CHAPTER 3. ELEMENTARY COMBINATORIAL METHODS

In Chapter 2 on probability spaces, we saw that when the equiprobable probability function is used (for reasons of physical symmetry or for other reasons), the probability of an event A in a probability space S is the ratio of the number of sample points in A to the number of sample points in S. Since sample points correspond to the ways in which the experiment can have an outcome, we now look at methods for counting the number of ways in which an experiment can have an outcome, and for counting from among these ways the number of ways in which a given event can occur. These methods are called combinatorial methods.

Example 1. Three books, which we call A, B, and C, are placed on a shelf. This is done in a "random way" so that the different possible arrangements of books are equally likely. What is the probability that books A and B are next to each other?

We can solve this problem by first listing all possible arrangements. We get ABC, ACB, BCA, BAC, CAB, CBA. There are six arrangements. We then note that in four of these arrangements, A and B are next to each other: ABC, BAC, CAB, CBA. Thus the desired probability is $\frac{4}{6} = \frac{2}{3}$.

Note. It is common, in describing experiments with finitely many outcomes, to use the word "random" to mean that the physical circumstances of the experiment are such that the equiprobable probability function can be used. Thus the above example could have been stated, "three books are arranged on a shelf in random order." Similarly, the experiment of rolling a fair die could be described briefly: "a number is chosen at random from the set $\{1,2,3,4,5,6\}$."

We now outline several basic combinatorial methods.

Method I. Making a list of all possible cases. This is how we approached Example 1. It is sometimes the only method available. It has, however, two disadvantages. First, there may be too many cases for this to be practicable. (If the above example had used 8 books instead of 3, there would be more than 40,000 cases (arrangements of books on the shelf.)) Second, there may be no obvious way to be sure that our list includes all the cases. These disadvantages suggest that we seek more systematic combinatorial methods.

Method II. Tree diagrams. In counting the number of ways in which an outcome can occur, it is often useful to analyze a problem by describing or inventing a process which (i) has the same outcomes, and (ii) occurs as a sequence of steps in time (though the physical experiment itself may not occur as this sequence of steps). Recall Example 1 above. One way of analyzing this into steps in time is as follows. Choose three empty positions on the shelf for the 3 books. Call them, from left to right, positions 1, 2, and 3. Then do the following steps.

Step 1. Choose one of the three books and place it in position 1.

Step 2. Choose one of the two remaining books and place it in position 2.

Step 3. Place the remaining book in position 3.

The ways in which these steps can be carried out are given in the following diagram.

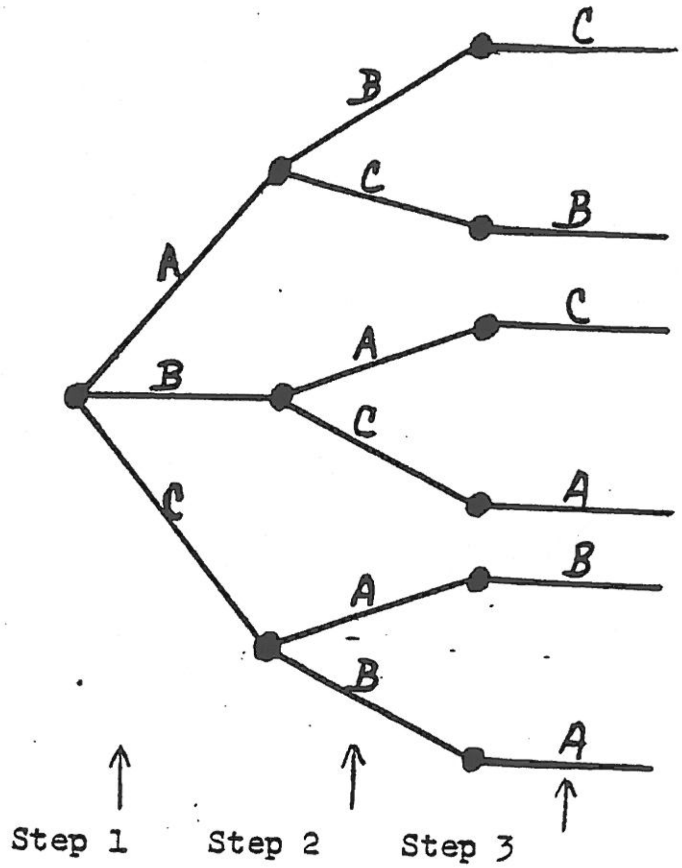


Figure 4.1

Because of its branchings as we go through the steps from left to right, this diagram is called a tree diagram. The number of cases is the number of branches or paths through the tree. In the above diagram there are six paths. A tree diagram is more systematic than a mere listing of cases, and it has the added advantage of making sure that we cover all possible cases.

In Example 1 we can also use a tree diagram to count the number of ways in which the books can be placed so that A and B are next to each other. We get:

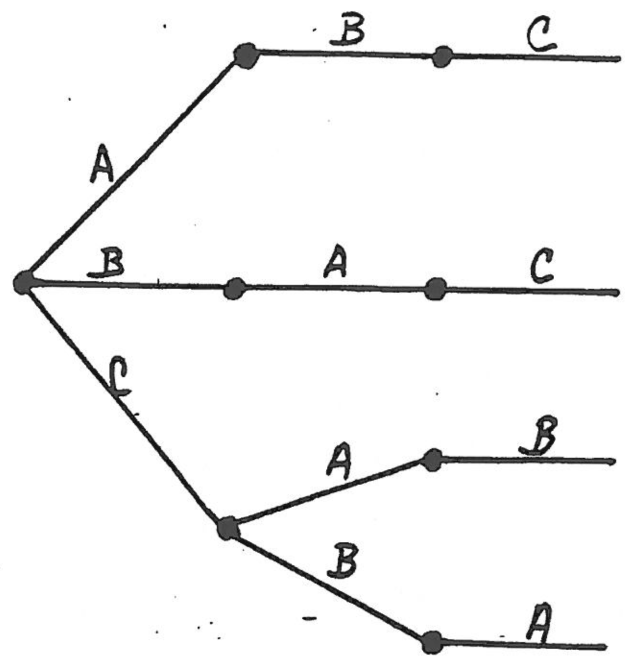


Figure 4.2

Here there are four paths.

The idea of a tree diagram is fundamental in many combinatorial problems, even though the actual diagram may be too large and complex to draw. (Recall that for 8 books we would get more than 40,000 branches.)

Method III. The multiplication principle. If we can analyze a problem by finding or inventing a process which occurs as a sequence of steps, which we call step 1, step 2, ..., step k, and if step 1 always occurs in exactly r_1 ways, step 2 in exactly r_2 ways, ..., step k in exactly r_k ways, then the number of ways in which the entire process can occur is given by the product $r_1 \times r_2 \times \ldots r_k$. This is called the multiplication principle. For example in Figure 4.1, step 1 always occurs in 3 ways, step 2 always occurs in 2 ways, and step 3 always occurs in 1 way. Hence the entire process can occur in $3 \cdot 2 \cdot 1 = 6$ ways.

Example 2. I wish to travel from Boston to New York City by way of Hartford. There are 3 routes from Boston to Hartford and 5 routes from Hartford to New York. How many routes are there for the entire trip? Here the multiplication principle gives the immediate answer $3 \cdot 5 = 15$.

Note that there are problems which can be solved by tree diagrams but for which the multiplication principle does not apply. Figure 4.2 gives an illustration of this. Here, step 2 sometimes occurs in 2 ways and sometimes in only 1 way, so that the basic condition for the multiplication principle does not hold.

The multiplication principle is the most common and useful of combinatorial methods. It tells us immediately, for example, that the number of ways of placing 8 books on a shelf must be exactly $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$. Here, the idea of a tree diagram shows us what factors to use. We do not actually draw the diagram; instead, we merely calculate the product.

Method IV. The addition principle. In counting the number of ways in which an event can occur, it is sometimes helpful to break the event into several disjoint sets and then to do a separate calculation for each of these sets. The total number of ways for the event to occur will then be the sum of the numbers of ways for each of the disjoint sets. We call this the addition principle.

Example 3. In Massachusetts, 35 basic symbols are used on license plates, the 10 digits and the 25 letters of the alphabet other than O. How many different plates can be formed using 3 or fewer symbols? To solve this problem, we consider three separate cases: The 1-symbol plates, the 2-symbol plates, and the 3-symbol plates. In each case we apply the multiplication principle. This gives:

(i) 35 plates for the 1-symbol case;

(ii) $35 \cdot 35 = 1225$ plates for the 2-symbol case;

(iii) $35 \cdot 35 \cdot 35 = 42,875$ plates for the 3-symbol case.

Hence, by the addition principle, we have $35 + 1225 + 42,875 = 44,135$ plates of 3 symbols or less.

Method V. Standard versions of the multiplication principle. Certain special versions of the multiplication principle occur often, and it is useful to develop and learn standard formulas for these versions. We give some of these below.

Standard formula 1. Permutations. It is evident, from the example of 8 books on a shelf given above, that the number of ways of arranging n books in order on a shelf must be

$$n(n-1)\,(n-2)\,\ldots\,3\cdot2\cdot1\ .$$

This product is called n factorial, and is usually abbreviated $n!$. The number of arrangements of n objects in order is often called the number of permutations of n objects. We thus have the formula:

the number of permutations of n objects = $n!$

Note that the quantity $n!$ increases extraordinarily rapidly with n. Thus

$$\begin{aligned} 5! &= 120 \\ 10! &= 3{,}628{,}800 \\ 20! &= 2.4 \times 10^{18} \\ 30! &= 2.7 \times 10^{32} \\ 50! &= 3.0 \times 10^{64}\ . \end{aligned}$$

(The number of ways of arranging 50 books on a shelf, for example, exceeds the number of atoms in the visible universe.)

Note. It is useful in mathematics to take a product with no factors to be 1. Hence we take $0! = 1$.

Standard formula 2. Ordered samples without replacement. Assume that we are given 10 different books. In how many ways can we select and arrange 4 out of the 10 books on a shelf. We analyze in steps as follows. Choose 4 positions on the shelf.

Step 1. Choose 1 of the 10 books for the first position.

Step 2. Choose one of the 9 remaining books for the second position.

Step 3. Choose one of the 8 remaining for the third position.

Step 4. Choose one of the 7 remaining for the fourth position.

Applying the multiplication principle, we get $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ ways of selecting and arranging the books. Note that the product on the left can be more briefly indicated as $\frac{10!}{6!}$, since $10 \cdot 9 \cdot 8 \cdot 7$ is the remaining product after the indicated division. More generally, if we are given n books, the number of ways of selecting and arranging r out of the n books on a shelf will be

$$n(n-1)(n-2)\ldots(n-r+1).$$

This result can be conveniently abbreviated as

$$\frac{n!}{(n-r)!}.$$

This formula is called the number of <u>permutations of n objects taken r at a time</u>. It is also called the number of <u>ordered samples of n objects taken r at a time without replacement</u>. Here "ordered" refers to the fact that when the same r books appear in two different orders, they count as two different arrangements, and "without replacement" refers to the fact that once a book is chosen at a certain step, it is not available to be chosen again at a later step. (It is not "replaced" in the stock of books from which we are choosing.)

Standard formula 3. Ordered samples with replacement. How many four letter blocks can be formed using the first 10 letters of the alphabet? AJAC, ACAJ, BBDF are examples of such blocks. As with the bookshelf, we can think of 4 positions to be filled. Any one of 10 letters can go into the first position; any one of 10 letters can go into the second position; and so forth. Applying the multiplication principle, we find that the number of blocks is $10^4 = 10,000$. More generally, given n letters, the number of r letter blocks that can be formed is

$$n^r .$$

This formula is sometimes called the number of ordered samples of n objects taken r at a time with replacement. Here "with replacement" refers to the fact that if we use a letter at one step, it remains available for use at a later step. (It is "replaced" in the original stock of letters from which we choose.)

Standard formula 4. Unordered samples without replacement (combinations). Assume that from a personal library of 10 different books, I wish to choose a subset of four books to take with me on a trip. In how many different ways can I choose such a subset? We solve this problem by first asking a different question: In how many ways can I choose four books and place them on a shelf. Let us call this number y. (By standard formula 2 above we know that $y = \frac{10!}{6!}$.) Now each subset can appear on the shelf in $4!$ different orders. (By standard formula 1.) Hence, if x is the number of different subsets (the number we are looking for), we have

$y = x4!$. Thus $x = \frac{y}{4!} = \frac{10!}{4!5!} = 210$. More generally, the number of ways of choosing a subset of r books from a set of n books must be

$$\frac{n!}{r!(n-r)!}$$

This formula is often given the special abbreviation

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

This abbreviation $\binom{n}{r}$ is called the binomial coefficient n over r (for reasons that we see below).

The above formula is also sometimes called the number of combinations of n objects taken r at a time, or the number of unordered samples of n objects taken r at a time without replacement.

Standard formula 5. Unordered samples with replacement. There are three candidates, A, B, and C, to be president of a small club. In the election 4 votes are cast, and a final tally is made of the number of votes received by each candidate. Such a tally can be given in the form of an ordered triple of non-negative integers. For example, (1,2,1) states that A and C have each received one vote and that B has received 2 votes, while (2,2,0) states that A and B have each received 2 votes while C has received no votes. How many different final tallies are possible in an election with 3 candidates in which 4 votes are cast? We can list the possible tallies (Method I) and we get: (4,0,0), (3,1,0), (3,0,1), (2,2,0), (2,1,1), (2,0,2), (1,3,0), (1,2,1), (1,1,2), (1,0,3), (0,4,0), (0,3,1), (0,2,2), (0,1,3), (0,0,4). There are thus 15 tallies. We can obtain a formula for this result by thinking of each tally as a block of four letters formed from the 3 letters A, B, C where the number of occurrences of each letter in the block is the same as the number of votes for that candidate in the tally. Thus (1,2,1) can be thought of as ABBC, and (0,1,3) can be thought of as BCCC. We then take four <u>stars</u> (*) to represent the positions of the four letters, a <u>bar</u> (|) to represent the division between A's and B's, and a bar to represent the division between B's and C's. Then each possible block corresponds to a different way of placing the 4 stars and 2 bars in order. For example the block AABB corresponds to **|**| and the block AAAC corresponds to ***||*. Thus each different block (and hence each different tally) corresponds to a different set of positions available for the stars and the bars. By standard formula 4, the number of possibilities must be $\binom{6}{4} = \frac{6!}{4!2!} = 15$, agreeing with the result given by our Method I listing. More generally, if we ask how many different final tallies are possible in an election with n candidates in which r votes are cast, we obtain the formula

$$\binom{r+n-1}{r} = \frac{(r+n-1)!}{r!(n-1)!} .$$

This formula is sometimes called the number of <u>unordered samples of n objects taken r at a time with replacement</u>. This concept has the same relation to the concept of ordered sample with replacement (formula 3) that the concept of combination (formula 4) has to the concept of permutation (formula 2). The phrase "unordered samples" is used because we can think of the selection process as follows (using the above example): from an unlimited supply of A's, B's, and C's, choose a set of four letters all at once. This set is the unordered sample (here $n = 3$ and $r = 4$). The number of occurrences of each letter corresponds to the number of votes for that letter in a tally. We can count the number of distinct possible such sets by arranging each set in a block, as above, with A's preceding B's and B's preceding C's, and then counting the number of distinct possible blocks. As we have seen, this number is given by the formula above.

<u>Note</u>. It is clear, from the above discussion, that the above result can be briefly restated as follows. Let n and r be given. Consider ordered n-tuples of non-negative integers where the sum of the integers in each n-tuple is r. The number of such n-tuples is $\binom{r+n-1}{r}$.

<u>Remark</u>. The word "sample" has been used in two different ways. In Chapter 2, we used "sample space" to mean the set representing

the basic outcomes of an experiment. Here in Chapter 4, we have used it to mean a possible subset or arrangement in a combinatorial problem. The two uses are unrelated, and should not be confused by the reader.

Some further examples. The student should keep in mind that the multiplication principle (Method III) and tree diagrams (Method II) are more fundamental than the standard formulas (Method V). If one cannot think of a standard formula to apply to a given problem, one should go back to the multiplication principle, and, if this fails, one should use a tree diagram. Finally, if a tree diagram fails, one should resort to a listing of possible cases (Method I). We now look at further examples where we shall see that it is sometimes helpful to use a combination of different formulas and methods in the same problem.

Example 4. Five cards are taken from a shuffled bridge deck. What is the probability of finding three cards of one denomination (or "kind") (for instance, 3 jacks) and of having the remaining two cards be of another kind (for instance, 2 eights)? A set of five cards is called a poker hand. A poker hand with three of one kind and two of another is called a full house. We therefore seek the probability of a full house when a poker hand of five cards is taken at random from a bridge deck.

As our probability space, we use the set of all possible poker hands (that is, of all possible five card subsets of a deck of 52 cards), and we use the equiprobable assignment of probability values. The number of points in the probability space is $\binom{52}{5}$ = 2,598,960, by standard formula 4. The number of full house hands can be got by the following analysis into steps.

Step 1. Select one of the 13 possible kinds for the group of three cards.

Step 2. Select three of the four possible cards of that kind to go into the hand.

Step 3. Select one of the 12 remaining kinds for the group of two cards.

Step 4. Select two of the four cards of this latter kind to go into the hand.

Applying the multiplication principle and using the standard formula for Steps 2 and 4, we get that the number of full houses is:

$$13 \binom{4}{3} 12 \binom{4}{2} = 13 \cdot 4 \cdot 12 \cdot 6 = 3744$$

Hence the desired probability is $\frac{3744}{2,598,960} = 0.0014.$

Example 5. If we take a poker hand as in Example 4, what is the probability of having two cards of one kind, two cards of a second kind, and the remaining card of a third kind? Such a hand is called two pairs. Here we can analyze into steps as follows.

Step 1. Select two of the 13 possible kinds for the two pairs.

Step 2. Select two of the four possible cards of the higher kind.

Step 3. Select two of the four possible cards of the lower kind.

Step 4. Select a single card of one of the remaining 11 kinds.

Applying the multiplication principle and using the appropriate standard formulas, we get that the number of hands which are two pairs is:

$$\binom{13}{2}\binom{4}{2}\binom{4}{2}\; 11\binom{4}{1} = 78 \cdot 6 \cdot 6 \cdot 11 \cdot 4 = 123,552.$$

Hence the desired probability is $\frac{123,552}{2,598,960} = 0.048$

(The reader should note that the probability of one pair - 2 cards of one kind and 3 cards of three other kinds - is much higher. We get the number of such hands to be

$$13\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 1,098,240;$$

hence the probability is $\frac{1,098,240}{2,598,960} = 0.42.$)

It is sometimes possible to analyze a given problem in an unexpected way that permits the use of a standard formula. The reader will find that ability to make such an analysis improves with practice. The following is an illustration.

Example 6. Three boxes are placed in a row on a table. For each of four identical marbles (in turn), a box is chosen at random and the marble is placed in that box. How many different final arrangements of marbles in boxes are possible? We analyze as follows.

Call the boxes A, B, C. Then the choice of a box for each marble can be thought of as a vote for that box, and the choices for all four marbles make a tally of four votes for those three boxes. Hence standard formula 5 applies and the number of arrangements is

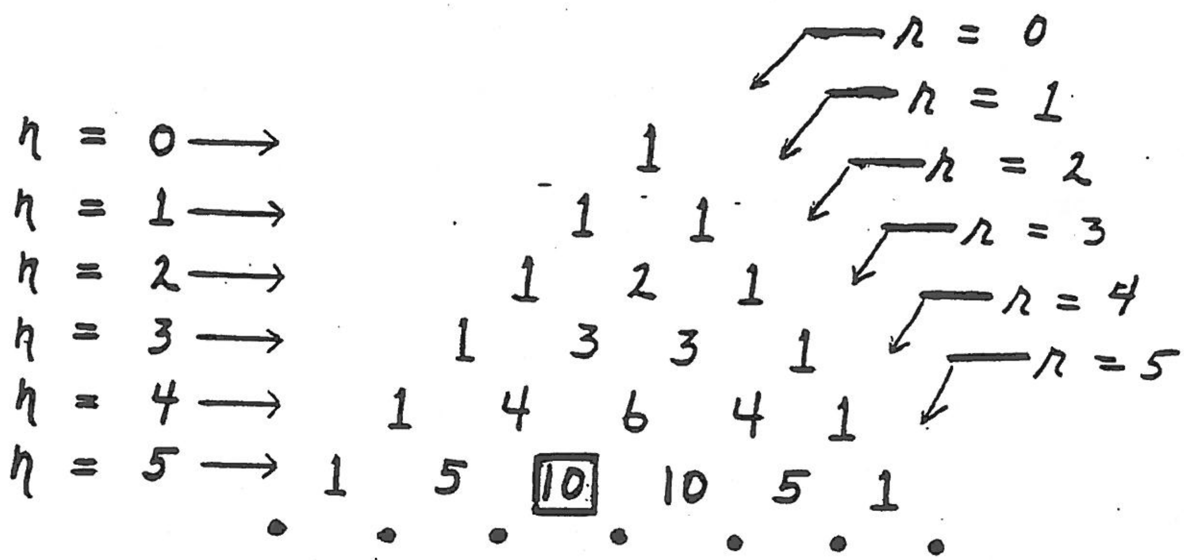
$$\binom{4+3-1}{4} = \binom{6}{4} = 15 .$$

More generally, in combinatorial work, a real or imaginary row of boxes of this kind is called a row of cells. Thus the number of different ways in which r identical objects can be placed in n cells is $\binom{r+n-1}{r}$.

Choosing a sample space. In Chapter 2, we noted that in the same given probability problem, there may be several different ways to choose the sample space. We see another example of this in the case of poker hands. We may think of each hand as a 5-card subset of the 52-card deck. This gives a sample space with $\binom{52}{5} = 2{,}598{,}960$ points. (We used this in Examples 4 and 5 above.) Or we may think of each hand as five cards dealt in a certain order. This gives a sample space with $\frac{52!}{47!} = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311{,}875{,}200$ points. If we had used the latter space in Example 4, the event full house would have been correspondingly larger. There would have been 449,280 full house hands, giving a probability value $\frac{449,280}{311,875,200} = 0.0014$, which is the same final value as before.

<u>Binomial coefficients</u>. There are several simple facts about binomial coefficients that are often helpful in combinatorial work. They include the following:

(a) <u>Pascal's triangle</u>. In the following diagram, values of n give the rows and values of r give the diagonals. The value of $\binom{n}{r}$ appears at the intersection of row n and diagonal r.



Thus $\binom{5}{2} = 10$ appears as marked with a square. A <u>special property</u> of this diagram is that each entry is the sum of the two adjacent entries immediately above it. Thus $10 = 4 + 6$. This property gives us an easy way to get further rows. The row for $n = 6$, for example, is immediately seen to be 1,6,15,20,15,6,1. Pascal's triangle is useful as a way of recalling the values of binomial coefficients for small values of n. The special property can be proved by verifying the formal identity $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

(b) Symmetry. It is immediate, both from the formula and from Pascal's triangle, that we always have

$$\binom{n}{r} = \binom{n}{n-r} .$$

Thus for example, $\binom{52}{5} = \binom{52}{47}$.

(c) Calculation. It is sometimes helpful to note that $\binom{n}{r}$ can be obtained by taking the product of the first r factors in n! and dividing this product by r! . Thus, for example, $\binom{5}{2} = \frac{5 \cdot 4}{2 \cdot 1} = 10$ and $\binom{12}{1} = \frac{12}{1} = 12$.

(d) Binomial theorem. The following well-known identity can be proved by mathematical induction, and is known as the binomial theorem:

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 \quad \ldots + \binom{n}{n} b^n$$
$$= \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j .$$

This identity is the reason for the name "binomial coefficient."

(e) Other identities. Useful identities can be derived from the binomial theorem. For example, putting $a = 1$ and $b = 1$, we get

$$2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} .$$

Hence the sum of entries in row n of Pascal's triangle must be 2^n. Returning to the meaning of $\binom{n}{r}$ as given by standard formula 4, we see from this identity that the total number of subsets (of all sizes) of a set with n elements must be 2^n. (We could have obtained this latter fact directly by numbering the objects in the set from 1 to n and then analyzing the making of a subset into the following n steps:

Step 1. Decide whether or not object 1 goes into the subset.

Step 2. Decide whether or not object 2 goes into the subset.

....

Step n. Decide whether or not object n goes into the subset.

As each step can be done in two ways, the multiplication principle gives the result 2^n for the total number of possible subsets that can be found in this way. Note that the empty set and the original set itself are included among the 2^n subsets.)

Standard Formula 6. Multinomial coefficients. We give one further standard formula that is important and commonly used.

Assume that we have ten books to place on a shelf. Five of the books are identical copies of a certain book which we call A. Three are identical copies of another book, B. And the two remaining are identical copies of a third book, C. How many different arrangements are possible? We analyze into steps as follows.

<u>Step 1</u>. Choose five positions for the copies of A, and place them. This can be done in $\binom{10}{5}$ ways.

<u>Step 2</u>. Choose three of the remaining five positions for the copies of B and place them. This can be done in $\binom{5}{3}$ ways.

<u>Step 3</u>. Place the copies of C in the remaining two positions. This can be done in one way.

Applying the multiplication principle, we get:

$$\binom{10}{5}\binom{5}{3} 1 = \frac{10!}{5!5!} \cdot \frac{5!}{3!2!} = \frac{10}{5!3!2!} = 2,520 .$$

More generally, if we want to arrange r objects in a row when k_1 of the objects are of one kind and identical, k_2 are of a second kind and identical, ... , and k_n are of an n^{th} kind and identical, then this can be done in

$$\frac{r!}{k_1!k_2!\ldots k_n!} \text{ ways,}$$

(where we always have $k_1 + k_2 + \ldots + k_n = r$). This formula is sometimes given the special abbreviation:

$$\frac{r!}{k_1!k_2!\ldots k_n!} = \binom{r}{k_1,k_2,\ldots,k_n}.$$

Thus $\frac{10!}{5!3!2!}$ becomes $\binom{10}{5,3,2}$. This abbreviation is called a <u>multinomial coefficient</u>. When $n = 2$, the multinomial coefficient $\binom{r}{k_1,k_2}$ is the same as the binomial coefficient $\binom{r}{k_1}$, since $k_1 + k_2 = r$ (and hence $\frac{r!}{k_1!k_2!} = \frac{r!}{k_1!(n-k_1)!}$.)

Multinomial coefficients appear in the <u>multinomial theorem</u> (an identity similar to the binomial theorem):

$(a_1 + a_2 + \ldots + a_n)^r =$ the sum of all distinct terms of the form

$$M(k_1,\ldots,k_n)a_1^{k_1}a_2^{k_2} \ldots a_n^{k_n},$$

where all possible choices of $k_1,k_2,\ldots,k_n \geq 0$ occur such that $k_1 + k_2 + \ldots + k_n = r$, and where the coefficient $M(k_1,\ldots,k_n)$ is $\binom{r}{k_1,k_2,\ldots,k_n}$.

(<u>Note</u>. Using standard formula 5, we immediately see that there must be $\binom{r+n-1}{r}$ different terms in this sum since $\binom{r+n-1}{r}$ is the number of distinct n-tuples of non-negative integers $(k_1,\ldots,k_n)$ such that $k_1 + k_2 + \ldots + k_n = r$.)

Stirling's formula. In hand calculations with factorials, it is often helpful to use the following approximation.

$$n! \approx e^{-n}\, n^{n+\frac{1}{2}} \sqrt{2\pi} = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\ .$$

This approximation is known as Stirling's formula. As n increases, the relative error of this approximation goes to zero. That is to say, the ratio of the approximate value to the exact value goes to 1. More specifically, for all n, this ratio can be shown to lie between $e^{-1/12n}$ and 1. (Thus, for $n \geq 10$, the approximation is accurate to within 1 percent since $e^{-1/12n} > 0.99$.) As an example, we use Stirling's formula to get a value for $\binom{52}{13}$, the number of possible bridge hands, as follows:

$$\binom{52}{13} = \frac{52!}{13!39!} = \frac{e^{-52}\, 52^{52\frac{1}{2}} \sqrt{2\pi}}{e^{-13}\, 13^{13\frac{1}{2}} \sqrt{2\pi}\ e^{-39}\, 39^{39\frac{1}{2}} \sqrt{2\pi}}$$

Canceling common factors, we get

$$\left(\frac{4}{3}\right)^{39\frac{1}{2}} \frac{4^{13}}{\sqrt{26\pi}} = 6.4 \times 10^{11}\ ,$$

with accuracy to 2 significant figures. As we shall later see, Stirling's formula has theoretical uses. It is also useful with electronic calculators, since many calculators cannot directly get $n!$ for $n \geq 70$.

For 5!, Stirling's formula gives the approximate value 118.02 for a relative error of 1.7 percent from the correct value 120. (Here $e^{-1/12n} = 0.983$.) For 10!, Stirling's formula gives the approximate value 3.60×10^6 for a relative error of 0.8 percent from the correct value 3,628,800. For 52!, Stirling's formula gives the approximate value $8.053 \ldots \times 10^{67}$ for a relative error of 0.2 percent from the correct value $8.066 \ldots \times 10^{67}$. (Here $e^{-1/12n} = 0.998$.)

<u>Proof of Stirling's formula</u>. The formula can be proved by noting that the value of $\log (n!)$ is approximately given by the area of a certain region under the curve $y = \log x$ and by computing this area by calculus. The proof is elementary but lengthy. In outline, it goes as follows. $\log k$ is approximately given by the area (call it A_k) under the curve $y = \log x$ between $x = k - \frac{1}{2}$ and $x = k + \frac{1}{2}$. Hence $\log (n!) = \log 1 + \log 2 + \ldots + \log n = \log 2 + \ldots + \log n$ is approximately given by the area under $y = \log x$ between $x = 3/2$ and $x = n + 1/2$. Thus

$$\log (n!) = E_n + \int_{3/2}^{n+(1/2)} \log x \, dx,$$

where E_n is the total error accumulated in replacing $\log k$ by A_k for each $k \leq n$. Evaluating the integral, we obtain

$$\log (n!) = E_n + (n+\tfrac{1}{2}) \log (n+\tfrac{1}{2}) - n - \tfrac{1}{2} - \tfrac{3}{2} \log \tfrac{3}{2} + \tfrac{3}{2} .$$

Next, we can show (using the infinite series for $\log (1+x)$) that

$$(n+\tfrac{1}{2}) \log (n+\tfrac{1}{2}) = (n+\tfrac{1}{2}) \log n + \tfrac{1}{2} + D_n$$

where $D_n \to 0$ as $n \to \infty$. Hence,

$$\log (n!) = (n+\tfrac{1}{2}) \log n - n + C_n + D_n$$

where $C_n = E_n + \frac{3}{2} - \frac{3}{2} \log \frac{3}{2}$. Finally, by careful analysis, we can show that as $n \to \infty$, $C_n \to \frac{1}{2} \log 2\pi$. Hence, $\log (n!) = (n+\frac{1}{2}) \log n - n + \frac{1}{2} \log 2\pi + B_n$ where $B_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $n! = n^{n+(1/2)} e^{-n} \sqrt{2\pi}\, e^{B_n} \approx n^{n+(1/2)} e^{-n} \sqrt{2\pi}$. We can also show, by series methods, that $0 < B_n < \frac{1}{12n}$, and this gives the limit of error stated above for Stirling's formula.

EXERCISES FOR CHAPTER 3.

Remark. Correct use of elementary combinatorial methods requires practiced insight. At first, the student will find that certain common kinds of mistakes occur. In particular, certain outcomes will have been counted more than once, and certain other outcomes, intended to be included, will have been omitted altogether. Other common mistakes include: neglecting to subtract the intersection term in applications of the addition law (see third fact on page 47, and see also the inclusion/exclusion principle in Exercise 3-15 below); using the multiplication principle when the number of ways of doing a certain step varies from outcome to outcome; and confusing samples with replacement and samples without replacement.

3-1. (a) How many distinct blocks of 5 letters can be formed from the 26 letters of the alphabet when repetition is allowed?

(b) How many such blocks can be formed if exactly two of the five letters in each block are vowels? (Take the vowels to be A, E, I, O, and U; repetition is again allowed.)

3-2. If four identical apples and three identical pears are arranged in a row, and if all distinct arrangements are equally probable, what is the probability that the three pears will be included in the first five positions (reading from left to right)?

3-3. You are dealt a poker hand from a shuffled deck. You receive two kings, a seven, a nine, and a five. You discard the seven, nine, and five, and draw three more cards from the remaining deck. What is the probability that you get either one or both of the remaining kings?

3-4. (a) At State Institute of Technology, freshmen are required to take the following courses: a choice of one of three possible year-courses in English, of one of two possible year-courses in mathematics, of one of three possible year-courses in physics, of one of two possible year-courses in chemistry, and of one of nine possible year electives in other subjects. Such a program of five year-courses is called a standard program. How many distinct standard programs are possible?

(b) After a major study of its first-year curriculum, the faculty establishes the following revised requirements for mathematics, physics, and chemistry during the two terms of the freshman year. A freshman is is required to take one of two possible first-term courses in mathematics (Math A_1 or Math B_1 (harder)), one of three possible first-term courses in physics (Physics A_1, Physics B_1 (harder), or physics C_1 (hardest)), and one of two possible first-term courses in chemistry Chem A_1 or Chem B_1 (harder)). In the second term, a freshman is required to take one of two possible second-term courses in mathematics (Math A_2 or B_2), one of three possible second-term courses in physics (Physics A_2, B_2, or C_2), and one of two possible second-term courses in chemistry (Chem A_2 or B_2). A freshman who is in the A_1 course in a given subject cannot move to the B_2 or C_2 course in the same subject in the second term, and a freshman in Physics B_1 cannot move to Physics C_2 in the second term. A freshman must be in Math B_1 to take Physics B_1 or C_1 at the same time and in Math B_2 to take Physics B_2 or C_2. A freshman must be in Physics B_1 or C_1 to take Chem B_1 and must be in Physics B_2 or C_2 to take Chem B_2. How many different first-term programs are possible in mathematics, physics, and chemistry?

(c) Under the requirements given in (b), how many different programs for the full year are possible in mathematics, physics, and chemistry?

3-5. (a) A class of 12 students occupies assigned desks in a classroom with 18 desks. How many ways are there of assigning individual students to individual desks?

(b) There will be 6 unassigned desks in the classroom. How many different sets of 6 unassigned desks are possible?

(c) Each student in the class may choose one of five given topics for a term project. In how many different ways can the students choose their topics?

(d) In preparing to grade the project reports, the teacher makes an over-all tally of the numbers of reports on each of the five topics. How many different tallies are possible?

3-6. (a) An eight-oared racing shell has eight rowing positions in a line from bow to stern. Four positions have oars on the starboard side and four have oars on the port side. A given crew of eight rowers has four <u>starboard</u> rowers capable of rowing in any of the four starboard positions and four <u>port</u> rowers capable of rowing in any of the four port positions. The coach decides to hold a time-trial (timed practice run) for each different possible arrangement of the eight rowers (with starboard rowers assigned to starboard positions and port rowers assigned to port positions). How many time-trials must be scheduled?

(b) The coach later discovers that one of the port rowers is capable of rowing in starboard positions and that two of the starboard rowers are capable of rowing in port positions. How many additional time trials must be scheduled?

(c) The next year, the coach has five port rowers and seven starboard rowers out for the eight-oared crew. No rower is capable of rowing both starboard and port. The coach again wishes to have a time-trial for each possible arrangement. How many must be scheduled?

3-7. (a) A hockey coach has 3 goalies, 6 defensemen, and 12 forwards on his squad. At the beginning of a game, he must put 1 goalie, 2 defensemen, and 3 forwards on the ice. How many different groups of six players can he put on the ice?

(b) There are two defense positions (left defense and right defense) and three forward positions (left wing, center, and right wing). If each forward is capable of playing in any of the three forward positions and each defenseman is capable of playing in either defense position, in how many ways can the coach choose six starting players and assign the six positions to them?

3-8. (a) How many distinct ways are there of ordering the six letters of the word VERMONT?

(b) How many for the nine letters of the word SASSAFRAS?

(c) How many for the eleven letters of the word MISSISSIPPI?

3-9. A <u>bridge hand</u> consists of 13 cards drawn at random without replacement from a bridge deck of 52 cards.

(a) What is the probability that a bridge hand will consist of all cards of the same suit?

(b) What is the probability that a hand will include 5 spades, 4 hearts, 2 diamonds, and 2 clubs?

(c) What is the probability that a hand will include 4 spades, 3 hearts, 3 diamonds, and 3 clubs?

(d) What is the probability that there is some suit with 4 cards and that there are 3 cards from each of the remaining suits?

(e) A _yarborough_ at bridge is a hand containing no aces and no card higher than _nine_. It is named for the Earl of Yarborough who was willing to bet 1000 to 1 odds against its occurrence. Estimate the true odds against drawing a yarborough.

3-10. A poker hand is drawn at random without replacement from a bridge deck of 52 cards. Calculate the probability of getting:

(a) _Three of a kind_. A hand is called a _three of a kind_ if there are three cards of one kind and one card of each of two other kinds.

(b) _Four of a kind_. A hand is called a _four of a kind_ if there are four cards of one kind.

(c) _Flush_. A hand is called a _flush_ if all five cards are of the same suit.

(d) _Straight_. A hand is called a _straight_ if it contains cards of five different kinds, where the kinds are five adjacent, successive kinds in the sequence: _ace_, _two_,...,_ten_, _jack_, _queen_, _king_, _ace_.

(e) _Straight flush_. A hand is called a _straight flush_ if it is both a straight and a flush.

3-11. (a) A club contains 5 men and 7 women. Five members are chosen at random to form a committee. What is the probability that the committee includes 2 men and 3 women?

(b) A club contains n_1 men and n_2 women. Give a general formula for the probability that a randomly chosen group of r people will include r_1 men and r_2 women (where $r_1 + r_2 = r$). This formula is called the <u>hypergeometric formula</u>. It is considered further in Chapter 12.

3-12. Let A be the set $\{1,2,\ldots,m\}$ and let B be the set $\{1,2,\ldots,n\}$. A <u>mapping</u> from A to B is an assignment which associates with each member of A a unique member of B. The same member of B may be associated with more than one member of A, but each member of A has only one member of B associated with it.

(a) For given m and n, how many distinct mappings are there from A to B?

(b) An <u>injection</u> from A to B (for the case where $n \geq m$) is a mapping from A to B in which no member of B is associated with more than one member of A. For given m and n, with $n \geq m$, how many distinct injections are there from A to B?

3-13. Given $r \geq n$, in how many different ways can r identical objects be placed in n cells so that each cell is non-empty?

3-14. A bookcase has r shelves. You have n different books to place in the bookcase. Each shelf can hold up to n books. The books on each shelf are placed as close to the left end of the shelf as possible. How many different arrangements of the books can be made?

3-15. The addition law of probability states that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. This law can be generalized to n events $A_1, A_2, \ldots, A_n$ as follows.

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \sum_{i<\ldots<\ell} P(A_i \cap A_j \cap A_k \cap A_\ell) + \ldots$$

$$+ (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n).$$

This is the <u>inclusion/exclusion principle</u>. Exercise 2-4 was to show this for the case $n = 3$.

(a) Assuming this law for $n = 3$, prove that it holds for $n = 4$. (This will show how a proof for the full result, by mathematical induction, can be obtained.)

(b) The inclusion/exclusion principle is especially useful when it is known that the value of $P(A_i)$ is the same for all i, that the value of $P(A_i \cap A_j)$ is the same for all $i < j$, that the value of $P(A_i \cap A_j \cap A_k)$ is the same for all $i < j < k, \ldots$. Show that in this case

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = nP(A_1) - \binom{n}{2} P(A_1 \cap A_2)$$

$$+ \binom{n}{3} P(A_1 \cap A_2 \cap A_3) - \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n).$$

(c) A card is drawn from a shuffled bridge deck and then replaced. This is repeated six times. What is the probability that all four suits appear during the six draws?
(_Hint_. Let A_1 be the event that _spades_ do _not_ appear, and let A_2, A_3, and A_4 be the corresponding events for _hearts_, _clubs_, and _diamonds_. The probability that we seek is $1 - P(A_1 \cup A_2 \cup A_3 \cup A_4)$. Use the formula in (b) to calculate $P(A_1 \cup A_2 \cup A_3 \cup A_4)$ with $P(A_1) = 3^6/4^6$, $P(A_1 \cap A_2) = 2^6/4^6$, $P(A_1 \cap A_2 \cap A_3) = 1/4^6$, and $P(A_1 \cap A_2 \cap A_3 \cap A_4) = 0$.)

(d) Six cards are drawn without replacement from a shuffled bridge deck. Find the probability that all four suits appear.

(e) A bridge deck is shuffled and all 52 cards are laid out in a line face up. A second deck is then shuffled and its 52 cards are laid out face up in a line next to the first line. We say that a _match_ occurs at the i^{th} position if the cards in the i^{th} position in the two lines are the same. What is the probability that at least one match occurs? (_Hint_. Let A_i be the event that a match occurs at the i^{th} position, and use (b) to calculate $P(A_1 \cup A_2 \cup \ldots \cup A_{52})$.)

3-16. The four players at a bridge table are known as North, East, South, and West. A shuffled deck is dealt to the four players.

(a) What is the probability that each of the four players receives all cards of the same suit?

(b) What is the probability that North receives all cards of the same suit? (This is the same as Exercise 3-9a.)

(c) What is the probability that at least one player receives all cards of the same suit? (Hint. Use the inclusion/exclusion principle of Exercise 3-15.)

(d) Give plausible guesses, based on your personal observation as to the frequency with which bridge is played, as to whether or not the events in (a) and in (c) have occurred anywhere during this century.

3-17. (a) What is the probability that a bridge hand has no aces?

(b) The probability in (a) is obviously the same as the probability that 39 cards, drawn without replacement, will include all four aces. We now draw 39 cards with replacement from a repeatedly shuffled bridge deck. What is the probability that all four aces appear during the draws?

(Note. These results are a good illustration of the different effects of assuming or not assuming replacement.)

3-18. (a) Three cards are drawn, without replacement, from a shuffled bridge deck. What is the probability that at least two are of the same suit? (Hint. Consider ordered samples.)

(b) Three cards are drawn, in turn, with replacement from a shuffled bridge deck. What is the probability that at least two are of the same suit?

(c) What is the probability, in a poker hand, that exactly one suit will be missing? (Note. This result has been used as the basis of a sucker bet, as have the results in (a) and (b) and in Exercises 3-15 and 3-17.)