I. Homotopy

Definition 1.1. Two maps \( f_i : X \to Y, i = 0,1 \), are called homotopic if there exists a map \( F : X \times I \to Y \) such that \( F(x,i) = f_i(x) \) for \( i = 0,1 \) and all \( x \in X \). Here \( I = [0,1] \).

Denote this relation by \( f_0 \sim f_1 \).

Proposition 1.2. \( \sim \) is an equivalence relation between maps \( X \to Y \).

Proposition 1.3. \( \sim \) is preserved under composition.

Definition 1.4. \( b : I \to X \) is a path from \( b(0) \) to \( b(1) \). \( X \) is path-connected if ....

Proposition 1.5. Every path-connected space is connected. There is a compact, connected subset of \( \mathbb{R}^2 \) which is not path-connected.

Proposition 1.6. \( \mathbb{R}^n \), \( S^n \), and \( I^n \) are path-connected if \( n \geq 0 \) except for \( S^0 \).

Definition 1.7. \( f : X \to Y \) is a homotopy equivalence if \( \exists \) \( g : Y \to X \) such that \( gf \sim id_X \) and \( fg \sim id_Y \). \( g \) is called a homotopy inverse to \( f \).

Proposition 1.8. A homotopy inverse to a homotopy equivalence is a homotopy equivalence. The composition of two homotopy equivalences is a homotopy equivalence.

Definition 1.9. \( X \) and \( Y \) are of the same homotopy type if \( \exists \) a homotopy equivalence \( f : X \to Y \).

Proposition 1.10. Being of the same homotopy type is an equivalence relation.

Proposition 1.11. \( \exists \) \( X \) and \( Y \) of the same homotopy type but \( X \) and \( Y \) are not homeomorphic.
Definition 1.12. X is **contractible** if X is of the same homotopy type as a one point space P.

Proposition 1.13. $\mathbb{R}^n$ and $I^n$ are contractible if $n \geq 0$.

Proposition 1.14. X is contractible if and only if all maps $X \to X$ are homotopic.

Definition 1.15. $f: (X,A) \to (Y,B)$ if $f: X \to Y$ and $f(A) \subseteq B$. Define the notion of homotopy between maps of pairs.

Proposition 1.16. \( \simeq \) is an equivalence relation.

II. The Fundamental Group

Construction 2.1. If $x \in X$, let $c_x: I \to X$ be the path defined by $c_x(t) = x$ for all $t \in I$. If $b$ is a path, let $b^{-1}(t) = b(1-t)$ for all $t \in I$. If $b$ and $d$ are paths in $X$ such that $b(1) = d(0)$, define $b \ast d$ by $b \ast d(t) = b(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $d(2t-1)$ for $\frac{1}{2} \leq t \leq 1$. Let $[b]$ denote the homotopy class of all paths in $X$ which are homotopic to $b$, considered as maps $b: (I,\{0\},\{1\}) \to (X,\{b(0)\},\{b(1)\})$.

Proposition 2.2. If $b$ is a path in $X$, then $[b \ast b^{-1}] = [c_b(0)]$ and $[b^{-1} \ast b] = [c_b(1)]$.

Proposition 2.3. If $b$, $d$, and $e$ are paths in $X$ such that $b(1) = d(0)$ and $d(1) = e(0)$, then $[(b \ast d) \ast e] = [b \ast (d \ast e)]$.

Proposition 2.4. If $[b] = [b']$ and $[d] = [d']$ and $b(1) = d(0)$, then $[b \ast d] = [b' \ast d']$.

Definition 2.5. A path $b$ in $X$ is a **loop** at $x_0 \in X$ if $b(0) = b(1) = x_0$. $[b]$ denotes the homotopy class of loops homotopic to $b$.

Theorem 2.6. The homotopy classes of loops at $x_0 \in X$
form a group with the product \([b_1] \ast [b_2] = [b_1 \ast b_2]\). This group is called the fundamental group or first homotopy group of \(X\) with \(x_0\) as base point and is denoted by \(\pi_1(X,x_0)\).

Theorem 2.7. Let \(f: \langle X,x_0 \rangle \to \langle Y,y_0 \rangle\). Define \(f_\#: \pi_1(X,x_0) \to \pi_1(Y,y_0)\) by \(f_\#([b]) = [fb]\). Then \(f_\#\) is a homomorphism.

Theorem 2.8. \((id_X)_\# = id\). If \(f: \langle X,x_0 \rangle \to \langle Y,y_0 \rangle\) and \(g: \langle Y,y_0 \rangle \to \langle Z,z_0 \rangle\), then \(g_\# f_\# = (gf)_\#\).

Theorem 2.9. If \(f_0 \simeq f_1: \langle X,x_0 \rangle \to \langle Y,y_0 \rangle\), then \((f_0)_\# = (f_1)_\#\).

Definition 2.10. \(f: \langle X,x_0 \rangle \to \langle Y,y_0 \rangle\) is a homotopy equivalence relative to the base point if \(\exists g: \langle Y,y_0 \rangle \to \langle X,x_0 \rangle\) such that \(gf \simeq id_{\langle X,x_0 \rangle}\) and \(fg \simeq id_{\langle Y,y_0 \rangle}\).

Proposition 2.11. If \(f\) is a homotopy equivalence relative to the base point, then \(f_\#\) is an isomorphism.

Corollary 2.12. \(\pi_1(\mathbb{R}^n,0) = \{1\}\) and \(\pi_1(I^n,0) = \{1\}\) if \(n \geq 2\).

Theorem 2.13. \(\pi_1(X \times Y, (x_0,y_0)) \cong \pi_1(X,x_0) \times \pi_1(Y,y_0)\).

Proposition 2.14. Let \(U \subset X\) be the path component of \(x_0\) in \(X\) and let \(j: \langle U,x_0 \rangle \to \langle X,x_0 \rangle\) be the inclusion map. Then \(j_\#: \pi_1(U,x_0) \to \pi_1(X,x_0)\) is an isomorphism.

Theorem 2.15. Let \(j: I \to S^1\) be defined by \(j(t) = e^{2\pi it}\), and let \(s_0 = j(0) = 1\). Define \(j_\#: \pi_1(S^1,s_0) \to \pi_1(X,x_0)\) by \(j_\#([b]) = [bj]\). Then \(j_\#\) is a bijection.

Construction 2.16. Let \(b\) be a path in \(X\). Define \(b_\#: \pi_1(X,b(0)) \to \pi_1(X,b(1))\) by \(b_\#(q) = [b^{-1}] \ast q \ast [b]\).

Proposition 2.17. \(b_\#\) is a homomorphism. If \(d\) is a path
such that \( d(0) = b(1) \), then \( d \ast b = (b \ast d) \ast. \) If \([b] = [b']\),
then \( b \ast = b' \ast. \) \( (c_x)\ast \) is the identity.

**Theorem 2.18.** \( b \ast \) is an isomorphism.

**Lemma 2.19.** Let \( h_0 \sim h_1 : X \to Y \). Then \( \exists b \), a path in \( Y \),
such that \( b \ast h_0 = h_1 \ast. \)

**Proposition 2.20.** Let \( f : X \to Y \) be a homotopy equivalence. Then \( f \ast : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \) is an
isomorphism. (Compare 2.11.)

**Definition 2.21.** \( X \) is simply connected (or 1-connected)
if \( X \) is path connected and \( \pi_1(X, x_0) = \{1\} \) for some \( x_0 \in X \).

**Proposition 2.22.** If \( X \) is simply connected, then
\( \pi_1(X, x_0) = \{1\} \) for all \( x_0 \in X \).

**Proposition 2.23.** Every contractible space is simply
connected.

**Theorem 2.24.** Let \( b \) be a path from \( x_0 \) to \( x_1 \). Then \( b \ast : \pi_1(X, x_0) \to \pi_1(X, x_1) \) is independent of the choice of \( b \)
if and only if \( \pi_1(X, x_0) \) is abelian.

**Definition 2.25.** An \( H\)-space is a pair \( (H, e) \), where \( H \) is
a space, \( e \in H \), and we are given a map \( m : H \times H \to H \) such
that \( m(\cdot, e) \sim m(e, \cdot) \sim \text{id}(H, e) \).

**Example 2.26.** A topological group is an \( H\)-space.

**Theorem 2.27.** Let \( (H, e) \) be an \( H\)-space. Then \( \pi_1(H, e) \) is
abelian.

III. **Covering Spaces**

**Definition 3.1.** A space \( X \) is called **locally pathwise
connected** if for every \( x \in X \) and every open set \( U \) with \( x \in U \),
there exists a pathwise connected open set \( V \) such that \( x \in V \)
C U.

Definition 3.2. Let \( f: Y \to X \) and let \( U \) be an open set of \( X \). Then \( U \) is evenly covered by \( f \) if there is an indexing set \( J \) and for every \( j \in J \), an open set \( U_j \subseteq Y \) such that

(i) \( \bigcup_{j \in J} U_j = f^{-1}(U) \),
(ii) if \( j \neq k \), \( U_j \cap U_k = \emptyset \),
(iii) \( f|_{U_j}: U_j \to U \) is a homeomorphism for \( j \in J \).

Definition 3.3. A map \( \pi: \tilde{X} \to X \) is a covering of \( X \) if

(i) \( \pi \) is onto,
(ii) \( \tilde{X} \) is pathwise connected and locally pathwise connected,
(iii) for every \( x \in X \), \( \exists \) an open \( U \) which is evenly covered by \( \pi \) and \( x \in U \).

Proposition 3.4. The following are covering spaces.

(i) \( \text{id}_X: X \to X \), if \( X \) is pathwise connected and locally pathwise connected,
(ii) \( \pi: \mathbb{R}^1 \to S^1 \) given by \( \pi(r) = e^{2\pi ir} \),
(iii) \( \pi_n: S^1 \to S^1 \) given by \( \pi_n(e^{ir}) = e^{inr} \), for \( n \neq 0 \) an integer,
(iv) \( \pi: S^n \to \mathbb{R}P^n = \{ \text{pairs of antipodal points of } S^n \} \) with the quotient topology (\( \mathbb{R}P^n \) is called real \( n \)-dimensional projective space).

Notation. l.c. stands for pathwise connected and locally pathwise connected.

Proposition 3.5. Let \( U \subseteq X \) be evenly covered by \( f: Y \to X \) and let \( Y \) be l.c. Then \( U \) is locally pathwise connected.

Proposition 3.6. Let \( \pi: \tilde{X} \to X \) be a covering. Then \( \tilde{X} \) is l.c.
Proposition 3.7. Let \( U \subset X \) be evenly covered by \( f: Y \to X \). Let \( Z \) be connected and let \( g: Z \to U \). Let \( y \in Y \), \( z \in Z \) be such that \( f(y) = g(z) \). Then \( \exists g': Z \to Y \) such that \( g'(z) = y \) and \( fg' = g \).

Proposition 3.8. Let \( \pi: X^\wedge \to X \) be a covering. Let \( b \) be a path in \( X \) and let \( x^\wedge \in X^\wedge \) be such that \( \pi(x^\wedge) = b(0) \). Then \( \exists \) path \( b^\wedge \) in \( X^\wedge \) such that \( x^\wedge = b^\wedge(0) \) and \( \pi b^\wedge = b \).

Proposition 3.9. Let \( \pi: X^\wedge \to X \) be a covering. Let \( Z \) be pathwise connected and let \( g_i: Z \to X^\wedge \) be maps such that \( \pi g_0 = \pi g_1 \) and \( g_0(z) = g_1(z) \) for some \( z \in Z \). Then \( g_0 = g_1 \).

Proposition 3.10. Let \( \pi: X^\wedge \to X \) be a covering, let \( g: I^2 \to X \) be a map, and let \( x^\wedge \in X^\wedge \) be such that \( \pi(x^\wedge) = g(0,0) \). Then \( \exists \) \( g^\wedge: I^2 \to X^\wedge \) such that \( g^\wedge(0,0) = x^\wedge \) and \( \pi g^\wedge = g \).

Proposition 3.11. Let \( \pi: X^\wedge \to X \) be a covering. Then \( \pi_\#: \pi_1(X^\wedge, x^\wedge) \to \pi_1(X, \pi(x^\wedge)) \) is one to one.

IV. Classification of Coverings with Base Point

Definition 4.1. \( \pi: (X^\wedge, x^\wedge) \to (X, x) \) is a covering with base point if \( \pi: X^\wedge \to X \) is a covering.

Definition 4.2. Two coverings \( \pi_1: (X^\wedge_1, x^\wedge_1) \to (X, x) \) are homeomorphic if \( \exists \) a homeomorphism \( h: (X^\wedge_1, x^\wedge_1) \to (X^\wedge_2, x^\wedge_2) \) such that \( \pi_2 h = \pi_1 \).

Proposition 4.3. Being homeomorphic is an equivalence relation on the set of coverings with base point.

Proposition 4.4. Let \( \pi_1 \) and \( \pi_2 \) be homeomorphic coverings. Then \( (\pi_1)_\#(\pi_1(X^\wedge_1, x^\wedge_1)) = (\pi_2)_\#(\pi_1(X^\wedge_2, x^\wedge_2)) \subseteq \pi_1(X, x) \).
Proposition 4.5. Let $\pi$ be a covering and let $g: (Z, z) \to (X, x)$ be a map such that $\text{Im } g_# \subseteq \text{Im } \pi_#$. Let $b$ and $d$ be paths in $Z$ such that $b(0) = d(0) = z$ and $b(1) = d(1) = z'$ . Let $b^*$ and $d^*$ be the unique paths in $X^*$ such that $\pi b^* = gb$, $\pi d^* = gd$, and $b^*(0) = d^*(0) = x^*$. Then $b^*(1) = d^*(1)$.

Proposition 4.6. Let $\pi$ be a covering, let $Z$ be 1.c., and let $g: (Z, z) \to (X, x)$ be a map such that $\text{Im } g_# \subseteq \text{Im } \pi_#$. Then $\exists! g^*: (Z, z) \to (X^*, x^*)$ such that $\pi g^* = g$.

Proposition 4.7. Two coverings $\pi_1$ and $\pi_2$ are homoeomorphic if and only if $\text{Im } (\pi_1)_# = \text{Im } (\pi_2)_#$.

Definition 4.8. $X$ is semi-locally 1-connected if for every $x \in X$ there is an open set $U$, $x \in U$, such that $\pi_1(U, x) \to \pi_1(X, x)$ is the trivial homomorphism.

Definition 4.9. Let $G \subseteq \pi_1(X, x)$ be a subgroup. Two paths $b$ and $d$ are $G$-homotopic if $b(0) = d(0) = x$, $b(1) = d(1)$, and $[b * d^{-1}] \in G$.

Proposition 4.10. $G$-homotopy is an equivalence relation on paths in $X$ which start at $x$.

Construction 4.11. Let $X_G$ denote the set of $G$-homotopy classes of paths in $X$ which start at $x$. Let $\mathfrak{K}: X_G \to X$ be defined by $\mathfrak{K}(\langle b \rangle) = b(1)$. Let $V \subseteq X$ be open and path connected. Let $b$ be a path in $X$ such that $b(0) = x$ and $b(1) \in V$. Define $U(b, V) \subseteq X_G$ to consist of the $G$-homotopy classes of paths of the form $b * d$ where $d(0) = b(1)$ and $d(t) \in V$. Take the $U(b, V)$ as a subbase for the open sets of $X_G$.

Theorem 4.12. Let $X$ be 1.c. and semi-locally 1-connected. Then the homeomorphism classes of coverings of $(X, x)$ are in one to one correspondence (i.e. $\exists$ a bijection)
with the subgroups of $\pi_1(X,x)$.

V. Classification of Coverings and Their Translations.

Definition 5.1. Two coverings $\pi_i : X^i \to X$, $i = 1, 2$, are homeomorphic if there is a homeomorphism $h : X^1 \to X^2$ such that $\pi_2 h = \pi_1$.

Proposition 5.2. Being homeomorphic is an equivalence relation on the set of coverings of $X$.

Definition 5.3. Two subgroups $H, K$ of a group $G$ are conjugate if there is a $g \in G$ such that $K = g^{-1}Hg$.

Proposition 5.4. Being conjugate is an equivalence relation on the set of subgroups of $G$.

Proposition 5.5. Let $\pi$ be a covering and let $x^i \in X^i$ be such that $\pi(x^i) = x$, $i = 1, 2$. Then $\pi_#(\pi_1(x^1, x^1))$ and $\pi_#(\pi_1(x^2, x^2))$ are conjugate in $\pi_1(X,x)$.

Proposition 5.6. Let $\pi$ be a covering, let $x^1 \in X^1$, and let $G \subset \pi_1(X, \pi(x^1))$ be conjugate with $\text{Im } \pi_#$. Then $\exists x^1 \in X^1$ such that $\pi_#(\pi_1(x^1, x^1)) = G$.

Theorem 5.7. Let $\pi_1$ be coverings and let $x^1 \in X^i, i$ be such that $\pi_1(x^1) = \pi_2(x^2)$. Then $\pi_1$ and $\pi_2$ are homeomorphic if and only if $\text{Im } (\pi_1)_#$ and $\text{Im } (\pi_2)_#$ are conjugate in $\pi_1(X, \pi_1(x^1))$.

Theorem 5.8. The homeomorphism classes of coverings of a 1.c. and semi-locally 1-connected space $X$ are in one to one correspondence with the conjugacy classes of subgroups of $\pi_1(X,x)$.

Definition 5.9. A translation of a covering $\pi$ is a homeomorphism $h : X^1 \to X^1$ such that $\pi h = \pi$. Let $T(\pi)$ denote
the set of translations of $\pi$.

Proposition 5.10. $T(\pi)$ is a group under composition of translations.

Definition 5.11. Let $H \leq G$ be a subgroup of $G$. The elements $g \in G$ such that $g^{-1}Hg = H$ form a subgroup $N(H)$, the normalizer of $H$ in $G$.

Proposition 5.12. $H$ is normal in $N(H)$.

Construction 5.13. Let $\pi$ be a covering and let $H = \pi_H(\pi_1(X^\infty, x^\infty))$. Define a function $t: N(H) \to T(\pi)$ as follows. Let $q \in N(H)$. Choose a path $b^\infty$ in $X^\infty$ such that $b^\infty(0) = x^\infty$ and $[\pi b^\infty] = q$. $t(q)$ is the unique translation such that $t(q)(x^\infty) = b^\infty(1)$.

Theorem 5.14. $t$ is well defined, $t$ is a homomorphism onto $T(\pi)$, and $\ker t = H$.

Corollary 5.15. $N(H)/H$ is isomorphic to $T(\pi)$.

VI. Some Fundamental Groups.

Definition 6.1. A covering $\pi$ is called universal if $X^\infty$ is simply connected.

Corollary 6.2. Let $\pi$ be a universal covering. Then $\pi_1(X, x)$ is isomorphic to $T(\pi)$.

Proposition 6.3. Example 3.4 (ii) is a universal covering.

Theorem 6.4. $\pi_1(S^1, s)$ is isomorphic to the additive group of integers, $\mathbb{Z}$, for any $s \in S^1$.

Example 6.5. Describe the covering space corresponding to each subgroup of $\pi_1(S^1, s)$.

Definition 6.6. $T^n = S^1 \times \ldots \times S^1, n$ times, is the
n-dimensional torus.

Theorem 6.7. $\pi_1(T^n,t) \cong Z \oplus \ldots \oplus Z$, n times.

Theorem 6.8. Let $X$ be 1.c. and semi-locally 1-connected, let $A$, $B \subset X$ be open subsets which are 1.c. and simply-connected, and let $A \cap B$ be pathwise connected and $A \cup B = X$. Then $X$ is simply-connected.

Theorem 6.9. $S^n$ is simply-connected for all $n > 1$.

Theorem 6.10. Example 3.4 (iv) is a universal covering and hence $\pi_1(\mathbb{R}P^n,p)$ is isomorphic to $2/22$ if $n > 1$.

VII. Higher Homotopy Groups.

Definition 7.1. $Y^X$ denotes the function space of continuous maps $f: X \rightarrow Y$ with the compact-open topology. If $B \subset Y$ and $A \subset X$, then $(Y, B, X, A)$ is $Y^X$ is the subspace of functions such that $f(A) \subset B$.

Theorem 7.2. Define $\alpha: Z^X \times Y \rightarrow (Z^Y)^X$ by $[[\alpha(f)](x)](y) = f(x, y)$. If $Y$ is locally compact and Hausdorff, then $\alpha$ is a bijection.

Definition 7.3. $\langle X, x_0 \rangle^{(1, 0)}$ is the space of paths on $X$ starting at $x_0$. $\langle X, x_0 \rangle^{(1, 0)}$ is the space of loops on $X$ at $x_0$ and denoted by $\Omega(X, x_0)$. Define $\Omega^n(X, x_0) = \Omega(\Omega^{n-1}(X, x_0), x_{n-1})$, where $x_{n-1}$ is the constant loop at $x_{n-2}$.

Definition 7.4. $\pi_n(X, x_0) = \pi_1(\Omega^{n-1}(X, x_0), x_{n-1})$.

Theorem 7.5. $\Omega(X, x_0)$ is an H-space.

Corollary 7.6. $\pi_n(X, x_0)$ is abelian if $n > 1$.

Construction 7.7. Let $f: \langle X, x_0 \rangle \rightarrow \langle Y, y_0 \rangle$. Define $f_1: \Omega(X, x_0) \rightarrow \Omega(Y, y_0)$ by $f_1(b) = fb$. Define $f_n: \langle \Omega^n(X, x_0), x_n \rangle \rightarrow \langle \Omega^n(Y, y_0), y_n \rangle$ by induction. Define $f_\# : \pi_n(X, x_0) \rightarrow$
\[ \pi_n(\langle Y, y_0 \rangle) \text{ by } (f_{n-1})# : \pi_1(\Omega^{n-1}(\langle X, x_0 \rangle, x_{n-1})) \rightarrow \pi_1(\Omega^{n-1}(\langle Y, y_0 \rangle, y_{n-1})). \]

Theorem 7.8. \( \text{id}_# \) is the identity. If \( f : \langle X, x_0 \rangle \rightarrow \langle Y, y_0 \rangle \) and \( g : \langle Y, y_0 \rangle \rightarrow \langle Z, z_0 \rangle \), then \( g#f# = (gf)#. \)

Proposition 7.9. If \( f \sim g : \langle X, x_0 \rangle \rightarrow \langle Y, y_0 \rangle \), then \( f# = g# \).

Corollary 7.10. If \( \langle X, x_0 \rangle \) is contractible, then \( \pi_n(\langle X, x_0 \rangle) = 0 \) for \( n > 1 \).

Theorem 7.11. \( \pi_n(\langle X, x_0 \rangle) \cong \pi_{n-1}(\Omega(\langle X, x_0 \rangle, x_1)). \)

Definition 7.12. Let \( I^* \subset I^n \) denote the subspace of all \( n \)-tuples for which at least one coordinate is 0 or 1. \( \pi(I^n, I^*; X, x_0) \) denotes the set of homotopy classes of maps from \( \langle I^n, I^* \rangle \) to \( \langle X, x_0 \rangle \).

Theorem 7.13. There is a natural 1-1 correspondence between \( \pi_n(\langle X, x_0 \rangle) \) and \( \pi(I^n, I^*; X, x_0) \).

VIII. Fibre Spaces.

Definition 8.1. Let \( p : E \rightarrow B. \) \( p \) is a fibre map if for every l.c. space \( X \) the outer triangle in the diagram below can be filled in by the dotted arrow so that the triangles commute:

\[
\begin{array}{ccc}
E & \xrightarrow{g} & X \\
\downarrow & & \downarrow \\
B & & B \\
\end{array}
\]

where \( i_0(x) = (x, 0). \) \( (E, p, B) \) is called a fibre space, \( E \) is the total space, \( B \) the base space, and \( p \) the projection. If \( B \) has a base point \( b_0 \), then \( F = p^{-1}(b_0) \) is the fibre.
Proposition 8.2. If $B$ is pathwise connected and $E \neq \emptyset$, then $p$ is onto.

Proposition 8.3. Let $E = F \times B$ and let $p$ be the projection onto the second factor. Then $(E, p, B)$ is a fibre space.

Proposition 8.4. A covering space $(X^\pi, \pi, X)$ is a fibre space.

Theorem 8.5. Let $A, B \subseteq Y$. Let $E(Y; A, B) \subseteq Y^I$ be the subspace of those maps such that $f(0) \in A$ and $f(1) \in B$. Define $p: E(Y; A, B) \to A \times B$ by $p(f) = (f(0), f(1))$. Then $p$ is a fibre map.

Corollary 8.6. $p: (X, x_0)^{(I, 0)} \to X$ by $p(f) = f(1)$ is a fibre space with fibre $\Omega(X, x_0)$.

IX. Exact Sequences.

Definition 9.1. A sequence $\ldots \to A \to B \to C \to \ldots$ of abelian groups and homomorphisms is exact if $\text{Im } f = \text{Ker } g$.

Proposition 9.2. $0 \to A \to 0$ is exact if and only if $A = 0$.

Proposition 9.3. $0 \to A \to B \to 0$ is exact if and only if $f$ is an isomorphism.

Proposition 9.4. $0 \to A \to B \to C \to 0$ is exact if and only if $g^\pi: B/f(A) \to C$ is an isomorphism and $f$ is injective.

"Five Lemma" 9.5. Let

$$
\begin{array}{cccc}
A & B & C & D & E \\
\quad & f & g & h & i \\
\quad & | & | & | & | \\
lp_1 & lp_2 & lp_3 & lp_4 & lp_5
\end{array}
$$
be a commutative diagram with exact rows. If \( p_1, p_2, p_4, \) and \( p_5 \) are isomorphisms, so is \( p_3 \).

X. Relative Homotopy Groups and Exact Sequences.

Definition 10.1. Let \( I^{n-1} \subset I^n \) be the subspace where \( t_n = 0 \). Let \( J^{n-1} \subset I^n \) be the closure of \( I^n - I^{n-1} \). Let \( x_0 \in A \subset X \). For \( n > 2 \), define \( \pi_n(X,A) \) to be the set of homotopy classes of maps from \( (I^n, I^{n-1}, J^{n-1}) \to (X,A,x_0) \).

Proposition 10.2. Let \( Z \subset X \) be defined by \( Z = \{ f \mid f(0) = x_0, f(1) \in A \} \). Then \( \pi_n(X,A) \simeq \pi_{n-1}(Z,c) \).

Corollary 10.3. \( \pi_n(X,A) \) is abelian for \( n > 2 \).

Definition 10.4. Let \( f : (X,A,x_0) \to (Y,B,y_0) \). Define \( f^\# : \pi_n(X,A,x_0) \to \pi_n(Y,B,y_0) \) in the usual way.

Theorem 10.5. \( f^\# \) is a homomorphism, \( (id)^\# = id \), and \( g^\# f^\# = (gf)^\# \).

Definition 10.6. \( \partial : \pi_n(X,A) \to \pi_{n-1}(A,x_0) \) is defined by \( \partial([f]) = [f[I^{n-1}]] \).

Theorem 10.7. Let \( i : (A,x_0) \to (X,x_0), j : (X,x_0) \to (X,A) \).

Then

... \( \to \pi_n(A,x_0) \to \pi_n(X,x_0) \to \pi_n(X,A) \to \pi_{n-1}(A,x_0) \to ... \)

\( \partial \quad i^\# \quad j^\# \quad \partial \quad i^\# \)

is exact. (This is called the exact sequence of a pair.)

Theorem 10.8. Let \( (E,p,B) \) be a fibre space, \( b_0 \in B \), \( p^\# : (E,F) \to (B,b_0) \). Then \( p^\# : \pi_n(E,F) \to \pi_n(B,b_0) \) is an isomorphism for \( n \geq 2 \).

Theorem 10.9. Let \( (E,p,B) \) be a fibre space, \( b_0 \in B \), \( e_0 \in
Let \( (X^\land, \pi, X) \) be a covering space. Then \( \pi_{\#} : \pi_n(X^\land, X^\land) \to \pi_n(X, \pi(X^\land)) \) is an isomorphism for \( n \leq 2 \).

Corollary 10.11. \( \pi_n(S^1, x_0) = 0 \) if \( n > 1 \).

Corollary 10.12. \( \pi_n(S^n, x_0) \cong \pi_n(\mathbb{R}P^n, (x_0, -x_0)) \), if \( n > 1 \).

11. Polyhedra

Definition 11.1. A simplicial complex \( K \) (of dimension at most 2) is a collection of vertices \( \{V_i\} \), a collection of edges \( \{V_i, V_j\} \), and a collection of triangles \( \{V_i, V_j, V_k\} \) satisfying the following condition: given a set in the collection, then a non-empty subset of it is in the collection.

(Intuitive) Definition 11.2. A topological space \( X \) is a polyhedron if it can be broken up into vertices, edges, and triangles as in definition 11.1. More precisely, if \( X \) is homeomorphic to \( |K| \), the geometric realization of \( K \).

Proposition 11.3. \( I^n \), \( S^n \), and \( R^n \) are polyhedra (if \( n \leq 2 \)).

Definition 11.4. Given a simplicial complex \( K \). An edge path in \( K \) is a sequence of vertices of \( K \), \( w = V_0 \ldots V_k \), such that each successive pair of vertices is an edge or repeats itself. If \( w' = V_k \ldots V_1 \), then \( w \ast w' = V_0 \ldots V_k \), the product of edge-paths. \( w^{-1} = V_k \ldots V_0 \) is the inverse edge-path to \( w \).

Definition 11.5. We define an equivalence relation on
edge-paths with the same beginning and end points as follows. If \( w = v_0 \ldots v_i v_j \ldots v_k \), then \( w \sim w' = v_0 \ldots v_i v_j \ldots v_k \). If \( v_i, v_j, v_k \) is a triangle in \( K \), then \( w = v_0 \ldots v_i v_j v_k \sim w' = v_0 \ldots v_i v_j v_k \). Let \([w]\) denote the equivalence class of \( w \).

**Proposition 11.6.** Let \( \pi(K,V) \) denote the set of equivalence classes of edge-paths in \( K \) which start and end at \( V \). Then \( \pi(K,V) \) is a group under the operation \([w] * [w'] = [w * w']\). This is called the edge-path group of \( K \).

**Theorem 11.7.** Let \( X \) be a polyhedron with corresponding simplicial complex \( K \). Then \( \pi(K,V) \cong \pi_1(X,V) \).

**Definition 11.8.** \( K \) is **connected** if \( |K| \) is path-connected. \( K \) is **simply connected** if \( |K| \) is simply connected.

**Proposition 11.9.** Given \( K \). Then there exists a simply connected subcomplex \( L \) (having only vertices and edges) of \( K \) which contains all the vertices of \( K \). Such a subcomplex is a maximal tree.

**Definition 11.10.** Given \( K \) and a subcomplex \( L \). Let \( G = G(K,L) \) be the group generated by \( g_{ij} \), one generator for each edge \( v_i, v_j \) in \( K \) subject to the relations \( g_{ij} = 1 \) if \( v_i, v_j \) is in \( L \) and \( g_{ij}g_{jk} = g_{ik} \) if \( v_i, v_j, v_k \) is in \( K \).

**Theorem 11.11.** If \( L \) is a maximal tree, then \( \pi(K,V) \cong G(K,L) \).

**Corollary 11.12.** If \( K \) is a finite simplicial complex, then \( \pi_1(|K|,V) \) is finitely generated and finitely presented.

**Corollary 11.13.** If \( K \) has no triangles, then \( \pi_1(|K|,V) \) is free.

**Proposition 11.14.** \( \pi_1 \) of a figure 8 is free on two
generators.

Theorem 11.15. A free group on two generators contains a free group on $n$ generators.