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RELATIONS AMONG THE STIEFEL-WHITNEY CLASSES OF MANIFOLDS

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1. Introduction.

It is known that there are no relations among the Stiefel-Whitney classes which hold for tangent bundles of all closed C^∞-manifolds. However, if we hold the dimension of the manifolds fixed, there are many relations. In section 3, we describe without proofs some results relating to a complete determination of such relations. The answers are by no means complete.

"Whenever there are relations, there are secondary objects". In section 4 we describe some secondary characteristic classes defined for a class of n-plane bundles including tangent bundles of closed C^∞- manifolds. At present, it is not clear whether or not these secondary characteristic classes have any applications.

The algebraic notions of section 2 are useful in formulating the results of sections 3 and 4. They may be of interest in their own right.

All cohomology groups and rings have mod 2 coefficients.

2. Some Algebra.

Let A denote the mod 2 Steenrod algebra. Let $\chi : A \to A$ denote the canonical anti-automorphism of A.

Definition. An algebra H over Z_2 is called a left-right algebra over A if A operates on H on the left[2] and on the right and if the following condition relates the two operations:

$$(2.1) \qquad (h)\chi(S_q^i) = \sum_{j=0}^{i} u_j \cdot S_q^{i-j}(h), \quad (\text{for } S_q^i \text{ read } Sq^i)$$

where $u_j = (1)\chi(S_q^j)$ and $h \in H$.

An example. Let H be an n-dimensional algebra over Z_2 with left A operations and satisfying Poincaré duality. (E.g. $H^*(M)$, M a closed manifold.) H can be made into a right algebra over A by defining

$$(h_j)a \cdot h_{n-i-j} = h_j \cdot a(h_{n-i-j})$$

for all $h_{n-i-j} \in H^{n-i-j}$, where $a \in A^i$ and $h_j \in H^j$. (See Adams [1].)

Lemma 2.2. With this definition, H is a left-right algebra over A .

Such a left-right algebra will be called a P-algebra.

Proposition 2.3. If H is a P-algebra, $u_i = \overline{W}_i$.

The following theorem gives further examples of left-right algebras over A .

Theorem 2.4. Let H be a left algebra over A . Let u_i, $i \geq 0$ be given with $u_0 = 1$. Then formula (2.1) makes H into a left-right algebra over A if and only if u_i satisfy the Wu formulae, i.e.

$$S_q^r(u_i) = \sum_{t=0}^{r} \binom{i-r+t-1}{t} u_{r-t} \cdot u_{i+t} .$$

Corollary 2.5. A vector bundle over a CW-complex K makes $H^*(K)$ into a left-right algebra over A by setting $u_i = \overline{W}_i(\xi)$.

Corollary 2.6. Let H be a left-right algebra over A . Then there is a unique homomorphism τ_H^* of left-right algebras over A , $\tau_H^* : H^*(BO_\infty) \longrightarrow H$. If H is n-dimensional, then τ_H^* may be chosen so that $\tau_H^* : H^*(BO_n) \longrightarrow H$.

Note that the right operations in a left-right algebra over A do not satisfy the same axioms as the left operations. The analog of the Cartan formula is theorem 2.7.

Theorem 2.7. $(h_1 \cdot h_2)S_q^i = \sum_{j=0}^{i} (h_1)S_q^{i-j} \cdot \chi(S_q^j)(h_2)$.

3. Relations among Stiefel-Whitney classes.

Definition. Let $S \subset H^*(BO_n)^3$. Define $I_n(S,\text{alg.}) = \bigcap_{\tau_H^*} \text{Ker } \tau_H^*$, where H runs through all n-dimensional P-algebras with $\tau_H^*(S) = 0$. Define $I_n(S,\text{geom.}) = \bigcap_{\tau_M} \text{Ker } \tau_M^*$, where the tangent map $\tau_M : M^n \to BO_n$ runs through all closed C^∞-manifolds of dimension n with $\tau_M^*(S) = 0$.

There are three questions which may be asked:

1) What is $I_n(S,\text{geom.})$?

2) What is $I_n(S,\text{alg.})$?

3) When is $I_n(S,\text{geom.}) \subset I_n(S,\text{alg.})$?

(Clearly $I_n(S,\text{geom.}) \supset I_n(S,\text{alg.})$.)

The most interesting cases are when $S = \emptyset$ or $\{W_1\}$. We remark that it is not true that $I_n(S,\text{alg.})$ is the ideal generated by S and $I_n(\emptyset,\text{alg.})$.

Proposition 3.1. 3) above is not true in general. A counter-example can be constructed when $n = 32$ and $\{W_1,W_2,W_4,W_8,W_{24}\} = S$

The following theorem is due to Dold [2] and solves part of 1).

Theorem 3.2. $I_n(\emptyset,\text{geom.})^n$ = the Z_2-module generated by $(S_q^i + v_i \cup)(H^{n-i}(BO_n))$, $i = 1, \ldots, n$, where $v_i = (1)S_q^i$.

Corollary 3.3. $I_n(\emptyset,\text{geom.})^n = I_n(\emptyset,\text{alg.})^n$.

The following theorem is due, independently, to E. Brown and R. Stong (to appear).

Theorem 3.4. $I_n(\emptyset,\text{geom.})^i = 0$ if $i \leqq \frac{n}{2}$.

Since it is clear that $\tau_H^*(v_i) = 0$ for all n-dimensional P-algebras H and all $i > \frac{n}{2}$, our first conjecture might be that $I_n(\emptyset,\text{alg.}) = J_n$, where J_n is the left-right ideal generated by v_i for $i > \frac{n}{2}$. (The left ideal is automatically a left-right ideal.) Direct computation shows that this is so for $n \leqq 5$ and, except for dimension n, for $n = 6$. Further computations shows that $(W_2 \cdot W_3)S_q^2 \in I_8(\emptyset,\text{alg.}) - J_8$.

E. Brown (unpublished) has given a computable description of $I_n(\emptyset,\text{alg.})$.

Theorem 3.5. Let $L \in H^{n-k}(Z_2,n-k)$ be the canonical generator. Then $R \in I_n(\emptyset,\text{alg.})^k$ if and only if $R \otimes L$ belongs to the Z_2-module generated by

$$(S_q^i + v_i \otimes 1 \cup)(H^{n-i}(BO_n \times K(Z_2,n-k))) \subset H^n(BO_n \times K(Z_2,n-k)), \quad i=1,\ldots,n.$$

Using this theorem, one can prove that $\bar{W}_9 = 0$ for orientable 12-manifolds (there is an analogous theorem for $S = \{W_1\}$), and this result does not follow from the techniques of Massey [3] (see a forthcoming paper of Massey and Peterson).

However, the computations needed to apply this theorem are long, and one hopes to find a simpler description of $I_n(\emptyset, \text{alg.})$. To this end, define F_n to be the Z_2-module generated by $(H^j(BO_n))S_q^i$, for $i > n-i-j$.
$\tau^*_H((h_j)S_q^i) \cdot h_{n-i-j} = \tau^*_H(h_j) \cdot S_q^i(h_{n-i-j}) = 0$ for all $h_{n-i-j} \in H^{n-i-j}$, H a P-algebra, and all $h_j \in H^j(BO_n)$.
Thus $F_n \subset I_n(\emptyset, \text{alg.})$.

Theorem 3.6. F_n is a left-right ideal and $F_n^n = I_n(\emptyset,\text{alg.})^n$.

Conjecture. $F_n = I_n(\emptyset,\text{alg.})$.

4. Secondary characteristic classes.

Because of the relations between the Stiefel-Whitney classes of tangent bundles, we may define "secondary characteristic classes" for manifolds.

Let $v: BO_n \longrightarrow \prod_{i=\left[\frac{n}{2}\right]+1}^{n} K(Z_2, i)$ be defined by $v(L_i) = v_i \in H^i(BO_n)$. Let $\pi : P_n \longrightarrow BO_n$ be the fibre space induced by v from the path space over $\prod_{i=\left[\frac{n}{2}\right]+1}^{n} K(Z_2, i)$.

Let ξ be an n-plane bundle over K and $f = f(\xi): K \longrightarrow BO_n$ be its classifying map. Assume that $v_i(\xi) = f^*(v_i) = 0$ for $\frac{n}{2} < i \leq n$. Then there exists a map $\hat{f} : K \longrightarrow P_n$ such that $\pi\hat{f} \simeq f$. Note that the homotopy class of $\hat{f}$ is not necessarily uniquely determined by that of f. If $\lambda \in H^*(P_n)$, and $\lambda \notin \pi^*(H^*(BO_n)) \simeq H^*(BO_n)/J_n$, then $\{\hat{f}^*(\lambda)\} \subset H^*(K)$ is a secondary characteristic class of ξ. We now describe how to construct such λ.

Theorem 4.1. The following are relations in $H^*(BO_n)$:

i) $\left(\sum_{i=0}^{k-1} S_q^{2i} v_{4k-1-2i}\right) S_q^2 = 0$ (n=4k or 4k+1),

ii) $\left(\sum_{i=0}^{k-1} S_q^{2i+1} v_{4k-2i}\right) S_q^2 = 0$ (n=4k+2 or 4k+3),

iii) $\left(\sum_{i=0}^{k} S_q^{i} v_{2k+1-i}\right) S_q^1 = 0$ (n=2k+1),

iv) $(v_5)S_q^3 = 0$, (n=7,8,or 9) and $(v_7)S_q^4 = 0$, (n=10,...,13).

Let $(\sum_{i>\frac{n}{2}} a_i v_i)S^r_q = 0$ be such a relation, where $a_i \in A^{s+1-r-i}$. Since $\pi^*(v_i) = 0$ for $\frac{n}{2} < i \leq n$, we may form the functional operation $(\sum_{i>\frac{n}{2}} a_i v_i)_\pi S^r_q \subset H^s(P_n)$.

Theorem 4.2. There exists an element $\lambda \in (\sum_{i>\frac{n}{2}} a_i v_i)_\pi S^r_q$ such that $j^*(\lambda) = \sum a_i \, {}^1L_i \in H^s(\prod_{i=[\frac{n}{2}]+1}^{n} K(Z_2, i-1))$, where ${}^1L_i$ is the generator of $H^{i-1}(Z_2, i-1)$ and j is the inclusion of the fibre into P_n. Furthermore, if $\hat{f}_k \colon K \longrightarrow P_n$ are such that $\pi f_k \simeq f$, $k=1,2$, then there exists an element $h \in H^s(K)$ such that $\hat{f}^*_2(\lambda) - \hat{f}^*_1(\lambda) = (h)S^r_q$.

Thus choosing such a λ, we may define $\phi_\lambda(\xi) \in H^s(K) / (H^{s-r}(K))S^r_q$. Of particular interest is the case when $K = M^n$, a closed C^∞-manifold and ξ is the tangent bundle.

Proposition 4.3. If λ corresponds to the relations in theorem 4.1, then $\phi_\lambda(\tau)$ is defined modulo the zero subgroup.

Theorem 4.4. One can choose λ for relation i) of theorem 4.1 so that $\phi_\lambda(\xi) = 0$ when ξ is orientable.

Using results of J. Stasheff (unpublished) one can prove the following theorem.

Theorem 4.5. If λ corresponds to the relations in theorem 4.1, then $\phi_\lambda(\tau)$ is independent of the differentiable structure on M; in fact, ϕ_λ can be defined for closed topological manifolds.

Open questions. 1) Find a λ so that $\phi_\lambda(\tau)$ is not identically a polynomial in $W_1, \ldots, W_n$[4]. 2) Determine the structure of $H^*(P_n)$[5].

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References.

1. J. F.Adams, On formulae of Thom and Wu, Proc.London Math. Soc. 11 (1961), 741-752.

2. A.Dold, Vollständigkeit der Wuschen Relationen zwischen der Stiefel-Whitneyschen Zahlen Differenzierbaren Mannigfalligkeiten, Math.Z. 65 (1956), 200-206.

3. W.S.Massey, On the Stiefel-Whitney classes of a manifold, Amer.J.Math. 82 (1960), 92-102.

Footnotes.

[1] The author is an Alfred P.Sloan fellow and was partially supported by the U.S. Army Research Office.

[2] The usual axioms for A operating on $H^*(X)$ on the left are assumed (e.g. the Cartan formula).

[3] For this section, $H^j(BO_n)$ will be considered to be 0 for $j>n$.

[4] Clearly $\phi_\lambda(\overline{\pi})$ is not a polynomial in $W_1, \ldots, W_n$.

[5] The elements described above generate $H^s(P_n)$ for $s \leq n \leq 6$.