Homology operations derived from the Dickson's invariants

By

Huỳnh Mùi

Department of Mathematics

University of Hanoi

ĐAI HOC TONG HOP, Hanoi

1. Introduction. In / 7 / and / 8 /, we have introduced the cohomology operations derived from the modular invariants to give a new interpretation of the Milnor basis of the Steenrod algebras. By taking the dual of this analogy, in the present paper, we shall introduce the homology operations in terms of the modular invariants which lead to a simple description of the homology coalgebras of the infinite loop space $\Omega^\infty \Sigma^\infty X$ in terms of the homology coalgebras H_*X.

Till now, the coalgebra structure of $H_*\Omega^\infty \Sigma^\infty X$ can be only expressed in terms of allowable iterated Dyer-Lashopf operations via a modulus of the Adem relations. As a matter of fact, it was unable to formulate explicitely by this way the comultiplication of $H_*\Omega^\infty \Sigma^\infty X$. Our homology operations are eventually certain linear combinations of iterated Dyer-Lashopf operations. The modular invariants offer among many things such good combinations to overcome the Adem phenomenon.

We shall focus our attention to the homology Hopf algebras $H_* \mathfrak{S}_\infty \int X$ since one can obtain the desired results from here for $H_*\Omega^\infty \Sigma^\infty X$ in reading to / 4 / and / 1 /. (see also for related results in [12]). The present work is a generalization of / 2 / where the Hopf algebra $H_*\mathfrak{S}_\infty$ is computed.

For the sake of simplicity, we shall only consider the case where the coefficients are taken in $\mathbb{Z}/p\mathbb{Z}$ with $p = 2$. The case $p > 2$ will be treated in a later paper. Further, it will be proved in [9] that the homology coalgebras $H_*\Omega^q\Sigma^q X$ for finite q can be expressed in terms of the homology operations defined here and the unstable Browder operations.

2. The natural homomorphisms $D_K : H_*X \longrightarrow H_*\mathfrak{S}_\infty\int X$.

Let G be a subgroup of the symmetric group $\mathfrak{S}_m = \mathfrak{S}(S)$ on a set S of m elements and E_G' a G-free contractible space. Let $(X,*)$ be a based space and $X^m = X^S$ on which G operates by permuting the factors. Let $G\int X = E_G \underset{G}{\times} X^S$. Here we are working in the category of compactly generated Hausdorff spaces. The space $\mathfrak{S}_\infty\int X$ mentioned in the introduction is the direct limit of $\mathfrak{S}_m\int X$ with respect to the canonical inclusions $\mathfrak{S}_m\int X \subset \mathfrak{S}_{m+1}\int X$.

Let $\mathfrak{S}_{2^n}$ denote the symmetric group of degree 2^n on the point set of the vector space $V = \mathbb{Z}_2^{\,n}$ and E^n the elementary abelian 2-subgroup of $\mathfrak{S}_{2^n}$ consisting of all translations on V. Then, recall that we have the maps

$$H^*X \xrightarrow{P_n} H^*\mathfrak{S}_{2^n}\int X \xrightarrow{d_n} H^*E^n \otimes H^*X$$

where P_n is the Steenrod power map and d_n the homomorphism induced by the diagonal map $X \longrightarrow X^V$, the inclusion $E^n \subset \mathfrak{S}_{2^n}$ and the Künneth formula (see e.g. / 7 /, / 8 /).

Let $Q_{n,o}, \dots, Q_{n,n-1}$ be the well known Dickson's invariants of $GL_n = GL(n, \mathbb{Z}_2)$ in H^*E^n. As a consequence of Theorems 4.9 and 6.2 in [5], we have then the following.

Theorem 2.1. $\mathrm{Im}\, d_n = Z_2\left[Q_{n,o}, \dots, Q_{n,n-1}\right] \otimes d_n P_n H^*X$.

As easily observed from A.4 in / 5 /, an inverse image of $Q_{n,s}$ by d_n can be chosen as the image $W_{n,s}$ of the $(2^n - 2^s)$-th Stiefel-Whitney class of the natural representation $\mathfrak{S}_{2^n} \subset O(2^n)$.

Hence we have

$$H^* \mathfrak{S}_{2^n} \int X = \mathrm{Ker}\, d_n \oplus \mathbb{Z}_2 \left[W_{n,o}, \dots, W_{n,n-1} \right] \otimes \underline{P}_n H^* X .$$

Here $\underline{P}_n H^* X$ means the free module generated by $P_n u$ with u running over a homogeneous basis of $H^* X$.

Let us consider the homology H_* as the dual of the cohomology H^* and let $\langle \cdot , \cdot \rangle$ be the dual pairing. For every sequence $K = (k_o, \dots, k_{n-1})$ of non negative integers with $n \geq o$, we put $W^K = W_{n,o}^{k_o} \cdots W_{n,n-1}^{k_{n-1}}$ with $W^\emptyset = 1$. Then we define the natural map

$$W_K : H_* X \longrightarrow H_* \mathfrak{S}_{2^n} \int X$$

by the relations

$$\langle W_K x , \mathrm{Ker}\, d_n \rangle = 0, \quad \langle W_K x , W^H \otimes P_n u \rangle = \delta_{KH} \langle x, u \rangle .$$

where δ_{KH} denotes the Kronecker symbol. P_n is not a homomorphism but $d_n P_n$ is a homomorphism, hence so is obviously the map W_K. Further P_n and d_n are natural in X, hence so is W_K.

<u>Definition 2.2</u>. <u>For every</u> $K = (k_o, \dots, k_{n-1})$, <u>we define the natural homomorphism</u>

$$D_K : H_i X \longrightarrow H_{i+d(K)} \mathfrak{S}_{2^n} \int X$$

<u>of degree</u> $d(K) = \sum_{s=o}^{m-1} k_s (2^n - 2^s)$ <u>by the formulas</u> $D_K x = o$ <u>if</u> $k_o < i$ <u>and</u>

$$D_K x = W_{(k_o - i, k_1, \dots, k_{n-1})}(x), \text{ if } k_o \geq i.$$

<u>If</u> $n = o$, <u>then</u> $K = \emptyset$, <u>we define</u> $D_\emptyset = id$.

The following important property of D_K can be proved easily from the definition.

<u>Theorem 2.3. Let</u> Δ <u>denote the comultiplication in homology. Then for every</u> $x \in H_* X$, <u>we have</u>

$$\Delta(D_H x) = \sum_{K+L=H} \sum_{(x)} D_K x' \otimes D_L x'' \quad \underline{\text{with}} \quad \Delta x = \sum_{(x)} x' \otimes x'' .$$

Of particular importance is the case where X is a $\mathfrak{C}_\infty$ - space, i.e. a space together with a natural morphism $\theta : \mathfrak{S}_\infty \int X \longrightarrow X$, (e.g. $X = \mathfrak{S}_\infty \int Y$). In this case, we have the homology operations, also denoted by D_K, which are given by $D_K = \theta_* D_K$.

3. The homology Hopf algebras $H_* \mathfrak{S}_\infty \int X$.

According to the well known Steenrod decomposition theorem, we have the monomorphisms $i^*_{m,n} : H_* \mathfrak{S}_m \int X \longrightarrow H_* \mathfrak{S}_n \int X$ given by the canonical inclusion $\mathfrak{S}_m \int X \subset \mathfrak{S}_n \int X$ for $m \leq n$. Let $H_* \mathfrak{S}_\infty \int X = \lim H_* \mathfrak{S}_m \int X$ be equiped with the multiplication given by the homomorphisms

$$H_* \mathfrak{S}_m \int X \otimes H_* \mathfrak{S}_n \int X \longrightarrow H_* \mathfrak{S}_{m+n} \int X \tag{3.1}$$

as usual. Together with the homology comultiplication, $H_* \mathfrak{S}_\infty \int X$ becomes a Hopf algebra as proved in Nakaoka / 10 /.

<u>Theorem 3.2. Let</u> x_i, $i \in I$ <u>denote a homogenous basis of</u> $H_* X$ <u>and consider</u> $H_* X \subset H_* \mathfrak{S}_\infty \int X$. <u>Then we have</u>

$$H_* \mathfrak{S}_\infty \int X = \mathbb{Z}_2 \left[D_K x_i \; ; \; i \in I, \; K = (k_0, \ldots, k_{n-1}), k_0 > \dim x_i, n \geq 0 \right]$$

<u>as Hopf algebras with the comultiplication given by</u>

$$\Delta(D_H x) = \sum_{K+L=H} \sum_{(x)} D_K x' \otimes D_L x'' \quad \underline{\text{with}} \quad \Delta x = \sum_{(x)} x' \otimes x'' .$$

<u>Moreover, the comultiplication formula can be reduced by the relations</u>

$$D_{(\dim x, k_1, \ldots, k_{n-1})}(x) = (D_{(k_1 + \dim x, k_2, \ldots, k_{n-1})}(x))^2 .$$

If X is an one point space, $H_* \mathfrak{S}_\infty \int X$ is the homology Hopf algebra of the infinite symmetric group, and the theorem has been proved by Nguyễn H.V. Hưng in / 2 / where one can found an essential proof for the later part of the theorem. Hung's computation of $H_* \mathfrak{S}_\infty$ based on a combination of the modular invariant theory and the Nakamura's cellular decomposition of the configuration spaces. This method can be used also to prove Theorem 3.2. But we shall briefly present here a relatively simple algebraic approach to this theorem.

For a given natural number m, we let $A(m)$ be the set of all infinite sequences of non negative integers $M = (m_0, m_1, \dots)$ such that $m = \sum_{n \geq 0} m_n 2^n$. Let S be a set of m elements and $\mathfrak{S}_m = \mathfrak{S}(S)$. Then, for every $M = (m_n) \in A(m)$, S can be regarded as the disjoint union $S = \coprod_{n \geq 0} \coprod_{j=1}^{m_n} V_j^n$ of the vector space V_j^n of dimension n. We define the subspaces

$$\text{(3.3)} \qquad E(M)\int X = \prod_{n,j} E(V_j^n)\int X = \prod_{n \geq 0} (E^n \int X)^{m_n}$$

$$E(M,X) = \prod_{n,j} E(V_j^n) \times X = \prod_{n \geq 0} (E^n \times X)^{m_n} \subset E(M)\int X$$

of $\mathfrak{S}_m \int X$. Here and from now on, we write $BG \times X$ briefly by $G \times X$ for $G \subset \mathfrak{S}_m$ and consider $E^n \times X \subset E^n \int X$ via the diagonal $X \longrightarrow X^{V^n}$.

<u>Lemma 3.4. The homomorphism</u> $\bigoplus_{M \in A(m)} H_* E(M,X) \longrightarrow H_* \mathfrak{S}_m \int X$ <u>given by the inclusions</u> $E(M,X) \subset \mathfrak{S}_m \int X$ <u>is surjective</u>.

The lemma follows by a generalization of the proof of II.2.8 and II.3.8 in / 5 /. The construction of $E(M,X)$ and 3.1 lead to the following.

<u>Lemma 3.5.</u> <u>The algebra</u> $H_* \mathfrak{S}_\infty \int X$ is generated by the image of $H_*(E^n \times X) \longrightarrow H_* \mathfrak{S}_\infty X$, $n \geq 0$.

If X is an one point space, this result can be found in / 11 /, with a suggestion for the proof is that one must use the geometrical computation of Nakaoka on $H_* \mathfrak{S}_\infty$ in / 10 /. It seems that we may not obtain an easy proof by this way.

Let $P^n : H_* X \longrightarrow H_* E^n \int X = H_*(E^n, H_* X^V)$ be the homorphism given by $P^n x = x^{2^n} \in H_0(E^n, H_* X^V) = H_0 E^n \otimes H_*(X^V)_{E^n}$. Here $\dim V = n$ and E^n is the group of all translations on V as in Section 2. By 2.1 and 3.5, we have

<u>Lemma 3.6.</u> <u>We have the injections</u>

$$(H_* E^n)_{GL_n} \otimes P^n H_* X \subset H_* E^n \int X \longrightarrow H_* \mathfrak{S}_\infty \int X, \quad n \geq 0,$$

<u>and their images generate the algebra</u> $H_* \mathfrak{S}_\infty \int X$.

This result proves that the elements $D_K H_* X$ generate the algebra $H_* \mathfrak{S}_\infty \int X$.

<u>Lemma 3.7.</u> <u>We have the injections</u>

(i) $\quad (H_* X^m)_{\mathfrak{S}_m} \longrightarrow H_* \mathfrak{S}_m \int X, \quad m \geq 0$;

(ii) $\quad \bigotimes_{n=0}^{t} ((H_* E^n)_{GL_n} \otimes P^n H_* X) \longrightarrow H_* \mathfrak{S}_{2^t - 1} \int X, \quad t \geq 0$.

The injectivity of the maps in (i) follows from a similar argument in II.3.2 [5]. We take this opportunity to say that II.3.10 in [5] is <u>untrue</u>. It is true with an addition assumption on the sequence $M \in A(m)$ that the subgroup $\mathfrak{S}_M$ defined as in p. 347 [5] satisfies the condition $[\mathfrak{S}_m : \mathfrak{S}_M] = 1 \bmod 2$. This is the case for $m = 2^t - 1$ and $M = (1, \ldots, 1, 0, \ldots) \in A(2^t - 1)$.

From this fact follows the injectivity of the maps in 3.7.ii.

By the injectivity of the maps in (1), one can prove easily that any set of linear independent elements in H_*X generate a free algebra in $H_*\mathfrak{S}_\infty\int X$. On the other hand, (ii) shows that K 1 the generators of $H_*\mathfrak{S}_\infty\int X$ given in 3.2 are linearly independent, hence they are algebraically independent in $H_*\mathfrak{S}_\infty\int X$. Consequently, the monomorphism $H_*\mathfrak{S}_\infty\int X \longrightarrow H_*\mathfrak{S}_\infty\int\mathfrak{S}_\infty\int X$ shows that they are algebraically independent in $H_*\mathfrak{S}_\infty\int X$. We obtain Theorem 3.2.

Now, the Steenrod decomposition theorem says that

$$H_*\mathfrak{S}_m\int X = \bigoplus_{0\leq k\leq m} H_*(\mathfrak{S}_k\int X, \mathfrak{S}_{k-1}\int X)$$

for $m \geq 0$ (see e.g. [6]). This leads to a multiplicity μ defined on $H_*\mathfrak{S}_\infty\int X$ such that $\mu(a) \leq m$ for $a \in H_*\mathfrak{S}_m\int X$ and $\mu(ab) = \mu(a) + \mu(b)$ for $a, b \in H_*\mathfrak{S}_\infty\int X$.

Let us define μ by $\mu(1) = 0$, $\mu(x) = 1$ if $x \in H_*(X,*)$ and $\mu(D_K x) = 2^n$ if $x \in H_*X$ and K is of length n.
Then the homology coalgebra $H_*\mathfrak{S}_m\int X$ is the subcoalgebra of $H_*\mathfrak{S}_\infty\int X$ consisting of all elements with multiplicity $\leq m$. Theorem 3.2 also determined the coalgebra structure of $H_*\mathfrak{S}_m\int X$.

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