Problems from the
Indiana Topology Seminar
Opera-Topology Miniconference

March 23-24, 1974

Bloomington, Indiana

Session chaired by

Frank Peterson

The general format follows that of the Madison conference, though certain sections are not represented. No problems on H-spaces are included since a list in that area is being prepared by Jim Stasheff. We have taken the liberty of including a number of problems proposed by people not at the conference but of interest to those there. Our thanks to Frank Peterson and to all those who submitted problems, and our apologies in advance to those whose problems have been inadvertently misrecorded.

Claude Schochet

Larry Smith

scribes

A. Manifolds

1. (Mahowald): If a closed spin manifold M^n has k linearly independent vector fields, then the L class of M satisfies certain divisibility conditions. Specifically, the class $\prod_{\text{primes}} p^{[j/p-1]} L_j \in H^{4j}(M;Z)$ is divisible by $2^{((4j-n+k+\epsilon)/2)+\delta}$, where $\epsilon = 0,1,2,3$ is chosen such that $4j-n+k+\epsilon \equiv 0\,(4)$ and δ depends upon the parity of $(4j-n+k+\epsilon)/2$.

2. (Mahowald): Let M^{2n+1} be a closed connected manifold, τ its tangent bundle with Thom space $T(\tau)$, and $i \in \pi_{2n+1}(T(\tau))$ the canonical class.

Conjecture: The Whitehead product $[i,i] \neq 0$ if and only if $\chi_2(M) \neq 0$, where

$$\chi_2(M) = \sum_{i=0}^{n} \dim H_i(M;Z/2)$$

is the mod 2 semi characteristic.

3. (D. Ravanel): Compute the action of the Dyer-Lashof algebra on $H^*(BG;Z/p)$. [May has known the answer for several years and has lectured on it at Cambridge and Chicago; to appear in "The homology of E_∞ ring spaces." Tsuchiya has published calculations based upon incorrect mixed Adem relations.]

4. (D. Ravanel): Are the exotic characteristic classes for spherical fibrations (defined by Ravanel in Comm. Math. Helv. 47) primitive? This would give an explicit formula for the Kervaire invariant part of putting a topological structure on a spherical fibration.

5. (M. Hirsch): Every π manifold M^n embeds in $\mathbb{R}^m$ where $m \geq 3(n+1)/2$. (This is very closely related to a hard homotopy problem, viz. if $f: S^{n+k} \to S^n$ is stably trivial, then $\Sigma^{[(k-n)/2]}f$ is already trivial.)

B. Cobordism Theories

6. Determine the odd torsion in Ω_*^{SPL} .

7. Determine Ω_*^{Sp} . (N.B. A warning should be attached to this problem by the surgeon general.) S. Kochman claims to compute E_∞ of the Adams spectral sequence for $\pi_*(MSp)$.

8. N. Ray has constructed an infinite family $\varphi_{8k+5} \in \Omega_{8k+5}^{Sp}$ of manifolds indecomposable under the ring structure. Represent these classes geometrically. (N.B. The paper: J. C. Alexander, A Family of ... , Am. J. of Math., 94(1972), pp. 699-710 purporting to solve this problem contains a gap in that lemma 3.1 is false.)

9. (W. Browder - Madison): Find cobordism theories in which the absolute Kervaire invariant can be computed by simple formulas. For example, let

$$\varphi\colon BO \to \mathop{\times}_{i=1}^{\infty} K(Z/2;2^i)$$

be the map with $\varphi^* i_{2^j} = w_{2^j}$, and let B be the fibre of φ with $f: B \to BO$ the natural map, and consider bordism of manifolds with a (B,f) structure. Then the absolute Kervaire invariant K is defined

$$K: \Omega_{2n}^{(B,f)} \to Z/2 .$$

Conjecture: K is given by the characteristic class associated to an element $k = 1 + k_1 + k_2 + \ldots + k_n \in H^{2n}(B; Z/2)$, if $n \neq 2^j - 1$. Peterson has shown that the conjecture is true for $n = 5$.

10. Find a necessary and sufficient condition for a vector bundle to be orientable for complex cobordism theory. Gallant has shown that the orientation fibration $P \to BSO$ introduced by Patterson and Stong splits into two pieces

$$P_1 \times P_2 \xrightarrow{P_1 \times (\mathrm{const})} BSO \times pt$$

and has closely examined P_1. [May, Quinn, and Ray have, in a conceptual sense, solved this problem for any sufficiently good ring theory. (cf., "E_∞ ring spectra")].

11. (I. Kozma): Does there exist a complex K such that $H^*(K;Z) \neq 0$ but $MU^*(K) = 0$? Such a complex must of necessity be tall and fat!

D. Homotopy Theory

12. Solve the Arf invariant problem. Specifically do there exist elements $\theta_j \in \pi^S_{2^j-2}$ with Arf invariant one if $j \geq 6$? (Milgram's proof that θ_6 exists has fallen through.)

13. Find computable invariants that detect infinite families in coker J .

14. (L. Smith): For each prime $p > 3$ does coker J contain a subcomplex B carrying the β sequence and nothing else, or the β and ϵ sequences and nothing else?

15. (J. Cohen - Madison): Let M be a module over the mod p Steenrod algebra $\mathfrak{a}^*(p)$. Define

$$\text{rank } M = \sum_{i=0}^{\infty} M_i \; .$$

Peter Hoffman shows that there are only a finite number of indecomposable modules of a fixed rank that can be realized by a space with no p torsion in its homology. Can the "no p torsion" condition be removed?

Addendum (L.S.): List the modules allowed by Hoffman.

16. Let p be a prime and $V(n)$ the spaces introduced by L. Smith that satisfy

$$\tilde{H}^*(V(n);Z/p) \simeq E[Q_1,Q_2,\ldots,Q_n]$$

as a module over the mod p Steenrod algebra $\mathfrak{A}^*(p)$, where $Q_i \in \mathfrak{A}^{2p^i-1}(p)$ are the Milnor primitives.

Conjectures:

(a) $V(\infty)$ cannot exist. More strongly,

(b) $V(0),\ldots,V(p-2)$ exist, but $V(p-1)$ does not exist. The first open case is

(c) $V(4)$ exists for some primes.

17. What other modules over $\mathcal{G}^*(p)$ can be realized?

18. (Barratt): Nishida has shown that each element of π_*^S is nilpotent. Is it possible that the p component $\pi_*^S(p)$ is nilpotent? That is, does there exist an integer $d(p)$ such that $x^{d(p)} = 0 \ \forall \ x \in \pi_n^S(p)$; $n > 0$? It is unknown if

$$x^4 = 0 : \forall \ x \in \pi_n^S(2) \ ; \ n > 0$$

or if $$x^{2p^2} = 0 : \forall \ x \in \pi_n^S(p) \ ; \ n > 0 \ , \ p > 2 \ .$$

19. (Lin): If $\alpha \in \pi_*^S$ and

$$0 \in \underbrace{\{\alpha, \alpha, \ldots, \alpha\}}_{k}$$

for all k then is it true that $\alpha = 0$?

20. (B. Gray): Let A be the algebra of integral stable cohomology operations, and $I \subset A$ the ideal of elements of positive grading.

<u>Conjecture</u>: All possible matrix Massey products

$$\langle M_0, \ldots, M_s \rangle$$

contain 0 , where M_i has entries in I .

<u>Restatement</u>: The category of torsion-free spectra is equivalent to a suitably defined category of free chain complexes.

21. Eccles - Kahn - Priddy have shown

$$\Sigma^\infty \Omega^\infty \Sigma^\infty X \equiv \text{giant bouquet}$$

and Snaith shows

$$\Sigma^\infty \Omega^k \Sigma^k X \equiv \text{giant bouquet}$$

Prove one (and hopefully only one!) of the following

<u>Conjecture (Mahowald)</u>: There is no $N < \infty$ such that $\Sigma^N \Omega^k \Sigma^k X \equiv$ giant bouquet .

<u>Conjecture (Barratt)</u>: $\Sigma^k \Omega^k \Sigma^k X \equiv$ this bouquet .

[Note: Fred Cohen has a new splitting of $\Sigma\Omega^k \Sigma^k X$.]

22. (J. Harper): The natural map

$$\Omega S^{n+1} \to K(Z;n)$$

has been shown incompressable by Weingram. What about the maps

$$\Omega(S^{n+1} \cup_p e^{n+1}) \to K(Z/p;n)$$

$$\Omega S^{n+1} \to K(Z/p;n) \quad (n \text{ even})$$

23. (J. C. Moore): For a complex X let

$$P(X;z) = \sum_{i=1}^{\infty} \dim H_i (X;Q) z^i$$

For X finite and 1 connected, $P(\Omega X;z)$ is a rational function of z with a pole at $z = -1$ of order equal to the order of the 0 of $P(X,z)$ at $z = -1$.

(<u>N.B.</u> One easily checks this for X of cat ≤ 2 , or H spaces. Clark & Smith have shown that if Moore's conjecture holds for A and B it holds for $A \vee B$.)

24. (D. Sullivan): For a 1 connected finite complex X let

$$Q(X;z) = \sum_{i=0}^{\infty} \dim(\pi_i X \otimes Q) z^i$$

Then $Q(X,1) < \infty \Rightarrow Q(X,-1) \leq 0$.

25. (D. Sullivan): There are no essential maps

$$\mathbb{R}P(\infty) \to X = \text{finite complex} .$$

More generally there are no essential maps

$$BG \to X = \text{finite complex}$$

for G a compact Lie group.

It is perhaps worthwhile noting in this connection that a conjecture of Segal seems relevant.

Conjecture (Segal): Let π be a finite group and $A(\pi)$ its Burnside ring. Then

$$A(\pi)^{\wedge} \simeq \pi_S^0(B\pi) .$$

tom Dieck extends the definition of the Burnside ring to all compact Lie groups, and it follows easily from his definition that $A(S^1) = 0$. Hence if Segal's conjecture is true for compact connected groups, $\pi_S^0(CP(\infty)) = 0$, so that there would be no stable essential maps $BS^1 \to X =$ finite complex.

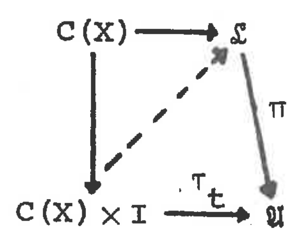
It is also worth noting that Lin has proved there is no <u>stable</u> essential map

$$KZ/2 \to X$$

where $KZ/2$ is the mod 2 Eilenberg MacLane spectrum and X is the suspension spectrum of a finite complex.

E. Miscellaneous

26. (Kaminker and Schochet): Let $\mathcal{L} \xrightarrow{\pi} \mathfrak{A}$ be the projection of the bounded operators on a Hilbert space to the Calkin algebra. Does π have the C*-algebra covering homotopy property? i.e., can one complete the diagram



where all maps are (continuous families of) C*-injections and X is finite dimensional compact metric. [True for the spaces $[0,1]$, S^1 , finite set, totally disconnected by work of John Conway.] If true would be basic to the Brown-Douglas-Fillmore work in operator theory.

27. (Barratt): Is there a discrete, cancellation monoid M embedded in its universal group UM for which $H_*(UM) \neq H_*(M)$? Barratt notes that if M is free or abelian or if $UM = M \cdot M^{-1}$ then $H_*(UM) = H_*(M)$. There are noncancellation monoids M with $H_*(UM) \neq H_*(M)$.