\pi_{i+3}(S^{2pm+3})$ *are epic, then* $g_{m*} : {}_p\pi_i(Y^{2pm-2}) \longrightarrow {}_p\pi_i(Q_2^{2m-1})$ *is also epic.*

We refere an image $g_{m*}(\gamma) \in \pi_i(Q_2^{2m-1})$ of $\gamma \in \pi_i(Y^{2pm-2})$ to be Moore represented, or simply, *M-represented* by γ.

Denote the adjoint of $\overline{g} : Y^{2p} \to \Omega S^3$ by

(2.3.5) $$G_1 : Y^{2p+1} = S^{2p} \cup_p e^{2p+1} \longrightarrow S^3 .$$

This map G_1 will be used instead of $g_1 : Y^{2p-2} \to Q_2^1$ of (2.3.3).

An image of $G_{1*}(\gamma) \in \pi_{i+1}(S^3)$ of $\gamma \in \pi_{i+1}(Y^{2p+1})$ is called to be *purely M-represented* by γ.

2.4 Primary computations

For the sake of simplicity, we omit the symbol of the p-component as folllows.

(2.4.1) $\pi_i(S^{2m+1}) =_p \pi_i(S^{2m+1}),\quad \pi_i(\widetilde{S^3}) =_p \pi_i(\widetilde{S^3})\quad$ *and* $\quad\pi_i(Q_2^{2m-1}) = {}_p\pi_i(Q_2^{2m-1})$

We apply the exactness of EHP-sequence (2.1.3).

The computation of unstable groups is done by induction on k for each k-stem groups $\pi_{2m+1+k}(S^{2m+1})$ by using the following diagram called *computing diagram.*

$$
\begin{array}{ccccccccccc}
 & \pi_{2+k}(S^3) & \xrightarrow{E^2} & \pi_{4+k}(S^5) & \xrightarrow{E^2} & \pi_{6+k}(S^7) & \xrightarrow{E^2} & \pi_{8+k}(S^9) & \xrightarrow{E^2} & \cdots & \\
 & & \nwarrow P & & \nwarrow P & & \nwarrow P & & \nwarrow P & & \\
\pi_{3+k}(Q_1^2) & & \pi_{2+k}(Q_2^3) & & \pi_{4+k}(Q_2^5) & & \pi_{6+k}(Q_2^7) & & & & \\
\cong\nwarrow H & & \nwarrow H & & \nwarrow H & & \nwarrow H & & & & \\
 & \pi_{3+k}(S^3) & \xrightarrow{E^2} & \pi_{5+k}(S^5) & \xrightarrow{E^2} & \pi_{7+k}(S^7) & \xrightarrow{E^2} & \pi_{9+k}(S^9) & \xrightarrow{E^2} & \cdots & \\
 & & \nwarrow P & & \nwarrow P & & \nwarrow P & & \nwarrow P & & \\
 & & \pi_{3+k}(Q_2^3) & & \pi_{5+k}(Q_2^5) & & \pi_{7+k}(Q_2^7) & & & &
\end{array}
$$

We shall compute k-stem unstable groups $\pi_{n+k}(S^n)$ for $k < pq - 2$, where $q = 2(p-1)$.

First we quote the results of the stable group.

(2.4.2) $$\pi^S_{rq-1} = <\alpha_r> \cong \mathbf{Z}/p \quad for \quad 1 \le r \le p-1$$

$$\pi^S_k = 0 \quad otherwise\ for \quad 0 < k < pq-2\,.$$

Here the element α_1 is the stable class of $\alpha_1(3) \in \pi_{2p}(S^3)$, and for $r > 1$, α_r is defined inductively by

(2.4.3) $$\alpha_r \equiv \{\alpha_1, p\iota, \alpha_{r-1}\} \qquad \bmod\ \alpha_1 \circ \pi^S_{(r-1)q}\,.$$

The invariant of these range are all stable type and computed by Proposition 2.1. In the computation diagram, we remove trivial group $\pi_i(Q_2^{2m-1}) = 0$ and replace non-trivial group of invariants by its generator.

For the case $k = q - 1 = 2p - 3$, the diagram becomes

$$
\begin{array}{cccccccc}
Q^1(\iota) & & & & & & & \\
\nwarrow H & & & & & & & \\
 & \pi_{q+2}(S^3) & \xrightarrow{E^2} & \pi_{q+4}(S^5) & \xrightarrow{E^2} & \pi_{q+6}(S^7) & \xrightarrow{E^2} & \cdots\,,
\end{array}
$$

where E^2 are isomorphic since $H(\pi_{2m+q}(S^{2m+1}) = 0$ and $P(\pi_{2m+q}(Q_2^{2m+1}) = 0$ for $m \ge 1$.

Define $\alpha_1(n) \in \pi_{q+n-1}(S^n)$ by

$$\alpha_1(n) = E^{n-3}\alpha_1(3) \quad for \quad n > 3\,.$$

Then $\pi_{q+2m}(S^{2m+1})$ is generated by $\alpha_1(2m+1)$ for $m \ge 1$.

Next unstable groups appear for $k = 2q-2$ and $k = 2q-1$ and the diagram is

$$
\begin{array}{lllllll}
Q^1(\alpha_1) & & & & & & \\
\quad \nwarrow_H & & & & & & \\
 & \pi_{2q+1}(S^3) & \xrightarrow{E^2} & \pi_{2q+3}(S^5) & \xrightarrow{E^2} & \pi_{2q+5}(S^7) & \xrightarrow{E^2} \quad \cdots, \\
 & & \nwarrow_P & & & & \\
\overline{Q}^1(\alpha_1) & & Q^2(\iota) & & & & \\
\quad \nwarrow_H & & \quad \nwarrow_H & & & & \\
 & \pi_{2q+2}(S^3) & \xrightarrow{E^2} & \pi_{2q+4}(S^5) & \xrightarrow{E^2} & \pi_{2q+6}(S^7) & \xrightarrow{E^2} \quad \cdots .
\end{array}
$$

By Theorem 2.4, $\alpha_2(3) = \{\alpha_1(3), p\iota_{2p}, E\alpha_1(2p-1)\} \in \pi_{2q+2}(S^3)$ corresponds to $\overline{Q}^1(\alpha_1)$. Also the stable class of $\alpha_2(3)$ is α_2. It follows that E^2 of the bottom line are isomorphic and that $H(\pi_{2q+4}(S^5)) = 0$. Then $P \neq 0$ and $\pi_{2q+3}(S^5) = 0$, which is also a consequence of $\pi^S_{2q-2} = 0$.

If $p > 3$, we can continue this sort of discussions, before new stable generator $\beta_1 \in {}_p\pi^S_{pq-2}$ appears.

Define $\alpha_r(n) \in \pi_{n+rq-1}(S^n)$ for $n \geq 3$ and $1 \leq r < p$ inductively by

$$\alpha_r(3) = \{\alpha_1(3), p\iota_{2p}, \alpha_{r-1}(2p)\} \quad \textit{and} \quad \alpha_r(n) = E^{n-3}\alpha_r(3).$$

Then we have

Theorem 2.5 *The non-trivial unstable k-stem groups for $k < pq-2$ are the followings.*

$${}_p\pi_{2m+rq}(S^{2m+1}) = \langle \alpha_r(2m+1) \rangle \cong \mathbf{Z}/p \quad \textit{for} \quad 1 \leq r \leq p-1 \quad \textit{and} \quad 1 \leq m,$$

$${}_p\pi_{2pm-1}(S^{2m+1}) = \langle P(Q^{m+1}(\iota)) \rangle \cong \mathbf{Z}/p \qquad \textit{for} \quad 1 < m < p,$$

$${}_p\pi_{2m+rq-1}(S^{2m+1}) = \langle P(\overline{Q}^{m+1}(\alpha_{r-m-1})) \rangle \cong \mathbf{Z}/p \quad 1 < r < p \quad \textit{and} \quad 1 \leq m < r-1.$$

Next consider lower dimensional homotopy of Moore spece $Y^{n+1} = S^n \cup_p e^{n+1}$ which is a mapping cone of a map $f_p : S^n \to S^n$ of degree p : odd prime. Let

(2.4.4) $$S^n \xrightarrow{f_p} S^n \xrightarrow{i} Y^{n+1} \xrightarrow{\pi} S^{n+1} \xrightarrow{f_p} S^{n+1} \xrightarrow{i} Y^{n+2} \xrightarrow{\pi} \cdots$$

be cofibering sequence. Since the map $G : S^{2p} \to S^3$ represents the generator $\alpha_1(3) \in \pi_{2p}(S^3)$ of order p, there exists an extension of G over Y^{2p+1} :

(2.4.5) $$\overline{G} : Y^{2p+1} \longrightarrow S^3 \quad \textit{such that} \quad G = \overline{G} \circ i.$$

Consider the composition $f_p \circ \overline{G} : Y^{2p+1} \to S^3 \to S^3$. S^3 is an H-space and Y^{2p+1} is a co-H-space. Then $f_p \circ \overline{G}$ is homotopic to p times of $\overline{G}$ which is homotopic zero since p times of the identity of Y^{2p+1} is the same. So, we have a coextension A of $\overline{G}$:

$$A : Y^{2p+2} \longrightarrow Y^4 \quad \textit{such that} \quad \pi \circ A = S\overline{G} : Y^{2p+2} \longrightarrow S^4 \textit{ and } \pi \circ A \circ i = SG \,.$$

We denote suspensions of A and their classes by

$$(2.4.6) \qquad A(n) : Y^{n+q} \longrightarrow Y^n \quad \textit{and} \quad \alpha(n) \in [Y^{n+q}, Y^n]$$

for $n \geq 4$, $q = 2(p-1)$, and call them Adams maps and Adams classes respectively.

The r fold composition of $\alpha(n) = \alpha^1(n)$ is defined inductively by

$$\alpha^r(n) = \alpha^{r-1}(n+q) \circ \alpha(n) \in [Y^{n+rq}, Y^n] \quad \textit{for} \quad n \geq 4 \textit{ and } r > 1 \,,$$

and the r-th member $\alpha_r(n)$ of α-series is defined by

$$\alpha_r(n) = i \circ \alpha^r(n) \circ \pi \in \pi_{n+kq-1}(S^n) \quad \textit{for} \quad n \geq 4 \textit{ and } r \geq 1 \,,$$

where i and π denote, by the same symbol, the classes of $i : S^{n+kq-1} \to Y^{n+kq}$ and $\pi : Y^n \to S^n$ respectively. Here, we must add the definition of $\alpha_r(3)$, $r > 1$ by

$$\alpha_r(3) = \overline{G}_*(\alpha^{r-1} \circ \pi) \,.$$

Since the class $\alpha^{r-1}(n+q) \circ \pi$ represents a coextension $S^{n+kq-1} \to Y^{n+q}$ of $\alpha_{r-1}(n+q-1)$, we have for $n \geq 3$

$$(2.4.7) \quad \alpha_r(n) \equiv \{\alpha_1(n), p\iota_{n+q-1}, \alpha_{r-1}(n+q-1)\} \qquad \text{mod } \alpha_1(n) \circ \pi_{n+rq-1}(S^{n+q-1}) \,.$$

This is compatible with the definition (2.4.3) and we have

$$(2.4.8) \qquad \alpha_r = E^\infty \alpha_r(n) \in \pi^S_{rq-1} \qquad \textit{for} \quad n \geq 3 \,.$$

Let $[\mathbf{Y}, \mathbf{Y}]_k = \lim_n [Y^{n+k}, Y^n]$ be the stable self homotopy groups of Moore spectrum $\mathbf{Y} = \{Y^{n+1}\}$ which is computed by the following two exact sequecnes :

$$\pi^S_k \xrightarrow{f_{p*}} \pi^S_k \xrightarrow{i_*} \pi_k(\mathbf{Y}) \xrightarrow{\pi_*} \pi^S_{k-1} \xrightarrow{f_{p*}} \pi^S_{k-1} \,,$$
$$\pi_{k-1}(\mathbf{Y}) \xrightarrow{f_p^*} \pi^S_{k-1}(\mathbf{Y}) \xrightarrow{i^*} [\mathbf{Y}, \mathbf{Y}]_k \xrightarrow{\pi^*} \pi_k(\mathbf{Y}) \xrightarrow{f_p^*} \pi_k(\mathbf{Y}) \,,$$

where $\pi_k^S = [\mathbf{S},\mathbf{S}]_k$ and $\pi_k(\mathbf{Y}) = [\mathbf{S},\mathbf{Y}]_k$ for the sphere spectrum $\mathbf{S} = \{S^n\}$.

Since f_{p*} and f_p^* are p times we have the following split exact seqeunces :

(2.4.9)
$$0 \longrightarrow \pi_k^S \otimes \mathbf{Z}/p \xrightarrow{i_*} \pi_k(\mathbf{Y}) \xrightarrow{\pi_*} \mathrm{Tor}(\pi_{k-1}^S, \mathbf{Z}/p) \longrightarrow 0\,,$$
$$0 \longrightarrow \pi_{k-1}(\mathbf{Y}) \xrightarrow{\pi^*} [\mathbf{Y},\mathbf{Y}]_k \xrightarrow{i^*} \pi_k(\mathbf{Y}) \longrightarrow 0\,.$$

Since the identity class $1_\mathbf{Y} \in [\mathbf{Y},\mathbf{Y}]_0$ is of order p, $\pi_k(\mathbf{Y})$ and $[\mathbf{Y},\mathbf{Y}]_k$ are $\mathbf{Z}/p$-modules.

For $0 < r < p$, $\pi_{rq-1}^S =< \alpha_r >$ and $\alpha_r = \pi\alpha^r i$, then it follows from (2.4.10)

$$\pi_{rq-1}^S(\mathbf{Y}) =< \delta\alpha^r i > \quad \textit{and} \quad \pi_{rq}^S(\mathbf{Y}) =< \alpha^r i >$$

where $\delta = i \circ \pi \in [\mathbf{Y},\mathbf{Y}]_{-1}$.

Proposition 2.3 $[\mathbf{Y},\mathbf{Y}]_0 =< 1_\mathbf{Y} >\cong \mathbf{Z}/p \quad [\mathbf{Y},\mathbf{Y}]_{-1} =< \delta >\cong \mathbf{Z}/p$.

For $0 < r < p$,

$$[\mathbf{Y},\mathbf{Y}]_{rq} =< \alpha^r >\cong \mathbf{Z}/p\,, \quad [\mathbf{Y},\mathbf{Y}]_{rq-1} =< \delta\alpha^r, \alpha^r\delta >\cong \mathbf{Z}/p \oplus \mathbf{Z}/p\,,$$
$$[\mathbf{Y},\mathbf{Y}]_{rq-2} =< \delta\alpha^r\delta >\cong \mathbf{Z}/p\ .$$

The other groups $[\mathbf{Y},\mathbf{Y}]_k$ *are trivial for* $k < pq-3$.

Since Y^n is $(n-2)$-connected we have

(2.4.10)
$$E^\infty : [Y^{n+k}, Y^n] \cong [\mathbf{Y},\mathbf{Y}]_k \quad \textit{for} \quad n > k+3\,.$$

We shall use the following convension.

If a stable class $\xi \in [\mathbf{Y},\mathbf{Y}]_k$ is an image of E^∞, then $\xi(n) \in [Y^{n+k}, Y^n]$ indicates an element satisfying $E^\infty(\xi(n)) = \xi$.

Similarly, for $\xi \in \pi_k^S$, $\xi(n) \in \pi_{n+k}(S^n)$ satisfies $E^\infty(\xi(n)) = \xi$.

This notation $\xi(n)$ is not necessarily unique, but we can choose $\{\xi(n)\}$ such that $E(\xi(n)) = \xi(n+1)$ whenever they exist. Sometimes, instead of $\xi(n)$ we use simply ξ, ξ in S^n or ξ in Y^n.

3 Iterated Suspension and Lemmas

3.1 Homology of iterated suspension fibre

The $2k$-fold iterated suspension

$$E^{2k} : \pi_i(S^{2m-1}) \longrightarrow \pi_{i+2k}(S^{2m-1+2k})$$

is equivalent to the homomorphism induced by the canonical inclusion

$$i^{2k} : S^{2m-1} \longrightarrow \Omega^{2k} S^{2m-1+2k}.$$

We give a homotopy fibre of i^{2k} as the path space

$$Q_{2k}^{2m-1} = \Omega(\Omega^{2k} S^{2m-1+2k}, S^{2m-1}).$$

The fibering $Q_{2k}^{2m-1} \to S^{2m-1} \to \Omega^{2k} S^{2m-1+2k}$ induces the following exact sequence including E^{2k} :

$$\text{(3.1.1)} \qquad \cdots \xrightarrow{E^{2k}} \pi_{i+2k+1}^{2m-1+2k} \xrightarrow{H^{(2k)}} \pi_i(Q_{2k}^{2m-1}) \xrightarrow{P^{(2k)}} \pi_i^{2m-1} \xrightarrow{E^{2k}} \pi_{i+2k}^{2m-1+2k} \xrightarrow{H^{(2k)}} \cdots,$$

where we write simply $\pi_j^n = \pi_j(S^n)$.

From the triple $(\Omega^{2k+2h} S^{2m-1+2k+2h}, \Omega^{2k} S^{2m-1+2k}, S^{2m-1})$, we have a fibering

$$\text{(3.1.2)} \qquad Q_{2k}^{2m-1} \xrightarrow{i} Q_{2k+2h}^{2m-1} \xrightarrow{j} \Omega^{2k} Q_{2h}^{2m-1+2k}.$$

Then the following sequence is exact

$$\text{(3.1.3)} \qquad \cdots \xrightarrow{\partial} \pi_i(Q_{2k}^{2m-1}) \xrightarrow{i_*} \pi_i(Q_{2k+2h}^{2m-1}) \xrightarrow{j_*} \pi_{i+2k}(Q_{2h}^{2m-1+2k})$$
$$\xrightarrow{\partial} \pi_{i-1}(Q_{2k}^{2m-1}) \xrightarrow{i_*} \cdots.$$

It is well known [4] that mod p homology ring $H_*(\Omega^r S^{n+r})$ is a free commutative algebra over $\mathbf{Z}/p$ and the injection of $\Omega^r S^{n+r}$ into the infinite loop space $Q(S^n) = \lim_s \Omega^s(S^{n+s})$ induces an injective homomorphism of rings. Dyer-Lashof operation, modified in [20],

$$Q^j : H_i(\Omega^r S^{n+r}) \longrightarrow H_{i+jq}(\Omega^r S^{n+r}), \quad q = 2(p-1)$$

is defined such that it is compatible with the homology suspension and $Q^j(x) = x^p$ for x of degree $2j$.

Starting from $H_*(S^{2m+1}) = \Lambda(u)$, we have $H_*(\Omega S^{2m+1}) = \mathbf{Z}/p[u]$ and

$$H_*(\Omega^2 S^{2m+1}) = \Lambda(u, Q^m u, Q^{pm} u, \cdots) \otimes \mathbf{Z}/p[\Delta Q^m u, \Delta Q^{pm} Q^m u, \cdots]$$

for each fundamental class u and homology Bockstein Δ, and then

$$\begin{aligned} H_*(\Omega^3 S^{2m+1}) = \mathbf{Z}/p[u, Q^m u, \Delta Q^{pm-1} \Delta Q^m u, Q^{pm} Q^m u, \Delta Q^{p(pm-1)} Q^{pm-1} \Delta Q^m u \cdots] \\ \otimes \Lambda(\Delta Q^m u, Q^{pm-1} \Delta Q^m u, \Delta Q^{pm} Q^m u, Q^{p(pm-1)} Q^{pm-1} \Delta Q^m u \cdots). \end{aligned}$$

Furthemore, we have for $k < pm$,

$$\begin{aligned} H_*(\Omega^{2k+1} S^{2m+2k-1}) = \mathbf{Z}/p[u, Q^m u, Q^{m+1} u, \cdots, Q^{m+k-1} u, \Delta Q^{pm-1} \Delta Q^m u, \cdots] \\ \otimes \Lambda(\Delta Q^m u, \Delta Q^{m+1} u, \cdots, \Delta Q^{m+k-1} u, Q^{pm-1} \Delta Q^m u, \cdots). \end{aligned}$$

The mod p homology of the homotopy fibre

$$Q_{2k}^{2m-1} = \Omega(\Omega^{2k} S^{2m+2k-1}, S^{2m-1})$$

of the $2k$-fold iterated suspension $i^{2k} : S^{2m-1} \to \Omega^{2k} S^{2m+2k-1}$ is computed by the Wang exact sequence associated with the fibering

$$\Omega^{2k} S^{2m+2k-1} \xrightarrow{i} Q_{2k}^{2m-1} \longrightarrow S^{2m-1}.$$

Then i induces a surjection of H_* with the kernel (u). Thus $H_*(Q_{2k}^{2m+2k-1})$ has the induced ring structure and the following results hold.

(3.1.4) $$H_*(Q_2^{2m-1}) = \mathbf{Z}/p[u, \Delta v_1, v_2, \Delta w, \cdots] \otimes (\Delta u, v_1, \Delta v_2, \cdots)$$

for $u \in H_{2pm-2}, v_l \in H_{2p^2m-2p-1}, v_2 \in H_{2p^2m-2}$ *and* $w \in H_{2p^3m-2p^2-1}$.

(3.1.5) $$H_*(Q_{2k}^{2m-1}) = \mathbf{Z}/p[u_0, u_1, \cdots, u_{k-1}, \Delta v, \cdots] \otimes (\Delta u_0, \Delta u, u_1, \cdots, \Delta v_{k-1}, \cdots)$$

for $u_i \in H_{2pm-2+iq}$, $(0 \leq i \leq k-1)$ *and* $v \in H_{2p^2m-2p-1}$.

Let

$$\wp^i_* : H_n(X) \longrightarrow H_{n-iq}(X)$$

be the dual Steenrod operation mod p. Then we have Nishida's relation [20] between Q^j and $\wp^i_*$. In particular,

$$
\begin{aligned}
&(3.1.6) \quad & \wp^1_* Q^{s+1} &= sQ^s, \quad \wp^1_*\Delta Q^{s+1} = (s+1)\Delta Q^s + Q^s\Delta \quad (s>0)\\
&(3.1.7) \quad & \wp^p_* Q^{s+p} &= -\binom{s(p-1)}{p} Q^s + Q^{s+1}\wp^1_*\\
&(3.1.8) \quad & \wp^p_*\Delta Q^{s+p} &= -\binom{s(p-1)-1}{p}\Delta Q^s + \Delta Q^{s+1}\wp^1_* + \binom{s(p-1)-1}{p-1} Q^s\Delta
\end{aligned}
$$

Applying (3.1.6) we have

$$
\begin{aligned}
\wp^1_*(\Delta Q^{pm}Q^m u) &= (pm\Delta Q^{pm-1} + Q^{pm-1}\Delta)Q^m u = Q^{pm-1}\Delta Q^m u,\\
\wp^1_*(Q^{m+i}u) &= (m+i-1)Q^{m+i-1}u,\\
\wp^1_*(\Delta Q^{m+i}u) &= ((m+i)\Delta Q^{m+i-1} + Q^{m+i-1}\Delta)u = (m+i)\Delta Q^{m+i-1}u.
\end{aligned}
$$

By the naturality of $\wp^1_*$, the same relations hold for the correponding elements

$$
\begin{aligned}
u_i = i_*(Q^{m+i}u) &\in H_{2mp-2+iq}(Q^{2m-1}_{2k}; \mathbf{Z}/p) \quad (0 \geq i < k)\\
\textit{and} \qquad v = i_*(Q^{pm}Q^m u) &\in H_{2p^2m-2}(Q^{2m-1}_2; \mathbf{Z}/p).
\end{aligned}
$$

Then we have the following theorem.

Theorem 3.1 (1) *For degree* $< 2p^3m - 2p^2 - 4$ *and* $u_0 \in H_{2pm-2}$, $v \in H_{2p^2m-2}$,

$$H_*(Q^{2m-1}_2; \mathbf{Z}/p) = \mathbf{Z}/p[u_0, \Delta\wp^1_*\Delta v, v] \otimes \Lambda(\Delta u_0, \wp^1_*\Delta v, \Delta v)\,.$$

(2) *For degree* $< 2p^2m - 2p - 3$ *and* $u_i \in H_{2pm-2+iq}$ $(0 \leq i \leq k-1)$,

$$H_*(Q^{2m-1}_{2k}; \mathbf{Z}/p) = \mathbf{Z}/p[u_0, u_1, \cdots, u_{k-1}] \otimes \Lambda(\Delta u_0, \Delta u_1, \cdots, \Delta u_{k-1})\,,$$

where the following relations hold for $0 \leq i \leq k-1$:

$$\wp^1_* u_i = (m+i-1)u_{i-1}, \quad \wp^1_*\Delta u_i = (m+i)\Delta u_{i-1}\,.$$

3.2 Simple unstable elements

In the exact couple associated to EHP-sequences, the first differential is

$$d_1 = H \circ P : \pi_{i+3}(Q_2^{2m+1}) \longrightarrow \pi_i(Q_2^{2m-1}).$$

Suppose that the d_1-image $d_1(x)$ of an element $x \in \pi_{i+3}(Q_2^{2m+1})$ is non-trivial and let $\xi = P(x) \in \pi_{i+3}(S^{2m+1})$. Then we have

$$(3.2.1) \qquad E^2(\xi) = 0 \quad \textit{and} \quad \xi \notin \operatorname{Im} E^2.$$

Such an element ξ is called as a *simple unstable* element.

In the case $m \not\equiv 0, 1 \pmod p$, we say this element ξ *removable* in the sense that the collection $\{x, \xi = P(x), H(\xi) \neq 0\}$ can be removed in the computation of unstable groups, since the cancellation of this collection is independent both of EHP-sequence and IJΔ-sequence.

The above differential d_1 is induced by the inclusion

$$d : \Omega^3 Q_2^{2m+1} \longrightarrow Q_2^{2m-1}$$

of the homotopy fibre of $i : Q_2^{2m-1} \to Q_4^{2m-1}$. Consider the map

$$g_m : Y^{2pm-2} \longrightarrow Q_2^{2m-1}$$

of (2.3.3). We choose g_m such that its induced map g_{m*} of mod p homology carries the orientation of the top cell to the class u_0 of Theore 3.1. Consider $g_{m+1} : Y^{2p(m+1)-2} \longrightarrow Q_2^{2m+1}$ similarly, and let

$$\Omega_0^3 g_{m+1} : Y^{2p(m+1)-5} = Y^{2pm+q-3} \longrightarrow \Omega^3 Q_2^{2m+1} \quad (q = 2(p-1))$$

be the map adjoint to g_{m+1}.

Since g_m is mod p equivalence up to degree $< 4pm - 5$, then there exists a map

$$h_m : Y^{2pm+q-3} \longrightarrow Y^{2pm-2}$$

for $m > 1$ such that the following diagram is homotopy commutatrive.

$$(3.2.2)\qquad \begin{array}{ccc} Y^{2pm+q-3} & \xrightarrow{h_m} & Y^{2pm-2} \\ \downarrow{\scriptstyle \Omega_0^3 g_{m+1}} & & \downarrow{\scriptstyle g_m} \\ \Omega^3 Q_2^{2m+1} & \xrightarrow{d} & Q_2^{2m-1} \end{array}$$

Let

$$K(m,2) = Y^{2pm-2} \cup_{h_m} CY^{2pm+q-3}$$

be the mapping cone of the map h_m. Then g_m can be extended to a map $\overline{g}_m : K(m,2) \to Q_4^{2m-1}$, and we obtain the following homotopy commutative diagram.

$$(3.2.3)\qquad \begin{array}{ccccc} Y^{2pm-2} & \xrightarrow{i'} & K(m,2) & \xrightarrow{\pi'} & Y^{2pm+q-2} \\ \downarrow{\scriptstyle g_m} & & \downarrow{\scriptstyle \overline{g}_m} & & \downarrow{\scriptstyle \Omega_0^2 g_{m+1}} \\ Q_2^{2m-1} & \xrightarrow{i} & Q_4^{2m-1} & \xrightarrow{j} & \Omega^2 Q_2^{2m+1} \end{array}$$

where π' shrinks the subcomplex Y^{2pm-2} of $K(m,2)$.

These maps g_m, g_{m+1} and $\overline{g}_m$ are unique up to homotopy. The map $\overline{g}_m$ is also mod p equivalence up to degree $< 4pm-5$. When $m > 1$, the homotopy class

$$\eta_m = [h_m] \in [Y^{2pm+q-3}, Y^{2pm-2}] \cong [Y^{q-1}, Y^0]^S$$

is uniquely determined. Then we have the following theorem.

Theorem 3.2 *For $m > 1$ the map $h_m : Y^{2pm+q-3} \to Y^{2pm-2}$ represents*

$$\eta_m = ((m+1)\delta\alpha - m \cdot \alpha\delta)(2pm-2) \in [Y^{2pm+q-3}, Y^{2pm-2}].$$

This follows from the homology structure of Q_4^{2m-1} given in Theorem 3.1 (2). For the details see Proposotion 4.5 of [37]. Desuspensions of η_m are denoted as follows:

$$(3.2.4)\qquad \eta_m^{(t)} = ((m+1)\delta\alpha - m \cdot \alpha\delta)(2pm-2-t), \quad \eta'_m = \eta_m^{(1)}.$$

In the case $m = 1$ we replace Q_2^1 by the 3-connective fibre $\widetilde{S^3}$ of S^3 since $Q_1^2 \simeq \Omega^3\widetilde{S^3}$. The space Q_4^1 is also replaced by the 3-connective fibre $\widetilde{\Omega^2 S^5}$ of $\Omega^2 S^5$. Then $\Omega(\Omega^2\widetilde{S^5}, \widetilde{S^3})$ is homotopy equivalent to $Q_2^3 = \Omega(\Omega^2 S^5, S^3)$, and we have a fibering

$$Q_2^3 \xrightarrow{\tilde{d}} \widetilde{S^3} \xrightarrow{\tilde{i}} \Omega^2\widetilde{S^5}.$$

The mod p cohomology structure of $\widetilde{S^5}$ is verified from the fibering

$$\widetilde{S^5} \longrightarrow S^5 \longrightarrow K(\mathbf{Z},5)$$

and

$$H^*(K(\mathbf{Z},5);\mathbf{Z}/p) = \Lambda(u, \wp^1 u, \wp^2 u, \wp^p\wp^1 u, \cdots) \otimes \mathbf{Z}/p[\Delta\wp^1 u, \Delta\wp^2 u, \Delta\wp^p\wp^1 u, \cdots]$$

for the fundamental class $u \in H^5$ and the cohomology Bockstein Δ.

By Adem relation we have

$$\wp^1(\wp^1 u) = 2\wp^2 u \qquad \mathit{and} \qquad \wp^1(\Delta\wp^1 u) = \Delta\wp^2 u\,.$$

Let $w_5 \in H^{2p+2}(\widetilde{S^5};\mathbf{Z}/p)$ be the cohomology suspension of $\wp^1 u \in H^{2p+3}$, then by Serre spectral seqeuence we have

$$H^*(\widetilde{S^5};\mathbf{Z}/p) = \mathbf{Z}/p[w_5, \wp^1 w_5, \wp^p w_5, \cdots] \otimes \Lambda(\Delta w_5, \Delta\wp^1 w_5, \Delta\wp^p w_5, \cdots)$$

for $* < p(2p+2)$ with $\Delta\wp^1 w_5 = 2\wp^1\Delta w_5$, and further, for $w_3 \in H^*(\Omega^2\widetilde{S^5};\mathbf{Z}/p)$,

(3.2.5) $$H^*(\Omega^2\widetilde{S^5};\mathbf{Z}/p) = \mathbf{Z}/p[w_3, \wp^1 w_3] \otimes \Lambda(\Delta w_3, \Delta\wp^1 w_3) \quad \mathit{for} \quad * < 2p^2\,.$$

with $\Delta\wp^1 w_3 = 2\wp^1\Delta w_3$.

Similarly,

$$H^*(\Omega S^3;\mathbf{Z}/p) = \mathbf{Z}/p[w_3] \otimes \Lambda(\Delta w_3) \quad \mathit{for} \quad * < 2p^2\,.$$

Let $\widetilde{G}_1 : Y^{2p+1} \to \widetilde{S}^3$ be a lift of $G_1 : Y^{2p+1} \to S^3$, then $\widetilde{G}_1$ is a mod p equiavalence for dimension $< 4p+1$. Approximating $\widetilde{d} : Q_2^3 \to \widetilde{S}^3$ through g_2 and $\widetilde{G}_1$, we have a map $\widetilde{h}_1 : Y^{2p+q} \to Y^{2p+1}$ such that the following diagram homotopy commutes :

(3.2.6) $$\begin{array}{ccccc} Y^{2p+q} & \xrightarrow{\widetilde{h}_1} & Y^{2p+1} & \xrightarrow{i'} & \widetilde{K} \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle \widetilde{G}_1} & & \downarrow{\scriptstyle \widetilde{G}_2} \\ Q_2^3 & \xrightarrow{\widetilde{d}} & \widetilde{S^3} & \xrightarrow{i} & \Omega^2\widetilde{S^5} \end{array}$$

where $\widetilde{K} = Y^{2p+1} \cup CY^{2p+q}$ is a mapping cone of $\widetilde{h}_1$ and $\widetilde{G}_2$ is a mod p equivalence up to dimmension $< 4p+1$.

Then similar to Theorem 3.2 the following theorem holds for the class $\widetilde{\eta}_1$ of $\widetilde{h}_1$.

Theorem 3.3 *The map* $\widetilde{h}_1 : Y^{4p-2} \longrightarrow Y^{2p+1}$ *represents*

$$\widetilde{\eta}_1 = (2\delta\alpha - \alpha\delta)(2p+1) \in [Y^{4p-2}, Y^{2p+1}] .$$

As an application of Theorems 3.2 and 3.3 to the simple unstable elements, we have

Proposition 3.1 (1) *Let* $m > 1$. *If an element* $x \in \pi_{i+3}(Q_2^{2m+1})$ *is M-represented by* $E^3\gamma$ *for an element* $\gamma \in \pi_i(Y^{2p(m+1)-5})$, *then* $d_1(x) = HP(x) \in \pi_i(Q_2^{2m-1})$ *is M-represented by*

$$h_{m*}(\gamma) = \eta_m \circ \gamma \in \pi_i(Y^{2pm-2}) .$$

(2) *Let* $m = 1$. *If an element* $x \in \pi_i(Q_2^3)$ *is M-represented by* $\gamma \in \pi_i(Y^{4p-2})$, *then* $P(x) \in \pi_i(S^3)$ *is purely M-represented by*

$$\widetilde{h}_{1*}(\gamma) = \widetilde{\eta}_1 \circ \gamma \in \pi_i(Y^{2p+1}) .$$

For a stable element $\xi \in \pi_k^S$, we denote by $\xi(n)$ an element in $\pi_{n+k}(S^n)$ satisfying $E^\infty(\xi(n)) = \xi$. We define *unstableness* $u(\xi)$ of a stble element $\xi \in \pi_k^S$ by

(3.2.7) $$u(\xi) = \mathrm{Min}\{n | \exists\, \xi(n) \;:\; E^\infty(\xi(n)) = \xi\} .$$

Using the notations $Q^m(\xi)$ and $\overline{Q}^m(\xi)$ of section 3.2, we have the following lemmas.

Lemma 3.1 *If* $u(\xi) < 2p(m+1) - 5$ *for* $\xi \in \pi_k^S$ *then*

$$d_1(Q^{m+1}(\xi)) = PH(Q^{m+1}(\xi)) = (m+1)Q^m(\alpha_1\xi) .$$

When $m = 1$ *the above holds for* ξ *of* $u(\xi) \leq 4p - 3$.

Lemma 3.2 *If* $u(\xi) < 2p(m+1) - 5$ *and* $p\xi = 0$ *for* $\xi \in \pi_k^S$ *then*

$$d_1(\overline{Q}^{m+1}(\xi)) = PH(\overline{Q}^{m+1}(\xi)) = m \cdot \overline{Q}^m(\alpha_1\xi) .$$

When $m = 1$ *the above holds for* ξ *of* $u(\xi) \leq 4p - 3$.

Lemma 3.3 *If* $u(\xi) < 2p(m+1) - 5$, $p\xi = 0$ *and* $\alpha_1\xi = 0$ *for* $\xi \in \pi_k^S$ *then*

$$d_1(\overline{Q}^{m+1}(\xi)) = PH(\overline{Q}^{m+1}(\xi)) = Q^m(\eta) \quad \textit{for} \quad \eta \in (m+1)\{\alpha_1, p, \xi\} - m\{p, \alpha_1, \xi\} .$$

When $m = 1$ *the above holds for* ξ *of* $u(\xi) \leq 4p - 3$.

3.3 Lemmas for p times and Δ

Consider the cofibering of the mapping cone $K(m,2) = Y^{2pm-2} \cup_{h_m} CY^{2pm+q-3}$:

(3.3.1) $$Y^{2pm-2} \xrightarrow{i'} K(m,2) \xrightarrow{\pi'} Y^{2pm+q-2}.$$

Let $1_K : K(m,2) \to K(m,2)$ be the identity of $K(m,2)$.

Theorem 3.4 *The p times $p \cdot 1_K : K(m,2) \to K(m,2)$ of the identity 1_K is homotopic to the composition*

$$i' \circ \alpha(2pm-2) \circ \pi' : K(m,2) \longrightarrow Y^{2pm+q-2} \longrightarrow Y^{2mp-2} \longrightarrow K(m,2).$$

This theorem is essentially proved by Gray[7]. Since $[Y^n, Y^n] \cong \mathbf{Z}/p$, p times of identity of Y^n is homotopic to zero. This gives a homotopy

$$p \cdot 1_k \simeq i' \circ f \circ \pi'$$

for some $f : Y^{2pm+q-2} \to Y^{2pm-2}$, and a homotopy equivalence between the mapping cone of two maps. The mapping cone of $p \cdot 1_K$ is the smash product $Y^2 \wedge K(m,2)$. The mapping cone of $i' \circ f \circ \pi'$ contains the mapping cone of f as subcomplex. Cnsidering $\wp^1$ operations in $Y^2 \wedge K(m,2)$ by Cartan formula, one obtains the $\wp^1$ in the mapping cone of f and the theorem is proved.

Let

$$h'_m : Y^{2pm+q-4} \longrightarrow Y^{2pm-3} \quad \textit{and} \quad K'(m,2) = Y^{2pm-3} \cup_{h'_m} CY^{2pm+q-4}$$

be desuspensions of h_m and $K(m,2)$ and consider the suspension

$$E : \pi_{i-1}(K'(m,2)) \longrightarrow \pi_i(K(m,2)).$$

If $x \in \mathrm{Im}E$ then $p \cdot x = (p \cdot 1K)_*(x)$. So, as a corollary of Theorem 3.4 we have

Proposition 3.2 *Asuume that an element $\gamma \in \pi_i(K(m,2))$ is a suspension image then there holds the equality*

$$p(\gamma) = i'_*(\alpha(2pm-2) \circ \pi'_*(\gamma)).$$

Applying this to the commutative diagram

$$\begin{array}{ccccc}
\pi_{i+3}(S^{2m+1}) & \xrightarrow{E^2} & \pi_{i+5}(S^{2m+3}) & \xrightarrow{=} & \pi_{i+5}(S^{2m+3}) \\
\downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H^{(4)}} & & \downarrow{\scriptstyle H} \\
\pi_i(Q_2^{2m-1}) & \xrightarrow{i_*} & \pi_i(Q_4^{2m-1}) & \xrightarrow{j_*} & \pi_{i+2}(Q_2^{2m+1}) \\
\downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P^{(4)}} & & \downarrow{\scriptstyle P} \\
\pi_i(S^{2m-1}) & \xrightarrow{=} & \pi_i(S^{2m-1}) & \xrightarrow{E^2} & \pi_{i+2}(S^{2m+1})
\end{array} \tag{3.3.2}$$

we have the following two lemmas which generalize Theorems 5.3, 5.4 of [38].

Lemma 3.4 *Assume that an element x of $\pi_{i+2}(Q_2^{2m+1})$ is M-represented by $E^4(\gamma)$ with $h'_{m*}(\gamma) = 0$ then there exists an element $\xi \in \pi_i(S^{2m-1})$ such that*

$$p \cdot \xi = P(y) \quad and \quad E^2\xi = P(x)$$

for an element $y \in \pi_i(Q_2^{2m-1})$ M-represented by $\alpha(2pm-2) \circ E^2\gamma$.

Proof. By the notations of (3.2.3), $x = \Omega_0^2 g_{m+1*}(E^2\gamma)$. By the assumption there exists a coextension $\overline{\gamma} \in \pi_i(K(m,2))$ of x such that $\pi'_*(\overline{\gamma}) = E^2\gamma$ and $\overline{\gamma}$ is a suspension image. Proposition 3.2 implies

$$p(\overline{\gamma}) = i'_*(\alpha(2pm-2) \circ E^2\gamma) .$$

Put $\overline{x} = \overline{g}_{m*}(\overline{\gamma})$ and $y = g_{m*}(\alpha(2pm-2) \circ E^2\gamma)$ for the maps of (3.2.3). The commutativity of (3.2.3) implies

$$p(\overline{x}) = i_*(y) \quad and \quad j_*(\overline{x}) = x \, . \tag{3.3.3}$$

Then the lemma follows from the commutativity of (3.3.2). □

Lemma 3.5 *Assume that the Hopf invariant $H(\xi)$ of an element $\xi \in \pi_{i+5}(S^{2m+3})$ is M-represented by $E^4(\gamma)$ with $h'_{m*}(\gamma) = 0$, then there exists an element $\eta \in \pi_{i+3}(S^{2m+1})$ such that*

$$p \cdot \xi = E^2\eta$$

and $H(\eta)$ is M-represented by $\alpha(2pm-2) \circ E^2(\gamma)$.

Proof. Put $x = H(\xi) \in \pi_{i+2}(Q_2^{2m+1})$, then (3.3.3) holds for $y = g_{m*}(\alpha(2pm-2) \circ E^2\gamma)$ and $\overline{x} \in \pi_i(Q_4^{2m-1})$. Since

$$j_*(\overline{x} - H^{(4)}(\xi)) = j_*(\overline{x}) - H(\xi) = x - x = 0\,,$$

there exists $z \in \pi_i(Q_2^{2m-1})$ satisfying $\overline{x} = H^{(4)}(\xi) + i_*(z)$. By Lemma 2.2 $pz = 0$. Then $i_*(y) = p(\overline{x}) = H^{(4)}(p\xi)$ and $P(y) = P^{(4)}(i_*(y)) = P^{(4)}H(4)(p\xi) = 0$. By the exactness of EHP-sequence, there exists $\eta' \in \pi_{i+3}(S^{2m+1})$ such that $H(\eta') = y$. Then

$$H^{(4)}(p\xi - E^2\eta') = i_*(y) - i_*H(\eta') = i_*(y) - i_*(y) = 0$$

and $p\xi - E^2\eta' = E^4\eta_0$ for some $\eta_0 \in \pi_{i+1}(S^{2m-1})$. By putting $\eta = \eta' + E^2\eta_0$ the lemma is established. □

Next we consider the homomorphism

$$\Delta : {}_p\pi_{i+2}(S^{2pm+1}) \longrightarrow {}_p\pi_i(S^{2pm-1})\,.$$

This homomorphism is induced by the map ∂_p in the following fibre sequence of (2.1.9).

$$\Omega^4 S^{2pm+1} \xrightarrow{\partial_p} \Omega^2 S^{2pm-1} \longrightarrow Q_2^{2m-1} \longrightarrow \Omega^3(S^{2pm+1}) \xrightarrow{\partial_p} \Omega(S^{2pm-1})$$

By Lemma 2.1, the restriciton of the first ∂_p on S^{2pm-3} is a map of degree p. Then by taking path spaces we have a map $\overline{\partial}_p : Q_4^{2pm+1} \to Q_2^{2pm-1}$ which induces a homomorphism $\overline{\Delta} : \pi_{i-1}(Q_4^{2pm-3}) \to \pi_{i-1}(Q_2^{2pm-3})$ such that the following diagram commutes.

(3.3.4)
$$\begin{array}{ccccccc}
\pi_i(S^{2pm-3}) & \xrightarrow{E^4} & \pi_{i+4}(S^{2pm+1}) & \xrightarrow{H^{(4)}} & \pi_{i-1}(Q_4^{2pm-3}) & \xrightarrow{P^{(4)}} & \pi_i(S^{2pm-3}) \\
\downarrow{\cdot p} & & \downarrow{\Delta} & & \downarrow{\overline{\Delta}} & & \downarrow{\cdot p} \\
\pi_i(S^{2pm-3}) & \xrightarrow{E^2} & \pi_{i+2}(S^{2pm+1}) & \xrightarrow{H} & \pi_{i-1}(Q_2^{2pm-3}) & \xrightarrow{P} & \pi_i(S^{2pm-3})
\end{array}$$

Since the maps $\overline{g}_{pm-1} : K(pm-1,2) \to Q_4^{2pm-3}$ and $g_{pm-1} : Y^{2p(pm-)-2} \to Q_2^{2pm-2}$ are mod p equivalence up to degree $4p(pm-1)-5$, there exists a map D' such that the diagram

(3.3.5)
$$\begin{array}{ccc}
K(pm-1,2) & \xrightarrow{D'} & Y^{2p(pm-1)-2} \\
\downarrow{\overline{g}_{pm-1}} & & \downarrow{g_{pm-1}} \\
Q_4^{2pm-3} & \xrightarrow{\overline{\partial}_p} & Q_2^{2pm-3}
\end{array}$$

is homotopy commutative.

Since the restricition of D' on the subcomplex $Y^{2p(pm-1)-2}$ is a mapping of degree p, it is null homotopic. So we can take D' such as

$$D' = D \circ \pi' : K(pm-1,2) \longrightarrow Y^{2p(pm-1)-2+q} \longrightarrow Y^{2p(pm-1)-2}\,.$$

Proposition 3.3 *The above map $D : Y^{2p(pm-1)-2+q} \longrightarrow Y^{2p(pm-1)-2}$ represents*

$$\alpha(2p(pm-1)-2)$$

up to non-zero coefficient.

This lemma is Lemma 9.2 of [38]. One may prove this by use of Theorem 3.1(1).

Lemma 3.6 *If Hopf invariant $H(\xi) \in \pi_{i+1}(Q_2^{2pm-1})$ of $\xi \in \pi_{i+4}(S^{2pm+1})$ is M-represented by $E^2\gamma$ for an element $\gamma \in \pi_{i-1}(Y^{2p(pm-1)-4})$ satisfying*

$$h_{pm-1*}(\gamma) = 0\,.$$

Then $H(\Delta(\xi))$ is M-represented by

$$\alpha(2p(pm-1)-2) \circ \gamma$$

up to non-zero coefficient.

The following case is a special cse of the abovve lemma but it frequently ocuurs.

Lemma 3.7 *For $\xi \in \pi_{i+4}(S^{2pm+1})$, $\gamma \in \pi_s^S$ and $\delta \in \pi_t^S$, there hold the following relations up to non-zero coeffieients.*

(1) $\quad H(\xi) = Q^{pm}(\gamma)$ *implies* $H(\Delta(\xi)) = \overline{Q}^{pm-1}(\alpha\gamma) = HP(\overline{Q}^{pm}(\xi))$.

(2) $\quad H(\xi) = Q^{pm}(\gamma)$ *and* $\Delta(\xi) = P(\overline{Q}^{pm}(\gamma))$ *implies*

$$H(\xi \circ \delta(i+4)) = Q^{pm}(\gamma\delta) \quad \text{and} \quad \Delta(\xi \circ \delta(i)) = P(\overline{Q}^{pm}(\gamma\delta)).$$

Proof. (1) follows from Lemma 3.6 by considering the injection image of $S^{2pm} \subset Y^{2mp+1}$. (2) follows from the naturalities with respect to H, Δ and P. □

3.4 Short range unstable elements

We consider the second differential

$$d_2 : \mathrm{Ker} d_1 \longrightarrow \mathrm{Coker} d_1 .$$

More precisely we consider an element $\xi \in \pi_i(S^{2m+1})$ satisfying the following condition.

$$E^2(\xi) \neq 0, \quad E^4(\xi) = 0 \quad \textit{and} \quad \xi \notin \mathrm{Im} E^2 . \tag{3.4.1}$$

This is equivalent to

$$H(\xi) \neq 0 \quad \textit{and} \quad E^2(\xi) = P(x) \neq 0 \quad \textit{for some} \quad x \in \pi_{i+2}(Q_2^{2m+3}) . \tag{3.4.2}$$

Such a set of elements $\{\xi, E^2\xi\}$ is called as two stage unstable elements or *secondary unstable elements.*

The following diagram is commutative.

$$\begin{array}{ccccc} \pi_i(S^{2m+1}) & \xrightarrow{E^2} & \pi_{i+2}(S^{2m+3}) & \xleftarrow{P} & \pi_{i+2}(Q_2^{2m+3}) \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H^{(4)}} & & \downarrow{\scriptstyle =} \\ \pi_{i-3}(Q_2^{2m-1}) & \xrightarrow{i_*} & \pi_{i-3}(Q_4^{2m-1}) & \xleftarrow{\partial} & \pi_{i+2}(Q_2^{2m+3}) \end{array} \tag{3.4.3}$$

Next, concerning the results (3.1.5) we have the following homotopy commutative diagram :

$$\begin{array}{ccccc} Y^{2pm-2} & \xrightarrow{i'} & K(m,2) & \xleftarrow{\widetilde{h}_{m+1}} & Y^{2pm+4p-5} \\ \downarrow{\scriptstyle g_m} & & \downarrow{\scriptstyle \bar{g}_m} & & \downarrow{\scriptstyle \Omega_0^3 g_{m+2}} \\ Q_2^{2m-1} & \xrightarrow{i} & Q_4^{2m-1} & \xleftarrow{d} & \Omega^3 Q_2^{2m+3} \end{array} \tag{3.4.4}$$

where $\widetilde{h}_{m+1}$ is coextension of $h_{m+1}^{(2)} : Y^{2pm+4p-6} \to Y^{2pm+2p-4}$.

Proposition 3.4 *Assume that an invariant $x \in \pi_{i+2}(Q_2^{2m+3})$ is M-represented by $E^6\gamma$ for $\gamma \in \pi_{i-4}(Y^{2m+4p-8})$ and $\eta_{m+1*}^{(3)}(\gamma) = 0$, then*

$$i_*\{\eta_m, \eta_{m+1}^{(2)}, E^3\gamma\}_1 = \partial(x) .$$

The complex $K(m,2) = Y^{2pm-2} \cup e^{2pm+2p-5} \cup e^{2pm++2p-4}$ contains a mapping cone

$$C^{2pm+2p-5}_{(m+1)\alpha} = S^{2pm-3} \cup e^{2pm+2p-5}$$

of $(m+1)\alpha_1(2pm-3)$ as subcomplex. Moreover the restriction of $\widetilde{h}_{m+1}$ on $S^{2pm+4p-6}$ is homotopic to a map

$$\widetilde{h} : S^{2pm+4p-6} \longrightarrow C^{2pm+2p-5}_{(m+1)\alpha}$$

which is a coextension of $(m+2)\alpha_1(2pm+2p-6)$. Then as a corollary of Proposition 3.4 we have

Lemma 3.8 *Assume* $m \not\equiv -1, -2 \pmod p$ *and that an invariant* $x \in \pi_{i+2}(Q_2^{2m+3})$ *is M-represented by* $E^6(i_*\gamma)$ *for* $\gamma \in \pi_{i-4}(S^{2m+4p-9})$ *and* $\alpha_1(2m+4p-9) \circ \gamma = 0$, *then there exists an element* $\xi \in \pi_i(S^{2m+1})$ *such that*

$$H(\xi) \in I\{\alpha_1(2pm-1), \alpha_1(2pm+2p-4), E^3\gamma\} \quad \text{and} \quad E^2(\xi) = P(x)$$

up to non-zero coefficient.

Similarly, concerning $K(m,2)/C^{2pm+2p-5}_{(m+1)\alpha}$ we have

Lemma 3.9 *Assume* $m \not\equiv 0, -1 \pmod p$ *and that an invariant* $x \in \pi_{i+2}(Q_2^{2m+3})$ *is M-represented by* $E^6\gamma$ *for* $\gamma \in \pi_{i-3}(Y^{2m+4p-8})$ *and* $\eta^{(3)}_{m+1} \circ \gamma = 0$, *then there exists an element* $\xi \in \pi_{i+1}(S^{2m+1})$ *such that*

$$H_p(\xi) = JH(\xi) \in \{\alpha_1(2pm+1), \alpha_1(2pm+2p-2), E^3\pi_*\gamma\}$$

up to non-zero coefficient.

Let β_1 be a generator of $\pi^S_{pq-2} \cong \mathbf{Z}/p$. β_1 is given by a long bracket

$$\beta_1 = \{\alpha_1, \alpha_1, \cdots, \alpha_1\}$$

consists of p number of α_1 and it is detected by the secondary operation of Adem relation

$$\wp^{p-1}\wp^1 = 0\,.$$

In unstable case $\beta_1(n) \in \pi_{n+pq-2}(S^n)$ of $E^\infty\beta_1(n) = \beta_1$ is defined for $n \geq 2p-1$, which is stable for $n \geq 2p+1$ and $\beta_1(2p-1)$ is of order p^2.

We quote from [38] the following two lemmas (Theorems 10.3, 10.8) on a little longer series of unstable elements.

Lemma 3.10 *Let $l \geq 1$ and $m = pl$. Then there exists an element $v(2m+1)$ of $\pi_{2pm+pq-2}(S^{2m+1})$ such that*

$$H(v(2m+1)) = I(Q_m(\beta_1)) \quad and \quad E^{2(p-2)}v(2m+1) = P(I(\alpha_1(2mp+pq-1))$$

up to non-zero coefficient, where $v(2m+2p-1) = E^{2p-2}v(2m+1)$.

Let $\overline{\beta} \in [Y^{n+pq-1}, S^n]$ and $\widetilde{\beta} \in \pi_{n+pq-1}(Y^{n+1})$ be an extension and a coextension of β_1 respectively for large n. Let

$$C_{\overline{\beta}} = S^n \cup e^{n+pq-1} \cup e^{n+pq} \quad and \quad C_{\widetilde{\beta}} = S^n \cup e^{n+1} \cup e^{n+pq}$$

be mapping cones of $\overline{\beta}$ and $\widetilde{\beta}$ respectively. Then β_1 is also characterized by the following property.

(3.4.5) $$\wp^p \neq 0 \quad on \quad C_{\overline{\beta}} \quad and \quad C_{\widetilde{\beta}}.$$

We have the other generator β_2 of $\pi^S_{(2p+1)q-2} \cong \mathbf{Z}/p$, which is detected by the secondary operation associated to Adem relation

$$\wp^p\wp^{p+1} = \wp^{2p+1} + \wp^{2p}\ wp^1.$$

Lemma 3.11 *Assume that $m = pn$ for an integer $n \not\equiv p-2 \pmod p$ and $n > 1$. Then there exists an element $v_1(2m+1) \in \pi_{2pm+(2p+1)q-2}(S^{2m+1})$ such that*

$$H(v_1(2m+1)) = \overline{Q}^m(\beta_2) \quad and \quad v_1(2m+2p+3) = P(Q^{m+p+1}(\beta_1))$$

up to non-zero coefficient, where $v(2m+2p+3) = E^{2p+2}v(2m+1)$.

3.5 Lemmas for α_1 times

For $m \geq 1$ consider a sphere-bundle over sphere

$$S^{2m+1} \xrightarrow{i} B_m(\alpha) \xrightarrow{p} S^{2m+2p-1}$$

having the characteristic class $\alpha_1(2m+1)$. We consider the boundary homomorphism ∂_α in the homotopy exact sequence associated with the sphere-bundle

$$\cdots \xrightarrow{i_*} \pi_{i+1}(B_m(\alpha)) \xrightarrow{p_*} \pi_{i+1}(S^{2m+2p-1}) \xrightarrow{\partial_\alpha} \pi_i(S^{2m+1}) \xrightarrow{i_*} \pi_i(B_m(\alpha)) \xrightarrow{p_*} \cdots,$$

then we have

Proposition 3.5 $\quad \partial_\alpha(E^2\gamma) = \alpha_1(2m+1) \circ E\gamma \quad for \quad \gamma \in \pi_{i-1}(S^{2m-1+q})$.

Oka[21] constructed maps

(3.5.1) $$f : S^2(B_m(\alpha)) \longrightarrow B_{m+1}(\alpha) \qquad for \quad m \geq 1$$

which induces isomorphisms of $H_i(\ )$ for $i < 4m+2p$. Let $QB_m(\alpha)$ be a homotopy fibre of the adjoint of f, then we have a fibering

(3.5.2) $$QB_m(\alpha) \longrightarrow B_m(\alpha) \xrightarrow{i} \Omega^2 B_{m+1}(\alpha),$$

and the following EHP-sequence for $B_m(\alpha)$:

(3.5.3) $$\cdots \xrightarrow{E^2} \pi_{i+3}(B_{m+1}(\alpha)) \xrightarrow{H} \pi_i(QB_m(\alpha)) \xrightarrow{P} \pi_i(B_m(\alpha)) \xrightarrow{E^2} \pi_{i+2}(B_{m+1}(\alpha)) \xrightarrow{H} \cdots$$

Furethermore we have the following commutative and exact diagram.

(3.5.4)

$$\begin{array}{ccccccc}
\pi_{i+2}(B_m(\alpha)) & \xrightarrow{p_*} & \pi_{i+2}(S^{2m+2p-1}) & \xrightarrow{\partial_\alpha} & \pi_{i+1}(S^{2m+1}) & \xrightarrow{i_*} & \pi_{i+1}(B_m(\alpha)) \\
\downarrow E^2 & & \downarrow E^2 & & \downarrow E^2 & & \downarrow E^2 \\
\pi_{i+4}(B_{m+1}(\alpha)) & \xrightarrow{p_*} & \pi_{i+4}(S^{2m+2p+1}) & \xrightarrow{\partial_\alpha} & \pi_{i+3}(S^{2m+3}) & \xrightarrow{i_*} & \pi_{i+3}(B_{m+1}(\alpha)) \\
\downarrow H & & \downarrow H & & \downarrow H & & \downarrow H \\
\pi_{i+1}(QB_m(\alpha)) & \xrightarrow{p_*} & \pi_{i+1}(Q_2^{2m+2p-1}) & \xrightarrow{\partial_\alpha} & \pi_i(Q_2^{2m+1}) & \xrightarrow{i_*} & \pi_i(QB_m(\alpha)) \\
\downarrow P & & \downarrow P & & \downarrow P & & \downarrow P \\
\pi_{i+1}(B_m(\alpha)) & \xrightarrow{p_*} & \pi_{i+1}(S^{2m+2p-1}) & \xrightarrow{\partial_\alpha} & \pi_i(S^{2m+1}) & \xrightarrow{i_*} & \pi_i(B_m(\alpha))
\end{array}$$

Consider the following composition :

$$J \circ \partial_\alpha \circ I : \pi_{i+3}(S^{2p(m+p)-1}) \longrightarrow \pi_{i+1}(Q_2^{2m+2p-1}) \longrightarrow \pi_i(Q_2^{2m+1}) \longrightarrow \pi_{i+3}(S^{2p(m+1)+1}).$$

The following theorem is due to Oka[21].

Theorem 3.5 *Up to non-zero coefficients the relation*

$$J\partial_\alpha I(E^3\gamma) = \beta_1(2p(m+1)+1) \circ E^3\gamma.$$

holds for any $\gamma \in \pi_i(S^{2pm+p-1})$.

The boundary homomorphism $\partial_\alpha : \pi_{i+1}(Q_2^{2m+2p-1}) \to \pi_i(Q_2^{2m+1})$ is induced by a map $d_\alpha : \Omega Q_2^{2m+2p-1} \to Q_2^{2m+1}$. Approximate the map by Morre spaces, then we have a commutative diagram

(3.5.5)
$$\begin{array}{ccc} Y^{2p(m+p)-3} & \xrightarrow{\beta_{(1)}} & Y^{2p(m+1)-2} \\ \downarrow{\scriptstyle \Omega_0 g_{m+p}} & & \downarrow{\scriptstyle g_{m+1}} \\ \Omega Q_2^{2m+2p-1} & \xrightarrow{d_\alpha} & Q_2^{2m+1} \end{array}$$

for $m \geq 1$, where $\beta_{(1)}$ satisfies

(3.5.6)
$$\pi\beta_{(1)}i = \beta_1(2p(m+1)-2).$$

Applyng this to (3.5.4) we have the following lemmas.

Lemma 3.12 *Assume that Hopf invariant $H(\xi)$ of an element $\xi \in \pi_{i+4}(S^{2m+2p+1})$ is M-represented by $E\gamma$ for $\gamma \in \pi_i(Y^{2p(m+p)-3})$. If $m \geq 1$ then the Hopf invariant $H(\partial_\alpha(\xi))$ is M-represented by $\beta_{(1)} \circ \gamma$.*

If the invariants are stabel type the lemma is applied as follows.

(3.5.7)
$$H(\xi) = Q^{m+p}(\gamma) \quad \textit{implies} \quad H(\partial_\alpha(\xi)) = \overline{Q}^{m+1}(\beta_1\gamma).$$

Lemma 3.13 *Let $x \in \pi_{i+1}(Q_2^{2m+2p-1})$ is M-represented by $E\gamma$ and $y \in \pi_i(Q_2^{2m+1})$ is represented by $\beta_{(1)}\gamma$. If $m \geq 1$ then $P(y) = \partial_\alpha P(x)$.*

4 Unstable Alpha families

4.1 Alpha type invariants

First, recall the relation of Yamamoto[45] in the algebra $[\mathbf{Y},\mathbf{Y}]_* = \sum_k [\mathbf{Y},\mathbf{Y}]_k$ over $\mathbf{Z}/p$ of stable self homotopy of Moore space.

$$2\alpha\delta\alpha = \alpha^2\delta + \delta\alpha^2 \tag{4.1.1}$$

From this relation and $\delta\delta = 0$ we have

$$\alpha^s\delta\alpha^t = t\cdot\alpha^{s+t-1}\delta\alpha + (1-t)\alpha^{s+t}\delta = s\cdot\alpha\delta\alpha^{s+t-1} + (1-s)\delta\alpha^{s+t} \tag{4.1.2}$$

$$\alpha^s\delta\alpha^t\delta = \delta\alpha^t\delta\alpha^s = t\cdot\alpha^{s+t-1}\delta\alpha\delta$$

Next, we recall e-invariant of Adams[1]

$$e : \pi^S_{rq-1} \longrightarrow \mathbf{Q}/\mathbf{Z}\ .$$

In particular,

Theorem 4.1 *Let $r = ap^\nu$ for an integer $a \not\equiv 0 \pmod p$.*

(1) $e(\alpha_r) \equiv -1/p \pmod 1$.

(2) $e(\pi^S_{rq-1}) \subset \mathbf{Z}(1/p^{\nu+1})/\mathbf{Z}$.

(3) $e\{p,\alpha_{r-1},\alpha_1\} = b/p^{\nu+1}$ *for some* $b \not\equiv 0 \pmod p$.

From (1) and (2) of of the above theorem, it follows that if $r \not\equiv 0 \pmod p$ then

$$\alpha_r = \pi\alpha^r i = i^*\pi_*(\alpha^r)$$

is not divisible by p, and $\delta\alpha^r\delta \neq 0$ by (2.4.10).

By (4.1.2) $\delta\alpha^r\delta = r\cdot\alpha^{r-1}\delta\alpha\delta$. Then $\alpha^{r-1}\delta\alpha\delta \neq 0$ for $r \not\equiv 0 \pmod p$ and thus

$$\alpha^s\delta\alpha\delta \neq 0 \quad \textit{for all} \quad s \geq 0\,. \tag{4.1.3}$$

Consequently Yamamoto[45] obtained

Proposition 4.1 *The subalgebra of $[\mathbf{Y},\mathbf{Y}]_*$ generated by α and δ has an additive base*

$$\{1_{\mathbf{Y}}, \delta, \alpha^k, \alpha^k\delta, \alpha^{k-1}\delta\alpha, \alpha^{k-1}\delta\alpha\delta; k = 1,2,\cdots\}\ .$$

The results (4.1.3) implies the following

Proposition 4.2 *There exists an element* $\widetilde{\alpha}_r \in \pi^S_{rq-1}$ *uniquely modulo* $p \cdot \pi^S_{rq-1}$, *such that* $i\widetilde{\alpha}_r = (\alpha^{r-1}i)\alpha_1$, $\widetilde{\alpha}_r\pi = \alpha_1(\pi\alpha^{r-1})$, $i\widetilde{\alpha}_r\pi = \alpha^{r-1}\delta\alpha\delta = \delta\alpha\delta\alpha^{r-1} \neq 0$ *and* $\widetilde{\alpha}_r \in \{p, \alpha_{r-1}, \alpha_1\} = \{\alpha_1, \alpha_{r-1}, p\}$.

Proof. We know that $\alpha_{r-1}\alpha_1 = 0$ since α_1 and hence $\alpha_1\alpha_{r-1} = \alpha_{r-1}\alpha_1$ are J-images. Then $\pi(\alpha^{r-1}\delta\alpha i) = \alpha_{r-1}\alpha_1 = 0$. By the exactness of the sequence (2.4.9), there exists $\widetilde{\alpha}_r$ satisfying $i\widetilde{\alpha}_r = \alpha^{r-1}\delta\alpha i = (\alpha^{r-1}i)\alpha_1$. Then $i\widetilde{\alpha}_r\pi = \alpha^{r-1}\delta\alpha\delta = \delta\alpha\delta\alpha^{r-1} \neq 0$. Since $\alpha^{r-1}i$ is a coextension of α_{r-1}, $i\widetilde{\alpha}_r = (\alpha^{r-1}i)\alpha_1$ belongs to $i\{p, \alpha_{r-1}, \alpha_1\}$. Since $\mathrm{Ker} i_* = p\pi^S_{rq-1}$, $\widetilde{\alpha}_r$ belongs to $\{p, \alpha_{r-1}, \alpha_1\}$. Dually we have the others. □

(2) and (3) of Theorem 4.1 show that the order of $\widetilde{\alpha}_r$ is a multiple of $p^{\nu+1}$.

Next, consider unstable version. $\alpha(n) \in [Y^{n+q}, Y^n]$, $q = 2(p-1)$ is defined in (2.4.7) for $n \geq 4$ and $\delta(n) \in [Y^{n-1}, Y^n]$ for $n \geq 3$. They are connected by $E\alpha(n) = \alpha(n+1)$, $E^\infty\alpha(n) = \alpha$, $E\delta(n) = \delta(n+1)$ and $E^\infty\delta(n) = \delta$.

Unstable versions of products are also defined naturally. For example, $\delta\alpha(n) = \delta(n+q) \circ \alpha(n)$, $\alpha\delta(n) = \alpha(n-1) \circ \delta(n)$ and so on.

Lemma 4.1 *The relation* (4.1.1) *holds for dimension* $n \geq 6$, *that is,*

$$2\alpha\delta\alpha(n) = (\alpha^2\delta + \delta\alpha^2)(n)\,.$$

Thus the relations in (4.1.2) *hold for unstable case of dimension* $n \geq 6$.

The proof is seen in Proposition 4.2 of [38].

For $r \geq 0$, we define an invariant

$$A_r(2m-1) \in \pi_{2pm+rq-3}(Q_2^{2m-1})$$

to be M-represented by

$$\text{(4.1.4)} \qquad \alpha^r i(2pm-2) : S^{2pm+rq-3} \to Y_p^{2pm+rq-2} \to Y_p^{2pm-2} \; (\xrightarrow{g_m} Q_2^{2m-1})\,.$$

In the case $r = 0$, $A_0(2m-1) = I(\iota_{2pm-1}) \neq 0$ for the identity class $\iota_{2pm-1} \in \pi_{2pm-1}(S^{2pm-1})$. We denote this element by

$$w = I(\iota_{2pm-1}) = A_0(2m-1) \in \pi_{2pm-3}(Q_2^{2m-1}). \tag{4.1.5}$$

In the case $r > 0$, $J(A_r(2m-1)) = E^3(\pi\alpha^r i)(2pm+1)$, that is,

$$J(A_r(2m-1)) = \alpha_r(2pm+1) \neq 0 \quad in\ \pi_{2m+rq}(S^{2pm+1}). \tag{4.1.6}$$

Proposition 4.3 *The invariants $A_r(2m-1)$ are non-trivial for $r \geq 0$.*

We define also another invariant

$$a_r(2m-1) \in \pi_{2pm+rq-4}(Q_2^{2m-1}) \quad for \quad r > 0$$

to be M-represented by

$$\alpha^{r-1} i\alpha_1(2pm-2) : S^{2pm+rq-4} \to S^{2pm+(r-1)q-3} \to Y_p^{2pm-2}\ (\xrightarrow{g_m} Q_2^{2m-1}). \tag{4.1.7}$$

4.2 Simple unstable alpha families

By Proposition 3.1, d_1-image of $A_{r-1}(2m+1)$ is M-represented by

$$\eta_m \circ (\alpha^{r-1} i) = ((m+1)\delta\alpha - m\alpha\delta)(\alpha^{r-1} i) = (m+1)\delta\alpha^r - m\alpha\delta\alpha^{r-1} i \quad on\ Y^{2pm-2}.$$

Since $\delta\alpha^r i = (r\alpha^{r-1}\delta\alpha + (1-r)\alpha^r\delta)i = r\alpha^{r-1}\delta\alpha i$ and $\alpha\delta\alpha^{r-1} i = (r-1)\alpha^{r-1}\delta\alpha i$,

$$\eta_m \circ (\alpha^{r-1}\ i\alpha_1) = (m+r)\alpha^{r-1}\delta\alpha i = (m+r)\alpha^{r-1}\alpha_1 \quad on\ Y^{2pm-2}.$$

Thus we have

Proposition 4.4 $d_1(A_{r-1}(2m+1)) = HP(A_{r-1}(2m+1)) = (m+r)a_r(2m-1)$.

We denote P-image of the invariant $A_{r-m-1}(2m+1)$ by

$$\alpha_r^*(2m+1) = P(A_{r-m-1}(2m+1)) \in \pi_{2m+rq-1}(S^{2m+1}) \quad for \quad 1 \leq m < r. \tag{4.2.1}$$

In meta-stable range, $a_r(2m-1) \neq 0$ since $J(a_r(2m-1)) = i\tilde{\alpha}_r(2pm-1)$ is not divisible by p. Then $\alpha_r^*(2m+1)$ is a simple unstable element when $m + r \not\equiv 0 \pmod p$.

In order to estimate the non-triviality of $a_r(2m-1)$, we prepare the following

Lemma 4.2 *Assume that elements $\xi \in \pi_i(S^{2m-1})$ and $\eta \in \pi_j(S^i)$ satisfy*

$$p\xi = 0, \qquad p\eta = 0 \quad and \quad E^2(\xi \circ \eta) = 0\,,$$

then $\{p\iota_{2m+1}, E^2\xi, E^2\eta\}$ and $\{E^2\xi, E^2\eta, p\iota_{j+2}\}$ are E^2-images.

Proof. Let $C_\eta = S^i \cup e^{j+1}$ be a mapping cone of η and $\overline{\xi} : S^2C_\eta \to S^{2m+1}$ be an extension of $E^2\xi : S^{i+2} \to S^{2m+1}$.

Let $\mu^p \simeq i^2 \circ r_m : \Omega^2 S^{2m+1} \to \Omega^2 S^{2m+1}$ be the deformation of (2.1.12).

For the adjoint $\Omega_0^2\overline{\xi}$ of $\overline{\xi}$ we have

$$\mu^p \circ \Omega_0^2\overline{\xi} \simeq i^2 \circ r_m \circ \Omega_0^2\overline{\xi} : C_\eta \longrightarrow \Omega^2 S^{2m+1} \longrightarrow \Omega^2 S^{2m+1}\,.$$

Let $g = r_m \circ \Omega_0^2\xi : C_\eta \to S^{2m-1}$ then the adjoint of $i^2 \circ g$ is $E^2 g$.

The map $\mu^p \circ \Omega_0^2\overline{\xi}$ is the adjoint of

$$S^2C_\eta \xrightarrow{\overline{\xi}} S^{2m+1} \overset{i^2}{\subset} S^2\Omega^2 S^{2m+1} \xrightarrow{S^2\mu^p} S^2\Omega^2 S^{2m+1} \xrightarrow{e} S^{2m+1}\,,$$

where e is the evaluation and $f_p = e \circ S^2\mu^p \circ i^2 : S^{2m+1} \to S^{2m+1}$ is a map of degree p. $E^2 g \simeq f_p \circ \overline{\xi}$ and $f_p \circ \overline{\xi} : S^2C_\eta \to S^{2m+1} \to S^{2m+1}$ represents $-\pi^*\{p\iota_{2m+1}, E^2\xi, E^2\eta\}$ for $\pi^* : \pi_{j+3}(S^{2m+1}) \to [Y^{j+3}, S^{2m+1}]$. Thus $E^2\{g\} \in -\{p\iota_{2m+1}, E^2\xi, E^2\eta\}_2$. The indeterminacy of the bracket is $p \cdot \pi_{j+3}(S^{2m+1}) + E^3\pi_{i+1}(S^j)$. Since $p(\pi_{j+3}(S^{2m+1})) \subset E^2(\pi_{j+1}(S^{2m-1}))$ by Lemma 2.2, the above bracket is an E^2-image.

The second bracket $\{E^2\xi, E^2\eta, p\iota_{j+2}\}$ is represented by the composition $\overline{\xi} \circ \widetilde{p\iota}$: $S^{j+3} \to S^2C_\eta \to S^{2m+1}$, where $\widetilde{p\iota}$ is a coextension of $p\iota_{j+2}$. The $H(\{E^2\xi, E^2\eta, p\iota_{j+2}\})$ is a p-times and zero by Theorem 2.3. □

Proposition 4.5 *The invariants $a_r(2m-1)$ are non-trivial for $r > 0$.*

Proof. $J(a_r(2m-1)) = E^3((\pi\alpha^{r-1}i\alpha_1)(2pm-2)) = \alpha_{r-1}\alpha_1(2pm+1)$.

If $\alpha_{r-1}\alpha_1(2pm+1) \neq 0$ then $a_r(2m-1) \neq 0$.

So, we may assume that $\alpha_{r-1}\alpha_1(2pm+1) = 0$. Since the homomorphism J in (2.1.9) is induced by a map $j : Q_2^{2m-1} \to \Omega^3 S^{2pm+1}$ and $j \circ g_m$ is homotopic to $i^3 \circ \pi$: $Y^{2pm-2} \to S^{2pm-2} \to \Omega^3 S^{2pm+1}$, we see that $J(a_r(2m-1)) = 0$.

Then under the homomorphism I of (2.1.9), $a_r(2m-1)$ is the image of an element $\widetilde{\alpha}$ in $\{p, \alpha_{r-1}, \alpha_1\}(2pm+1)$. By Lemma 4.2, $\widetilde{\alpha}$ is an E^2-image of $\widetilde{\alpha}_r(2pm-1)$ satisfying $E^\infty(\widetilde{\alpha}_r(2pm-1)) = \widetilde{\alpha}_r$.

If $a_r(2m-1) = I(\widetilde{\alpha}) = 0$ then $E^2\widetilde{\alpha} = E^2\Delta(\xi) = p\xi$ for some ξ. But this contradicts to non-divisibility of $\widetilde{\alpha}_r$. Thus $a_r(2m-1) \neq 0$. □

The results of Theorem 2.5 is extended as follows.

Theorem 4.2 *Assume that* $r \not\equiv 0 \pmod p$.

(1) *For* $m \geq 1$, $\alpha_r(2m+1)$ *generates direct summand of* $\pi_{2m+rq}(S^{2m+1})$ *isomorphic to* $\mathbf{Z}/p$. $E^2\alpha_r(2m+1) = \alpha_r(2m+3)$ *and* $H_p(\alpha_r(3)) = \alpha_{r-1}(2p+1)$ *if* $r > 1$.

(2) *For* $1 \leq m < r$, $\alpha_r^*(2m+1)$ *generates a direct summand of* $\pi_{2m+rq-1}(S^{2m+1})$ *isomorphic to* $\mathbf{Z}/p$. $H(\alpha_r^*(2m+1)) = r \cdot a_{r-m}(2m-1)$.

Proof. (1) follows from that α_r of $r \not\equiv 0$ is no divisible by p. (2) follows from Propositions 4.4 and 4.5. □

The computing diagram for the above result is presented as follows.

$$
\begin{array}{llllllll}
n = & 3 & 5 & 7 & \cdots & 2r-5 & 2r-3 & 2r-1 \quad \textit{stable} \\
 & a_{r-1} & a_{r-2} & a_{r-3} & & a_3 & a_2 & a_1 \\
 & \nwarrow_H & \nwarrow_H & \nwarrow_H & & \nwarrow_H & \nwarrow_H & \nwarrow_H \\
k=rq-2 & \alpha_r^*(3) & \alpha_r^*(5) & \alpha_r^*(7) & \cdots & \alpha_r^*(2r-5) & \alpha_r^*(2r-3) & \alpha_r^*(2r-1) \\
 & \quad\nwarrow_P & \quad\nwarrow_P & & & \nwarrow_P & \nwarrow_P & \nwarrow_P \\
 & A_{r-1} & A_{r-2} & A_{r-3} & & A_2 & A_1 & A_0 \\
 & \nwarrow_H & & & & & & \\
k=rq-1 & \alpha_r(3) = & \alpha_r(5) = & \alpha_r(7) = & \cdots = & \alpha_r(2r-5) = & \alpha_r(2r-3) = & \alpha_r(2r-1) \overset{E^\infty}{\to} \alpha_r
\end{array}
$$

4.3 Unstable alpha families

Here we consider $(rq-1)$-stem and $(rq-2)$-stem groups of the case $r \equiv 0 \mod p$. Proposition 4.5 of this case is

$$HP(A_{r-m-1}(2m+1)) = H(\alpha_r^*(2m+1)) = 0\,. \tag{4.3.1}$$

It follows from Lemmas 3.5, 3.6

Lemma 4.3 *Let* $r \equiv 0 \pmod p$ *and* $m \geq 1$.

(1) *There exists an element* $\alpha^{*\prime}_r(2m-1) \in \pi_{2m-3+rq}(S^{2m-1})$ *such that*

$$E^2\alpha^{*\prime}_r(2m-1) = \alpha^*_r(2m+1) \quad \textit{and} \quad p \cdot \alpha^{*\prime}_r(2m-1) = \alpha^*_r(2m-1)\,.$$

(2) *If* $H(\xi) = A_{r-m-1}(2m+1)$ *for an element* ξ *of* $\pi_{2m+2+rq}(S^{2m+3})$ *then there exists an element* ξ' *of* $\pi_{2m+rq}(S^{2m+1})$ *such that*

$$H(\xi') = A_{r-m}(2m-1) \quad \textit{and} \quad E^2\xi' = p\xi\,.$$

Let $r = ap^{\nu(r)}$ for $a \not\equiv 0 \pmod p$. Then α_r is divisible by $p^{\nu(r)}$. Denote by $\alpha_r^{(s)}$ an element satisfying $p^s\alpha_r^{(s)} = \alpha_r$ for $1 \leq s \leq \nu(r)$.

Gray[6] gave unstable elements $\alpha_r^{(s)}(2m+2s+1)$ which converge to $\alpha_r^{(s)}$.

We consider an element $\alpha_r^{*(s)}(2m-1)$ of $\pi_{2m-3+rq}(S^{2m-1})$, if it exists, to be satisfy

$$(4.3.2) \quad E^{2s}\alpha_r^{*(s)}(2m-1) = \alpha^*_r(2m+2s-1) \quad \textit{and} \quad p^s\alpha_r^{*(s)}(2m-1) = \alpha^*_r(2m-1)\,.$$

Now, we quote the following results of Gray[7].

Lemma 4.4 *Let* $r = ap^\nu$ *for* $a \not\equiv 0 \pmod p$, $\nu = \nu(r)$, *then there exists an element* $\alpha_r^{*(\nu)}(2r-2\nu-1) \in \pi_{2pr-2\nu-3}(S^{2r-2\nu-1})$ *such that*

$$E^{2\nu}\alpha_r^{*(\nu)}(2r-2\nu-1) = \omega = PI(\iota_{2pr-1}) \in \pi_{2pr-3}(S^{2r-1})\,.$$

Moreover, $H(\alpha_r^{*(\nu)}(2r-2\nu-1)) = Q^{2r-2\nu-1}(\alpha^J_{\nu+1})$ *for a generator* $\alpha^J_{\nu+1}$ *of* $\mathrm{Im}J \cap \pi^S_{\nu+1}$.

For convenience of discussion we propose

Assertion A *In Lemma 4.4 we can take* $\widetilde{\alpha}_{\nu+1}$ *instead of* $\alpha^J_{\nu+1}$ *up to non-zero coefficient.*

For $r < p(p-1)$ we see that ${}_p\pi^S_{rq-1}$ has a single generator, hence Assertion A holds. For the case $r = p(p-1)$, ${}_p\pi^S_{rq-1}$ has another generator $\alpha_1\beta_1^{p-1}$. Since $Q^{2r-2\nu-1}(\alpha_1\beta_1^{p-1}) = HP(Q^{2r-2\nu})$, we can replace $\alpha^J_{\nu+1}$ in Lemma 4.4 by $\widetilde{\alpha}_{\nu+1}$. The next not alpha type generator of ${}_p\pi^S_{rq-1}$ is $\alpha_1\beta_1^{p-2}\beta_2$. Thus Assertion A holds for $r < p^2+1$, and so on.

We do not know any counter example for Assertion A.

The following theorem show the relations betwen Grays element in [7] and our invariants in EHP-sequence. Rcall (4.2.1) : $\alpha_r^*(2m+1) = P(A_{r-(m+1)}(2m+1))$.

Theorem 4.3 *Let* $r = ap^\nu$ *for* $a \not\equiv 0 \pmod p$ *and* $\nu = \nu(r) > 0$. *There are elements*

$$\alpha_r^{*(s)}(2m+1) \in \pi_{2m-1+rq}(S^{2m+1}) \quad \textit{for} \quad s \leq \nu \quad \textit{and} \quad s < m \leq r-s$$

and $\quad \alpha_r^{(s)}(2m+1) \in \pi_{2m+rq}(S^{2m+1}) \quad$ *for* $\quad s \leq \nu \quad$ *and* $\quad s < m$

satisfying the following properties up to non-zero coefficients.

(0) $\alpha_r^{*(0)}(2m+1) = \alpha_r^*(2m+1)$ *and* $\alpha_r^{(0)}(2m+1) = \alpha_r(2m+1)$.

(1) $p \cdot \alpha_r^{*(s)}(2m+1) = \alpha_r^{*(s-1)}(2m+1)$ *and* $E^2\alpha_r^{*(s)}(2m+1) = \alpha_r^{*(s-1)}(2m+3)$
for $1 \leq s \leq \nu$ *and* $s < m \leq r-s$.

(2) $p \cdot \alpha_r^{(s)}(2m+1) = \alpha_r^{(s-1)}(2m+1)$ *and* $E^2\alpha_r^{(s)}(2m+1) = \alpha_r^{(s)}(2m+3)$
for $1 \leq s \leq \nu$ *and* $s < m$.

(3) $H(\alpha_r^{(m)}(2m+1)) = A_{r-m}(2m-1)$ *for* $1 \leq m \leq \nu+1$,
$P(A_{r-m-1}(2m+1)) = E^{2\nu}\alpha_r^{*(\nu)}(2m+1)$ *for* $\nu+1 \leq m < r$.

(4) *The orders of* $\alpha_r^{*(s)}(2m+1)$ *and* $\alpha_r^{(s)}(2m+1)$ *are both* p^{s+1}.

(5) *If Assertion A holds for r, in particular if* $r \leq p^{p^2}$, *then*
$H(\alpha_r^{*(s)}(2m+1)) = a_{r-m}(2m-1)$ *for* $s = \mathrm{Min}(\nu, m+1)$ *and* $1 \leq m < r-\nu$.

Proof. If the element $\xi \in \pi_{2m+2+rq}(S^{2m+3})$ of Lemma 4.3(2) exists, then we get successively $\xi, \xi', \cdots, \xi^{(m-1)}$ and the last element $\xi^{(m-1)} \in \pi_{2+rq}(S^3)$ satisfies $E^{2m-2}\xi^{(m-1)} = p^{m-1}\xi$ and $H(\xi^{(m-1)}) = A_{r-1}(1) = H(\alpha_r(3))$. Thus $\xi^{(m-1)}$ is of order p, and the order of ξ is p^{m+1}. It follows from Theorem 4.1 that $m \leq \nu$. This shows that if $m > \nu$ such an element ξ does not exist, and

(4.3.3) $$\alpha_r^*(2m+1) = P(A_{r-m-1}(2m+1)) \neq 0 \quad \textit{for} \quad m > \nu.$$

Next the above results implies that for larger values of m, $\alpha_r^{*(s)}(2m+1)$ is of order p^{s+1} if it exists. Lemma 4.4 and Lemma 4.3 show the existence of $\alpha_r^{*(\nu)}(2m+1)$ for $r - \nu \geq m > \nu$. If (4.3.3) holds for $m = \nu$ then there exists $\alpha_r^{*(\nu)}(2\nu+1)$ of order $p^{\nu+1}$ which contradicts to Theorem 2.3. Thus $P(A_{r-\nu-1}(2\nu+1)) = 0$. This means the existence of $\alpha_r^{(\nu)}(2\nu+3)$ with $H(\alpha_r^{(\nu)}(2\nu+3)) = A_{r-\nu-1}(2\nu+1)$. Then the existence of $\alpha_r^{(m-1)}(2m+1))$ for $1 \leq m \leq \nu+1$ follows.

Consequently (1) to (4) are established. (5) follows from Lemma 4.4. □

We define by A_i^{2m+1} a subgroup of ${}_p\pi_i(S^{2m+1})$ generated by $\alpha_r^{(s)}$ or $E^{2j}\alpha_r^{*(s)}$. Thus

$$
\begin{array}{lrcl}
(4.3.4) & A_{2m+rq}^{2m+1} & = & <\alpha_r^{(s)}(2m+1)> \quad \mathit{for} \quad s = \mathrm{Min}(\nu, m) \\
 & A_{2m+rq-1}^{2m+1} & = & <\alpha_r^{*(s)}(2m+1)> \ \mathit{for}\ s = \mathrm{Min}(\nu, r-m, m) \ \ \mathit{and} \ \ m < r-\nu \\
 & A_i^{2m+1} & = & 0 \quad \mathit{otherwise.}
\end{array}
$$

Also we define by AQ_i^{2m-1} a subgroup of $\pi_i(Q_2^{2m-1})$ generated by $A_{r-m}(2m-1)$ or $a_{r-m}(2m-1) \in \mathrm{Im}P$. Thus

$$
\begin{array}{lrcl}
(4.3.5) & AQ_{rq-1}^{2m-1} & = & <A_{r-m}(2m-1)> \quad \mathit{for} \quad 1 \leq m \leq r \\
 & AQ_{rq-2}^{2m-1} & = & <a_{r-m}(2m-1)> \quad \mathit{for} \quad 1 \leq m < r-\nu \\
 & AQ_i^{2m-1} & = & 0 \quad \mathit{othersise.}
\end{array}
$$

We call subsystem of computing diagram consists of A_i^{2m+1} and AQ_i^{2m-1} the system of *unstable alpha families.*

Proposition 4.6 *The homomorphisms E^2, H and P in EHP-sequence are closed on unstable alpha families up to a k-stem groups for $k < p^{p^2+1}q-2$.*

The computing diagram of unstable alpha family of the case $\nu(r) = 1$ is as follows.

$$
\begin{array}{lccccccccc}
n = & 3 & 5 & 7 & \cdots & 2r-5 & 2r-3 & 2r-1 & \mathit{stable} \\
 & a_{r-1} & a_{r-2} & a_{r-3} & & a_3 & a_2 & (a_1) & \\
 & \nwarrow H & \nwarrow H & \nwarrow H & & \nwarrow H & \nwarrow H & & \\
k=rq-2 & \alpha_r^*(3) \hookrightarrow & \alpha_r^{*\prime}(5) \to & \alpha_r^{*\prime}(7) \to & \cdots \to & \alpha_r^{*\prime}(2r-5) \to & \alpha_r^{*\prime}(2r-3) \to & \alpha_r^*(2r-1) & \\
 & & & \nwarrow P & & \nwarrow P & \nwarrow P & \nwarrow P & \\
 & A_{r-1} & A_{r-2} & A_{r-3} & & A_2 & A_1 & A_0 & \\
 & \nwarrow H & \nwarrow H & & & & & & \\
k=rq-1 & \alpha_r(3) \hookrightarrow & \alpha_r'(5) = & \alpha_r'(7) = & \cdots = & \alpha_r'(2r-5) = & \alpha_r'(2r-3) = & \alpha_r'(2r-1) & \overset{E^\infty}{\to} \alpha_r'
\end{array}
$$

In this case the invariant $a_1 = a_1(2r-3)$ is out of the unstable alpha family. If $\nu(r) = 2$, then a_1 and a_2 are out of the family.

4.4 Removability and Residue

We have seen that the unstable alpha families are close with respect to the EHP-sequence. However, the EHP-sequence on these systems is not necessarily exact.

Proposition 4.7 *The exactness of EHP-sequence on unstble alpha families breaks on the following two points:*

(1) *The invariants $a_r(2m+1)$ for $r-\nu(r)<m<r$ are not H-images of unstable alpha families. We must exclude them.*

(2) *The unstable elements $\alpha_r^*(2r-2s-1)$ for $s\equiv 0 \pmod p$ and $s<\nu(r)$ is a kernel of Δ and generates not alpha type invariant.*

We shall excluding unstable alpha families and make a computing diagram mod A. In that case above two sort of elements must be added as residue elements.

On the other hand, compatibility of the unstable alpha families with respect to IJΔ-sequence, especially the homomorphism $\Delta:\pi_{i+2}(S^{2pm+1})\to\pi_i(S^{2pm-1})$ have to be checked.

Consider the case $r\not\equiv 0 \pmod p$ of Theorem 4.2. Since $E^2: A^{2pm-1}_{2pm+rq-2}\to A^{2pm+1}_{2pm+rq}\cong \mathbf{Z}/p$ is isomorphic and $\Delta\circ E^2=p\cdot$, we have $\Delta(A^{2pm+1}_{2pm+rq})=0$. Then IJ$\Delta$-sequence produces two invariants $A_r(2m-1)$ and $a_r(2m-1)$.

For the generator $\alpha_r^*(2pm-1)$ and $\alpha_r^*(2pm+1)$ of the $(rq-2)$-stem groups satisfy $H(\alpha_r^*(2pm-1))=a^*_{r-m}(2pm-3)$ and $H(\alpha_r^*(2pm+1))=a^*_{r-m-1}(2pm-1)$. By the definition of a_k^* we have $a^*_{r-m}=\alpha\circ a^*_{r-m-1}$. Then it follows from Lemma 3.7 that $\Delta(\alpha_r^*(2pm+1))\equiv\alpha_r^*(2pm-1) \mod \mathrm{Ker}H=\mathbf{Im}E^2$. This shows that *Delta* cancells the generators of the $(kq-2)$-stem groups.

For the case $r\equiv 0 \pmod p$, the results are similar but rather easier by using higher order generators.

There is an exceptional case that $E^2: A^{2pm-1}_{2pm+\epsilon+rq-2}\to A^{2pm+1}_{2pm+\epsilon+rq}$ $(\epsilon=0,-1)$ is injective but $\mathrm{Coker}E^2\cong\mathbf{Z}/p$. In this case correspondent Δ are both epic, and produce one invariant for each.

5 Old Table of 3-primary Groups

5.1 3-primary Groups, stable to unstable

The results of 3-primary k-th homotopy groups for $k \leq 45$ was given in Toda[37],[38] and announced for $k \leq 55$ by Maruyama and Mimura in 1998.

We begin to quote from Oka[22] the following list of 3-primary stable homotopy groups ${}_3\pi_k^S$ with generators for $k < 62$ excluding the alpha type generators α_r, α'_{3s}, α''_{9t}. For the notations of generators, we use those given in [22].

(5.1.1)
List of ${}_3\pi_k^S =< generator >$, *relations up to sign*

(1) ${}_3\pi_{10}^S =< \beta_1 >$, $\beta_1 = \{\alpha_1, \alpha_1, \alpha_1\}$.
(2) ${}_3\pi_{13}^S =< \alpha_1\beta_1 >$.
(3) ${}_3\pi_{20}^S =< \beta_1^2 >$.
(4) ${}_3\pi_{23}^S =< \alpha_1\beta_1^2 >$.
(5) ${}_3\pi_{26}^S =< \beta_2 >$.
(6) ${}_3\pi_{29}^S =< \alpha_1\beta_2 >$.
(7) ${}_3\pi_{30}^S =< \beta_1^3 >$, $\beta_1^3 = \{\alpha_1, 3, \beta_2\}$.
(8) ${}_3\pi_{36}^S =< \beta_1\beta_2 >$.
(9) ${}_3\pi_{37}^S =< \epsilon' >$, $\epsilon' = \{\alpha_1, \alpha_1, \beta_1^3\}$.
(10) ${}_3\pi_{38}^S =< \epsilon_1 >$, $\epsilon_1 = \{\alpha_1, 3, \beta_1^3, \alpha_1\}$.
(11) ${}_3\pi_{39}^S =< \alpha_1\beta_1\beta_2 >$.
(12) ${}_3\pi_{40}^S =< \beta_1^4 >$, $\alpha_1\epsilon' = \beta_1^4$.
(13) ${}_3\pi_{42}^S =< \epsilon_2 >$, $\epsilon_2 = \{\alpha_1, 3, \epsilon_1\} = 2\{3, \alpha_1, \epsilon_1\}$.
(14) ${}_3\pi_{45}^S =< \varphi >$, $\varphi = \{\alpha_1, \alpha_1, \epsilon_1\}$, $p\varphi = \alpha_1\epsilon_2$.
(15) ${}_3\pi_{46}^S =< \beta_1^2\beta_2 >$.
(16) ${}_3\pi_{47}^S =< \beta_1\epsilon' >$.
(17) ${}_3\pi_{49}^S =< \alpha_1\beta_1^2\beta_2 >$.
(18) ${}_3\pi_{50}^S =< \beta_1^5 >$.
(19) ${}_3\pi_{52}^S =< \beta_2^2 >$, $\beta_2^2 = \{\alpha_1, \alpha_1, \varphi\}$.
(20) ${}_3\pi_{55}^S =< \alpha_1\beta_2^2 >$.

From now on we restrict ouself to the computation of 3-primary unstable groups mod A, and use the notation

$$\pi_i^n = {}_3\pi_i(S^n)/A_i^n, \qquad Q_i^n = {}_3\pi_i(Q_2^n)/AQ_i^n \quad and \quad \pi_k^S = {}_3\pi_k^S/A_k^S$$

for the sake of simplicity.

The computation of the 3-primary components of the k-stem groups π_{n+k}^n for odd n will be done usually by induction on k and n for the following sequence of double suspensisons E^2

$$\pi_{3+k}^3 \xrightarrow{E^2} \pi_{5+k}^5 \xrightarrow{E^2} \cdots \xrightarrow{E^2} \pi_{2m-1+k}^{2m-1} \xrightarrow{E^2} \pi_{2m+1+k}^{2m+1} \xrightarrow{E^2} \cdots \xrightarrow{S^\infty} \pi_k^s, \tag{5.1.2}$$

where we assume that the stable group π_k^S and the unstable groups π_{2m+1+j}^{2m+1} for $j < k$ are already fixed.

Each double suspension E^2 is verified by using EHP-sequence (2.1.1) of $p = 3$ which is rewritten as the following exact sequence :

$$\cdots \xrightarrow{H} Q_{2m-1+k}^{2m-1} \xrightarrow{P} \pi_{2m-1+k}^{2m-1} \xrightarrow{E^2} \pi_{2m+1+k}^{2m+1} \xrightarrow{H} Q_{2m-2+k}^{2m-1} \xrightarrow{P} \cdots . \tag{5.1.3}$$

The computations of the k-stem groups are done by use of the following

Computing Diagram mod A

$$\begin{array}{ccccccccccccc}
 & \pi_{2+k}^3 & \xrightarrow{E^2} & \pi_{4+k}^5 & \xrightarrow{E^2} & \pi_{6+k}^7 & \xrightarrow{E^2} & \cdots & \xrightarrow{E^2} & \pi_{2m-2+k}^{2m-1} & \xrightarrow{E^2} & \pi_{2m+k}^{2m+1} & \xrightarrow{E^2} \ \cdots \\
 & & \nwarrow P & & \nwarrow P & & & & & & \nwarrow P & & \\
Q_k^1 & & & Q_{2+k}^3 & & Q_{4+k}^5 & & \cdots & \cdots & & & Q_{2m-2+k}^{2m-1} & \cdots \\
 & \nwarrow H & & & \nwarrow H & & \nwarrow H & & & & & \nwarrow H & \\
 & \pi_{3+k}^3 & \xrightarrow{E^2} & \pi_{5+k}^5 & \xrightarrow{E^2} & \pi_{7+k}^7 & \xrightarrow{E^2} & \cdots & \xrightarrow{E^2} & \pi_{2m-1+k}^{2m-1} & \xrightarrow{E^2} & \pi_{2m+1+k}^{2m+1} & \xrightarrow{E^2} \ \cdots \\
 & & \nwarrow P & & \nwarrow P & & & & & & \nwarrow P & & \\
Q_{1+k}^1 & & & Q_{3+k}^3 & & Q_{5+k}^5 & & \cdots & \cdots & & & Q_{2m-1+k}^{2m-1} & \cdots \\
 & \nwarrow H & & & \nwarrow H & & \nwarrow H & & & & & \nwarrow H & \\
 & \pi_{4+k}^3 & \xrightarrow{E^2} & \pi_{6+k}^5 & \xrightarrow{E^2} & \pi_{8+k}^7 & \xrightarrow{E^2} & \cdots & \xrightarrow{E^2} & \pi_{2m+k}^{2m-1} & \xrightarrow{E^2} & \pi_{2m+2+k}^{2m+1} & \xrightarrow{E^2} \ \cdots
\end{array}$$

Here the group Q_{2m-1+k}^{2m-1} has to be fiexed in advance by the IJΔ-sequence of Theorem 2.2. For $k < 107$, Q_{2m-1+k}^{2m-1} contains extra elements

$$w = IP(\iota) \in Q_{18m-5}^{2m-1}, \qquad a_1 \in Q_{18m-8}^{6m-3}, \qquad a_2 \in Q_{54m-8}^{18m-5} \tag{5.1.4}$$

of section 4.5. Let $\overline{Q}^{2m-1}_{2m-1+k}$ be the quotient over the subgroup spanned by (5.1.4). Then the invariants mod A is obtained by the following exact sequence.

$$(5.1.5)\quad \cdots \xrightarrow{J} \pi^{6m+1}_{2m+1+k} \xrightarrow{\Delta} \pi^{6m-1}_{2m-1+k} \xrightarrow{I} \overline{Q}^{2m-1}_{2m-1+k} \xrightarrow{J} \pi^{6m+1}_{2m+k} \xrightarrow{\Delta} \pi^{6m-1}_{2m-2+k} \xrightarrow{I} \cdots .$$

In the case $m = 1$, $H : \pi^3_{3+k} \to Q^1_k$ is bijective for $k \neq 0$. Then the group π^3_{3+k} will be computed directly by the following sequence

$$(5.1.6)\qquad \cdots \xrightarrow{H_p} \pi^7_{4+k} \xrightarrow{\Delta} \pi^5_{2+k} \xrightarrow{G} \pi^3_{3+k} \xrightarrow{H_p} \pi^7_{3+k} \xrightarrow{\Delta} \pi^5_{1+k} \xrightarrow{G} \cdots ,$$

derived from Theorem 2.4. Remark that the sequence (5.1.6) is exact except the case $k = 11$.

In the computating diagram, many numbers of generators in Q^{2m-1}_i are given as stable type invariants

$$Q^m(\xi) \in Q^{2m-1}_{6m-1+k} \quad or \quad \overline{Q}^m(\xi) \in Q^{2m-1}_{6m+k} \quad for \quad \xi \in \pi^S_k$$

given in (2.4.1) or (2.4.2). We represent such generators by corresponding roman charactors. The following (5.1.7) is the list of stable type invariants correponding to the stable generators in (5.1.1).

(5.1.7) *Invariants* $Q^m(\xi)$ *and* $\overline{Q}^m(\xi)$ *for* $\xi \in \pi^S_k$, $k \leq 55$.

k	0	3	7	10	13	20	23	26	29	30	36	37	38
ξ	ι	α_1	α_2	β_1	$\alpha_1\beta_1$	β_1^2	$\alpha_1\beta_1^2$	β_2	$\alpha_1\beta_2$	β_1^3	$\beta_1\beta_2$	ϵ'	ϵ_1
$Q^m(\xi)$	i	a	a_2	b	ab	b^2	ab^2	b_2	ab_2	b^3	bb_2	e'	e_1
$\overline{Q}^m(\xi)$				B	AB	B^2	AB^2	B_2	AB_2	B^3	BB_2	E'	E_1

k	39	40	42	45	45	46	47	49	50	52	55
ξ	$\alpha_1\beta_1\beta_2$	β_1^4	ϵ_2	φ	$\alpha_1\epsilon_2$	$\beta_1^2\beta_2$	$\beta_1\epsilon'$	$\alpha_1\beta_1^2\beta_2$	β_1^5	β_2^2	$\alpha_1\beta_2^2$
$Q^m(\xi)$	abb_2	b^4	e_2	ph	$\times$	b^2b_2	be'	ab^2b_2	b^5	b_2^2	ab_2^2
$\overline{Q}^m(\xi)$	AAB_2	B^4	E_2	$\times$	AE_2	B^2B_2	BE'	AB^2B_2	B^5	B_2^2	AB_2^2

The first differentials

$$d_1 = H \circ P : \pi_{i+3}(Q_2^{2m+1}) \longrightarrow \pi_{i+3}(S^{2m+1}) \longrightarrow \pi_i(Q_2^{2m-1})$$

of these stable type invariants are verified by use of Lemmas 3.1, 3.2 and 3.3. The results depend on the values of m (mod 3) as is seen below.

Proposition 5.1 *The image $d_1(x) = HP(x) \in Q_{6m+k}^{2m-1}$, up to sign, of each invariant $x \in Q_{6m+3+k}^{2m+1}$ is given as follows.*

k	10	11	20	21	26	27	36	37	37	38	39	43
x	b	B	b^2	B^2	b_2	B_2	bb_2	BB_2	e'	E'	E_1	E_2
$m \equiv 0$	ab	0	ab^2	0	ab_2	b^3	abb_2	0	b^4	0	e_2	0
$m \equiv 1$	ab	AB	ab^2	AB^2	ab_2	AB_2	abb_2	ABB_2	b^4	B^4	0	AE_2
$m \equiv 2$	0	AB	0	AB^2	0	AB_2	0	ABB_2	0	B^4	e_2	AE_2

k	46	47	47	48	52	53
x	b^2b_2	B^2B_2	be'	BE'	b_2^2	B_2^2
$m \equiv 0$	ab^2b_2	0	b^5	0	ab_2^2	0
$m \equiv 1$	ab^2b_2	AB^2B_2	b^5	B^5	ab_2^2	AB_2^2
$m \equiv 2$	0	AB^2B_2	0	B^5	0	AB_2^2

$d_1(x) = 0$ *for the remainning invariants :*

(5.1.8) $\quad x = ab,\ ab^2,\ ab_2,\ b^3,\ e_1,\ abb_2,\ b^4,\ e_2,\ ab_2^2,\ b^5,\ ab_2^2,$

(5.1.9) $\quad x = AB,\ AB^2,\ AB_2,\ B^3,\ ABB_2,\ B^4,\ AB^2B_2,\ B^5,\ AB_2^2.$

Proof. If $\alpha_1\xi$ is not divisible by 3, then for $x = Q^{m+1}(\xi)$, $d_1(x) = (m+1)Q^m(\alpha_1\xi) \neq 0$ by Lemma 3.1. This is applied to invariants $x = b, b^2, b_2, bb_2, e', b^2b_2, be', b_2^2$, where we note that $\alpha_1\epsilon' = \beta_1^4$.

Similarly, it follows from Lemma 3.2 that for $X = B, B^2, B_2, BB_2, E', BE', B_2^2$ we have $d_1(X) = \pm AX$ if $m \equiv 1, 2$ (mod 3). But in the case $m \equiv 0$ (mod 3), $d_1(X)$ is not necessarily zero and may be a form $Q^m(\eta)$ for some η. $d_1(B) = 0$ since $\pi_{14}^S = 0$. Then by the naturality, $d_1(X) = 0$ for $X = B^2, BB_2, BE'$. Also for the case $m \equiv 0$ (mod 3), $d_1(E') = 0$ since $\pi_{41}^S = 0$ and $d_1(B_2^2) = 0$ since $\pi_{56}^S = 0$. The relation $d_1(B_2) = b^3$ follows from Lemma 3.3 and the relation in (7) of (5.1.1).

From the relations in (13) of (5.1.1) and Lemma 3.3, it follows $d_1(E_1) = Q^m(\eta)$ for

$$\eta = (m+1)\{\alpha_1, 3, \epsilon_1\} - m\{3, \alpha_1, \epsilon_1\} = ((m+1) - 2m)\epsilon_2 = (1-m)\epsilon_2.$$

Thus the results for E_1 are obtained.

For $m \equiv 0 \pmod 3$, we have similarly $d_1(E_2) = Q^m(\eta)$ for $\eta = \{\alpha_1, 3, \epsilon_2\}$. Asuume that $\eta \neq 0$, then $\eta = \pm\beta_1^2\beta_2$ by (15) of (5.1.1). By (13) of (5.5.1)

$$\alpha_1\eta = \{\alpha_1, \alpha_1, 3\}\epsilon_2 = -\alpha_2\epsilon_2 = \alpha_2\{3, \alpha_1, \epsilon_1\} = \alpha_3\epsilon_1 = 3\alpha'\epsilon_1 = 0$$

which contradicts to $\alpha_1\eta = \pm\alpha_1\beta_1^2\beta_2 \neq 0$. Thus $d_1(E_2) = Q^m(\eta) = 0$.

$d_1 = 0$ for the elements x in (5.1.8) and (5.1.9) is proved without difficulties. □

Next consider the short range unstable elements $\{H(\xi) = x \neq 0,\ E^2\xi = P(y) \neq 0\}$. Assume that stable elements γ and δ satisfy

(5.1.10) $$\delta \in \{\alpha_1, \alpha_1, \gamma\}$$

From Lemmas 3.8 and 3.9 we have the following propositions.

Proposition 5.2 *Let $m \equiv 0 \pmod 3$. For the elements in (5.1.10) let $x = Q^{m+2}(\gamma)$ and $y = Q^m(\delta)$ then there exists an element $\xi \in \pi_i(S^{2m+1})$ such that*

$$H(\xi) = y \quad and \quad E^2\xi = P(x).$$

Proposition 5.3 *Let $m \equiv 1 \pmod 3$. For the elements in (5.1.10) let $X = \overline{Q}^{m+2}(\gamma)$ and $Y = \overline{Q}^m(\delta)$. Assume that the relation $d_1(X) = HP(X) = 0$ is obtained as in Lemma 3.9, for example by a computation in stable range. Then there exists an element $\xi \in \pi_i(S^{2m+1})$ such that*

$$H(\xi) = Y \quad and \quad E^2\xi = P(X).$$

The elements γ, δ in (5.1.10) and the related invariants are listed as follows.

(5.1.11)

γ	α_1	$\alpha_1\beta_1$	$\alpha_1\beta_1^2$	$\alpha_1\beta_2$	β_1^3	ϵ_1	$\alpha_1\beta_1\beta_2$	β^4	φ
δ	β_1	β_1^2	β_1^3	$\beta_1\beta_2$	ϵ'	φ	$\beta_1^2\beta_2$	$\beta_1\epsilon'$	β_2^2
x	a	ab	ab^2	ab_2	b^3	e_1	abb_2	b^4	ph
y	b	b^2	b^3	bb_2	e'	ph	b_2^2	be'	b_2^2
X		AB	AB^2	AB_2	B^3		ABB_2	B^4	
Y		B^2	B^3	BB_2	E'		B^2B_2	BE'	

5.2 Computation diagram mod A of lower stems

In order to compute groups π^n_{n+k} succesively, we use computing diagrams mod A, omitting the symbols E^2, H and P, replacing Q^n_{n+k} by its generators. We also replace π^n_{n+k} by a collection of symbols which indicate direct summands such as

(5.2.1) $$\bullet \cong \mathbf{Z}/3, \quad \circ \cong \mathbf{Z}/9 \quad and \quad \triangleright \cong \mathbf{Z}/27.$$

The symbol $=$ at E^2 means bijective such as

(5.2.2) $$\bullet = \bullet, \qquad \circ = \circ \qquad and \qquad \triangleright = \triangleright.$$

The synbol $\to$ at E^2 means not bijective and non-trivial such as

(5.2.3) $$\bullet \to \circ, \quad \circ \to \triangleright, \quad \circ \to \bullet, \quad \triangleright \to \circ \quad \circ \to \circ \quad and \quad \triangleright \to \triangleright,$$

where the first two are injective, the next two are surjective and the last two are p times.

First observe the 10-stem groups ${}_3\pi_{n+10}(S^n)$:

$${}_3\pi_{13}(S^3) \xrightarrow{E^2} {}_3\pi_{15}(S^5) \xrightarrow{E^2} {}_3\pi_{17}(S^7).$$

This is represented as

$$\bullet \longrightarrow \circ \longrightarrow \bullet.$$

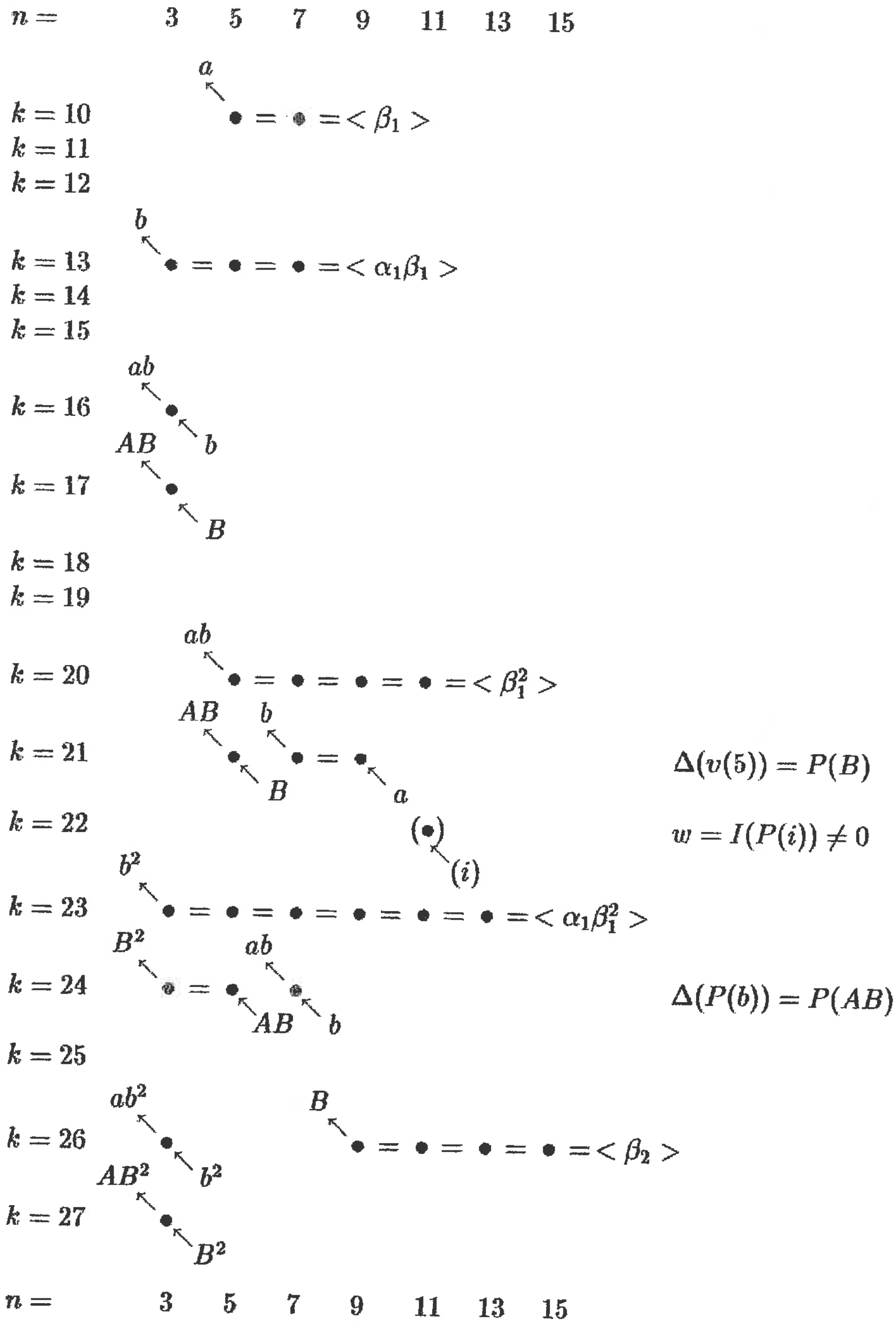
The equality $\Delta \circ E^2 = p\cdot$ of Lemma 2.1 implies that

$$Q^1_{13} = < b > \cong \mathbf{Z}/3 \quad and \quad Q^1_{14} = 0.$$

This is the only case we know that (5.1.5) cannot be applied.

Then computing diagram mod A up to 27-stem is presented as follows.

Theorem 5.1 *The k-stem groups mod A π^n_{n+k} for $k \leq 27$ and odd n are computed as follows.*



All invariants of Q_i^{2m-1} for $m \geq 2$ are in metastable range, and generators are obtained by Theorem 2.2 or Proposition 2.1 as is shown in the above diagram.

The generators of $Q_i^1 \cong \pi_{3+i}^3$ are verified by (5.1.6) from the the results of Δ.

Since $E^2 : \pi_{5+k}^5 \to \pi_{7+k}^7$ are isomorpisms of $\bullet$ for $k = 13, 20, 23$, then from $\Delta \circ E = p \cdot$ it follows $\Delta = 0 : \pi_{7+k}^7 \to \pi_{5+k}^5$. So, the generators of $\pi_{3+k}^3 \cong Q_k^1$ and $\pi_{4+k}^3 \cong Q_{1+k}^1$ are obtained by Theorem 2.4 as is shown in the diagram.

$E^2 = 0 : \pi_{5+k}^5 \to \pi_{7+k}^5$ for $k = 21, 24$. It follows from Theorem 4.3 that $\Delta : \pi_{7+k}^7 \to \pi_{5+k}^5$ are surjective hence isomorphic. Thus we have no more generators of π_{3+k}^3.

By the exactness of EHP-sequence, the only possibility of the case $k = 16$ is $HP(b) = ab$. By composing β_1, it follows from the naturality that $HP(b^2) = ab^2$ for the case $k = 26$. The cases $k = 17$ and $k = 27$ are similar.

For the stable generators $\alpha_1\beta_1$, β_1^2, $\alpha_1\beta_1^2$ and β_2, by the exactness of EHP-sequence, we see the only possibilities for the cases $k = 13, 20, 23, 26$. The cases $k = 21$ and $k = 24$ are also uniquely determined from $\pi_k^S = 0$.

In the above computation we need only the exactness of EHP-sequence and the informations on invariants. However, the computaions in the sequel have some ambiguity and we need several lemmas of section 3 in order to determine the exact sequences.

So, we shall divide our computing diagram into three parts, simple unstable elements, short range unstable elements and remaining part containing long unstable series which may converge to a stable class.

For example, the computation diagram in Proposition 5.1 is divided into the following three parts (1), (2) and (3).

(1) *Removable simple unstable elements.*

$(n,k) =$	$(3,16)$	$(3,17)$	$(3,26)$	$(3,27)$
	ab	AB	ab^2	AB^2
	$\bullet$ b	$\bullet$ B	$\bullet$ b^2	$\bullet$ B^2

Here the symbol $\nwarrow$ is omitted. Each collection consists of $\{x, P(x) \in \pi_{n+k}^n, HP(x) \neq 0\}$ for a pair $\{x, d_1(x)\}$ listed in Proposition 5.1, $m \equiv 1 \pmod 3$.

(2) *Removable short range unstable elements.*

$n=$	3	5	7	9	11	13	15
$k=21$		$\overset{AB}{\bullet}$ $\overset{b}{\underset{B}{\hookleftarrow}}$	$\bullet$ $=$	$\underset{a}{\bullet}$			
$k=24$	$\overset{B^2}{\bullet}$ $=$	$\bullet$ $\overset{ab}{\underset{AB}{\hookleftarrow}}$	$\underset{b}{\bullet}$				

(3) *Remaining parts.*

$n=$	3	5	7	9	11	13	15	
$k=10$		$a \nwarrow \bullet$ $=$	$\bullet$	$=< \beta_1 >$				
$k=13$	$b \nwarrow \bullet$ $=$	$\bullet$ $=$	$\bullet$	$=< \alpha_1\beta_1 >$				
$k=20$		$ab \nwarrow \bullet$ $=$	$\bullet$ $=$	$\bullet$ $=$	$\bullet$	$=< \beta_1^2 >$		
$k=23$	$b^2 \nwarrow \bullet$ $=$	$\bullet$ $=$	$\bullet$ $=$	$\bullet$ $=$	$\bullet$ $=$	$\bullet$	$=< \alpha_1\beta_1^2 >$	
$k=26$				$B \nwarrow \bullet$ $=$	$\bullet$ $=$	$\bullet$ $=$	$\bullet$	$=< \beta_2 >$

In (2), the symbols $\nwarrow$ are also deleted for the sake of simplicity.

The symbols $\bullet \hookleftarrow \bullet$ at k-stem groups between $n = 2m-1$ and $n = 2m+1$ indicate that the homomorphism

$$\Delta : \pi^{2m+1}_{2m+1+k} \longrightarrow \pi^{2m-1}_{2m-1+k}, \quad \textit{where} \quad m \equiv 0 \pmod{p},$$

cancells two cyclic groups of order p correponding to two $\bullet$.

For the case $k = 21$, $\Delta \neq 0$ follows from Lemma 3.7. By the naturality, we have $\Delta \neq 0$ for the case $k = 24$ composing α_1 from the right.

From this diagram we have the following results on the Hopf invariants of origins of stable classes:

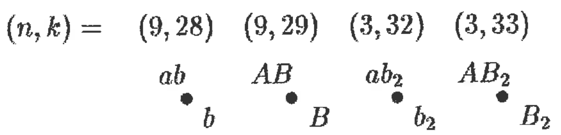
(5.2.4) $\quad H(\beta_1(5)) = I(\alpha_1(9))$ *and* $H_p(\beta_2(9)) = J(H(\beta_2(9))) = \beta_1(25)$.

5.3 3-primary k-stem Groups for $28 \leq k \leq 35$

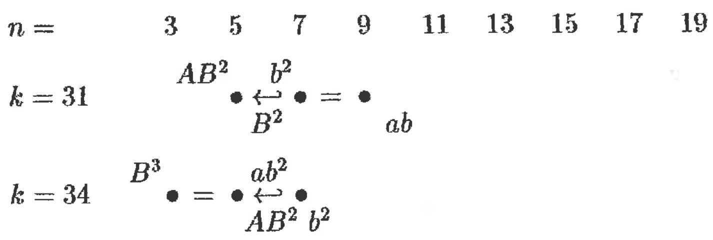
We consider the computing diagram mod A for $28 \leq k \leq 35$. In this case, the invariants are stable type except $w = IP(i) \in Q^3_{3+28}$.

Theorem 5.2 *The mod A k-stem groups π^n_{n+k} for $28 \leq k \leq 35$, n : odd, are obtained as the direct sum of the following three parts.*

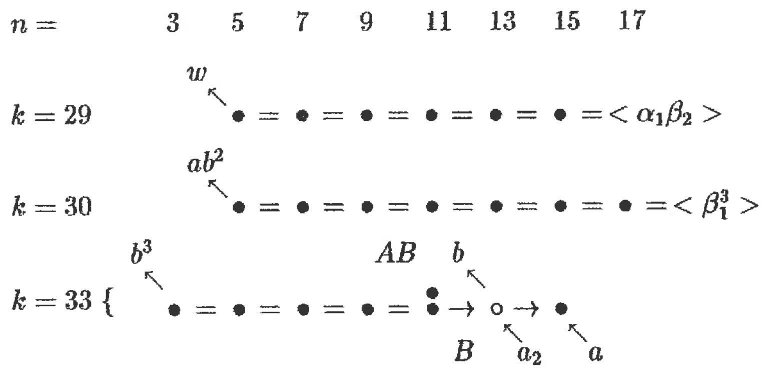
(1) *Removable simple unstable elements.*



(2) *Removable short range unstable elements.*



(3) *Remaining part.*



Let v be a generator of $\pi^{13}_{13+33} \cong \mathbf{Z}/9$ then up to non-zero coefficient,

$$H(v) = b, \quad 3v = \alpha_1\beta_1^3(13) \quad \textit{and} \quad \Delta v = \alpha_1\beta_1^3(11) \pm P(B).$$

Proof. The first two of (1) follows from Proposition 5.1. Let $\xi = \alpha_1(3) \circ \alpha_1\beta_2(6)$, then $H\xi = ab_2$. $E^2\xi = 0$ and $P(b_2) = \xi$ follows from $\alpha_1(5) \circ \alpha_1(8) = 0$. By Theorem 2.4, $H(-\xi) = AB_2$ for $\xi \in \{\alpha_1, 3, \alpha_1\beta_2\}$ in S^3. In S^5, $\xi \in \alpha_2\beta_2 \equiv -\alpha_1\{\alpha_1, 3, \beta_2\} \equiv -\alpha_1\beta_1^3 \equiv 0$. Thus $P(B_2) \equiv \xi \mod \alpha_1\beta_1^3$.

(2) is proved by composing β_1 from the right to (2) of $k = 21, 24$.

(3). The results of $k = 29$ and $k = 30$ are unique solutions. For the case $k = 33$ we should make the following general statement. □

Lemma 5.1 *Let $l \geq 1$. Then the followings hold up to non-zero coefficients.*

(1) *There exists elements $v(6l+1) \in \pi^{6l+1}_{18l+10}$ satisfying,*

$$H(v(6l+1)) = Q^{3l}(\beta_1) = b, \quad and \quad v(6l+3) = P(Q^{3l+2}(\alpha_1)) = P(a)$$

where $v(6l+3) = E^2v(6l+1)$. Thus $v(6l+3)$ is of order 3 and $E^2v(6l+3) = 0$.

Furthermore we have

$$\Delta(v(6l+1)) \equiv P(\overline{Q}^{3l-1}(\beta_1)) = P(B) \qquad \mathrm{mod}\ \ E^2\pi^{6l-3}_{18l+6}\,.$$

(2) *If $l \not\equiv 2$ (mod 3) then the order of $v(6l+1)$ is 3 .*

(3) *If $l \equiv 2$ (mod 3) and $E^\infty : \pi^{6l-3}_{18l+7} \to \pi^S_{12l+10}$ is surjective, then the order of $v(6l+1)$ is 9 and*

$$\Delta(v(6l+1)) = P(B) + E^2v'\,,$$

where v' is an element of $\pi_{18l+6}(S^{6l-3})$ satisfying $E^4v' = 3v(6l+1)$.

Proof. Since $\beta_1 = \{\alpha_1, \alpha_1, \alpha_1\}$, (1) follows from Proposition 5.2 the existence of $\xi = v(6l+1)$ with $H(\xi) = b$ and $E^2(\xi) = P(a)$. By Lemma 3.7,(1) and Lemma 3.2 $H\Delta(v(6l+1)) = \overline{Q}^{3l-1}(\alpha_1\beta_1) = AB = HP(B)$, and the last statemnt follows.

In the case (2), $a_2 = a_2(6l+1) = H(\alpha^*_{3l+3}(6l+3))$. Then $P(a_2) = 0$ and $E^2 : \pi^{6l+1}_{18l+10} \to \pi^{6l+3}_{18l+12}$ is injective. Thus $3v(6l+1) = 0$.

In the case(3), a_2 is out of alpha invariants and $P(a_2) \neq 0$. a is M-represented by αi an a_2 is M-represented by $\alpha^2 i$. Then by Lemma 3.4 $P(a_2) = 3v(6l+1)$. Thus $v(6l+1)$ is of order 9. Let $\Delta(v(6l+1)) = P(B) + E^2v'$. By Lemma 2.1, $3v(6l+1) = E^2\Delta v(6l+1) = E^2P(B) + E^4v' = E^4v'$. □

5.4 3-primary k-stem Groups for $36 \le k \le 45$

In the result of the 33-stem groups, we see a long series of unstable elements

$$\alpha_1\beta_1^3(3) \overset{E^2}{\mapsto} \alpha_1\beta_1^3(5) \overset{E^2}{\mapsto} \alpha_1\beta_1^3(7) \overset{E^2}{\mapsto} \alpha_1\beta_1^3(9) \overset{E^2}{\mapsto} \alpha_1\beta_1^3(11) \overset{E^2}{\mapsto} \alpha_1\beta_1^3(13) \overset{E^2}{\mapsto} 0 \tag{5.4.1}$$

where $\alpha_1\beta_1^3(3) = \alpha_1(3) \circ \beta_1^3(13)$ for a generator $\beta_1^3(13)$ of π_{43}^{13} with stable limit β_1^3. Then we have not stable type invariants

$$ab^3 = I(\alpha_1\beta_1^3(6m-1)) \quad \textit{and} \quad AB^3 \textit{ with } J(AB^3) = \alpha_1\beta_1^3(6m+1)$$

for $m = 1$ and $m = 2$.

In the computing diagram mod A for $36 \le k \le 45$, invariants of not stable type are $w \in Q_{5+42}^5$ and the above ab^3, AB^3.

Theorem 5.3 *The mod A k-stem groups π_{n+k}^n for $36 \le k \le 45$, n : odd, are obtained as the direct sum of the following three parts.*

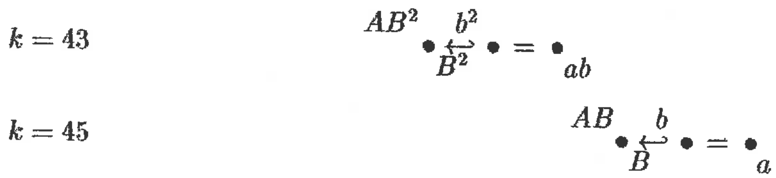
(1) *Removable simple unstable elements*

$(n,k) =$	$(3,36)$	$(3,37)$	$(9,38)$	$(9,39)$	$(15,40)$	$(15,41)$	$(3,42)$	$(3,43)$
	ab^3	AB^3	ab^2	AB^2	ab	AB	abb_2	ABB_2
	•	•	•	•	•	•	•	•
	b^3	B^3	b^2	B^2	b	B	bb_2	BB_2

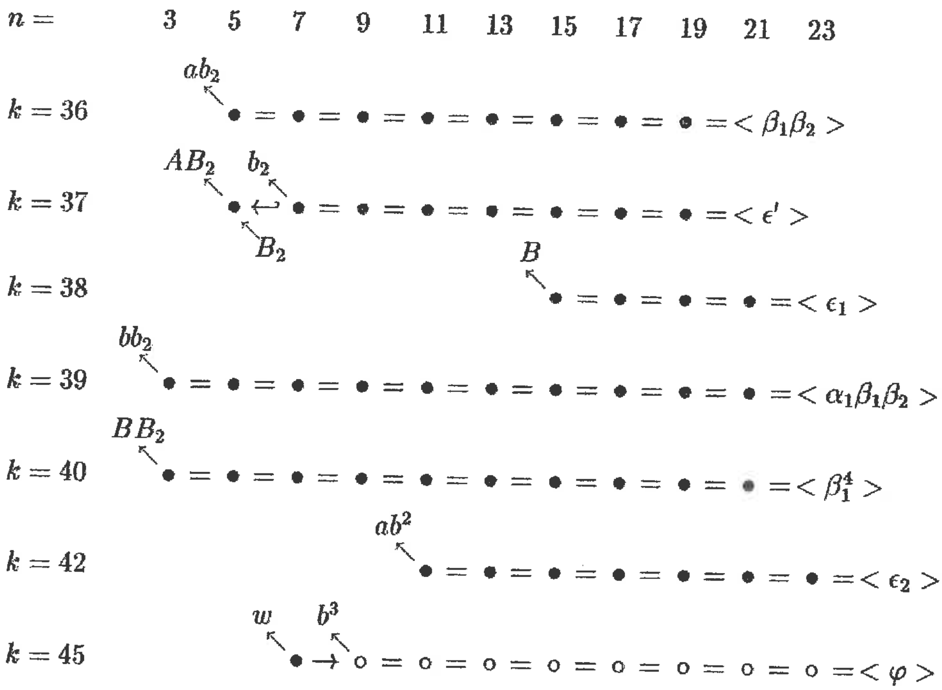
$(n,k) =$	$(3,43)$	$(3,44)$	$(9,44)$	$(9,45)$
	b^4	B^4	ab_2	AB_2
	•	•	•	•
	e'	E'	b_2	B_2

(2) *Removable short range unstable elements*

$n =$	3	5	7	9	11	13	15	17	19	21	23
$k = 36$				B^2 • =	• $\overset{ab}{\underset{AB}{\leftrightarrows}}$	• b					
$k = 40$	• $\overset{ab^3 \quad ab_2}{\underset{AB_2}{\leftrightarrows}}$	• b_2									
$k = 41$	• $\overset{AB^3 \quad b^3}{\underset{B^3}{\leftrightarrows}}$	• B_2									



(3) *Remaining part*



Proof. (1) From the relations $d_1(b^2) = ab^2$ and $d_1(B^2) = AB^2$ in (1) of Theorem 5.2, we have $d_1(b^3) = ab^3$ and $d_1(B^3) = AB^3$ by composing β_1 from the right. The remaining relations on d_1 follow from Proposition 5.1.

(2) The short range unstable elements in $k = 36, 43, 45$ are established by Propositions 5.2,5.3. The removability of the collections in $k = 43, 45$ follows from Lemma 3.7,(1) and one in $k = 36$ from Lemma 3.7,(2).

In $k = 40$, $d_1(AB_2) = d_1(A_1)\circ\beta_2 = a_2\circ\beta_2$ is represented by $\alpha_2\beta_2 = -\{\alpha_1, \alpha_1, 3\}\beta_2 = \alpha_1\{\alpha_1, 3, \beta_2\} = \alpha_1\beta_1^3$ in S^9. Thus $d_1(AB_2) = ab^3$. The other 3 simple unstable elements in $k = 40, 41$ are established by Proposition 5.1 and naturality. The removability follows from Lemma 2.4.

(3) $d_1(B_2) = AB_2$ follows from Proposition 5.1. Then the results of (3) are uniquely determined by the exactness of EHP-sequence. The removability in k=37 follows from Lemma 2.4. □

It follows from (3) of the theorem

(5.4.2) $\qquad u(\epsilon') = 7, \quad u(\epsilon_1) = 15, \quad u(\epsilon_2) = 11, \quad u(\varphi) = 9,$

(5.4.3) $\quad u(\beta_1\beta_2) = 5,\ u(\alpha_1\beta_1\beta_2) = 3,\ u(\beta_1^4) = 3$ *and* $u(3\varphi) = u(\alpha_1\epsilon_2) = 7\,.$

Here we refer to applications of ∂_α of Oka.

In Theorem 5.1 apply ∂_α to the short range unstable elements in the 21-stem groups, then we obtain those in the 24-stem groups :

$$P(a) = E^2(v),\ H(v) = b \quad \Longrightarrow \quad P(AB) = E^2(\partial_\alpha(v)),\ H(\partial_\alpha(v)) = B^2\,.$$

We see similar situations from the k-stem groups to the $(k+3)$-stem groups for $k = 31$ and $k = 33$. However, for elements of π_{7+k}^7 we cannot apply Lemmas 3.12, 3.13 since the case $m = 0$ is excluded. In the case $m = 0$, the map $\beta_{(1)}$ in (3.5.5) does not exist, but Theorem 3.5 still holds. So we prepare the following

Lemma 5.2 *Assume that Hopf invariant $H(\xi)$ of an element $\xi \in \pi_{i+4}(S^{2p+1})$ is M-represented by γ for $\gamma \in \pi_{i+1}(Y^{2p^2-2})$. Then up to non-zero coefficient there holds*

$$JH(\partial_\alpha(\xi)) = H_p(\partial\alpha(\xi)) = \overline{\beta}_1 \circ E^2\gamma\ ,$$

where $\overline{\beta}_1 : Y^{2p^2} \to S^{2p+1}$ is an extension of $\beta_1(2p+1)$.

Proof. ∂_α is induced by $d_\alpha : \Omega S^{2p+1} \to S^3$ and H_p is induced by $h_p : \Omega S^3 \to \Omega S^{2p+1}$. Then $H_p\partial_\alpha$ is induced by $h_p \circ \Omega d_\alpha : \Omega^2 S^{2p+1} \to \Omega S^{2p+1}$, whose restriction on S^{2p-1} is homotopic to zero. Thus $h_p \circ \Omega d_\alpha$ induces a map

$$h' : Q_2^{2p-1} = \Omega(\Omega^2 S^{2p+1}, S^{2p-1}) \longrightarrow \Omega^2 S^{2p+1}\,.$$

Let $\overline{\beta}_1 : Y^{2p^2} \to S^{2p+1}$ be adjoint to $h' \circ g_{2p-1} : Y^{2p^2-2} \to Q_2^{2p-1} \to \Omega^2 S^{2p+1}$, then Theorem 3.5 shows that $\overline{\beta}_1 | S^{2p^2-1} = \beta_1(2p+1)$. Then the lemma follows. □

For example, apply the lemma to $\epsilon'(7) \in \pi_{7+37}^7$ with $H(\epsilon'(7)) = b_2$ and put $\beta_1^4(3) = \partial_\alpha(\epsilon'(7))$ and $\beta_1^4(5) = E^2(\beta_1^4(3))$. Then we have

(5.4.4) $\qquad \beta_1^4(5) = \alpha_1(5) \circ \epsilon'(8) = \beta_1(5) \circ \beta_1^3(15) \pm P(AB_2)\,.$

5.5 3-primary k-stem Groups for $46 \le k \le 55$

For $46 \le k \le 55$, the invariants in the computing diagram mod A are all stable type.

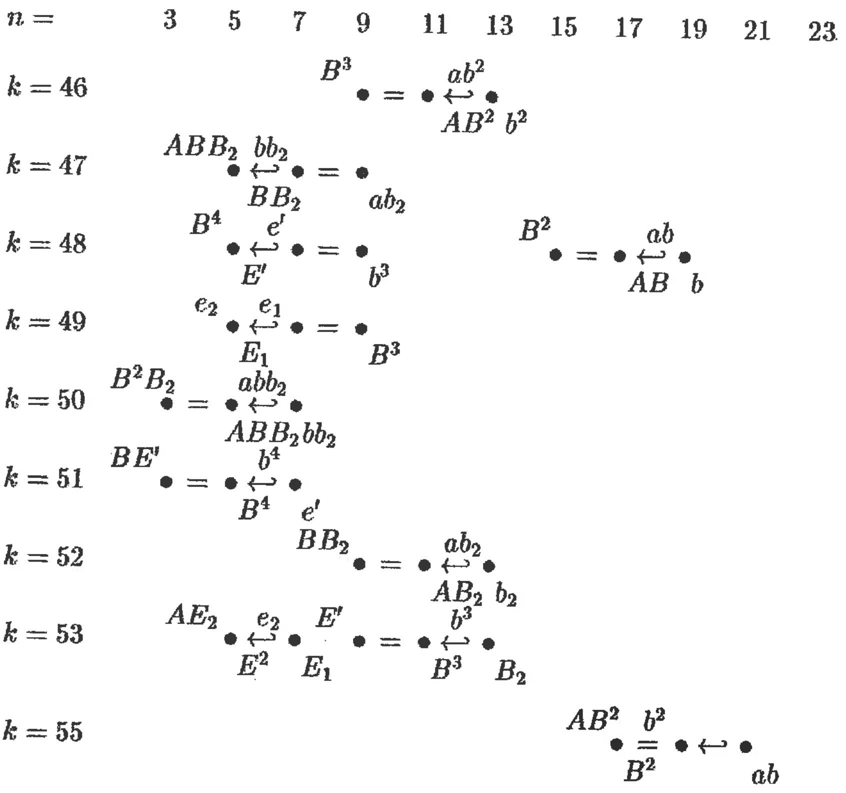
Theorem 5.4 *The mod A k-stem groups π^n_{n+k} for $46 \le k \le 55$, n : odd, are obtained as the direct sum of the following three parts.*

(1) *Removable simple unstable elements*

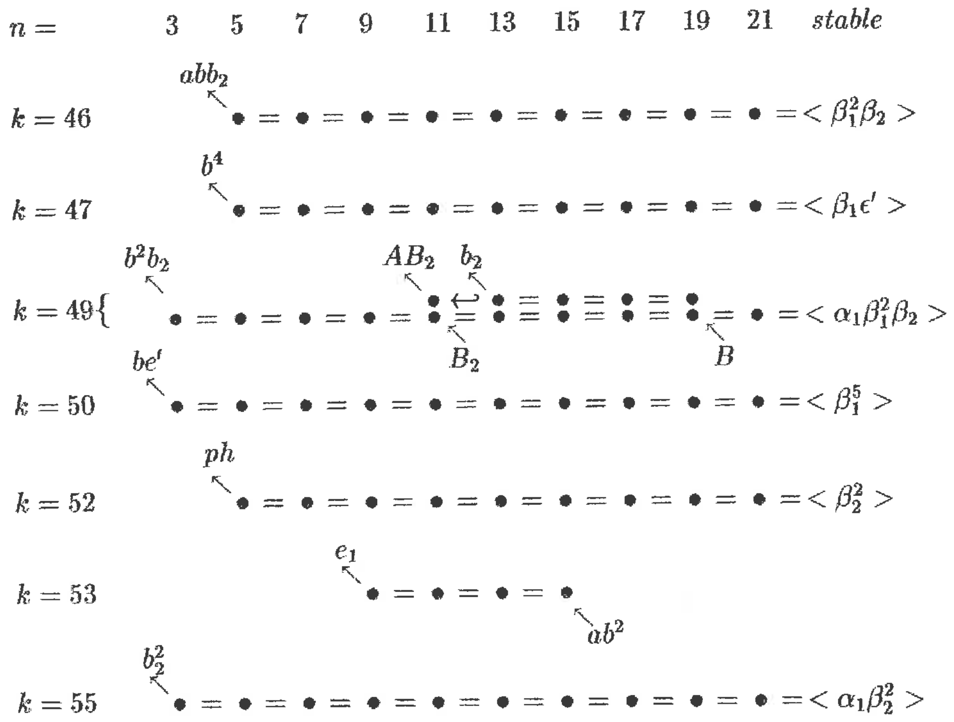
$(n,k) =$	$(3,49)$	$(15,50)$	$(15,51)$	$(3,52)$	$(3,53)$	$(21,52)$	$(21,53)$
	AE_2	ab^2	AB^2	ab^2b_2	AB^2B_2	ab	AB
	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	E_2	b^2	B^2	b^2b_2	B^2B_2	b	B

$(n,k) =$	$(3,53)$	$(3,54)$	$(9,54)$	$(9,55)$	$(9,55)$
	b^5	B^5	abb_2	ABB_2	b^4
	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	be'	BE'	bb_2	BB_2	e'

(2) *Removable short range unstable elements*



(3) *Remaining part*



Proof. (1) The first relation $d_1(E_2) = AE_2$ cannot establish by Proposition 5.1 or Lemma 3.2 because $u(E_2) = 11$ and E_2 is not M-representable. Except this all the other simple unstable elements are established by Proposition 5.1.

(2) Except the cases $k = 49, 50, 51$, short range ubstable elements and their removability are established by Propositions 5.2, 5.3 and Lemma 2.4.

The case $k = 50$ is established from the case $k = 24$ of Theorem 5.1 by composing β_2 from the right.

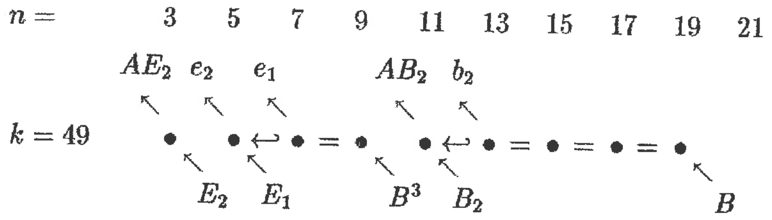
Consider the case $k = 51$. Let η be the element in $k = 48$ such that $H(\eta) = e'$ and $E^2\eta = P(B^3)$. By Lemma 5.2 $\xi = \partial_\alpha(\eta)$ satisfy $H(\xi) = BE'$ and $E^2\xi = \alpha_1(5) \circ E\eta$. Then $E^5\xi = \alpha_1(8) \circ E^4\eta = \alpha_1(8) \circ E^2P(B^3) = 0$, and $E^4\xi = 0$ by (2.1.2). If $E^2\xi = 0$ then $\xi = \pm P(ph)$. But $HP(ph)$ is purely M-represented by $i_*\alpha_1\varphi(7) = i_*E^2(\alpha_1(5)\varphi(8)) \in i_*E^2\pi^5_{5+48} = 0$. Thus $E^2\xi \neq 0$, $E^4\xi = 0$ and $E^2\xi = \pm P(B^4)$. The removability follows from one in $k = 41$ of Theorem 5.3 (2).

The case $k = 49$ is remained.

(3) Let $\xi = \alpha_1(7) \circ \varphi(10) \in \pi^7_{7+48}$. Since $\xi = E(\alpha_1(6) \circ \varphi(9))$, it follows from (2.1.2) and k=48 of (2) that $\xi \in E^2\pi^5_{5+48} = 0$. Thus $\beta_2^2(5) \in \{\alpha_1(5), \alpha_1(8), \varphi(11)\}$ is defined. Then $\beta_2^2(5)$ gives an origin of the stable class $\beta_2^2 = \{\alpha_1, \alpha_1, \varphi\}$. The other origins of stable classes of (2) are obviously established.

Long range unstable elements in $k = 53$ is a unique solution.

Consequently only the case $k = 49$ is remained as follows in question where the sequence $\{\alpha_1\beta_1^2\beta_2(2m+1)\}$ is removed.



The above groups and E^2 are unique solutions. The removability follows from Lemma 2.4 and Lemma 3.7. □

Note that the long range unstable elements in $k = 49$ is given by Lemma 3.11.

Also we see there an example of Proposition 3.4 for $x = B^3 \in \pi^9_{9+49}$ such that $\partial(B^3) = i_*(e_1)$. Proposition 3.4 states that $e_1 \in \{\eta_3, \eta_4^{(2)}, B^3\}$. Since $\eta_m = \eta_{m+3}$ in stable range, $e_1 \in \{\eta_m, \eta_{m+1}, B^3\}$ holds for $m \equiv 0 \pmod 3$. Thus we have

Lemma 5.3 *If* $m \equiv 0 \pmod 3$, *there exists* $v \in \pi^{2m+1}_{6m+38}$ *satisfying*

$$H(v) = e_1 \quad \text{and} \quad E^2(v) = P(B^3).$$

We denote the long range unstable elements in $k = 49$ and $k = 53$ as follows.

(5.5.1) $v_1(13) \in \pi^{13}_{13+49}$ *with* $H(v_1(13)) = b_2$ *and* $P(B) = E^6 v_1(13) = v_1(19)$,

(5.5.2) $v_2(9) \in \pi^9_{9+53}$ *with* $H(v_2(9)) = e_1$ *and* $P(ab^2) = E^6 v_2(9) = v_2(15)$.

By Theorem 2.2, these elements induce the following not stable type invariants.

(5.5.3) $u_1 = u_1(5) = I(v_1(17)) \in Q^5_{5+59}$ *and* $U_1 = U_1(5) \in Q^5_{5+60}$, $J(U_1) = v_1(19)$,

(5.5.4) $u_2 = u_2(3) = I(v_2(11)) \in Q^3_{3+59}$ *and* $U_2 = U_2(3) \in Q^3_{3+60}$, $J(U_2) = v_2(13)$.

(3) Let $\xi = \alpha_1(7) \circ \varphi(10) \in \pi^7_{7+48}$. Since $\xi = E(\alpha_1(6) \circ \varphi(9))$, it follows from (2.1.2) and k=48 of (2) that $\xi \in E^2\pi^5_{5+48} = 0$. Thus $\beta^2_2(5) \in \{\alpha_1(5), \alpha_1(8), \varphi(11)\}$ is defined. Then $\beta^2_2(5)$ gives an origin of the stable class $\beta^2_2 = \{\alpha_1, \alpha_1, \varphi\}$. The other origins of stable classes of (2) are obviously established.

Long range unstable elements in $k = 53$ is a unique solution.

Consequently only the case $k = 49$ is remained as follows in question where the sequence $\{\alpha_1\beta^2_1\beta_2(2m+1)\}$ is removed.

$$
\begin{array}{lcccccccccc}
n = & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 \\
 & AE_2 & e_2 & e_1 & & AB_2 & b_2 & & & & \\
 & \nwarrow & \nwarrow & \nwarrow & & \nwarrow & \nwarrow & & & & \\
k = 49 & \bullet & \bullet \hookleftarrow & \bullet = & \bullet & \bullet \hookleftarrow & \bullet = & \bullet = & \bullet = & \bullet & \\
 & & \nwarrow & \nwarrow & & \nwarrow & \nwarrow & & & \nwarrow & \\
 & & E_2 & E_1 & & B^3 & B_2 & & & B &
\end{array}
$$

The above groups and E^2 are unique solutions. The removavility follows from Lemma 2.4 and Lemma 3.7. □

Note that the long range unstable elements in $k = 49$ is given by Lemma 3.11.

Also we see there an example of Proposition 3.4 for $x = B^3 \in \pi^9_{9+49}$ such that $\partial(B^3) = i_*(e_1)$. Proposition 3.4 states that $e_1 \in \{\eta_3, \eta^{(2)}_4, B^3\}$. Since $\eta_m = \eta_{m+3}$ in stable range, $e_1 \in \{\eta_m, \eta_{m+1}, B^3\}$ holds for $m \equiv 0 \pmod 3$. Thus we have

Lemma 5.3 *If $m \equiv 0 \pmod 3$, there exists $v \in \pi^{2m+1}_{6m+38}$ satisfying*

$$H(v) = e_1 \quad \textit{and} \quad E^2(v) = P(B^3).$$

We denote the long range unstable elements in $k = 49$ and $k = 53$ as follows.

(5.5.1) $v_1(13) \in \pi^{13}_{13+49}$ *with* $H(v_1(13)) = b_2$ *and* $P(B) = E^6 v_1(13) = v_1(19)$,

(5.5.2) $v_2(9) \in \pi^9_{9+53}$ *with* $H(v_2(9)) = e_1$ *and* $P(ab^2) = E^6 v_2(9) = v_2(15)$.

By Theorem 2.2, these elements induce the following not stable type invariants.

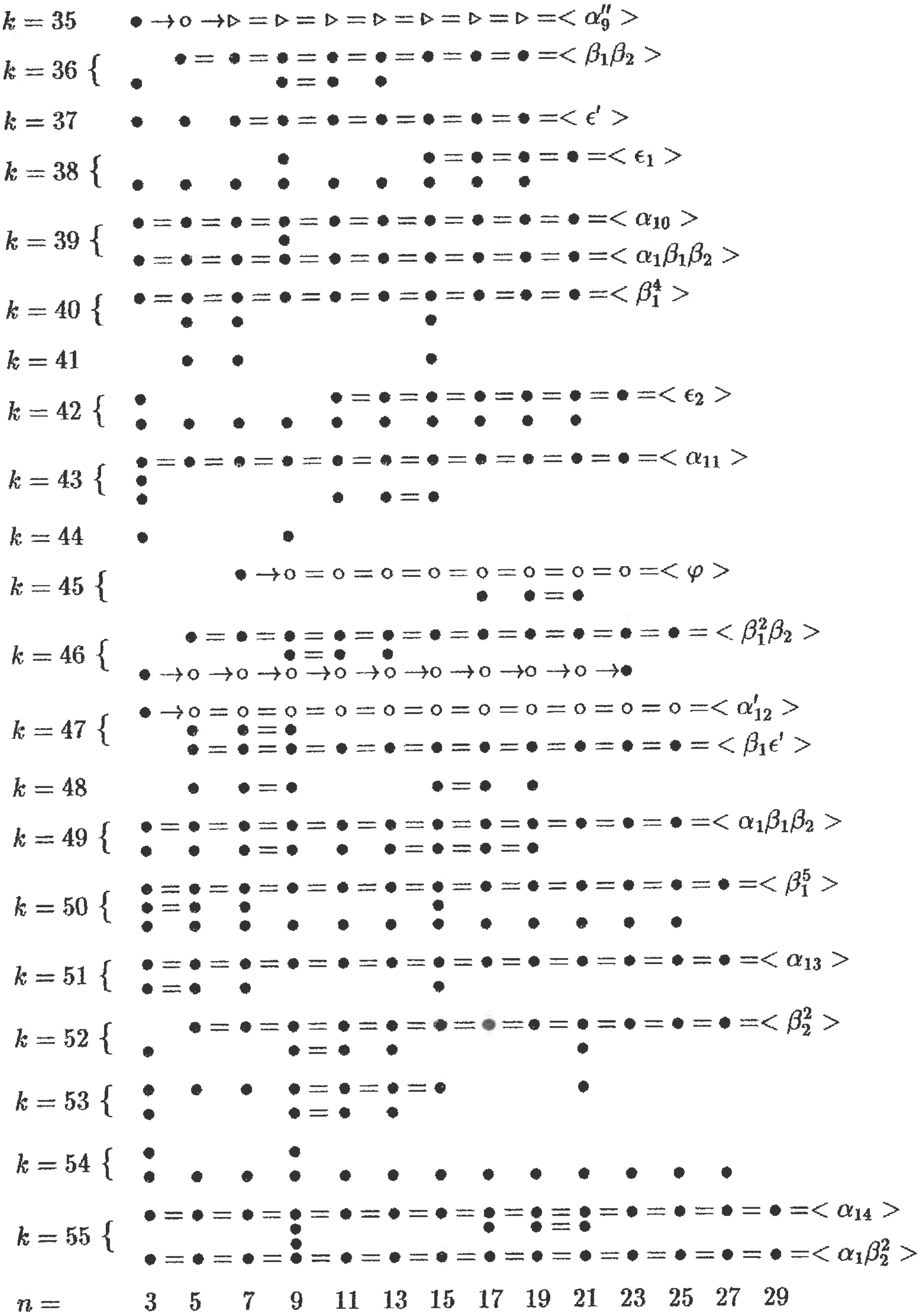
(5.5.3) $u_1 = u_1(5) = I(v_1(17)) \in Q^5_{5+59}$ *and* $U_1 = U_1(5) \in Q^5_{5+60}$, $J(U_1) = v_1(19)$,

(5.5.4) $u_2 = u_2(3) = I(v_2(11)) \in Q^3_{3+59}$ *and* $U_2 = U_2(3) \in Q^3_{3+60}$, $J(U_2) = v_2(13)$.

5.6 Table of 3-primary k-stem Groups for $k \leq 55$

Summerizing the results, we have the following table of ${}_3\pi_{n+k}(S^n)$ for odd n and $k \leq 55$.

$n=$	3	5	7	9	11	13	15	17	19	21	23	25	27	29
$k=3$	• $=<\alpha_1>$													
$k=6$	•													
$k=7$	•	= • $=<\alpha_2>$												
$k=10$	•	→ ∘	→ • $=<\beta_1>$											
$k=11$	•	→ ∘	= ∘ $=<\alpha_3'>$											
$k=13$	•	= •	= • $=<\alpha_1\beta_1>$											
$k=14$	•	•	•											
$k=15$	•	= •	= •	= • $=<\alpha_4>$										
$k=16$	•													
$k=17$	•													
$k=18$	•	•	•	•										
$k=19$	•	= •	= •	= •	= • $=<\alpha_5>$									
$k=20$		•	= •	= •	= • $=<\beta_1^2>$									
$k=21$		•	•	= •										
$k=22$	•	→ ∘	→ ∘	→ ∘	→ •									
$k=23$ {	•	→ ∘	= ∘	= ∘	= ∘	= ∘ $=<\alpha_6'>$								
	•	= •	= •	= •	= •	= • $=<\alpha_1\beta_1^2>$								
$k=24$	•	= •	•											
$k=26$ {	•			•	= •	= •	= • $=<\beta_2>$							
	•	•	•	•	•	•								
$k=27$ {	•	= •	= •	= •	= •	= •	= • $=<\alpha_7>$							
	•													
$k=28$				•										
$k=29$ {		•	= •	= •	= •	= •	= • $=<\alpha_1\beta_2>$							
				•										
$k=30$ {		•	= •	= •	= •	= •	= • $=<\beta_1^3>$							
	•	•	•	•	•	•	•							
$k=31$ {	•	= •	= •	= •	= •	= •	= •	= • $=<\alpha_8>$						
		•	•	= •										
$k=32$	•													
$k=33$ {	•				•									
	•	= •	= •	= •	= •	→ ∘	→ •							
$k=34$ {	•	= •	•											
	•	→ ∘	→ ▷	→ ▷	→ ▷	→ ▷	→ ∘	→ •						


k = 35
k = 36
k = 37
k = 38
k = 39
k = 40
k = 41
k = 42
k = 43
k = 44
k = 45
k = 46
k = 47
k = 48
k = 49
k = 50
k = 51
k = 52
k = 53
k = 54
k = 55
n = 3 5 7 9 11 13 15 17 19 21 23 25 27 29
< α_9'' >
< $\beta_1\beta_2$ >
< ϵ' >
< ϵ_1 >
< α_{10} >
< $\alpha_1\beta_1\beta_2$ >
< β_1^4 >
< ϵ_2 >
< α_{11} >
< φ >
< $\beta_1^2\beta_2$ >
< α_{12}' >
< $\beta_1\epsilon'$ >
< $\alpha_1\beta_1\beta_2$ >
< β_1^5 >
< α_{13} >
< β_2^2 >
< α_{14} >
< $\alpha_1\beta_2^2$ >

6 New Tables of 3-primary Groups

6.1 Stable elements and Stable type invariants

For the results of the stable k-stem groups for $55 < k \leq 81$, we quote from Oka[22], Nakamura[18],Tangora[33] and Ravenel[25].

(6.1.1)

List of ${}_3\pi_k^S = <$ *generator* $>$, *relations up to sign*

(1) ${}_3\pi_{62}^S = < \beta_1\beta_2^2 >$.

(2) ${}_3\pi_{65}^S = < \alpha_1\beta_1\beta_2^2 >$.

(3) ${}_3\pi_{68}^S = < \lambda >$, $\lambda = \{\beta_2, \epsilon_1, \alpha_1\}$.

(4) ${}_3\pi_{72}^S = < \beta_1^2\beta_2^2 >$, $\beta_1^2\beta_2^2 = \{\alpha_1, 3, \lambda\}$.

(5) ${}_3\pi_{74}^S = < \beta_5 >$.

(6) ${}_3\pi_{75}^S = < \mu >$, $\mu = \{\alpha_1, \alpha_1, \lambda\}$, $3\mu = \alpha_1\beta_1^2\beta_2^2$.

(7) ${}_3\pi_{78}^S = < \beta_2^3 >$, $\beta_2^3 = \beta_1\lambda = \alpha_1\mu$.

(8) ${}_3\pi_{81}^S = < \gamma_2, \mu_2 >$, $\mu_2 = \{\alpha_1, \alpha_1, \beta_5\}$.

We use notations of invariant similar to (5.1.7).

(6.1.2) *Invariants* $Q^m(\xi)$ *and* $\overline{Q}^m(\xi)$ *for* $\xi \in \pi_k^S$, $55 < k < 81$.

k	62	65	68	72	74	75	75	78
ξ	$\beta_1\beta_2^2$	$\alpha_1\beta_1\beta_2^2$	λ	$\beta_1^2\beta_2^2$	β_5	μ	$\alpha_1\beta_1^2\beta_2^2$	β_2^3
$Q^m(\xi)$	bb_2^2	abb_2^2	l	$b^2b_2^2$	b_5	m	$\times$	b_2^3
$\overline{Q}^m(\xi)$	BB_2^2	ABB_2^2	L	$B^2B_2^2$	B_5	$\times$	$AB^2B_2^2$	B_2^3

6.2 3-primary k-stem Groups for $56 \leq k \leq 61$

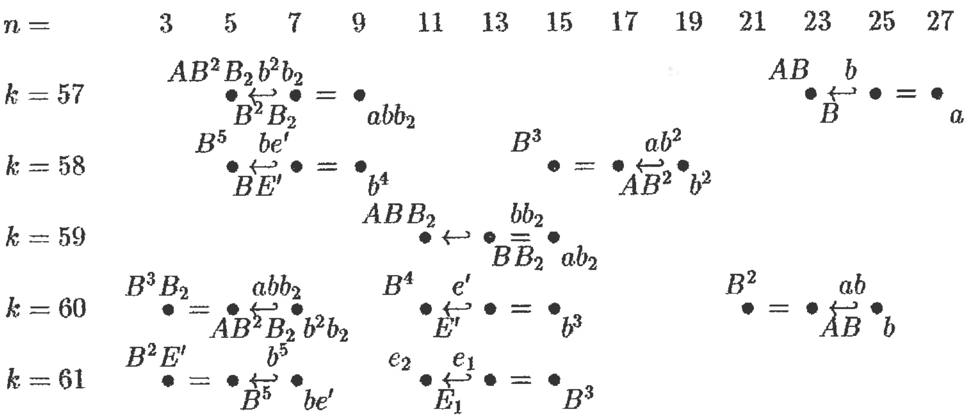
First remark that all invariants in Q^n_{n+k} for $55 \leq k \leq 61$, odd $n > 1$, and $Q^1_k \cong {}_3\pi_{3+k}(S^3)$ of $k < 59$ are fixed by the previous results.

Theorem 6.1 *The mod A k-stem groups* π^n_{n+k} $(n : odd)$ *for* $56 \leq k \leq 61$ *are obtained as the direct sum of the following three parts.*

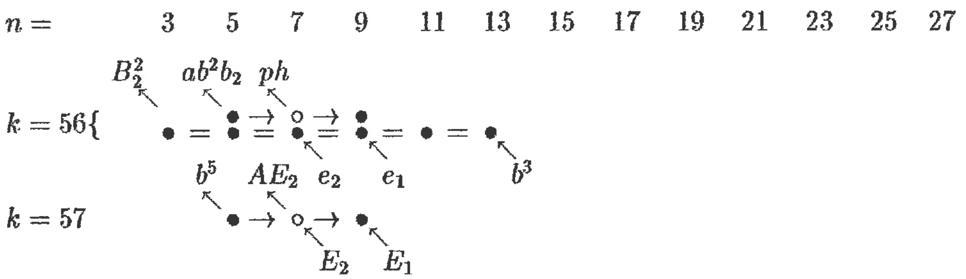
(1) *Removable simple elements (with stable type invariants)*

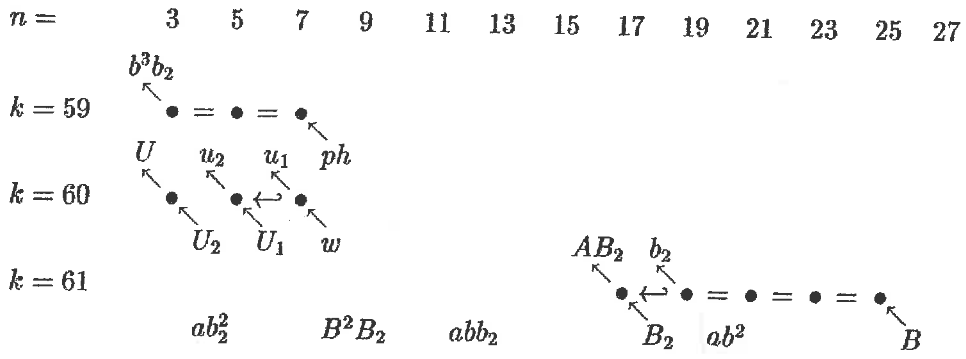
$(n,k) =$	$(15,56)$	$(15,57)$	$(3,58)$	$(3,59)$	$(9,61)$
	ab_2 • b_2	AB_2 • B_2	ab_2^2 • b_2^2	AB_2^2 • B_2^2	AB_2 • B_2

(2) *Removable short range elements (with stable type invariants)*



(3) *Remaining parts*





Proof. We start to establish the 58-stem groups. Since $u(\beta_2^2) = 5$ the simple element of $(n, k) = (3, 58)$ in (1) is removable. Two blocks in $k = 58$ of (2) are fixed since no other invariant which may vanishes any element of these blocks. The stable group π_{58}^S is trivial. Thus the 58-stem groups are the direct sum of the corresponding groups in (1) and (2).

Similarly the groups of the 56-stem and 57-stem groups in (1) and (2) can be removed. In order to fix the 56- and 57-stem groups it remains to fix those in (3) from stable results $\pi_{56}^S = \pi_{57}^S = 0$.

From the results of the 58-stem group, $P(E_2) \neq 0$ and $P(E_1) \neq 0$. By Proposition 5.1 $HP(E_1) = 0$ at $(n, k) = (7, 57)$. It follows then that π_{7+57}^7 is of order 9. But this groups is cyclic by Lemma 3.2 since $i\epsilon_2 = i\{p, \alpha_1, \epsilon_1\} = \alpha \circ i \circ \epsilon_1$. Thus the 57-stem groups are fixed.

By Lemma 2.4, the group π_{59}^3 is generated by an element

$$v(3) \in \{\alpha_1(3), p\iota_6, \beta_2^2(6)\}_1 \quad \textit{with} \quad H(v(3)) = B_2^2 .$$

We denote $v(n) = E^{n-3}v(3)$.

Since $H(\beta_1(5)) = a = I(\alpha_1(5))$, we have $H(\beta_1^3\beta_2(5)) = ab^2b_2$. Then

$$\pi_{61}^5 = < v(5), \beta_1^3\beta_2 > \cong \mathbf{Z}/3 \oplus \mathbf{Z}/3 .$$

Apply Lemma 3.8 to the relation $\varphi \in \{\alpha_1, \alpha_1, \epsilon_1\}$ of (5.1.1),(14), then we have an element $\xi(7) \in \pi_{63}^7$ such that

$$H(\xi(7)) = ph = I(\varphi(17)) \quad \textit{and} \quad P(e_1) = E^2\xi(7) = \xi(9) .$$

Also applying Lemma 3.4 to $\alpha i\epsilon_1 = i_*\{3,\alpha_1,\epsilon_1\} = \pm i_*\epsilon_2$ we have

$$3\xi(7) = \pm P(e_2).$$

Thus $\xi(7)$ is of order 9 and generates the kernel of $E^4 : \pi^7_{63} \to \pi^{11}_{67}$. The kernel of $E^6 : \pi^5_{61} \to \pi^{11}_{67}$ is generated by an element of order 3 which is mapped to $3\xi(7)$ under E^2.

Consequently the result in $k = 56$ of (3) follows from the following lemma.

Lemma 6.1 *$v(13) = \beta_1^3\beta_2(13)$ up to sign.*

Proof. In the 52-stem groups, we see that $\beta_2^2(9) \neq \beta_2(9)\circ\beta_2(35)$ but $\beta_2^2(13) = \beta_2(13)\circ\beta_2(39)$. Then, up to sign,

$$v(10) \in \{\alpha_1(10), p, \beta_2^2(13)\} \supset \{\alpha_1(10), p, \beta_2(13)\}\circ\beta_2(40) \ni \beta_1^3(10)\beta_2(40) = \beta_1^3\beta_2(10).$$

The difference $v(10) - \beta_1^3\beta_2(10)$ belongs to $\alpha_1(10)\circ\pi^{13}_{66}$. Since $\pi^{17}_{70} = 0$ it follows $E(v(13) - \beta_1^3\beta_2(13)) \in \alpha_1(14)\circ\pi^{17}_{70} = 0$, and the lemma is proved by (2.1.2). □

Note that up to sign

$$P(e_2) = v(7) \pm \beta_1^3\beta_2(7). \tag{6.2.1}$$

Next we consider the cases $k = 59, 60, 61$ of (3).

New invariants come from $\Delta : \pi^7_{63} \to \pi^5_{61}$ and $\Delta : \pi^7_{64} \to \pi^5_{62}$. Since $E^2\circ\Delta = p\cdot$, the image of first Δ is $< v(5) \pm \beta_1^3\beta_2(5) >$ and the second Δ is surjective. The kernel of the first Δ is $< v(7), \beta_1^3\beta_2 >$ an the second one is $< \beta_1^2\epsilon'(7) >$, since $H(\beta_1^2\epsilon(5)) = I(\alpha_1\beta_1\epsilon'(9)) = I(\beta_1^5)$. Then we have not stable type invariants

$$b^3b_2 \in Q^1_{59}, \quad B^3B_2,\ U \in Q^1_{60} \quad \textit{and} \quad B^2E' \in Q^1_{61}$$

with $b^3b_2 = I'\beta_1^3\beta_2(5)$, $J(B^3B_2) = \beta_1^3\beta_2(7)$, $J(U) = v(7)$ and $J(B^2E') = \beta_1^2\epsilon'(7)$.

The first short range unstable elements in $k = 60, 61$ of (2) is obtained from those in $k = 50, 51$ of Theorem 5.5,(2) by composing β_1 from the right respectively. The second short range unstable elements in$k = 60, 61$ of (2) is obtained by Lemma 3.8 and Lema 5.2 respectively.

Then (3) of $k = 59, 60, 61$ is established by EHP-sequence which complete the proof of Theorem. □

In the above proof we find the following relations.

Proposition 6.1 *Up to sign the following relations hold.*

(1) $H(v(3)) = B_2^2, \quad H(\beta_1^3\beta_2(5)) = ab^2b_2, \quad H(\xi(7)) = ph,$

$3\xi(7) = P(e_2) = v(7) \pm \beta_1^3\beta_2(7), \quad \xi(9) = P(e_1), \quad P(b^3) = v(13) = \pm\beta_1^3\beta_2(13).$

(2) *There exists* $\xi'(7) \in \pi_{64}^7$ *such that* $H(\xi'(7)) = AE_2$ *and*

$H(\beta_1^2\epsilon'(5)) = b^5, \quad 3\xi'(7) = \beta_1^2\epsilon'(7) = P(E_2), \quad \xi'(9) = P(E_1).$

Apply Lemma 3.5 to $H(\xi(7)) = ph$, then Proposition 6.1 (1) shows that $H(\beta_1^3(5)) = ab^2b_2$ is M-presented by $\alpha i(10) \circ \varphi(13) \in \{3\iota_9, \alpha_1(9), \varphi(12)\}$.

Similarly, $H(\xi'(7)) = AE_2$ shows that $H(\beta_1^2\epsilon'(5)) = b^5$ is M-presented by $\alpha(10) \circ \widetilde{\alpha_1\epsilon_2}$ where $\widetilde{\alpha_1\epsilon_2} \in \pi_{69}(Y^{14})$ is a coextension of $\alpha_1\epsilon_2(13) = 3\varphi(13)$.

These are represented by stable class as follows.

Proposition 6.2 *Up to sign the following relations hold.*

(1) $$\alpha_1\beta_1^2\beta_2 = \{3, \alpha_1, \varphi\}$$

(2) $$i_*\beta_1^5 = \{\alpha i, 3, \alpha_1\epsilon_2\} \quad i.e. \quad \beta_1^5 \in \{3, \alpha_1, 3, \alpha_1\epsilon_2\}.$$

6.3 3-primary k-stem Groups for $62 \le k \le 70$

The invariants in Q^n_{n+k} for $62 \le k \le 71$ are stable type except four invariants related to simple elements at $(n,k) = (3,62)$, $(3,63)$.

Theorem 6.2 *The mod A k-stem groups π^n_{n+k} (n : odd) for $62 \le k \le 70$ are obtained as the direct sum of the following three parts.*

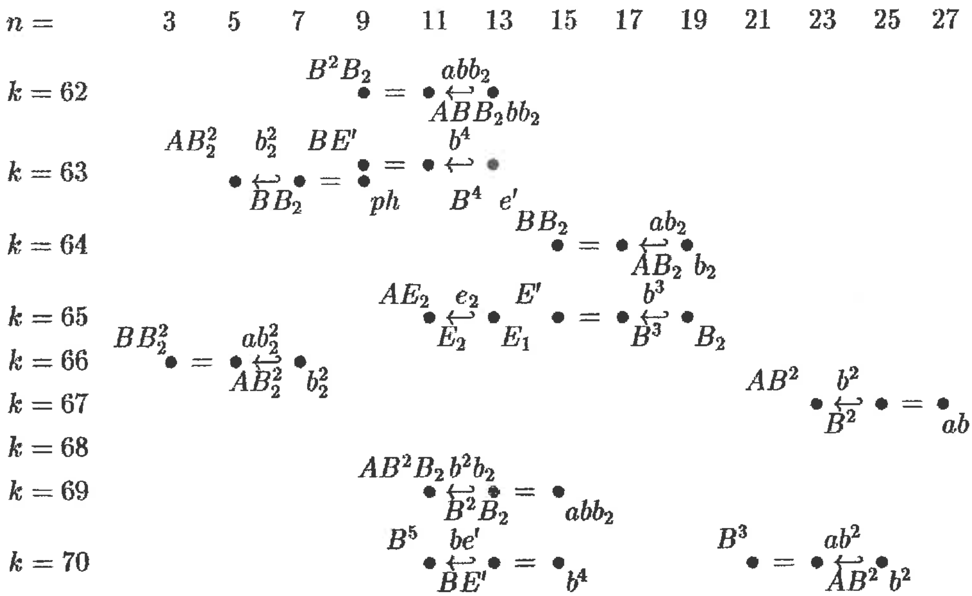
(1) *Removable simple unstable elements*

$(n,k)=$	$(3,62)$	$(21,62)$	$(3,63)$	$(21,63)$	$(9,64)$	$(27,64)$	$(9,65)$	$(9,65)$
	ab^3b_2	ab^2	AB^3B_2	AB^2	ab^2b_2	ab	b^5	AB^2B_2
	b^3b_2	b^2	B^3B_2	B^2	b^2b_2	b	be'	B^2B_2

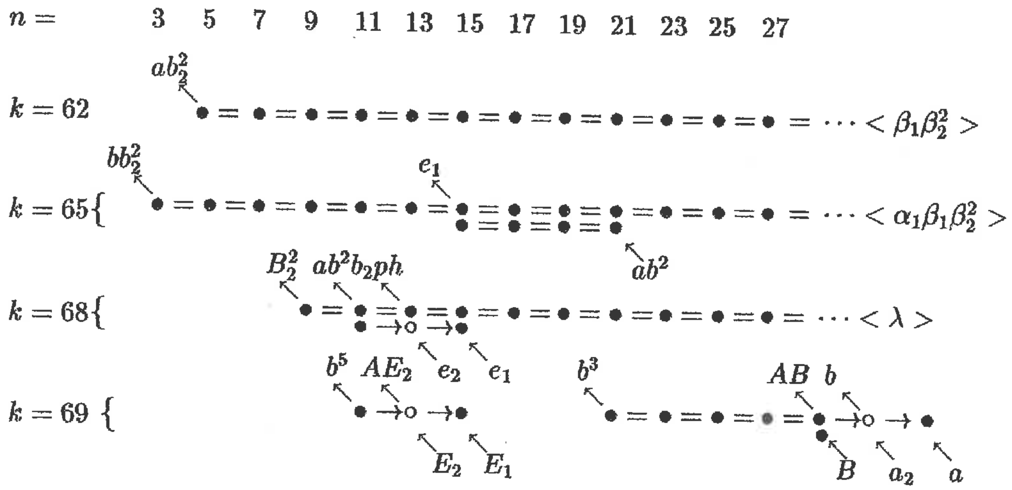
$(n,k)=$	$(27,65)$	$(9,66)$	$(15,66)$	$(15,67)$	$(15,67)$	$(3,68)$	$(15,68)$	$(21,68)$
	AB	B^5	abb_2	ABB_2	b^4	abb_2^2	B^4	ab_2
	B	BE'	bb_2	BB_2	e'	bb_2^2	E'	b_2

$(n,k)=$	$(3,69)$	$(21,69)$	$(9,70)$
	ABB_2^2	AB_2	ab_2^2
	BB_2^2	B_2	b_2^2

(2) *Removable short range elements*



(3) *Remaining parts*



Proof. Since $\beta_1^4 = \alpha_1\epsilon'$ we may regard that $b^4 = ae'$ and $b^5 = abe'$. All collections of (1) are of type $\{x, P(x), HP(x) = \pm ax\}$, $P(x) \in \pi_*(S^{6k+3})$. The collections in the cases $(n,k) = (3,62),(3,63)$ are obtained from corresponding collections at $(n,k) = (3,52),(3,53)$ of Theorem 5.4,(1), by composing β_1. The other collections in (1) are also simple and removable by Proposition 5.2.

Except short range collection in $k = 66$ of (2), the other short range collections in (2) are all established by Proposition 5.3.

Consider the element $\xi \in \pi_{43}^3$ of (5.4.2). Since $H(\xi) = BB_2$ we have $H(\xi \circ \beta_2(43)) = BB_2^2$. Since $\pi_{73}^{17} = 0$ by Theorem 6.1, it follows from (5.4.2)

$$E^4(\xi \circ \beta_2(43)) = \beta_1^4\beta_2(7) = \beta_1(7) \circ \beta_1^3\beta_2(17) \in \beta_1(7) \circ \pi_{73}^{17} = 0\,.$$

Thus $E^2(\xi \circ \beta_2(43)) = \pm P(AB_2^2)$. By Lemma 2.4 $\Delta(P(b_2^2)) = \pm\xi \circ \beta_2(43)$, and (2) has been established.

$\beta_1\beta_2^2(5)$ and $\alpha_1\beta_1\beta_2^2(3)$ are defined. So, k=62 and k=65 of (3) are completed.

Next observe unstable groups π^n_{n+67} and π^n_{n+70}, n : odd. From the stable results $\pi^S_{67} = \pi^S_{70} = 0$, we see that these unstable groups are consist of collections in (1) and (2). It follows that B_2^2, ab_2^2, ph are H-images and E_2, E_1, B, a_2, a are mapped injectively under P.

Also we see that $\pi^n_{n+68} = 0$ for $n = 5, 7$ and $\pi^n_{n+69} = 0$ for $n = 5, 7, 9$.

Since $HP(b^5) = ab^5 = 0$ we have $P(b^5) \in E^2\pi^7_{7+68} = 0$, and b^5 is an H-image.

Also we see by Lemma 3.9 that

$$P(AE_2) = E^2\eta \text{ for some } \eta \text{ and } H_p(\eta) = \{\alpha_1, \alpha_1, \alpha_1\epsilon_2\}(15) = \beta_1\epsilon_2(15).$$

In stable range, $\beta_1\epsilon_2 \in \pi^S_{52} =< \beta_2^2 >$, $\alpha_1\beta_2^2 \neq 0$ and $\alpha_1\beta_1\epsilon_2 = 3\beta_1\varphi = 0$. Thus $H_p(\eta) = 0$ and $P(AE_2) \in E^4\pi^7_{7+68} = 0$.

Then the first extension $\bullet \to \circ \to \bullet$ in $k = 69$ of (3) is established by Lemma 3.7.

Let $\xi \in \pi^{13}_{13+68}$ satisfy $H(\xi) = ph$. Since $HP(e_2) = ae_2 = 3ph = 0$, we have $E^2\xi \neq 0$. By Lemma 3.7 and $\varphi \in \{\alpha_1.\alpha_1, \epsilon_1\}$, we have $P(e_1) = E^2\xi$.

By Lemma 3.7 and Proposition 6.2, there exists $\xi' \in \pi^{11}_{11+68}$ satisfying $3\xi = E^2\xi'$ and $H(\xi') = ab^2b_2$. Thus the order of ξ is 9 and $E^4\xi = 0$.

λ is given by $\{\beta_2, \epsilon_1, \alpha_1\}$. Since $\beta_2(11) \circ \epsilon_1(37) \in \pi^{11}_{11+64} = 0$ there exists $\lambda(11) \in \{\beta_2(11), \epsilon_1(37), \alpha_1(75)\}$ with $E^\infty\lambda(11) = \lambda$. Since $E^\infty\xi' = 0$, $\lambda(9)$ with $E^2\lambda(9) = \lambda(11)$ exists. $\lambda(9)$ must satisfy $H(\lambda(9)) = \pm B_1^2$. Then $k = 68$ of (3) is established, and we have $P(b^3) = 0$. Thus $k = 69$ of (3) is established by Lemma 3.7. $\square$.

6.4 3-primary k-stem Groups for $71 \leq k < 80$

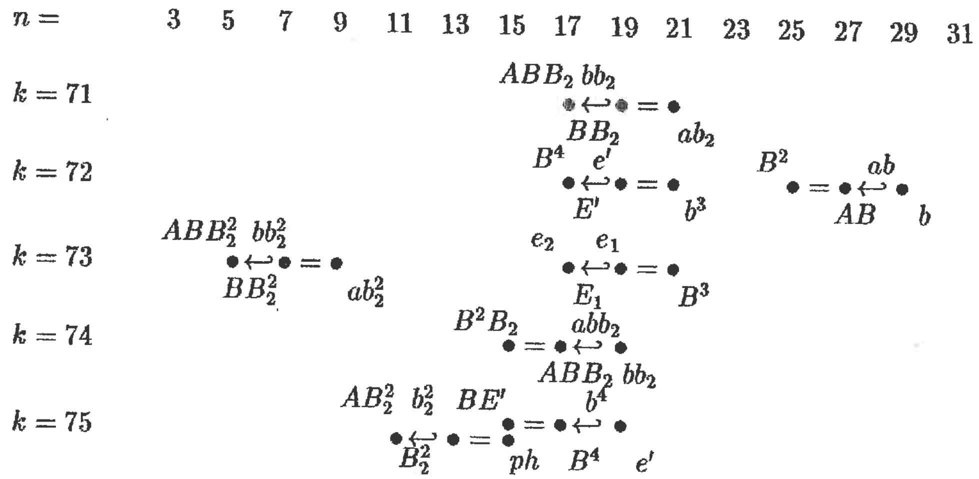
In the next theorem all invariants are stable type.

Theorem 6.3 *The mod A k-stem groups π^n_{n+k} (n : odd) for $71 \leq k \leq 75$ are obtained as the direct sum of the following three parts.*

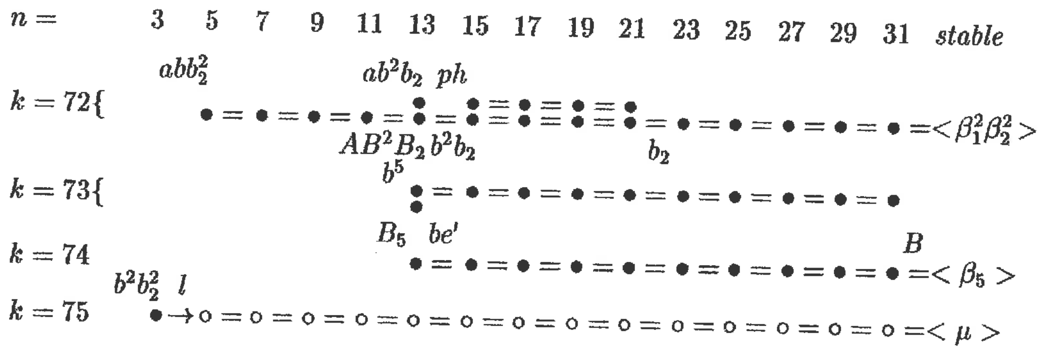
(1) *Removable simple unstable elements*

$(n,k) =$	$(9,71)$	$(15,73)$	$(27,75)$	$(27,75)$
	AB_2^2	AE_2	ab^2	AB^2
	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	B_2^2	E_2	b^2	B^2

(2) *Removable short range unstable elements*



(3) *Remaining parts*



Proof. From the results of $k = 70$, we see that the H-images of the 71-stem groups are AB_2^2, ABB_2 and bb_2. Then the stable result $\pi_{71}^S = 0$ implies that the unstable 71-stem groups are direct sum of ones in (1) and (2).

Consider the 72-stem groups. Since $H(\beta_1^2\beta_2^2) = abb_2^2$, the series $\{\beta_1^2\beta_2^2(2m+1), m > 1\}$ can be removed. $P(ph) = 0$ since $\pi_{13+71}^{13} = 0$. Then only possible element which cancells with ph is b_2. Thus the 72-stem groups are fixed.

The 73-stem, 74-stem and 75-stem groups are fixed without difficulty. □

Theorem 6.4 *The mod A k-stem groups* π_{n+k}^n *(n : odd) for* $76 \leq k \leq 79$ *have a direct summands given by the direct sum of the following three parts.*

(1) *Removable simple unstable elements*

$(n,k)=$	(15,76)	(33,76)	(15,77)	(15,77)	(33,77)	(15,78)	(21,78)	(3,79)	(21,79)	(21,79)
	ab^2b_2	ab	b^5	AB^2B_2	AB	B^5	abb_2	$AB^2B_2^2$	ABB_2	b^4
	•	•	•	•	•	•	•	•	•	•
	b^2b_2	b	be'	B^2B_2	B	BE'	bb_2	$B^2B_2^2$	BB_2	e'

(2) *Removable short range unstable elements*

$n=$	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33
$k=76$										BB_2 • =	• AB_2 ↩	ab_2 • b_2				
$k=77$								AE_2 • E_2 ↩	e_2 • E_1	E' • =	• B^3 ↩	b^3 • B_2				
$k=78$				BB_2^2 • =	• AB_2^2 ↩	ab_2^2 • b_2^2										
$k=79$														AB^2 • B^2 ↩	b^2 • =	• ab

(3) *Removable long range unstable elements*

$n=$	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	stable
$k=78$	m • =	• =	• =	• =	• =	• =	• =	• =	• =	• =	• =	• =	• =	• =	• =	$<\beta_2^3>$
$k=79$			l • =	• =	• =	• B_2^2										

Proof. In $k=78$, $\beta_2^3(5)=\beta_2^2(5)\circ\beta_2(57)$ exists and $H(\beta_2^3(5))=0$. Then the 78-stem groups are fixed, and $P(l)=0$. Since $\pi_{79}^S=0$, l has to be cancelled with B_2^2. The others are easily obtained. □

Now, the following groups $*$ and invariants remain to be fixed.

$n=$	3	5	7	9	11	13	15	17	19	21	23	25	27
	$B^2B_2^2$	L, U_3'	abb_2^2, u_2	u_1									
$k=76$	*	*	*	*	*	*	*	*	*				
		U_3	U_2	U_1	w				e_1				
		ABB_2^2	b_2^2										
$k=77$		*	*	*	*	*	*	*	*	*	*	*	*
													ab^2

where not stable type invariants are given as follows :

$$J(U_3') = P(e_2), \quad J(U_3) = P(E_2), \quad , J(U_2) = v_2(19), \quad J(U_1) = v_1(25),$$

and $$u_2 = I(v_2(17)), \quad u_1 = I(v_1(23)), \quad w = I(P(i)).$$

First we have $HP(w) = u_1$ by Lemma 3.13.

We consider two cases : Case (I) $P(U_3) = 0$ and Case (II) $P(U_3) \neq 0$.

In the first case, there exists an element ξ satisfying $H\xi = U_3$ which must be cancelled by the invariant ab^2. Then the groups $*$ in the 77-stem groups are all isomorphic to $\mathbf{Z}/3$ and connected by isomorphisms E^2. It follows that the invariants $U_3, U_2, U_1, w, ABB_2^2, b_2^2$ are mapped injectively by P and there exists an element η' with $H(\eta') = B^2B_2^2$. Applying Lemmas 3.7 and 3.9 to the relation $\beta_1^2\beta_2^2 = \{\alpha_1, 3, \lambda\}$, we have the existence of η satisfying $p\eta = E^2(\eta') = P(ABB_2^2) \neq 0$ up to sign.

In the second case, Lemma 3.9 shows that $P(ABB_2^2) = 0$ and the existence of an element η with $H(\eta) = ABB_2^2$ whcih must be cancelled by ab^2.

By the exactness of EHP-sequence we have the following posibilities.

Proposition 6.3 *We have the following variations.*

$n =$	3	5	7	9	11	13	15	17	19	21	23	25	27
Case (I)													
$k = 76\{$	● →	○ ● →	● ◇ }⇒	● ● =	● =	● =	● =	● =	●				
$k = 77$		● =	● =	● =	● =	● =	● =	● =	● =	● =	● =	● =	●
Case (II)													
$k = 76\{$	●	● ● =	● ◇ }⇒	● ● =	● =	● =	● =	● =	●				
$k = 77$			● =	● =	● =	● =	● =	● =	● =	● =	● =	● =	●

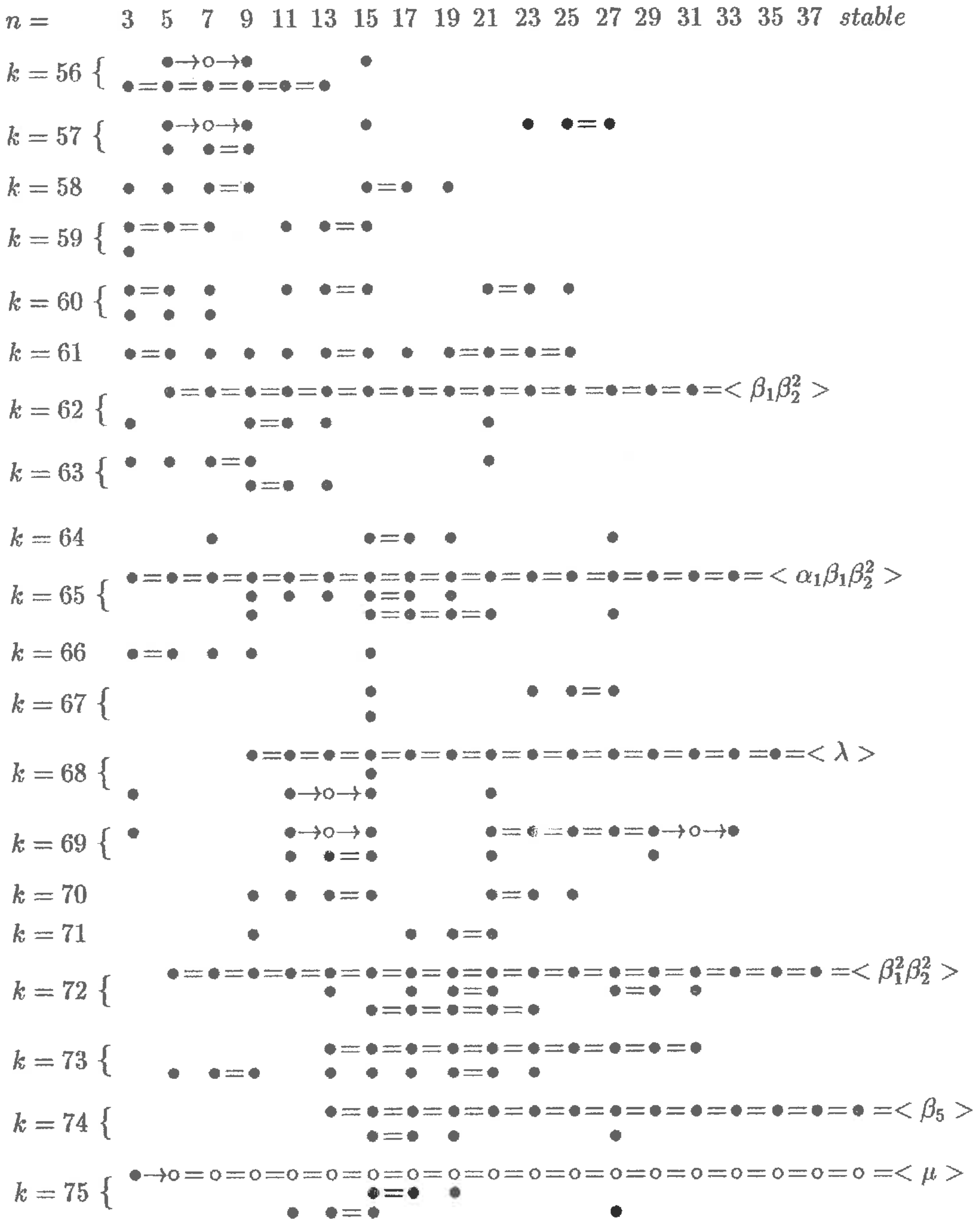
Here $\Rightarrow$ means that the image of $E^2 : \pi^7_{7+76} \to \pi^9_{9+76}$ is the upper factor, and $\diamond$ is a group of 9 elements.

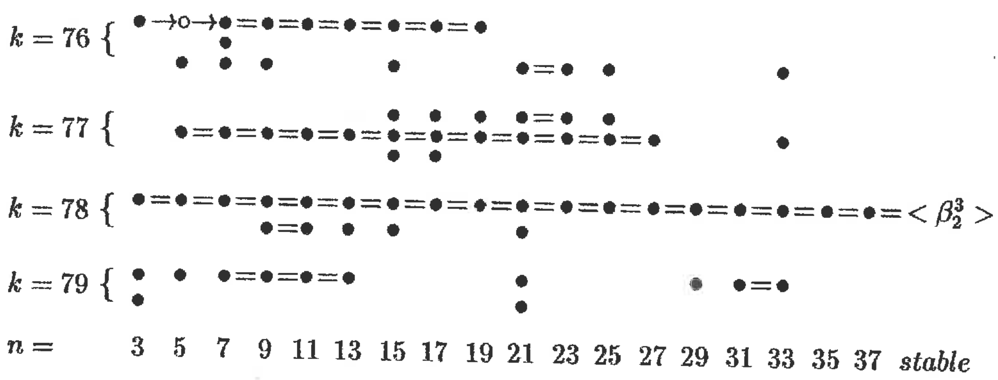
Finally, we remark the above result implies that $H(\pi^3_{3+79})$ has two generators, stable and non-stable type. Also the groups Q^3_{3+79} has similar two generatiors. Thus we have.

(6.4.1) $$\pi^3_{3+69} \cong \mathbf{Z}/3 \oplus \mathbf{Z}/3 \quad and \quad E^2(\pi^3_{3+79}) = 0.$$

6.5 Table of 3-primary k-stem Groups for $56 \leq k < 80$

The followings are the table of 3-primary k-stem groups mod A .





The results for $k = 76$ and $k = 77$ are not yet fixed and we show an example. The variations are seen in Proposition 6.3.

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