CHAPTER XII CLASS GROUPS

Modern Classical Algebra

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XII CLASS GROUPS

1. The Grothendieck Group

Consider an additive category $\mathcal{E}$ of finitely generated left Λ-modules over the ring Λ which satisfies the additional condition:

(1) with each two modules A and B in $\mathcal{E}$, $A \oplus B$ is also in $\mathcal{E}$; and so are the maps $A \longrightarrow A \oplus B$, $A \oplus B \longrightarrow A$ and $A \oplus B \longrightarrow B \oplus A$.

The isomorphisms in $\mathcal{E}$ establish an equivalence relation between the modules in $\mathcal{E}$. We assume now that also the following condition holds:

(2) The category $\bar{\mathcal{E}}$ of isomorphism classes of modules in $\mathcal{E}$ defined by this equivalence relation is a set.

The last condition is for instance always fulfilled if all isomorphisms between modules in $\mathcal{E}$ belong to $\mathcal{E}$.

If we do not specify the morphisms for a category of modules, we mean that all the morphisms between modules of $\mathcal{E}$ belong to $\mathcal{E}$. Let $F\bar{\mathcal{E}}$ be the free abelian group generated by $\bar{\mathcal{E}}$ over the integers, and I the ideal in $F\bar{\mathcal{E}}$ generated by the elements $[A] - [B] + [C]$ where $[A],[B],[C] \in \bar{\mathcal{E}}$ and A, B and C are related by the exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

with morphisms f,g belonging to $\mathcal{E}$. The group

$$K(\mathcal{E}) = F\bar{\mathcal{E}}/I$$

is the Grothendieck group associated to the category $\mathcal{C}$.

Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are additive categories satisfying (1,2). We do not assume that they are defined over the same ring. Consider an additive covariant functor T from $\mathcal{C}_1$ into $\mathcal{C}_2$. Recall that if A and B are modules in $\mathcal{C}_1$, then

$$TA \oplus B = TA \oplus TB \quad . \quad \text{(chapter VIII, proposition 5.5).}$$

T is an exact functor if

$$0 \longrightarrow TA \xrightarrow{Tf} TB \xrightarrow{Tg} TC \longrightarrow 0$$

is exact whenever

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact in $\mathcal{C}_1$. In this case T induces a morphism $K(\mathcal{C}_1) \longrightarrow K(\mathcal{C}_2)$ which we will denote also by T.

Let us consider an example. First note that whenever $\mathcal{C}_1$ and $\mathcal{C}_2$ are additive categories over the same ring such that $\mathcal{C}_1 \subset \mathcal{C}_2$, then we have an obvious morphism

$$K(\mathcal{C}_1) \longrightarrow K(\mathcal{C}_2) \quad .$$

Now suppose Λ and Γ are rings, and $\varphi : \Lambda \longrightarrow \Gamma$ is a morphism of rings. φ associates to each left Γ-module in a canonical way a left Λ-module by "restriction" of the operation of Γ to the operation of Λ.

(φ^*) If Γ is finitely generated as a left Λ-module, then φ induces an exact functor

$$\varphi^* : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$$

whenever $\mathcal{C}_1$ is an additive category over Γ and $\mathcal{C}_2$ an additive category over Λ containing $\mathcal{C}_1$ (modulo an identification via the morphism φ). Thus φ induces a morphism of groups

$$\varphi^*: K(\mathcal{C}_1) \longrightarrow K(\mathcal{C}_2) .$$

(φ_*) Let $\mathcal{C}_1$ be an additive category over Γ and $\mathcal{C}_2$ an additive category over Λ. Suppose that either Γ is flat as a left Λ-module or that each module in $\mathcal{C}_2$ is flat over Λ. If $\Gamma \otimes_\Lambda A$ is in $\mathcal{C}_1$ for each module A in $\mathcal{C}_2$, and $1_\Gamma \otimes_\Lambda f$ is in $\mathcal{C}_1$ for each morphism f in $\mathcal{C}_2$, then φ induces an exact functor

$$\varphi_* : \mathcal{C}_2 \longrightarrow \mathcal{C}_1$$

$$A \longrightarrow \Gamma \otimes_\Lambda A ,$$

and thus a morphism of groups

$$\varphi_* : K(\mathcal{C}_2) \longrightarrow K(\mathcal{C}_1) .$$

2. The Ideal and Projective Class Groups

Throughout this section R is a commutative ring.

Let $\mathcal{P}$ be the category of all finitely generated projective modules over R. In $\mathcal{P}$ we define an equivalence relation: two projective modules P and P' in $\mathcal{P}$ are equivalent if there are finitely generated free modules F and F' such that

$$P \oplus F \cong P' \oplus F' \quad .$$

The set of equivalence classes in $\mathcal{P}$ is a group PCG(R) , the projective class group of the ring R. The zero element is [F] where F is free; the inverse of [P] is [P'] where P' is a module in $\mathcal{P}$ such that $P \oplus P'$ is free.

The coherent projective class group CPCG(R) is defined in a similar way taking for $\mathcal{P}$ the additive class of all finitely generated coherent projective modules [chapter 5, § 5].

Closely related to the coherent projective class group is the ideal class group of R. In the class $\mathcal{P}_o$ of all coherent projective modules of rank 1 (which are finitely generated [chapter 5, prop 5,12]) we consider the equivalence relation defined by isomorphisms; let ICG(R) be the set of equivalence classes of isomorphic modules in $\mathcal{P}_o$. Define multiplication in ICG(R) by

$$[I]\cdot[J] = [I \otimes_R J] \quad .$$

[R] is clearly an identity in ICG(R). We show now that ICG(R) is in fact a group.

Recall that a module A over R is coherent projective of rank 1 if and only if the canonical morphism

$$\mathrm{Hom}_R(A,R) \otimes_R A \longrightarrow R$$

is an isomorphism. This shows that $\mathrm{Hom}_R(I,R)$ is in $\mathcal{P}_o$ whenever I is, and $[\mathrm{Hom}_R(I,R)]$ is an inverse for [I].

Now let us look at the special case where R is an integral domain. We denote by K the field of fractions of R. If I is in $\mathcal{P}_o$ then $K \otimes_R I$ is K-free with one generator, hence

$$K \otimes_R I \cong K \quad .$$

Therefore we can consider I as a R-submodule of K. Since I is finitely generated over R, there is a non-zero element $r \in R$ such that $r \cdot I \subset R$, i.e. I is isomorphic to an ideal. On the other hand, each non-zero R-module of K is clearly of rank 1. Since R is a coherent ring [chap. 7, proposition 2.3], every projective R-submodule of K is coherent and finitely generated, hence a fractional ideal (which is by definition a R-submodule I of K such that $r \cdot I \subset R$ for some non-zero element r of R). Therefore we may assume that $\mathcal{P}_o$ is the class of non-zero projective R-submodules of K or, what amounts to the same, the class of non-zero projective fractional ideals of R.

It is immediate that every morphism $I \longrightarrow K (I \in \mathcal{P}_o)$ is multiplication with an element k of K. Thus two fractional ideals $I, J \in \mathcal{P}_o$ are isomorphic if and only if there is $k \in K$ such that

$$I = k \cdot J \quad .$$

This shows also that

$$\mathrm{Hom}_R(I,R) = I^{-1} = \{k \in K \mid k \cdot I \subset R\}.$$

Since clearly $I \cdot J = I \otimes_R J$, we have the following result: $\mathcal{P}_o$ becomes a group if we define multiplication by $I \cdot J$ for $I, J \in \mathcal{P}_o$. Let $\mathcal{J}_o$ be the subgroup of non-zero principal fractional ideals of R. We have a canonical isomorphism of groups

$$ICG(R) = \mathcal{P}_o / \mathcal{J}_o \ .$$

In general we have always an epimorphism

$$i : CPCG(R) \longrightarrow ICG(R)$$

defined as follows. Let $p \in CPCG(R)$ and pick $P \in p$ with rank $P = n$, and define $ip = E(P)_n$ where $E(P)_n$ is the homogeneous part of degree n of the exterior algebra $E(P)$ of P. Recall that if A is a module of rank n , and B is a module of rank m , then $E(A \oplus B)_{n+m} = E(A)_n \otimes_R E(B)_m$. This shows that i is well defined and that it is a morphism of groups. (Remember that $E(F)_n = R$ for a free module F of rank n). In particular $CPCG(R) = PCG(R)$ whenever R is a coherent ring; so in this case we have an epimorphism

$$i : PCG(R) \longrightarrow ICG(R) \ .$$

<u>Proposition 2.1</u> Suppose R is a Dedekind domain; then

$$i : PCG(R) \longrightarrow ICG(R)$$

is an isomorphism.

This is immediate from the fact that a projective module of rank n is isomorphic to a module of the form $F \oplus I$ where F is free of rank $n - 1$ and I an ideal which is uniquely determined up to isomorphism (by proposition 5.6 of chap. VII).

Let $\mathcal{P}$ be the category of all coherent projective modules of finite rank over the commutative ring R. Define a map

$$\varphi : \mathcal{P} \longrightarrow Z, \quad \varphi P = \text{rank } P$$

where Z are the integers. If $P, Q \in \mathcal{P}$ have rank n and m respectively, then $E(P \oplus Q)_{m+n} = E(P)_n \otimes_R E(Q)_m \neq 0$ [chap. 5, prop 5.5]; hence φ is additive. Therefore φ passes to a morphism of groups

$$\varphi : K(\mathcal{P}) \longrightarrow Z$$

(every short exact sequence in $\mathcal{P}$ being split exact). On the other hand there is an obvious morphism

$$\eta : K(\mathcal{P}) \longrightarrow CPCG(R)$$

sending the class of $P \in \mathcal{P}$ in $K(\mathcal{P})$ into the class in $CPCG(R)$ determined by P.

<u>Proposition 2.2</u>. The morphism

$$\nu = \varphi \times \eta : K(\mathcal{P}) \longrightarrow Z \times CPCG(R)$$

is an isomorphism.

We define an inverse $\sigma : Z \times CPCG(R) \longrightarrow K(\mathcal{P})$ by setting

$$\sigma(n, [P]) = [F_n] + [P] - [F_{\text{rank } P}] \ ;$$

F_k denotes a free module of rank k. Suppose $P, Q \in [P] \in CPCG(R)$; by definition there are free modules F, G in $\mathcal{P}$ such that $P \oplus F \cong Q \oplus G$.

Now in $K(\mathcal{P})$ we have

$$[P] - [F_{\text{rank } P}] - [Q] + [F_{\text{rank } Q}]$$
$$= [P] - [F_{\text{rank } P}] - [Q \oplus G] + [G] + [F_{\text{rank } Q}]$$
$$= [P] - [F_{\text{rank } P}] - [P \oplus F] + [G] + [F_{\text{rank } Q}]$$
$$= [G \oplus F_{\text{rank } Q}] - [F \oplus F_{\text{rank } P}] = 0 \quad ,$$

since a consideration of ranks shows $G \oplus F_{\text{rank } Q} \cong F \oplus F_{\text{rank } P}$.

3. A Relation between Grothendieck Groups over the same Ring

Consider two additive categories $\mathcal{A}$ and $\mathcal{B}$ of left Λ-modules, Λ being a ring, and suppose that $\mathcal{B} \subset \mathcal{A}$. If

$$(+) \qquad 0 \longrightarrow B_n \xrightarrow{\varphi_n} \cdots \longrightarrow B_1 \xrightarrow{\varphi_1} A \longrightarrow 0$$

is an (exact) resolution for $A \in \mathcal{A}$ with modules B_i and morphisms φ_i in $\mathcal{B}$, then we can form another resolution

$$0 \longrightarrow B_n \longrightarrow \cdots \longrightarrow B_{i+1} \xrightarrow{(\varphi_{i-1},0)} B_i \oplus C \xrightarrow{\varphi_i \oplus 1_C} B_{i-1} \oplus C$$
$$\xrightarrow{(\varphi_{i-1},0)} B_{i-2} \cdots \longrightarrow B_1 \longrightarrow A \longrightarrow 0$$

for each module $C \in \mathcal{B}$. Any resolution which can be obtained by a finite number of such steps is called a modification of (+) modulo $\mathcal{B}$.

For simplicity we assume now that $\mathcal{A}$ containes all the morphisms between modules in $\mathcal{A}$, and all their kernels. The category $\mathcal{A}$

is said to be $\mathcal{B}$-resolutive if there is an integer m, $0 < m \leqq \infty$ such that every module $A \in \mathcal{A}$ has a resolution (+) with modules $B_i \in \mathcal{B}$ satisfying

(1) $n \leqq m$; $n < \infty$

(2) B_i is projective for $0 < i < m$.

<u>Lemma 3.1</u>

Any two resolutions of $A \in \mathcal{A}$ in $\mathcal{B}$ satisfying (1) and (2) have isomorphic modifications modulo $\mathcal{B}$ which again satisfy (1) and (2) (isomorphic as graded modules).

Let (B_i) and (B'_j) be two such resolutions of A in $\mathcal{B}$. We proceed by induction on m. The lemma being trivial for $m = 1$, we suppose that $m > 1$ and that the lemma holds for resolutions of length $< m$. Since B_1 and B'_1 are projective, there are morphisms $\tau_1 : B_1 \longrightarrow B'_1$ and $\tau'_1 : B'_1 \longrightarrow B_1$ such that the diagram

$$\begin{array}{ccccc} B_1 & \xrightarrow{\varphi_1} & A & \longrightarrow & 0 \\ \tau_1 \downarrow \uparrow \tau'_1 & & \downarrow \sigma_0 & & \\ B'_1 & \xrightarrow[\varphi'_1]{} & A & \longrightarrow & 0 \end{array}$$

is commutative: $\sigma_0 = 1_A$ is the identity. Our first modification is

$$0 \longrightarrow B_n \cdots \xrightarrow{(\varphi_3,0)} B_2 \oplus B'_1 \xrightarrow{\varphi_2 \oplus 1_{B'_1}} B_1 \oplus B'_1 \xrightarrow{(\varphi_1,0)} A \longrightarrow 0$$

$$0 \longrightarrow B'_n \cdots \xrightarrow[(\varphi'_3,0)]{} B'_2 \oplus B_1 \xrightarrow[\varphi'_2 \oplus 1_{B_1}]{} B'_1 \oplus B_1 \xrightarrow[(\varphi'_1,0)]{} A \longrightarrow 0$$

and we define σ_1 by the matrix

$$\begin{pmatrix} \tau_1 & 1-\tau_1 \circ \tau_1' \\ 1 & -\tau_1' \end{pmatrix} .$$

It is easily checked that σ_1 is an isomorphism and makes the diagram commutative. Now apply the induction hypothesis to the isomorphic modules $\ker(\varphi_1, 0)$ and $\ker(\varphi_1', 0)$ and their resolutions derived from the above diagram.

Lemma 3.2 Let

$$(*) \qquad 0 \longrightarrow A_1 \xrightarrow{i} A_2 \xrightarrow{j} A_3 \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$, and suppose that A_1 and A_3 have resulutions $B_1^1 = (B_k^1)$ and $B^3 = (B_k^3)$ in $\mathcal{B}$ respectively satisfying (1) and (2). Then A_2 has a resulution $B^2 = (B_k^2)$ in $\mathcal{B}$ satisfying (1) and (2); moreover we have an exact sequence

$$0 \longrightarrow B^1 \longrightarrow B^2 \longrightarrow B^3 \longrightarrow 0$$

of graded modules lying over $(*)$.

Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_1^1 & \xrightarrow{i_1} & B_1^2 & \xrightarrow{j_1} & B_1^3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_1^1} & & \downarrow{\scriptstyle \varphi_1^2} & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle \varphi_1^3} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{i} & A_2 & \xrightarrow{j} & A_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We define $B_k^2 = B_k^1 \oplus B_k^3$, $i_k : B_k^1 \longrightarrow B_k^2$ and $j_k : B_k^2 \longrightarrow B_k^3$ being the canonical morphisms. If $m \geqq 1$, B_1^3 is projective; therefore there is a morphism g such that $j \circ g = \varphi_1^3$. Define $\varphi_1^2 = i \circ \varphi_1^1 \oplus g$. Clearly the diagram commutes and $0 \longrightarrow \ker(\varphi_1^1) \longrightarrow \ker(\varphi_1^2) \longrightarrow \ker(\varphi_1^3) \longrightarrow 0$ is exact. Note that $\ker(\varphi_1^1)$ and $\ker(\varphi_1^3)$ have resolutions of length $< m$ and that the lemma is trivial for $m = 1$; thus we can proceed by induction on m.

<u>Proposition 3.3</u> Assume the additive category $\mathcal{A}$ containes all the morphisms between modules in $\mathcal{A}$, and all their kernels. Let $\mathcal{B}$ be an additive category contained in $\mathcal{A}$. If $\mathcal{A}$ is $\mathcal{B}$ -resolutive then the canonical morphism

$$\varphi : K(\mathcal{B}) \longrightarrow K(\mathcal{A})$$

(sending the class of $A \in \mathcal{B}$ in $K(\mathcal{B})$ into the class of A in $K(\mathcal{A})$) is an isomorphism.

We define an inverse $\eta : K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ in the following way:

Let $A \in \mathcal{A}$ and (B_i) be a resolution for A in $\mathcal{B}$ satisfying (1) and (2): then define

$$\eta[A] = \sum_i (-1)^i [B_i].$$

Lemma 3.1 shows that $\eta[A]$ does not depend on the choice of the resolution (B_i) ; Lemma 3.2 shows that $\eta[A]$ is independent of the choice of the representative $A \in [A]$. Clearly η is an inverse to φ.

Let us look at some examples:

Example 3.4 Suppose R is an integral noetherian domain. Consider the category $\mathcal{E}$ of all finitely generated R-modules. It containes the category $\mathcal{T}$ of all finitely generated torsion free R-modules. $\mathcal{E}$ is $\mathcal{T}$-resolutive, for every $A \in \mathcal{E}$ has a resolution

$$0 \longrightarrow B \longrightarrow F \longrightarrow A \longrightarrow 0$$

with F free of finite rank and B torsion free. Thus

$$K(\mathcal{E}) = K(\mathcal{T}) \quad .$$

Example 3.5 Let Λ be a left noetherian ring with the following property. Every finitely generated left Λ-module has a finite projective resolution by finitely generated left Λ-projectives. For instance every left noetherian ring of finite homological dimension has this property. If $\mathcal{E}$ denotes the class of all finitely generated left Λ-modules and $\mathcal{P}$ the class of all projective modules in $\mathcal{E}$, then $\mathcal{E}$ is $\mathcal{P}$-resolutive, thus

$$K(\mathcal{E}) = K(\mathcal{P}) \quad .$$

Example 3.6 In particular, if R has finite homological dimension (and is noetherian), then

$$K(\mathcal{E}(R[x_1,\ldots,x_n])) = K(\mathcal{P}(R[x_1,\ldots,x_n])) \quad ,$$

because $R[x_1,\ldots,x_n]$ has finite homological dimension. (By the Hilbert syzygy theorem, chapter IX, theorem 3.4).

4. Some Diagrams

For any ring Λ , $G(\Lambda)$ denotes the Grothendieck group $K(\mathcal{C})$ associated with the category $\mathcal{C}$ of all finitely generated left Λ-modules.

Theorem 4.1 Let R be a commutative noetherian ring, Λ a left noetherian ring which is also a R-algebra, and S a submonoid of R . Then we have an exact sequence

$$(+)\quad \Sigma' G(R/p \otimes_R \Lambda) \xrightarrow{\alpha=\Sigma'(\varphi_p \otimes 1_\Lambda)^*} G(\Lambda) \xrightarrow{(j_S \otimes 1_\Lambda)=\beta} G(R_S \otimes_R \Lambda) \longrightarrow 0$$

where the sum runs through the set of all prime ideals in R which intersect S ; $\varphi_p : R \longrightarrow R/_p$ and $j_S : R \longrightarrow R_S$ are the canonical morphisms.

Proof: First note that $R/p \otimes_R \Lambda$ and $R_S \otimes_R \Lambda$ are again left noetherian; this follows in the first case from $R \otimes_R \Lambda = \Lambda \longrightarrow R/p \otimes_R \Lambda$ being an epimorphism, and in the second case it is immediate from the characterisation of the ideals in R_S [chapt. 2, prop. 2.3].
$R/p \otimes_R \Lambda$ is finitely generated as a left Λ-module, and $R_S \otimes_R \Lambda$ is flat over Λ because $(R_S \otimes_R \Lambda) \otimes_\Lambda A = R_S \otimes_R (\Lambda \otimes_\Lambda A) = R_S \otimes_R A$ for a left Λ-module A , and R_S is flat over R . Therefore the maps in (+) are well defined. $\beta\circ\alpha = 0$ since $p \cap S \neq \emptyset$; therefore we obtain a morphism

$$\nu : G(\Lambda)/\mathrm{im}\,\alpha \longrightarrow G(R_S \otimes_R \Lambda) \quad .$$

We construct now an inverse δ for ν ; this will show that ν is an isomorphism, i.e. that (+) is exact. Let A be any finitely generated left $R_S \otimes_R \Lambda$-module. Pick any resolution

$$0 \longrightarrow N \longrightarrow F \longrightarrow A \longrightarrow 0$$

with F free over $R_S \otimes_R \Lambda$ and of finite rank. Now choose a free Λ-module F_o of the same rank as F ; thus $R_S \otimes_R F_o \cong F$, and we have a natural morphism $F_o \xrightarrow{\varphi} F$. $N_o = \varphi^{-1}(N)$ has the property that $R_S \otimes_R N_o = N$ (which follows as in the case of ideals in the proposition cited above). In order to have more freedom later on we choose any submodule N_o^* of N_o such that $R_S \otimes_R N_o^* = N$. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & A & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle\varphi} & & & & \\ 0 & \longrightarrow & N_o^* & \longrightarrow & F_o & \longrightarrow & A_o = F_o/N_o^* & \longrightarrow & 0 \end{array}$$

and would like to define

(++) $$\delta[A] = [A_o] = [F_o] - [N_o^*] \bmod(\operatorname{im} \alpha) .$$

1) Definition (++) is independent of the choice of F:

Suppose

$$0 \longrightarrow N' \longrightarrow F' \longrightarrow A \longrightarrow 0$$

is another choice; then this resolution and the first one have isomorphic modifications. Hence we may assume that

$0 \longrightarrow N' \longrightarrow F' \longrightarrow A \longrightarrow 0$ has the form

$0 \longrightarrow N \oplus F'' \longrightarrow F \oplus F'' \longrightarrow A \longrightarrow 0$ with F'' being free

(lemma 1.3). Now

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \oplus F'' & \longrightarrow & F \oplus F'' & \longrightarrow & A & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & N_o^* \oplus F_o'' & \longrightarrow & F_o \oplus F_o'' & \longrightarrow & A_o & \longrightarrow & 0 \end{array}$$

has the desired property, and will yield the same A_o.

(2) Definition (++) is independent of the choice of N_o^* :

It suffices to show that

$$[N_o^*] = [N_o] \bmod (\operatorname{im} \alpha) \quad .$$

By hypothesis $R_S \otimes_R N_o^* = R_S \otimes_R N_o$, thus $R_S \otimes_R N_o/N_o^* = 0$.

So it suffices to show that $R_S \otimes_R V = 0$ for a finitely generated left Λ-module V implies $[V] \in \operatorname{im} \alpha$. Let $\mathfrak{n} = \operatorname{ann}_R V$ be the annihilator of V considered as a R-module. Each element in V is annihilated by some element in S. Since V is finitely generated, $\mathfrak{n} \cap S \neq \emptyset$. Now write $\sqrt{\mathfrak{n}} = \bigcap_{i=1}^{k} p_i$, where $\{p_i\}$ is a finite collection of prime ideals in R. Clearly $p_i \cap S \neq \emptyset$. We have $(\sqrt{\mathfrak{n}})^m \subset \mathfrak{n}$ for some integer $m > 0$. The sequence

$$0 \longrightarrow (\sqrt{\mathfrak{n}})V \longrightarrow V \longrightarrow V/\sqrt{\mathfrak{n}}\, V \longrightarrow 0$$

is exact; therefore we find by induction on m that

$$[V] = \sum_{\text{fin}} [W_i]$$

where W_i are finitely generated left Λ-modules with $\text{ann}_R W_i \supset \sqrt{\mathfrak{a}}$. Now assume that the annihilater of V contains $\sqrt{\mathfrak{a}}$. We have an exact sequence

$$0 \longrightarrow p_i V \longrightarrow V \longrightarrow V/p_1 V \longrightarrow 0 \quad ;$$

so by induction on the number k of primes p_i we find

$$[V] = \sum_{\text{fin}} [U_j] \quad ,$$

where U_j are finitely generated left Λ-modules with $\text{ann}_R U_j = p_i$ for some i (depending on j).

3) Finally we have to check that for an exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ of finitely generated left $R_S \otimes_R \Lambda$-modules

$$[A_o] - [A'_o] - [A''_o] = 0 \quad .$$

We know (lemma 3.2) that there is an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

A simple diagram chasing shows that

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & \nearrow N''^{*}_{o} & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N'_o & \longrightarrow & N_o & \longrightarrow & N_o/N'_o & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F'_o & \longrightarrow & F_o & \longrightarrow & F''_o & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'_o & \longrightarrow & A_o & \longrightarrow & A''_o & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \searrow & \\
 & & 0 & & 0 & & 0 & & F''_o/N''_o
\end{array}$$

is exact, whence

$$\delta([A] - [A'] - [A'']) = [A_o] - [A'_o] - [A''_o] = 0 \quad .$$

Suppose that the R-algebra Λ is flat as a R-module; then $R/p \otimes_R \Lambda$ is flat as a R/p-module, and $R_S \otimes_R \Lambda$ is flat as a R_S-module. Let ε be the canonical morphism $R \longrightarrow \Lambda$ (resp. $R/p \longrightarrow R/p \otimes_R \Lambda$, resp. $R_S \longrightarrow R_S \otimes_R \Lambda$).

<u>Corollary 4.2</u> Suppose that in addition to the hypothesis of the theorem, Λ is flat as a R-module. Then the morphisms

$$\varepsilon_* : G(R) \longrightarrow G(\Lambda), \quad \varepsilon_* : G(R_S) \longrightarrow G(R_S \otimes_R \Lambda)$$

$$\varepsilon_* : G(R/p) \longrightarrow G(R/p \otimes_R \Lambda)$$

are well defined (φ_* in section 1.). Furthermore, the following diagram is commutative and has exact rows:

$$\begin{array}{ccccccc}
\Sigma' \, G(R/p \otimes_R \Lambda) & \xrightarrow{\Sigma'(\varphi_p \otimes 1_\Lambda)^*} & G(\Lambda) & \xrightarrow{(j_S \otimes 1_\Lambda)_*} & G(R_S \otimes_R \Lambda) & \longrightarrow & 0 \\
\uparrow \varepsilon_* & & \uparrow \varepsilon_* & & \uparrow \varepsilon_* & & \\
\Sigma' \, G(R/p) & \xrightarrow{\Sigma' \varphi_p{}^*} & G(R) & \xrightarrow{j_{S*}} & G(R_S) & \longrightarrow & 0
\end{array} \quad .$$

Before we can deduce the diagram which will be a main tool in the proof of Swan's theorem on decompositions of projective modules over group rings of Dedekind domains we need some lemmas.

Lemma 4.3 Suppose Λ is a ring and $\mathcal{E}$ a category of finitely generated left Λ-modules which satisfies the following conditions:

(a) Each $M \in \mathcal{E}$ admits a finite Jordan-Hoelder composition series $0 = M_o \subset M_1 \subset \cdots \subset M_n = M$.

(b) The factors M_{k+1}/M_k of any such decomposition series belong to $\mathcal{E}$.

Then $K(\mathcal{E})$ is free abelian with one generator for each isomorphism class of simple left Λ-modules in $\mathcal{E}$.

Proof: Let F be the free abelian group generated by the set of isomorphism classes of simple modules in $\mathcal{E}$. Define a morphism $K(\mathcal{E}) \xrightarrow{\varphi} F$ by sending $[M]$ into $\Sigma[M_{k+1}/M_k]$, where $0 = M_o \subset \cdots \subset M_n = M$ is any composition series. The uniqueness of the quotients M_{k+1}/M_k of a composition series shows that the map is well defined from $\mathcal{E}$ into F. It passes to a morphism from $K(\mathcal{E})$ into F because every short exact sequence can be refined to a Jordan-Holder composition series. Since $[M] = \Sigma[M_{k+1}/M_k]$ in $K(\mathcal{E})$, φ has an obvious inverse; hence φ is an isomorphism.

Lemma 4.4 Let R be a Dedekind domain and Λ a finitely generated R-algebra. If I is any ideal in R, then $R/I \otimes_R \Lambda$ satisfies the decending chain condition. Hence, $G(R/I \otimes_R \Lambda)$ is free abelian with one generator for each isomorphism class of finitely generated simple $R/I \otimes_R \Lambda$-modules. Let $\mathcal{J}$ be the category of finitely generated left Λ-modules which are torsion modules over R; then $K(\mathcal{J})$ is free abelian with one generator for each isomorphism class of simple left Λ-modules in $\mathcal{J}$. In particular, the natural injection $G(R/I \otimes_R \Lambda) \longrightarrow K(\Lambda)$ is a monomorphism.

It suffices to prove that R/I satisfies the descending chain condition, because this implies that every module in $\mathcal{J}$ (being a finitely generated R/I-module for some ideal I) admits a finite Jordan-Hoelder composition series. Let $I_1 \supset I_2 \supset \cdots \cdots \supset I$ be a descending chain in R/I ; the prime ideals occuring in the factorisation of I_j in R devide I . Since there can be only a finite number of such primes (counting multiplicities), the above chain must be finite.

<u>Lemma 4.5</u> Suppose I is a projective ideal in an integral domain R , and M a maximal ideal. Then there is an ideal J isomorphic to I such that $(J,M) = R$.

Proof: Since $I \otimes_R \mathrm{Hom}(I,R) \longrightarrow R$ is an isomorphism, there is $f \in \mathrm{Hom}(I,R)$ such that $f(I) \not\subset M$, I.e. $(f(I),M) = R$.

The setting for the next theorem is a Dedekind domain R with field of fractions K , and a finitely generated R-algebra Λ . Let us denote by $G'(\Lambda)$ the Grothendieck group associated to the category of all finitely generated left Λ-modules which are torsion free over R . As in example 3.4 we see that the natural morphism $G'(\Lambda) \longrightarrow G(\Lambda)$ is an isomorphism. Now let I be any ideal in R ; $\varphi_I : R \longrightarrow R/I$ denotes the canonical morphism. $\varphi_I \otimes 1_\Lambda$ defines a morphism

$$(\varphi_I \otimes 1_\Lambda)_* : G'(\Lambda) \longrightarrow G(R/I \otimes_R \Lambda)$$

according to (φ_*) in section 1, since the finitely generated torsion free modules over R are projective. Hence we have a morphism

$$G(\Lambda) \longrightarrow G(R/I \otimes_R \Lambda)$$

which we will denote by φ_{I*}.

<u>Theorem 4.6</u> There is a unique morphism $\psi_I\colon G(K \otimes_R \Lambda) \longrightarrow G(R/I \otimes_R \Lambda)$ which makes the diagram

$$\begin{array}{ccc} G(\Lambda) & \xrightarrow{\;j_*\;} & G(K \otimes_R \Lambda) \\ & {\scriptstyle\varphi_{I*}}\searrow \quad \swarrow{\scriptstyle\psi_I} & \\ & G(R/I \otimes_R \Lambda) & \end{array}$$

commutative. $j\colon R \longrightarrow K$ is the natural injection (we have written j_* instead of $(j \otimes 1_\Lambda)_*$).

Proof: The uniqueness is immediate since j_* is onto (theorem 4.1). The existance is equivalent with the statement

$$\varphi_{I*}[A] = 0 \quad \text{whenever } \operatorname{ann}_R A = p \text{ for some prime ideal } p \neq 0 \text{ in } R. \text{ (Theorem 4.1).}$$

Choose a resolution $0 \longrightarrow B \longrightarrow P \longrightarrow A \longrightarrow 0$ with P a finitely generated projective left Λ-module; thus $[A] = [P] - [B]$. We have thus to show that

$$[R/I \otimes_R P] - [R/I \otimes_R B] = 0$$

in $G(R/I \otimes_R \Lambda)$ or, equivalently, in $K(\mathcal{J})$ where $\mathcal{J}$ is the category of all finitely generated lefe Λ-modules which are torsion modules over R (lemma 4.4). We have exact sequences

$$0 \longrightarrow \operatorname{Tor}_1^R(R/I,A) \longrightarrow R/I \otimes_R B \longrightarrow R/I \otimes_R P \longrightarrow R/I \otimes_R A \longrightarrow 0$$

$$(+) \quad 0 \longrightarrow \operatorname{Tor}_1^R(R/I,A) \longrightarrow I \otimes_R A \longrightarrow R \otimes_R A \longrightarrow R/I \otimes_R A \longrightarrow 0$$

which give

(++) $[A] - [I \otimes_R A] = [R/I \otimes_R P] - [R/I \otimes_R B]$ in $K(\mathcal{J})$.

Note that we made use of the fact that $\mathrm{Tor}_1^R(R/I,A)$ is a left Λ-module. Choose an ideal J isomorphic to I such that $(J,p) = R$ (lemma 4.5). Replacing I by J in (+) gives $A = J \otimes_R A$ ($\mathrm{Tor}_1^R(R/J,A)$ and $R/J \otimes_R A$ are annihilated both by J and p, i.e. by R). Therefore $[A] - [I \otimes_R A] = [A] - [J \otimes_R A] = 0$. This shows that (++) vanishes in $K(\mathcal{J})$.

5. Applications to Polynomial Rings

First let us note a corollary to corollary 4.2:

Proposition 5.1 Suppose R is a commutative noetherian ring, Λ a left noetherian ring which is also a R-algebra and flat over R. Furthermore assume that the following conditions hold:

(1) R has finite Krull dimension.

(2) Whenever T is a field of the form $\mathbb{Q}(R/p)$ or R/p (where p is a prime resp. maximal ideal in R, and $\mathbb{Q}(\cdot)$ denotes the field of fractions), then

$$\varepsilon_* : G(T) \longrightarrow G(T \otimes_R \Lambda)$$

is an epimorphism.

Then

$$\varepsilon_* : G(R) \longrightarrow G(\Lambda)$$

is an epimorphism.

This is immediate from Corollary 4.2 using induction on the Krull dimension of R. For R an integral domain choose $S = R - \{0\}$; if R is any ring, take $S = \{0,1\}$ so that $R_S = 0 = R_S \otimes_R \Lambda$, thus reducing the problem (via the diagram in corollary 4.2) to the case of integral domains.

A supplemented algebra over a commutative ring R is a R-algebra Λ together with a left inverse $\pi : \Lambda \longrightarrow R$ for $\varepsilon : R \longrightarrow \Lambda$; π is called the projection. Clearly ε is a monomorphism in the case. Moreover

<u>Proposition 5.2</u> Let (Λ, R, π) be a supplemented algebra and suppose that Λ and R are left noetherian. If

(1) Λ is projective as a R-module and

(2) R considered as a left Λ-module has finite homological dimension,

then

$$\varepsilon_* : G(R) \longrightarrow G(\Lambda)$$

is a monomorphism.

Proof: We prove more; we show that ε_* has a left inverse η .

Define

$$\eta[A] = \Sigma(-1)^i [\mathrm{Tor}_i^{\Lambda}(R,A)]$$

for $[A] \in G(\Lambda)$. Note that the sum is finite since R admits a finite resolution by projective left Λ-modules, say of length n. We have to check that η is well defined. If

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

is an exact sequence of left Λ-modules, then

$$\begin{aligned} 0 \longrightarrow \mathrm{Tor}_n^\Lambda(R,A') \longrightarrow \mathrm{Tor}_n^\Lambda(R,A) \longrightarrow \mathrm{Tor}_n^\Lambda(R,A'') \longrightarrow \cdots \longrightarrow R \otimes_\Lambda A' \\ \longrightarrow R \otimes_\Lambda A \longrightarrow R \otimes_\Lambda A'' \longrightarrow 0 \end{aligned}$$

is exact, whence $\eta([A] - [A'] - [A'']) = 0$. Now let

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_o \longrightarrow R \longrightarrow 0$$

be a projective resolution of R over Λ. We may assume that the P_i are finitely generated over Λ. Since Λ is projective over R, so are the P_i. Therefore

$$\begin{array}{ccccccc} 0 \longrightarrow P_n \otimes_\Lambda (\Lambda \otimes_R A_o) & \longrightarrow \cdots \longrightarrow & R\otimes_\Lambda (\Lambda \otimes_R A_o) & = A_o \longrightarrow 0 \\ \| & & \| & \\ 0 \longrightarrow P_n \otimes_R A_o & \longrightarrow \cdots \longrightarrow & R \otimes_R A_o & = A_o \longrightarrow 0 \end{array}$$

is exact for every finitely generated R-module A_o. By the definition of Tor we get

$$\mathrm{Tor}_i^\Lambda(R,\Lambda\otimes_R A_o) = 0 \quad \text{for} \quad i > 0 ,$$

$$\mathrm{Tor}_o^\Lambda(R,\Lambda\otimes_R A_o) = R \otimes_\Lambda (\Lambda \otimes_R A_o) = A_o \quad .$$

This shows $\eta \circ \varepsilon = 1$.

We recollect:

Theorem 5.3 Let (Λ, R, π) be a supplemented algebra such that

(1) R and Λ are left noetherian,

(2) Λ is projective over R,

(3) R considered as a left Λ-module has finite homological dimension,

(4) R has finite Krull dimension,

(5) For any field T of the form $\mathbb{Q}(R/p)$ or R/p (where p is a prime resp. maximal ideal in R, and $\mathbb{Q}(\cdot)$ denotes the field of fractions) $\varepsilon_* : G(T) \longrightarrow G(T \otimes_R \Lambda)$ is an epimorphism.

Then

$$\varepsilon_* : G(R) \longrightarrow G(\Lambda)$$

is an isomorphism.

We apply now the above theorem to the polynomial rings over A noetherian ring $\mathcal{R}$ of finite Krull dimension. Condition (3) is fulfilled according to the Hilbert syzygy theorem (chapter IX, theorem 3.4). First let K be a field. $K[x_1]$ is a principal ideal domain, i.e. every ideal of $K[x_1]$ is free. Therefore every finitely generated projective module over $K[x_1]$ is free. Hence (cf example 3.6) $\varepsilon_* : G(K) \longrightarrow G(K[x_1])$ is an isomorphism.

Now we find by induction on n that

$$\varepsilon_* : G(K) \longrightarrow G(K[x_1,\ldots,x_n])$$

is an isomorphism. Therefore (5) is fulfilled with R, $\Lambda = R[x_1,\ldots,x_n]$; so we obtain:

Theorem 5.4 Let R be a noetherian ring of finite Krull dimension. Then

$$\varepsilon_* : G(R) \longrightarrow G(R[x_1,\dots,x_n])$$

is an isomorphism. In particular, for $R = K$ a field,

$$G(K[x_1,\dots,x_n]) = G(K) = Z \text{ , i.e.}$$

$$PCG(K[x_1,\dots,x_n]) = 0 \text{ .}$$

(Z denotes the integers).

This raises now the question whether every projective module over $K[x_1,\dots,x_n]$ is free. We have seen already that this is true for $n = 0$ and $n = 1$. In section 3 of chapter XIII we will see that this question can be answered positively also in the case where $n = 2$. It is still open to what extent it will be true in the general case.

6. The Grothendieck Ring associated with a Group Ring.

In this section R is always a commutative ring and π a finite group. $\mathcal{P}_o$ denotes the category of all finitely generated $R\pi$-modules which are projective over R. We define

$$G'(R\pi) = K(\mathcal{P}_o) \text{ .}$$

Let $A, B \in \mathcal{P}_o$: $A \overset{\pi}{\otimes}_R B$ is the tensor product $A \otimes_R B$ with diagonal action of π.

$$T : \begin{array}{l} \mathcal{P}_o \times \mathcal{P}_o \longrightarrow \mathcal{P}_o \\ (A,B) \longrightarrow A \overset{\pi}{\otimes}_R B \end{array}$$

defines an exact bilinear functor on $\mathcal{P}_o$. T induces in $G'(R\pi)$ a multiplication operation. $G'(R\pi)$ becomes thus a ring with unit $[R\pi]$. Hence

Proposition 6.1 $G'(R\pi)$ can be given the structure of a ring.

Proposition 6.2 Let Λ be any R-algebra. $G(\Lambda\pi)$ can be considered as a $G'(R\pi)$-module.

Proof: Let $\mathcal{C}(\Lambda\pi)$ be the category of finitely generated left $\Lambda\pi$-modules. Pick $A \in \mathcal{P}_o$;

$$B \longrightarrow A \overset{\pi}{\otimes}_R B = B \overset{\pi}{\otimes}_R A \ , \ B \in \mathcal{C}(\Lambda\pi)$$

is an exact additive functor on $\mathcal{C}(\Lambda\pi)$. It defines for $G(\Lambda\pi)$ the structure of a $G'(R\pi)$-module.

Now consider a subgroup $\sigma \subset \pi$ and let $i : \sigma \subset \pi$ be the inclusion map. For any ring Λ, $\Lambda\pi$ is a finitely generated free left $\Lambda\sigma$-module. Hence the maps

$$i^* : G(\Lambda\pi) \longrightarrow G(\Lambda\sigma)$$

$$i_* : G(\Lambda\sigma) \longrightarrow G(\Lambda\pi)$$

are defined (cf. (φ^*) and (φ_*) in section 1 of chapter XII), and similarly for $G'(R\pi)$.

Proposition 6.3 Let Λ be a R-algebra and $\sigma \subset \pi$ a subgroup of π. We have the following identities:

(1) $i^*(xy) = i^*(x)i^*(y)$ for $x \in G'(R\pi)$, $y \in G(\Lambda\pi)$

(2) $i_*(i^*(x)y) = x \cdot i_*(y)$ for $x \in G'(R\pi)$, $y \in G(\Lambda\sigma)$

(3) $i_*(x \cdot i^*(y)) = i_*(x) \cdot y$ for $x \in G'(R\sigma)$, $y \in G(\Lambda\pi)$

In particular, the image of $i_* : G'(R\sigma) \longrightarrow G'(R\pi)$ is an ideal in $G'(R\pi)$.

Proof: (1) is immediate. For (2) and (3) note that

$$R\pi \otimes_{R\sigma} (A \overset{\pi}{\otimes}_R B) \cong A \overset{\pi}{\otimes}_R (R\pi \otimes_{R\sigma} B)$$

for $A \in \mathcal{P}_o(R\pi)$, $B \in \mathcal{G}(\Lambda\sigma)$ (resp. $A \in \mathcal{P}_o(R\sigma)$, $B \in \mathcal{G}(\Lambda\pi)$ under the map

$$z \otimes (a \otimes b) \longrightarrow z \cdot a \otimes (z \otimes b) \quad z \in \pi, a \in A, b \in B \quad .$$

For any ring Λ let $\mathcal{P}(\Lambda)$ be the category of finitely generated left Λ-projectives. Define

$$P(\Lambda) = K(\mathcal{P}(\Lambda)) \quad .$$

A morphism of rings $\varphi : \Gamma \longrightarrow \Lambda$ induces a morphism

$$\varphi^* : P(\Lambda\pi) \longrightarrow P(\Gamma\pi) \quad .$$

Proposition 6.4 If $i : \sigma \subset \pi$ is a subgroup of π, then φ^* commutes with i^* and i_*.

This is trivial for i^* , and for i_* we have

$$\Lambda\pi \otimes_{\Lambda\sigma} (\Lambda\sigma \otimes_{\Gamma\sigma} A) = \Lambda\pi \otimes_{\Gamma\sigma} A = \Lambda\pi \otimes_{\Gamma\pi} (\Gamma\pi \otimes_{\Gamma\sigma} A), \; A \in P(\Gamma\sigma) \quad .$$

<u>Lemma 6.5</u> Let Λ be a ring, and A a left $\Lambda\pi$-module; then

$$\Lambda\pi \overset{\pi}{\otimes}_{\Lambda} A = \Lambda\pi \otimes_{\Lambda} A .$$

If A is free as a left Λ-module, then $\Lambda\pi \overset{\pi}{\otimes}_{\Lambda}$ is free as a left $\Lambda\pi$-module.

Proof: Define

$$\alpha : \Lambda\pi \otimes_{\Lambda} A \longrightarrow \Lambda\pi \overset{\pi}{\otimes}_{\Lambda} A$$

$$x \otimes a \longrightarrow x \otimes xa, \quad x \in \pi, \ a \in A$$

$$\beta : \Lambda\pi \overset{\pi}{\otimes}_{\Lambda} A \longrightarrow \Lambda\pi \otimes_{\Lambda} A$$

$$x \otimes A \longrightarrow x \otimes x^{-1}z, \ x \in \pi, \ a \in A.$$

Clearly $\alpha o \beta$ = identity, $\beta o \alpha$ = identity.

<u>Lemma 6.6</u> Let A, P be left $\Lambda\pi$-modules, A projective over Λ, and P projective over $\Lambda\pi$. Then $A \overset{\pi}{\otimes}_{\Lambda} P$ is $\Lambda\pi$-projective.

Proof: 1) Suppose A is Λ-free, and let P' be a left $\Lambda\pi$-projective such that $P \oplus P'$ is $\Lambda\pi$-free. $A \overset{\pi}{\otimes}_{\Lambda} P \oplus A \overset{\pi}{\otimes}_{\Lambda} P' = A \overset{\pi}{\otimes}_{\Lambda} (P \oplus P')$ is $\Lambda\pi$-free by lemma 6.5; so $A \overset{\pi}{\otimes}_{\Lambda} P$ is $\Lambda\pi$-projective. 2) In general, let A' be a left Λ-module such that $A \oplus A'$ is Λ-free. A'_{ε} denotes the module A' considered as a left $\Lambda\pi$-module, π acting trivially. Then

$$(A \oplus A'_{\varepsilon}) \overset{\pi}{\otimes}_{\Lambda} P = A \overset{\pi}{\otimes}_{\Lambda} P \oplus A'_{\varepsilon} \overset{\pi}{\otimes}_{\Lambda} P$$

is π-projective by 1).

<u>Proposition 6.7</u> If Λ is a R-algebra, then $P(\Lambda\pi)$ is a $G'(R\pi)$-module. For a subgroup $i : \sigma \subset \pi$ of π, i^* and i_* satisfy the

identies of proposition 6.3.

Proof: Let $A \in \mathcal{P}_o$, $P \in \mathcal{P}(\Lambda\pi)$. $A \otimes_R \Lambda$ is Λ-projective, hence $A \overset{\pi}{\otimes}_R P = A \overset{\pi}{\otimes}_R (\Lambda_\varepsilon \overset{\pi}{\otimes}_\Lambda P) = (A \overset{\pi}{\otimes}_R \Lambda_\varepsilon) \overset{\pi}{\otimes}_\Lambda P = (A \otimes_R \Lambda) \overset{\pi}{\otimes}_\Lambda P$ is $\Lambda\pi$-projective by lemma 6.6. Thus the bilinear functor

$$\mathcal{P}_o \times \mathcal{P}(\Lambda\pi) \longrightarrow \mathcal{P}(\Lambda\pi)$$

$$(A,P) \longrightarrow A \overset{\pi}{\otimes}_R P$$

is well defined. It is an exact functor; therefore $P(\Lambda\pi)$ becomes a $G'(R\pi)$-module. The rest follows as in proposition 6.3.

Consider a class M of subgroups $i : \sigma \subset \pi$ of π. Let Λ be any R-algebra. We denote by $G_M(\Lambda\pi)$ the submodule of $G(\Lambda\pi)$ generated by the modules $i_* G(\Lambda\sigma)$, $\sigma \in M$. $G_M(\Lambda\pi)$ is said to have exponent k in $G(\Lambda\pi)$ if $k \cdot G(\Lambda\pi) \subset G_M(\Lambda\pi)$. Similarly are defined $P_M(\Lambda\pi)$ and $G'_M(R\pi)$. $G'_M(R\pi)$ has exponent k in $G'(R\pi)$ if and only if $k \in G'_M(R\pi)$. We will show in the next section that for any ring Λ (considered as a Z-algebra) $G_M(\Lambda\pi)$ has exponent $(\#\pi)^2$ in $G(\Lambda\pi)$ if we take for M the class of all cyclic subgroups of π. This is a kind of induction theorem in that we need prove theorems only for cyclic subgroups of π.

Corollary 6.8.

(a) $$G'_M(R\pi) \cdot G(\Lambda\pi) \subset G_M(\Lambda\pi).$$

(b) If $G'_M(R\pi)$ has exponent k in $G'(R\pi)$, then $G_M(\Lambda\pi)$ has exponent k in $G(\Lambda\pi)$. (And similarly for $P(\Lambda\pi)$).

(a) follows from (3) of proposition 6.3, and (b) is a consequence of (a).

Remember that for a noetherian ring R of finite homological dimension $G'(R\pi) = G(R\pi)$ (proposition 3.3).

Proposition 6.9. Let R be a Dedekind domain and K its field of fractions. If $G_M(K\pi)$ has finite exponent k in $G(K\pi)$, then

(1) $G_M(R/p\pi)$ has exponent k in $G(R/p\pi)$ for every prime ideal in R.

(2) $G_M(R\pi)$ has exponent k^2 in $G(R\pi)$.

Proof: (1) We use the notation of theorem 4.6. ψ_p maps $G_M(K\pi)$ into $G_M(R/p\pi)$ since ψ_p commutes with all morphisms i_* (ψ_p commutes with i_* because j_* and φ_{p*} do by proposition 6.4). $j_*(1) = 1$ and $\varphi_{p*}(1) = 1$ imply $\psi_p(1) = 1$. By hypothesis $k \in G_M(K\pi)$; hence $k = \psi_p(k) \in G_M(R/p\pi)$.

(2) $j_* : G_M(R\pi) \longrightarrow G_M(K\pi)$ is an epimorphism because $j_* : G(R\pi) \longrightarrow G(K\pi)$ is an epimorphism and commutes with all morphisms i_*. Since $k \in G_M(K\pi)$, there is $x \in G_M(R\pi)$ with $j_*(x - k) = 0$. Therefore

$$x - k = \Sigma' \varphi_p^*(x_p)$$

the sum ranging over all non-zero prime ideals of R (theorem 4.1). By (1), $k \cdot x_p \in G_M(R/p\pi)$. Since φ_p^* preserves G_M, $k(x - k) \in G_M(R\pi)$.

7. Grothendieck Rings and Character Rings

Throughout this section π will denote a finite group and K a field of characteristic prime to the order $n = \#\pi$ of π.

<u>Lemma 7.1</u> $K\pi$ is semi-simple, i.e. every $K\pi$-module is projective

Proof: 1) K is projective over $K\pi$. Define

$$\varepsilon : K\pi \longrightarrow K, \qquad \sigma : K \longrightarrow K\pi$$

$$x \longrightarrow 1,\ x \in \pi \qquad 1 \longrightarrow \frac{1}{n}\cdot \sum_{x\in\pi} x.$$

$\varepsilon \circ \sigma = 1_K$; thus K is $K\pi$-projective.

2) Let A be a $K\pi$-module. $K\pi \overset{\pi}{\otimes}_K A$ is $K\pi$-free by lemma 6.5. Define

$$\delta : A \longrightarrow K\pi \overset{\pi}{\otimes} A$$

$$a \longrightarrow \sigma(1) \otimes a. \quad a \in A$$

$$\nu : K\pi \overset{\pi}{\otimes}_K A \longrightarrow A$$

$$x \otimes a \longrightarrow \varepsilon(x)\cdot a, \quad x \in \pi,\ a \in A.$$

$\nu \circ \delta = 1_A$; thus A is a direct summand of $K\pi \overset{\pi}{\otimes}_K A$:

<u>Corollary 7.2</u> If $[A] = [B]$ in $G(K\pi)$, then $A \cong B$.

$[A]$ is uniquely determined by the factors A_i occurring in a Jordan-Hoelder composition series of A (lemma 4.3). Since all A_i are projective, $A \cong \oplus A_i$.

We recall now some facts about group representations. We assume in the following that the characteristic of K is 0.

A finitely generated $K\pi$-module A is a finite dimensional vector space over K. Each $x \in \pi$ defines a linear transformation $x_A \in \mathrm{Hom}_K(A,A)$. $\Delta : x \longrightarrow x_A$ is a representation of the group π in $\mathrm{Hom}_K(A,A)$. On the other hand, a representation of Δ of π in the group of linear transformations $\mathrm{Hom}_K(A,A)$ of a finite dimensional vector space A makes A into a $K\pi$-module A_Δ : $x \cdot a = x_A(a)$ for $x \in \pi$, $a \in A$. Two representations $\Delta : x \longrightarrow x_A$ and $\Delta' = x \longrightarrow x'_A$ are equivalent if there is a transformation $T \in \mathrm{Hom}_K(A,A)$ such that $T x_A T^{-1} = x'_A$ for all $x \in \pi$; i.e. if and only if $A_\Delta = A_{\Delta'}$ as $K\pi$-modules.

The character χ_Δ of a representation Δ is defined as

$$\chi_\Delta(x) = \mathrm{Tr}\, x_A \quad ,$$

where Tr denotes the trace. Equivalent representations have the same character. Moreover

<u>Theorem 7.3</u> $\chi_\Delta = \chi_{\Delta'}$ if and only if Δ and Δ' are equivalent.

For a proof see for instance B. L. van der Waeden, Modern Algebra II, the theorem in § 125.

Instead of χ_Δ we write also χ_A, where A is the $K\pi$-module associated with Δ. We have

$$\chi_A + \chi_B = \chi_{A \oplus B}$$

$$\chi_A \cdot \chi_B = \chi_{A \otimes B} \quad (A \otimes B \text{ means } A \overset{\pi}{\otimes}_K B, \text{ here}).$$

A finitely generated $K\pi$-module A is a finite dimensional vector space over K. Each $x \in \pi$ defines a linear transformation $x_A \in \mathrm{Hom}_K(A,A)$. $\Delta : x \longrightarrow x_A$ is a representation of the group π in $\mathrm{Hom}_K(A,A)$. On the other hand, a representation of Δ of π in the group of linear transformations $\mathrm{Hom}_K(A,A)$ of a finite dimensional vector space A makes A into a $K\pi$-module $A_\Delta : x\cdot a = x_A(a)$ for $x \in \pi$, $a \in A$. Two representations $\Delta : x \longrightarrow x_A$ and $\Delta' = x \longrightarrow x'_A$ are equivalent if there is a transformation $T \in \mathrm{Hom}_K(A,A)$ such that $Tx_AT^{-1} = x'_A$ for all $x \in \pi$; i.e. if and only if $A_\Delta = A_{\Delta'}$ as $K\pi$-modules.

The character χ_Δ of a representation Δ is defined as

$$\chi_\Delta(x) = \mathrm{Tr}\ x_A \quad ,$$

where Tr denotes the trace. Equivalent representations have the same character. Moreover

<u>Theorem 7.3</u> $\chi_\Delta = \chi_{\Delta'}$ if and only if Δ and Δ' are equivalent. For a proof see for instance B. L. van der Waeden, Modern Algebra II, the theorem in § 125.

Instead of χ_Δ we write also χ_A, where A is the $K\pi$-module associated with Δ. We have

$$\chi_A + \chi_B = \chi_{A\oplus B}$$

$$\chi_A \cdot \chi_B = \chi_{A\otimes B} \quad (A\otimes B \text{ means } A\overset{\pi}{\otimes}_K B, \text{ here}).$$

Let $CR(K\pi)$ be the subring of the ring of all functions on π with values in K generated by all characters χ_A. $CR(K\pi)$ is called the character ring of $K\pi$.

<u>Proposition 7.4</u>

$$G(K\pi) \cong CR(K\pi)$$

under the morphism $[A] \longrightarrow \chi_A$.

This is immediate from corollary 7.2 and theorem 7.3.

We consider now representations of π into the rational numbers Q.

<u>Lemma 7.5</u> Suppose R is an integral domain and K its field of fractions with characteristic prime to the order of π. If A is a finitely generated $K\pi$-module, then there exists a $R\pi$-module B such that $K \otimes_R B = A$.

We know that A is a direct sum of simple modules (corollary 7.1). Let A therefore be simple, say with generator a. Define B to be the $R\pi$-submodule of A generated by a over $R\pi$. We have $K \otimes_R B = A$.

If $R = Z$ (the integers), B is free over Z. Hence the transformation x_A on $A (x \in \pi)$ is induced by a transformation of B, i.e. χ_A has integral values.

Corollary 7.6 The character ring of $Q\pi$ is a subring of the ring of all integral valued functions on π.

Lemma 7.7 Let π be a cyclic group and χ a rational character. If x and y are generators of π, then $\chi(x) = \chi(y)$.

Proof: Suppose χ is defined by the $Q\pi$-module B. Consider $A = \mathbb{C} \otimes_Q B$ ($\mathbb{C}$ the complex numbers). We can choose coordinates in A such that x_A is represented by a diagonal matrix; now all z_A $(z \in \pi)$ are represented by diagonal matrices, because x generated π. This shows that the representation of π defined by A is the direct sum of one-dimensional representations, say with characters χ_i. Since

$$[\chi_i(x)]^n = \chi_i(x^n) = 1,$$

$\chi_i(x)$ is a power ξ^{k_i} of a primitive n-th root of unity, ξ. $\chi(x^j) = \Sigma\, \chi_i(x^j)$; if $(j,n) = 1$, the map $\xi \longrightarrow \xi^j$ gives an automorphism α of $Q(\xi)$. Therefore

$$\alpha(\chi(x)) = \chi(x^j) = \chi(x),$$

χ being rational.

Corollary 7.8 Let χ be a rational character of π. If x and y generate the same cyclic subgroup of π, then $\chi(x) = \chi(y)$. Also $\chi(yxy^{-1}) = \chi(x)$ for $x,y \in \pi$.

Consider a subgroup $i : \sigma \subset \pi$ of π. Let A be a $K\sigma$-module. We want to express the character χ_{i_*A} of the induced representation i_*A in terms of χ_A. We have

$$K\pi \otimes_{K\sigma} A = \sum_{y\in Y} y\cdot A$$

where $\pi = \bigcup_{y\in Y} y\cdot\sigma$ is a decomposition of π into the right cosets of σ. Suppose $x \in \pi$; then

$$x\cdot\Sigma\, y\cdot A = \sum_{y^{-1}xy\in\sigma} yy^{-1}xyA \otimes x\cdot \sum_{y^{-1}xy\notin\sigma} yA.$$

The trace of the operator x on the second sum is 0 since x permutes all summands yA. Thus

(*) $$\chi_{i_*A}(x) = \frac{1}{\#\sigma}\cdot\sum_{y\in\pi}\chi_A(yxy^{-1}),$$

where we have set $\chi_A(y) = 0$ for $y \in \pi - \sigma$.

Theorem 7.9 (Artin) Let f be an integral valued function on π satisfying

(1) $f(yxy^{-1}) = f(x)$ for $x,y \in \pi$;

(2) if $x,y \in \pi$ generate the same cyclic subgroup of π, then $f(x) = f(y)$.

Then $n\cdot f$ is an integral linear combination of characters of π induced from the characters of trivial representations of subgroups of π.

Proof: For a cyclic subgroup $\tau \subset \pi$ of π define the function f_τ by induction on $\#\tau$:

(*): $$f_\tau = \#\tau - \sum_{i:\sigma\subset\tau} i_*(f_\sigma)\ ,\quad \text{where}\quad i_*f_\sigma = \Sigma\,\chi_{i_*A}\quad \text{if}\quad f_\sigma = \Sigma\, f_A.$$

By induction on $\#\tau$ we find that f_τ is an integral linear combination of characters of τ induced from trivial characters of subgroups of τ. Formula (*) shows that

$$f_\tau(x) = \{\begin{matrix} \#\tau & \text{if } x \text{ generates } \tau \\ 0 & \text{otherwise} \end{matrix}\}.$$

Define $g_\tau = i_* f_\tau (i : \tau \subset \pi$ is the inclusion map). We have

$$g_\tau(x) = \{\begin{matrix} [N_\tau : \tau] & \text{if } x \text{ is conjugate to a generator of } \tau \\ 0 & \text{otherwise} \end{matrix}\};$$

N_τ denotes the normalizor of τ. $[N_\tau : \tau] | n$. A simple argument shows that $n \cdot f$ is an integral linear combination of the g_τ whenever f satisfies (1) and (2).

This theorem shows together with corollaries 7.4, 7.6, and 7.7:

Theorem 7.10 Let C be the class of all cyclic subgroups of π. $G_C(Q\pi)$ has exponent $n = \#\pi$ in $G(Q\pi)$.

Corollary 7.11 Let Λ be any ring (considered as a Z-algebra). $G_C(\Lambda\pi)$ has exponent n^2 in $G(\Lambda\pi)$ (and similarly for $P(\Lambda\pi)$. If Λ is a field, we can replace n^2 by n.

This is an immediate consequence of corollary 6.8 and proposition 6.9. A field can always be considered as an algebra over Q or Z/p for some prime ideal $p \neq 0$ in Z.

Theorems of R. Brauer and E. Witt can be transformed into the language of Grothendieck groups in a similar way. The reader is referred to R. G. Swan, Annals of Mathematics 71 (1960).

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