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A topological generalization of the elliptic genus

by

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§0. Introduction

0.1 The quantum mechanics of relativistic one-dimensional continua (strings) moving in a general manifold has led, quite unexpectedly, to the discovery of new topological invariants of the underlying manifold, and has triggered an explosion of interest in connections between algebraic topology and conformal field theory, as seen in Landweber's conference notes [51]. The trajectory of a closed loop defines a map from an annulus to the base manifold; the physics of the matter comprising the loop can be encoded into a complex structure on that annulus, and elliptic cohomology can be interpreted as a functor defined on manifolds, taking values in the category of modules over some ring of functions (or modular forms) defined on the space of such annuli, as in [68].

However, from the point of view of physics, the motion of a simple loop is only one important special case, and we hope in the end to find a coherent theory which treats all Riemann surfaces on an equal footing. There has been considerable progress toward this goal in the absolute case (i.e., when the role of the base manifold is unimportant [9]), but so far elliptic cohomology has resisted attempts to encompass surfaces of genus greater than one.*

0.2 This paper is concerned with the topological notions required for such a

* but see [4].

generalization. One basic idea is suggested by the theory of moduli, where it has gradually become clear that it is often necessary to rigidify the objects of interest (by the specification of some (perhaps subtle) extra structure) before a satisfactory classification can be achieved. Thus our constructions define a rational cohomology functor which takes its values in the category of modules over a ring of modular functions on a space of Riemann surfaces endowed with projective connections; this amounts, roughly, to replacing the usual moduli object by its cotangent bundle.

There are several reasons why this sort of extra structure is reasonable in this context; as motivation we mention that a projective connection on a Riemann surface is understood by physicists as the stress-energy (pseudo-)tensor [8, §2] associated to a 'material' structure on the surface. From the mathematical point of view, on the other hand, such a projective connection permits us to (begin to) identify the universal cover of a Riemann surface with a standard disk (e.g., if the genus of the surface is > 1). This in turn leads to the second idea behind our constructions: that the (formal completion of the) universal covering map of a Riemann surface generalizes the exponential map, and can be used to endow such a Riemann surface with a one-dimensional formal group law which generalizes the formal completion of the Jacobian of an elliptic curve. Thus (rationally) the cohomology functor introduced here possesses a universal quadratic differential, analogous to the universal formal group law defined over the complex cobordism ring.

An analytic interpretation of the index associated to our cohomology functor as the partition function of a conformal field theory requires the construction of such a field theory, which is a very active subject of

research and is somewhat outside our field of competence. However, there are some very heuristic remarks below, sketching how the nonlinear sigma-model based on a manifold M can be used to begin to construct a conformal field theory. A general M will exhibit obstructions to completing this construction, which we expect to be manifested as failures of integrality in our generalized genus.

0.3 The four sections in this paper serve different purposes. Sections two and three treat projective connections on Riemann surfaces and generalized cohomology theories, and are written in an almost expository manner, on the grounds that those who know about one topic usually do not know a great deal about the other; though there is new material in each section. The first section is preliminary and contains some basic technical facts about the Schwarzian derivative operator, while the fourth uses those elementary facts, together with the constructions described in the intermediate sections, to produce our results.

Our main observation is that solutions of Hill's equations behave like $(-1/2)$ forms, while the invariant differential of a formal group law behaves like a 1-form, both well-known facts (to rather different groups of experts); but it takes some space to explain what "behaves like" means: in one context it refers to the Virasoro algebra, and in the other it refers to the Landweber-Novikov algebra, again two sides of one thing, each with its separate population of specialists.

Our principal conclusion is analogous to a theorem of Quillen; we construct a quadratic differential

$$u = \sum_{n \geq 0} u_n z^n dz^{\otimes 2}$$

with coefficients in the complex cobordism ring MU^* , which maps to the stress-energy-tensor defined by a holomorphic conformal field theory with central charge one. More precisely, to a family

$$\mathcal{C} \xrightarrow{\quad} \text{Spec } A$$

of curves with coordinatized point, such a field theory assigns a projective connection-form

$$T(z) dz^{\otimes 2} \in A[[z]] dz^{\otimes 2}.$$

Our result is that when A is an algebra over the rational numbers, there exists a unique ring homomorphism

$$\varphi: MU^* \longrightarrow A$$

such that $\varphi(u) = T(z) dz^{\otimes 2}$. When \mathcal{C} is the family of Jacobi quartics, we recover elliptic cohomology (over \mathbb{Q}), as

$$MU^*(-) \otimes_{\varphi} A.$$

Applied to the conformal field theory defined by Graeme Segal's charged chiral fermion [68, §8.15], this construction yields our generalization of elliptic cohomology.

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§1 Schwarzian derivatives

1.0 In this section we summarize some quite classical properties of the Schwarzian derivative, all easily verified by straightforward calculation, but we give references to modern accounts rather, with the intention of making the reader familiar with the associated literature.

Classically the Schwarzian derivative is an operation on analytic functions in a complex domain, but for our purposes we will be concerned mostly with its action upon formal power series with coefficients in an integral domain A in which 2 is a unit, with quotient field K of characteristic zero.

1.1 Proposition If $f'(z) \neq 0$, the formula

$$S\{f(z), z\} = -(f'(z))^{\frac{1}{2}} ((f'(z))^{-\frac{1}{2}})''$$

defines a map

$$S: A[[z]] \longrightarrow K((z)) := K[[z]][z^{-1}]$$

with the following properties:

$$1) \quad S\{f(z), z\} \in A[[z]]$$

ii) If $a, b, c, d \in A$ with $ad - bc \in A^\times$ and $cf(0) + d \in A^\times$, then

$$S\left\{\frac{af(z) + b}{cf(z) + d}, z\right\} = S\{f(z), z\}$$

iii) If $g \in A[[z]]$ with $g(0) = 0$, then

$$S\{f(g(z)), z\} = S\{f(g(z)), g(z)\} g'(z)^2 + S\{g(z), z\}.$$

proof Cf. [28, §9a] or [73, prop 1.2.3]. ■

Note that

$$S\{f(z), z\} = \frac{1}{2} \left[\left(\frac{f''}{f'}\right)' - \frac{1}{2} \left(\frac{f''}{f'}\right)^2 \right]$$

is one-half of the Schwarzian derivative as usually defined; in this we follow Segal [66, §7]. In fact Deligne observed, in lectures at the 1989 Bonn conference on loopspaces (cf. also [19, I§5]) that if we write

$$(T_n f)(z) = \frac{f^{(n)}(z)}{n!}$$

for the n^{th} Taylor coefficient of f at z , then

$$S\{f(z), z\} = 3(T_1 f)^{-2} \det \begin{bmatrix} T_3 f & T_2 f \\ T_2 f & T_1 f \end{bmatrix}.$$

1.2 Proposition Suppose that

$$f, \phi, T \in A[[z]],$$

with

$$f(0) = 0, \quad f'(0) = 1,$$

and

$$\phi''(z) + T(z)\phi(z) = 0.$$

Then the quantities

$$\tilde{\phi}(z) = (f'(z))^{-\frac{1}{2}} \phi(f(z))$$

and

$$\tilde{T}(z) = T(f(z))f'(z)^2 + S(f(z), z)$$

lie in $A[[z]]$ and satisfy the equations

$$\tilde{\phi}''(z) + \tilde{T}(z)\tilde{\phi}(z) = 0.$$

proof Cf. Schiffer and Hawley [65, II §1], or Segal [66, §8b], or Tjurin [73, prop 1.2.9]. ■

This proposition can be expressed more colloquially as the assertion that " ϕ transforms like a $(-\frac{1}{2})$ -density" if T "transforms like a projective connection." This will be central below.

1.3 Proposition Suppose that

$$T(z) = \sum_{n \geq 0} t_n \gamma_n(z)$$

and

$$\lambda(z) = \sum_{n \geq 0} \lambda_n \gamma_{n+1}(z), \text{ with } \lambda_0 = 1,$$

satisfy the equation

$$S(\lambda(z), z) = T(z),$$

where

$$\gamma_n(z) = \frac{z^n}{n!}$$

is the n^{th} divided power of z .

Then

$$\lambda_{n+2} \equiv \frac{1}{2} t_n \pmod{t_0, \dots, t_{n-1}}$$

in $\mathbb{Z}[\frac{1}{2}][t_0, t_1, \dots]$.

proof by induction. ■

Note that it is a corollary to this proposition that the ring

$$\mathbb{Z}[\frac{1}{2}][\lambda_1, \lambda_2, \dots]$$

is free, with basis $1, \lambda_1, \lambda_1^2, \dots$ over the ring $\mathbb{Z}[\frac{1}{2}][t_0, t_1, \dots]$.

1.4 Proposition Suppose, as in §1.2, that

$$\phi, T \in A[[z]]$$

satisfy the relation

$$\phi'' + T\phi = 0$$

with $\phi(0) = 1$. Then the 1-form $d\lambda(z) = \phi(z)^{-2} dz$ defines

$$\lambda(z) = (A \otimes \mathbb{Q})[[z]], \lambda(0) = 0,$$

satisfying the relation

$$S(\lambda(z), z) = T(z)$$

in $(A \otimes \mathbb{Q})[[z]]$.

proof Cf. Ihara [37, I§1.5, prop. 4], or Tjurin [73, prop. 1.2.7]. ■

At this point it may be appropriate to list some examples.

1.5.1

$$S(z^n, z) = -\frac{1}{4} (n^2 - 1) z^{-3},$$

cf. Tjurin [§1.2.15].

1.5.2

$$S(\log(1-z), z) = \frac{1}{4} (1-z)^{-2}$$

1.5.3

$$S(\text{Arctan } z, z) = -(1+z^2)^{-2} = \frac{1}{4} \sum [3(i\pm z)^{-2} + i(i\pm z)^{-1}]$$

1.5.4

$$S(\text{Arcsinh } z, z) = -\frac{1}{2}(1-\frac{1}{2}z^2)(1+z^2)^{-2} = \frac{1}{16} \sum [9(i\pm z)^{-2} + i(i\pm z)^{-1}]$$

1.5.5 The Jacobi sn-function is defined by the differential equation

$$\text{sn}'(z)^2 = \prod_{1 \leq i \leq 2(g+1)} (\text{sn}(z) - e_i)$$

and the boundary conditions

$$\text{sn}(0) = 0, \quad \text{sn}'(0) = 1.$$

when $g = 1$; when $g > 1$ the function sn is a kind of hyperelliptic generalization, (ch. [58], [72]). The inverse power series then has Schwarzian derivative

$$S(\text{Arcsn } z, z) = \frac{3}{16} \sum (z-e_i)^{-2} - \frac{1}{8} \sum_{i>j} (z-e_i)^{-1} (z-e_j)^{-1},$$

cf. [57 §2.10]. [If $\prod(X-e_i) = R(X) = 1 - 2\delta X^2 + \epsilon X^4$, then

$$S(\text{Arcsn } X, X) = (2\delta^2 - 3\epsilon)R(X)^{-2} + \delta R(X)^{-1}.]$$

1.6 The following examples are of a more sophisticated nature:

1.6.1 Ihara [38, §4 (Theorem 5)] has studied the following situation: given nonzero rational parameters $\rho_0, \rho_1, \rho_\infty$, define quantities α, β, γ and A, B, C by the equations

$$\begin{aligned} \rho_\infty^{-2} &= \alpha + 1 \\ \rho_1^{-2} &= \alpha + \beta + \gamma + 1 \\ \rho_0^{-2} &= \gamma + 1 \\ A &= \frac{1}{2}(1 - \rho_0 - \rho_1 - \rho_\infty) \\ B &= \frac{1}{2}(1 - \rho_0 - \rho_1 + \rho_\infty) \\ C &= 1 - \rho_0. \end{aligned}$$

The differential equation

$$\lambda'(z) = w^{\rho_0-1} (1-w)^{\rho_1} F(A, B, C; w)^{-2},$$

with F the hypergeometric function and $w = (1-c)^{-1}(z-c)$, defines a formal series satisfying the equation

$$S(\lambda(z), z) dz^{\otimes 2} = \frac{\alpha w^2 + \beta w + \gamma}{w^2(1-w)^2} dw^{\otimes 2}$$

by proposition 1.4 above.

1.6.2 It is worth emphasizing that the preceding Schwarzian derivatives are all rational in the complex plane; cf. [47, 63, 76, 80]. On the other hand, Krichever [48], following Hirzebruch [31], has studied the elliptic Baker-Akhiezer function

$$p(z, X) = \frac{\sigma(z-X)}{\sigma(z)\sigma(X)} \exp(z\zeta(X))$$

(defined in terms of Weierstrass elliptic functions); its Schwarzian derivative (with respect to the variable z) is algebraic (in the variables $X = \wp(x)$, $Z = \wp(z)$), although not rational, with leading term $\frac{3}{16} \sum (Z-e_i)^{-2}$.

1.6.3 In the final section of this paper, we will employ the preceding constructions to associate a one-dimensional formal group law

$$F_T(X, Y) = \lambda^{-1}(\lambda(X) + \lambda(Y))$$

to the solution λ of the Schwarzian equation

$$S(\lambda(z), z) = T(z).$$

Perhaps it would not be out of place to observe here that any such formal group over the p-adic integers can be constructed in this way, starting with a suitable rational function $T(z)$ defined over the p-adics. Indeed, the one-dimensional formal group laws over \mathbb{Z}_p are classified up to isomorphism by the Frobenius endomorphisms of their mod p reductions; such an endomorphism r generates the maximal ideal in the local \mathbb{Z}_p -algebra of endomorphisms of the group law, and the group law can be reconstructed (up to isomorphism) from the logarithm

$$\lambda_r(z) = z + \text{Norm} (1 - r^{-1}z^p)^{-1} - 1 \in \mathbb{Q}_p[[z]]$$

(cf. [30], [33]) where the norm is to be regarded as a map from the (totally ramified) quotient field of the endomorphism ring of the group law, to the prime field \mathbb{Q}_p . In particular, the coefficients of

$$S(\lambda_r(z), z) = T_r(z) \in \mathbb{Z}_p[[z]] \quad (p \text{ odd})$$

are rational functions of the eigenvalues of the Frobenius element r .

1.6.4 Finally, suppose that

$$\lambda(z) = \sum_{n \geq 1} a_{n-1} \frac{z^n}{n}, \quad \text{with } a_0 = 1;$$

then

$$S(\lambda(z), z) = \frac{1}{4} \left(\sum_{n \geq 0} a_n z^n \right)^{-2} \sum_{m, n \geq 1} c_{m, n} a_{m-1} a_{n-1} z^{m+n-4}$$

with

$$c_{m, n} = (m-n)^2 - mn + 1.$$

This will play an important role as a universal example, in §4.

§2 Projective connections

2.0 A projective structure on a Riemann surface C is an atlas of charts with coordinate transformations in the group $\text{PGL}(2, \mathbb{C})$ of complex fractional linear transformations; in other terms such a structure is a reduction of the principle bundle of $T_C^{\otimes 1}$ from germs of holomorphic functions with values in the linear group $\text{GL}(2, \mathbb{C})$, to locally constant function germs with values in $\text{PGL}(2, \mathbb{C})$. Classically such a structure is usually defined by an identification of the universal cover of C with the upper half-plane (when the genus of C exceeds one).

It is not the purpose of this section to give a systematic account of projective structures and connections on Riemann surfaces; Chapter nine of Gunning's text [26] and Chapter one of Tjurin's review article [73] are both masterful, easily accessible expositions. The first two subsections of this chapter summarize some basic facts and present Gunning's interpretation of projective connections in terms of affine connections on associated bundles. The third section summarizes the construction of some classical (Wirtinger) projective connections, following Tjurin and Fay; the fourth and fifth discuss Segal's construction of the projective connection defined by a conformal field theory, while the sixth and seventh define the moduli space of projective connections, following Hubbard, K. Saito, and Tjurin.

2.1 Now projective connections and projective structures are, with the proper notions of equivalence, synonymous; but their formal definitions are phrased in very different terms. Suppose the surface C possesses a coordinate atlas

(U_i, z_i) with transition functions

$$f_{ji} = z_i \circ z_j^{-1};$$

then the pullback

$$e_{ji} = z_j(S(f_{ji}(z), z)dz)^{\otimes 2} \in H^0(U_i \cap U_j; \Omega^{\otimes 2})$$

is a quadratic differential, holomorphic in the coordinate overlap $U_i \cap U_j$.

The composition law for Schwarzian derivatives in Proposition 1.1 implies that e obeys a cocycle identity

$$e_{ji} + e_{ik} + e_{kj} = 0$$

in the triple intersection $U_i \cap U_j \cap U_k$, which thus defines a class $[e]$ in the cohomology group $H^1(C, \Omega^{\otimes 2})$; but this group is zero if genus $C > 1$.

Consequently there exists on C a projective connection, i.e. a family of elements

$$T_i dz_i^{\otimes 2} \in H^0(U_i, \Omega^{\otimes 2})$$

such that

$$T_j dz_j^{\otimes 2} - T_i dz_i^{\otimes 2} = e_{ji}$$

in $U_i \cap U_j$. The more usual convention is to write this equation as

$$T(z_j) dz_j^{\otimes 2} = T(z_i) dz_i^{\otimes 2} + S(z_i, z_j) dz_j^{\otimes 2}.$$

If the f_{ji} are fractional linear transformations then $e_{ji} = 0$ by Proposition 1.1; conversely a solution to the equation

$$S(\bar{z}_i, z_j) = T_i$$

defines a new atlas (U_i, \bar{z}_i) with projective linear coordinate transformations, as in Tjurin's Proposition 1.3.4.

2.2 Since projective connections are relatively unfamiliar objects (to these authors, at least) it is of some interest to note that on a Riemann surface

with spin structure such connections are equivalent to the more usual affine connections, on certain related vector bundles. In this paragraph we summarize the relevant constructions, which are to be found in §9b of Gunning's book.

To begin, we recall that a spin structure on a Riemann surface is what is classically called a theta-characteristic (cf. [28]); such a structure is a choice of square root $\Omega^{\frac{1}{2}}$ of the canonical cotangent bundle of 1-forms (or equivalently a choice $\Omega^{-\frac{1}{2}}$ of square root of the tangent bundle). On a surface of genus g the set of spin structures is a principal homogeneous space of the group $H^1(C, \mathbb{F}_2)$ and thus contains 2^{2g} elements, of which $2^{g-1}(2^g+1)$ are spin boundaries [8], usually called even.

The bundle $J_1(\Omega^{-\frac{1}{2}})$ of 1-jets of $(-\frac{1}{2})$ -forms is the natural extension

$$0 \longrightarrow \Omega^1 \otimes \Omega^{-\frac{1}{2}} \longrightarrow J_1(\Omega^{-\frac{1}{2}}) \longrightarrow \Omega^{-\frac{1}{2}} \longrightarrow 0$$

(cf. [5]) which generates the group

$$\begin{aligned} \text{Ext}^1(\Omega^{-\frac{1}{2}}, \Omega^{\frac{1}{2}}) &\cong H^1(C, \text{Hom}(\Omega^{-\frac{1}{2}}, \Omega^{\frac{1}{2}})) \\ &\cong H^{1,1}(C). \end{aligned}$$

In local coordinates, a 1-jet can be written

$$(a_0 + a_1 z) dz^{-\frac{1}{2}},$$

and the space of such objects is isomorphic, under the action of (a double cover of) $Sl_2\mathbb{C}$ on z by fractional linear transformations, to the standard action of $Sl_2\mathbb{C}$ on \mathbb{C}^2 . Its determinant action is trivial, so $J_1(\Omega^{-\frac{1}{2}})$ is a candidate for a bundle with affine connection. Analogous to the projective connections in the preceding paragraph, such an affine connection on a vector

bundle E is a family of sections

$$A_i \in (\text{End}(E) \otimes \Omega^1)(U_i)$$

such that

$$A_i - A_j = \rho_{ij}^{-1} d \rho_{ij}$$

on $U_i \cap U_j$, where the ρ_{ij} are the transition functions for E . According to [27, §8c], such A_i exist if and only if there exist matrix-valued holomorphic functions F_i defined on the U_i such that $F_i^{-1} \rho_{ji} F_j$ is constant on the overlap $U_i \cap U_j$, in which case $A_i = F_i^{-1} d F_i$.

Now $J_1(\Omega^{-\frac{1}{2}})$ is the bundle A of Gunning's Theorem 21 [26, p. 197]; a projective connection on C is there shown to define matrices F_i satisfying the differential equation

$$F_i' = F_i \begin{bmatrix} 0 & -T_1 dz_1 \\ (dz_1)^{-1} & 0 \end{bmatrix}$$

in U_i , where the matrix is to be interpreted as an endomorphism of $J_1(\Omega^{-\frac{1}{2}})$ trivialized as $\Omega^{\frac{1}{2}} \otimes \Omega^{-\frac{1}{2}}$. It is an immediate consequence that

$$A_i = \begin{bmatrix} 0 & -T_1 dz_1^{\otimes 2} \\ 1 & 0 \end{bmatrix}$$

defines the desired affine connection, as a vector bundle morphism over U_i , from $J_1(\Omega^{-\frac{1}{2}})$ to $J_1(\Omega^{-\frac{1}{2}}) \otimes \Omega^1$. We summarize this as a

Proposition. There is a natural 1 - 1 correspondence between projective connections on the spin surface $(C, \Omega^{\frac{1}{2}})$ and affine connections on the 2-plane bundle $J_1(\Omega^{-\frac{1}{2}})$.

Note that because $\Omega^{\frac{1}{2}}$ is a subbundle of $J_1(\Omega^{-\frac{1}{2}})$ with positive Chern class,

$J_1(\Omega^{-\frac{1}{2}})$ is an unstable 2-plane bundle, in the sense of [59]. Indeed it is not stable even in the generalized sense of [32, §3.3 part iv]; consequently the functor $\Omega^{\frac{1}{2}} \mapsto J_1(\Omega^{-\frac{1}{2}})$ does not define a map from the moduli space of spin surfaces with projective connection, to any very familiar moduli space of rank two vector bundles; cf. also [7, §2.6].

2.3 In general it is a nontrivial task to exhibit a projective connection on a Riemann surface. Perhaps the most elegant construction is in terms of bidifferentials (cf. [65]) by which we understand a symmetric section B of the sheaf $\text{pr}_2^* \Omega^1 \otimes \text{pr}_1^* \Omega^1$ over $C \times C$. Such a section is said to be of the second kind (cf. [73, §1.3.6]) if it has at most a pole of order two along the diagonal; in this case it can be written locally (ie. in $U_i \times U_i$) in the form

$$(\beta(z_1 - z_2)^{-2} + b(z_1, z_2)) dz_1 \otimes dz_2,$$

with β an invariant, called the biresidue of B . If B is such a bidifferential, with residue $\beta = 1$, then

$$3 \lim_{z_1 \rightarrow z_2} b(z_1, z_2) dz_1^{\otimes 2}$$

is the connection-form for a projective connection in the coordinate patch U_i .

Explicit examples of such meromorphic bidifferentials can be given in terms of theta- or tau-functions (cf. [24, Cor 2.8], [12], [43], [45, §5.9]) as expressions of the form

$$B = \frac{\partial^2 \log \tau(z_1 - z_2)}{\partial z_1 \partial z_2} dz_1 \otimes dz_2.$$

To be more precise would require a substantial digression on such topics as

the prime-forms on a Riemann surface [46] and the Szegő kernel associated to an even spin structure [29], which would take us far afield. Very briefly, a prime form

$$E_{\sigma}(z_1, z_2) = \theta[\sigma](z_1 - z_2) h_{\sigma}(z_1) \otimes h_{\sigma}(z_2)$$

is associated to an odd spin-structure σ by the choice $h_{\sigma} \in H^0(\Omega^{-\frac{1}{2}})$ of a nontrivial section (which exists since the dimension of the space of sections is odd). The square of the Szegő kernel (or Green's function for the Dirac operator)

$$\begin{aligned} S_{\sigma}(z_1, z_2) &= \frac{\theta[\sigma](z_1 - z_2)}{\theta[\sigma](0) E_{\sigma}(z_1, z_2)} \\ &= (z_1 - z_2)^{-1} (dz_1 \otimes dz_2)^{\frac{1}{2}} + \dots \end{aligned}$$

associated to an even spin structure can be shown to be a bidifferential of the second kind, with biresidue one, invariantly associated to the spin structure ([24, Cor 2.12]).

The projective connection thus defined is called by Tjurin ([73, §1.3.5]) the Wirtinger connection (associated to the theta-characteristic); but he uses the same term for the average, over all even spin-structures, of their associated connections. For example, on a hyperelliptic curve

$$y^2 = \prod_{1 \leq i \leq 2(g+1)} (X - e_i)$$

an even spin structure can be defined by a partition of the Weierstrass points* e_i into two sets with $g+1$ elements; we can think of this as the partition defined by even and odd indices. Then

* Strictly speaking, the Weierstrass points lie over the e_i for the branched double cover defined by X .

$$\frac{3}{16} \left[d \log \Pi \left[\frac{X - e_{2i}}{X - e_{2i+1}} \right] \right]^2 - 3 \sum \frac{\partial^2 \log \theta[e]}{\partial x_i \partial x_j} \Big|_{x_i, x_j=0} \omega_i \omega_j$$

defines a connection in terms of a basis ω_i for the holomorphic differentials; here $\theta[e]$ is the theta function associated to the theta-characteristic, cf.

[29]. On an elliptic curve Fay calculates this more explicitly in terms of theta-constants $c_n = \frac{d^0 \log \theta[e]}{dz^n}(0)$, as

$$-2(c_2 + c_4^{-1} c_6) dx^{\otimes 2}$$

[24 p. 36]. Averaging over the spin structures yields the Wirtinger connection

$$\left[\frac{3}{16} \sum (x - e_i)^{-2} - \frac{1}{8} \sum_{i > j} (x - e_i)^{-1} (x - e_j)^{-1} \right] dx^{\otimes 2}$$

in a neighborhood of the point $[0, 1, 0]$, cf. [24, Cor. 2.5], and §1.5.5 above.

The theta-constant associated to an odd spin-structure σ satisfies $\theta[\sigma](0) = 0$, so in this case the formula for the Szegő kernel given above has no meaning. The Verlinde [75, §4] regularize that theta-quotient by l'Hopital's rule, and consider the expansion

$$S_{\sigma}(z_1, z_2; c) = \frac{\sum \partial_1 \theta[\sigma](z_1 - z_2) \omega_1(c)}{\sum \partial_1 \theta[\sigma](0) \omega_1(c)} E_{\sigma}(z_1, z_2)^{-1}$$

where the ω_i are a basis for the holomorphic differentials at c . It seems reasonable to hope that the square of their expression defines a connection differential as $z_1 \rightarrow z_2$, in a neighborhood of c .

2.4 In mathematics, projective connections on Riemann surfaces appear most

naturally in questions connected to problems of uniformization. It is striking, therefore, that projective connections appear naturally in the mathematical physics of models of general relativity on two-dimensional manifolds. In retrospect this connection can be seen even in Polyakov's earliest work on the conformal anomaly in string theory, but it takes a completely explicit form in [8], where it is shown that the analogue of the Einstein-Maxwell stress-energy tensor in two-dimensional conformal field theory is not (in general) a tensor, but rather a pseudotensor, or in more familiar language a projective connection. Since this work, the stress-tensor has become ubiquitous in string theory; cf. eg [2], [49], [71], and [78 (appendix)]. This may be understood as a very clear form taken by the Mach-Einstein principle of equivalence in two dimensions, in which the space-time geometry (ie. the uniformization) is determined by the physics, in terms of the stress-tensor.

In fact the central idea of this paper is that the geometric properties of conformal field theory embodied by invariants such as elliptic cohomology are consequences of the uniformization determined by the stress-tensor. To see this in general, without reference to any particular physical model, it is most helpful to look at the subject from the perspective of Graeme Segal's axiomatization of conformal field theory. [A rather truncated account of Segal's ideas is available in [25].] The basic object is the category having as objects collections of circles, i.e., parameterized compact 1-manifolds; a morphism from a collection C_0 to another such collection C_1 is to be a Riemann surface (i.e., compact two-manifold Σ with fixed conformal structure) such that $\partial\Sigma = C_0 \cup C_1^{\text{op}}$. (It eventually becomes helpful to allow these surfaces to

carry extra data; for our purposes the most important generalization is to spin surfaces. It is also necessary to introduce a natural central extension of the category, but that can be reinterpreted in terms of projective representations of the category, and will be less important here.)

A conformal field theory is a functor from such a category to the category of topological linear vector spaces and trace-class linear transformations, subject to certain intuitive axioms, e.g. that the functor is multiplicative in the sense that it send disjoint unions of circles to tensor products of vector spaces, and that it send circles with opposite orientations to dual vector spaces. Such a functor is said to be chiral if it varies holomorphically (or antiholomorphically) with changes in the complex (or conformal) structure on a morphism Σ ; it appears that such chiral functors are most naturally defined on the category of spin surfaces, cf. [88, §4.5].

The present paper had its genesis when the authors came to understand Segal's Proposition 9.4, in which it is shown that a (weakly) conformal field theory endows a surface with a canonical projective connection. It is perhaps worth noting that this is an unexpected solution of a quite classical problem, which was extensively studied by Poincaré and others (e.g. [80]) before the approach to uniformization via Dirichlet's principle became standard.

With the hope of making this paper somewhat more self-contained, we summarize Segal's idea very briefly, as follows. Suppose for simplicity that we are working with a chiral, holomorphic conformal field theory; let D be a disk, with parameterized boundary. D can then be interpreted as a morphism in Segal's category, from the empty collection C_0 containing no circles, to the collection C_1 containing one circle; we obtain a morphism

$$U_D: \mathbb{C} \longrightarrow H$$

called the vacuum-vector. The energy-momentum tensor measures the variation of U_D under change in conformal structure on D . Let

$$V = \text{Vect}(\partial D) / \text{Vect}(D)$$

be the tangent space to the moduli of conformal structures on D ; here $\text{vect}(D)$ (resp. $\text{vect}(\partial D)$) denotes the space of smooth vector fields on D (resp. ∂D).

We obtain a complex-linear map

$$\delta U_D: V \longrightarrow H$$

by varying the complex structure on D , and the image under δU_D of $v_0 = z^{-1}d/dz \in V$ is essentially the desired stress-tensor. It is more conveniently interpreted as the function which assigns to $\xi \in V$, the inner product of $U_D(\xi)$ and $U_D(v_0)$, i.e.,

$$\int_{\partial D} \xi(z) T(z) dz^{\otimes 2}.$$

We thus obtain a quadratic differential in D ; since a conformal field theory is in general a projective representation of Segal's category, a Schwarzian derivative (or anomaly) term appears in the transformation law for $T(z) dz^{\otimes 2}$ in overlapping coordinate patches, and we obtain a projective connection on a general surface.

2.5 The simplest example of a chiral field theory in Segal's charged fermion [68, §8.15], defined on the category of spin surfaces by the functor which sends a circle (with spin-structure) to the Fock space construction over the sections of its canonical line bundle. Even in this simplest case we are unable to write down the projective connection this field theory defines on a given spin surface; however it seems very reasonable to expect that this connection is (closely related to) the Wirtinger and Verlinde connections discussed in §2.3.

Now physicists are interested in a more general class of conformal field theories, called nonlinear sigma-models; but because the precise conditions for the existence of such a model are not yet clear, we will limit ourselves here to some heuristic remarks. What is wanted is a functor, defined on a suitable class of manifolds, with values in the category of conformal field theories. For example, to a complex manifold M we might try to associate a conformal field theory having as basic Hilbert space H , those holomorphic functions on the free loop space LM which are square-integrable with respect to some hypothetical Gaussian measure. A Riemann surface Σ with boundary $\partial \Sigma = C_0 \cup C_1^{\text{op}}$ defines a subspace

$$\text{Hol}(\Sigma, M) \longrightarrow \text{Map}(C_0, M) \times \text{Map}(C_1, M)$$

consisting of holomorphic maps from Σ to M , which is analogous to a lagrangian correspondence in the theory of geometric quantization. From such data one might expect to extract a trace-class morphism from $H^{\otimes n}$ to $H^{\otimes m}$, where n (resp. m) is the number of components of C_0 (resp. C_1).

For example, when $n = m = 1$ we obtain a trace-class map; its trace will be an analytic index of M , taking values in some ring of functions on the moduli of Σ . The interpretation of a map from an annulus to M as a path in the free loop space leads to the analytic index (cf. [11], [67]) and elliptic cohomology. In this case the analytic index is the partition function of the conformal field theory associated to M ; but because this index can be calculated in terms of characteristic numbers and other topological data [cf. §3.6], we might hope to gain some insight into the hypotheses needed to ensure the existence of our hypothetical nonlinear sigma-model. For example, the elliptic genus can be defined for any oriented manifold, but, in general it is neither integral nor fully modular. Similarly, we expect that the generalized elliptic genus defined below will display integrality obstructions to the existence of a conformal field theory associated to a general manifold M .

2.6 Having summarized some basic facts about projective connections on a fixed Riemannian surface, we turn to the theory of moduli for such objects. The space of projective connections on a surface C is a principal homogeneous space with respect to the $3(g-1)$ -dimensional space $H^0(C, \Omega^{\otimes 2})$ of quadratic differentials, so the space of projective connections is an affine bundle over the moduli space \mathfrak{M}_g of Riemann surfaces.

It will be convenient in what follows to think of \mathfrak{M}_g as K. Saito's space

$$\mathfrak{M}_g = \text{Hom}(\pi_1, \text{Sl}_2(\mathbb{R}))^{\text{irr}} / \text{Sl}_2(\mathbb{R})$$

of Fricke moduli for spin surfaces, where π_1 is the fundamental group of a compact Riemann surface of genus $g > 1$ and the superscript irr signifies the space of irreducible representations of π_1 . \mathfrak{M}_g is a 2^{2g-1} -fold cover of the usual moduli space [64, §5.3].

Following Hubbard [35, §3] and Tjurin [73, Prop. 3.2.1] the space of projective connections \mathfrak{T}_g on (variable) Riemann surfaces is a bundle

$$\mathfrak{T}_g \longrightarrow \mathfrak{M}_g$$

of $3(g-1)$ -dimensional affine spaces. Now in general the description of the n -dimensional affine group as a semi-direct product

$$0 \longrightarrow \mathbb{C}^n \longrightarrow A(n) \longrightarrow \text{Gl}_n(\mathbb{C}) \longrightarrow 1$$

defines a map

$$j: H^1(X, A(n)) \longrightarrow H^1(X, \text{Gl}_n(\mathbb{C}))$$

which sends an affine bundle A over X to an underlying vector $j(A)$; the fiber over $j(A)$, i.e., the set of affine structure on the underlying vector bundle, is the cohomology group $H^1(X; j(A))$, cf. [73, Prop. 2.1.1]. In the case at hand, $j(\mathfrak{T}_g)$ is the cotangent bundle of \mathfrak{M}_g , and the characteristic class defining the affine structure on \mathfrak{T}_g is the Weil-Petersson class in $H^1(\mathfrak{M}_g, \Omega^1)$, i.e., the by now familiar anomaly class, cf. [61].

Hubbard has given a helpful description of the tangent space to \mathfrak{M}_g .

analogous to the usual description (in terms of Kodaira-Spencer theory) of the tangent space to \mathfrak{M}_g at C as $H^1(C, T_C)$. Briefly, if (C, T) is a surface with projective connection there is a short exact sequence

$$0 \longrightarrow \hat{A}_{C,T} \longrightarrow T_C \xrightarrow{L_T} \Omega_C^{\otimes 2} \longrightarrow 0$$

of abelian sheaves on C , where L_T sends the germ χ of a holomorphic vector field on C to the Lie derivative $\mathcal{L}_\chi(T)$ of the projective connection. [Thus $\hat{A}_{C,T}$ is a local coefficient system, with fiber isomorphic to $sl_2(\mathbb{C})$; with respect to a T -flat coordinate w ,

$$L_T(p(w)\delta_w) = p''(w)dw^{\otimes 2}.$$

so $\hat{A}_{(C,T,c)}$ consists of germs of the form $(a_0 + a_1w + a_2w^2)\delta_w$. The tangent space to \mathfrak{X}_g at (C, T) is naturally isomorphic to $H^1(C, \hat{A})$, which is a kind of Eichler cohomology group, naturally isomorphic to the first cohomology of the fundamental group of C , with coefficients in a representation of $\pi_1 C$ in $sl_2(\mathbb{C})$. There is a local analytic homeomorphism mapping \mathfrak{X}_g to the space $\text{Hom}(\pi_1, PGL_2(\mathbb{C})^{irr}/PGL_2(\mathbb{C}))$ of Kleinian groups of genus g , but its properties are not very well understood, cf. [35].

2.7 We are finally ready to introduce the last piece of technology from the theory of moduli problems. By the Krichever construction on the moduli object \mathfrak{M}_g we mean a principal bundle

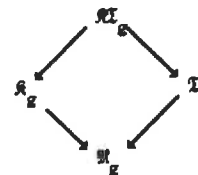
$$\mathfrak{K}_g \longrightarrow \mathfrak{M}_g$$

defined by triples (C, c, z) consisting of a spin surface C of genus g , a point c upon it, and a formal coordinate (ie. an isomorphism $\hat{\sigma}_{C,c} \longrightarrow \mathbb{C}[[z]]$) at the marked point, cf. [45], [69, §6]. The group of (complex) formal power series

$$f(z) = \sum_{i \geq 0} f_i z^{i+1}, \quad f_0 \neq 0.$$

is its structure group. [In the language of physics, the extra data in \mathfrak{K} is called 'the insertion of a coordinate' at c .]

The Krichever-Tjurin space, finally, is the fiber-product



of the Tjurin moduli space with the Krichever construction, over the moduli space \mathfrak{M}_g . It may be understood as the modular object for quadruples (C, T, c, z) , and the formula

$$(C, T, c, z) \longmapsto T(z) = \sum_{n \geq 0} t_n \gamma_n(z).$$

defines functions t_n on \mathfrak{K}_g , of weight $n + 2$ (in the sense that the coordinate transformation $z \longmapsto uz$, $u \in \mathbb{C}^*$, sends t_n to $u^{n+2} t_n$). The tangent spaces to \mathfrak{K}_g and \mathfrak{K}_g have descriptions analogous to those of \mathfrak{M}_g and \mathfrak{X}_g . In particular, the tangent space to \mathfrak{K}_g at (C, c, z) is naturally isomorphic to the cohomology group

$$\varinjlim H^1(C, \mathfrak{m}_c^n T_C)$$

of C with coefficients in the sheaf of germs of vector fields leaving c formally fixed to all orders; here \mathfrak{m}_c is the ideal in $\mathcal{O}_{C,c}$ of germs of functions vanishing at c , cf. [42, §4], [45, §2.19]. (This group can be seen to be isomorphic to the quotient

$$\mathbb{C}((z))\delta_z / H^0(T_C(*c))$$

of the formal germs of meromorphic vector fields at c , modulo global vector fields with a pole there.)

The composition

$$\Lambda \longrightarrow T_C \longrightarrow T_C/m_C^n T_C$$

defines a pro-complex of sheaves on C , for which we will use the abbreviation

$$\hat{\mathcal{C}}: 0 \longrightarrow \Lambda \longrightarrow \hat{T}_C \longrightarrow 0;$$

it is in some sense a complex of sheaves of Lie algebras. The commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \Lambda & \longrightarrow & \Lambda & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & m_C^n T_C & \longrightarrow & T_C & \longrightarrow & T_C/m_C^n T_C & \longrightarrow & 0 \end{array}$$

defines an exact sequence

$$0 \longrightarrow H^0(C, \Omega^{\otimes 2}) \longrightarrow H^0(C, \hat{\mathcal{C}}) \longrightarrow \varinjlim H^1(C, m_C^n T_C) \longrightarrow 0$$

(since the complex $0 \longrightarrow \Lambda \longrightarrow T_C \longrightarrow 0$ is quasi-isomorphic to the sheaf of quadratic differentials), and the morphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & \hat{T}_C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

of complexes yields a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(C, \Omega^{\otimes 2}) & \longrightarrow & H^0(C, \hat{\mathcal{C}}) & \longrightarrow & \varinjlim H^1(C, m_C^n T_C) & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(C, \Omega^{\otimes 2}) & \longrightarrow & H^1(C, \Lambda) & \longrightarrow & H^1(C, T_C) & \longrightarrow & 0 \end{array}$$

which exhibits $H^0(C, \hat{\mathcal{C}})$ as the fiber product of the tangent spaces $T_{\mathbb{A}^1}$ and $\hat{\mathbb{A}}_{\mathbb{A}^1}$ over \mathbb{A}^1 .

§3 Cohomology Theories

3.0 The functor \mathcal{O}_0 which assigns to the commutative ring A , the set of formal power series

$$f(z) = \sum_{i \geq 0} f_i z^{i+1} \in A[[z]]$$

with $f_0 = 1$, is an affine group-scheme, i.e., a group object in the category dual to that of commutative rings, cf. [56], [57]. Its representing object is the (dual) Landweber-Novikov algebra

$$S_* = \mathbb{Z}\langle s_1, \dots \rangle$$

of polynomials in generators s_i^* of degree $2i$. A ring homomorphism

$$f: S_* \longrightarrow A$$

corresponds to the series

$$\sum_{i \geq 0} f(s_i) z^{i+1}.$$

[The generators of S_* are traditionally denoted t_i , but in Prop. 1.3 we have used that notation for the coefficients of the stress-energy-tensor.]

Composition of formal power series in the set $\mathcal{O}_0(A)$ is represented by a ring homomorphism

$$S_* \longrightarrow S_* \otimes_{\mathbb{Z}} S_*$$

which makes S_* the (commutative, coassociative) Hopf algebra of functions on the (pro)algebraic group-scheme \mathcal{O}_0 .

This group object is very closely related to the group of diffeomorphisms of the circle; it is in fact a group of automorphisms of the formal line.

[There is an ungraded version of S_* , constructed by tensoring with the Hopf algebra $\mathbb{Z}[s_0, s_0^{-1}]$ which represents the multiplicative group, acting in S_* by sending a class s_n^I of degree $2|I|$ to $s_0^{|I|} s_n^I$; it represents the formal completion of the group of automorphisms of the line which leave the origin fixed, i.e. the group \mathcal{O} of series of the above form in which f_0 is merely required to be a unit in A .]

The group of diffeomorphisms of the circle is in a certain strong sense [61] not a (pro)algebraic group, but \mathcal{O} is nevertheless an analogue of a Bruhat subgroup (i.e. corresponding to 'upper-triangular' objects) in $\text{Diff } S^1$. It appeared in the preceding section as the structure group of the Krichever bundle

$$\mathbb{R}_G \longrightarrow \mathbb{M}_G,$$

but it also plays a role in our story as the group of multiplicative cohomology operations in complex cobordism. In this section we sketch this

interpretation, and we summarize the basic facts about elliptic cohomology as well, as a preliminary to the statement of a universal property characterizing complex cobordism in terms of chiral conformal field theories.

3.1 Just as complex K-theory is the simplest of all variants of K-theory, so complex cobordism is the (technically) most well-behaved cobordism functor. Briefly, a bordism theory is a covariant functor which assigns to a space X a set of equivalence classes $[g: M \longrightarrow X]$ of manifolds mapped to X , graded by the dimension of M ; complex cobordism is a special case in which the manifolds carry a complex structure on their stable tangent bundles, cf. [53]. Two mappings $f_i: M_i \longrightarrow X$, $i=0,1$, are equivalent, or cobordant, if

$$E_0 \cup E_1 : M_0 \cup M_1 \longrightarrow X$$

is the boundary of some

$$G: W \longrightarrow M$$

(equipped with compatible extra structure). This set of equivalence classes can be shown to be a graded group $MU_* X$, with sum defined by disjoint union; it is an analogue of the usual homology of X in which cycles are required to be smooth, not just simplicial [1], [16]. In fact $MU_*(\text{---})$ satisfies the usual axioms for a homology functor save only that the coefficient ring

$$MU_*(pt) = \mathbb{Z}[x_1, \dots]$$

is polynomial on one generator in each even dimension; it is thus much larger than the coefficient ring

$$H_*(pt) = \mathbb{Z}$$

of ordinary homology.

3.2 The functor $MU_*(\text{---})$ is representable in the homotopy category, and so for general reasons its natural endomorphisms are given by a universal coaction homomorphism

$$\psi: MU_*(\text{---}) \longrightarrow MU_*(\text{---}) \otimes_{MU_*(pt)} MU_*(\mathbb{M}U),$$

where $\mathbb{M}U$ is the spectrum representing the functor. The object on the right of the arrow can be shown to be $MU_*(\text{---}) \otimes_{\mathbb{Z}} S_*$, and it is usual to write

$$\psi(x) = \sum s_1(x) s_1^I, \quad s_1^I = s_1^{*1} \dots s_n^{*n}$$

as a sum of Landweber-Novikov operations s_1^I , with the variables s_n^I playing a bookkeeping role. [In fact the larger algebra $S_*(s_0^*, s_0^{*-1})$ acts in a natural way as well in an analogous $\mathbb{Z}/2\mathbb{Z}$ -graded context.] Alternately, one can think of (the ungraded module associated to) $MU_*(\text{---})$ as a linear representation of the group \mathcal{G} , in which $f \in \mathcal{G}_0(A)$ acts as the composition

$$\begin{array}{ccc} s_f: MU_*(\text{---}) \otimes A & \longrightarrow & MU_*(\text{---}) \otimes S_* \otimes A \\ & & \downarrow s_1 \longmapsto f_1 \\ & & MU_*(\text{---}) \otimes A \otimes A \longrightarrow MU_*(\text{---}) \otimes A \end{array}$$

For certain purposes it is convenient to use the cohomological analogue $MU^*(\text{---})$ of $MU_*(\text{---})$, in part because cartesian product of manifolds makes the former object into a (contravariant) ring-valued functor, which admits an analogous \mathcal{G} -action, now by (ungraded) ring automorphisms.

3.3 This action is highly nontrivial on the coefficient ring $MU_*(pt)$; in fact this representation is the restriction to \mathcal{G} of the metaplectic representation of the Virasoro algebra studied in [66] (cf. also [57]), and will be quite important in what follows.

To describe it more concretely we need some ideas of Quillen. The first point is that our group acts on the cobordism ring $MU^*CP(m)$ of complex projective space (of infinite dimension). This space classifies complex line bundles, and is thus an H-space (with composition-law derived from the tensor product of line bundles); its cobordism ring is (topologically) free on the Chern-Euler class of the universal line bundle, so

$$MU^*CP(m) \cong MU^*(pt)[[e]].$$

Expressing the characteristic class

$$e(L_1 \otimes L_2) = \sum_{i,j \geq 1} a_{ij} e(L_1)^i e(L_2)^j = F(e(L_1), e(L_2))$$

of a product bundle defines a (completed) Hopf algebra structure

$$\Delta: MU^*CP(m) \longrightarrow MU^*CP(m) \hat{\otimes}_{MU^*(pt)} MU^*CP(m)$$

on $MU^*CP(m)$, by $\Delta(e) = F(e \otimes 1, 1 \otimes e)$. Quillen shows that this group-law is in a certain sense universal; in combination with results of Lazard it is a consequence that the ring $MU^*(pt)$ is polynomial, as described above. It had been shown earlier by Miščenko that

$$F(z_1, z_2) = \log_U^{-1}(\log_U(z_1) + \log_U(z_2)),$$

where

$$\log_U(z) = \sum_{n \geq 1} \frac{CP(n-1)}{n} z^n;$$

for our purposes the associated (translation-invariant) differential

$$d \log_U(z) = \sum_{n \geq 0} CP(n) z^n dz \in \Omega_{MU^*(pt)}^1 MU^*CP(m)$$

will be equally important.

In these terms Quillen's result on the action of \mathcal{G}_0 on $MU^*(pt)$ can be stated as the formula

$$s_f(d \log_U(z)) = d \log_U(f(z)).$$

[The classes $\mathbb{C}P(n)$ are polynomial generators for $MU^*(pt) \otimes \mathbb{Q}$, so this equation determines the action on integral generators as well.

It is worth mentioning that the Lie algebra of \mathcal{O} is generated by the elements

$$v_k = z^{k+1} \frac{d}{dz}, \quad k \geq 0;$$

it is not hard to calculate from Quillen's formula that v_k acts on $\mathbb{C}P(n-1)$ like z^{-n} in the Fock representation of the Virasoro algebra, i.e.

$$v_k \mathbb{C}P(n-1) = -n \mathbb{C}P(n-k-1).$$

3.4 Elliptic cohomology is most easily characterized in terms of complex cobordism. The key fact from the classical theory of elliptic functions is that the Jacobi quartic

$$y^2 = 1 - 2\delta x^2 + \epsilon x^4 = R(x)$$

defines a group variety in the projective plane, with $[0,1,0]$ as the identity element, when the discriminant

$$\Delta = \epsilon(\delta^2 - \epsilon)^2 \neq 0.$$

There is a natural flat coordinate on (the universal cover of) this curve, such that

$$X = \operatorname{sn} z, \quad Y = \operatorname{sn}' z;$$

the classical addition law

$$\operatorname{sn}(z_1 + z_2) = \frac{\operatorname{sn}(z_1)\operatorname{sn}'(z_2) + \operatorname{sn}(z_2)\operatorname{sn}'(z_1)}{1 - \epsilon \operatorname{sn}^2(z_1)\operatorname{sn}^2(z_2)}$$

(cf. [52]) allows us to express the sum of $X_1 = \operatorname{sn}(z_1)$ and $X_2 = \operatorname{sn}(z_2)$ near the origin by Euler's formula

$$X_1, X_2 \mapsto E(X_1, X_2) = \frac{X_1 R^{\frac{1}{2}}(X_2) + X_2 R^{\frac{1}{2}}(X_1)}{1 - \epsilon X_1^2 X_2^2} \in \mathbb{Z}[\frac{1}{2}][\epsilon, \delta][[X_1, X_2]].$$

According to the theorem of Quillen and Lazard, there is thus a unique ring homomorphism

$$MU^*(pt) \longrightarrow \mathbb{Z}[\frac{1}{2}][\epsilon, \delta][\Delta^{-1}] = \operatorname{Ell}_\epsilon$$

mapping the coefficients of the universal group law F to those of E , and elliptic cohomology is the quotient of complex cobordism defined by

$$\operatorname{Ell}^*(X) = MU^*(X) \otimes_{MU^*(pt)} \operatorname{Ell}_\epsilon.$$

In this construction the invariant differential of §3.3 is mapped to the classical invariant differential

$$dz = \frac{dX}{Y} = R(X)^{-\frac{1}{2}} dX = \sum_{n \geq 0} P_n(\delta \epsilon^{-\frac{1}{2}}) \epsilon^{\frac{n}{2}} X^{2n} dX,$$

P_n denoting the n^{th} Legendre polynomial.

[That this functor is a cohomology theory is a very nontrivial consequence of Landweber's exact functor theorem [50]; for $\operatorname{Ell}_\epsilon$ is certainly not flat over $MU^*(pt)$. This result is analogous to the theorem of Conner and Floyd [18], that

$$K^*(X) = MU^*(X) \otimes_{MU^*(pt)} K^*(pt)$$

(where $K^*(pt)$ is a (highly non-flat) MU^* -module by means of the Todd genus homomorphism

$$MU^*(pt) \longrightarrow K^*(pt) = \mathbb{Z}[b, b^{-1}]$$

which maps $\mathbb{C}P(n)$ to b^n). In general, it is a consequence of the theory of one-dimensional formal group laws that (for any prime p) if

$$[p](z) = z +_p \dots +_p z \in MU^*(pt)[[z]]$$

denotes the p -fold sum with respect to the universal group law F , and if u_n is

the coefficient of z^{p^0} in $[p](z)$, then a ring homomorphism

$$\rho: MU^*(pt) \longrightarrow E^*$$

defines a cohomology functor by

$$E^*(X) = MU^*(X) \otimes_{MU} E^*$$

provided that the sequence

$$p, \rho(u_1), \rho(u_2), \dots \in E^*$$

is regular (i.e. $\rho(u_n)$ is not a zero-divisor in $E^*/(p, \rho(u_1), \dots, \rho(u_{n-1}))$). In the case of elliptic cohomology, the crucial fact (cf. [53]) is that when p is odd,

$$u_2 \equiv (-1)^{\frac{1}{2}(p-1)} c^{\frac{1}{2}(p^2-1)}$$

mod (p, u_1) .

A similar application of the exact functor shows that

$$Ell^*(X) \otimes_{Ell^*(pt)} \mathbb{Z}[\frac{1}{2}][[q]]$$

is a cohomology functor, where

$$Ell^*(pt) \longrightarrow \mathbb{Z}[\frac{1}{2}][[q]]$$

embeds the ring of Jacobi modular forms in the ring of formal characters of the circle, as $\Gamma_0(2)$ -equivariant forms on the upper half-plane. This $Ell^*(pt)$ can be interpreted as a ring of characters of Segal's complexification of $Diff S^1$ [68, §2.5]. In this case, the key congruence is

$$u_1 \equiv 2^{-(p-1)}$$

mod $(q) \subset \mathbb{Z}[\frac{1}{2}][[q]]$, cf. [79, Prop. 19].

Note finally that if every prime p is invertible in E^* , i.e. if E^* is a \mathbb{Q} -algebra, then the hypotheses of the exact functor theorem are (rather trivially) valid. This is a reformulation of an old result of Dold, that over the rationals all cohomology functors are scaled-up versions of ordinary cohomology. This corollary is the only version of the exact functor theorem that we will use in the arguments of this paper.

3.5 It is reasonable to interpret elliptic cohomology as a functor taking values in the category of coherent sheaves over the moduli scheme of Jacobi quartics, but it is also somewhat unnatural, in that the curves come with a fixed parameterization. One way to loosen this restriction is to look at the Krichever construction on $\text{Proj } Ell_*$, i.e. the space of data consisting of a quartic together with a coordinatized point upon it.

The Boardman-Hurewicz homomorphism

$$\begin{aligned} Ell_*(X) = \pi_*(X \wedge \mathcal{C}(I)) &\longrightarrow \pi_*(X \wedge \mathbb{R}P^1 \wedge \mathcal{C}(I)) \\ &\cong Ell_*(X \wedge \mathbb{R}P^1) \\ &\cong Ell_*(X) \otimes_{\mathbb{Z}} S_* \end{aligned}$$

represents the morphism

$$\text{Proj } Ell_* \otimes S_* \longrightarrow \text{Proj } Ell_*$$

defining this Krichever bundle, when X is a point; a ring homomorphism

$$Ell_* \otimes S_* \longrightarrow A$$

defines an elliptic curve over A , with modular invariant

$$j = 2^{-4} \frac{(1 + \frac{1}{3}\gamma)^3}{(1 - \gamma)^2}, \quad \gamma = e^{-1} \delta^2,$$

together with a parameter δ at the marked point. [cf. also [55], [23]].

In general $\text{Ell}_*(X \wedge \mathbb{R}P^1)$ defines a sheaf over this Krichever bundle: it is just the orbit (under \mathcal{C}_0) of the subspace of Jacobi curves in $\text{Proj } \text{MU}^*$.

3.6 The image of a complex-oriented manifold in $\text{Ell}^*(pt)$ is its elliptic genus; e.g. the elliptic genus of quaternionic projective n -space in ϵ^n . If M possesses in addition a spin structure, and its first Pontryagin class vanishes, then Witten [77], Alvarez, Windey [3] and others have shown how to construct a Dirac operator on the space of free loops in M , and Bott and Taubes [10] have proved Witten's conjecture that the equivariant index of this operator [cf. [54]] is in fact the modular form defined by its elliptic genus.

In this way an analytic and a topological index are seen to be the same. It is now a consequence of the topological Riemannian-Roch theorem [22, ch. I §D] of extraordinary cohomology theory, that this topological index can be calculated from ordinary cohomological invariants of M , by a formula involving a Hirzebruch multiplicative genus.

Indeed, if E^* is a torsion-free $\text{MU}^*(pt)$ -algebra satisfying the hypotheses of Landweber's Theorem, then the E^* -genus of M can be defined as the image of $1 \in E^0$ under the Gysin homomorphism $r_M^!$ associated to the projection

$$r_M: M \longrightarrow pt.$$

[The classical examples of this construction are the Todd, \hat{A} , and L-genus, to which we can now add the elliptic genus and its generalizations.] This invariant can be calculated by means of the Hirzebruch genus

$$\frac{x}{\exp_E(x)}$$

(where $\exp_E(x)$ is the power series inverse to

$$\log_E(X) = \sum_{n \geq 1} p_E(\mathbb{C}P^{(n-1)}) \frac{x^n}{n}$$

as

$$E\text{-genus}(M) = \langle \prod \frac{t_i}{\exp_E(t_i)}, [M] \rangle,$$

where the Chern classes of M are the elementary symmetric functions of the roots t_i . Thus $\exp_E(x) = 1 - e^{-x}$ when $E^* = K^*(pt)$; similarly, the \hat{A} -genus is defined by the hyperbolic sine, the L-genus is defined by the tangent function, and the elliptic genus by the Jacobi sn-function, motivating the examples in §1.5. The examples in §1.6 will be discussed in the next section.

§4 Results, and further questions

4.0 In section two of this paper we constructed for each integer $i \geq 0$ a (modular) function t_i , of weight $i+2$, on the Krichever-Saito-Tjurin object \mathcal{R}_g ; on the other hand in section one (prop. 1.3) we defined (for each integer $k \geq 2$) certain homogeneous polynomials λ_k in the t_i 's. In terms of these constructions we write

$$\lambda_T(z) = \sum_{n \geq 0} \lambda_n \gamma_{n+1}(z) \in \mathbb{Q}[t_i | i \geq 0][[\lambda_1]][[z]]$$

(with the convention that $\lambda_0 = 1$) and we define the formal group law

$$F_T(z_0, z_1) = \lambda_T^{-1}(\lambda_T(z_0) + \lambda_T(z_1)) \in \mathbb{Q}[t_i | i \geq 0][[\lambda_1]][[z_1, z_2]].$$

Note that solutions of Hill's equation possess a kind of

translation-invariance (with respect to the operation

$$\phi(x) \mapsto T_y \phi(x) = \left(\frac{\partial F(x,y)}{\partial x} \right)^{-1/2} \phi(F(x,y))$$

analogous to that studied in [13, §4] and [20].

Our principal result is as follows.

4.1 Proposition The classifying homomorphism

$$F_T: \text{MU}^*(\text{pt}) \longrightarrow \mathbb{Q}[t_i | i \geq 0][\lambda_1]$$

of this formal group law defines a \mathbb{C} -equivariant morphism

$$\mathcal{R}\mathbb{I} \longrightarrow \text{Proj MU}^*(\text{pt})$$

of schemes.

Proof: The element $f \in \mathbb{C}$ acts on the space of group laws, sending F to

$$F^f(z_1, z_2) = f^{-1}(F(f(z_1), f(z_2)));$$

while it acts on the KT-construction, sending the quadruple (C, T, c, z) to $(C, T, c, f(z))$. The action of f on the coefficients t_i of T is thus such that

$$\sum_{n \geq 0} \circlearrowleft_f(t_n) \tau_n(z) = T(f(z)) f'(z)^2 + S(f(z), z);$$

but, from proposition 1.2 it follows that

$$\circlearrowleft_f(\phi(z)) = f'(z)^{-1/2} \phi(f(z)),$$

and it is a consequence of proposition 1.4 that

$$\begin{aligned} \circlearrowleft_f(d\lambda_T(z)) &= \circlearrowleft_f(\phi(z)^{-2} dz) \\ &= \phi(f(z)) df(z) \\ &= d\lambda_T(f(z)). \end{aligned}$$

The assertion is thus a consequence of Quillen's theorem, as stated above in section 3.3. ■

4.2 Proposition. The quadratic differential

$$\begin{aligned} \nu &= \frac{1}{4} \left(\sum_{n \geq 0} \mathbb{C}P(n) z^n \right)^{-2} \sum_{m, n \geq 1} c_{m,n} \mathbb{C}P(m-1) \mathbb{C}P(n-1) z^{m+n-4} dz^{\otimes 2} \\ &= \frac{1}{4} [(4\mathbb{C}P(2) - 3\mathbb{C}P(1)^2) \\ &\quad + (12\mathbb{C}P(3) - 16\mathbb{C}P(2)\mathbb{C}P(1) + 6\mathbb{C}P(1)^3)z + \dots] \end{aligned}$$

$$\in \Omega_{\text{MU}^*(\text{pt})}^{\otimes 2} \text{MU}^*(\mathbb{C}P(m))$$

of §1.6.4 satisfies the identity

$$\circlearrowleft_f(\nu(z)) = \nu(f(z)) + S(f(z), z) dz^{\otimes 2}.$$

Proof. We have

$$(\nu) = S(\log_U(z), z) dz^{\otimes 2},$$

while

$$\circlearrowleft_f(d\log_U(z)) = d\log_U(f(z))$$

by Quillen; so the assertion follows from the composition law stated in §1.1. ■

4.3. This differential has a universal property analogous to the universal property of $d\log_U$, at least over the rational numbers.

To be more precise, a chiral holomorphic conformal field theory of central charge one defines a section T of the Tjurin bundle of projective connections; pulling this back over the Krichever construction, we define a

morphism

$$\mathbb{A}_g \xrightarrow{T} \mathbb{A}_g^* \longrightarrow \text{Proj } \text{MU}^*(\text{pt}) ;$$

by proposition 4.1. In dual terms, to an affine Spec A in \mathbb{A}_g we have associated a ring homomorphism

$$\text{MU}^*(\text{pt}) \longrightarrow \mathbb{Q}[t_1 | t_2 \geq 0][\lambda_1] \longrightarrow A \otimes S_g .$$

in which the second arrow expresses the stress-energy-tensor of the chiral theory in terms of the ring A of modular functions.

We can summarize this as

Corollary The projective connection $T(z)dz^{\otimes 2}$ over $A \otimes S_g$ is the pullback $T^*(v(z))$ of the universal quadratic differential defined over $\text{MU}^*(\text{pt})$. The associated cohomology functor $H^*(-, A \otimes S_g)$ is the sheafification of $\text{MU}^*(-)$ over the orbit of Spec A, and

$$\frac{x}{\lambda^{-1}(x)}$$

is the associated Hirzebruch genus. ■

4.4 Although we have emphasized closed surfaces in our discussion, there seems to be no special reason to limit ourselves to this case. One of the most interesting examples is the thrice-punctured Riemann sphere; Ihara's quadratic differential, as discussed in §1.8.1, corresponds to $\mathbb{P}_1 - (c,1,\infty)$ with a point inserted at 0; c is the cross-ratio $[c:1:0:\infty]$. A conformal field theory associated to this surface defines a linear transformation

$$H^{\otimes 2} \longrightarrow H^{\otimes 2}$$

with trace the analogue in this situation of the elliptic genus. The cohomological invariant

$$\langle \mathbb{H} \frac{t_1}{\lambda^{-1}(t_1)}, [M] \rangle$$

where now

$$d\lambda(x) = w^{\rho_0 - 1} (1-w)^{\rho_1} F(A,B,C,w)^{-2} dw$$

with $w = (1-c)^{-1}(x-c)$, is the topological candidate for its partition function. In the elliptic case, integrality of the genus corresponds to modularity with respect to q in the present case, integrality would correspond to rationality with respect to c .

This example bears upon the very interesting question of the integrality of the generalized elliptic genus. Although there is a well-developed theory of second-order linear algebraic differential equations (cf. [21],[34],[39]), it seems to be difficult to say very much about general integrality properties of solutions to the Fuchsian equation

$$\phi'' + T\phi = 0$$

under hypotheses of integrality on T and it is even harder to deduce the integrality of F. [A solution ϕ will have a nontrivial radius of p-adic convergence provided that T does, but this radius is not easy to predict, cf. [15].] Thus, for example, we know that the differential $d\lambda$ is suitably integral in the hypergeometric (cf. [44]) and hyperelliptic (cf. [36],[72]) cases, but this does not seem to be enough to prove that the corresponding group law is integral. We are therefore in no position to say much of interest about the finer exactness properties (ie. connected with Landweber's theorem) of the cohomology sheaves

$$H^*(-, A \otimes S_g)$$

over the Krichever construction \mathbb{A} over \mathbb{H} . [It is elementary, however, to

calculate that the derivative of $\lambda'(z)$ in the direction of the quadratic differential $q(z)dz^{\otimes 2}$ is

$$2\lambda'(z)^{-1} \int_0^z \lambda'(v) \left[\int_0^v q(u) \lambda'(u)^{-1} du \right] dv$$

and that the p-series has derivative

$$\partial_q([p](z)) = \lambda'([p](z))^{-1} \left[p(\partial_q \lambda(z)) - \partial_q \lambda([p](z)) \right].$$

However, we should note that Katsura, Shimizu and Ueno [43, §3] have studied the formal group defined over the integers by a Cartier curve in the Jacobian of a Riemann surface defined over \mathbb{Z} ; if we think of the uniformization of a surface C defined by a projective connection as a coordinatization of its universal cover \tilde{C} , then the composition

$$G_n \cong \tilde{C} \longrightarrow C \longrightarrow \text{Jacobian}$$

defines just such a Cartier curve. It is natural to expect this to generalize to curves with punctures, using the generalized Jacobians of [70].

4.5 We close with a remark about the special role played by the variable λ_1 in proposition 4.1. The Schwarz-Fuchs equation

$$S(\lambda(z), z) = T(z)$$

is of the third order, but the natural boundary conditions

$$\lambda(0) = 0, \lambda'(0) = 1$$

leave the coefficient λ_1 undetermined. Now in §3.3 we mentioned that $MU^*(pt) \otimes \mathbb{Q}$ carries a Virasoro algebra action extending that of the Landweber-Novikov algebra; in [66, §6] Segal decomposed this representation into irreducible components P_q , $q = 0, 1, \dots$, and showed that the Poincaré series of P_0 is $\prod_{n>1} (1-t^n)^{-1}$. It seems reasonable to expect that the algebra

$\mathbb{Q}[t_1 | 1 \geq 0]$ is precisely the representation P_0 .

In helpful conversation, I.M. Singer (cf. [4]) pointed out that (from the point of view of Calabi-Yau compactification) it is SU-cobordism (cf. [17], [60]) which appears most naturally in conformal field theory; this suggests a natural interpretation of $MSU^*(pt) \otimes \mathbb{Q}$ as P_0 .

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