Some Remarks on Homotopy Commutativity of The Classical Groups

by Franklin P. Peterson
Massachusetts Institute of Technology

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1. Introduction.

Let A and B be subsets of a topological group G. We define A and B to homotopy commute in G if $\phi : A \times B \to G$ is homotopic to the constant map at the identity of G, where $\phi(a,b) = aba^{-1}b^{-1}$. Various results are known; in particular, James and Thomas [2] study the case where $A = SO(m)$, $B = SO(n)$, and $G = SO(t)$ and prove that A and B do not homotopy commute in G when $t = m + n - 1$ for most values of m and n. They solve the unitary and symplectic cases completely.

In this paper we derive some mixed results. Let $G(m) = SO(2m + 1)$ or $Sp(m)$. We want to prove the following statement: $G(m)$ and $G(n)$ do not homotopy commute in $U(2m + 2n - 1)$, where $m, n \geq 1$. Unfortunately, we only prove this statement for many values of m and n (see corollaries 3.3 and 3.4). We conjecture that it is true for all values of m and n.

2. Notations.

For a topological group G, let $B(G)$ denote the base space of the universal bundle for G. Let $\tau: \pi(SX; B(G)) \to \pi(X; G) = \pi(X; {}^1B(G))$ be the usual isomorphism, where ${}^1Y$ denotes the i-fold space of loops on Y. For any space X, let $X^{(q)}$ denote the q^{th} part of the Postnikov system of X; i.e. there is a map $\pi_q : X \to X^{(q)}$ such that $\pi_{q\#}: \pi_i(X) \to \pi_i(X^{(q)})$ is an isomorphism for $i \leq q$ and $\pi_i(X^{(q)}) = 0$ for $i > q$.

In [3], it is shown that $B(U(t))^{(2t)} = B(U(\infty))^{(2t)} = {}^2B(U(\infty))^{(2t+2)}$

We denote by $\nu: B(U(t))^{(2t)} \times B(U(t))^{(2t)} \to (B(U(t))^{(2t)}$ the induced multiplication. Recall that $H^*(B(U(t)); Z)$ is a polynomial ring on generators $c_1, \ldots, c_t$, where $c_i \in H^{2i}(B(U(t)); Z)$. Also recall that $H^*(G(m); Z_p)$ is an exterior algebra on generators $X_i \in H^{4i-1}(G(m); Z_p)$, for $i=1,\ldots,m$, and p an odd prime.

Let $i_A: A \to G$, $i_B: B \to G$ be inclusions. Let ∇ denote the folding map. Stasheff [4] proves the following theorem.

Theorem 2.1. A and B homotopy commute in G if and only if $\nabla(\tau^{-1}(i_A) \vee \tau^{-1}(i_B)): SA \vee SB \to B(G)$ can be extended to $SA \times SB$. Our study will be based on this theorem.

3. The Main Theorems.

The following lemma is implicit in [3].

Lemma 3.1. There is a unique element $\bar{c}_{t+1} \in H^{2t+2}(B(U(t))^{(2t)}; Z_p) = H^{2t+2}(B(U(\infty))^{(2t)}; Z_p)$ for $p \leq t$, p a prime, such that $\pi_{2t}^*(\bar{c}_{t+1}) = c_{t+1} \pmod p \in H^{2t+2}(B(U(\infty)); Z_p)$.

Our main theorem is the following one and is proved in section 4.

Theorem 3.2. Let p be an odd prime, $p < 2m + 2n$. Assume that there is no map $f: SG(m) \# SG(n) \to B(U(\infty))^{(4m+4n-2)}$ such that $f^*(\bar{c}_{2m+2n}) = \pm\sigma^*(X_m) \otimes \sigma^*(X_n) \in H^{4m+4n}(SG(m) \# SG(n); Z_p)$. Then G(m) and G(n) do not homotopy commute in $U(2m + 2n - 1)$.

In order to derive corollaries of theorem 3.2, we study cohomology operations in $H^*(B(U(\infty))^{(4m+4n-2)}; Z_p)$; in particular we wish to find operations θ such that $\theta(u) = \bar{c}_{2m+2n}$ for some u such that $\theta(f^*(u)) = 0$ for all f. In section 4 we do this and give proofs for the following corollaries.

Corollary 3.3. Let p be an odd prime, let $r \geq 0$. Let

$$m + n = \begin{cases} 1 + (1 + p + \cdots + p^r)\left(\frac{p-1}{2}\right) & \text{or} \\ 2 + 3(1 + p + \cdots + p^r)\left(\frac{p-1}{2}\right) & \text{for } p > 3 \text{ or} \\ 2 + [3+2(1 + p + \cdots + p^r)]\left(\frac{p-1}{2}\right) & \text{for } p > 3 \text{ or} \\ 2 + [2 + 1 + p + \cdots + p^r]\left(\frac{p-1}{2}\right) & \text{or} \\ 3 + 5(1 + p + \cdots + p^r)\left(\frac{p-1}{2}\right) & \text{for } p > 5. \end{cases}$$

Then $G(m)$ and $G(n)$ do not homotopy commute in $U(2m + 2n - 1)$.

Corollary 3.4. For every $\epsilon > 0$, there exists an N such that if $m + n > N$, and $G(m)$ and $G(n)$ homotopy commute in $U(2m + 2n - 1)$, then either m or $n \leq \epsilon(m + n)$.

Corollary 3.5. $G(m)$ does not commute with itself in $U(4m - 1)$.

4. Proof of Theorem 3.2.

Let $i_m : G(m) \to U(2m + 2n - 1)$, $i_n : G(n) \to U(2m + 2n - 1)$, $m, n \geq 1$, be the standard inclusions. Assume there is a map $F : S(G(m)) \times S(G(n)) \to B(U(2m + 2n - 1))$ extending $\nabla(\tau^{-1}(i_m) \vee \tau^{-1}(i_n))$. Then $\pi_{4m + 4n - 2}F$ extends $\pi_{4m+4n-2}\nabla(\tau^{-1}(i_m) \vee \tau^{-1}(i_n)) : S(G(m)) \vee S(G(n)) \to B(U(2m + 2n - 1))^{(4m+4n-2)}$.

Note that $(\pi_{4m + 4n - 2}F)^*(\bar{c}_{2m + 2n}) = F^*(\pi_{4m + 4n - 2})^*(\bar{c}_{2m + 2n}) = F^*(0) = 0$. On the other hand, we shall now prove that any extension of $\pi_{4m + 4n - 2}\nabla(\tau^{-1}(i_m) \vee \tau^{-1}(i_n))$ sends $\bar{c}_{2m + 2n}$ into a non-zero element of $H^{4m + 4n}(S(G(m)) \times S(G(n)); Z_p)$.

Lemma 4.1. Let ν be the multiplication in $B(U(\infty))^{(4m + 4n - 2)}$. Let $c_i \in H^{2i}(B(U(\infty))^{(4m + 4n - 2)}; Z_p)$ denote the class such that $(\pi_{4m + 4n - 2})^*(c_i)$ is the i^{th} Chern class mod p, $i=1, \ldots, 2m + 2n - 1$. Then $\nu^*(\bar{c}_{2m + 2n}) = \bar{c}_{2m + 2n} \otimes 1 + 1 \otimes \bar{c}_{2m + 2n} + \sum_{i=1}^{2m+2n-1} c_i \otimes c_{2m+2n-i}$.

Proof: Apply $(\pi_{4m+4n-2})^* \otimes (\pi_{4m+4n-2})^*$ to the formula in the lemma and one obtains

$$\tilde{\nu}^*(c_{2m+2n}) = \sum_{i=0}^{2m+2n} c_i \otimes c_{2m+2n-i} \in H^{4m+4n}(B(U(\infty)) \times B(U(\infty)); Z_p).$$

where $\tilde{\nu}$ is the multiplication in $B(U(\infty))$. This equation is well known to be true. To conclude the lemma, we remark that $(\pi_{4m+4n-2})^*$ is an isomorphism mod p in dimensions $\leq 4m + 4n$.

Let $i: G \to H$ be a homomorphism of topological groups. Let $B(i): B(G) \to B(H)$ be the induced map on their classifying spaces. Let $P_i \in H^{4i}(B(G(\infty)); Z_p)$ denote the Pontrjogin classes reduced mod p. The statements in the following lemma are proved in [1].

Lemma 4.2. $B(i_m)^*(c_i) = 0$ if i is odd. $B(i_m)^*(c_{2i}) = \pm P_i$ if $i \leq m$.

The following lemma is easy to prove and its proof is left to the reader.

Lemma 4.3. Let $1: G \to G$. Let $u \in H^*(B(G); R)$. Then $\tau^{-1}(1)^*(u) = \pm\sigma^*({}^1u)$, where ${}^1u \in H^*(G; R)$ is the space of loops suspension of u, and $\sigma^*: H^*(G; R) \to H^*(S(G); R)$ is the usual suspension isomorphism.

We now return to the proof of theorem 3.2.

$$\pi_{4m+4n-2} \nabla(\tau^{-1}(i_m) \vee \tau^{-1}(i_n)) = \nabla(\pi_{4m+4n-2}\tau^{-1}(i_m) \vee \pi_{4m+4n-2}\tau^{-1}(i_n)).$$

This map has an obvious extension, namely

$\nu(\pi_{4m+4n-2}\tau^{-1}(i_m) \times \pi_{4m+4n-2}\tau^{-1}(i_n)) = g$. We now compute

$$g^*(\bar{c}_{2m+2n}),\ g^*(\bar{c}_{2m+2n}) =$$

$$(\pi_{4m+4n-2}\tau^{-1}(i_m))^* \otimes (\pi_{4m+4n-2}\tau^{-1}(i_n))^*(\bar{c}_{2m+2n} \otimes 1 + 1 \otimes \bar{c}_{2m+2n} + \sum_{i=1}^{2m+2n-1} c_i \otimes c_{2m+2n-i}) = \sum_{i=1}^{2m+2n-1} \tau^{-1}(i_m)^*(c_i) \otimes \tau^{-1}(i_n)^*(c_{2m+2n-i}).$$

By lemmas 4.2 and 4.3, $\tau^{-1}(i_m)^*(c_i) = 0$ if i is odd, and $\tau^{-1}(i_m)^*(c_{2i}) = \tau^{-1}(1)^* B(i_m)^*(c_{2i}) = \pm\sigma^*({}^1P_i)$ which is 0 if $i > m$ and not zero if $i \leq m$. Thus, the only non-zero term is when $i = 2m$, and we have $g^*(\bar{c}_{2m+2n}) = \pm\sigma^*({}^1P_m)\otimes\sigma^*({}^1P_n) \neq 0$ as ${}^1P_i$ can be taken to be $X_i \in H^{4i-1}(G(m); Z_p)$ for $i \leq m$.

Recall that $\pi(S(G(m)) \# S(G(n)); B(U(\infty))^{(4m+4n-2)}) \overset{\alpha}{\to} \pi(S(G(m))\times S(G(n)); B(U(\infty))^{(4m+4n-2)}) \overset{\beta}{\to} \pi(S(G(m))\vee S(G(n)); B(U(\infty))^{(4m+4n-2)})$ is exact. Also, $\beta([g]) = \beta([\pi_{4m+4n-2}F])$, hence there is an $f \in \pi(S(G(m)) \# S(G(n)); B(U(\infty))^{(4m+4n-2)})$ such that $[\pi_{4m+4n-2}F] = [g] + \alpha([f])$. Therefore, $(\pi_{4m+4n-2}F)^*(\bar{c}_{2m+2n}) = (\nu(g\times\alpha(f))\Delta)^*(\bar{c}_{2m+2n}) = \Delta^*(g\times\alpha(f))^*\nu^*(\bar{c}_{2m+2n}) = \pm\sigma^*({}^1P_m)\otimes\sigma^*({}^1P_n) + \alpha(f)^*(\bar{c}_{2m+2n}) + \sum_{i=1}^{2m+2n-1} g^*(c_i)\cup\alpha(f)^*(c_{2m+2n-i})$.

Note that $\alpha(f)^*(c_{2m+2n-i}) = \Sigma\, u_j \otimes v_j$, where $\dim u_j > 0$, $\dim v_j > 0$. Hence $g^*(c_i)\cup\alpha(f)^*(c_{2m+2n-i}) = 0 \in H^*(S(G(m))\times S(G(n)); Z_p)$. By hypothesis, $\alpha(f)^*(\bar{c}_{2m+2n}) \neq \mp\sigma^*({}^1P_m)\otimes\sigma^*({}^1P_n)$. Hence $(\pi_{4m+4n-2}F)^*(\bar{c}_{2m+2n}) \neq 0$ which is a contradiction.

5. Proofs of 3.3, 3.4, and 3.5.

We are going to need the following lemma which is proved in [5].

Lemma 5.1. *In $H^*(B(U(\infty)); Z_p)$, we have*
$P^s(c_k) = \binom{k-1}{s} c_{k+s(p-1)}$ + a polynomial in lower Chern classes.
In $H^*(B(G(\infty)); Z_p)$, we have $P^s(P_k) = \pm\binom{2k-1}{s} P_{k+s(\frac{p-1}{2})}$ + a polynomial in lower Pontrjagin classes.

In order to prove corollary 3.3, we shall show that $\bar{c}_{2m+2n} = P^I(u)$ + a polynomial in lower classes, where $P^I f^*(u) \neq 0$

for any map $f: S(G(m)) \# S(G(n)) \to B(U(\infty))^{(4m+4n-2)}$. Since $S(G(m)) \# S(G(n)) = S[G(m) \# S(G(n))]$, $f^*(\text{polynomial}) = 0$. The corollary will then follow from theorem 3.2.

When $m + n = 1 + (1 + p + \cdots + p^r)(\frac{p-1}{2})$, we have $\bar{c}_{2m+2n} = P^{p^r} P^{p^{r-1}} \cdots P^p P^1(c_2)$ + a polynomial. $f^*(c_2) = 0$ as $H^4(S(G(m)) \# S(G(n)); Z_p) = 0$. When $m+n = 2 + 3(1 + p + \cdots + p^r)(\frac{p-1}{2})$, we have $\bar{c}_{2m+2n} = P^{3p^r} P^{3p^{r-1}} \cdots P^{3p} P^3(c_4)$ + a polynomial if $p > 3$. $f^*(c_2)$ is a multiple of $\sigma^*({}^1P_1) \otimes \sigma^*({}^1P_1)$ and $P^3(\sigma^*({}^1P_1) \otimes \sigma^*({}^1P_1)) =$
$P^2(\sigma^*({}^1P_1)) \otimes P^1(\sigma^*({}^1P_1)) + P^1(\sigma^*({}^1P_1)) \otimes P^2(\sigma^*({}^1P_1)) =$
$\sigma^*(P^2({}^1P_1)) \otimes P^1(\sigma^*({}^1P_1)) + P^1(\sigma^*({}^1P_1)) \otimes \sigma^*(P^2({}^1P_1)) = 0$. The other three cases are similar and the proofs are omitted.

We could give more values of $m + n$ such that $G(m)$ and $G(n)$ do not homotopy commute in $U(2m + 2n - 1)$ by similar techniques, however, this does not seem to lead to a proof for all $m + n$ so we forego this. The lowest value of $m + n$ not covered by this corollary is $m + n = 18$.

We now turn to the proof of corollary 3.4. By lemma 5.1, $P^1(c_{2m+2n-p+1}) = (2m + 2n)\, c_{2m+2n}$ + a polynomial.

If $m + n \not\equiv 0 \pmod p$ and $f^*(\bar{c}_{2m+2n}) = \sigma^*({}^1P_m) \otimes \sigma^*({}^1P_n)$ for some f, we must have either $4m - 2(p-1) \geq 4$ or $4n - 2(p-1) \geq 4$. That is, either $2m \geq p + 1$ or $2n \geq p + 1$ where $p + 1 \leq 2m + 2n$. Using the prime number theorem we can prove that for every $\epsilon > 0$, we can find an N such that there is a prime with $2(1 - \epsilon)(m + n) \leq p + 1 \leq 2m + 2n$ if $m + n > N$. We can take

$\varepsilon < \frac{1}{2}$, so that $m + n \not\equiv 0 \pmod p$. Then if $2m \geq p + 1$, we have $2m + 2n - \varepsilon(2m + 2n) \leq 2m$. Hence $2n \leq 2\varepsilon(m + n)$ or $n \leq \varepsilon(m + n)$. To prove corollary 3.5, we note that for $\varepsilon = \frac{1}{3}$, we may take $N = 10$.

6. Concluding Remarks.

Since Sp(m) and Sp(n) homotopy commute in Sp(m + n), they also do in U(2m + 2n). Hence our result is best possible. Similarly, SO(2m + 1) and SO(2n + 1) homotopy commute in U(2m + 2n + 2), thus leaving two cases undecided. For SO(2m + 2) and SO(2n + 2), our technique only gives the same results as for SO(2m + 1) and SO(2n + 1). Since SO(2m + 2) and SO(2n + 2) homotopy commute in U(2m + 2n + 4), this leaves four cases undecided. For very low values of m and n, one can obtain further ad hoc results using dimensional arguments. For example, our theorem shows SO(3) and SO(4) do not homotopy commute in U(3). One can show that they do homotopy commute in U(5), leaving only one case undecided.

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