The cohomology of classifying spaces of H-spaces

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Introduction

It is well known that a topological group G has a classifying space B_G which is the base space of a principal fibration $\mathcal{E}_G \to B_G$ with G as its fibre, and $\mathcal{E}_G$ is contractible. In 1956, Milnor [8] gave a functorial construction of the bundle $\mathcal{E}_G$ as the join of an infinite sequence of copies of G. The successive finite joins give a filtration of $\mathcal{E}_G$ and also of B_G. Milnor observed that the exact couple obtained by taking the homology of the terms of the filtration gives a spectral sequence converging to $H(B_G)$. He described its E_1-term.

In 1959, Eilenberg and Moore constructed a new type of spectral sequence for fibrations [10]. In particular, for $\mathcal{E}_G \to B_G$, its E_2-term is $\mathrm{Tor}^{H(G)}(R,R)$ (R is the ground ring), and it converges to $H(B_G)$. Their construction uses the methods of relative homological algebra applied to the singular chains of G treated as a differential graded algebra.

In this paper we bring together the two methods. In particular we show that the E^2-term of Milnor's spectral sequence is $\mathrm{Tor}^{H(G)}(R,R)$. In fact the E^1-, E^2-terms are precisely the stages of the bar resolution over $H(G)$. The analogous results are obtained for cohomology: the E_2-term is $\mathrm{Ext}_{H(G)}(R,R)$.

This last result is incomplete because $\mathrm{Ext}_{H(G)}(R,R)$ has a multiplication which makes it a bigraded algebra, but there is no

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natural method of defining products in the spectral sequence of a filtered space. An attempt to do this for Milnor's B_G encounters the difficulty that $B_{G \times G} \neq B_G \times B_G$.

The main effort of this paper is devoted to resolving this difficulty. We develop geometric methods which parallel those of homological algebra. We define the concept of a geometric resolution over G . Milnor's $\mathcal{E}_G$ is one such resolution. In every case it is a principal G-bundle with a contractible total space, and it possesses a natural filtration. The base space is a classifying space for G . Its filtration yields a spectral sequence whose E_2-term is $\mathrm{Ext}_{H(G)}(R,R)$. As in homological algebra, there is a comparison theorem: any two resolutions are homotopically equivalent, hence they yield isomorphic spectral sequences beyond their first terms. In particular this is true of $B_{G \times G}$ and $B_G \times B_G$. This enables us to define products in the spectral sequence of B_G having all the desired properties, in particular $E_2 \approx \mathrm{Ext}_{H(G)}(R,R)$ is an isomorphism of bigraded algebras.

The advantage of our approach is that it gives a more direct and conceptual definition of the Eilenberg-Moore spectral sequence. The importance of the latter is that it is a more efficient tool for the study of the cohomology of B_G . In a subsequent paper we will apply the method to reprove results of Cartan on the cohomology with coefficients Z_p of the Eilenberg-MacLane spaces $K(Z_{p^r},n)$.

Moore pointed out [10] that his spectral sequence gives an easy proof of the theorem of Borel which states: <u>If</u> $H(G)$ <u>is an exterior</u>

algebra with generators of odd dimensions and R is a field then the cohomology $H^*(B_G)$ is a polynomial algebra on corresponding generators of one higher dimension. Moore argues that a brief computation shows that the E_2-term, $\mathrm{Ext}_{H(G)}(R,R)$, is already just such a polynomial algebra. Then all terms of E_2 of odd total degree are zero. Hence every $d_r = 0$, so $E_2 = E_\infty$. Since E_∞ is a polynomial algebra, it is algebraically free; and therefore $H^*(B_G) \approx E_\infty$ as an algebra (because any relation in $H^*(B_G)$ of pullbacks of the generators of E_∞ must also hold on the generators themselves).

Dold and Lashof [2] generalized Milnor's construction to the case where G is an associative H-space with a 2-sided unit. Our work is carried through with the same generality. In this way our results apply to loop spaces.

In dealing with non-compact G's, difficulties arise of a point-set nature which, in other contexts, are surmounted by assuming that everything in sight is a CW-complex. This will not do for us because we must deal with decomposition spaces. We avoid the difficulties by working in the category of compactly generated spaces, and most subspaces are required to be neighborhood deformation retracts. One effect of this is that our topology for $\mathcal{E}_G$ may not be the same as that of Milnor or Dold-Lashof. However the compact subsets are the same, hence the singular complex and the homology are unchanged.

The paper is arranged so that the development of ideas is not interrupted and delayed by lengthy proofs of propositions about standard concepts. The propositions are proved in the latter half of the paper beginning with §12. The final section shows how our machinery may be used to define the products in the Leray-Serre spectral sequence of a fibration whose base is a complex.

1. Compactly generated spaces, filtered spaces, and NDRs

1.1. Definition. A Hausdorff space is said to be compactly generated if each subset which intersects every compact set in a closed set is a closed set.

In the book of Kelley [6; pp. 230, 240], such a space is said to have the k-toplogy. He shows that these spaces include all locally compact spaces and all spaces satisfying the first axiom of countability (e.g. metric spaces). In the work of Weingram [12], the category of compactly generated spaces is shown to be convenient and useful. As we will restrict ourselves to this category in all subsequent sections, we shall review here some of its properties.

1.2. If a function $f: X \longrightarrow Y$ is continuous on each compact set in X, and X is compactly generated, then f is continuous.

For, if B is closed in Y, and C is compact in X, then $(f^{-1}B) \cap C$ is closed because $f|C$ is continuous; hence $f^{-1}B$ is closed because X is compactly generated.

1.3. If X is compactly generated and A is closed in X, then A, in its relative topology, is compactly generated.

The proof is straightforward.

1.4. Definition. Let F be a family of closed sets of a Hausdorff space X. We say that X has the topology of the union of F if X is the union of F, and each subset of X which intersects every set of F in a closed set is a closed set. (It is sometimes said that X has the weak topology determined by F.)

In case F is the family of all compact subsets, then X is compactly generated. Even more, if we assume only that each set of F is compactly generated (in its relative topology), then X is compactly generated.

1.5. <u>Definition.</u> A filtered space X consists of a space X and a sequence of closed subsets $X_o \subset X_1 \subset \ldots \subset X_n \subset \ldots$ such that $X = \bigcup_o^\infty X_n$. If Y is also a filtered space, a mapping $f: X \longrightarrow Y$ of filtered spaces is a mapping of spaces such that $fX_n \subset Y_n$ for all n.

1.6. Assume that $X = \bigcup_o^\infty X_n$ is a filtered space, that it has the topology of the union, and is compactly generated, then each compact subset of X lies in some X_n.

For, if a set A in X is not in any X, we may form a set S consisting of one point from each $A \cap (X - X_n)$. Any subset of S meets each X_n in a finite set, and therefore it is closed in X. Since $A \supset S$, and S is infinite, closed and discrete in X, it follows that A is not compact.

1.7. <u>Definition.</u> If X is a Hausdorff space, we obtain an <u>associated topology</u> X' on X by defining a subset to be closed if it meets each compact set of X in a closed set. Then X' is compactly generated and the identity mapping $X' \longrightarrow X$ is continuous. It is not hard to show that X' and X have the same compact sets.

1.8. <u>Definition.</u> If X and Y are compactly generated, the customary topology for $X \times Y$ may fail to be compactly generated, in which case $X \times Y$ will designate the associated topology which is compactly generated.

This product satisfies the axioms for a product in the category of compactly generated spaces. The crucial axiom is that a mapping $A \longrightarrow X \times Y$ is continuous if its projections are continuous. By 1.2, continuity depends only on the behavior on compact sets, but on these nothing has been changed.

It is notable that this change in the definition of product space affects all concepts based on products such as topological group, transformation group, and fibre bundle. A more detailed discussion of these questions is given by Weingram [12]. It should be kept in mind however that the product is as usual when X, Y are locally compact or metrizable. Even more we show in 12.3 that the product is as usual if X is locally compact and Y is compactly generated. In particular the concept of homotopy remains unchanged.

1.9. Definition. If X and Y are filtered spaces, then $X \times Y$ denotes the filtered space where

$$(X \times Y)_n = \bigcup_{i=0}^{n} X_i \times Y_{n-i} .$$

The interval $I = [0,1]$ is filtered by I_0 = the end points and $I_1 = I$. If $f_0, f_1 \colon X \longrightarrow Y$ are maps of filtered spaces, then a homotopy $f_0 \simeq f_1$ is a mapping of filtered spaces $F \colon I \times X \longrightarrow Y$ such that $F(0,x) = f_0 x$ and $F(1,x) = f_1 x$. Thus F is a homotopy of maps of spaces such that $F(I \times X_n) \subset Y_{n+1}$ for each n.

1.10. Definition. A closed subspace A of a space X is called a neighborhood deformation retract in X (briefly, an NDR in X) if there is a mapping $u \colon X \longrightarrow I$ and a homotopy $h \colon I \times X \longrightarrow X$ such that $A = u^{-1}(0)$, $h(0,x) = x$ for all $x \in X$, $h(t,x) = x$ for all $(t,x) \in I \times A$, and $h(1,x) \in A$ for all x such that $ux < 1$. The pair (X,A) is called an NDR pair.

The existence of u implies that A is a closed G_δ in X. If U is the open set where $ux < 1$, the mapping $g\colon U \longrightarrow A$ defined by $gx = h(1,x)$ retracts U into A; so A is a neighborhood retract in X. This in turn implies that (X,A) has the homotopy extension property.

Note that $(X,\emptyset)$ is an NDR pair for any X because we may set $ux = 1$ and $h(t,x) = x$ for all $x \in X$, $t \in I$. Similarly (X,X) is an NDR pair by setting $ux = 0$ and $h(t,x) = x$ for all $x \in X$, $t \in I$.

A characterization of an NDR in terms of the homotopy extension property has been given by G. S. Young [13].

2. Geometric resolutions

Henceforth G will denote an associative H-space with a two-sided unit e. A <u>right action</u> of G in a space X is a mapping $X \times G \longrightarrow X$ such that $xe = x$ for all x in X, and $x(g_1 g_2) = (xg_1)g_2$ for all x in X, and g_1, g_2 in G.

2.1. <u>Definition.</u> A resolution $\mathcal{E}$ of a point over G (briefly, a G-resolution) consists of a right action of G in a space $|\mathcal{E}|$, a filtration of $|\mathcal{E}|$ by closed G-invariant subspaces $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n \subset \cdots$, and of closed subsets $D_n \subset \mathcal{E}_n$, for each n, such that

(i) $\mathcal{E}_0$ is not empty, $|\mathcal{E}| = \cup_0^\infty \mathcal{E}_n$ and has the topology of the union,

(ii) for each n, $\mathcal{E}_n$ is contractible to a point in $\mathcal{E}_{n+1}$,

(iii) for each n, $\mathcal{E}_{n-1} \subset D_n$ and the action mapping $\psi_n\colon \mathcal{E}_n \times G \longrightarrow \mathcal{E}_n$ restricts to a relative homeomorphism

$$\varphi_n\colon (D_n, \mathcal{E}_{n-1}) \times G \longrightarrow (\mathcal{E}_n, \mathcal{E}_{n-1}),$$

(when $n = 0$, φ_0 is a homeomorphism $D_0 \times G \longrightarrow \mathcal{E}_0$),

(iv) for each n, there are mappings $u_n\colon D_n \longrightarrow I$ and $h_n\colon I \times D_n \longrightarrow D_n$ representing $\mathcal{E}_{n-1}$ as an NDR in D_n, and they are such that the unique functions $\bar{u}_n, \bar{h}_n$ which they induce via φ_n (see the commutative diagrams below) are continuous.

$$\begin{array}{ccc} D_n \times G & \xrightarrow{\text{proj.}} & D_n \\ \downarrow \varphi_n & & \downarrow u_n \\ \mathcal{E}_n & \xrightarrow{\bar{u}_n} & I \end{array} \qquad \begin{array}{ccc} I \times D_n \times G & \xrightarrow{h_n \times 1} & D_n \times G \\ \downarrow 1 \times \varphi_n & & \downarrow \varphi_n \\ I \times \mathcal{E}_n & \xrightarrow{\bar{h}_n} & \mathcal{E}_n \end{array}$$

If conditions iii and iv are omitted, $\mathcal{E}$ is called an <u>acyclic, filtered G-space.</u> If ii alone is omitted, $\mathcal{E}$ is called a <u>free, filtered G-space.</u>

Condition iv implies that $\mathcal{E}_{n-1}$ is an NDR in $\mathcal{E}_n$. Moreover $\bar{h}_n(t,xg) = \bar{h}_n(t,x)g$ for all $t \in I$, $x \in \mathcal{E}_n$, and $g \in G$. This can be restated: the homotopy $\bar{h}_n$ commutes with the action of G. Similarly $\bar{u}_n(xg) = \bar{u}_n x$ for all (x,g).

In case G is compact, there is no essential loss in restricting ourselves to resolutions in which each $\mathcal{E}_n$ is compact. In this case, $\mathcal{E}_n$ has the topology of the decomposition space of $D_n \times G$. It follows that any representation u_n, h_n of $(D_n, \mathcal{E}_{n-1})$ as an NDR pair will induce <u>continuous</u> $\bar{u}_n, \bar{h}_n$. Under these circumstances condition (iv) may be relaxed to: $\mathcal{E}_{n-1}$ is an NDR in D_n.

In interpreting condition iii, it should be noted that the open subsets $\mathcal{E}_n - \mathcal{E}_{n-1}$ of $\mathcal{E}_n$ and $(D_n - \mathcal{E}_{n-1}) \times G$ of $D_n \times G$ have relative topologies which are compactly generated. To see this, let x be a point of $\mathcal{E}_n - \mathcal{E}_{n-1}$. Choose a number ε so that $0 < \varepsilon < \bar{u}_n x$. It follows that the neighborhood $\bar{u}_n^{-1}(\varepsilon, 1]$ of x is such that its closure in $\mathcal{E}_n$ lies in $\mathcal{E}_n - \mathcal{E}_{n-1}$. The conclusion follows now from (2.1.

Since G may not be a group, the orbit of a point of $\mathcal{E}$ may not be a copy of G. However each point lies in a maximal orbit which is a copy of G because $\mathcal{E}$ is the union of the sets $\varphi_n(D_n - \mathcal{E}_{n-1}) \times G)$. These maximal orbits are closed sets.

2.2. <u>Definition.</u> The <u>base space</u> B of a G-resolution $\mathcal{E}$ is the decomposition space of $\mathcal{E}$ by its maximal G-orbits: $B = \mathcal{E}/G$. The projection $p: \mathcal{E} \longrightarrow B$ assigns to each x the maximal orbit containing x. We set $B_n = p\,\mathcal{E}_n$. The base space together with its filtration $\{B_n\}$ is called a <u>classifying space</u> for G.

Since $\mathcal{E}$ is compactly generated and B is a decomposition space of $\mathcal{E}$, it follows by 12.2 that B is compactly generated.

2.3. <u>Lemma.</u> Each B_n is closed in B. B_o is not empty, and p restricts to a homeomorphism $D_o \longrightarrow B_o$. B has the topology of the union $\bigcup_o^\infty B_n$. For each n, the projection p restricts to a relative homeomorphism $(D_n, \mathcal{E}_{n-1}) \longrightarrow (B_n, B_{n-1})$, and B_{n-1} is an NDR in B_n.

<u>Proof.</u> Since $p^{-1}B_n = \mathcal{E}_n$ is closed in $\mathcal{E}$, it follows that B_n is closed in B. Under the homeomorphism $\varphi_o: D_o \times G \longrightarrow \mathcal{E}_o$, $p|\mathcal{E}_o$ corresponds to the projection of $D_o \times G$ into D_o, so D_o is homeomorphic to B_o. If a set A in B meets each B_n in a closed set, then $(p^{-1}A) \cap \mathcal{E}_n = p^{-1}(A \cap B_n)$ is closed. Since $\mathcal{E}$ has the topology of the union, $p^{-1}A$ is closed in $\mathcal{E}$; hence A is closed in B, and B has the topology of the union.

The restriction p' of p to $(D_n, \mathcal{E}_{n-1})$ is obviously continuous and one-to-one from $D_n - \mathcal{E}_{n-1}$ to $B_n - B_{n-1}$. If A is closed in $D_n - \mathcal{E}_{n-1}$, then iii implies that $p^{-1}p'A = \varphi_n(A \times G)$ is closed in $\mathcal{E}_n - \mathcal{E}_{n-1}$; hence $p'A$ is closed in $B_n - B_{n-1}$. This proves the continuity of the inverse.

Since $\bar{u}_n$ and $\bar{h}_n$ commute with the action of G in $\mathcal{E}_n$, they induce functions $v_n: B_n \longrightarrow I$ and $k_n: I \times B_n \longrightarrow B_n$. Since $\bar{u}_n$ is continuous and B_n has the decomposition space topology, it follows that v_n is continuous. By 12.5, $I \times B_n$ has the decomposition space topology of $I \times \mathcal{E}_n$ under $1 \times p$; hence k_n is continuous. We omit the routine verification that v_n, k_n represent B_{n-1} as an NDR in B_n.

2.4. <u>Lemma.</u> If $\mathcal{E}$ is a G-resolution, then there is a homotopy $F: I \times \mathcal{E} \longrightarrow \mathcal{E}$ contracting $\mathcal{E}$ to a point e_o in $\mathcal{E}_o$ such that $F(I \times \mathcal{E}_n) \subset \mathcal{E}_{n+1}$ for each n.

<u>Proof.</u> By ii, there is a homotopy $F_o: I \times \mathcal{E}_o \longrightarrow \mathcal{E}_1$ contracting $\mathcal{E}_o$ to e_o. Let us make the inductive assumption that we have homotopies $F_i: I \times \mathcal{E}_i \longrightarrow \mathcal{E}_{i+1}$ for $0 \leqq i < n$ such that, for each i, F_i extends F_{i-1}, and F_i contracts $\mathcal{E}_i$ to e_o. To construct F_n, we note that it is already required to be F_{n-1} on $I \times \mathcal{E}_{n-1}$, and is specified on $0 \times \mathcal{E}_n$ by $F_n(0,x) = x$, and on $1 \times \mathcal{E}_n$ by $F_n(1,x) = e_o$. This mapping

$$h: (0 \times \mathcal{E}_n) \cup (I \times \mathcal{E}_{n-1}) \cup (1 \times \mathcal{E}_n) \longrightarrow \mathcal{E}_n$$

is homotopic in $\mathcal{E}_{n+1}$ to a constant by ii. As $\mathcal{E}_{n-1}$ is an NDR in

$\mathcal{E}_n$, and the endpoints $I_0 = \{0,1\}$ of I form an NDR in I, lemma 13.2 states that $(0 \times \mathcal{E}_n) \cup (I \times \mathcal{E}_{n-1}) \cup (1 \times \mathcal{E}_n)$ is an NDR in $I \times \mathcal{E}_n$. Hence this pair has the homotopy extension property. Since h is homotopic in $\mathcal{E}_{n+1}$ to an extendable map (a constant), it follows that h can be extended continuously to a mapping $F_n: I \times \mathcal{E}_n \longrightarrow \mathcal{E}_{n+1}$. Define $F: I \times \mathcal{E} \longrightarrow \mathcal{E}$ to be the union of the mappings $\{F_n\}$. By 12.4, $I \times \mathcal{E}$ has the topology of the union $\cup_0^\infty I \times \mathcal{E}_n$. This and the continuity of each F_n imply the continuity of F.

3. Milnor's resolution

In case G is a topological group, Milnor [8] defined $\mathcal{E}_n$ to be the join of n copies of G, and he obtained $\mathcal{E}$ as the union $\cup_0^\infty \mathcal{E}_n$ under natural imbeddings $\mathcal{E}_{n-1} \subset \mathcal{E}_n$. Dold and Lashof [2] gave a modification of the Milnor construction, suitable when G is an associative H-space. We will use the Dold-Lashof version modified only by the requirement that all topologies are compactly generated. When G is a compact group, all three definitions coincide.

We define a sequence of spaces $\mathcal{E}_n$ and right actions $\psi_n: \mathcal{E}_n \times G \longrightarrow \mathcal{E}_n$ inductively as follows. Set $\mathcal{E}_0 = G$, and let ψ_0 be the multiplication in G. Assuming $\mathcal{E}_{n-1}$ and ψ_{n-1} are defined, let D_n be the cone on $\mathcal{E}_{n-1}$. Define $\mathcal{E}_n$ to be the space obtained by adjoining $D_n \times G$ to $\mathcal{E}_{n-1}$ by the action mapping ψ_{n-1}. Thus $\mathcal{E}_n$ is the decomposition space of $D_n \times G$ obtained by collapsing $\mathcal{E}_{n-1} \times G$ under ψ_{n-1} into $\mathcal{E}_{n-1}$. Now $D_n \times G$ has the natural right action $1 \times \psi_0: D_n \times G \times G \longrightarrow D_n \times G$. This action commutes with the identifications ψ_{n-1} because we are assuming the associativity of the action in $\mathcal{E}_{n-1}$. Therefore the action in $D_n \times G$ induces an action ψ_n in $\mathcal{E}_n$ which extends ψ_{n-1}. The associativity of $1 \times \psi_0$ implies that of ψ_n. This completes the inductive step.

We set $\mathcal{E} = \cup_0^\infty \mathcal{E}_n$, we give it the topology of the union, and define $\psi: \mathcal{E} \times G \longrightarrow \mathcal{E}$ by $\psi | \mathcal{E}_n \times G = \psi_n$. Now $\mathcal{E} \times G$ has the topology of the union $\cup_0^\infty(\mathcal{E}_n \times G)$ by 12.4. Since each ψ_n is continuous, it follows from 1.2 that ψ is continuous.

3.1. <u>Lemma.</u> The preceding construction gives a G-resolution. It is a covariant functor of G.

<u>Proof.</u> Condition i of 2.1 follows by definition. Since D_{n+1} is the cone on $\mathcal{E}_n$, $\mathcal{E}_n$ is contractible to a point in D_{n+1}; hence ii holds. Since $\mathcal{E}_n$ is defined to be the adjunction space $\mathcal{E}_{n-1} \cup_\psi (D_n \times G)$, it follows directly that $D_n \times G - \mathcal{E}_{n-1} \times G$ maps topologically onto $\mathcal{E}_n - \mathcal{E}_{n-1}$.

To prove iv, start with a pair $u: I \longrightarrow I$ and $h: I \times I \longrightarrow I$ which represents the endpoint 0 of I as an NDR in I. We can suppose moreover that $u(1) = 1$, and $h(t,1) = 1$ for all t. (For example, we could take $ux = 2x$ for $x \in [0,1/2]$, $ux = 1$ for $x \in [1/2,1]$, and h would contract $[0,1/2]$ to 0 and expand $[1/2,1]$ to $[0,1]$.) Then $0 \times \mathcal{E}_{n-1}$ is an NDR in $I \times \mathcal{E}_{n-1}$ where $u'(\tau,x) = u(\tau)$ and $h'(t,\tau,x) = h(t,\tau)$. Collapsing $1 \times \mathcal{E}_{n-1}$ to a point, we obtain D_n as the decomposition space of $I \times \mathcal{E}_{n-1}$. The mappings u', h induce mappings u_n, h_n which represent $\mathcal{E}_{n-1}$ $(= 0 \times \mathcal{E}_{n-1})$ as an NDR in D_n. These induced mappings are automatically continuous because D_n and $I \times D_n$ are decomposition spaces of $I \times \mathcal{E}_{n-1}$ and $I \times I \times \mathcal{E}_{n-1}$, respectively, (see 12.5). Moreover, for the same reason, the mappings $\bar{u}_n, \bar{h}_n$ induced by u_n, h_n are continuous. This completes the proof.

4. The spectral sequence of a filtered space

Applying singular homology or cohomology to the filtration $\{X_n\}$ of a space X yields an exact couple and an associated spectral sequence [5; p. 232]. To fix the notation, using homology with coefficients in a ring R, set

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}), \qquad A^1_{p,q} = H_{p+q}(X_p).$$

Then the morphisms of the homology sequences of the pairs (X_p, X_{p-1}) for all p, define an exact couple

$$\begin{array}{ccc} A^1 & \xrightarrow{\ i\ } & A^1 \\ {\scriptstyle\partial}\nwarrow & & \swarrow{\scriptstyle j} \\ & E^1 & \end{array} \qquad (4.1)$$

of bigraded modules. We denote it and its successive derived couples by $AE^r(X)$, $r = 1,2,\ldots$. The sequence $\{E^r\}$ and derivations $d_r = j_r\partial_r$ form the spectral sequence of the filtered space.

Since $H_i = 0$ for $i < 0$, we have $E^1_{p,q} = 0 = A^1_{p,q}$ for $p+q < 0$. Moreover if $p < 0$, we have $X_p = \emptyset$, so $A^1_{p,q} = 0 = E^1_{p,q}$. An $x \in E^1_{p,q}$ is a d_r-cycle if $\partial x = i^r y$ for $y \in A^1_{p-r,q+r-1}$. ? Taking $r > p$, we obtain $\partial x = 0$, whence $x \in \operatorname{Im} j$. Thus $\operatorname{Im} j$ consists of the d_r-cycles for all r. A d_r-boundary is an element of the form ju such that $i^r u = 0$, i.e. $i^{r-1}u = \partial z$. Define $A^\infty_{p,q}$ to be the quotient of $A^1_{p,q}$ by elements u such that $i^r u = 0$ for some $r > 0$. It follows now that we have, for each s, the diagram

$$\begin{array}{ccccccccccc}
0 & \xrightarrow{i} & A^\infty_{o,s} & \xrightarrow{i} & A^\infty_{1,s-1} & \xrightarrow{i} & \cdots & \xrightarrow{i} & A^\infty_{p,s-p} & \xrightarrow{i} \cdots \longrightarrow & H_s(X) \\
 & & \downarrow j & & \downarrow j & & & & \downarrow j & & \\
 & & E^\infty_{o,s} & & E^\infty_{1,s-1} & & & & E^\infty_{p,s-p} & &
\end{array}$$

where each i and j is induced by the i and j of the initial couple. Moreover it is readily verified that

4.3. In 4.2, each i is a monomorphism, each j is an epimorphism, and $\operatorname{Im} i = \operatorname{Ker} j$.

It follows that 4.2 is a composition series for the direct limit (or union) of the A's .

4.4. <u>Theorem.</u> If X has the topology of the union $\bigcup_o^\infty X_p$, then $H_s(X)$ is the direct limit of 4.2, hence the E^∞-term of the spectral sequence is the bigraded module associated with the filtration of $H(X)$ by the images $H(X_p) \longrightarrow H(X)$ for <u>all</u> p .

The proof is given in §14.

The corresponding results for cohomology will require an additional condition. First let

(4.5) $$E_1^{p,q} = H^{p+q}(X_p, X_{p-1}) , \qquad A_1^{p,q} = H^{p+q}(X, X_{p-1}) .$$

The cohomology sequences of the triples (X, X_p, X_{p-1}) for <u>all</u> p combine to form an exact couple $AE_1(X)$.

(4.6) $$\begin{array}{ccc} A_1 & \xrightarrow{i} & A_1 \\ & {}_{\delta}\nwarrow \quad \swarrow_{j} & \\ & E_1 & \end{array}$$

It and its derived couples are denoted by $AE_r(X)$, $r \geqq 1$; and its terms $\{E_r(X), d_r\}$ form the cohomology spectral sequence of X. Again all (p,q)-terms are zero if $p+q < 0$, and $E_1^{p,q} = 0$ whenever $p < 0$. Clearly $A_1^{p,q} = H^{p+q}(X)$ for $p \leqq 0$.

Define $A_\infty^{p,q}$ to be the co-image of $i^p : H^{p+q}(X, X_{p-1}) \to H^{p+q}(X)$. If $i^p x = 0$, exactness gives a $y \in H^{p+q-1}(X_o)$ such that $\delta y = i^{p-1} x$, which implies that jx is a d_p-boundary. Therefore j induces a mapping $j : A_\infty^{p,q} \longrightarrow E_\infty^{p,q}$. This gives the diagram

$$(4.7)\qquad \begin{array}{ccccccccc} H^s(X) = A_\infty^{o,s} & \xleftarrow{i} & A_\infty^{1,s-1} & \xleftarrow{i} & \cdots & \xleftarrow{i} & A_\infty^{p,s-p} & \xleftarrow{i} & \cdots \\ \downarrow j & & \downarrow j & & & & \downarrow j & & \\ E_\infty^{o,s} & & E_\infty^{1,s-1} & & & & E_\infty^{p,s-p} & & \end{array}$$

in which it is readily proved that

4.8. Each i is a monomorphism, and $\operatorname{Im} i = \operatorname{Ker} j$.

Thus to know that 4.7 gives a decomposition series for $H^s(X)$, we need to have

(4.9) (a) each j is an epimorphism, (b) $\bigcap_{p=o}^{\infty} i^p A_\infty^{p,s-p} = 0$.

Consider the conditions

$$(4.10)\qquad \bigcap_{r=o}^{\infty} i^r H^t(X, X_{p-1+r}) = 0 \quad \text{in} \quad H^t(X, X_{p-1}).$$

For $(p,t) = (o,s)$, this condition is equivalent to 4.9b. We will show now that 4.10 for $t = s+1$ and all p implies 4.9a. Let $x \in E_\infty^{p,s-p}$. Then x is represented by $u \in H^s(X_p, X_{p-1})$ such that $\delta u \in H^{s+1}(X, X_p)$ is in the image of $H^{s+1}(X, X_{p+r})$ for every $r \geqq 0$.

By 4.10, we must have $\delta u = 0$. Hence $u = jv$ for some $v \in H^s(X, X_{p-1})$. Therefore j is an epimorphism.

4.11. Theorem. Let X have the topology of the union $\cup_0^\infty X_p$. If condition 4.10 holds for all (p,t) , then the E_∞-term of the spectral sequence is the graded module associated with the filtration of $H^*(X)$ by the images of $H^*(X, X_{p-1})$ for all p . Condition 4.10 holds for all (p,t) if the coefficient group is compact or if $E_2^{p,q} = 0$ whenever $q < 0$.

The proof is given in §14.

Remarks. In forming the exact couple for homology we could have used the homology sequences of the triples (X, X_p, X_{p-1}) instead of the pairs (X_p, X_{p-1}) so that $\bar{A}^1_{p,q} = H_{p+q}(X, X_{p-1})$. The E^1-terms remain the same. This exact couple maps into the former one under $\partial: \bar{A}^1_{p,q} \longrightarrow A^1_{p-1,q+1}$. Using this comparison, it can be shown that the E^r-terms are isomorphic for all r . Thus there is no essential difference in the two methods. An analogous remark applies to cohomology.

If we replace homology groups by homotopy groups with suitable modifications for dimensions 1 and 0 , we obtain entirely analogous results; in particular, 4.3 holds with π_i in place of H_i .

5. The E^2-term of the spectral sequence

5.1. <u>Theorem.</u> If $H(G)$ is R-free (e.g. if R is a field), the second terms of the spectral sequences of a G-resolution $\mathcal{E}$ are

$$E^2 \approx \mathrm{Tor}^{H(G)}(R,R) \quad \text{and} \quad E_2 \approx \mathrm{Ext}_{H(G)}(R,R) .$$

It should be emphasized that it is only claimed that these are isomorphisms of bigraded modules. The Tor-term is a co-algebra and the Ext-term is an algebra, but no such structures have been defined as yet in the E_2, E^2-terms. The main purpose of §§9 - 11 is to define such structures and show that they correspond under the above isomorphisms to the structures in Tor and Ext .

The proof to follow is incomplete because it uses the extra hypothesis that each $H(D_n, \mathcal{E}_{n-1})$ is R-free. If R is a field, it holds automatically. For a general R , it is true of the Milnor resolution as shown in 5.3 below. In 10. 5 , we will show that any two resolutions have isomorphic spectral sequences from the second terms on. Thus the truth of 5.1 for Milnor's resolution implies its truth in general.

<u>Proof.</u> Since $\mathcal{E}_{n-1}$ is contractible to a point in $\mathcal{E}_n$, the induced morphism of homology is zero, hence the homology sequence of $(\mathcal{E}_n, \mathcal{E}_{n-1})$ is a short exact sequence.

$$0 \longrightarrow \tilde{H}(\mathcal{E}_n) \xrightarrow{j} H(\mathcal{E}_n, \mathcal{E}_{n-1}) \xrightarrow{\partial} \tilde{H}(\mathcal{E}_{n-1}) \longrightarrow 0 .$$

This implies immediately the exactness of the sequence

$$\cdots \xrightarrow{d_1} H(\mathcal{E}_{n+1}, \mathcal{E}_n) \xrightarrow{d_1} H(\mathcal{E}_n, \mathcal{E}_{n-1}) \xrightarrow{d_1} \cdots$$

(5.2)
$$\cdots \xrightarrow{d_1} H(\mathcal{E}_1, \mathcal{E}_0) \xrightarrow{d_1} H(\mathcal{E}_0) \xrightarrow{\varepsilon} R \longrightarrow 0$$

where $d_1 = j\partial$ is the derivation in $E^1(\mathcal{E})$, and ε is induced by the mapping of $\mathcal{E}_0$ to a point.

The right action ψ of G in $\mathcal{E}$ maps each $\mathcal{E}_n$ into itself, and thereby induces a right $H(G)$-module structure in $H(\mathcal{E}_n)$ and $H(\mathcal{E}_n, \mathcal{E}_{n-1})$ using the cross-product α

$$H(\mathcal{E}_n, \mathcal{E}_{n-1}) \otimes H(G) \xrightarrow{\alpha} H((\mathcal{E}_n, \mathcal{E}_{n-1}) \times G) \xrightarrow{\psi_*} H(\mathcal{E}_n, \mathcal{E}_{n-1}) .$$

Since α is natural, it follows that the morphisms i, j of the exact couple $AE^1(\mathcal{E})$ are $H(G)$-mappings. Since the cross-product satisfies $\partial(x \times y) = \partial x \times y$ when x is a relative class and y is an absolute class, it follows that ∂ in $AE^1(\mathcal{E})$ is an $H(G)$-mapping. Thus 5.2 is an exact sequence of $H(G)$-modules.

By 2.1 iv, the mapping $\varphi_n: (D_n, \mathcal{E}_{n-1}) \times G \longrightarrow (\mathcal{E}_n, \mathcal{E}_{n-1})$ of 2.1 iii satisfies the conditions of 13.7; therefore φ_n induces isomorphisms

$$\varphi_{n*}: H(D_n, \mathcal{E}_{n-1}) \times G) \approx H(\mathcal{E}_n, \mathcal{E}_{n-1}) .$$

Define a right G-structure in $D_n \times G$ by $(x,g)g' = (x,gg')$ for all $x \in D_n$, and $g, g' \in G$. Then the associativity of G implies that $D_n \times G$ is a G-space and φ_n is a G-mapping. It follows that φ_{n*} is an isomorphism as $H(G)$-modules. Since $H(G)$ is R-free, the Künneth theorem states that the cross-product gives an isomorphism of $H(G)$-modules

$$\alpha\colon H(D_n, \mathcal{E}_{n-1}) \otimes H(G) \approx H((D_n, \mathcal{E}_{n-1}) \times G) .$$

We need here the extra hypothesis that $H(D_n, \mathcal{E}_{n-1})$ is R-free. It implies that $H(D_n, \mathcal{E}_{n-1}) \otimes H(G)$ is free as an H(G)-module. Then the isomorphism $\varphi_{n*}\alpha$ shows that $H(\mathcal{E}_n, \mathcal{E}_{n-1})$ is H(G)-free. This completes the proof that $\{E^1(\mathcal{E}), d_1\}$, i.e. the sequence 5.2, is a free resolution of R over H(G).

In the following diagram, μ is the natural

$$\begin{array}{ccccc} H(D_n, \mathcal{E}_{n-1}) & & & & H(\mathcal{E}_n, \mathcal{E}_{n-1}) \otimes_{H(G)} R \\ \downarrow h & & & \nearrow \mu & \\ H(D_n, \mathcal{E}_{n-1}) \otimes H(G) & \xrightarrow{\varphi_{n*}\alpha} & H(\mathcal{E}_n, \mathcal{E}_{n-1}) & \xrightarrow{p_*} & H(B_n, B_{n-1}) \end{array}$$

map into the quotient and h is defined by $hx = x \otimes 1$. Since $\varphi_{n*}\alpha$ is an isomorphism, it is a standard fact that $\mu\varphi_{n*}\alpha h$ is an isomorphism. Now $p_*\varphi_*\alpha h$ is also an isomorphism because $p|(D_n, \mathcal{E}_{n-1})$ is a relative homeomorphism of NDR's (see 2.3 and 13.7). It follows that μp_*^{-1} is an isomorphism. Since p_* commutes with d_1, it is an isomorphism of $\{E^1(B), d_1\}$ with the result of tensoring the resolution 5.2 with R over H(G). This proves that $E^2(B) \approx \mathrm{Tor}^{H(G)}(R,R)$.

Since $H(D_n, \mathcal{E}_{n-1}) \approx H(B_n, B_{n-1})$ the R-freeness of $H(D_n, \mathcal{E}_{n-1})$ implies that the natural mapping

$$\gamma\colon H^*(B_n, B_{n-1}) \longrightarrow \mathrm{Hom}\,(H(B_n, B_{n-1}), R)$$

is an isomorphism. In the diagram below μ is the

$$\begin{array}{ccc} \mathrm{Hom}_{H(G)}(H(\mathcal{E}_n, \mathcal{E}_{n-1}), R) & \xrightarrow{\mu} & \\ & & \mathrm{Hom}(H(\mathcal{E}_n, \mathcal{E}_{n-1}), R) \\ \mathrm{Hom}(H(B_n, B_{n-1}), R) & \xrightarrow{p^*} & \end{array}$$

inclusion. Using the isomorphism $\varphi_{n*}\mathcal{E}$, it is readily seen that p^* is an isomorphism with the image of μ. Therefore $p^*\gamma$ maps $\{E_1(B), d_1\}$ isomorphically onto the cochain complex obtained by applying $\mathrm{Hom}_{H(G)}(\ \ , R)$ to the resolution $\{E_1(\mathcal{E}), d_1\}$. This proves that $E_2(B) \approx \mathrm{Ext}_{H(G)}(R, R)$, and completes the proof of 5.1.

5.3. Theorem. In the case of Milnor's resolution $\mathcal{E}$ over G, the initial stage $\{E^1(\mathcal{E}), d_1\}$ is isomorphic to the bar resolution over the algebra $H(G)$ (see [7, p. 299]).

Before proceeding to the proof, let us recall some details of the bar resolution of R over a graded algebra A. Let $\bar{A}$ be the kernel of the augmentation $\varepsilon: A \longrightarrow R$. Then the n^{th} term X_n of the resolution is the tensor product $A \otimes \bar{A}^n = A \otimes \bar{A} \otimes \ldots \otimes \bar{A}$. A generator of X_n is written $a_o \otimes [a_1|\ldots|a_n]$, where $a_o \in A$ and $a_i \in \bar{A}$ for $i = 1, \ldots, n$. The action $\mu_n: A \otimes X_n \longrightarrow X_n$ is defined by

$$a(a_o \otimes [a_1|\ldots|a_n]) = (aa_o) \otimes [a_1|\ldots|a_n] .$$

The contracting homotopy $s_n: X_n \longrightarrow X_{n+1}$ is defined by

$$s_n(a_o \otimes [a_1|\ldots|a_n]) = \begin{cases} 0 & \text{if } a_o = 1, \\ 1 \otimes [a_o|a_1|\ldots|a_n] & \text{if } a_o \in \bar{A}. \end{cases} \tag{5.4}$$

There is a somewhat long formula for $\partial_n: X_n \longrightarrow X_{n-1}$, but all we need know is that ∂_n is an A-mapping and

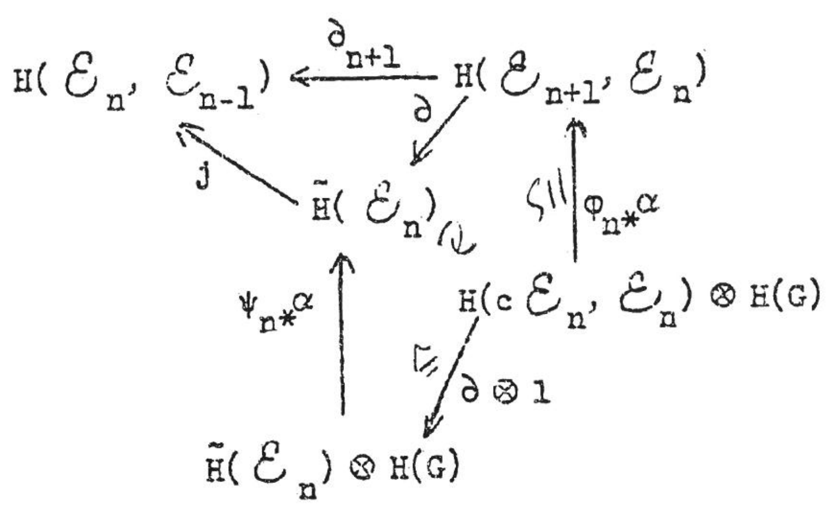
$$(5.5)\qquad s_{n-1}\partial_n + \partial_{n+1}s_n = \begin{cases} 1 & \text{for } n > 0\,, \\ 1 - \varepsilon & \text{for } n = 0\,. \end{cases}$$

5.6. <u>Lemma.</u> The bar resolution is characterized by $X_o = A$ and the inductive requirement that X_{n+1} is isomorphic to $A \otimes \operatorname{Ker} \partial_n$ as an A-module, and, under this isomorphism, ∂_{n+1} corresponds to the action $\mu_n\colon A \otimes \operatorname{Ker} \partial_n \longrightarrow \operatorname{Ker} \partial_n$ (when $n = 0$, $\operatorname{Ker} \partial_o$ is replaced by $\operatorname{Ker} \varepsilon$) .

<u>Proof.</u> If $u \in \operatorname{Ker} \partial_n$, then $\partial_n u = 0$, so 5.5 gives $\partial_{n+1}s_n u = u$. If $v \in R \otimes \bar{A}^{n+1}$, then $s_{n+1}v = 0$ by 5.4, so 5.5 gives $s_n \partial_{n+1} v = v$. Therefore s_n gives an isomorphism $\operatorname{Ker} \partial_n \approx R \otimes \bar{A}^{n+1}$ and ∂_{n+1} gives its inverse. Then $1 \otimes s_n$ gives an isomorphism of A-modules $A \otimes \operatorname{Ker} \partial_n \approx A \otimes \bar{A}^{n+1} = X_{n+1}$. If $u \in \operatorname{Ker} \partial_n$ and $a \in A$, we have
$\partial_{n+1}(1 \otimes s_n)(a \otimes u) = \partial_{n+1}(a \otimes s_n u) = a\partial_{n+1}(1 \otimes s_n u) = a\partial_{n+1}(1 \otimes s_n)u = au$.

<u>Proof of 5.3.</u> Milnor's resolution begins with $\mathcal{E}_o = G$ so $E^1_{o,q} = H_q(G)$. We have seen in the proof of 5.1 that $\tilde{H}(\mathcal{E}_n)$ maps isomorphically onto the kernel of $\partial_n = d_1$ (see 5.2). In the diagram 5.7, $\psi_{n*}\alpha$

(5.7)



gives the structure of $\tilde{H}(\mathcal{E}_n)$ as a right $H(G)$-module. Since D_{n+1} is the cone $c\,\mathcal{E}_n$, $\varphi_{n*}\alpha$ is the isomorphism occurring in the proof of 5.1. As observed there, it is an isomorphism of right $H(G)$-modules. Since $c\,\mathcal{E}_n$ is contractible, $\partial\colon H(c\,\mathcal{E}_n, \mathcal{E}_n) \approx \tilde{H}(\mathcal{E}_n)$, hence $\partial \otimes 1$ is an isomorphism of right $H(G)$-modules. The composite isomorphism $\varphi_{n*}\alpha(\partial \otimes 1)^{-1}$ represents $H(\mathcal{E}_{n+1}, \mathcal{E}_n)$ in the form $\operatorname{Ker} \partial_n \otimes H(G)$. The diagram is commutative because of the property of cross-products $\partial(u \times v) = \partial u \times v$. This shows that, under the isomorphism, ∂ corresponds to the action $\psi_{n*}\alpha$. Therefore $\{E^1(\mathcal{E}), d_1\}$ satisfies the conditions of 5.6; and this proves 5.3.

6. Products of resolutions

6.1. <u>Theorem.</u> If $\mathcal{E}$ and $\mathcal{E}'$ are resolutions over G and G', respectively, then the natural action of $G \times G'$ in $|\mathcal{E}|\times|\mathcal{E}'|$ and the subspaces

$$(\mathcal{E} \times \mathcal{E}')_n = \cup_{i=0}^{n} \mathcal{E}_i \times \mathcal{E}'_{n-i}$$

$$\mathcal{B}_n = (\mathcal{E} \times \mathcal{E}')_{n-1} \cup \cup_{i=0}^{n} D_i \times D'_{n-i}$$

constitute a $G \times G'$-resolution which we denote by $\mathcal{E} \times \mathcal{E}'$ and call the product resolution. Moreover, by 12.5, the base space of $\mathcal{E} \times \mathcal{E}'$ is the product $B \times B'$ of the base spaces.

<u>Proof.</u> Condition i of 2.1 follows from 12.4. To prove ii, choose contractions $F: I \times \mathcal{E} \to \mathcal{E}$ and $F': I \times \mathcal{E}' \to \mathcal{E}'$ as given by 2.4. Form the homotopy contracting $\mathcal{E} \times \mathcal{E}'$ to $e_o \times e'_o$ by, first, applying F to the $\mathcal{E}$-coordinate, keeping the $\mathcal{E}'$-coordinate fixed, and then applying F' to the $\mathcal{E}'$-coordinate, keeping the $\mathcal{E}$-coordinate fixed. This homotopy contracts $\mathcal{E}_i \times \mathcal{E}'_j$ to a point in the union of $\mathcal{E}_{i+1} \times \mathcal{E}'_j$ and $e_o \times \mathcal{E}'_{j+1}$. Hence it contracts $(\mathcal{E} \times \mathcal{E}')_n$ to a point in $(\mathcal{E} \times \mathcal{E}')_{n+1}$. This proves ii.

To prove iii, we observe that $\mathcal{B}_n - (\mathcal{E} \times \mathcal{E}')_{n-1}$ is the union of the disjoint open subsets

$$\mathcal{B}_n - (\mathcal{E} \times \mathcal{E}')_{n-1} = \cup_{i=0}^{n} (D_i - \mathcal{E}_{i-1}) \times (D'_{n-1} - \mathcal{E}'_{n-i-1}) .$$

Likewise

$$(\mathcal{E} \times \mathcal{E}')_n - (\mathcal{E} \times \mathcal{E}')_{n-1} = \cup_{i=0}^{n} (\mathcal{E}_i - \mathcal{E}_{i-1}) \times (\mathcal{E}'_{n-i} - \mathcal{E}'_{n-i-1}) .$$

The first line when multiplied by $G \times G'$ is carried into the second line by the action mapping. Since a product of homeomorphisms is a homeomorphism, and a disjoint union of homeomorphisms is a homeomorphism, this mapping is a homeomorphism. Hence iii holds.

To prove iv , we apply 13.1 to obtain that each pair $(D_i, \mathcal{E}_{i-1}) \times (D'_{n-i}, \mathcal{E}'_{n-i-1})$ is an NDR. By successive applications of 13.2, we piece these pairs together, and obtain finally that $(\mathcal{B}_n, (\mathcal{E} \times \mathcal{E}')_{n-1})$ is an NDR. We may perform the same argument on the pairs $(\mathcal{E}_i, \mathcal{E}_{i-1}) \times (\mathcal{E}'_{n-i}, \mathcal{E}'_{n-i-1})$ and their union $((\mathcal{E} \times \mathcal{E}')_n, (\mathcal{E} \times \mathcal{E}')_{n-1})$ taking care at each stage to use the representations as NDR's induced by the corresponding pairs above. Then iv must hold.

7. Mappings of resolutions

7.1. <u>Definition.</u> Let G, G' be associative H-spaces with units, and let $f\colon G \longrightarrow G'$ be a morphism (i.e. f is continuous, $fe = e'$, and $f(g_1 g_2) = (fg_1)(fg_2)$ for all $g_1, g_2 \in G$). Let $\mathcal{E} = \{\mathcal{E}_n\}$ be a filtered G-space, and $\mathcal{E}' = \{\mathcal{E}'_n\}$ a filtered G'-space. Then an f-mapping $\tilde{f}\colon \mathcal{E} \longrightarrow \mathcal{E}'$ is a mapping of filtered spaces (see 1.5) such that $\tilde{f}(xg) = (\tilde{f}x)(fg)$ for all $x \in \mathcal{E}$ and $g \in G$. If $\tilde{f}_0, \tilde{f}_1$ are two f-mappings $\mathcal{E} \longrightarrow \mathcal{E}'$, then an f-homotopy of $\tilde{f}_0$ into $\tilde{f}_1$ is a homotopy $F\colon I \times \mathcal{E} \longrightarrow \mathcal{E}'$ of $\tilde{f}_0$ into $\tilde{f}_1$ such that $F(I \times \mathcal{E}_n) \subset \mathcal{E}_{n+1}$ for each n, and

$$F(t, xg) = F(t,x)(fg) \quad \text{for all} \quad (t,x,g) \in I \times \mathcal{E} \times G .$$

The conditions on $\tilde{f}$ imply that it carries each maximal G-orbit into some maximal G'-orbit. Hence $\tilde{f}$ induces a mapping $\bar{f}\colon B \longrightarrow B'$ of base spaces such that $\bar{f}p = p'\tilde{f}$, and $\bar{f}B_n \subset B'_n$ for all n.

We may regard $I = [0,1]$ as a filtered space in which I_0 consists of the endpoints and $I_1 = I$ (see 1.9). It is in fact a resolution over the trivial group consisting of the identity alone. If $\mathcal{E}$ is a filtered G-space, we give to $I \times \mathcal{E}$ the product filtration (see 1.9) so that $(I \times \mathcal{E})_{n+1} = (I_0 \times \mathcal{E}_{n+1}) \cup (I \times \mathcal{E}_n)$. Then $I \times \mathcal{E}$ is a filtered G-space, and the f-homotopy F of $\tilde{f}_0, \tilde{f}_1$ is precisely a homotopy of $\tilde{f}_0$ into $\tilde{f}_1$ such that F is an f-mapping $I \times \mathcal{E} \longrightarrow \mathcal{E}'$. It follows from the result of the preceeding paragraph that $\bar{F}$ is a homotopy $\bar{f}_0 \simeq \bar{f}_1$ of maps of filtered spaces.

7.2. <u>Theorem.</u> If $f\colon G \longrightarrow G'$ is a morphism, $\mathcal{E}$ is a free filtered G-space, and G' is an acyclic filtered G-space (see 2.1), then there exists an f-mapping $\mathcal{E} \longrightarrow \mathcal{E}'$, and any two f-mappings are f-homotopic.

<u>Proof.</u> We will construct the f-mapping $\tilde{f}$ as the union of f-mappings $\tilde{f}_n\colon \mathcal{E}_n \longrightarrow \mathcal{E}'_n$ for $n \geqq 0$. For $n = 0$, we choose any mapping $\beta\colon D_o \longrightarrow \mathcal{E}'_o$ (e.g. a constant, the condition $\mathcal{E}'_o \neq \emptyset$ of 2.1 i is used here), and we set

$$\tilde{f}_o = \psi'_o(\beta \times f)\varphi_o^{-1}\colon \mathcal{E}_o \longrightarrow D_o \times G \longrightarrow \mathcal{E}'_o \times G' \longrightarrow \mathcal{E}'_o .$$

Assume that the $\tilde{f}_i$ have been properly constructed for $i < n$. Since $\mathcal{E}'_{n-1}$ is contractible in $\mathcal{E}'_n$, $\tilde{f}_{n-1}$ is homotopic to a constant in $\mathcal{E}'_n$. Since a constant map is extendable, and $\mathcal{E}_{n-1}$ has the homotopy extension property in D_n, it follows that $\tilde{f}_{n-1}$ can be extended to a mapping $\beta\colon D_n \longrightarrow \mathcal{E}'_n$. Let $\gamma\colon D_n \longrightarrow D_n$ be the end mapping of the homotopy h_n of 2.1 iv, that is $\gamma x = h_n(1,x)$. Similarly define $\bar{\gamma}\colon \mathcal{E}_n \longrightarrow \mathcal{E}_n$ by $\bar{\gamma}x = \bar{h}_n(1,x)$. Now define $\tilde{f}_n$ by

$$\tilde{f}_n x = \begin{cases} \tilde{f}_{n-1}\bar{\gamma}x & \text{if} \quad \bar{u}_n x < 1 , \\ \psi'_n(\beta \times f)(\gamma \times 1)\varphi_n^{-1}x & \text{if} \quad \bar{u}_n x > 0 . \end{cases}$$

Each of these two lines defines a continuous function because it is a composition of continuous functions (condition 2.1 iii asserts that φ_n^{-1} is continuous). Each line is defined on an open set of $\mathcal{C}_n$. On their overlap $\bar{u}_n^{-1}(0,1)$ the two lines agree; for, if $x = \varphi_n(y,g)$ where $0 < \bar{u}_n x < 1$, we have $\beta\gamma y = \tilde{f}_{n-1}\gamma y$ because γy

is in $\mathcal{E}_{n-1}$, so the second line reduces to

$$\psi'_n(\tilde{f}_{n-1}\gamma y, fg) = (\tilde{f}_{n-1}\gamma y)(fg) = \tilde{f}_{n-1}((\gamma y)g)$$

$$= \tilde{f}_{n-1}\psi_n(\gamma \times 1)(y,g) = \tilde{f}_{n-1}\bar{\gamma}x \ .$$

This shows that $\tilde{f}_n$ is uniquely defined and continuous. The proof that $\tilde{f}_n(xg) = (\tilde{f}_n x)(fg)$ is routine and is omitted. For $x \in \mathcal{E}_{n-1}$, we have $\bar{\gamma}x = x$; hence $\tilde{f}_n x = \tilde{f}_{n-1}x$. This completes the inductive step. The union $\tilde{f}$ of the mappings $\tilde{f}_n$ is continuous on $\mathcal{E}$ because $\mathcal{E}$ has the topology of the union.

To prove the second half of the theorem, let $\tilde{f}_o, \tilde{f}_1$ denote two f-mappings $\mathcal{E} \longrightarrow \mathcal{E}'$. We will construct F as the union of mappings $F_n: I \times \mathcal{E}_n \longrightarrow \mathcal{E}'_{n+1}$. Assume inductively that, for some n , F_{n-1} has been properly constructed. The two stage filtration $I_o \subset I$ is a resolution over the trivial group consisting of the identity element; hence, by 6.1, $I \times \mathcal{E}$ is also a G-resolution. Moreover the mappings $F_{n-1}: I \times \mathcal{E}_{n-1} \longrightarrow \mathcal{E}'_n$, $\tilde{f}_o: 0 \times \mathcal{E}_n \longrightarrow \mathcal{E}'_n$ and $\tilde{f}_i: 1 \times \mathcal{E}_n \longrightarrow \mathcal{E}'_n$ unite to give an f-mapping $\xi: (I \times \mathcal{E})_n \longrightarrow \mathcal{E}'_n$. By the proof of the first half of the theorem, ξ may be extended to an f-mapping $\zeta: (I \times \mathcal{E})_{n+1} \longrightarrow \mathcal{E}'_{n+1}$. The restrictions of ζ to $I \times \mathcal{E}_n$ is the required F_n . This completes the inductive step.

Since each F_n is an f-mapping, their union F is also an f-mapping. The continuity of F follows from the fact that $I \times \mathcal{E}$ has the topology of the union $\cup_o^\infty I \times \mathcal{E}_n$ (see 12.4). This completes the proof.

7.3. <u>Corollary.</u> Any two G-resolutions $\mathcal{E}, \mathcal{E}'$ are f-homotopically equivalent where f is the identity map $G \to G$; hence their base spaces are homotopically equivalent as filtered spaces. Thus the various classifying spaces of G belong to a single homotopy type.

We apply 7.2 to $\mathcal{E}, \mathcal{E}'$ and f to obtain an f-mapping, $\tilde{f}: \mathcal{E} \to \mathcal{E}'$, and then to $\mathcal{E}', \mathcal{E}, f$ to obtain an f-mapping $\tilde{g}: \mathcal{E}' \to \mathcal{E}$. Then $\widetilde{\widetilde{gf}}: \mathcal{E} \to \mathcal{E}$ and the identity are two f-mappings; so 7.2 provides an f-homotopy $\widetilde{\widetilde{gf}} \simeq 1$. Similarly there is an f-homotpy of $\widetilde{\widetilde{fg}}$ with the identity of $\mathcal{E}'$.

8. Classification of bundles

8.1. <u>Theorem.</u> If G is a topological group, any G-resolution $\mathcal{E}$ is a principal G-bundle over $B = \mathcal{E}/G$ with the action $\mathcal{E} \times G \longrightarrow \mathcal{E}$ as principal map.

<u>Proof.</u> By 2.1 iii, $D_o \times G \longrightarrow \mathcal{E}_o$ is a homeomorphism; it represents $\mathcal{E}_o \longrightarrow B_o$ as a product bundle. Assume inductively that $\mathcal{E}_{n-1} \longrightarrow B_{n-1}$ is a principal G-bundle having the action $\mathcal{E}_{n-1} \times G \longrightarrow \mathcal{E}_{n-1}$ as its principal map. By 2.1 iii, the homeomorphism φ_n of $(D_n - \mathcal{E}_{n-1}) \times G$ onto $\mathcal{E}_n - \mathcal{E}_{n-1}$ gives a product representation over the open set $B_n - B_{n-1}$ having the required principal map.

The crucial point is to find such a representation over a neighborhood in B_n of a point $x_o \in B_{n-1}$. By the inductive hypothesis, there is a neighborhood U of x_o in B_{n-1}, and a homeomorphism $\varphi\colon U \times G \longrightarrow p^{-1}U$ such that $p\varphi(x,g) = x$, and $\varphi(x,g)g' = \varphi(x,gg')$ for all $x \in U$, $g,g' \in G$. Referring to condition 2.1 iv, set $ry = \bar{h}_n(1,y)$ for all $y \in \mathcal{E}_n$ such that $\bar{u}_n y < 1$. Then r retracts a neighborhood of $\mathcal{E}_{n-1}$ into $\mathcal{E}_{n-1}$ and is a G-mapping. Let W denote the open set $r^{-1}p^{-1}U$ of $\mathcal{E}_n$. Then $V = pW$ is open in B_n, $W \cap \mathcal{E}_{n-1} = p^{-1}U$, and $V \cap B_{n-1} = U$.

Let $q\colon p^{-1}U \longrightarrow G$ denote the second factor of $\varphi^{-1}\colon p^{-1}U \longrightarrow U \times G$. Then $y = \varphi(py, qy)$ for all $y \in p^{-1}U$. We will construct G-mappings ξ, ζ as in the diagram:

$$\begin{array}{ccccc} W \times G & & & & \\ {\scriptstyle p \times 1}\downarrow & \searrow^{\xi} & & & \\ V \times G & \xrightarrow{\zeta} & W & \xrightarrow{(p,qr)} & V \times G \, . \end{array}$$

Set $\xi(y,g) = y(qry)^{-1}g$ (it is here that we need inverses in G). Suppose $py = py'$. This implies $y' = yg'$ for some g' , and then, since q and r are G-mappings, we have

$$\begin{aligned} \xi(y',g) = y'(qry')^{-1}g &= yg'(qr(yg'))^{-1}g \\ &= yg'g'^{-1}(qry)g = \xi(y,g) \, . \end{aligned}$$

It follows that ξ induces a mapping ζ such that $\xi = \zeta(p \times 1)$. Since, by 12.5, $V \times G$ is the decomposition space of $W \times G$ under $p \times 1$, it follows that ζ is continuous.

Since ξ is a G-mapping, so also is ζ . Now

$$p\xi(y,g) = p(y(qry)^{-1}g) = py \, ,$$

and

$$qr\xi(y,g) = qr(y(qry)^{-1}g) = (qry)(qry)^{-1}g = g \, .$$

Therefore ζ composed with (p,qr) is the identity of $V \times G$. On the other hand, if $y \in W$,

$$\xi(y,qry) = y(qry)^{-1}(qry) = y \, ,$$

and this shows that (p,qr) composed with ζ is the identity of W . Therefore ζ represents $W = p^{-1}V$ as a product, and ζ is a G-mapping. If $x \in U$, then

$$\begin{aligned} \zeta(x,g) = \xi(\varphi(x,g),g) &= \varphi(x,g)(qr\varphi(x,g))^{-1}g \\ &= \varphi(x,g)g^{-1}g = \varphi(x,g) \, . \end{aligned}$$

Hence ζ extends the representation φ. This completes the inductive step from $n-1$ to n.

We pass to the limit as follows. Suppose $x \in B$. For some k, we have $x \in B_k - B_{k-1}$. We set $V_k = B_k - B_{k-1}$ and define $\zeta_k : V_k \times G \longrightarrow \mathcal{E}_k - \mathcal{E}_{k-1}$ to be the map induced by $\varphi_k : (D_k - \mathcal{E}_{k-1}) \times G \longrightarrow \mathcal{E}_k - \mathcal{E}_{k-1}$. Starting with V_k, ζ_k and proceeding inductively as above, we construct, for each $n = k+1, k+2, \ldots,$ an open set V_n of B_n and a product representation

$$V_n \times G \xrightarrow[\zeta_n]{} p^{-1}V_n \xrightarrow[(p,q_n)]{} V_n \times G$$

so that $V_n \cap B_{n-1} = V_{n-1}$, $\zeta_n | V_{n-1} \times G = \zeta_{n-1}$, and $q_n | p^{-1}V_{n-1} = q_{n-1}$. Taking the union as $n \longrightarrow \infty$ gives mappings

$$V \times G \xrightarrow[\zeta]{} p^{-1}V \xrightarrow[(p,q)]{} V \times G .$$

It is clear that ζ and (p,q) are inverses, and are G-mappings. Since $V \times G$ has the topology of the union of the $V_n \times G$, it follows that ζ is continuous. Similarly q and hence (p,q) are continuous. Finally V is an open set because V intersects each B_n in an open set, and B has the topology of the union. Thus each point $x \in B$ lies in an open set V for which there is a G-homeomorphism $V \times G \longrightarrow p^{-1}V$. This completes the proof.

Since $\mathcal{E}$ is contractible, it follows from 8.1 that $\mathcal{E}$ is a universal bundle for G-bundles over complexes [11; p. 102]. Precisely, if K is any CW complex, then there is a one-to-one correspondence between equivalence classes of G-bundles over K and homotopy classes of maps of K into B, denoted by $[K,B]$. The correspondence is given by the bundle induced by a map $K \longrightarrow B$. This justifies calling B a classification space for G.

The filtration $\{B_n\}$ of B has geometric significance for the classification as follows.

8.2. <u>Theorem.</u> If G is a group, $\mathcal{E}$ is a G-resolution, and K is a finite complex of Lusternik-Schnirelmann reduced category r, then each G-bundle over K is induced by some mapping $K \longrightarrow B_r$, and, if two such bundles are equivalent, the corresponding maps are homotopic in B_{r+1}. Thus the image of $[K,B_r]$ in $[K,B_{r-1}]$ is in one-to-one correspondence with equivalence classes of G-bundles over K.

<u>Proof.</u> Let $p' : \mathcal{E}' \longrightarrow K$ be a G-bundle over K. Since the category of K is r, in some subdivision of K, there are subcomplexes $L_o, L_1, \ldots, L_r$ such that each is contractible to a point in K, and K is their union. Set $K_n = \cup_{i=o}^{n} L_i$, and let $\mathcal{E}'_n = p'^{-1}K_n$. Since the inclusion $L_i \subset K$ is homotopic to a constant, it follows that the part of the bundle over L_i has a representation as a product $f_i : L_i \times G \longrightarrow p'^{-1}L_i$ (see [11; Th.11.5]). Set

$$D'_n = \mathcal{E}'_{n-1} \cup f_n(L_n \times e) .$$

Since

$$D'_n - \mathcal{E}'_{n-1} = f_n((L_n - L_n \cap K_{n-1}) \times e) ,$$

the action mapping $(D'_n - \mathcal{E}'_{n-1}) \times G \longrightarrow \mathcal{E}'_n - \mathcal{E}'_{n-1}$ is equivalent to a restriction of f_n, and is therefore a homeomorphism. Thus condition 2.1 iii holds. As to iv, $L_n \cap K_{n-1}$ is a subcomplex, and we may choose a representation of it as an NDR in L_n. Using f_n this induces a representation of $\mathcal{E}'_{n-1}$ as an NDR in D'_n which satisfies iv. ?

In this way $\mathcal{E}'$ becomes a free, filtered G-space. By 7.2, there is a G-mapping $\tilde{f}: \mathcal{E}' \longrightarrow \mathcal{E}$ preserving the filtration. Then the induced mapping $\bar{f}: K \longrightarrow B$ is such that $\bar{f}K \subset B_r$ because $K = K_r$. Hence $\bar{f}$ induces the bundle $\mathcal{E}'$.

If ${}_0f, {}_1f: K \longrightarrow B_r$ induce equivalent bundles ${}_0\mathcal{E}'$ and ${}_1\mathcal{E}'$ over K, then the equivalence may be interpreted as a homotopy $I \times K \longrightarrow B$ of ${}_0f$ into ${}_1f$, and hence as a bundle $\mathcal{E}'$ over $I \times K$ whose restrictions to $0 \times K$ and $1 \times K$ are ${}_0\mathcal{E}'$ and ${}_1\mathcal{E}'$ respectively. Let ${}_i\tilde{f}: {}_i\mathcal{E}' \longrightarrow \mathcal{E}_r$ denote the natural bundle map over ${}_if$, $i = 0,1$. Filter $I \times K$ by $(I \times K)_n = (I_0 \times K_n) \cup (I \times K_{n-1})$ where I_0 denotes the endpoints of I. Then $I \times K_n = (I \times K)_{n-1} \cup (I \times L_n)$. Since $I \times L_n$ is contractible to a point in $I \times K$, we may define, as above, a free filtered G-space structure on $\mathcal{E}'$ over the filtration of $I \times K$.

We must show that the given maps ${}_0\tilde{f}, {}_1\tilde{f}$ may be extended to a G-mapping $\tilde{f}: \mathcal{E}' \longrightarrow \mathcal{E}_{r+1}$ which preserves the filtration. As in the proof of 7.2, we construct $\tilde{f}$ stepwise according to the filtration index. Since $\mathcal{E}'_0 = {}_0\mathcal{E}'_0 \cup {}_1\mathcal{E}'_0$, $\tilde{f}$ is already specified on $\mathcal{E}'_0$. Suppose inductively that $\tilde{f}$ has been extended to a G-mapping $\mathcal{E}'_{n-1} \longrightarrow \mathcal{E}_{n-1}$. As in the proof of 6.2, we first extend $\tilde{f}$ to a G-mapping $\mathcal{E}'_{n-1} \cup p'^{-1}(I \times L_{n-1}) \longrightarrow \mathcal{E}_n$. Then we define $\tilde{f}$ to be ${}_0\tilde{f}$ on ${}_0\mathcal{E}'_n$ and ${}_1\tilde{f}$ on ${}_1\mathcal{E}'_n$. Since the three extensions occur on three disjoint open sets, the resulting map is uniquely defined. This completes the inductive step. Since $\mathcal{E}' = \mathcal{E}'_{r+1}$, we obtain the required $\tilde{f}$ after $r+1$ steps. Then $\tilde{f}$ induces a homotopy $\bar{f}: I \times K \longrightarrow B_{r+1}$ of ${}_0f$ into ${}_1f$. This completes the proof.

8.3. <u>Corollary.</u> Under the hypotheses of 8.2, the image of $[K, B_r]$ in $[K, B_{r+1}]$ is in 1-1 correspondence with $[K, B]$.

This follows since any map $K \longrightarrow B$ induces a G-bundle over K which in turn is induced by a map into B_r.

Since the i-sphere has reduced category 1, we have

8.4. <u>Corollary.</u> For $r \geqq 1$, the image of $\pi_i(B_r)$ in $\pi_i(B_{r+1})$ maps isomorphically onto $\pi_i(B)$.

Because of the fact that $\pi_i(B) \approx \pi_{i-1}(G)$, this last result is of little use for computing $\pi_i(B)$; but it does show that the exact couple obtained from the filtration of B by using homotopy groups becomes trivial after one derivation. In Milnor's resolution, B_1 is just the suspension of G; hence it is clear that $\pi_i(B)$ is generated by spheres in B_1. It is perhaps less clear that any homotopy relation among them in B, occurs already in B_2.

<u>Remark.</u> In case G is an associative H-space, Dold and Lashof [2; p. 293] have shown that, for their version of Milnor's resolution, $\mathcal{E} \longrightarrow B$ is a principal quasifibration. It is natural to conjecture that the same is true of any G-resolution.

9. Cross-products and spectral sequences

Let X, Y be filtered spaces. We assume henceforth that the pairs (X_n, X_{n-1}) and (Y_n, Y_{n-1}) are NDR's for all n, and that X, Y have the topologies of the unions of the X_n, Y_n respectively. We proceed to show that the spectral sequences based on homology of X, Y and $X \times Y$ are related by a cross-product operation

(9.1) $$\varphi\colon E^r(X) \otimes E^r(Y) \longrightarrow E^r(X \times Y), \qquad r \geqq 1.$$

To simplify the notation, let $X_{p,1}$ denote the pair (X_p, X_{p-1}). Recall that the product of pairs (X,A) and (Y,B) is the pair $(X \times Y, A \times Y \cup X \times B)$. The product φ of E^1-terms is defined by the diagram 9.2

(9.2) $$\begin{array}{ccc} E^1_{p,s}(X) \otimes E^1_{q,t}(Y) = H_{p+s}(X_{p,1}) \otimes H_{q+t}(Y_{q,1}) & & \\ \Big\downarrow \varphi & \searrow^{\alpha} & \\ & & H_m(X_{p,1} \times Y_{q,1}) \\ & \swarrow_{k_*} & \\ E^1_{p+q,s+t}(X \times Y) = H_m((X \times Y)_{p+q,1}) & & \end{array}$$

where $m = p+q+s+t$, k denotes the inclusion of one pair in the other, α is the standard cross-product for homology $\alpha(u \otimes v) = u \times v$ (all coefficients are in a ring R), and φ is defined to be $k_*\alpha$. We write $u \cdot v$ for $\varphi(u \otimes v)$.

9.3. <u>Theorem.</u> The above product φ of E^1-terms induces a product φ of E^r-terms for all $r \geqq 1$ such that:

i. If $u \in E^r_{p,s}(X)$, $v \in E^r_{q,t}(Y)$, and $1 \leqq r < \infty$, then

$$d_r(u \cdot v) = d_r u \cdot v + (-1)^{p+s} u \cdot d_r v.$$

ii. The induced product of E^∞-terms coincides, under the isomorphisms with the associated graded modules, with the product induced by the cross-product $H(X) \otimes H(Y) \longrightarrow H(X \times Y)$.

iii. The products are natural with respect to mappings of filtered spaces $f: X \longrightarrow X'$ and $g: Y \longrightarrow Y'$, that is

$$(f \times g)_*(u \cdot v) = f_* u \cdot g_* v .$$

iv. If $T: X \times Y \longrightarrow Y \times X$ is given by $T(x,y) = (y,x)$, then

$$T_*(u \cdot v) = (-1)^{mn} v \cdot u , \qquad \dim u = m, \quad \dim v = n .$$

v. The products satisfy the associative law $(u \cdot v) \cdot w = u \cdot (v \cdot w)$ where u, v are as above, and $w \in E^r$ of some third filtered space.

vi. If X is a single point x_o, and $1 \in E^r_{o,o}(x_o)$ corresponds to $1 \in R$ under the canonical isomorphism $R \approx H_o(x_o) = E^r(x_o)$, and $g: Y \longrightarrow x_o \times Y$ is defined by $gy = (x_o, y)$, then

$$g_* v = 1 \cdot v \qquad \text{for all } v \in E^r(Y) .$$

The proofs of the theorems of this section are given in §16.

9.4. <u>Theorem.</u> If the modules $E^r_{p,q}(X)$ (or $E^r_{p,q}(Y)$) are free for all p,q,r, then, for each r, the multiplication φ of 9.1 is an isomorphism of bigraded modules:

$$E^r_{p,s}(X \times Y) \approx \sum_{i=o}^{p} \sum_{j=-i}^{p+s-i} E^r_{i,j}(X) \otimes E^r_{p-i,s-j}(Y) .$$

We now state analogous results for cohomology. These are not trivial consequences since there seems to be no strict duality from which they can be derived.

The product φ of E_1-terms is defined by the

$$\begin{array}{ccc}
E_1^{p,s}(X) \otimes E_1^{q,t}(Y) = H^{p+s}(X_{p,1}) \otimes H^{q+t}(Y_{q,1}) & \xrightarrow{\alpha} & H^m(X_{p,1} \times Y_{q,1}) \\
(9.5) \quad \downarrow \varphi & & h^* \nearrow \approx \\
E_1^{p+q,s+t}(X \times Y) = H^m((X \times Y)_{p+q,1}) & \xleftarrow{k^*} & H^m((X \times Y)_{p+q}, \bigcup_{j \neq p} X_j \times Y_{p+q-j})
\end{array}$$

diagram 9.5, i.e. $\varphi = k^* h^{*-1} \alpha$. Expressed otherwise $u \cdot v = k^* h^{*-1}(u \times v)$. Note that h^* is an isomorphism because h is an excision.

9.6. <u>Theorem.</u> The product φ of E_1-terms induces a product φ of E_r-terms for all $r \geqq 1$ such that:

i. If $u \in E_r^{p,s}(X)$, $v \in E_r^{q,t}(Y)$, and $1 \leqq r < \infty$, then

$$d_r(u \cdot v) = d_r u \cdot v + (-1)^{p+s} u \cdot d_r v .$$

ii. The induced product of E_∞-terms coincides, under the isomorphisms with the associated graded modules, with the product induced by the cross-product $H^*(X) \otimes H^*(Y) \longrightarrow H^*(X \times Y)$.

iii. The products are natural with respect to mappings of filtered spaces $f: X \longrightarrow X'$ and $g: Y \longrightarrow Y'$, that is

$$(f \times g)^*(u' \cdot v') = f^* u' \cdot g^* v' .$$

iv. If $T: X \times Y \longrightarrow Y \times X$ is given by $T(x,y) = (y,x)$, then

$$T^*(u \cdot v) = (-1)^{mn} v \cdot u , \quad \dim u = m, \quad \dim v = n .$$

v. The products satisfy the associative law $(u \cdot v) \cdot w = u \cdot (v \cdot w)$ where u, v are as above and $w \in E_r$ of some third filtered space.

vi. If X is a single point x_o, and $1 \in E_r^{o,o}(X)$ corresponds to the unit of R under the canonical isomorphism $R \approx H^o(x_o) = E_r^{o,o}(x_o)$, and $g: x_o \times Y \longrightarrow Y$ is the projection $g(x_o,y) = y$, then

$$g^* v = 1 \cdot v \qquad \text{for all } v \in E_r(Y) .$$

9.7. <u>Corollary.</u> If $1 \in E_r^{o,o}(X)$ denotes the image $f^* 1$ where f collapses X to a point x_o, and if $h: X \times Y \longrightarrow Y$ is the projection, then

$$h^* v = 1 \cdot v \qquad \text{for all } v \in E_r(Y) .$$

This follows from iii and vi because $g(f \times 1) = h$.

9.8. <u>Theorem.</u> If the modules $E^1_{p,q}(X)$ and $E_r^{p,q}(X)$ are free for all p,q,r, and each $E^1_{p,q}(X)$ has a finite basis then, for each r, the multiplication

$$E_r^{p,s}(X \times Y) \approx \sum_{i=o}^{p} \sum_{j=-i}^{p+s-i} E_r^{i,j}(X) \otimes E_r^{p-i,s-j}(Y) .$$

Proofs will be found in §16.

10. Morphisms of spectral sequences induced by mappings

The results of this section are little more than corollaries of preceding theorems. We shall state, as one, propositions for homology and cohomology which are dual, and give only the proof for homology.

10.1. <u>Lemma.</u> If X is a filtered space, $F: I \times X \longrightarrow X$ is the projection, and $g_o, g_1: X \longrightarrow I \times X$ are defined by $g_o x = (0,x)$, $g_1 x = (1,x)$, then, for each $r \geqq 2$, F_* is an isomorphism of exact couples $AE^r(I \times X) \approx AE^r(X)$, and its inverse is $g_{o*} = g_{1*}$. Also, for $r \geqq 2$, F^* is an isomorphism $AE_r(X) \approx AE_r(I \times X)$ with inverse $g_o^* = g_1^*$.

<u>Proof.</u> It should be emphasized that $I \times X$ has the product filtration, and F, g_o, g_1 preserve filtrations. Recall that the unit interval $I = [0,1]$ has the filtration where I_o consists of the end points and $I_1 = I$. The E^1-term of its exact couple has just two non-zero groups: $E^1_{1,o} \approx R$, and $E^1_{o,o} \approx R + R$. Moreover d_1 maps $E^1_{1,o}$ monomorphically onto a direct summand of $E^1_{o,o}$. It follows that, for all $r \geqq 2$, E^r has only the one non-zero component $E^r_{o,o} \approx R$. It follows now from 9.5 that

$$E^2_{p,q}(I \times X) \approx E^2_{o,o}(I) \otimes E^2_{p,q}(X) \approx E^2_{p,q}(X) .$$

Let $z_o \in E^2_{o,o}(I)$ be represented by the 0-cycle on the vertex 0 having coefficient 1. Then the above isomorphism sends $u \in E^2(X)$ into $z_o \cdot u$. It is also evident that $g_{o*}u = z_o \cdot u$; hence g_{o*} gives the isomorphism. Since $Fg_o = Fg_1$ is the identity of X, we have

$g_{o*} = F_*^{-1} = g_{1*}$. So F^* gives an isomorphism of E^2-terms. That it gives an isomorphism of A^2-terms follows from the lemma below. Evidently the validity of the theorem for $r = 2$ implies its validity for $r > 2$.

10.2. <u>Lemma.</u> Let AE and $A'E'$ be exact couples such that $A_{p,q} = 0 = A'_{p,q}$ whenever $p < 0$, and let f be a morphism $AE \longrightarrow A'E'$ such that $f\colon E \approx E'$, then also $f\colon A \approx A'$ and f is an isomorphism of exact couples.

<u>Proof.</u> Suppose $x_o \in A_{p,q}$ and $fx_o = 0$. Then $fjx_o = j'fx_o = 0$. As the kernel of $f|E_{p,q}$ is zero, we have $jx_o = 0$; hence, by exactness, $x_o = iu$ where $u \in A_{p-1,q+1}$. Then $i'fu = fiu = fx_o = 0$; hence, by exactness, $fu = \partial y'$ for $y' \in E'_{p,q+1}$. Since f maps $E_{p,q+1}$ onto $E'_{p,q+1}$, there is a $y \in E_{p,q+1}$ such that $fy = y'$. Set $x_1 = u - \partial y$. It follows now that $ix_1 = x_o$ and $fx_1 = 0$. Repeat the preceding argument with x_1 in place of x_o , obtaining $x_2 \in A_{p-2,q+2}$ such that $ix_2 = x_1$ and $fx_2 = 0$. Continue, inductively, obtaining $x_k \in A_{p-k,q+k}$ for each $k > 0$ such that $ix_k = x_{k-1}$ and $fx_k = 0$. Taking $k = p+1$, the hypothesis $A_{-1,q+p+1} = 0$ implies $x_{p+1} = 0$, whence $x_o = i^{p+1}x_{p+1} = 0$. This shows that f is monomorphic on each A-term.

Thus f imbeds AE in $A'E'$. Form the quotient $A'E'/f(AE)$. Since the quotient of an exact sequence by an exact sequence is also exact, it follows that this quotient is an exact couple $A''E''$. Clearly its E''-terms are zero. By exactness, $i''\colon A''_{p,q} \approx A''_{p+1,q-1}$ for all p,q . Since $A''_{-1,p+q+1} = 0$, it follows that every $A''_{p,q} = 0$. This completes the proof.

10.3. Theorem. If $f_o, f_1 : X \to Y$ are mappings of filtered spaces which are homotopic as maps of filtered spaces (i.e. the homotopy $I \times X \to Y$ preserves filtrations), then, for each $r \geqq 2$, f_o and f_1 induce the same morphism of exact couples $AE^r(X) \to AE^r(Y)$ and the same morphism $AE_r(Y) \to AE_r(X)$.

Proof. If F is the homotopy, and g_o, g_1 are as in 10.1, then $f_o = Fg_o$, $f_1 = Fg_1$; hence $g_{o*} = g_{1*}$ for $r \geqq 2$ implies $f_{o*} = f_{1*}$ because $f_{i*} = F_* g_{i*}$ for $i = 0,1$.

10.4. Theorem. Let $f: G \to G'$ be a morphism of associative H-spaces, let $\mathcal{E}$ be a G-resolution, and $\mathcal{E}'$ a G'-resolution, then any two f-mappings $\tilde{f}_o, \tilde{f}_1 : \mathcal{E} \to \mathcal{E}'$ induce the same morphisms of the associated exact couples

$$\bar{f}_{o*} = \bar{f}_{1*} : AE^r(B_G) \longrightarrow AE^r(B'_{G'}) \qquad \text{for } r \geqq 2 ,$$

$$\bar{f}_o^* = \bar{f}_1^* : AE_r(B'_{G'}) \longrightarrow AE_r(B_G) \qquad \text{for } r \geqq 2 .$$

Proof. By 7.2, there is an f-homotopy $F: I \times \mathcal{E} \to \mathcal{E}'$ of $\tilde{f}_o$ into $\tilde{f}_1$. Then $\bar{F}: I \times B \to B'$ is a homotopy of $\bar{f}_o$ into $\bar{f}_1$ as maps of filtered spaces. The conclusion follows now from 10.3.

10.5. Corollary. The associated exact couples and spectral sequences of any two G-resolutions are canonically isomorphic from the AE^2-terms on. In particular they are isomorphic to those of Milnor's resolution.

Proof. If we apply 7.2 to the case of two G-resolutions $\mathcal{E}, \mathcal{E}'$ with f the identity map of G, we obtain f-mappings $\tilde{f}: \mathcal{E} \to \mathcal{E}'$ and $\tilde{g}: \mathcal{E}' \to \mathcal{E}$. Then $\tilde{g}\tilde{f}$ and the identity are two f-mappings $\mathcal{E} \longrightarrow \mathcal{E}$. By 10.4, we have that $(\bar{g}\,\bar{f})_* = \bar{g}_*\bar{f}_*$ is the identity map of $AE^r(B)$ for $r \geqq 2$. Similarly $\bar{f}_*\bar{g}_*$ is the identity of $AE^r(B')$.

10.6. Corollary. For each $r \geqq 2$, the associated exact couples $AE^r(B_G)$ and $AE_r(B_G)$ are covariant and contravariant functors, respectively, from the category of associative H-spaces G and their continuous morphisms to the category of exact couples and their morphisms.

In view of 10.5, this is a routine argument.

11. Products and coproducts in the spectral sequences of resolutions

Let $\mathcal{E}$ be a G-resolution and B its base space. Let $\Delta: G \longrightarrow G \times G$ be the diagonal. By 6.1, $\mathcal{E} \times \mathcal{E}$ is a $G \times G$-resolution. Since Δ is a morphism of H-spaces, theorem 7.2 states that there is a Δ-mapping of resolutions $\tilde{\Delta}: \mathcal{E} \longrightarrow \mathcal{E} \times \mathcal{E}$. By 10.4, the induced morphisms $\bar{\Delta}^*: E_r(B \times B) \longrightarrow E_r(B)$ do not depend on the choice of $\tilde{\Delta}$ when $r \geq 2$. We define the (cup-) <u>product</u> in each term $E_r(B)$ of the cohomology spectral sequence to be the composition

$$(11.1) \qquad E_r(B) \otimes E_r(B) \xrightarrow{\varphi} E_r(B \times B) \xrightarrow{\bar{\Delta}^*} E_r(B)$$

where φ is the product of 9.6. We abbreviate $\bar{\Delta}^*\varphi(u \otimes v)$ by uv. Thus $uv = \bar{\Delta}^*(u \cdot v)$. When $E^r(B)$ is free for all r, then the product φ for homology is an isomorphism by 9.4. In this case we define the <u>coproduct</u> for homology by

$$(11.2) \qquad E^r(B) \xrightarrow{\bar{\Delta}_*} E^r(B \times B) \xrightarrow{\varphi^{-1}} E^r(B) \otimes E^r(B) .$$

Consider now the special case where G is a <u>commutative</u> H-space. Then the multiplication $\mu: G \times G \longrightarrow G$ is a morphism of H-spaces, so it induces morphisms of the spectral sequences for $G \times G$ and G. We define the <u>product</u> in the homology spectral sequence by

$$(11.3) \qquad E^r(B) \otimes E^r(B) \xrightarrow{\varphi} E^r(B \times B) \xrightarrow{\bar{\mu}_*} E^r(B) .$$

We abbreviate $\bar{\mu}_*\varphi(u \otimes v)$ by uv. Thus $uv = \bar{\mu}_*(u \cdot v)$. Assuming also that $E^1(B)$ is free of finite type, and each $E_r(B)$ is free, then 9.8 asserts that the product φ for cohomology is an isomorphism. In this case we define the <u>coproduct</u> for cohomology by

$$(11.4) \qquad E_r(B) \xrightarrow{\bar{\mu}^*} E_r(B \times B) \xrightarrow{\varphi^{-1}} E_r(B) \otimes E_r(B) .$$

It should be emphasized that these operations do not depend on the choices of $\tilde{\Delta}$ and $\tilde{\mu}$ when $r \geqq 2$, but they do depend on the choices when $r = 1$.

We shall summarize the main properties of the two products and two coproducts in a series of propositions whose proofs are given later on in this section. In each proposition the words <u>products</u> and <u>coproducts</u> should be modified by the phrase "when defined."

11.5. <u>Lemma.</u> If $2 \leqq r \leqq \infty$, then the products (coproducts) are commutative and associative, and they have units (co-units).

11.6. <u>Lemma.</u> If $1 \leqq r < \infty$, then d_r commutes with the products (coproducts), i.e. d_r is a derivation (coderivation).

11.7. <u>Lemma.</u> If $1 \leqq r < s \leqq \infty$, then the natural map of a submodule of $E_r(B)$ onto $E_s(B)$ $(E^r(B)$ onto $E^s(B))$ preserves products and coproducts.

11.8. <u>Lemma.</u> The isomorphism of $E_\infty(B)$ $(E^\infty(B))$ with the associated graded module of the filtration of $H^*(B)$ $(H(B))$ preserves products and coproducts.

In the case of primary interest to us, $H(G)$ is free (so that 5.1 holds), and then $E_2^{p,q}(B) = 0$ for $q < 0$ by 5.3 and 10.5. It follows from 4.11 that the spectral sequence converges. Thus the cohomology part of 11.8 needs the hypothesis: $H(G)$ is free or R is compact.

11.9. <u>Lemma.</u> If $f: G \longrightarrow G'$ is a morphism of H-spaces, then the induced morphisms $\bar{f}_*, \bar{f}^*$ of spectral sequences (see 10.4) preserve products and coproducts.

11.10. <u>Lemma.</u> Whenever the product and the coproduct of $E_r(B)$ $(E^r(B))$ are defined, then $E_r(B)$ $(E^r(B))$ is a Hopf algebra for $r \geqq 2$.

11.11. <u>Lemma.</u> When $H(G)$ is free, then the isomorphisms of 5.1

$$E^2(B) \approx \mathrm{Tor}^{H(G)}(R,R)\ , \text{ and } \ E_2(B) \approx \mathrm{Ext}_{H(G)}(R,R)$$

preserve products and coproducts.

<u>Proof of 11.5.</u> Define $T: G \times G \longrightarrow G \times G$ by $T(g_1, g_2) = (g_2, g_1)$. Then T is a morphism of H-spaces. Since $T\Delta = \Delta$, we have by 10.6 that $\bar{T}_*\bar{\Delta}_* = \bar{\Delta}_*$ and $\bar{\Delta}^*\bar{T}^* = \bar{\Delta}^*$ for $r \geqq 2$. Let T' interchange the factors of $E_r(B) \otimes E_r(B)$ $(E^r(B) \otimes E^r(B))$, thus

$$T'(u \otimes v) = (-1)^{mn}\, v \otimes u\ , \qquad m = \deg u, \quad n = \deg v\ .$$

By 9.3 iv and 9.6 iv, we have $\varphi T' = \bar{T}_*\varphi$ $(= \bar{T}^*\varphi)$. It follows that

$$\bar{\Delta}^*\varphi T' = \bar{\Delta}^*\bar{T}^*\varphi = \bar{\Delta}^*\varphi\ , \quad \text{and}$$

$$T'\varphi^{-1}\bar{\Delta}_* = \varphi^{-1}\bar{T}_*\bar{\Delta}_* = \varphi^{-1}\bar{\Delta}_*\ .$$

This proves commutativity for the cohomology product and the homology coproduct.

Assuming G commutative, then $\mu T = T$. By 10.6, this implies $\bar{\mu}_*\bar{T}_* = \bar{\mu}_*$ and $\bar{T}^*\bar{\mu}^* = \bar{\mu}^*$ for $r \geqq 2$. Hence

$$\bar{\mu}_* \varphi T' = \bar{\mu}_* \bar{T}_* \varphi = \bar{\mu}_* \varphi \quad \text{and}$$

$$T' \varphi^{-1} \bar{\mu}^* = \varphi^{-1} \bar{T}^* \bar{\mu}^* = \varphi^{-1} \bar{\mu}^* .$$

This proves commutativity for the homology product and the cohomology coproduct.

To prove associativity, note first that Δ and μ are associative: $(\Delta \times 1)\Delta = (1 \times \Delta)\Delta$, $\mu(\mu \times 1) = \mu(1 \times \mu)$. By 10.6 this implies the associativity of the induced morphisms of spectral sequences. By 9.3 v and 9.6 v, each φ is associative; and by 9.3 iii and 9.6 iii, each φ is natural. Combining these facts gives the associativity of products and coproducts, e.g. for the cohomology product:

$$(\bar{\Delta}^* \varphi)(1 \otimes (\bar{\Delta}^* \varphi)) = (\bar{\Delta}^* \varphi)(1 \otimes \bar{\Delta}^*)(1 \otimes \varphi) = \bar{\Delta}^*(1 \times \bar{\Delta})^* \varphi(1 \otimes \varphi)$$

$$= \bar{\Delta}^*(\bar{\Delta} \times 1)^* \varphi(\varphi \otimes 1) = (\bar{\Delta}^* \varphi)(\bar{\Delta}^* \otimes 1)(\varphi \otimes 1) = (\bar{\Delta}^* \varphi)(\bar{\Delta}^* \varphi \otimes 1) .$$

The naturality of φ is used in steps two and four.

Let e be the unit element of G. Then e is an associative and commutative H-space. We may form the e-resolution consisting of e alone, and its base space is e. The associated spectral sequences collapse into $E_r^{o,o}(e) = R = E^r_{o,o}(e)$. The inclusion and projection

$$e \xrightarrow{h} G \xrightarrow{k} e$$

are maps of associative H-spaces. They induce morphisms

$$R \xrightarrow{\bar{k}^*} E_r(B) \xrightarrow{\bar{h}^*} R , \qquad R \xrightarrow{\bar{h}_*} E^r(B) \xrightarrow{\bar{k}_*} R$$

defining the required units and co-units. To prove that they are such, let

$$f = (k \times 1)\Delta\colon G \longrightarrow e \times G , \qquad g = \mu(h \times 1)\colon e \times G \longrightarrow G .$$

Then, by 9.6 vi, we have, for $u \in E_r(B)$ and $r \geqq 2$, that

$$(\bar{k}^* 1)u = \bar{\Delta}^*(\bar{k}^* 1)\cdot u = \bar{\Delta}^*(\bar{k} \times 1)^*(1\cdot u) = \bar{f}^*(1\cdot u) = u ,$$

and, when the coproduct 11.4 is defined, also

$$(\bar{h}^* \otimes 1)\varphi^{-1}\bar{\mu}^* u = \varphi^{-1}(\bar{h} \times 1)^* \bar{\mu}^* u = \varphi^{-1}\bar{g}^* u = \varphi^{-1}(1\cdot u) = 1 \otimes u .$$

Therefore $\bar{k}^* 1$ is a unit for $E_r(B)$, and $\bar{h}^*$ is its co-unit. The analogous argument for $E^r(B)$ is similar. This completes the proof of 11.5.

<u>Proof of 11.6.</u> If we define d_r in $E_r \otimes E_r$ by the usual rule $d_r(u \otimes v) = d_r u \otimes v + (-1)^m u \otimes d_r v$ where $m = \deg u$, then we can interpret 9.3 i and 9.6 i as stating that d_r commutes with the φ's . Since d_r also commutes with morphisms of spectral sequences induced by mappings, it must also commute with $\bar{\Delta}^*$, $\bar{\Delta}_*$, $\bar{\mu}^*$, and $\bar{\mu}_*$. It will therefore commute with the compositions 11.1-11.4 which define the products and coproducts.

<u>Proof of 11.7.</u> The natural map λ of a submodule of E_r onto E_s commutes with morphisms of spectral sequences induced by mappings, hence with $\bar{\Delta}^*$, $\bar{\Delta}_*$, $\bar{\mu}^*$, and $\bar{\mu}_*$. Since the products φ of E_r and E_s terms are induced by the product φ of E_1 terms, it follows that φ also commutes with λ . Therefore λ will commute with the compositions 11.1-11.4 which define the products and coproducts.

<u>Proof of 11.8.</u> Let ξ denote the isomorphism $E_\infty \approx E_0 H^*$ which is the associated graded module of the filtered H^* . It is

easily seen that ξ commutes with morphisms of spectral sequences induced by mappings of filtered spaces. We can interpret 9.6 ii as stating that ξ commutes with φ. It follows that ξ commutes with the compositions 11.1 and 11.4 defining the product and coproduct in E_∞ and E_0H^*. There is a similar argument for homology.

<u>Proof of 11.9.</u> Since $(f \times f)\Delta = \Delta' f$, we have by 10.6 that $\overline{\Delta}^*(\overline{f} \times \overline{f})^* = \overline{f}^*\overline{\Delta'}^*$. By 9.5 iii, we have $(\overline{f} \times \overline{f})^*\varphi = \varphi(\overline{f}^* \otimes \overline{f}^*)$. Combining these gives

$$\overline{\Delta}^*\varphi(\overline{f}^* \otimes \overline{f}^*) = \overline{f}^*\overline{\Delta'}^*\varphi ,$$

and this is equivalent to $(\overline{f}^*u')(\overline{f}^*v') = \overline{f}^*(u'v')$. Again by 10.6, we have $(\overline{f} \times \overline{f})_*\overline{\Delta}_* = \overline{\Delta}'_*\overline{f}_*$, and, by 9.3 iii, $\varphi(\overline{f}_* \otimes \overline{f}_*) = (\overline{f} \times \overline{f})_*\varphi$. If φ is an isomorphism, this last gives $(\overline{f}_* \otimes \overline{f}_*)\varphi^{-1} = \varphi^{-1}(\overline{f} \times \overline{f})_*$, and combining yields

$$(\overline{f}_* \otimes \overline{f}_*)\varphi^{-1}\overline{\Delta}_* = \varphi^{-1}\overline{\Delta}'_*\overline{f}_*$$

which shows that $\overline{f}_*$ preserves the coproduct in the homology spectral sequence.

When G and G' are commutative, then 10.6 can be applied to the relation $f\mu = \mu'(f \times f)$ to obtain $\overline{f}_*\overline{\mu}_* = \overline{\mu}'_*(\overline{f} \times \overline{f})_*$ and $\overline{\mu}'^*\overline{f}^* = (\overline{f} \times \overline{f})^*\overline{\mu}^*$. We may now reason as above to obtain:

$$\overline{f}_*\overline{\mu}_*\varphi = \overline{\mu}'_*(\overline{f} \times \overline{f})_*\varphi = \overline{\mu}'_*\varphi(\overline{f}_* \otimes \overline{f}_*) \text{ , and}$$

$$\varphi^{-1}\overline{\mu}'^*\overline{f}^* = \varphi^{-1}(\overline{f} \times \overline{f})^*\overline{\mu}^* = (\overline{f}^* \otimes \overline{f}^*)\varphi^{-1}\overline{\mu}^* .$$

<u>Proof of 11.10.</u> To prove that $E_r(B)$ is a Hopf algebra when G is commutative, we must show that the coproduct mapping $\varphi^{-1}\overline{\mu}^*$ is a morphism of algebras $E_r(B) \longrightarrow E_r(B) \otimes E_r(B)$. By 11.9, $\overline{\mu}^*$ is a

morphism of algebras (preserves products). It suffices then to show that φ preserves products. This is equivalent to proving the commutativity of the diagram 11.12 where the exponents mean tensor powers

(11.12)
$$\begin{array}{ccccccc}
E_r(B)^4 & \xrightarrow{1\otimes T\otimes 1} & E_r(B)^4 & \xrightarrow{\varphi\otimes\varphi} & E_r(B^2)^2 & \xrightarrow{\bar{\Delta}^*\otimes\bar{\Delta}^*} & E_r(B)^2 \\
\downarrow{\scriptstyle \varphi\otimes\varphi} & & & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\
 & & & \overset{(1\times\bar{T}\times 1)^*}{\nearrow} & E_r(B^4) & \overset{(\bar{\Delta}\times\bar{\Delta})^*}{\searrow} & \\
E_r(B^2)^2 & \xrightarrow{\varphi} & E_r(B^4) & & \xrightarrow{\bar{\Delta}_2^*} & & E_r(B^2)
\end{array}$$

when applied to modules and cartesian powers when applied to spaces. The diagonal map for $G \times G$ is denoted by Δ_2, and T is the interchange of two factors of a product. The top line (bottom line) defines the product in $E_r(B)^2$ $(E_r(B^2))$. Commutativity in the triangle follows from $\Delta_2 = (1 \times T \times 1)(\Delta \times \Delta)$ and 10.6. Commutativity of the right square follows from 9.6iii. As for the hexagonal part, start with a generator of the form $a \otimes b \otimes c \otimes d$ in $E_r(B)^4$. Moving over and down gives $(-1)^{pq}(a\cdot c)\cdot(b\cdot d)$ in $E_r(B^4)$. The other way around gives $(1 \times \bar{T} \times 1)^*((a\cdot b)\cdot(c\cdot d))$. The associative law for φ and naturality (see 9.6v and iii) show that this last reduces to $a\cdot\bar{T}^*(b\cdot c)\cdot d$. Then the commutative law 9.6iv gives $(-1)^{pq}a\cdot(c\cdot b)\cdot d$ which reassociates into $(-1)^{pq}(a\cdot c)\cdot(b\cdot d)$. This proves that $E_r(B)$ is a Hopf algebra.

To prove that $E^r(B)$ is a Hopf algebra we must show that $\varphi^{-1}\bar{\Delta}_*$ preserves products. By 11.9, this is true of $\bar{\Delta}_*$. To show that φ preserves products, we construct the diagram analogous to 11.12

in which the index r is raised, the stars are lowered, and Δ is replaced by μ. Its commutativity is proved just as before but using 9.3 in place of 9.6. It is essential here that the product μ_2 in $G \times G$ be defined by $(g_1, g_2)(g_3, g_4) = (g_1 g_3, g_2 g_4)$ so that $\mu_2 = (\mu \times \mu)(1 \times T \times 1)$.

<u>Proof of 11.11.</u> Since $H(G)$ is free, the Künneth theorem asserts that $\alpha\colon H(G) \otimes H(G) \approx H(G \times G)$. Then $\alpha^{-1}\Delta_*$ defines a coproduct for $H(G)$, and $H(G)$ becomes a Hopf algebra.

Let us recall a theorem of homological algebra [7; p.232,4.5]. If A is a Hopf algebra, then the product in $\mathrm{Ext}_A(R,R)$ can be obtained by "pulling back" the external cross-product via the morphism induced by the coproduct $\Delta\colon A \longrightarrow A \otimes A$, thus

$$\mathrm{Ext}_A(R,R) \otimes \mathrm{Ext}_A(R,R) \xrightarrow{\alpha} \mathrm{Ext}_{A \otimes A}(R,R) \xrightarrow{\Delta^*} \mathrm{Ext}_A(R,R) .$$

Since Δ is a morphism of algebras Δ^* is well defined.

Taking $A = H(G)$ and using a resolution $X = \{H(\mathcal{E}_n, \mathcal{E}_{n-1})\}$, the above prescription says to choose a Δ-mapping of resolutions $h\colon X \longrightarrow X \otimes X$, and then compute products on the cochain level by $uv = h^{\#}(u \otimes v)$. Now a geometric mapping of resolutions $\tilde{\Delta}\colon \mathcal{E} \longrightarrow \mathcal{E} \times \mathcal{E}$ induces just such an h, and the resulting $\bar{h}^*$ is our $\bar{\Delta}^*$. This shows that $E_2 \approx \mathrm{Ext}_{H(G)}(R,R)$ preserves products.

When A is a commutative algebra, the multiplication $\mu\colon A \otimes A \longrightarrow A$ is a morphism of algebras, and we obtain an internal product for $\mathrm{Tor}^A(R,R)$ as the composition

$$\mathrm{Tor}^A(R,R) \otimes \mathrm{Tor}^A(R,R) \xrightarrow{\alpha} \mathrm{Tor}^{A \otimes A}(R,R) \xrightarrow{\mu_*} \mathrm{Tor}^A(R,R) .$$

The same reasoning as above shows that, when $A = H(G)$, this corresponds precisely to the product we have defined in $E^2(B)$.

It is not easy to locate in the literature explicit definitions of the coproducts in Tor and Ext. If we take as their definitions the compositions

$$\mathrm{Tor}^A(R,R) \xrightarrow{\Delta_*} \mathrm{Tor}^{A\otimes A}(R,R) \xrightarrow{\alpha^{-1}} \mathrm{Tor}^A(R,R) \otimes \mathrm{Tor}^A(R,R),$$

$$\mathrm{Ext}_A(R,R) \xrightarrow{\mu^*} \mathrm{Ext}_{A\otimes A}(R,R) \xrightarrow{\alpha^{-1}} \mathrm{Ext}_A(R,R) \otimes \mathrm{Ext}_A(R,R),$$

where restrictions are imposed to insure that the α's have inverses [1; p.209, 3.1], then it follows as above that these correspond precisely to the coproducts we have defined in $E^2(B)$ and $E_2(B)$.

12. Further properties of compactly generated spaces

12.1. If X is compactly generated, and the open set U of X is such that each $x \in U$ has a neighborhood V whose closure $\overline{V}$ lies in U (e.g. if X is a regular space), then, U in its relative topology, is compactly generated.

For suppose $B \subset U$ and B meets each compact set of U in a closed set. Let x be a limit point of B in U. By assumption there is a neighborhood V of x such that $\overline{V} \subset U$. If C is compact in X, then $\overline{V} \cap C$ is a compact set of X in U. Since it is also compact in the relative topology of U, it follows that $B \cap \overline{V} \cap C$ is closed first in U, then in $\overline{V} \cap C$, and finally in X. As C is any compact set of X, it follows that $B \cap \overline{V}$ is closed in X. As x is a limit point of $B \cap \overline{V}$, we must have $x \in B \cap \overline{V}$, so $x \in B$; hence B is closed in U.

12.2. If $f: X \longrightarrow Y$ is continuous, $fX = Y$, X is compactly generated, and Y has the decomposition space topology induced by f (i.e. if $B \subset Y$ and $f^{-1}B$ is closed, then B is closed), then Y is compactly generated.

Suppose B meets each compact set of Y in a closed set. Let C be a compact set in X. Then fC is compact, $B \cap fC$ is closed, $f^{-1}(B \cap fC)$ is closed, and $f^{-1}(B \cap fC) \cap C$ is closed. Since this last set coincides that $f^{-1}B \cap C$, it follows that $f^{-1}B$

meets each compact set of X in a closed set; hence $f^{-1}B$ is closed in X. As Y has the decomposition topology, this implies that B is closed in Y. Hence Y is compactly generated.

12.3. If X is locally compact, and Y is compactly generated, then the standard topology of $X \times Y$ is compactly generated.

Let A be a subset of $X \times Y$ which meets each compact set in a closed set, and let (x_o, y_o) be a point of its complement. By local compactness, x_o has a neighborhood whose closure N is compact. Since $N \times y_o$ is compact, we have $A \cap (N \times y_c)$ is closed. It follows that x_o has a smaller neighborhood U such that $\bar{U} \times y_o$ does not meet A. Let B denote the projection in Y of $A \cap (\bar{U} \times Y)$. If C is a compact set in Y, then $A \cap (\bar{U} \times C)$ is compact, and therefore $B \cap C$ is closed. As Y is compactly generated, B must be closed in Y. Since y_o is not in B, it follows that $U \times (Y-B)$ is a neighborhood of (x_o, y_o) not meeting A. This proves that A is closed. Hence $X \times Y$ is compactly generated.

12.4. Let $\{X_n\}$ and $\{Y_n\}$ be filtrations of X and Y, respectively, by closed sets such that X, Y have the topologies of their respective unions (all compactly generated). Then $X \times Y$ has the topology of the union of the sets

$$F_n = \cup_{i=o}^{n} X_i \times Y_{n-i}, \qquad n = 0, 1, \ldots .$$

Suppose $A \subset X \times Y$ meets each F_n in a closed set. Let C be a compact set in $X \times Y$. The projections C', C'' of C in X, Y, respectively, are compact. By 1.7, there are integers p and q such that $C' \subset X_p$ and $C'' \subset Y_q$. Then $C \subset F_{p+q}$. Since $A \cap F_{p+q}$ is closed, and C is closed, we have that $A \cap C = (A \cap F_{p+q}) \cap C$ is closed. As this holds for every compact C, and $X \times Y$ is compactly generated, it follows that A is closed in $X \times Y$. Hence $X \times Y$ has the topology of the union.

12.5. Let $f: X \longrightarrow X'$ and $g: Y \longrightarrow Y'$ represent X', Y' as decomposition spaces, all compactly generated (see 12.2). Then $X' \times Y'$ is the decomposition space of $f \times g: X \times Y \longrightarrow X' \times Y'$.

Now $f \times g$ factors into the composition $(f \times 1)(1 \times g)$, and a composition of representations as a decomposition space is another such. Therefore it suffices to prove the special case where $Y = Y'$ and g is the identity. Suppose then that $A \subset X' \times Y$ and $(f \times 1)^{-1}A$ is closed in $X \times Y$. Let C be a compact set of $X' \times Y$. Let D, E be its projections in X', Y respectively. Then $D \times E$ is compact. If we can show that $A \cap (D \times E)$ is closed, it will follow that $A \cap C$ is closed and the proposition will be proved. Since $(f \times 1)^{-1}(D \times E) = f^{-1}D \times E$ is closed in $X \times Y$, it follows that $(f \times 1)^{-1}(A \cap (D \times E))$ is closed in $f^{-1}D \times E$. Substituting X, X', Y for $f^{-1}D, D, E$, respectively, we have reduced the proof to the case where X' and Y are compact. By 12.3, the products have the customary topologies.

Suppose then that $W \subset X' \times Y$, $(f \times 1)^{-1}W$ is open in $X \times Y$, and $(x'_o, y_o) \in W$. Choose $x_o \in X$ such that $fx_o = x'_o$. Since (x_o, y_o) is in the open set $(f \times 1)^{-1}W$, and Y is compact, there is a neighborhood V of y_o such that $x_o \times \overline{V} \subset (f \times 1)^{-1}W$. Let U be the set $x \in X$ such that $(fx) \times \overline{V} \subset W$. To see that U is open in X, let $x_1 \in U$. We can cover $x_1 \times \overline{V}$ by products of open sets contained in $(1 \times f)^{-1}W$. As $x_1 \times \overline{V}$ is compact, we can select a finite such covering. The intersection of their X factors gives a neighborhood N of x_1 such that $N \times \overline{V} \subset (f \times 1)^{-1}W$. Therefore U is open. By its definition, $U = f^{-1}fU$; hence fU is open in X', because X' has the decomposition topology. Then $(x'_o, y_o) \in (fU) \times V \subset W$. This shows that W is open in $X' \times Y$, and completes the proof of 12.5.

<u>Remark.</u> It is not too difficult to construct counter-examples to the conclusions of 12.4 and 12.5 using the customary product topology. Since both propositions are essential to us, this explains our working within the category of compactly generated spaces.

13. Further properties of NDR's

It was observed in §1 that an NDR in X is a neighborhood retract in X. This somewhat weaker property behaves badly when forming products. As an example, let X be the transfinite line $[0,\Omega]$ and let A be the point Ω. Then A is a neighborhood retract in X but not an NDR because it is not a G_δ in X. Let $Y = [0,1]$ and let B be the point 1, so that B is an NDR in Y. It is well known that if the point $A \times B$ is deleted from $X \times Y$, then the resulting space is not normal in that $(X-A) \times B$ and $A \times (Y-B)$ have no separation. This implies that $X \times B \cup A \times Y$ is not a neighborhood retract in $X \times Y$.

13.1. <u>Lemma.</u> If (X,A) and (Y,B) are NDR pairs, then so also is $(X,A) \times (Y,B)$.

<u>Proof.</u> Let $u\colon X \longrightarrow I$ and $h\colon I \times X \longrightarrow X$ represent A as an NDR in X, and let $v\colon Y \longrightarrow I$ and $j\colon I \times Y \longrightarrow Y$ represent B as an NDR in Y. Define $w\colon X \times Y \longrightarrow I$ by $w(x,y) = (ux)(vy)$. Clearly $w^{-1}(0)$ is $X \times B \cup A \times Y$. Define the homotopy $k\colon I \times X \times Y \longrightarrow X \times Y$ by

$$k(t,x,y) = \begin{cases} (x,y) & \text{if } x \in A \text{ and } y \in B \\ \left(h(t,x), j\left(\frac{ux}{vy}t, y\right)\right) & \text{if } vy \geqq ux \text{ and } vy > 0, \\ \left(h\left(\frac{vy}{ux}t, x\right), j(t,y)\right) & \text{if } ux \geqq vy \text{ and } ux > 0. \end{cases}$$

The domains of definition of the last two lines intersect in the relatively closed subset where $vy = ux > 0$, and both lines reduce to $(h(t,x), j(t,y))$. Thus they define a continuous function on

$I \times (X \times Y - A \times B)$. Hence the proof of continuity of k reduces to proving continuity at a point (t,x,y) in $I \times A \times B$. It suffices to prove that the components of k are continuous. So let U be an open set of X containing x , and V an open set of Y containing y . Since $x \in A$, we have $I \times \{x\} \subset h^{-1}U$. Since I is compact, and $h^{-1}U$ is open, there is an open set S of X containing x such that $I \times S \subset h^{-1}U$. Similarly there is an open set T of Y containing y such that $I \times T \subset j^{-1}V$. It follows that k maps $I \times S \times T$ into $U \times V$. This shows that k is continuous.

When $t = 0$, all three lines defining k reduce to (x,y). When $x \in A$ we have $ux = 0$, so $k(t,x,y)$ is given by line 1 or 2, and line 2 reduces to $(h(t,x),j(0,y)) = (x,y)$. Hence $k(t,x,y) = (x,y)$ whenever $x \in A$. Similarly $k(t,x,y) = (x,y)$ whenever $y \in B$.

Now let $t = 1$ and suppose $0 < w(x,y) < 1$. There are two similar cases according as $0 < ux < 1$ or $0 < vy < 1$. Consider the first. In the subcase $ux \leqq vy$, $k(1,x,y)$ is given by line 2. Since $h(1,x) \in A$, it follows that $k(1,x,y)$ is in $A \times Y$. In the subcase $vy < ux$ we must use line 3. Since $j(1,y) \in B$, it follows that $k(1,x,y)$ is in $X \times B$. This completes the proof.

13.2. Lemma. Let a space be the union $X \cup Y$ of closed subsets X,Y . Let A,B be NDR's in X,Y respectively such that $A \cap B = X \cap Y$. Then $A \cup B$ is an NDR in $X \cup Y$.

<u>Proof.</u> Let u, v, h and j be as in the proof of 13.1. Define $w: X \cup Y \longrightarrow I$ by $w|X = u$ and $w|Y = v$. Define $k: I \times (X \cup Y) \longrightarrow X \cup Y$ by $k|I \times X = h$ and $k|I \times Y = j$. Since u and v are zero on $A \cap B = X \cap Y$, w is well defined. Since h and j agree on $I \times (A \cap B) = I \times (X \cap Y)$, k is well defined. The proof that w, k represent $A \cup B$ as an NDR is routine and is omitted.

13.3. <u>Lemma.</u> If $A \subset B \subset X$, A is an NDR in B, and B is an NDR in X, then A is an NDR in X.

<u>Proof.</u> Let u, h be a representation of A as an NDR in B, and let v, k be a representation of B as an NDR in X. Since (X, B) has the homotopy extension property, the homotopy $h: I \times B \longrightarrow B$ extends to a homotopy $h': I \times X \longrightarrow X$ such that $h'(0, x) = x$ for all $x \in X$. Define a homotopy $m: I \times X \longrightarrow X$ by

$$m(t,x) = \begin{cases} k(2t, x) & \text{for } 0 \leqq t \leqq 1/2 , \\ h'(2t-1, k(1,x)) & \text{for } 1/2 \leqq t \leqq 1 . \end{cases}$$

Extend $u: B \longrightarrow I$ to a mapping $u': X \longrightarrow I$. Define $w: X \longrightarrow I$ by letting wx be the smaller of $vx + u'k(1,x)$ and 1. It is readily verified that w, m represent A as an NDR in X.

13.4. <u>Lemma.</u> If (X, A) and (Y, B) are NDR pairs, then so also are the nine non-trivial pairs formed from the five spaces $X \times Y$, $X \times B \cup A \times Y$, $X \times B$, $A \times Y$ and $A \times B$.

Proof. Lemma 13.1 applies directly to the five pairs $(X,A) \times (Y,B)$, $(X,\emptyset) \times (Y,B)$, $(X,A) \times (Y,\emptyset)$, $(X,A) \times (B,\emptyset)$ and $(A,\emptyset) \times (Y,B)$. If we apply 13.3 to $X \times Y \supset X \times B \supset A \times B$, it follows that $(X \times Y, A \times B)$ is an NDR. Choose a representation u,h of $(X \times B, A \times B)$ as an NDR. Extend over $A \times (Y-B)$ by letting u be zero and h be constant there. These extensions represent $(X \times B \cup A \times Y, A \times Y)$ as an NDR. Symmetrically $(X \times B \cup A \times Y, X \times B)$ is an NDR. Finally we apply 13.3 to $X \times B \cup A \times Y \supset A \times Y \supset A \times B$ to obtain that $(X \times B \cup A \times Y, A \times B)$ is an NDR.

13.5. Lemma. Let $f\colon (X,A) \longrightarrow (Y,B)$ be a relative homeomorphism, and let (X,A) and (Y,B) have representations u,h and v,k, respectively, as NDR pairs such that $vf = u$ and $k(1 \times f) = fh$, then f induces isomorphisms of the singular homology and cohomology groups.

Proof. Let $U = u^{-1}[0,1/2)$ and $V = v^{-1}[0,1/2)$. Then U,V are open, $U = f^{-1}V$, and f defines a relative homeomorphism $f_1\colon (X,U) \longrightarrow (Y,V)$. Since A is closed and U is open, the inclusion $(X-A,U-A) \subset (X,U)$ is a proper excision for singular theory; hence it induces isomorphisms. Similarly $(Y-B,V-B) \subset (Y,V)$ induces isomorphisms. Since f_1 restricts to a homeomorphism $(X-A,U-A) \longrightarrow (Y-B,V-B)$ of the excised pairs, it follows that f_{1*} is an isomorphism.

14. The convergence of the spectral sequence

Proof of 4.3. Since a singular simplex is compact, it follows from 1.6 that the singular complex $S(X)$ is the union of the subcomplexes $S(X_p)$ for $p \geqq 0$. An element $x \in H_s(X)$ is represented by an s-cycle y which, being finite, must lie in some $S(X_p)$. It determines then an element of $H_s(X_p)$, and then of $A^{\infty}_{p,s-p}$ whose image in $H_s(X)$ is x. Thus the direct limit maps onto $H_s(X)$. Suppose that $u \in A^{\infty}_{p,s-p}$ maps into zero in $H_s(X)$. It is represented by an s-cycle v of $S(X_p)$ which bounds a chain w in $S(X)$. As w is also finite, it lies in some $S(X_{p+r})$; this implies that $i^r u = 0$, and hence $u = 0$. This completes the proof.

Proof of 4.11. The first conclusion was proved in §4 when we showed that 4.10 implies 4.9. Assume next that the coefficient group is compact. Let $x \in H^t(X,X_{p-1})$ lie in the image of $H^t(X,X_k)$ for all $k \geqq p$. By exactness this implies that x restricts to 0 in each $H^t(X_k,X_{p-1})$. Let u be a cocycle representing x. Then, for each $k \geqq 0$, $u|S(X_k)$ is the coboundary of a cochain of $S(X_k)$ which is zero on $S(X_{p-1})$. Let V_k denote the set of all such cochains. It is a coset of the group of cocycles of $S(X_k)$ which are zero on $S(X_{p-1})$. Since the coefficient group is compact, the groups of cochains and cocycles are compact, hence V_k is compact. The sequence $V_p \longleftarrow V_{p+1} \longleftarrow \cdots \longleftarrow V_k \longleftarrow \cdots$, where the maps are induced by inclusions of subcomplexes, is an inverse system of compact sets. As is well known, there are elements $v_k \in V_k$, $k \geqq p$, such that $v_{k+1}|S(X_k)$ is v_k. Define v to be the cochain of $S(X)$ which restricts to v_k on $S(X_k)$ for each $k \geqq p$. As X has the topology of the union, $S(X) = \cup_0^\infty S(X_k)$, hence v is well defined. It follows now that $\delta v = u$ because this holds on each $S(X_k)$. Therefore $x = 0$, hence 4.10 holds when the coefficient group is compact.

Assume now that $E_2^{p,q} = 0$ when $q < 0$. Let x and u be as above. We will construct a sequence of cochains v_k of $S(X_k)$ for $k \geqq t$ such that $\delta v_k = u|X_k$, v_k is zero on X_{p-1}, and $v_k|X_{k-1} = v_{k+1}|X_{k-1}$ for $k \geqq t$. Once this is done, we define v to be v_{k+1} on X_k for each k, and obtain thus a cochain such that $\delta v = u$; hence $x = 0$. Begin by selecting v_t to be any cochain of X_t which is zero on X_{p-1} and such that $\delta v_t = u|X_t$. Assume inductively that $v_t, v_{t+1}, \ldots, v_k$ have been properly selected. Extend v_k to a cochain v_k' of X_{k+2}, and let w be a cochain of X_{k+2} which is zero on X_{p-1} and such that $\delta w = u|X_{k+2}$. Let $z = \delta(v_k' - y)$. Since z is zero on X_k, it represents $\bar{z} \in H^t(X_{k+2}, X_k)$. By exactness it restricts to a class $z' \in H^t(X_{k+1}, X_k)$ such that $d_1 z' = 0$. Since $k+1 > t$, we have $E_2^{k+1,t-k-1} = 0$; hence $z' = d_1 y$ for some $y \in H^{t-1}(X_k, X_{k-1})$. Choose a cochain b of X_{k+1} which is zero on X_{k-1}, and is such that $b|X_k$ represents y. Then δb represents $d_1 y = z'$ on X_{k+1}. Since $z|X_{k+1}$ also represents z', there is a cochain c of X_{k+1} which is zero on X_k such that $\delta c = \delta b - z|X_{k+1}$. Now set

$$v_{k+1} = v_k'|X_{k+1} - b + c .$$

A brief calculation gives $\delta v_{k+1} = \delta w | X_{k+1} = u | X_{k+1}$. Since b, c are zero on X_{k-1} , we have $v_{k+1} | X_{k-1} = v_k | X_{k-1}$. This completes the proof.

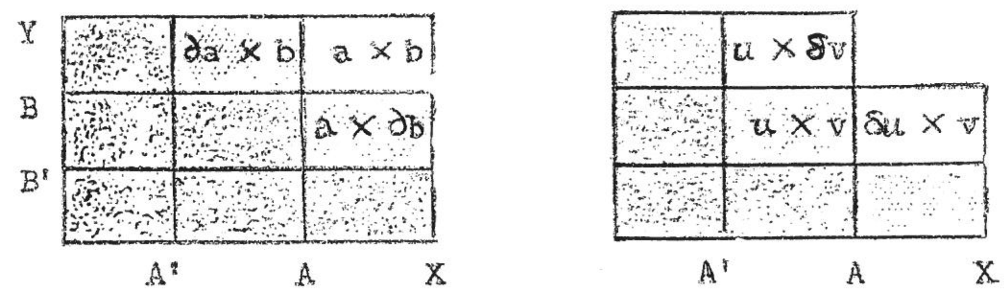
<u>Example.</u> We give now an example of a locally finite countable complex X , and a filtration by subcomplexes such that 4.9b does not hold if integer coefficients are used. Let $\{S_p\}$ be a sequence of 1-spheres, let C_p denote the mapping cylinder of a map of degree 2 of S_p onto S_{p+1} , and set $X_p = \cup_0^p C_i$, $X = \cup_0^\infty C_i$. Let $u \in H^2(X)$ be represented by the 2-cocycle v which has the value 1 on each C_i . Let w_p be the 1-cochain having the value $2^{p-i}+1$ on S_i for $i = 0, 1, \ldots, p$. Since $\partial C_i = 2S_{i+1} - S_i$, it follows that δw_p has the value 1 on C_i for $i < p$. Hence $v - \delta w_p$ is zero on X_p . Therefore u is in the image of $H^2(X, X_p)$ for every p . Now we cannot have $v = \delta w$ in X , for otherwise the value of w on S_0 would need to be divisible by arbitrarily large powers of 2 . Therefore $u \neq 0$.

15. The boundary of a product

In order to discuss products in spectral sequences, we will need a formula for the boundary of a product which on the chain level is the standard formula $\partial(a \times b) = \partial a \times b + (-1)^p a \times \partial b$ $(p = \dim a)$. We will state and prove corresponding formulas on the level of homology and cohomology.

Let (X,A,A') be an NDR triple, i.e. $A' \subset A \subset X$ and (X,A) and (A,A') are NDR pairs. Let (Y,B,B') be an NDR triple. By 13.4, the various pairs we can form in $X \times Y$, using products of these spaces and their unions, are likewise NDRs. The two theorems we wish to prove are indicated schematically by the two figures in 15.1. We picture

(15.1)



X, A and A' as intervals with A' on the left of A, and A on the left of X. Picturing (Y,B,B') similarly, then the various subsets of $X \times Y$ are indicated by unions of rectangular blocks. Thus the heavily shaded portion in the left figure denotes the union of $A' \times Y$, $A \times B$ and $X \times B'$. Let $a \in H_p(X,A)$ and $b \in H_q(Y,B)$.

Then $a \times b \in H_{p+q}((X,A) \times (Y,B))$. It has a boundary

$$\partial(a \times b) \in H_{p+q-1}(X \times B \cup A \times Y,\ X \times B' \cup A \times B \cup A' \times Y) .$$

We will show that it is a sum of two terms as suggested by the figure. On the right of 15.1 is the figure for cohomology where $u \in H^p(A,A')$ and $v \in H^q(B,B')$.

The precise formulation of the result for homology

(15.2)

$$\begin{array}{ccccc}
 & & H((X,A) \times (Y,B)) & & \\
 & \swarrow^{\partial_1} & \downarrow \partial & \searrow^{\partial_2} & \\
H(X \times B \cup A \times Y, X \times B \cup A' \times Y) & & & & H(X \times B \cup A \times Y, X \times B' \cup A \times Y) \\
\uparrow k_1 \approx & \nwarrow^{j_1} & & \nearrow^{j_2} & \uparrow k_2 \approx \\
 & & H(X \times B \cup A \times Y,\ X \times B' \cup A \times B \cup A' \times Y) & & \\
 & \nearrow^{i_1} & & \nwarrow^{i_2} & \\
H((A,A') \times (Y,B)) & & & & H((X,A) \times (B,B'))
\end{array}$$

is based on the diagram 15.2 where the morphisms i,j,k are induced by inclusions. Taking homology with coefficients in a ring R , the product $a \times b$ lies in the group at the top of the diagram. Taking $\partial a \in H_{p-1}(A,A')$, then $\partial a \times b$ lies in the group at the lower left of 15.1. Similarly $a \times \partial b$ is in the group on the lower right. The desired result is

15.3. <u>Theorem.</u> $\partial(a \times b) = i_1(\partial a \times b) + (-1)^p\, i_2(a \times \partial b)$.

<u>Proof.</u> Now i_1, j_2 are morphisms of the homology sequence of a triad (see [3; p. 35]), hence $\operatorname{Im} i_1 = \operatorname{Ker} j_2$. Similarly, $\operatorname{Im} i_2 = \operatorname{Ker} j_1$. Moreover k_1, k_2 are isomorphisms since the corresponding inclusions are excisions. It follows now, as in [3; p. 32],

that i_1, i_2 give an injective representation of the group in the middle of 15.1 as a direct sum. Then the commutativity of the diagram gives the relation

$$(15.4) \qquad \partial(a \times b) = i_1 k_1^{-1} \partial_1 (a \times b) + i_2 k_2^{-1} \partial_2 (a \times b) \; .$$

Thus the proof is reduced to showing

$$(15.5) \qquad k_1^{-1} \partial_1 (a \times b) = \partial a \times b \; , \qquad k_2^{-1} \partial_2 (a \times b) = (-1)^p \, a \times \partial b \; .$$

We will prove only the first relation since obvious modifications will provide a proof of the second.

Consider first the special case where A' and B are empty. Then k_1 is an identity map, and we need only prove that $\partial_1(a \times b) = \partial a \times b$. Denoting the singular chain complex of X by SX , we have the diagram

$$(15.6) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & SA \otimes SY & \xrightarrow{i'} & SX \otimes SY & \xrightarrow{j'} & (SX/SA) \otimes SY & \longrightarrow & 0 \\ & & \uparrow\downarrow & & \lambda\uparrow\downarrow\mu & & \uparrow\downarrow & & \\ 0 & \longrightarrow & S(A \times Y) & \xrightarrow{i} & S(X \times Y) & \xrightarrow{j} & S(X \times Y)/S(A \times X) & \longrightarrow & 0 \end{array}$$

where the horizontal rows are exact (SY is free), and the vertical chain maps are the natural homotopy equivalences constructed by Eilenberg and Zilber [4]. Choose cycles $a_1 \in SX/SA$ and $b_1 \in SY$ representing a and b , and choose $a_2 \in SX$ such that $j'a_2 = a_1$. Then ∂a_2 in SA represents ∂a . Now, by definition, $a \times b$ is represented by $\mu(a_1 \otimes b_1)$. Since $j\mu(a_2 \otimes b_1) = \mu(a_1 \otimes b_1)$, it follows that $\partial_1(a \times b)$ is represented by

$$\partial\mu(a_2 \otimes b_1) = \mu\partial(a_2 \otimes b_1) = \mu(\partial a_2 \otimes b_1) \; ,$$

and the latter represents $\partial a \times b$. Hence $\partial_1(a \times b) = \partial a \times b$.

Consider next the special case where A' is empty and B is a single point. The inclusion $g: Y \longrightarrow (Y,B)$ and the identity map of (X,A) induce a mapping of the first case into the second. Since g_* is an epimorphism, there is a $b' \in H_q(Y)$ such that $g_*b' = b$. Referring to

$$\begin{array}{ccccc} H((X,A) \times Y) & \xrightarrow{\partial_1} & H(A \times Y) & & \\ \downarrow g_1 & & \downarrow g_2 & \searrow g_3 & \\ H((X,A) \times (Y,B)) & \xrightarrow{\partial_1} & H(X \times B \cup A \times Y, X \times B) & \xleftarrow{k_1} & H(A \times (Y,B)) \end{array} \qquad (15.7)$$

the diagram 15.7 where g_1, g_2, g_3 and k_1 are induced by inclusions, we have $g_1(a \times b') = a \times b$ by the naturality of the cross-product. Commutativity in the diagram gives

$$\partial_1(a \times b) = \partial_1 g_1(a \times b') = g_2\partial_1(a \times b') = k_1 g_3(\partial a \times b') = k_1(\partial a \times b) .$$

As k_1 is an isomorphism, this case is proved.

Consider next the case where A' is empty. Let Y' be obtained from Y by collapsing B to a point B' , and let $h: (Y,B) \longrightarrow (Y',B')$ be the collapsing map. Then $1 \times h$ maps the present case into the preceding, and we have

$$(1 \times h)_* k_1^{-1}\partial_1(a \times b) = k_1'^{-1}\partial_1(a \times h_*b) = \partial a \times h_*b = (1 \times h)_*(\partial a \times b) .$$

Since $1 \times h$ is a relative homeomorphism $A \times (Y,B) \longrightarrow A \times (Y',B')$ of NDR's, lemma 13.7 asserts that $(1 \times h)_*$ is an isomorphism. Applying its inverse to the last equation gives the required result.

Finally, consider the general case. The inclusion $f: A \longrightarrow (A,A')$ gives a mapping of the preceding case into this one, and we have

$$k_1^{-1}\partial_1(a \times b) = (f \times 1)_* k_1'^{-1}\partial_1(a \times b) = (f \times 1)_*(\partial' a \times b)$$
$$= (f_*\partial' a) \times b = \partial a \times b \,.$$

This completes the proof of theorem 15.3.

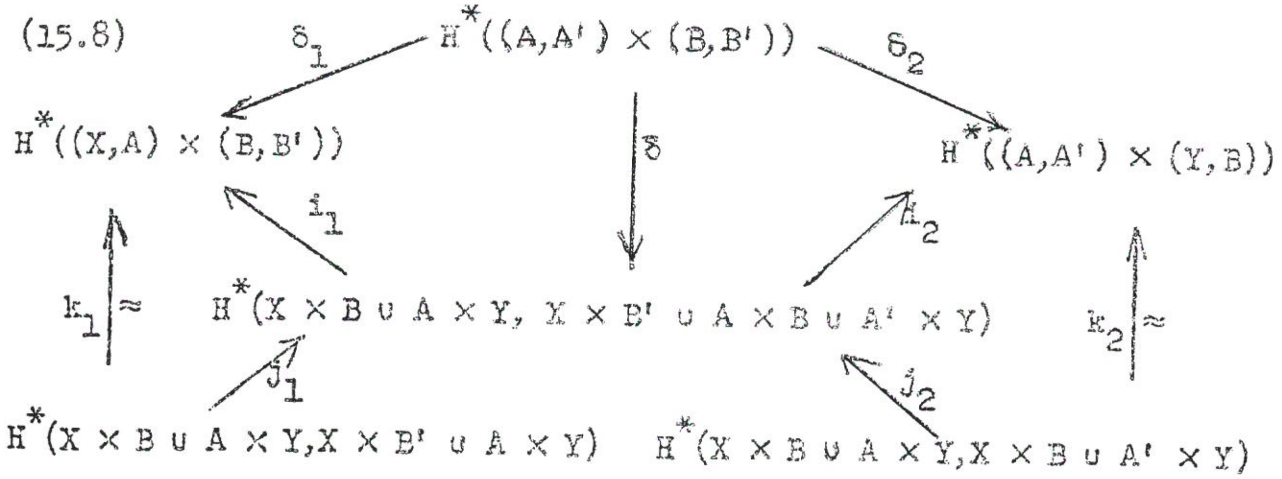
The theorem for cohomology is only roughly dual to the preceding. A first difference occurs in the definition of the cross-product of classes $u \in H^p(A,A')$ and $v \in H^q(B,B')$. The definition is based on the diagram of mappings of chain complexes:

$$(SA/SA') \otimes (SB/SB') = SA \otimes SB/(SA \otimes SB' \cup SA' \otimes SB)$$
$$\lambda \downarrow \uparrow \mu$$
$$S(A \times B)/S(A \times B' \cup A' \times B) \xleftarrow{g} S(A \times B)/S(A \times B') \cup S(A' \times B)$$

Since λ, μ are natural transformations, they induce a homotopy equivalence of quotients as indicated. On passing to cohomology, we find that the map g, induced by inclusions, has the wrong direction for converting $\mu^*(u \otimes v)$ into a product $u \times v$ in $H^*((A,A') \times (B,B))$. One needs to know that g^* is an isomorphism so that we may set $u \times v = g^{*-1}\mu^*(u \otimes v)$. Since we are dealing with NDR pairs, the triad $(A \times B; A \times B', A' \times B)$ is proper, and the inclusion of $S(A \times B') \cup S(A' \times B)$ in $S(A \times B' \cup A' \times B)$ is a homotopy equivalence [3; p. 204]. Hence g^* is an isomorphism.

The theorem for cohomology is based on the diagram 15.8 where the morphisms i, j, k are induced

(15.8)



by inclusions, and the coboundary operators are those of certain triads [3; p. 37] (e.g. δ_1 is that of the triad $(X \times B; A \times B, X \times B' \cup A' \times B)$). The product $u \times v$ lies in the group at the top of the diagram, and $\delta u \times v$, $u \times \delta v$ are on the upper left and right respectively.

15.9. <u>Theorem.</u> $\delta(u \times v) = j_1 k_1^{-1}(\delta u \times v) + (-1)^p\, j_2 k_2^{-1}(u \times \delta v)$.

<u>Proof.</u> We have $\operatorname{Im} j_2 = \operatorname{Ker} i_1$ because i_1, j_2 are morphisms of the cohomology sequence of the triad $(X \times B \cup A \times Y;\ X \times B,\ X \times B' \cup A \times B \cup A' \times Y)$. <u>Similarly</u>, $\operatorname{Im} j_1 = \operatorname{Ker} i_2$. Since we are dealing with NDR's, k_1 and k_2 are isomorphisms induced by excisions. It follows that j_1, j_2 give an injective representation as a direct sum, and

$$\delta(u \times v) = j_1 k_1^{-1} i_1 \delta(u \times v) + j_2 k_2^{-1} i_2 \delta(u \times v)$$
$$= j_1 k_1^{-1} \delta_1(u \times v) + j_2 k_2^{-1} \delta_2(u \times v) .$$

Thus we are reduced to proving

(15.10) $$\delta_1(u \times v) = \delta u \times v , \quad \delta_2(u \times v) = (-1)^p u \times \delta v .$$

We will prove only the first, as the second is similar.

Consider first the special case where A' and B' are empty. We need now the diagram obtained from 15.6 by replacing Y by B throughout. Denote the dual cochain complex by $(SA)^*$, etc. Choose cocycles $u_1 \in (SA)^*$, $v_1 \in (SB)^*$ representing u, v respectively. Let $u_2 \in (SX)^*$ be a cochain extending u_1 so that $\delta u_2 \in (SX/SA)^*$ represents δu. Then $\lambda^*(u_1 \otimes v_1)$ represents $u \times v$, and $\lambda^*(u_2 \otimes v)$ is a cochain extending it to $S(X \times B)^*$. Hence $\delta\lambda^*(u_2 \otimes v)$ represents $\delta_1(u \times v)$. But

$$\delta\lambda^*(u_2 \otimes v) = \lambda^*\delta(u_2 \otimes v) = \lambda^*(\delta u_2 \otimes v)$$

also represents $\delta u \times v$. Thus 15.10 holds in this case.

Consider next the case where A' is empty and B' is a point. Geometrically, the first case maps into the second, but the algebraic map of cohomology sends the second into the first. If $g\colon B \longrightarrow (B,B')$ is the inclusion, we obtain directly from case 1 that

$$(1 \times g)^*\delta_1(u \times v) = (1 \times g)^*(\delta u \times v) .$$

To prove that $(1 \times g)^*$ is a monomorphism, consider the mappings

$$(X,A) \xrightarrow{f} (X \times B' \cup A \times B, A \times B) \xrightarrow{h} (X \times B, A \times B) \xrightarrow{k} (X,A)$$

where f is the section through the point B', h is the inclusion and k the projection. Since khf is the identity and f is a relative homeomorphism, it follows that h induces an epimorphism of cohomology. This and exactness of the cohomology sequence of the obvious triple imply that the mapping $(1 \times g)^*$ of the sequence is a monomorphism. This proves 15.10 in case 2.

Consider next the case where A' is empty. Let $f\colon (B,B') \longrightarrow (\bar{B},b)$ collapse B' to a point b. The resulting map of the cohomology diagram sends case 2 into case 3. Since f^* is an isomorphism, there is a $\bar{v} \in H^q(\bar{B},b)$ such that $f^*\bar{v} = v$. By case 2 we have $\delta_1(u \times \bar{v}) = \delta u \times \bar{v}$. We apply $(1 \times f)^*$ to both sides, and deduce readily that $\delta_1(u \times v) = \delta u \times v$.

Finally consider the general case. The inclusion $f\colon A \longrightarrow (A,A')$ leads to a geometric map of case 3 into case 4, and hence induces an algebraic map of case 4 into case 3. Now $\delta_1(f^*u \times v) = \delta f^*u \times v$ by case 3. Moreover the definition of coboundary gives $\delta f^*u = \delta u$ in $H^*(X,A)$. For the same reason, $\delta_1(f \times 1)^*(u \times v) = \delta_1(u \times v)$. The last three relations combine to give 15.10. This completes the proof of theorem 15.9.

16. Proofs of properties of the cross-products

<u>Proof of 9.3.</u> Once 9.3i is proved for $r = 1$, it follows by standard arguments that the product of E^1-terms induces a product of E^2-terms. If this in turn satisfies 9.3i for $r = 2$, there is an induced product of E^3-terms; and so forth. Assume inductively that a product of the E^r-terms is induced for some r. Let $u' \in E^1_{p,s}(X)$, $v' \in E^1_{q,t}(Y)$ satisfy $d_h u' = 0 = d_h v'$ for $h < r$ so that u', v' represent classes u, v in the E^r-terms. We must show that $d_h(u' \cdot v') = 0$ for $h < r$, and that 9.3i holds.

Consider the diagram 16.1 where all maps

$$(16.1)\qquad \begin{array}{ccccc} H(X_p, X_{p-r}) & \xrightarrow{\partial''} & H(X_{p-r}) & \xrightarrow{j} & H(X_{p-r}, X_{p-r-1}) \\ \downarrow g & & \downarrow i^{r-1} & & \\ H(X_p, X_{p-1}) & \xrightarrow{\partial} & H(X_{p-1}) & & \\ & \partial' \searrow & \swarrow & & \\ & & H(X_{p-1}, X_{p-r}) & & \end{array}$$

are induced by inclusions or are boundary operators. Now $d_h u' = 0$ for $h < r$ implies that $\partial u'$ is in the image of i^{r-1}; hence, by exactness, $\partial' u' = 0$. This and exactness for the triple (X_p, X_{p-1}, X_{p-r}) gives an element $u'' \in H(X_p, X_{p-r})$ such that $gu'' = u'$. It follows that $j\partial'' u''$ represents $d_r u$. Similarly there is a $v'' \in H(Y_q, Y_{q-r})$ which maps into v' and whose boundary in $H(Y_{q-r}, Y_{q-r-1})$ represents $d_r v$.

We have in $X \times Y$ the diagram 16.2. If we start with $u'' \times v''$ in

$$\begin{array}{ccc} H((X_p,X_{p-r}) \times (Y_q,Y_{q-r}) & \xrightarrow{\partial} & H(X_p \times Y_{q-r} \cup X_{p-r} \times Y_q) \\ \downarrow & & \downarrow \\ H(X_{p,1} \times Y_{q,1}) & & H((X \times Y)_{p+q-r}) \\ \downarrow & & \downarrow i^{r-1} \\ H((X \times Y)_{p+q,1}) & \xrightarrow{\partial} & H((X \times Y)_{p+q-1}) \end{array} \tag{16.2}$$

the upper left corner and move down and over, we obtain successively $u' \times v'$, $u' \cdot v'$ and $\partial(u' \cdot v')$. Moving over and down gives the same element; hence $\partial(u' \cdot v')$ is in the image of i^{r-1}, and therefore $d_h(u' \cdot v') = 0$ for all $h < r$. Moreover the same argument shows that $\partial(u'' \times v'')$, on the upper right, maps into a representative of $d_r(u \cdot v)$ in $H((X \times Y)_{p+q-r,1})$.

We now apply 15.3 to the product $u'' \times v''$ where (X,A,A') of 15.3 is (X_p,X_{p-r},X_{p-r-1}) and (Y,B,B') is (Y_q,Y_{q-r},Y_{q-r-1}). By hypothesis and 13.4 these are NDR triples. By 15.3, we have

$$\partial(u'' \times v'') = i_1(\partial u'' \times v'') + (-1)^{p+s} i_2(u'' \times \partial v'')$$

in

$$H((X_p \times Y_{q-r}) \cup (X_{p-r} \times Y_q), (X_p \times Y_{q-r-1}) \cup (X_{p-r} \times Y_{q-r}) \cup (X_{p-r-1} \times Y_q)) .$$

As noted above, its image in $H((X \times Y)_{p+q-r,1})$ represents $d_r(u \cdot v)$. It is easily checked that the images of the two terms on the right represent $d_r u \cdot v$ and $(-1)^{p+s} u \cdot d_r v$ respectively. This proves 9.3i.

In case u and v are in E^{∞}, we choose representatives u', v' in E^1. Following the argument above, we take $r > p, q$, and find $u'' \in H(X_p)$ and $v'' \in H(Y_q)$ such that $ju'' = u'$ and $jv'' = v'$. The inclusion $k: X_p \times Y_q \longrightarrow (X \times Y)_{p+q}$ gives an element $k_*(u'' \times v'')$ such that $jk_*(u'' \times v'') = k_*(u' \times v') = u' \cdot v'$ (see 9.2). Taking images in $H(X)$, $H(Y)$ and $H(X \times Y)$ of u'', v'' and $k_*(u'' \times v'')$, it follows that the products in the E^{∞}-terms and in the spaces correspond. This proves 9.3ii.

To prove naturality under mappings $f: X \longrightarrow X'$ and $g: Y \longrightarrow Y'$, we use $f_* \otimes g_*$ and $(f \times g)_*$ to obtain a mapping of the diagram 9.2 for X, Y into the corresponding diagram for X', Y'. It is well known that α is natural, i.e. $(f \times g)_*(u \times v) = f_* u \times g_* v$. Since the k's are inclusions, the other square is also commutative. This gives $(f \times g)_*(u \cdot v) = f_* u \cdot g_* v$ at the E^1-level. It follows that the same relation holds at the E^r-level since the products for E^r are induced by those for E^1. Hence 9.3 iii holds.

It is well known that the product α satisfies the commutative law $T_*(u \times v) = (-1)^{(p+s)(q+t)} v \times u$. We apply to this relation the mapping k of 9.2, and use $k_* T_* = T_* k_*$ to obtain the commutative law for $T_*(u \cdot v)$ at the E^1-level. This implies the same relation in each E^r by passing to classes. Thus 9.3 iv holds.

The diagram for proving associativity has nine groups in a 3×3 pattern. One of the four squares is commutative because the cross-product is known to be associative. Two other squares are commutative because the cross-product is natural. The remaining square is commutative because its morphisms are induced by inclusions. This proves 9.3 v.

The case $r = 1$ of 9.3 vi implies, on passing to classes, the cases $r > 1$. In the case $r = 1$, the product $1 \cdot v$ is given by 9.2 in which k_* is the identity because $p = 0$. Thus we must show that $g_* v = 1 \times v$. But this is trivially true on the chain level. This completes the proof of 9.3.

<u>Proof of 9.4.</u> We need first the fact that the morphisms

$$H(X_{i,1} \times Y_{p-i,1}) \longrightarrow H((X \times Y)_{p,1}), \qquad i = 0,1,\ldots,p,$$

form an injective representation of the group on the right as a direct sum. In [3; p. 77], a direct sum theorem is proved for a complex and subcomplexes. The cellular structure is used only to insure that certain triads are proper. Thus the theorem still holds under the weaker hypothesis that these triads are NDR's. If the K, L, and K_i of the theorem are taken to be $(X \times Y)_p$, $(X \times Y)_{p-1}$, and $X_i \times Y_{p-i}$, then the weaker hypotheses are fulfilled, and the conclusion follows.

Since every $E^1_{p,s}(X) = H_{p+s}(X_{p,1})$ is free, the Künneth theorem for a product space [1; p. 373] asserts that the cross-product morphisms

$$\alpha_t : H_t(X_{i,1}) \otimes H_{n-t}(Y_{p-i,1}) \longrightarrow H_n(X_{i,1} \times Y_{p-i,1})$$

for $t = 0,1,\ldots,n$, give an injective representation as a direct sum. Combining these two direct sum representations gives the assertion of the theorem for $r = 1$.

Assume inductively that the assertion holds for some $r \geqq 1$. We write $E^r(X) = \Sigma_i K_i$ as a direct sum of free chain complexes each having d_r as a boundary operator; a typical K_i has the form

$$\cdots \xrightarrow{d_r} E^r_{p+r,q-r+1} \xrightarrow{d_r} E^r_{p,q} \xrightarrow{d_r} E^r_{p-r,q+r-1} \xrightarrow{d_r} \cdots .$$

Then $E^{r+1}(X) = H(E^r(X)) = \Sigma_i H(K_i)$. Similarly, we write $E^r(Y) = \Sigma_j L_j$ as a direct sum of chain subcomplexes. It follows that

$$E^r(X \times Y) \approx E^r(X) \otimes E^r(Y) = \Sigma_{i,j} K_i \otimes L_j$$

is again a direct sum of subcomplexes $K_i \otimes L_j$. Hence

$$E^{r+1}(X \times Y) = H(E^r(X \times Y) = \Sigma_{i,j} H(K_i \otimes L_j) .$$

Since every $E^{r+1}_{p,q}(X)$ is free, so is every $H_t(K_i)$. Since K_i and $H(K_i)$ are free, the Künneth theorem asserts that $H(K_i) \otimes H(L_j) \approx H(K_i \otimes L_j)$. Combining these isomorphisms yields the assertion of the theorem for $r+1$. This proves the inductive step, and completes the proof of the theorem.

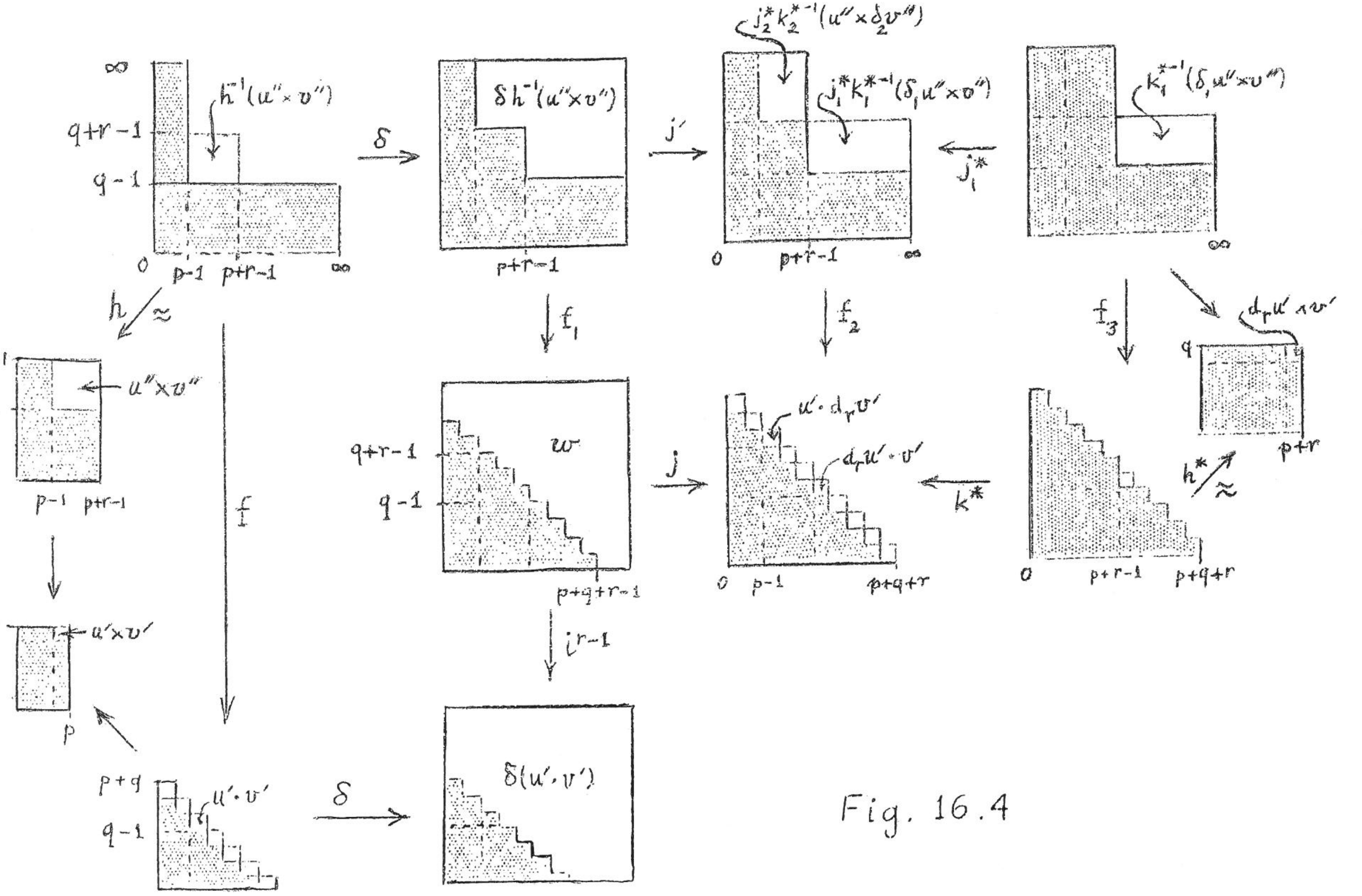
<u>Proof of 9.6.</u> If we can prove 9.6 i for $r = 1$, it follows that φ induces a product of E_2-terms, etc. Assume, inductively, that a product of E_r-terms is induced for some $r \geqq 0$. Let $u' \in E_1^{p,s}(X)$, $v' \in E_1^{q,t}(Y)$ satisfy $d_h u' = 0 = d_h v'$ for $h < r$ so that u', v' represent classes u, v in the E_r-terms. We must show that $d_h(u' \cdot v') = 0$ for $h < r$, and that 9.6 i holds.

Consider the diagram 16.3 induced by inclusions.

$$
(16.3)\qquad
\begin{array}{ccccc}
H^*(X_{p+r-1}, X_{p-1}) & \xrightarrow{\delta'} & H^*(X, X_{p+r-1}) & \xrightarrow{j} & H^*(X_{p+r}, X_{p+r-1}) \\
\downarrow g & & \downarrow i^{r-1} & & \\
H^*(X_p, X_{p-1}) & \xrightarrow{\delta} & H^*(X, X_p) & & \\
& \delta'' \searrow & \swarrow i' & & \\
& & H^*(X_{p+r-1}, X_p) & &
\end{array}
$$

Now $d_h u' = 0$ for $h < r$ implies $\delta u' \in \operatorname{Im} i^{r-1}$; hence, by exactness, $\delta'' u' = i' \delta u' = 0$. Exactness for the triple $(X_{p+r-1}, X_p, X_{p-1})$ now gives an element $u'' \in H^*(X_{p+r-1}, X_{p-1})$ such that $g u'' = u'$. It follows that $j \delta' u'' = d_r u'$ represents $d_r u$. Similarly there is a $v'' \in H^*(Y_{q+r-1}, Y_{q-1})$ which restricts to v' and whose coboundary in $H^*(Y_{q+r,1})$ represents $d_r v$.

To simplify the following argument we appeal to the diagram 16.4. Each of the twelve figures of the diagram stands for the cohomology of a pair of subspaces of $X \times Y$. The bounding edge indicates the larger space of the pair (following the scheme described in 15.1), and the shaded area indicates the smaller one. For example, on the lower left we have $H^*((X \times Y)_{p+q,1})$, and just to the left of center we have $H^*(X \times Y, (X \times Y)_{p+q+r-1})$. All of the arrows denote morphisms induced by inclusions except for the two coboundaries of triples.



∞
q+r−1
q−1
0
p−1
p+r−1
∞
h⁻¹(u″×v″)
δ
δh⁻¹(u″×v″)
p+r−1
j′
j₂*k₂*⁻¹(u″×δ₂v″)
j₁*k₁*⁻¹(δ₁u″×v″)
0
p+r−1
∞
j₁*
k₁*⁻¹(δ₁u″×v″)
∞
h
≈
f₁
f₂
f₃
d_r u′×v′
q
q+r−1
q−1
u″×v″
p−1
p+r−1
w
q+r−1
q−1
p+q+r−1
j
u′·d_r v′
d_r u′·v′
0
p−1
p+q+r
k*
0
p+r−1
p+q+r
h*
≈
p+r
f
j^{r−1}
q
u′×v′
p
p+q
q−1
u′·v′
δ
δ(u′·v′)

Fig. 16.4

To show that $d_h(u'\cdot v') = 0$ for $h < r$, we must show that $\delta(u'\cdot v') = i^{r-1}w$ for some w as indicated. Using the quadrilateral on the extreme left of 16.4, we see that $fh^{-1}(u'' \times v'') = u'\cdot v'$. By the commutativity of the diagram, it follows that

$$w = f_1 \delta h^{-1}(u'' \times v'')$$

satisfies the required condition. Moreover jw represents $d_r(u\cdot v)$.

We must now show that jw is a sum of two terms representing $d_r u\cdot v$ and $(-1)^{p+s} u\cdot d_r v$ respectively. By 15.9, we have the decomposition

$$j'\delta h^{-1}(u'' \times v'') = j_1^* k_1^{*-1}(\delta_1 u'' \times v'') + (-1)^{p+s} j_2^* k_2^{*-1}(u'' \times \delta_2 u'') .$$

Apply f_2 to this relation, and use $f_2 j' = jf_1$ to obtain

$$jw = f_2 j_1^* k_1^{*-1}(\delta_1 u'' \times v'') + (-1)^{p+s} f_2 j_2^* k_2^{*-1}(u'' \times \delta_2 v'') .$$

The diagram on the right of 16.4 shows that the first term reduces to $d_r u' \cdot v'$. An analogous diagram shows that the second term reduces to $(-1)^{p+s} u'\cdot d_r v'$. This completes the proof of 9.6i.

To prove 9.6ii, let $u \in E_1^{p,s}(X)$ and $v \in E_1^{q,t}(Y)$ be d_i-cycles for every i. Then $\delta u \in H^{p+s+1}(X, X_p)$ is in the image $H^{p+s+1}(X, X_{p+i})$ for every i. In order to formulate 9.6ii we must suppose that we are in the situation where the spectral sequences converge, namely, condition 4.10 holds. It follows that $\delta u = 0$. Then exactness gives $u' \in H^{p+s}(X, X_{p-1})$ such that $ju' = u$. Similarly there is a $v' \in H^{q+t}(Y, Y_{q-1})$ such that $jv' = v$. In the diagram 16.5, all morphisms are induced by inclusions.

$$
\begin{array}{ccccc}
H^*((X,X_{p-1})\times(Y,Y_{q-1})) & \xrightarrow{f} & H^*(X\times Y,(X\times Y)_{p+q-1}) & \xrightarrow{i} & H^*(X\times Y) \\
\downarrow j\times j & \searrow & & & \searrow j \\
H^*(X_{p,1}\times Y_{q,1}) & \underset{h^*}{\xleftarrow{\approx}} & H^*((X\times Y)_{p+q},\bigcup_{j\neq p} X_j\times Y_{p+q-j}) & \xrightarrow{k^*} & H^*((X\times Y)_{p+q,1})
\end{array} \tag{16.5}
$$

Starting in the upper left corner with the element $u'\times v'$, and moving down and over gives $u\times v$ and $k^*h^{*-1}(u\times v) = u\cdot v$. Therefore $jf(u'\times v') = u\cdot v$. Since $if(u'\times v') = iu'\times iv'$, the assertion 9.6ii follows.

To prove naturality under mappings $f\colon X\longrightarrow X'$ and $g\colon Y\rightarrow Y'$, we use $f_*\otimes g_*$ and morphisms induced by restrictions of $f\times g$ to obtain a mapping of the diagram 9.5 for X',Y' into the same for X,Y. It is well known that α is natural, i.e. $(f\times g)^*(u'\times v') = f^*u'\times g^*u'$. Since the h's and k's are induced by inclusions, the other squares are also commutative. This gives $(f\times g)^*(u'\cdot v') = f^*u'\cdot g^*v'$ at the E_1-level. By passing to classes, the same inclusion follows at the E_r-level. Hence 9.6 iii hold.

It is well known that α satisfies the commutative law $T^*(u\times v) = (-1)^{(p+s)(q+t)}v\times u$. We apply to this relation the mappings h^{*-1} and k^* of 9.5. Using obvious commutativities, we obtain $T^*(u\cdot v) = (-1)^{(p+s)(q+t)}v\cdot u$ at the E_1-level. Passing to classes gives the same result at each E_r-level. This proves 9.6iv.

The diagram for proving associativity has sixteen groups in a 4×4 pattern. One of the nine squares is commutative because α is associative. Four other squares are commutative because α is natural. The remaining four squares are commutative because its morphisms are induced by inclusions. This proves 9.6 v.

In the case where X is a single point, we have $X = X_o = X_i$ for $i \geqq 0$. It follows that three of the pairs in 9.5 coincide, and the mappings h and k are identities. Thus, for the E_1-level, we have $1 \cdot v = 1 \times v$. Now $g^* v = 1 \times v$ is an immediate consequence of the cochain definition of α. This proves 9.6vi when $r = 1$. Its truth for $r > 1$ follows by passing to classes. This completes the proof of Theorem 9.6.

<u>Proof of 9.8.</u> We need first the fact that the morphisms

$$H^*((X \times Y)_{m,1}) \longrightarrow H^*(X_{p,1} \times Y_{m-p,1}), \quad p = 0, 1, \ldots, m$$

induced by inclusions form a projective representation of the module on the left as a direct sum. This is proved by the argument dual to that of the first paragraph of the proof of 9.4.

Consider now the diagram

$$\begin{array}{ccc} H^*(X_{p,1}) \otimes H^*(Y_{q,1}) & \xrightarrow{\alpha} & H^*(X_{p,1} \times Y_{q,1}) \\ \downarrow{\scriptstyle \lambda \otimes 1} & & \downarrow{\scriptstyle \mu} \\ \mathrm{Hom}(H(X_{p,1}), R) \otimes H^*(Y_{q,1})) & \xrightarrow{\nu} & \mathrm{Hom}(H(X_{p,1}), H^*(Y_{q,1})) \end{array}$$

where λ, μ, ν are the indicated natural transformations of functors. Since $H(X_{p,1})$ is free, we have by [1; p.373, Cor 5.2] that λ and μ are isomorphisms. Since $H(X_{p,1})$ is free and finitely generated, ν is also an isomorphism, because it is clearly an isomorphism when $H(X_{p,1})$ is a copy of R, and we may take finite direct sums on both sides. Since $\mu\alpha = \nu(\lambda \otimes 1)$, it follows that α is an isomorphism. Combining this with the direct sum representation above (with $q = m-p$), we obtain 9.8 for $r = 1$.

The inductive proof for $r > 1$ proceeds now exactly as in the last part of the proof of 9.4.

17. Products in the Leray spectral sequence of a fibration

Let $p: X \longrightarrow K$ be a fibre space where K is a CW-complex. We will describe briefly how the work we have done on filtered spaces and products enable us to construct products in the spectral sequence of the fibration.

Let K_n denote the n-skeleton of K, and set $X_n = p^{-1}K_n$. Denote by $\{E_r(X)\}$ the cohomology spectral sequence of this filtration of X. It is not difficult to show that $E_1^{s,t}$ is isomorphic to the module of s-cochains of K with local coefficients $H^t(F)$ where F is the fibre. Moreover d_1 is just the coboundary in K. It follows that $E_2^{s,t} \approx H^s(K;H^t(F))$.

Suppose $q: Y \longrightarrow L$ is another fibration where L is a complex, and that $f: X \longrightarrow Y$ is a map of fibre spaces. By a standard approximation theorem, the induced mapping $\bar{f}: K \longrightarrow L$ is homotopic to a skeletal mapping $\bar{g}: K \longrightarrow L$. There is a covering homotopy of f into a mapping $g: X \longrightarrow Y$ which preserves filtrations, and thereby induces a morphism $g^*: E_r(Y) \longrightarrow E_r(X)$. By the same approximation theorem, any two approximations $\bar{g}_0, \bar{g}_1$ are homotopic via a skeletal map $I \times K \longrightarrow L$. It follows that the covering maps g_0, g_1 are homotopic as maps of filtered spaces. Therefore $g_0^* = g_1^*$ when $r \geqq 2$, and hence it may be denoted by f^*. It is an easy exercise to verify the functorial properties of this induced morphism of spectral sequences.

Now we are able to compare the spectral sequences of X based on two triangulations of the base, say, K and K'. Letting the f above be the identity map of X, and using the functorial properties of f^*, we conclude that the two spectral sequences are canonically isomorphic from E_2 on. In this way the spectral sequence from E_2 on becomes a contravariant functor defined on the category of fibrations whose base spaces are triangulable.

When K is triangulable so also is $K \times K$, hence the diagonal $\Delta: X \longrightarrow X \times X$ is a mapping in the category. Using the product φ of 9.6, we define the (cup-)product in the spectral sequence $\{E_r(X)\}$ for $r \geqq 2$ to be the composition

(17.1) $$E_r(X) \otimes E_r(X) \xrightarrow{\varphi} E_r(X \times X) \xrightarrow{\Delta^*} E_r(X) .$$

By paraphrasing the arguments of §11, one proves that this product is associative, commutative and unitary, each d_r is a derivation, the isomorphism $E_\infty \approx E_0(H^*(X))$ preserves products, and the products commute with morphisms induced by mappings of fibre spaces.

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