SOME REMARKS ON $\text{Tor}_2 \mathbb{R}^\mathbb{P}$

R.O. Nadiradze

Tbilisi Mathematical Institute
Academy of Sciences of the Georgian SSR
Z.Rukhadze Str. 1, Tbilisi 380093, USSR

The $\mathbb{R}^\mathbb{P}$-theory of cobordisms appeared in [4] and was developed in [11], [8], [3] and other works.

A series of free generators was constructed by Stong [14]. Great difficulty was encountered in treating the question on the 2-primary torsion.


By analogy with [10] here we construct a series of finite order elements and relations in the $\mathbb{R}^\mathbb{P}$-theory.

1. Let

$$e(\Theta \bar{\gamma}) = x + y + \sum \phi_k y^k,$$

where $\Theta$ is a canonical complex linear fibering over $\mathbb{R}P(\infty)$, $\bar{\gamma}$ a canonical symplectic fibering over $\mathbb{H}P(\infty)$, $e$ - Euler's class in the $\mathbb{R}^\mathbb{P}$-theory, $x = e(\Theta)$, $y = e(\bar{\gamma})$ and $\phi_k \in \mathbb{R}^\mathbb{P}_* \mathbb{R}P(\infty)$.

**Remark 1.** Let $\varphi: \mathbb{R}P(1) \to \mathbb{R}P(\infty)$ be a canonical imbedding.

Then

$$\nu^k \phi_k = \Theta_k \cdot S,$$

where $\Theta_k \in \mathbb{R}^\mathbb{P}_* \mathbb{R}P(1)$ are Ray's elements [12] and $S$ is the generator $\mathbb{R}^\mathbb{P}_* \mathbb{R}P(1)$.

**Remark 2.** It follows from [13], [2], [10] that for $k \geq 0$, classes $\phi_{2k+1}$ depend functionally on classes $\phi_1, \phi_2, \phi_3, \ldots, \phi_k$.
while classes $\psi_1, \psi_2, \psi_3, \ldots, \psi_k$ are functionally independent.

By virtue of the property of two-valued formal groups and Conner-Floyd classes in the $\Omega^\infty_{\mathcal{Z}_p}$-theory [1] we have

**Lemma 1.** \[ 4 e(\Theta \otimes 5) = 2 \Theta_1(x, y), \]
\[ 6 e^2(\Theta \otimes 5) = \Theta_2(x, y) + 2 \Theta_2(x, y) = \sum x_i x_j \]

where $x_i, x_j \in \Omega^\infty_{\mathcal{Z}_p}$,
\[ 4 e^3(\Theta \otimes 5) = 2 \Theta_1(x, y) \cdot \Theta_2(x, y), \]
\[ e^4(\Theta \otimes 5) = \Theta_2(x, y), \]

where $\Theta_1(x, y)$ and $\Theta_2(x, y)$ are coefficients of the two-valued formal group.

**Remark 3.** In the $\Omega^\infty_{\mathcal{Z}_c}$-theory of cobordisms
\[ 2 e(\Theta \otimes 5) = \Theta_1(x, y), \]
\[ e^2(\Theta \otimes 5) = \Theta_2(x, y). \]

**Corollary 1.**
\[ 6 \psi_{2k}^2 = -12 \psi_4 \psi_k x - 12 \sum_{i=1}^{\infty} \psi_i \psi_j + \sum_{i=1}^{\infty} x_{i, 4k} x^i, \]
\[ 6 \psi_1^2 = -6 - 12 \psi_1 - 12 \psi_2 + \sum_{i=0}^{\infty} x_{i, 2} x^i. \]
Definition. \[ R(\omega) = \langle \Phi_1^{k_0} \Phi_2^{k_1} \cdots \Phi_n^{k_1}, \mathcal{R}(4m+3) \rangle = \mathcal{D}_{\mathcal{S}p} \Phi_1^{k_0} \Phi_2^{k_1} \cdots \Phi_n^{k_1} | \mathcal{R}(4m+3) \rangle, \]

where \( k_0 > 0, k_i > 0 \) are integers, \( \mathcal{D}_{\mathcal{S}p} \) the duality operator of Atiyah-Poincaré [7], \( \mathcal{E} \) the argumentation,

\[ 1 < i_1 < i_2 < \cdots < i_n; \]

\[ \omega = (k_0, k_1, \cdots, k_n, 4m+3). \]

Remark 4. By virtue of Remark 3 we consider in the \( \mathcal{O}_{\mathcal{S}c}^* \) -theory the following elements \( R(\omega) [10] \):

\[ \omega = (k, 1, 1, \cdots, 1, 4m+1) \quad \text{where} \quad k = 0, 1. \]

Remark 5. From the formula \( e^4(\Theta \Theta^5) = \Theta_2^2(x, y) \) it follows that \( \Phi_{2i} \) are decomposable elements in the ring \( \mathcal{O}_{\mathcal{S}p}^* \mathcal{R}(\infty) \). Therefore in what follows we shall always consider the case \( k \leq 3 \).

Theorem 1. If \( k_i > 1 \) for some \( i > 0 \), then

\[ 2 R(\omega) = K(i, \omega), \]

where \( K(i, \omega) \) are decomposable elements in the ring \( \mathcal{O}_{\mathcal{S}p}^* \).

Remark 6. To construct elements \( K(i, \omega) \) we need the definition of \( R(\omega) \), Corollary 1 and the formula

\[ \mathcal{D}_{\mathcal{S}p} \langle x \rangle | \mathcal{R}(4m+3) = [I_{P}(4m+3)] \]

in the \( \mathcal{O}_{\mathcal{S}p}^* \) -theory.

Remark 7. The elements \( R(\omega) \) and \( K(i, \omega) \) are defined correctly and Theorem 1 is the analogue of the Massey product in the \( \mathcal{O}_{\mathcal{S}p}^* \) -theory. The Massey product in the cobordism theory has hitherto been the only means for constructing new elements in
the $\Omega^{*}_{L}$-theory [6],[8],[3].

**Corollary 2.** The element $K^{111}_{111}$ in the dimension 111 coincides with Kochman's element [8].

1. $K^{111}_{111} = \mathcal{R}(\frac{1}{2}, \frac{1}{8}, \frac{1}{16}, 15) = \langle \eta^2, \eta^2 \eta^4, \eta^3 \rangle$.

2. $4K^{111}_{111} = 0$.

**Hypothesis.** 1. Among $K_{i,\infty}$ there exist non-trivial elements.

2. $2K^{111}_{111} \neq 0$

**Remark 8.** Kochman [8], Ray, Ivanovski, Vershinin, Botvinik put forward a hypothesis that fourth order appear for the first time in dimension 111.

2. **Definition.** A non-trivial element on which all Landweber-Novikov operations [5],[9] act trivially is called the primitive element in the ring.

Examples of primitive elements:

1. $\vartheta_1 \in \Omega^{1}_{L}$

2. $\vartheta_1^2 \in \Omega^{2}_{L}$

3. $\vartheta_2 \vartheta_4^2 \in \Omega^{31}_{L}$

**Theorem 2.** For any primitive element $\varepsilon$ in the ring $\Omega^{*}_{L}$ there exists an element $\delta \in \Omega^{*}_{L}$ such that

$4\varepsilon = 8\delta$.

**Remark 9.** The theorem is proved using Steenrod-Dick operations, an exact sequence of the covering in the $\Omega^{*}_{L}$-theory, transfer properties for the covering and elements of the two-valued formal group theory.

**Definition.** $OM$ is a set of maximum order elements in $\Omega^n_{L}$, where $n$ is the minimum dimension in which there exist elements of order higher than 4.
Corollary 2. If in $\Omega^M$ there exists a primitive element, then

$$4 \text{Tors} \Omega^*_\mathfrak{s} \mathfrak{p} = 0.$$ 

Hypothesis. 1. $\Omega^M$ contains a primitive element

2. $2 \text{Tors} \Omega^*_\mathfrak{s} \mathfrak{p} \subset \Theta_1 \Omega^*_\mathfrak{s} \mathfrak{p} \cdot$

3. $2 \text{Im} (\text{Tors} \Omega^*_\mathfrak{s} \mathfrak{p} \rightarrow \Omega^*_\mathfrak{sc}) = 0.$

Remark 10. As reported by B.I. Botvinik, $16 \text{Tors} \Omega^*_\mathfrak{s} \mathfrak{p} = 0.$

References

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