NEW MODEL STRUCTURES FROM OLD

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Abstract. If two model structures $\mathfrak{M}_1, \mathfrak{M}_2$ on the same underlying category satisfy simply stated conditions, then it is possible to obtain a third model structure whose weak equivalences are just those morphisms which are weak equivalences in both $\mathfrak{M}_1$ and $\mathfrak{M}_2$. We show how this can be used to construct new model structures on some categories of complexes and double complexes.

The aim of this paper is to prove the simple but surprising Theorem 1 and to give some examples of the model structures which it allows us to produce.

Theorem 1. *Let $\mathcal{M}$ be a category. Let $\mathcal{C}_0$, $\mathcal{C}_2$, $\mathcal{W}'$ be respectively the cofibrations, acyclic cofibrations and weak equivalences of a model structure on $\mathcal{M}$ and let $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{W}''$ be the cofibrations, acyclic cofibrations and weak equivalences of another model structure on $\mathcal{M}$. If $\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \mathcal{C}_3$, then there is a model structure on $\mathcal{M}$ with cofibrations $\mathcal{C}_0$, acyclic cofibrations $\mathcal{C}_3$ and weak equivalences $\mathcal{W} = \mathcal{W}' \cap \mathcal{W}''$.*

The theorem gives us a model structure whose weak equivalences are the intersection of two given classes of weak equivalences for which model structures are known. We will give an example of its application where the two classes are the horizontal and vertical quasi-isomorphisms in a category of double complexes.

We will also use the theorem to show that the category of positively-graded complexes of modules over a Frobenius ring has a model structure whose cofibrations are the levelwise injections and whose fibrations are the levelwise surjections. The construction allows us to characterise the weak equivalences in this structure as the intersection of two more familiar classes of weak equivalences: the quasi-isomorphisms and the levelwise stable equivalences.

The paper is divided into three sections, being the proof of Theorem 1 followed by the two separate applications. All three sections can be read independently given the statement of Theorem 1 above.

Composite model structures

In this section we prove Theorem 1. The proof is short and elementary, but it is not required reading for the rest of the paper. We begin by presenting our preferred axiomatisation for model structures and showing that it is equivalent to standard ones (see e.g. [Hirschhorn] or [Hovey 1998]; the only difference is that there is no need for us to require any completeness condition on the underlying category) and to some other criteria which we will find useful.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Definition. *A pair* (c, f) *of morphisms in a category* $\mathcal{M}$ has the lifting property *(or* c has the left lifting property (l.l.p.) with respect to f *or* f has the right lifting property (r.l.p.) with respect to c*) if whenever* x*,* y *are morphisms of* $\mathcal{M}$ *such that* $fy = xc$*, there exists* z *such that* $fz = x$ *and* $y = zc$*.*

A pair $(\mathcal{C}, \mathcal{F})$ *of subcategories, or collections of morphisms, of a category* $\mathcal{M}$ has the lifting property *(or* $\mathcal{C}$ has the l.l.p. with respect to $\mathcal{F}$ *or* $\mathcal{F}$ has the r.l.p. with respect to $\mathcal{C}$*) if every* $c \in \mathcal{C}$ *has the left lifting property with respect to every* $f \in \mathcal{F}$*.*

Notation. *Given a collection of morphisms* $\mathcal{Z}$ *of a category* $\mathcal{M}$*, we write* $\rho(\mathcal{Z})$ *(resp.* $\lambda(\mathcal{Z})$*) for the collection of all morphisms which have the right (resp. left) lifting property with respect to every* $z \in \mathcal{Z}$*.*

It is easily verified that $\rho(\mathcal{Z})$ and $\lambda(\mathcal{Z})$ are subcategories of $\mathcal{M}$ which contain all objects and are closed under retracts, and that ρ and λ form a Galois correspondence.

Definition. *A* weak factorisation system (w.f.s.) *on a category* $\mathcal{M}$ *is a pair* $(\mathcal{C}, \mathcal{F})$ *of subcategories of* $\mathcal{M}$ *such that (i)* $\mathcal{C} = \lambda(\mathcal{F})$ *and* $\mathcal{F} = \rho(\mathcal{C})$*, and (ii) every morphism in* $\mathcal{M}$ *factorises as* fc*,* $f \in \mathcal{F}$*,* $c \in \mathcal{C}$*.*

Obviously, a w.f.s. is uniquely determined by specifying either one of $\mathcal{C}$ or $\mathcal{F}$.

Definition. *A* model structure *on a category* $\mathcal{M}$ *is a pair* $\big[(\mathcal{C}, \mathcal{F}^w),\ (\mathcal{C}^w, \mathcal{F})\big]$ *of w.f.s. such that (i) the inclusion holds* $\mathcal{C} \supset \mathcal{C}^w$ *(and hence* $\mathcal{F}^w \subset \mathcal{F}$*) and (ii) the collection* $\mathcal{W} = \mathcal{F}^w\mathcal{C}^w$ *of morphisms of the form* fc*,* $f \in \mathcal{F}^w$*,* $c \in \mathcal{C}^w$ *satisfies the condition*

$$\mathcal{W} \text{ is a subcategory which is closed under retracts and two-out-of-three.} \qquad (W)$$

$\mathcal{W}$ *is called the subcategory* of weak equivalences *of the model structure.*

Proposition. *Let* $(\mathcal{C}, \mathcal{F}^w)$*,* $(\mathcal{C}^w, \mathcal{F})$ *be w.f.s. on a category* $\mathcal{M}$*, and* $\mathcal{W}$ *a subcategory of* $\mathcal{M}$*. The following are equivalent:*

1. $\big[(\mathcal{C}, \mathcal{F}^w),\ (\mathcal{C}^w, \mathcal{F})\big]$ *is a model structure with weak equivalences* $\mathcal{W}$*;*
2. $\mathcal{W}$ *satisfies axiom (W),* $\mathcal{C} \cap \mathcal{W} = \mathcal{C}^w$ *and* $\mathcal{F} \cap \mathcal{W} = \mathcal{F}^w$*;*
3. $\mathcal{W}$ *satisfies axiom (W),* $\mathcal{C} \cap \mathcal{W} = \mathcal{C}^w$ *and* $\mathcal{W} \supseteq \mathcal{F}^w$*;*
4. $\mathcal{W}$ *satisfies axiom (W),* $\mathcal{W} \supseteq \mathcal{C}^w$ *and* $\mathcal{F} \cap \mathcal{W} = \mathcal{F}^w$*;*
5. $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *satisfies the usual model-category axioms of factorisation, lifting, retraction and two-out-of-three.*

$(1 \Rightarrow 2)$ $\mathcal{W}$ satisfies axiom (W) by assumption: we must check $\mathcal{C} \cap \mathcal{W} = \mathcal{C}^w$, and the other condition will hold by duality. By definition $\mathcal{W} \supseteq \mathcal{C}^w$, and by assumption $\mathcal{C} \supseteq \mathcal{C}^w$. Conversely, if $c \in \mathcal{C} \cap \mathcal{W}$, write $c = fw$, $f \in \mathcal{F}^w$, $w \in \mathcal{C}^w$. Then the lifting property for (c, f) gives z such that $zc = w$ and $fz = \mathrm{id}$. Consequently c is a retract of $w \in \mathcal{C}^w$ and hence $c \in \mathcal{C}^w$.

$(3 \Rightarrow 1)$ We must show that $\mathcal{W} = \mathcal{F}^w\mathcal{C}^w$; from the assumptions it is clear that $\mathcal{W} \supseteq \mathcal{F}^w\mathcal{C}^w$. Conversely, for any $w \in \mathcal{W}$ write $w = fc$, where $f \in \mathcal{F}^w$, $c \in \mathcal{C}$. Then $f \in \mathcal{W}$, so by two-out-of-three we have $c \in \mathcal{W}$. Then $c \in \mathcal{C} \cap \mathcal{W} = \mathcal{C}^w$, so $w = fc$ is the required factorisation.

$(1\&2 \Rightarrow 5)$ From (2), the factorisation and lifting in our definition coïncide with the required ones. Retraction holds for $\mathcal{C}$ (resp. $\mathcal{C}^w$, $\mathcal{F}$, $\mathcal{F}^w$) because it is of the form $\lambda(\mathcal{F}^w)$ (resp. $\lambda(\mathcal{F})$, $\rho(\mathcal{C}^w)$, $\rho(\mathcal{C})$.) Axiom (W) provides the rest.

$(5 \Rightarrow 2)$ and $(2 \Rightarrow 3)$ are easy, and (4) is dual to (3). ∎

Theorem 1 (restated). *Suppose* $\big[(\mathcal{C}_0,\mathcal{F}_0),(\mathcal{C}_2,\mathcal{F}_2)\big]$ *and* $\big[(\mathcal{C}_1,\mathcal{F}_1),(\mathcal{C}_3,\mathcal{F}_3)\big]$ *are model structures on* $\mathcal{M}$ *with weak equivalences* $\mathcal{W}'$, $\mathcal{W}''$ *respectively. If* $\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \mathcal{C}_3$*, then* $\big[(\mathcal{C}_0,\mathcal{F}_0),(\mathcal{C}_3,\mathcal{F}_3)\big]$ *is a model structure with weak equivalences* $\mathcal{W} = \mathcal{W}' \cap \mathcal{W}''$.

Proof $\mathcal{W} = \mathcal{W}' \cap \mathcal{W}''$ certainly satisfies the closure property (W) since $\mathcal{W}'$ and $\mathcal{W}''$ do. Using characterisation (2) of model structures,

$$\mathcal{C}_0 \cap \mathcal{W} = \mathcal{C}_0 \cap \mathcal{W}' \cap \mathcal{W}'' = \mathcal{C}_2 \cap \mathcal{W}'' \subseteq \mathcal{C}_1 \cap \mathcal{W}'' = \mathcal{C}_3,$$

but $\mathcal{C}_2 \supseteq \mathcal{C}_3$ and $\mathcal{W}'' \supseteq \mathcal{C}_3$ so $\mathcal{C}_0 \cap \mathcal{W} = \mathcal{C}_3$. An analogous argument shows that $\mathcal{F}_3 \cap \mathcal{W} = \mathcal{F}_0$, so the result follows. ∎

We will call $\big[(\mathcal{C}_0,\mathcal{F}_0),(\mathcal{C}_3,\mathcal{F}_3)\big]$ the *composite* of the other two model structures. It has $\big[(\mathcal{C}_0,\mathcal{F}_0),(\mathcal{C}_2,\mathcal{F}_2)\big]$ as a Bousfield localization and $\big[(\mathcal{C}_1,\mathcal{F}_1),(\mathcal{C}_3,\mathcal{F}_3)\big]$ as a Bousfield colocalization, and it is cofibrantly (resp. fibrantly) generated if both of these are. As a corollary, we have:

Corollary. *The relation* $\geq_m$ *defined by*

$$(\mathcal{C},\mathcal{F}) \geq_m (\mathcal{C}',\mathcal{F}') \text{ iff } \big[(\mathcal{C},\mathcal{F}),(\mathcal{C}',\mathcal{F}')\big] \text{ is a model structure}$$

is a partial order on w.f.s. on $\mathcal{M}$*, which is refined by (i.e., is compatible with and has fewer comparable pairs than) the inclusion order* $(\mathcal{C},\mathcal{F}) \geq_i (\mathcal{C}',\mathcal{F}')$ iff $\mathcal{C} \supseteq \mathcal{C}'$.

In general this refinement is strict.

Double complexes

In this section we will demonstrate the application of Theorem 1 to prove:

Theorem 2. *Let* $\mathcal{A}$ *be a Grothendieck category with generator* U*. There is a cofibrantly generated model structure on* $\mathsf{Ch}(\mathsf{Ch}(\mathcal{A}))$ *whose weak equivalences are precisely those maps which are both a horizontal and a vertical quasi-isomorphism.*

The method is to apply a construction of Hovey for producing model structures on $\mathsf{Ch}(\mathcal{A}')$ for a Grothendieck category $\mathcal{A}'$, which delivers the quasi-isomorphisms as weak equivalences while giving us flexibility in choosing the cofibrations. We do this in two different ways, taking $\mathcal{A}'$ once to be the horizontal complexes on $\mathcal{A}$ and once the vertical ones, and apply Theorem 1 to combine the two resulting model structures.

The relevant theorem is:

Theorem. *Suppose* $\mathcal{M}$ *is a set of monomorphisms whose codomains* F *generate* $\mathcal{A}'$ *and satisfy (i)* $\mathcal{M}$ *contains every* $0 \hookrightarrow F$*, and (ii) whenever* $X^\bullet \in \mathsf{Ch}(\mathcal{A}')$ *is levelwise* $\mathcal{M}$*-flasque,* $X^\bullet$ *acyclic implies* $\mathcal{A}'(F,X^\bullet)$ *acyclic. Then the* $D^i\mathcal{M}$ *generate the acyclic cofibrations, and the* $D^i\mathcal{M}$ *together with the* $S^{i-1}F \hookrightarrow D^iF$ *all the cofibrations, of a model structure whose weak equivalences are the quasi-isomorphisms.* ([Hovey 2001], Theorem 1.7)

Let S^n, D^n be the usual sphere and disk functors, that is,

$$S^nA = \cdots \to 0 \to A \to 0 \to \cdots; \qquad D^nA = \cdots \to 0 \to A \xrightarrow{\cong} A \to 0 \to \cdots$$

ambiguously denoting the functors either from $\mathcal{A}$ to horizontal complexes, or from vertical complexes to double complexes. Let $\bar{S}^m, \bar{D}^m$ denote the sphere and disk functors either from $\mathcal{A}$ to vertical complexes or from horizontal complexes to double complexes: clearly S^n, D^n commute with $\bar{S}^m, \bar{D}^m$. Then we generate our first model structure by taking $\mathcal{M} = \{0 \hookrightarrow D^n U\}$ in the above theorem. The conditions of the theorem are satisfied since $D^n U$ is a projective complex: writing C_1 for the resulting set of generating cofibrations and C_3 for the set of generating acyclic cofibrations, we get

$$C_1 = \{\bar{S}^{m-1} D^n U \hookrightarrow \bar{D}^m D^n U\} \cup C_3, \qquad C_3 = \{0 \hookrightarrow \bar{D}^m D^n U\}.$$

For the second model structure, take $\mathcal{M} = \{0 \hookrightarrow \bar{D}^m U,\ \bar{S}^{m-1} \hookrightarrow \bar{D}^m U\}$. Again the theorem applies, and writing C_0 for the generating cofibrations and C_2 for the generating acyclic cofibrations, the result is

$$C_0 = \{S^{n-1}\bar{D}^m U \hookrightarrow D^n \bar{D}^m U\} \cup C_2,\ C_2 = \{0 \hookrightarrow D^n \bar{D}^m U, D^n \bar{S}^{m-1} \hookrightarrow D^n \bar{D}^m U\}.$$

Evidently $C_0 \supset C_1 = C_2 \supset C_3$ so Theorem 1 will apply to the two cofibrantly generated model structures. Thus C_0 generates the cofibrations, and C_3 the acyclic cofibrations, of a model structure satisfying the conditions of Theorem 2.

Chain complexes on Frobenius rings

In this section we will use Theorem 1 to show:

Theorem 3. *Let R be a Frobenius ring. There is a model structure on $\mathsf{Ch}_+(R\text{-}\mathsf{mod})$ such that the cofibrations are precisely the levelwise injections and the fibrations are precisely the levelwise surjections. The weak equivalences are precisely those homology equivalences which are also levelwise stable equivalences.*

The Reedy construction on abelian categories. Recall Reedy's construction ([Reedy]) of a model structure $\mathbf{s}\mathcal{M}$ on the category $\mathsf{s}\mathcal{M}$ of simplicial objects over a model category $\mathcal{M}$, such that the weak equivalences in $\mathbf{s}\mathcal{M}$ are levelwise. In the case where $\mathcal{M}$ is an abelian category $\mathcal{A}$, the Dold–Kan equivalence ([Dold], [Kan])

$$\mathsf{s}\mathcal{A} \underset{\Gamma}{\overset{N}{\rightleftarrows}} \mathsf{Ch}_+(\mathcal{A})$$

induces a model structure $N\mathbf{s}\mathcal{A}$ on $\mathsf{Ch}_+(\mathcal{A})$ whose weak equivalences are levelwise. It turns out that:

Proposition. *The cofibrations of $N\mathbf{s}\mathcal{A}$ are precisely the levelwise cofibrations.*

Proof Let $f_\bullet : A_\bullet \to B_\bullet$ be a map in $N\mathbf{s}\mathcal{A}$, Applying Reedy's criterion for cofibrations, we must for each $n \geq 0$ determine whether the latching map $L_n \to \Gamma(B_\bullet)_n$ is a cofibration in R-**mod**, where the nth latching object is

$$L_n = \operatorname{colim}\Big(\coprod_{s:[n]\to[n-1]} \Gamma\big(A_\bullet\big)_{n-1} \amalg \coprod_{s:[n]\to[n-2]} \Gamma\big(B_\bullet\big)_{n-2} \overset{\rightrightarrows}{\vdots} \Gamma\big(A_\bullet\big)_n \amalg \coprod_{s:[n]\to[n-1]} \Gamma\big(B_\bullet\big)_{n-1}\Big)$$

$$= \operatorname{coker}\Big(\bigoplus_{\substack{s:[n]\to[n-1]\\ s':[n-1]\to[k]}} A_k \oplus \bigoplus_{\substack{s:[n]\to[n-2]\\ s':[n-2]\to[k]}} B_k \longrightarrow \bigoplus_{s':[n]\to[k]} A_k \oplus \bigoplus_{\substack{s:[n]\to[n-1]\\ s':[n-1]\to[k]}} B_k\Big).$$

Here the map $\bigoplus B_k \to \bigoplus B_k$ is given by an alternating sum of those naturally induced by degeneracies $\sigma : [n-1] \twoheadrightarrow [n-2]$; the map $\bigoplus A_k \to \bigoplus A_k$ is similar over degeneracies $\sigma : [n] \twoheadrightarrow [n-1]$ (which reduces to taking an alternating sum for each fixed k); the map $\bigoplus A_k \to \bigoplus B_k$ is induced by $f_\bullet$; and the remaining component is zero. All these maps preserve the composite $s'' := s's : [n] \twoheadrightarrow [k]$ and consequently the map splits as a direct sum

$$L_n = \bigoplus_{s'':[n]\twoheadrightarrow[k]} \operatorname{coker}\Big(\bigoplus_{[n]\to[n-1]\to[k]} A_k \oplus \bigoplus_{[n]\to[n-2]\to[k]} B_k \longrightarrow A_k \oplus \bigoplus_{[n]\to[n-1]\to[k]} B_k \Big) = A_n \oplus \bigoplus_{\substack{s'':[n]\to[k] \\ k<n}} B_k.$$

The map $L_n \to \Gamma(B_\bullet)_n$ is just

$$f_n \oplus \mathrm{id}_{\bigoplus_{\substack{s'':[n]\to[k]\\ k<n}} B_k} : A_n \oplus \bigoplus_{\substack{s'':[n]\to[k]\\ k<n}} B_k \longrightarrow B_n \oplus \bigoplus_{\substack{s'':[n]\to[k]\\ k<n}} B_k.$$

A direct sum of maps is a cofibration iff all summands are (this is easily seen from the characterization of cofibrations by a lifting property) and all identity maps are cofibrations, so the nth Reedy cofibration condition is satisfied iff f_n is a cofibration in $\mathcal{A}$. ∎

Thus the model structure on $N\mathsf{s}\mathcal{A}$ can be stated simply by saying that the cofibrations and acyclic cofibrations are both levelwise.

The stable structure on modules over a Frobenius ring. Let R be a Frobenius ring. There is a model structure R-**mod** on the category of R-modules, such that the cofibrations are the injections, the fibrations are the surjections and the weak equivalences are the stable equivalences. (See [Hovey 1998] for details.)

Proposition. *Then in $N\mathsf{s}R$-**mod**, the acyclic cofibrations are precisely the levelwise injections with levelwise projective cokernel.*

Proof Given the preceding Proposition, we must prove that a levelwise injection has projective levelwise cokernel iff it is a levelwise stable equivalence. Hence it suffices to verify that an injection has projective cokernel iff it is a stable equivalence. For this, let

$$A \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} C$$

be exact, and first suppose C is projective: then the sequence is right split, so it is left split and any left splitting is clearly the required stable inverse to f. Conversely, suppose $\iota : B \to A$ is a stable inverse to f. Then $\iota f - \mathrm{id}_A$ factors through a projective, say $\iota f = pe + \mathrm{id}_A$. Now e is a map to an injective (since all projectives are injective) and hence it factors through the injection f, say $e = jf$. Then

$$\mathrm{id}_A = \iota f - pe = \iota f - pjf = (\iota - pj)f$$

so f has a left inverse, i.e. the sequence is left split. Hence $A \oplus C \cong B$ which by hypothesis is stably equivalent to A, and so C must be projective. ∎

The composite model structure. For any ring R there is a well-known model structure on $\mathsf{Ch}_+(R\text{-}\mathsf{mod})$ in which the weak equivalences are the homology equivalences and the cofibrations are the levelwise injections with projective levelwise cokernel. If R is Frobenius, the cofibrations are precisely the acyclic cofibrations in $N\mathsf{s}R$-**mod**, and so Theorem 1 applies. The result is the model structure claimed in Theorem 3.

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