VIII HOMOLOGICAL ALGEBRA

Modern Classical Algebra

VIII. HOMOLOGICAL ALGEBRA

In this chapter, some basic concepts of homological algebra will be defined and some of their elementary properties developed. The duality of the concepts "projective module" and "injective module" will be systematically employed and, where applicable, dual propositions will be stated.

We will need to modify some of the definitions concerning graded modules. We will mean by a graded module a sequence indexed on the integers rather than on the positive integers. If X and Y are graded modules, a morphism of degree n, $f\colon X\longrightarrow Y$, is a sequence of morphisms $f^i\colon X^i\longrightarrow Y^{i+n}$.

A will be a ring, not necessarily commutative. All modules will be assumed left modules unless otherwise specified.

1. Differential operators and resolutions.

Definitions 1.1: A differential operator, or complex, is a pair (X,d) where X is a graded module and d: X \longrightarrow X is a morphism of degree +1 such that $d^{q+1}d^q=0$ for all q. We define further the q^{th} cocycle, $Z^q(X)$, as $\ker(d^q)$, the q^{th} coboundary, $B^q(X)$, as $\operatorname{im}(d^{q-1})$, and the q^{th} cohomology as $Z^q(X)/B^q(X)$.

We introduce the convention $X_N = X^{-N}$, $d_N = d^{-N}$, $Z_N = Z^{-N}$, $B_N = B^{-N}$, $H_N = H^{-N}$. Thus, when (X,d) is written as a complex with subscripts, $d_q \colon X_q \longrightarrow X_{q-1}$ and $d_q d_{q+1} = 0$; in this case $Z_q(X)$ is called the q^{th} cycle, $B_q(X)$, the q^{th} boundary, $H_q(X)$ the q^{th} homology.

(X,d) is called a right complex if $X^N=0$ for all N<0; it is called a left complex if $X_N=0$ for all N<0. Thus a right

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Delimition 1.1: A differential exercisor, or complex, is a pair (X, \mathbf{d}) where X is a graded module end $\mathbf{d}: X = -\infty E$ is a morphism of degree $-\infty E$ is a morphism of degree +1 and that $\mathbf{d}^{(k)} \mathbf{d}^{(k)} = 0$ for all \mathbf{q} . We define further the $\mathbf{q}^{(k)}$ co-cycle, $E^{(k)}(X)$, as $\operatorname{Rer}(\mathbf{d}^{(k)})$, the $\mathbf{q}^{(k)}$ obliquation as $E^{(k)}(X)$, as $\operatorname{Im}(E^{(k-1)})$, $\operatorname{Im}(E^{(k)}(X))$.

We introduce the convention $Y_1 = \mathbb{R}^{-1}$, $d_{Y} = d^{-1}$, $d_{Y} = d^{-1}$, $d_{Y} = \mathbb{R}^{-1}$, $d_{Y} = \mathbb{R}^{-1}$; then, when (x, d) is written as a conclusively stip subscript, $d_{Q}: \mathbb{R}_{q} \xrightarrow{\mathrm{cond}} \mathcal{S}_{q} \xrightarrow{\mathrm{cond}} \mathcal{S}_{q} \xrightarrow{\mathrm{cond}} \mathcal{S}_{Q}(\mathbb{X})$ is called the q^{th} dyale, $d_{Q}(\mathbb{X})$, the q^{th} boundary, $H_{Q}(\mathbb{X})$ the q^{th} boundary. $H_{Q}(\mathbb{X})$ the q^{th}

 $(X,d) \text{ is edilist a wight complex if } X_k^2 = 0 \text{ for all } N \leq 0 \text{ };$ it is belief a left, daugler if $X_k = 0$ for all $M \leq 0$. Given a wight

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complex has the form $\dots \to 0 \to 0 \xrightarrow{d^{-1}} x^0 \xrightarrow{d^0} x^1 \xrightarrow{d^1} x^2 \xrightarrow{d^2} \dots$, a left complex $\dots \to x_2 \xrightarrow{d_2} x_1 \xrightarrow{d_1} x_0 \xrightarrow{d_0} 0 \xrightarrow{d^{-1}} 0 \to \dots$.

For notational convenience, a complex (X,d) will hereafter be denoted simply by X.

Definitions 1.2: A left complex over a module A is a left complex X together with an epimorphism $\epsilon\colon X_0\longrightarrow A$ such that $\ldots\longrightarrow X_2\longrightarrow X_1\longrightarrow X_0\longrightarrow A\longrightarrow 0$ is a 0-sequence (the composition of any two consecutive morphisms is zero). A left complex is called a left resolution if $\ldots\longrightarrow X_2\longrightarrow X_1\longrightarrow X_0\longrightarrow A\longrightarrow 0$ is exact. A projective (free) resolution of A is a left resolution such that for each q, X_0 is projective (free).

The concepts of right complex over A, right resolution, and injective resolution are analogously defined. In this case, $0 \longrightarrow A \xrightarrow{\epsilon} x^1 \longrightarrow x^2 \longrightarrow \dots \text{ is a zero (resp., exact) sequence.}$

<u>Definitions 1.3</u>: Let X and Y be complexes. A translation $f: X \longrightarrow Y$ is a morphism of degree O which commutes with the boundary maps, that is, for which

$$\begin{array}{c} x_{q} \xrightarrow{f_{q}} y_{q} \\ \downarrow^{d_{q}} \downarrow^{d_{q}} \\ x_{q-1} \xrightarrow{f_{q-1}} y_{q-1} \end{array}$$

is a commutative diagram for all q. If f is a translation, f induces morphisms $H_N(f)\colon H_N(X) \longrightarrow H_N(Y)$, since $f(Z_N(X)) \subset Z_N(Y)$ and $f(B_N(X)) \subset B_N(Y)$.

If $f: X \longrightarrow Y$ and $g: X \longrightarrow Y$ are translations, a homotopy between f and g is a morphism D: $X \longrightarrow Y$ of degree -1 such that

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 $f_q - g_q = d_{q+1}D_q + D_{q-1}d_q$. The relation of homotopy is an equivalence equation.

<u>Proposition 1.4</u>: If two translations of X into Y are homotopic, then the corresponding morphisms $H_{\mathbb{N}}(X) \longrightarrow H_{\mathbb{N}}(Y)$ coincide.

Proof: If f and g are the homotopic translations, then f-g is null homotopic (homotopic to zero). Let $x_N \epsilon Z_N(X)$.

$$(f-g)(x_N) = d_{n+1}D_N(x_N) + D_{N-1}d_N(x_N)$$

= $d_{N+1}D_N(x_N) \in B_N(Y)$.

Therefore f-g induces a zero map $H_{\mathbb{N}}(X) \longrightarrow H_{\mathbb{N}}(Y)$.

We now obtain a series of propositions concerning projective resolutions of modules. We will then give the dual results concerning injective resolutions.

Proposition 1.5: Every module has a projective resolution.

Proof: Given a module A, construct exact sequences

where the X_{i} are projective. Define d_{N} as the composition

$$X_{N} \longrightarrow Z_{N-1} \longrightarrow X_{N-1}$$
.

Then $\dots \longrightarrow X_{\mathbb{N}} \longrightarrow \dots \longrightarrow X_{\mathbb{I}} \longrightarrow X_{\mathbb{O}} \longrightarrow A \longrightarrow 0$ is a projective resolution of A.

Corollary 1.6: If Λ is left Noetherian and A is a finitely generated left module, then A has a free resolution X where each X_N is finitely generated.

Proof: X_O may be chosen free and finitely generated. Then Z_O is finitely generated, and X_1 may be chosen free with a finite base, etc.

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Proposition 1.4: If the thenelations of X into Y are hemotopies. then the corresponding morphique $N_{M}(X)$ -- $N_{M}(Y)$ collectes.

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$$: (X)^{n} \mathbb{R}^{n} (^{n}x)^{n} \mathbb{Q}^{n+n} + (^{n}x)^{n} \mathbb{Q}^{n+n} \mathbb{Q}^{n} (^{n}x)^{n} \mathbb{Q}^{n+n})$$

Therefore f.g duduces a core map IIg(X) -- > Ag(Y).

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Then ... - A to a production of A. ... A to a production of A. ...

Corollary 4.6: In A te flere Masthartan and a is a standally generated left module, then A her a free requireden X where each $X_{\rm H}$ is finitely generated.

Proof: X may be chosen free and finitely generated. Then Z is finitely generated, and X, may be chosen from with a finite base, etc. <u>Proposition 1.7</u>: Let A and B be modules, let X be a projective resolution of A and Y a left resolution of B. If $f: A \longrightarrow B$ is a morphism, then there exists a translation $\tilde{f}: X \longrightarrow Y$ such that

$$\begin{array}{c} X_{\circ} \xrightarrow{\widetilde{f}_{\circ}} Y_{\circ} \\ \epsilon \downarrow \qquad \downarrow \epsilon \\ A \xrightarrow{f} B \end{array}$$

commutes. T is said to be over f.

Proof: Since X_o is projective, there exists $\tilde{f}_o: X_o \longrightarrow Y_o$ such that $\epsilon \tilde{f}_o = f \epsilon$. Since $\epsilon \tilde{f}_o d_1 = f \epsilon d_1 = 0$, $\tilde{f}_o d_1(X_1) \subset \ker(\epsilon) = \operatorname{im}(d_1)$, and there exists $\tilde{f}_1: X_1 \longrightarrow Y_1$ such that $\tilde{f}_o d_1 = d_1 \tilde{f}_1$. Proceeding inductively, we obtain the proposition.

<u>Proposition 1.8</u>: Under the hypothesis of proposition 1.7, if \tilde{f} and \tilde{g} both lie over f, then \tilde{f} and \tilde{g} are homotopic.

Proof:

$$\begin{split} &\mathbf{f}\varepsilon = \varepsilon \tilde{\mathbf{f}}_{0} = \varepsilon \tilde{\mathbf{g}}_{0} \text{ , so } \varepsilon (\tilde{\mathbf{f}}_{0} - \tilde{\mathbf{g}}_{0}) = 0 \text{ . Hence } \operatorname{im}(\tilde{\mathbf{f}}_{0} - \tilde{\mathbf{g}}_{0}) \subset \ker(\varepsilon) = \operatorname{im}(\mathbf{d}_{1}) \text{ ,} \\ &\text{and, since } \mathbf{X}_{0} \text{ is projective, there exists } \mathbf{D}_{0} \colon \mathbf{X}_{0} \longrightarrow \mathbf{Y}_{1} \text{ such that} \\ &\mathbf{d}_{1} \mathbf{D}_{0} = \tilde{\mathbf{f}}_{0} - \tilde{\mathbf{g}}_{0} \text{ . Now consider } \tilde{\mathbf{f}}_{1} - \tilde{\mathbf{g}}_{1} - \mathbf{D}_{0} \mathbf{d}_{1} \colon \mathbf{X}_{1} \longrightarrow \mathbf{Y}_{1} \text{ ,} \end{split}$$

 $\begin{array}{ll} d_1(\tilde{f}_1-\tilde{g}_1-D_0d_1) = \tilde{f}_0d_1-\tilde{g}_0d_1-d_1D_0d_1 = 0 \text{ , so } \operatorname{im}(\tilde{f}_1-\tilde{g}_1-D_0d_1) \subset \ker(d_1)=\operatorname{im}(d_2) \\ \text{and there exists } D_1\colon X_1 \longrightarrow X_2 \text{ such that } d_2D_1 = \tilde{f}_1-\tilde{g}_1-D_0d_1 \end{array}.$

Proposition 1.7: Let a and B be absulant, lett X be a projective nesolution of A and T c left resolution of B. If f: A -> B is normalism, then there exists a tempelschion f: X -> T such that

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From: Since X_0 to projective, there exists $X_0: X_0 \longrightarrow Y_0$, such that $X_0 = fe$. Since $\mathbf{e}_{X_0}^{\mathbb{Z}} d_{X_0}^{\mathbb{Z}} = fe$. Since $\mathbf{e}_{X_0}^{\mathbb{Z}} d_{X_0}^{\mathbb{Z}} = f_0$, for $f_0 = f_0$, $f_0 = f_0$. From the end there exists $f_0 = f_0 = f_0$. From this inductively, we obtain the proposition.

Proposition 1.8: Under the hypothesis of proposition 1.7, if I and g both lie over I, then I can g are hospingie.

Proof:

 $\frac{\partial_{1}(\tilde{\mathbf{i}}_{1} - \tilde{\mathbf{i}}_{1} - \mathbf{j}_{0}a_{1})}{\partial_{1} - \tilde{\mathbf{i}}_{0}a_{1} - \tilde{\mathbf{i}}_{0}a_{1} - \tilde{\mathbf{i}}_{0}a_{1} - \tilde{\mathbf{i}}_{0}a_{1} - \tilde{\mathbf{i}}_{0}a_{1}}) \left(\frac{\partial_{1}(\tilde{\mathbf{i}}_{1} - \tilde{\mathbf{i}}_{1} - \tilde{\mathbf{i}}_{0}a_{1})}{\partial_{1}(\tilde{\mathbf{i}}_{1} - \tilde{\mathbf{i}}_{0}a_{1})} \right) \left(\frac{\partial_{1}(\tilde{\mathbf{i}}_{1} - \tilde{\mathbf{i}}_{1} - \tilde{\mathbf{i}}_{0}a_{1})}{\partial_{1}(\tilde{\mathbf{i}}_{1} - \tilde{\mathbf{i}}_{0}a_{1})} \right) \left(\frac{\partial_{1}(\tilde{\mathbf{i}}_{1} - \tilde{\mathbf{i}}_{1} - \tilde{\mathbf{i}}_{0}a_{1})}{\partial_{1}(\tilde{\mathbf{i}}_{1} - \tilde{\mathbf{i}}_{0}a_{1})} \right) \left(\frac{\partial_{1}(\tilde{\mathbf{i}}_{1} - \tilde{\mathbf{i}}_{0}a_{1})}{\partial_{1}(\tilde{\mathbf{i}}_{0} - \tilde{\mathbf{i}}_{0}a_{1})} \right) \left(\frac{\partial_{1}(\tilde{\mathbf{i}}_{1} - \tilde{\mathbf{i}}_{0}a_{1})}{\partial_{1}(\tilde{\mathbf{i}}_{0} - \tilde{\mathbf{i}}_{0}a_{1})} \right) \left(\frac{\partial_{1}(\tilde{\mathbf{i}}_{0} - \tilde{\mathbf{i}}_{0}a_{1})}{\partial_{1}(\tilde{\mathbf{i}}_{0} - \tilde{\mathbf{i}}_{0}a_{1})} \right) \left(\frac{\partial_{1}(\tilde{\mathbf{i}_{0} - \tilde{\mathbf{i}}_{0}a_{1})}{\partial_{1}(\tilde{\mathbf{i}_{0} - \tilde{\mathbf{i}}_{0}a_{1})} \right) \left(\frac{\partial_{1}(\tilde{\mathbf{i}_{0} - \tilde{\mathbf{i}_{0}a_{1})}}{\partial_{1}(\tilde{\mathbf{i}_{0} - \tilde{\mathbf{i}_{0}a_{1}})} \right) \left(\frac{\partial_{1}(\tilde{\mathbf{i}_{0} - \tilde{\mathbf{i}_{0}a_{1})}}{\partial_{1}(\tilde{\mathbf{i}_{0} - \tilde{\mathbf{i}_{0}a_{1})}} \right) \left(\frac{\partial_{1}(\tilde{\mathbf{i}_{0} - \tilde{\mathbf{i}_{0}a_{1})}}{\partial_{1}(\tilde{\mathbf{i}_{0} - \tilde{\mathbf{i}_{0}a_{1})}} \right) \left(\frac{\partial_{1}(\tilde{\mathbf{i}_{0} - \tilde{\mathbf{i}_{0}a_{1})}}{\partial_{1}(\tilde{\mathbf{i}_{0} - \tilde{\mathbf{i}_{0}a_{1})}} \right) \left(\frac{\partial_{1}(\tilde{\mathbf{i}_{0} - \tilde{\mathbf{i}_{0}a_{1})}}{\partial_{1}(\tilde$

Proceeding inductively, the result is obtained.

Proposition 1.5': Every module has an injective resolution.

Proof: Given a module A, construct exact sequences

$$0 \longrightarrow A \longrightarrow Y^{\circ} \longrightarrow Z^{\circ} \longrightarrow 0$$

$$0 \longrightarrow Z^{\circ} \longrightarrow Y' \longrightarrow Z' \longrightarrow 0$$

$$0 \longrightarrow Z^{N-1} \longrightarrow Y^{N} \longrightarrow Z^{N} \longrightarrow 0$$

where the Y¹ are injective (proposition 3.10 of ch.3). Define d^N as the composition $Y^N \longrightarrow Z^N \longrightarrow Y^{N+1}$. Then

 $0 \longrightarrow A \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow ...$ is an injective resolution of A.

Note that there is no statement dual to corollary 1.6.

<u>Proposition 1.7'</u>: Let A and B be modules, let X be a right resolution of A and Y an injective resolution of B. If $f: A \longrightarrow B$ is a morphism, then there exists a translation $f: A \longrightarrow B$ over f.

Proof: Since Y^O is injective, there exists $\tilde{f}^O \colon X^O \longrightarrow Y^O$ such that $\tilde{f}^O \in = ef$. Since $d^O \tilde{f}^O \in = d^O ef = 0$, $d^O \tilde{f}^O$ induces a morphism $X^O/im(e) \longrightarrow Y^1$. Since $0 \longrightarrow X^O/im(e) \xrightarrow{d^O} X^1$ is exact, there exists $\tilde{f}^1 \colon X^1 \longrightarrow Y^1$ such that $\tilde{f}^1 d^O = d^O \tilde{f}^O$. Proceeding inductively, we obtain the proposition.

Proposition 1.8: Under the hypotheses of proposition 1.7', if \tilde{f} and \tilde{g} both lie over f, then \tilde{f} and \tilde{g} are homotopic.

Proof: $\begin{array}{c}
A & \xrightarrow{f} & B \\
X^{\circ} & \xrightarrow{f^{\circ}, g^{\circ}} & Y^{\circ} \\
\downarrow^{d^{\circ}} & \downarrow^{d^{\circ}}
\end{array}$

 $\begin{array}{l} \varepsilon f = \tilde{f}^{\circ} \varepsilon = \tilde{g}^{\circ} \varepsilon \;, \; so \quad (\tilde{f}^{\circ} - \tilde{g}^{\circ}) \varepsilon = 0 \;. \; \; \text{Hence} \quad \tilde{f}^{\circ} - \tilde{g}^{\circ} \quad \text{induces a morphism} \\ X^{\circ} / \mathrm{im}(\varepsilon) \longrightarrow Y^{\circ} \;, \; \text{and, since} \;\; 0 \longrightarrow X^{\circ} / \mathrm{im}(\varepsilon) \longrightarrow X^{\downarrow} \quad \text{is exact, there} \\ \text{exists a morphism} \quad D^{\downarrow} \colon X^{\downarrow} \longrightarrow Y^{\circ} \quad \text{such that} \quad D^{\downarrow} d^{\circ} = \tilde{f}^{\circ} - \tilde{g}^{\circ} \;. \end{array}$

Now consider $\tilde{r}^{l} - \tilde{g}^{l} - d^{O}D^{l}: X^{l} \longrightarrow Y^{l}$.

 $(\tilde{f}^1-\tilde{g}^1-d^0D^1)d^0=d^0\tilde{f}^0-d^0\tilde{g}^0-d^0D^1d^0=0$. Hence $\tilde{f}^1-\tilde{g}^1-d^0D^1$ induces a a morphism $X^1/\text{im}(d^0)\longrightarrow Y^1$, and, since $0\longrightarrow X^1/\text{im}(d^0)\xrightarrow{d^1}X^2$ is exact, there exists $D^2\colon X^2\longrightarrow Y^1$ such that $D^2d^1=\tilde{f}^1-\tilde{g}^1-d^0D^1$. Proceeding inductively, the result is obtained.

2. Resolutions of sequences.

Here we obtain some results concerning translations of resolutions over the modules of short exact sequences. We will prove our statements only for left resolutions, since the method of proof for right resolutions is step-by-step dualization just as in the proofs given above.

We first obtain a general lemma that will be of great importance in later applications.

<u>Lemma 2.1:</u> Suppose $0 \longrightarrow X' \xrightarrow{i} X \xrightarrow{j} X'' \longrightarrow 0$ is an exact sequence of complexes and translations. Then there is a canonical exact sequence

$$\dots \longrightarrow H_q(X^*) \xrightarrow{H_q(1)} H_q(X) \xrightarrow{H_q(j)} H_q(X^*) \xrightarrow{8} H_{q-1}(X^*) \longrightarrow \dots$$

$$\delta \text{ is called the connecting morphism.}$$

Proof: We are given a commutative diagram with exact rows and whose columns are O-sequences:

$$0 \longrightarrow X'_{N+1} \xrightarrow{1} X_{N+1} \xrightarrow{j} X''_{N+1} \longrightarrow 0$$

$$0 \longrightarrow X'_{N} \xrightarrow{1} X_{N} \xrightarrow{j} X''_{N} \longrightarrow 0$$

$$0 \longrightarrow X'_{N-1} \xrightarrow{1} X_{N-1} \xrightarrow{j} X''_{N-1} \longrightarrow 0$$

i) Definition of δ : Let $x^n \in Z_N(X^n)$. Let $x \in X_N$ be such that $j(x) = x^n$. jd(x) = 0, so $d(x) \in im(i)$. Let $x^i \in X_{N-1}$ be such that $i(x^i) = d(x)$. $di(x^i) = 0$, so $d(x^i) = 0$. Define $\delta(\overline{x}^n) = \overline{x}^i$. Now assume that $y^n \in Z_N(X^n)$ is such that $y^n = x^n \mod B_N(X^n)$. Choose any y and y^i such that $j(y) = y^n$, $i(y^i) = d(y)$. We must show that $x^i \equiv y^i \mod B_{N-1}(X^i)$. $x^n - y^n \in B_N(X^n)$, say $x^n - y^n = d(z^n)$. Let $z \in X_{N+1}$ be such that $j(z) = z^n$. $j(x-y-d(z)) = x^n - y^n - d(z^n) = 0$, so $x-y-d(z) \in im(i)$, say $i(z^i) = x-y-d(z)$, $z^i \in X_N^i$.

$$i(x^{i}-y^{i}-d(z^{i})) = d(x)-d(y)-id(z^{i})$$

= $d(x-y-i(z^{i})) = dd(z) = 0$.

Since i is a monomorphism, $x^*-y^*-d(z^*)=0$, $x^*\equiv y^*$ mod $B_{N-1}(X^*)$. δ is thus well-defined, and is clearly a morphism.

- ii) $H_N(X^*) \longrightarrow H_N(X) \longrightarrow H_N(X^*)$ is exact: Clearly $\ker(H_N(j)) \supset \operatorname{im}(H_N(i))$ since ji = 0. Let $\operatorname{xe} Z_N(X)$ be such that $H_N(j)(\overline{x}) = 0$. $j(x) \in B_N(X^*)$. Let $\operatorname{ye} X_{N+1}$ be such that $\operatorname{d} j(y) = j(x)$. $j(x-\operatorname{d}(y)) = j(x) \operatorname{d} j(y) = 0$, so $x-\operatorname{d}(y) \in \operatorname{im}(i)$, say $i(z^*) = x-\operatorname{d}(y)$. $\operatorname{d} i(z^*) = \operatorname{d} x-\operatorname{d} y = 0$, so $\operatorname{d}(z^*) = 0$. Thus $H_N(i)(\overline{z}^*) = \overline{x}$, and $\ker(H_N(j)) \subset \operatorname{im}(H_N(i))$.
- $\begin{array}{lll} &\text{iii)} & \text{$H_N(\textbf{X}) \longrightarrow \textbf{$H_N(\textbf{X}^n) \longrightarrow \textbf{$H_{N-1}(\textbf{X}^t)$ is exact:}} \\ &\text{$\texttt{x} \epsilon \textbf{$Z_N(\textbf{X})$ implies $\delta \textbf{$H_N(\textbf{j})(\overline{\textbf{x}}) = 0$ by construction of δ , so \\ &\text{$\texttt{ker}(\delta) \supset \text{im}(\textbf{$H_N(\textbf{j}))$. Let $$\texttt{$x^a \epsilon \textbf{$Z_N(\textbf{X}^n)$ and $\delta(\overline{\textbf{x}^n}) = 0$. Let $\textbf{j}(\textbf{x}) = \textbf{x}^n$,} \\ &\text{$\texttt{i}(\textbf{x}^t) = \textbf{d}(\textbf{x})$. $\texttt{$x^t \epsilon \textbf{$B_{N-1}(\textbf{X}^t)$, say $\textbf{x}^t = \textbf{d}(\textbf{y}^t)$. Now } \\ &\text{$\texttt{d}(\textbf{x} \text{-} \textbf{i}(\textbf{y}^t)) = \textbf{d}(\textbf{x}) \textbf{i}(\textbf{x}^t) = 0$, so $\textbf{x} \text{-} \textbf{i}(\textbf{y}^t) \epsilon \textbf{$Z_N(\textbf{X})$. $H_N(\textbf{j})(\overline{\textbf{x} \text{-} \textbf{i}(\textbf{y}^t)}) = $H_N(\textbf{j})(\overline{\textbf{x}}) = \overline{\textbf{x}^n$, and $\text{im}(\textbf{H}_N(\textbf{j})) \supset \text{ker}(\delta)$.} \end{array}$

if Defination of 6: Let $\mathbf{x}^n\mathbf{z}\in_{\mathbb{F}}(X^n)$. Let $\mathbf{x}\mathbf{x}^n\mathbf{z}$ be such that $\mathbf{j}(\mathbf{x})=\mathbf{x}^n$, $\mathbf{j}\mathbf{d}(\mathbf{x})=0$, so $\mathbf{d}(\mathbf{x})$. Let $\mathbf{x}^n\mathbf{z}\in_{\mathbb{F}^n}(X^n)$ be such that $\mathbf{j}(\mathbf{x}^n)=0$, so $\mathbf{d}(\mathbf{x}^n)=0$. Lettine $\mathbf{b}(\mathbf{x}^n)=\mathbf{x}^n$ for assistation that $\mathbf{y}^n\mathbf{z}\in_{\mathbb{F}^n}(X^n)$ is such that $\mathbf{y}^n=\mathbf{x}^n\log_{\mathbb{F}^n}(X^n)$. Choose that $\mathbf{y}^n=\mathbf{y}^n=\mathbf{y}^n$, $\mathbf{j}(\mathbf{y}^n)=\mathbf{d}(\mathbf{y})$. We must show that $\mathbf{j}(\mathbf{y})=\mathbf{y}^n$, $\mathbf{j}(\mathbf{y}^n)=\mathbf{d}(\mathbf{y})$. We must show that $\mathbf{j}(\mathbf{y})=\mathbf{y}^n$, $\mathbf{j}(\mathbf{y}^n)=\mathbf{d}(\mathbf{y})$. We must show that $\mathbf{j}(\mathbf{y})=\mathbf{y}^n\mathbf{z}\in_{\mathbb{F}^n}(X^n)$, soy $\mathbf{x}^n\mathbf{y}^n=\mathbf{d}(\mathbf{z}^n)$. Let $\mathbf{z}^n\in_{\mathbb{F}^n}(\mathbb{F}^n)$ be such that $\mathbf{j}(\mathbf{z})=\mathbf{z}^n$. $\mathbf{j}(\mathbf{x}^n\mathbf{y}^n+\mathbf{d}(\mathbf{z}))=\mathbf{x}^n\mathbf{j}^n\mathbf{d}(\mathbf{z}^n)$ as $\mathbf{z}^n\in_{\mathbb{F}^n}(\mathbb{F}^n)$ and $\mathbf{j}(\mathbf{z})=\mathbf{z}^n$. Let $\mathbf{z}^n\in_{\mathbb{F}^n}(\mathbb{F}^n)$ is $\mathbf{z}^n\mathbf{j}^n\mathbf{d}(\mathbf{z}^n)$ and $\mathbf{z}^n\mathbf{j}^n\mathbf{d}(\mathbf{z}^n)$ is $\mathbf{z}^n\mathbf{j}^n\mathbf{d}(\mathbf{z}^n)$ in $\mathbf{z}^n\mathbf{j}^n\mathbf{d}(\mathbf{z}^n)$ in $\mathbf{z}^n\mathbf{j}^n\mathbf{d}(\mathbf{z}^n)$ is $\mathbf{z}^n\mathbf{j}^n\mathbf{d}(\mathbf{z}^n)$ and $\mathbf{z}^n\mathbf{j}^n\mathbf{d}(\mathbf{z}^n)$ is $\mathbf{z}^n\mathbf{j}^n\mathbf{d}(\mathbf{z}^n)$.

 $i(x^*-y^*-\delta(c^*)) = d(x)-\delta(y)-ic(x^*)$

= $d(\mathbf{x} \cdot y \cdot \lambda(\mathbf{x}^{T}))$ = $dd(\mathbf{x}) \cdot = 0$.

Stace i is a monomorphism, $x^*-y^*-d(x^*)=0$, $x^*\approx y^*$ and $B_{N-1}(X^*)$. So is take well-defined, and is closely a morphism.

(Learly her($H_{M}(X)$) $\supset H_{M}(X)$ since M = G. Let $x \in X_{M}(X)$ be such that $H_{M}(A)(X) = O$. $J(x) \in H_{M}(X)$. Let $y \in H_{M+1}$ be such that

 $d_{2}(y) = j(x)$. j(x-d(y)) = j(x) - dj(y) = 0, so x-d(y)s in(i), say

 $\hat{\mathbf{r}}(u^*) = w \cdot a(y)$. Al(z^*) = dw-ddy = 0 , so A(z^*) = 0 . Thus

 $\vec{E}_{ij}(1)(\vec{z}^{\,i}) = \vec{x}$, and $\ker(\vec{E}_{ij}(\hat{z})) \subset \operatorname{Im}(\vec{E}_{ij}(1))$.

 $H_{N}(X) \longrightarrow H_{N}(X^{n}) \longrightarrow H_{N-1}(X^{n}) \text{ for example, the second second of the second se$

 $\operatorname{ve} Z_{W}(\mathbb{X})$ implies $\operatorname{dil}_{\mathbb{Z}}(\mathfrak{Z})(\overline{\mathbb{X}})=0$ by construction of \mathfrak{d} , so

 $\ker(S) \supseteq \operatorname{fu}(\mathbb{H}_{\mathbb{F}}(\mathfrak{Z})) \text{ . Let } \operatorname{sea}_{\mathbb{W}}(\mathbb{X}^n) \text{ and } \mathbb{S}(\mathbb{R}^n) = 0 \text{ . Let } \mathfrak{Z}(x) = x^n \text{ ,}$

 $\mathbb{E}(\mathbf{x}^t) = \mathbb{E}(\mathbf{x}^t)$, $\mathbf{x}^t \mathbf{\epsilon} \mathbb{E}_{\mathcal{Y} \subseteq \mathbb{L}}(\mathbb{R}^t)$, say $\mathbf{x}^t = \mathcal{E}(\mathbf{y}^t)$. Now

 $a(x-i(y^i)) = a(x) - i(x^i) = G$, so $x-i(y^i) \in \mathbb{F}_q(X)$. $\mathbb{F}_q(X) = a(x^i) =$

 $\hat{H}_{K}(\hat{\beta})(\vec{x}) = \vec{x}^{n}$, and $\text{Anc}(\hat{H}_{M}(\beta)) \supset \text{Herr}(\delta)$.

iv) $H_{N}(X^{v}) \longrightarrow H_{N-1}(X^{v}) \longrightarrow H_{N-1}(X)$ is exact:

If $x^* \in Z_{N-1}(X^*)$ and $\overline{x}^* = \delta(\overline{x}^n)$, then $i(x^*) = d(x)$ for some $x \in X_N$, and $H_N(i)(\overline{x}^*) = 0$, so $\ker(H_N(i)) \supset \operatorname{im}(\delta)$. Let $x^* \in Z_{N-1}(X^*)$ and $H_N(i)(\overline{x}^*) = 0$. $i(x^*) \in B_{N-1}(X)$, say $d(x) = i(x^*)$. Let $x^* = j(x)$. $d(x^n) = dj(x) = jd(x) = ji(x^*) = 0$. $\delta(\overline{x}^n) = \overline{x}^*$ by construction of δ , and $\operatorname{im}(\delta) \supset \ker(H_N(i))$. This completes the proof.

Corollary 2.2: If $0 \longrightarrow X^{i} \longrightarrow X \longrightarrow X^{m} \longrightarrow 0$ is an exact sequence of complexes and any two of X^{i} , X, X^{m} are exact, then so is the third.

Definitions 2.3: Let $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$ be an exact sequence. An exact sequence $0 \longrightarrow X' \xrightarrow{i} X \xrightarrow{j} X'' \longrightarrow 0$ of complexes, where X', X, X'' are left complexes (or left resolutions, etc.), over A', A, A'' and \tilde{i} , \tilde{j} are morphisms over i, j is called a left complex (or left resolution, etc.), over $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$. Right complexes over exact sequences are analogously defined.

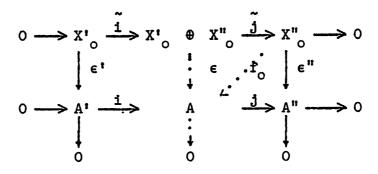
<u>Proposition 2.4</u>: If $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ is a left complex over $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ and if X' and X'' are projective complexes, then so is X.

Proof: For all N O \longrightarrow X'N \longrightarrow XN \longrightarrow X'N \longrightarrow O is split exact, XN is isomorphic X'N \oplus X'N, hence is a direct summand of a free module.

Corollary 2.5: If X' and X'' are projective resolutions of A' and A'', then X is a projective resolution of A.

Proposition 2.6: Let $0 \to A^1 \xrightarrow{i} A \xrightarrow{j} A^n \to 0$ be an exact sequence. Let X^i be a left resolution of A^i , X^n a projective complex over A^n . Then there exists a left complex X over A and maps i, j over i, j such that $0 \to X^i \xrightarrow{i} X \xrightarrow{j} X^n \to 0$ is a left complex over $0 \to A^i \xrightarrow{i} A \xrightarrow{j} A^n \to 0$. Proof: Set $X_N = X'_N \oplus X''_N$; let $\tilde{i}_N \colon X'_N \longrightarrow X_N$ and $\tilde{j}_N \colon X_N \longrightarrow X''_N$ be the canonical injection and projection. We must define a differential operator on X such that the desired commutativity relations are satisfied.

i) Consider degree 0.



Since X''_{o} is projective, there exists $f_{o} \colon X''_{o} \longrightarrow A$ such that $jf_{o} = \varepsilon''$. Define $\varepsilon \colon X'_{o} \oplus X''_{o} \longrightarrow A$ by $\varepsilon(x',x'') = i\varepsilon'(x') + f_{o}(x'')$. Then $\varepsilon i(x') = i\varepsilon'(x')$, $j\varepsilon(x',x'') = jf_{o}(x'') = \varepsilon'' j(x',x'')$, and the diagram commutes. We must show that ε is an epimorphism. Let $x\varepsilon A$. Let $x''\varepsilon X''_{o}$ be such that $j(x) = \varepsilon''(x'')$. Let $y = x-f_{o}(x'')$. $j(y) = j(x) - jf_{o}(x'') = j(x) - \varepsilon''(x'') = 0$, so $y\varepsilon$ im(i). Let $x'\varepsilon X''_{o}$ be such that $i\varepsilon'(x') = y$. Then

$$\epsilon(x^{i},x^{i}) = i\epsilon^{i}(x^{i}) + f_{o}(x^{i}) = y + f_{o}(x^{i}) = x$$
.

ii) For $N \ge 1$, let $f_N: X_N^n \longrightarrow X_{N-1}^n$ be, for the moment, arbitrary morphisms. Define $d_N(x^i, x^n) = (d_N^i x^i + f_N^i x^n, d_N^i x^n)$. Clearly

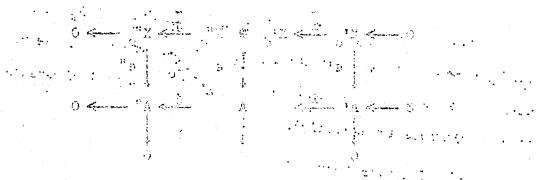
$$0 \longrightarrow X_{N-1} \longrightarrow X^{N-1} \longrightarrow X_{N-1} \longrightarrow 0$$

$$0 \longrightarrow X_{N-1} \longrightarrow X^{N} \longrightarrow X_{N-1} \longrightarrow 0$$

is commutative for all $N \ge 1$. We will define the f_N so that $\epsilon d_1 = 0$, $d_{N-1} d_N = 0$ for N > 1 .

Front: Set $X_{\Gamma} = X_{\Gamma}^{\dagger} \oplus X_{\Gamma}^{\dagger}$; let $\tilde{X}_{\Gamma}^{\dagger} : X_{\Gamma}^{\dagger} \longrightarrow \tilde{X}_{\Gamma}^{\dagger}$ and $\tilde{X}_{\Gamma}^{\dagger} : X_{\Gamma}^{\dagger} \longrightarrow \tilde{X}_{\Gamma}^{\dagger}$ be the demandral injection and projection. We nucl define a differential appearance on X such that the desired coefficient that the abstraction.

i) - Janeider dogwoo - C .



Equation is a projective, there writes $T_1: \mathbb{R}^n , \dots \geq A$ and that $f_{0}^{n} = e^n$. Before $e: X^n : \Theta^n X^{n+1} \xrightarrow{n} \mathbb{R}^n A \to g = (x^n), x^n) = 1e^n(x^n) + f_{0}(x^n)$. Then $ef_{0}(x^n) : f_{0}(x^n) : g^n g(x^n) : g^$

("x, x") - ie'(x') + f ("x) = ("x, +x's)

.11) From E. 2. 1, 1 or z_{ij} : $X^{ij}_{ij} = -\infty$ is z_{ij} ; be, for the moment, arbitrary surgicions. Define $d_{ij}(x^i, x^{ij}) = (\tilde{v}^i)_{ij} x^i + \tilde{z}_{ij} x^{ij}, \tilde{v}^{ij}_{ij} x^{ij})$. Clearly

in commutablish dust old - 1 s is will destine vid spans. Of - 1 so that.

 $q^{N-T}q^{N} = q^{N-T}(q_{1}^{N} + t_{1}^{N}, q_{1}^{N}) = (q_{1}^{N-T}q_{1}^{N} + q_{1}^{N-T}t_{1}^{N} + t_{1}^{N-T}q_{1}^{N}, q_{1}^{N-T}q_{1}^{N})$ $\epsilon q^T = \epsilon(q_1^T + \zeta^T, q_1^T) = \epsilon \epsilon_1 q_1^T + \epsilon q_1^T + \zeta^Q q_1^T = \epsilon \epsilon(q_1^T + \zeta^Q q_1^T)$

• $(0^{N_1}p^{T-N}J + N^{T-N}p) =$

Thus we wish $-f_0^{-1} = i \epsilon^i f_1$, $-f_{N-1}^{-1} f_N = i'_{N-1} f_N$.

$$0 = \lim_{t \to \infty} b^{t} \Rightarrow - \lim_{t \to \infty} x$$

$$\int_{0}^{t} \int_{0}^{t} \int_{0}^{t} d^{t} d$$

since X_1^{L} is projective, there exists $f_1:X_1^{L}\longrightarrow X_1^{C}$ such that commutes, giving im(- f_0d^n) \subseteq ker(j), and the row is exact. Hence,

respace 1.5 = -1.5 = -1.5 = 1.5 =

$$X \stackrel{\mathsf{T}_{\mathsf{L}} d^{\mathsf{H}} \mathcal{L}}{\overset{\mathsf{L}}{\overset{\mathsf{L}}{\overset{\mathsf{L}}{\overset{\mathsf{L}}}}} \overset{\mathsf{L}}{\overset{\mathsf{L}}} \overset{\mathsf{L}}} \overset{\mathsf{L}}{\overset{\mathsf{L}}} \overset{\mathsf{L}}{\overset{\mathsf{L}}} \overset{\mathsf{L}}{\overset{\mathsf{L}}} \overset{\mathsf{L}}{\overset{\mathsf{L}}} \overset{\mathsf{L}}} \overset{\mathsf{L}}{\overset{\mathsf{L}}} \overset{\mathsf{L}}} \overset{\mathsf{L}}{\overset{\mathsf{L}}} \overset{\mathsf{L}}} \overset{\mathsf{L}}{\overset{\mathsf{L}}} \overset{\mathsf{L}}} \overset{\mathsf{L}}{\overset{\mathsf{L}}} \overset{\mathsf{L}}} \overset{\mathsf{L}}{\overset{\mathsf{L}}} \overset{\mathsf{L}}{\overset{\mathsf{L}}} \overset{\mathsf{L}}} \overset{\mathsf{L}}{\overset{\mathsf{L}}} \overset{\mathsf{L}}} \overset{\mathsf{L}} \overset{\mathsf{L}}} \overset{\mathsf{L}} \overset{\mathsf{L}}} \overset{\mathsf{L}} \overset{\mathsf{L}}} \overset{\mathsf{L}}} \overset{\mathsf{L}} \overset{\mathsf{L}} \overset{\mathsf{L}}} \overset{\mathsf{L}} \overset{\mathsf{L}}} \overset{$$

tnat d'lle -fld". Finally, commutes and the row is exact, so there exists $f_2\colon X^n \longrightarrow X^i$ such

$$X_{n} = \frac{\alpha_{n} N}{\alpha_{n} N^{+T}} \times \frac{N^{-T}}{\alpha_{n} N^{-T}} \times \frac{N^{-S}}{\alpha_{n} N^{-T}} \times \frac{N^{-S}}{\alpha_{n} N^{+T}} = 0$$

Tuts completes the proof. extate $t_{M+1}: X_{M+1} \longrightarrow X_{M}$ such that $d'_{M+1} = t_{M}d'_{M+1}$. commutes, by induction, and the row is exact by hypothesis, so there

tive resolutions of A^{1} and A^{11} , then X will necessarily be a Corollary 2.7: In the proposition above, if X' and X" are projec-

projective resolution of A.

$$\begin{split} \mathbf{e} \mathbf{d}_{1} &= \mathbf{e} (\mathbf{d}^{*}_{1} + \mathbf{f}_{1}, \mathbf{d}^{n}_{1}) = \mathbf{f} \mathbf{e}^{*} \mathbf{d}^{*}_{1} + \mathbf{f} \mathbf{e}^{*} \mathbf{f}_{1} + \mathbf{f}_{0} \mathbf{d}^{n}_{1} + \mathbf{f}$$

.使."

Thus we wish -10^{4} = 10^{4} -10^{4} -10^{4} $= 0^{4}$ -10^{4} . Now

commutes, giving $4m(-r_0^3a_1^n) \subseteq \ker(\mathfrak{F})$, and the row is exact. Hence, since \mathbb{X}_1^n is projective, there exists $f_1\colon\mathbb{X}_1^n\longrightarrow\mathbb{X}_0^n$ such that is $f_1=-f_0^3a_1^n$.

committee and the row is exact, co there exists $r_2\colon X''_2\longrightarrow X'_1$ such that $\delta^*_1 r_2=-r_1 d''_2$. Finally,

commutes, by isduction, and the row is exact by hypothesis, so there exists if we are the constituted in the compilities the jaren.

Corollary 2.7: In the proposition shows, if X' and X" are projective resolutions of A' and A", then X will necessarily be a projective resolution of A.

Proposition 2.8: Let

$$0 \longrightarrow A' \xrightarrow{1} A \xrightarrow{j} A'' \longrightarrow 0$$

$$\downarrow g' \qquad \downarrow g \qquad \downarrow g''$$

$$0 \longrightarrow B' \xrightarrow{k} B \xrightarrow{\ell} B'' \longrightarrow 0$$

be a commutative diagram with exact rows. Let $0 \to X' \xrightarrow{\tilde{i}} X \xrightarrow{\tilde{j}} X'' \to 0$ be a split exact left complex over $0 \to A' \xrightarrow{\tilde{i}} A \xrightarrow{\tilde{j}} A'' \to 0$ and $0 \to Y' \xrightarrow{\tilde{k}} Y \xrightarrow{\tilde{j}} Y'' \to 0$ be a split exact left complex over $0 \to B' \xrightarrow{\tilde{k}} B \xrightarrow{\tilde{j}} B'' \to 0$. Further, let X'' be a projective complex and let Y' be exact. Then if $\tilde{g}': X' \to Y'$ and $\tilde{g}'': X'' \to Y''$ are translations over g' and g'', there exists a translation $\tilde{g}: X \to Y$ over g such that

$$0 \longrightarrow X' \xrightarrow{\tilde{1}} X \xrightarrow{\tilde{j}} X'' \longrightarrow 0$$

$$\downarrow_{\tilde{g}'} \qquad \downarrow_{\tilde{g}} \qquad \downarrow_{\tilde{g}''} \qquad 0$$

$$0 \longrightarrow Y' \xrightarrow{\tilde{k}} Y \xrightarrow{\tilde{\ell}} Y'' \longrightarrow 0$$

is a commutative diagram of complexes and translations.

Proof: Writing X_N as $X^!_N \oplus X^n_N$ and Y_N as $Y^!_N \oplus Y^n_N$, we see that $\tilde{g}_N(x^!,x^n) = (\tilde{g}^!_N(x^!_N) + q_N(x^n_N), \tilde{g}^n_N(x^n_N))$ is necessary for $0 \longrightarrow X^!_N \longrightarrow X_N \longrightarrow X^n_N \longrightarrow 0$ $\downarrow \tilde{g}^!_N \qquad \downarrow \tilde{g}^n_N \qquad \downarrow \tilde{g}^n_N$

 $0 \longrightarrow X_i^M \longrightarrow X^M \longrightarrow X_i^M \longrightarrow 0$

to commute, where $q_N: X^n \longrightarrow Y^n$ is to be determined. The problem

$$\tilde{X} \xrightarrow{\tilde{g}} \tilde{Y}$$

$$\epsilon \downarrow \qquad \qquad \downarrow \epsilon$$

$$A \xrightarrow{g} B$$

is to choose the q_N such that

commutes. We write d',d,d",e',e,e" and δ',δ,δ'' , ϵ' , ϵ , ϵ , ϵ'' for the differentiation and augmentation morphisms of X',X,X" and Y',Y,Y".

O em nu em X em Q. dell capar dorne units meablig anignamum e eq be a split exact left complex over 0 -> 4 1 A 1 A 0 and 0 -> B: ... B - B F -> 0': Nurther, let X be a projective con.... plex and let Y' be exact. Then in g': X' -> Y' and 'E": X" -> Y'' are transletions over go show go, above extate a translabion S: X -- Y over w such that

is a contraterity dispress of completes and translations.

The off Sections X was a way as a X section the off we see that $\frac{n}{2}(x^n,x^n) = (\frac{n}{2}^n(x^n) + \frac{1}{2}(x^n), \frac{n}{2}^n(x^n))$ is necessary OSTINE OF THE ONE OF TO1

to commute, where $q_{H}\colon X_{W}^{n} \dashrightarrow X_{V}$ is to be determined. The problem is to observe the opening that

consentes. We write e^{i} , e^{i} for the differentiation and augmentential participations of X^{*}, X, X^{*} and Y^{*}, Y, Y^{*} . Let $f_o \colon X''_o \longrightarrow A$, $f_N \colon X''_N \longrightarrow X'_{N-1}$ and $\phi_o \colon Y''_o \longrightarrow B$, $\phi_N \colon Y''_N \longrightarrow Y'_{N-1}$ be morphisms as in the proof of the previous proposition. (These necessarily exist, since the sequences of complexes are split exact.)

By hypothesis, then, we have relations:

$$\begin{split} \mathbf{e} &= \mathbf{i} \mathbf{e}^{!} + \mathbf{f}_{0}, \ \mathbf{i} \mathbf{e}^{!} \mathbf{f}_{1} + \mathbf{f}_{0} \mathbf{d}^{"}_{1} = 0 \ , \quad \mathbf{d}^{!}_{N-1} \mathbf{f}_{N} + \mathbf{f}_{N-1} \mathbf{d}^{"}_{N} = 0 \ , \\ \mathbf{e} &= \mathbf{k} \mathbf{e}^{!} + \mathbf{\phi}_{0}, \ \mathbf{k} \mathbf{e}^{!} \mathbf{\phi}_{1} + \mathbf{\phi}_{0} \delta^{"}_{1} = 0 \ , \quad \delta^{!}_{N-1} \mathbf{\phi}_{N} + \mathbf{\phi}_{N-1} \delta^{"}_{N} = 0 \ \text{and} \\ \mathbf{g}^{!} \mathbf{e}^{!} &= \mathbf{e}^{!} \widetilde{\mathbf{g}}^{!}_{0}, \ \widetilde{\mathbf{g}}^{!}_{N-1} \mathbf{d}^{!}_{N} = \delta^{!}_{N} \widetilde{\mathbf{g}}^{!}_{N} \ , \quad \mathbf{g}^{"} \mathbf{e}^{"} = \mathbf{e}^{"} \widetilde{\mathbf{g}}^{"}_{0}, \quad \widetilde{\mathbf{g}}^{"}_{N-1} \mathbf{d}^{"}_{N} = \delta^{"}_{N} \widetilde{\mathbf{g}}^{"}_{N} \ . \end{split}$$

We wish to obtain ge = $\epsilon \tilde{g}_0$, $\tilde{g}_{N-1} d_N = \delta_N \tilde{g}_N$. These relations then take the forms

i)
$$k \epsilon' q_0 = -\phi_0 \tilde{g}''_0 + g f_0$$

$$(\epsilon \tilde{g}_0 = \epsilon (\tilde{g}'_0 + q_0, \tilde{g}''_0) = k \epsilon' \tilde{g}'_0 + k \epsilon' q_0 + \phi_0 \tilde{g}''_0;$$
 $ge = gie' + g f_0 = k g'e' + g f_0 = k \epsilon' \tilde{g}'_0 + g f_0).$

Finally, then, we observe that in the following diagrams, the rows are exact and the diagrams commute, so that, by the projectivity of X'', the desired morphisms are obtained:

$$-\phi_{0}\tilde{g}_{0}^{"}+gf_{0}$$

$$-\phi_{0}\tilde{g}_{0}^{"}+gf_{0}$$

$$\vdots -\ell\phi_{0}\tilde{g}_{0}^{"}+\ell gf_{0} = -\epsilon^{"}\tilde{g}_{0}^{"}+g^{"}f_{0} = -\epsilon^{"}\tilde{g}_{0}^{"}+g^{"}e^{"}=0$$

$$Y'_{0} \xrightarrow{k\epsilon'} B \xrightarrow{\ell} B"$$

Tet for $T'_0 \longrightarrow F$, $T''_{N-1} \longrightarrow F'_{N-1}$ and $\Phi_0: T''_0 \longrightarrow F$, $\Psi_N: V''_N \longrightarrow V'_{N-1} \quad \text{ne morphisms an in the proof of the previous proposition. (These necessarily thus, since the sequences of complemes are option exact.)$

By bypothegis, than, we have relations:

We wish so quistin $g_0 = e_{K_0}$. $e_{K_0} d_K = h_{\| V_{K_0} \|}$. Whose relations then take the form

 $\frac{1}{2} + \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{1}{2}$

Finally, then, we observe tout in the following diagrams, the rows are exact and the the constate, so that, by the projectivity of Z^n , the desired correlates are obtained:

$$\tilde{g}'_{0}f_{1}+q_{0}d''_{1}-\phi_{1}\tilde{g}''_{1} \qquad \vdots \qquad ke'\tilde{g}'_{0}f_{1}+ke'q_{0}d''_{1}-ke'\phi_{1}\tilde{g}''_{1} \\
Y'_{1} \xrightarrow{\delta'_{1}} Y'_{0} \xrightarrow{ke'_{1}} B = kg'e'f_{1}-\phi_{0}\tilde{g}''_{0}d''_{1}+gf_{0}d''_{1}+\phi_{0}\delta''_{1}\tilde{g}''_{1} \\
= g(ie'f_{1}+f_{0}d''_{1})+\phi_{0}(\delta''_{1}\tilde{g}''_{1}-\tilde{g}''_{0}d''_{1}) = 0$$

$$= \underbrace{\tilde{g}_{1}}_{N-1} \underbrace{f_{N-1}}_{Q_{1}} \underbrace{f_{N-1}} \underbrace{f_{N-1}}_{Q_{1}} \underbrace{f_{N-1}}_{Q_{1}} \underbrace{f_{N-1}}_{Q_{1}}$$

<u>Proposition 2.4'</u>: If $0 \to Y' \to Y \to Y'' \to 0$ is a right complex over the exact sequence $0 \to A' \to A \to A'' \to 0$ and if Y' and Y'' are injective complexes, then so is Y.

Corollary 2.5': If Y' and Y" are injective resolutions of A' and A'', then Y is an injective resolution of A.

<u>Proposition 2.6'</u>: Let $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ be an exact sequence. Let Y' be an injective complex over A', Y'' a right resolution of A''. Then there exists a right complex Y over A and maps \tilde{i}, \tilde{j} over i, j such that $0 \to Y' \xrightarrow{\tilde{i}} Y \xrightarrow{\tilde{j}} Y'' \to 0$ is a right complex over $0 \to A' \xrightarrow{\tilde{i}} A \xrightarrow{\tilde{j}} A'' \to 0$.

Corollary 2.7': In the proposition above, if Y' and Y" are injective resolutions of A' and A", then Y will necessarily be an injective resolution of A.

 $0 = (\begin{bmatrix} 1 & 9 & 2 & -1 & 3 & 1 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \end{bmatrix})^{2k} + (\begin{bmatrix} 1 & 9 & 1 & 1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \\ 1 & 9 & 2 & -1 & 3 & 3 \\ 1 & 9 & 2 & 2 & 3 \\ 1$

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0 = "," 2(No 1-2, 12 + 12, 12 - 30) - (N, 01 - 13, 13, 12 - 13, 12

ireposition 2.5% If $0 \to 2^* \to Y \to Y^* \to 0$, is a right complex over the exact sequence $0 \to X^* \to X \to X^* \to 0$ and if and if sective templates, when so is Y.

Corollary 2.51: If Y' and Y" are injective resolutions of A' and A'; then Y is an injective resolution of A.

Proposition 2.61: 1.35 0 -> 21 $\stackrel{?}{\longrightarrow}$ 2 $\stackrel{?}{\longrightarrow}$ 4" -> 0 be an encet sequence. Let Y' be in this chive complex even A', Y'' a right resolution of A". Then there exists a right complex to complex to $\stackrel{?}{\longrightarrow}$ Y'' $\stackrel{?}{\longrightarrow}$ 0 is a right complex even $0 \rightarrow A'$ $\stackrel{?}{\longrightarrow}$ A' $\stackrel{\longrightarrow}$ A' $\stackrel{?}{\longrightarrow}$ A' $\stackrel{?}{\longrightarrow}$ A' $\stackrel{?}{\longrightarrow}$ A' $\stackrel{?}{\longrightarrow}$

Corellary 2.71: In whe proposition above, if Y' and Y" are injective resolutions of A' and .", then Y will proposed by an injective resolution of A.

Proposition 2.81: Let

$$0 \longrightarrow A' \xrightarrow{1} A \xrightarrow{j} A'' \longrightarrow 0$$

$$\downarrow g' \qquad \downarrow g \qquad \downarrow g''$$

$$0 \longrightarrow B' \xrightarrow{k} B \xrightarrow{\ell} B'' \longrightarrow 0$$

be a commutative diagram with exact rows. Let $0 \to X^{!} \xrightarrow{\tilde{1}} X \xrightarrow{\tilde{j}} X'' \to 0$ be a split exact right complex over $0 \to A^{!} \xrightarrow{\tilde{1}} A \xrightarrow{\tilde{j}} A'' \to 0$ and $0 \to Y^{!} \xrightarrow{\tilde{k}''} Y \xrightarrow{\tilde{k}''} Y'' \to 0$ be a split exact right complex over $0 \to B^{!} \xrightarrow{k} B \xrightarrow{\ell} B'' \to 0$. Further, let X'' be exact and let $Y^{!}$ be an injective complex. Then if $\tilde{g}^{!} \colon X^{!} \to Y^{!}$ and $\tilde{g}'' \colon X'' \to Y^{!}$ are translations over $g^{!}$ and $g^{''}$, there exists a translation $\tilde{g} \colon X \to Y$ over g such that

$$0 \longrightarrow X' \xrightarrow{\tilde{I}} X \xrightarrow{\tilde{J}} X'' \longrightarrow 0$$

$$\downarrow \tilde{g}' \qquad \downarrow \tilde{g} \qquad \downarrow \tilde{g}''$$

$$0 \longrightarrow Y' \xrightarrow{\tilde{k}} Y \xrightarrow{\tilde{\ell}} Y'' \longrightarrow 0$$

is a commutative diagram of complexes and translations.

3. Construction of Tor(A,B)

Let A be a right A-module, B a left A module. Then $A \otimes B$ is an Abelian group (see ex. 10, 11 ch. 3). Recall that if $0 \longrightarrow A^1 \longrightarrow A \longrightarrow A^{"} \longrightarrow 0$ is exact, then $A^1 \otimes B \longrightarrow A \otimes B \longrightarrow A^{"} \otimes B \longrightarrow 0$ is exact, and if $0 \longrightarrow B^1 \longrightarrow B \longrightarrow B^{"} \longrightarrow 0$ is exact, then so is $A \otimes B^1 \longrightarrow A \otimes B \longrightarrow A \otimes B^{"} \longrightarrow 0$. In this section we will construct objects by means of which the behavior of tensored exact sequences on the left may be studied.

Let X be a complex of right Λ -modules, Y a complex of left Λ -modules. $X \otimes Y$ is a graded module with $(X \otimes Y)_N = \bigoplus_{i+j=N} X_i \otimes Y_j$. For notational convenience, we write $X_i \otimes Y_j = X^i \otimes^j Y$.

Proposition 2.01: Let

be a computative diagram with exect rows: Let-

is a commutative disgrem of completes and translations.

3. Construction of Ebr(A,B)

Let A be a sight A-module, B a left A module. Then A \otimes B is an Abelian giving (see ex. 10, 11 ab. 3). Recall that if $0 \longrightarrow A' \longrightarrow A'' \longrightarrow 0$ is exact, then

At due coaxe at 0 - 3 "A" and 1 and 4 - 8 9 'A

0 -> E' -> B -> B" -> 0 % exact, then so is . ?

A \otimes B' \longrightarrow A \otimes B \longrightarrow A' \otimes B'' \longrightarrow 0. In this rection we will construct objects by means of which the behavior of tensored exact sequences on the left may be rtudied.

Let X be a complex of right A-modules. Y a complex of lart A-modules. X $\mathbb C$ Y is a gracel module with $(X \otimes Y)_N = 0$ X, $\mathbb C$ Y, when notational convenience, we write $\mathbb X_* \oplus \mathbb Y_* = \mathbb X^{\frac 1 + \frac 1 +$

Now let X',X" be further complexes of right A-modules, Y' and Y" of left A-modules. Let $f\colon X\longrightarrow X'$ be a morphism of degree p, $g\colon Y\longrightarrow Y'$ be a morphism of degree q. Define $f^i \not \otimes^j g = (-1)^{iq} f_i \otimes g_j$ and $(f\otimes g)_N = \theta_{i+j=N} f^i \not \otimes^j g$. Suppose $f'\colon X'\longrightarrow X''$ and $g'\colon Y'\longrightarrow Y''$ are morphisms of degrees p' and q'. Then f'f and g'g are morphisms of degree p+p' and q+q'. Further, we have $f'f^i \not \otimes^j g'g = (-1)^{i(q+q')} f'_{i-p} f_i \otimes g'_{j-q} g_j$ $= (-1)^{i(q+q')} (f'_{i-p} \otimes g'_{j-q}) (f_i \otimes g_j)$ $= (-1)^{i(q+q')} (-1)^{-(i-p)q'} (f'^{i-p} \not \otimes^j -q_g) (-1)^{-iq} (f^i \not \otimes^j g)$ $= (-1)^{pq'} (f'^{i-p} \not \otimes^j -q_g) (f^i \not \otimes^j g)$ $= (-1)^{pq'} (f'^{i-p} \not \otimes^j -q_g) (f^i \not \otimes^j g)$

We now return to the consideration of $X \otimes Y$. We will define a differential operator on $X \otimes Y$ by $d = d_X \otimes i_Y + i_X \otimes d_Y$ where d_X and d_Y are the differential operators on X and Y and i_X and i_Y are the identities of X and Y. d is of degree l, and dd = 0 since $dd = d_X d_X \otimes i_Y i_Y + i_X d_X \otimes d_Y i_Y + d_X i_X \otimes i_Y d_Y + i_X i_X \otimes d_Y d_Y$ $= i_X d_X \otimes d_Y i_Y + d_X i_X \otimes i_Y d_Y$ $= (-1)(i_X \otimes d_Y)(d_X \otimes i_Y) + (d_X \otimes i_Y)(i_X \otimes d_Y) = 0$.

Thus $X \otimes Y$ is given the structure of a complex.

Before defining Tor(A,B), we prove the Lemma 3.1: Suppose X and Y are left complexes, X_q is flat for all q and Y is exact. Then $X \otimes Y$ is exact. Proof: i) Suppose $Y_q \neq 0$ for q=s and q=s+l only. Then $0 \longrightarrow Y_{s+1} \stackrel{d}{\longrightarrow} Y_s \longrightarrow 0$ is exact. Let $f_s \colon Y_s \longrightarrow Y_{s+1}$ be the inverse isomorphism to d_{s+1} , $f_q = 0$, $q \neq s$. Then $df + fd = i_y$. Define D: $X \otimes Y \longrightarrow X \otimes Y$ by $D = i_y \otimes f$.

 $= (-1)^{1/(2+2^{-1})}(x_{1-2} \otimes x_{1-2})(x_{2} \otimes x_{3})$ $= (-1)^{1/(2+2^{-1})}(x_{1-2} + (1-2)^{1/2}(x_{2} + (2-2)^{1/2}(x_{3}))(x_{3} + (2-2)^{1/2}(x_{3} + (2-2)^$

We now resturn to the equalisantial of X G Y. We will define a differential eparator on X (4% by d = \hat{a}_X G Y. \hat{a}_X where \hat{a}_y where \hat{a}_y are the differential eparators on X and Y and \hat{a}_X and \hat{a}_Y are the identities of \hat{X} and \hat{Y} . A is or degree 1, and \hat{a}_S and \hat{a}_S of \hat{a}_S degree \hat{a}_S and $\hat{$

 $= 2 \frac{1}{2} \frac{1}{2}$

Thus X C Y is given the statestairs of a complete.

Peters dainaing the (A.B), we prove that

Lemma 3.1: Suppose X row, Y are Tell complement X_q is flat for all q and Y is exobly then X_q Y is exact.

From () Suppose $V_0 \neq 0$ for q=s and q=s+1 anly. This is $0 \rightarrow V_{g+1} = 0$ is small. Let $V_0 \rightarrow V_{g+1} = 0$ be the inverse isomorphism to $V_{g+1} = 0$, $q\neq s$. Then $ds+sV_0=t_0$. Define v.

Then
$$dD = d_{x}i_{x} \otimes i_{y}f + i_{x}i_{x} \otimes d_{y}f$$

 $Dd = i_{x}d_{x} \otimes fi_{y} + i_{x}i_{x} \otimes fd_{y}$
 $dD + Dd = i_{x}i_{x} \otimes d_{y}f + i_{x}i_{x} \otimes fd_{y}$
 $= i_{x} \otimes (d_{y}f + fd_{y}) = i_{x} \otimes i_{y} = i_{x \otimes y}$.

Hence the identity of $X \otimes Y$ is homotpic to the zero map and $H_N(X \otimes Y) = 0$ for all N.

II) Proceeding inductively, assume $Y_q = 0$ for q < r and for q > N+1 > r. Define Y' by $Y'_q = 0$ for $q \ne N$, N+1, $Y'_N = B_N(Y)$, $Y'_{N+1} = Y_{N+1}$; $H_q(X \otimes Y') = 0$ for all q by step i). Define Y'' such that $0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ is exact. Since $H_q(Y') = H_q(Y) = 0$ for all q, $H_q(Y'') = 0$ for all q. $Y''_q = 0$ if q < r and if q > N, so, by induction, $H_q(X \otimes Y'') = 0$ for all q. Since X is flat, $0 \longrightarrow X \otimes Y' \longrightarrow X \otimes Y \longrightarrow X \otimes Y'' \longrightarrow 0$ is exact (and is a sequence of translations), and $H_q(X \otimes Y) = 0$ for all q.

iii) Now assume the original hypotheses. For $s \ge 0$, $(X \otimes Y)_s = \overset{s}{0} \quad X_i \otimes Y_{s-i} \quad \text{Define } Y' \quad \text{by } Y'_q = Y_q \quad \text{if } q > s+1 \;, \\ Y'_{s+1} = B_{s+1}(Y) \;, \quad Y'_q = 0 \quad \text{if } q < s+1 \;. \quad \text{Define } Y'' \quad \text{such that} \\ 0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0 \quad \text{is exact.} \quad H_q(Y') = 0 \quad \text{for all } q \quad \text{by} \\ \text{construction since } H_q(Y) = 0 \quad \text{for all } q \;. \quad \text{Hence } H_q(Y'') = 0 \quad \text{for all } q \;. \\ Y''_q = 0 \quad \text{for } q < 0 \;, \quad Y''_q = 0 \quad \text{for } q > s+1 \;, \quad \text{so by step ii)} \\ H_q(X \otimes Y'') = 0 \quad \text{for all } q \;. \quad H_g(X \otimes Y') = 0 \quad \text{by construction, hence} \\ 0 \longrightarrow H_g(X \otimes Y) \longrightarrow 0 \quad \text{is exact,} \quad H_g(X \otimes Y) = 0 \;. \quad \text{Since } s \quad \text{was} \\ \text{arbitrary,} \quad H_q(X \otimes Y) = 0 \quad \text{for all } q \;. \quad \text{for all } q \;. \\ \end{array}$

Note that the hypotheses and proof of the lemma are symmetric in X and Y: If X is exact and Y is flat, $H_q(X \otimes Y) = 0$ for all q.

Hence the identity of X % Y is bomothic to the new map and $E_{\mu}(Z \oplus Y) = 0$ for all $E_{\mu}(Z \oplus Y)$

II) Froceeding investigaly, assume $Y_0 = 0$ for q < r and for q > 20 for all q > 20 fo

isi) Now assume the injectables. For a ≥ 0 , $(X \otimes Y)_g = \frac{1}{6} \cdot R_1 \otimes R_2 \cdot R_2 \cdot R_3 \cdot R_4 \cdot R_4 \cdot R_5 \cdot R_4 \cdot R_5 \cdot R_$

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Definition 3.2: Let A be a right Λ -module, B a left Λ -module, X a projective resolution of A, and Y a projective resolution of B. Define $\operatorname{Tor}_{\mathbb{N}}(A\otimes B)=H_{\mathbb{N}}(X\otimes Y)$. We must prove that $\operatorname{Tor}_{\mathbb{N}}(A\otimes B)$ is independent of the choice of the projective resolutions X and Y. Instead of proving this directly, we first prove

<u>Proposition 3.3</u>: $H_q(A \otimes Y)$, $H_q(X \otimes Y)$, and $H_q(X \otimes B)$ are isomorphic for all q, where A and B are regarded as complexes concentrated in degree O with d=O.

Proof: $Y \xrightarrow{\epsilon} B \longrightarrow 0$ may be regarded as a translation $(\epsilon_q = 0 \text{ for } q \neq 0)$. Define Y^i by $Y^i_q = Y_q$ for $q \neq 0$, $Y^i_0 = \ker(\epsilon_0)$. Then $0 \longrightarrow Y^i \longrightarrow Y \longrightarrow B \longrightarrow 0$ is an exact sequence of translations. $H_q(Y^i) = 0$ for all q since $\ker(\epsilon_0) = B_0(Y) = Y^i_0$. By lemma 3.1, $h_q(X \otimes Y^i) = 0$ for all q. By lemma 2.1, there is an exact sequence

$$\begin{split} &\dots \longrightarrow \text{O} = \text{H}_q(\text{X}\!\!\otimes\!\text{Y}^*) \longrightarrow \text{H}_q(\text{X}\!\!\otimes\!\text{Y}) \longrightarrow \text{H}_q(\text{X}\!\!\otimes\!\text{B}) \longrightarrow \text{H}_{q-1}(\text{X}\!\!\otimes\!\text{Y}^*) = \text{O} \longrightarrow \dots \\ \text{so } \text{H}_q(\text{X} \otimes \text{Y}) \text{ is isomorphic to } \text{H}_q(\text{X} \otimes \text{B}) \text{ for all } \text{q} \text{.} \\ \text{H}_q(\text{X} \otimes \text{Y}) \text{ is isomorphic to } \text{H}_q(\text{A} \otimes \text{Y}) \text{ for all } \text{q} \text{.} \end{split}$$

Proposition 3.4: Tor_Q(A,B) is independent of the choice of X and Y. Proof: Let X and X' be projective resolutions of A. By proposition 1.5, there exists $f: X \longrightarrow X'$ and $g: X' \longrightarrow X$ lying over the identity i_A of $A \longrightarrow A$. By proposition 1.8, $gf: X \longrightarrow X$ lying over i_A is homotopic to $i_X: X \longrightarrow X$, say $dD + Dd = i_X - gf$. Then $i_X \otimes i_B - gf \otimes i_B = (i_X - gf) \otimes i_B = (dD + Dd) \otimes i_B = (dD + Dd) \otimes i_B = dD \otimes i_B + Dd \otimes i_B$

= $(d \otimes i_{R})(D \otimes i_{R})+(D \otimes i_{R})(d \otimes i_{R})$, and,

Definition 3.8: Not A be a right A-sodule, B a left A-module. If a projective resolution of A, and Y a projective resolution of B. Define Tory(A C B) = $\Pi_{ij}(Z \in Y)$. We must prove that story(A (B) is independent of the choice of the projective resolutions. If and Y. Instead of proving this directly, we first prove

Propositive Right H_Q(A G X), H_Q(R C X), and H_Q(X G B) are isonorphic for fill q / whose A and B are reserved as completes concentrated in degree 0 with A = 0.

Proof Y = 0 for $0 \neq 0$. Define Y' by $Y'_0 = Y_0$ for $0 \neq 0$. Then $Y'_0 = Y_0 = Y_0$ is an exact sequence of translations. If $(Y'_0) = 0$ for all $0 \neq 0$ is an exact sequence by lemma 3.1, . b $(Y'_0) = 0$ for all $0 \neq 0$ forms as a exact sequence.

... $\Rightarrow 0 = H_{q}(XY^{1}) + \Rightarrow H_{q}(XX) \Rightarrow H_{q}(XX) + \Rightarrow H_{q}(XX^{1}) = 0 + 2^{-1}$.

so $H_{q}(X \otimes Y)$ is isomorphic to $H_{q}(X \otimes B)$. Son all q. Similarly, $H_{q}(X \otimes Y)$ is isomorphic to $H_{q}(X \otimes Y)$ for all q.

Proposition 3.4: For (A.B) is independent of the choice of X and Y. Froof: Let X and Y' be projective resolutions of A. Fr

proposition 1.5, there exists it, $X \longrightarrow X'$ and g: $X' \longrightarrow X$ lying two: the identity A_K of $A \longrightarrow K$. By proportional 1.6, $G(X, Z \longrightarrow K')$ lying over A_K is bosomorphic to: $A_K : X \longrightarrow X$, say $A \ni A : A_K : A_$

3. = (40 + 100) = .5

= ab @ ig + Da @ ig

= (2 = 13)(0 & 13)+(0 & 13)(4 = 13), and,

: • :

since $d\otimes l_B$ is the differential operator of $X\!\otimes B$, $D\otimes l_B$ is a homotopy between $i_X \otimes l_B$ and $gf\otimes l_B$. $gf\otimes l_B = (g\otimes l_B)(f\otimes l_B)$, so $H_q(gf\otimes l_B) = H_q(g\otimes l_B)H_q(f\otimes l_B)$, and, arguing similarly for fg, since $H_q(i_X\otimes l_B)$ is the identity on $H_q(X\!\otimes B)$ and $H_q(i_X, \otimes l_B)$ is the identity on $H_q(X^{l}\otimes B)$, we have that $H_q(g\otimes l_B)$ and $H_q(f\otimes l_B)$ are inverse isomorphisms. Thus $H_q(X\!\otimes B)$ and $H_q(X^{l}\otimes B)$ are canonically isomorphic for all q. Arguing similarly, $Tor_q(A,B)$ is independent of the choice of Y.

Proposition 3.5: Let $0 \longrightarrow A' \xrightarrow{k} A \xrightarrow{\ell} A'' \longrightarrow 0$ be an exact sequence of right Λ -modules and B a left Λ -module. Then there exists a canonical exact sequence

$$\longrightarrow \operatorname{Tor}_{\mathbb{N}}(A^{*}B) \longrightarrow \operatorname{Tor}_{\mathbb{N}}(A,B) \longrightarrow \operatorname{Tor}_{\mathbb{N}}(A^{*},B) \longrightarrow \operatorname{Tor}_{\mathbb{N}-1}(A^{*},B) \longrightarrow \dots$$

$$\longrightarrow \operatorname{Tor}_{\mathbb{N}}(A^{*},B) \longrightarrow \operatorname{Tor}_{\mathbb{N}}(A,B) \longrightarrow \operatorname{Tor}_{\mathbb{N}}(A^{*}B) \longrightarrow 0.$$

Proof: Let X' and X" be projective resolutions of A' and A". By proposition 2.6 and corollary 2.7, there exists a projective resolution X of A and morphisms k and l over k and l such that $0 \to X' \xrightarrow{\tilde{k}} X \xrightarrow{\tilde{l}} X'' \to 0$ is a projective resolution of $0 \to A' \xrightarrow{k} A \xrightarrow{l} A'' \to 0$. $0 \to X' \otimes B \to X \otimes B \to X' \otimes B \to 0$ is exact, so by lemma 2.1 there is an exact sequence i) ... $\to H_N(X' \otimes B) \to H_N(X \otimes B) \to H_N(X' \otimes B) \to H_N(X' \otimes B) \to 0$.

To complete the proof, it suffices to show that the connecting morphism δ is independent of the choice of X. Suppose $0 \longrightarrow X^{*} \longrightarrow Y \longrightarrow X^{"} \longrightarrow 0$ also lies over $0 \longrightarrow A^{*} \longrightarrow A \longrightarrow A^{"} \longrightarrow 0$. Then by proposition 2.8, lying over the commutative diagram

since $d^{n}(g)$ is the differential operator of NEB , Diff. is a bouncepy between $A_{ij}(g)$ and $B_{ij}(g)$, $E_{ij}(g)$, $E_{ij}(g$

Proposition 3.7: Let $0 \to A^{1} \to A^{-1} \to A^{0} \to 0$, be an exact sequence of right A-modules and B a left A-module. Then there exists a canonical exact sequence .

From: Let X' and X' be projective resolutions of had A". By proposition 2.6 and corollery 2.7, there exists a projective resolution X of A and corplished X on S over X and 3 such that $0 \to X' \xrightarrow{E} X \xrightarrow{E} X'' \to 0$ is a projective resolution of $0 \to X' \xrightarrow{E} X \xrightarrow{E} X'' \to 0$ is a projective resolution of $0 \to X' \xrightarrow{E} X \xrightarrow{E} X'' \to 0$.

... $\leftarrow (823)_{1.0} \times (2)_{1.0} \times (2)_{1.0$

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

$$\downarrow^{i}_{A'} \qquad \downarrow^{i}_{A} \qquad \downarrow^{i}_{A''}$$

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

is a commutative diagram

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

$$\downarrow^{i}X' \qquad \downarrow^{f} \qquad \downarrow^{i}X'' \qquad 0$$

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

The latter diagram induces a translation of the sequence i).

Thus

$$H_{q}(X \otimes B) \xrightarrow{\delta(X)} H_{q-1}(X \otimes B)$$

$$\downarrow H_{N}(i_{X_{1}} \otimes i_{B}) \qquad \downarrow H_{N-1}(i_{X_{1}} \otimes i_{B})$$

$$H_{q}(X \otimes B) \xrightarrow{\delta(X)} H_{q-1}(X \otimes B)$$

is a commutative diagram whose columns are identity morphisms, and δ is independent of the choice of X.

Note that the proof holds equally well for an exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0 \quad \text{of left Λ-modules and a right Λ-module A , yielding a canonical exact sequence$

Proposition 3.6: Tor (A,B) is isomorphic to ASB.

Proof: Let X be a projective resolution of A. $X_1 \otimes B \xrightarrow{d_1 \otimes i_B} X_0 \otimes B \xrightarrow{e \otimes i_B} A \otimes B \longrightarrow 0 \text{ is exact, so } H_0(X \otimes B) = X_0 \otimes B/\text{im}(d_1 \otimes i_B)$ is isomorphic to $A \otimes B$.

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The latter diagram induced a translation of the sequence il.

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$$H_{q}(x^{*}33) \xrightarrow{G(3)} H_{q+1}(x^{*}34)$$

$$\lim_{q \to \infty} (x^{*}34) \xrightarrow{G(3)} H_{q+1}(x^{*}34)$$

is a negarbative diagrem whose columns are identity morphisms, and to a tode pardent of the chains of .

Note that the proof holds squally well for an rimer sequence 0 --> 3' --> 3 --> 5" --> 5 --> 6 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 6 --> 7 --> 7 --> 6 --> 7 --> 7 --> 7 --> 7 --> 7 --> 6 --> 7 -

$$\dots \Leftrightarrow \operatorname{deg}(A, B^*) \to \operatorname{deg}(A, B^*) \to \operatorname{deg}(A, B^*) \to \operatorname{deg}(A, B^*) \to \dots$$

$$\dots \Leftrightarrow \operatorname{deg}(A, B^*) \to \operatorname{deg}(A, B^*) \to \operatorname{deg}(A, B^*) \to 0 \quad \dots$$

Proposition (1.6: $w_{c_0}(1,3)$ is tomorphic to Min .

From: Low X has a projective recolution of A. $\frac{d_1CC_3}{d_1CC_3} \times_{\mathbb{C}^3} X \otimes \mathbb{R} \xrightarrow{\mathbb{C}^3} \mathbb{C}^3 \times_{\mathbb{C}^3} \times_{\mathbb{C}^3} \mathbb{C}^3 \times_{\mathbb$

Proposition 3.7: The following are equivalent:

- i) A is flat
- ii) $Tor_1(A,B) = 0$ for all B.
- iii) $Tor_q(A,B) = 0$ for all $q \ge 1$ and for all B.
- iv) $\operatorname{Tor}_{q}(A,B) = 0$ for all $q \ge 1$ and for all finitely generated B.
- v) $Tor_q(A,B) = 0$ for all $q \ge 1$ and for all cyclic B.
- vi) $\operatorname{Tor}_q(A,\Lambda/I)=0$ for all $q\geq 1$ and for all ideals I .

Proof: i)==>iii) Let B be a module and X a projective resolution of B. ... $\longrightarrow A\otimes X_N \longrightarrow ... \longrightarrow A\otimes X_O \longrightarrow A\otimes B \longrightarrow 0$ is exact. iii)==>ii) is immediate.

ii)==>i) Let $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ be exact. $0 = \text{Tor}_1(A,B'') \longrightarrow A\otimes B' \longrightarrow A\otimes B \longrightarrow A\otimes B'' \longrightarrow 0$ is exact. iii) ==> vi) is obvious.

iv) ==> iii): Let B be a module. Let the finitely generated submodules of B be indexed by I where we define $i \le j$ if $B_i \subseteq B_j$. Let B denote the direct system of the finitely generated submodules of B so obtained. Then $\lim_{\longrightarrow} B = B$. $\lim_{\longrightarrow} (A \otimes B) = A \otimes \lim_{\longrightarrow} B = A \otimes B$ by proposition 5.12 of chapter 3. If X is a projective resolution of A, ... $\longrightarrow X_N \otimes B \longrightarrow ... \longrightarrow X_N \otimes B \longrightarrow 0$ is an exact sequence of direct systems, so that ... $\longrightarrow X_N \otimes B \longrightarrow ... \longrightarrow X_N \otimes B \longrightarrow 0$ is exact, by proposition 5.16 of chapter 3.

v) => iv): Let B have N generators, and assume the result for modules with N-1 generators. Let $0 \longrightarrow B^! \longrightarrow B \longrightarrow B^{"} \longrightarrow 0$ be exact, where $B^{"}$ has one generator, $B^{!}$ has N-1 generators. Then $\cdots \longrightarrow 0 = \operatorname{Tor}_q(A,B^!) \longrightarrow \operatorname{Tor}_q(A,B) \longrightarrow \operatorname{Tor}_q(A,B^{"}) = 0 \longrightarrow \cdots$ is exact, $\operatorname{Tor}_q(A,B) = 0$ for $q \ge 1$.

Proposition 3.7: The Solleving are equivalent:

- Josh at A (20 ... 2
- . E. IES with $O = (\widetilde{A}, \widetilde{B})$ and (ii)
- is the real flow of the real of the all B.
- yiestain ile vol bas $|1| \leq p$ lie vol G = (8,A) von (+8). E isotopologi
 - . A, with the set of the set of $\phi = (\Re_A A)_{\alpha} \text{mod}$ (v
 - . I sinch the rot opp () in the the the theology (iverself) and for all then in

Troof: i) -2141) Lot 3 be a module and X a projective resolution of B. ... -> ASX --> ... -> ASK --> 0 (is exact.

11) - 20) 100 17 - 31 - 8 - 8 - 18 (12-10)

0 = 30x (A, B") - + 103' -- 100 -- ACB" -- 0 15 brack.

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ty) => iti): Let D be a module. Let the firstsless keaperaged submodules of D coindered in I where we dofine i < i if D D B.

Let B? denote the otherst cyclem of the finitely generated submodules of B en obtained. Then I in 18 = D. I in (SD) = AO Lin B = AOD by proposition 1.12 of chapter 3. If I is a projective resolution of A. ... -> I SB -> ... -> B SB -> ... -> B SB -> 0 is an exact sequence of direct average at that ... -> B SB -> 0 is an exact sequence.

v) ==> iv): Let B drove W generators, and reserve this possible for wearful it. I generators. Let $0 \longrightarrow \mathbb{S}^1 \to \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \to \mathbb{R}^2$. The exact, where \mathbb{R}^n has car generator, \mathbb{S}^1 has \mathbb{R}^n has car generator, \mathbb{S}^1 has \mathbb{R}^n dependently \mathbb{R}^n in \mathbb{R}^n and \mathbb{R}^n in \mathbb{R}^n

vi) ==> v): Let B have one generator. Then there exists I such that $0 \longrightarrow I \longrightarrow \Lambda \longrightarrow B \longrightarrow 0$ is exact, B is isomorphic to Λ/I . Of course, the proposition holds also if the roles of the first and second variables in Tor(A,B) are interchanged.

Remark 3.8: If Λ is a commutative ring, then ASB and XSY are Λ -modules, so that $Tor_{\alpha}(ASB)$ will be a Λ -module.

For further use, we note the following:

<u>Proposition 3.9</u>: If A is a finitely generated module over a commutative Noetherian local ring Λ with maximal ideal M, the following are equivalent:

- i) A is free
- ii) A is projective
- ii) A is flat
- iv) $Tor_1(A, \Lambda/M) = 0$.

Proof: i) ==> ii) ==> iv) are clear.

iv) ==> i) A/MA = $\Lambda/M\triangle A$ is a finitely generated vector space over Λ/M . Choose $x_1, \ldots, x_N \in A$ such that $\overline{x_1}, \ldots, \overline{x_N}$ generate A/MA. Let F be free with N generators e_1, \ldots, e_N . Define f: F \longrightarrow A by $f(e_1) = x_1$. Since $\Lambda/M\supseteq F \xrightarrow{1} \Lambda/M\supseteq A$ is an epimorphism, so is $F \xrightarrow{f} A$ by proposition 4.2 of chapter 5. Let $B = \ker(f)$. Since Λ is Noetherian, B is finitely generated. $0 \longrightarrow B \longrightarrow F \longrightarrow A \longrightarrow 0$ is exact, hence

 $0 = \text{Tor}_{1}(A_{1}, \Lambda/M) \longrightarrow B \otimes \Lambda/M \longrightarrow F \otimes \Lambda/M \longrightarrow A \otimes \Lambda/M \longrightarrow 0$ is exact. Thus $B \otimes \Lambda/M = 0$; B = 0 by proposition 4.1, chapter 5. Thus A is isomorphic

to F.

such that $0 \iff v \in \mathbb{N}$ have one generator, lines there exists I such that $0 \iff I \iff k \iff k \iff k$ is exact, it is isomorphis to A/I, of course, the proposition holds also if the roles of the fibre and sound variables in (0,r(k)) are inversing poles.

Remora 2.6: If A is a communitive wing, when 103 and NE are A-modules, as that for (ME) will be a A-module.

No further use, we note the fall wings.

Exceptition 3.9: It A is a finitely generated madulo over a cornetably Matherina lovel ring A with maningl Aperl 14, the following are equivalent:

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define A is

. O = (M/A,A) rich (vi

Proof: 1) ==> 11) Co (11 Co (1 tree)

17) -2 1) A/MA = K/Miny to a first oly depended vector space

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Since A is Dotharian, B is indhely gamerabed.

0 - B - F if - A - O is exact, hence:

 $0 = \text{Tor}_{\lambda}(A_{\lambda}^{-}, A/M) \longrightarrow \text{BS}(A/M) \longrightarrow \text{To}(A/M) \longrightarrow \text{Notion of the enact. Thus }$ BS, A/M = 0; B = 0 by proposition k.1, the complete S. Thus A is isomorphic to F.

4. Construction of Ext(A,B).

In this section all modules will be assumed to be left Λ -modules. If A and B are modules, $\operatorname{Hom}(A,B)$ is an Abelian group. Recall that if $0 \to A^{!} \to A \to A^{"} \to 0$ and $0 \to B^{!} \to B \to B^{"} \to 0$ are exact sequences, then $0 \to \operatorname{Hom}(A^{"},B) \to \operatorname{Hom}(A,B) \to \operatorname{Hom}(A^{!},B)$ and $0 \to \operatorname{Hom}(A,B^{!}) \to \operatorname{Hom}(A,B) \to \operatorname{Hom}(A,B^{"})$ are exact. Here we will construct objects by means of which we may study the behavior of the latter sequences on the right. The constructions of this section will closely parallel those of the previous section, and most proofs will be outlined only.

Let X and Y be complexes, X written with subscripts, Y with superscripts. Hom(X,Y) is a graded module with Hom(X,Y) = II Hom(X₁,Y^j). Observe that if $f \in Hom(X,Y)^N$, then $f_1: X^{-1} = X_1 \rightarrow Y^{N-1}$; i+j=N that is, f is a morphism of degree N. For notational convenience we write $Hom(X,Y) = Hom(X_1,Y^j)$.

Now let X', X'', Y', Y'' also be complexes. Let $f: X \longrightarrow X'$ and $g: Y \longrightarrow Y'$ be morphisms of degrees p and q. Define i,j $Hom(f,g) = (-1)^{iq} Hom(f_{i+p},g^j)$ $(Hom(f,g):Hom(X_i',Y^j) \longrightarrow Hom(X_{i+p},Y^{i,j+q}))$, and define $Hom(f,g)^N = \prod_{i \neq j = N} Hom(f,g)$. Thus Hom(f,g) is a morphism of degree p+q. Suppose $f': X' \longrightarrow X''$ and $g': Y' \longrightarrow Y''$ are morphisms of degrees p' and q'. f'f and g'g are morphisms of degrees p+p' and q+q'. We have

$$\begin{array}{l} \textbf{i,j} \\ \textbf{Hom}(\textbf{f'f,g'g}) = (-1)^{\textbf{i}(\textbf{q+q'})} & \textbf{Hom}(\textbf{f'}_{\textbf{i+p}} \textbf{f}_{\textbf{i+p'+p}}, \textbf{g'}^{\textbf{j+q}} \textbf{g'}^{\textbf{j}}) \\ \\ = (-1)^{\textbf{i}(\textbf{q+q'})} & \textbf{Hom}(\textbf{f}_{\textbf{i+p'+p}}, \textbf{g'}^{\textbf{j+q}}) & \textbf{Hom}(\textbf{f'}_{\textbf{i+p'}}, \textbf{g'}^{\textbf{j}}) \\ \\ = (-1)^{\textbf{i}(\textbf{q+q'})} & \textbf{(-1)}^{-(\textbf{i+p'})} \textbf{q'} & \textbf{i+p'}, \textbf{j+q} \\ \\ = (-1)^{\textbf{p'q'}} & \textbf{Hom} & (\textbf{f,g'}) & \textbf{Hom}(\textbf{f',g)} \\ \end{array}$$

We will now define a differential operator on $\operatorname{Hom}(X,Y)$ by $d = \operatorname{Hom}(d_X,i_Y) + \operatorname{Hom}(i_X,d_Y) \ .$

$$\begin{split} \mathrm{dd} &= \mathrm{Hom}(\mathrm{d}_{x}, \mathbf{i}_{y}) \ \mathrm{Hom}(\mathrm{d}_{x}, \mathbf{i}_{y}) + \mathrm{Hom}(\mathrm{d}_{x}, \mathbf{i}_{y}) \ \mathrm{Hom}(\mathrm{i}_{x}, \mathrm{d}_{y}) \\ &+ \mathrm{Hom}(\mathrm{i}_{x}, \mathrm{d}_{y}) \ \mathrm{Hom}(\mathrm{d}_{x}, \mathbf{i}_{y}) + \mathrm{Hom}(\mathrm{i}_{x}, \mathrm{d}_{y}) \ \mathrm{Hom}(\mathrm{i}_{x}, \mathrm{d}_{y}) \\ &= \mathrm{Hom}(\mathrm{d}_{x}, \mathbf{i}_{y}) \ \mathrm{Hom}(\mathrm{i}_{x}, \mathrm{d}_{y}) + \mathrm{Hom}(\mathrm{i}_{x}, \mathrm{d}_{y}) \ \mathrm{Hom}(\mathrm{d}_{x}, \mathbf{i}_{y}) \\ &= \mathrm{Hom}(\mathrm{i}_{x}, \mathrm{d}_{y}) - \mathrm{Hom}(\mathrm{d}_{x}, \mathrm{d}_{y}, \mathrm{d}_{y}) = 0 \ . \end{split}$$

As in the construction of Tor , we will employ

Lemma 4.1: Suppose X is a left complex and Y is a right complex.

Then: i) If X is projective and Y is exact, Hom(X,Y) is exact.

ii) If X is exact and Y is injective, Hom(X,Y) is exact.

Proof: Note that Hom(X,Y) is a right complex. The proof is similar to that of lemma 3.1, using propositions 2.10 and 3.5 of chapter 3 and lemma 2.1 in the second and third steps.

<u>Definition 4.2</u>: Let A and B be left A-modules, X a projective resolution of A and Y an injective resolution of B. Define $\operatorname{Ext}^{\mathbb{N}}(A,B) = \operatorname{H}^{\mathbb{N}}(\operatorname{Hom}(X,Y)).$

Proposition 4.3: $H^{q}(Hom(A,Y))$, $H^{q}(Hom(X,Y))$ and $H^{q}(Hom(X,B))$ are isomorphic for all q.

Proof: Define Y' by $Y^{1q} = Y^q$ for $q \neq 0$, $Y^{0} = Y^0/im(\varepsilon)$. Then $0 \longrightarrow B \longrightarrow Y \longrightarrow Y' \longrightarrow 0$ is an exact sequence of translations, $H^q(Y^1) = 0$ for all q, hence by lemma 4.1 $H^q(Hom(X,Y^1)) = 0$ for all q. Since by proposition 2.10 of chapter 3,

 $0 \longrightarrow \operatorname{Hom}(X,B) \longrightarrow \operatorname{Hom}(X,Y) \longrightarrow \operatorname{Hom}(X,Y') \longrightarrow 0 \text{ is exact, by lemma 2.1}$ $0 \longrightarrow \operatorname{H}^{\operatorname{q}}(\operatorname{Hom}(X,B)) \longrightarrow \operatorname{H}^{\operatorname{q}}(\operatorname{Hom}(X,Y)) \longrightarrow 0 \text{ is exact for all } \operatorname{q}.$ Arguing similarly using proposition 3.5 of chapter 3, $\operatorname{H}^{\operatorname{q}}(\operatorname{Hom}(X,Y))$ and $\operatorname{H}^{\operatorname{q}}(\operatorname{Hom}(A,Y))$ are isomorphic for all $\operatorname{q}.$

Proposition 4.4: Ext^q(A,B) is independent of the choice of X and Y.

Proof: The proof is similar to that of proposition 3.4.

Propositions 1.5 and 1.8 are used to prove independence of X, 1.5'

and 1.8' for Y.

Proposition 4.5: i) Let $0 \to A^* \to A \to A'' \to 0$ be an exact sequence and B a module. Then there exists a canonical exact sequence $0 \to \operatorname{Ext}^0(A'',B) \to \operatorname{Ext}^0(A,B) \to \operatorname{Ext}^0(A',B) \to \dots$ $\longrightarrow \operatorname{Ext}^{N-1}(A',B) \to \operatorname{Ext}^N(A'',B) \to \operatorname{Ext}^N(A,B) \to \operatorname{Ext}^N(A,B) \to \dots$

ii) Let $0 \to B' \to B \to B'' \to 0$ be an exact sequence and A a module. Then there exists a canonical exact sequence $0 \to \operatorname{Ext}^{O}(A,B') \to \operatorname{Ext}^{O}(A,B) \to \operatorname{Ext}^{O}(A,B'') \to \dots$ $\to \operatorname{Ext}^{N-1}(A,B'') \to \operatorname{Ext}^{N}(A,B') \to \operatorname{Ext}^{N}(A,B) \to \operatorname{Ext}^{N}(A,B'') \to \dots$

Proof: The proof is similar to that of proposition 3.5:

- i) follows using proposition 2.6, corollary 2.7, proposition 2.10 of chapter 3, lemma 2.1, and proposition 2.8.
- ii) follows using proposition 2.6', corollary 2.7', proposition 3.5 of chapter 3, lemma 2.1, and proposition 2.8'.

Proposition 4.6: Ext^O(A,B) is isomorphic to Hom(A,B).

Proof: Let Y be an injective resolution of B.

$$0 \longrightarrow \operatorname{Hom}(A,B) \xrightarrow{\operatorname{Hom}(i_A,\varepsilon)} \operatorname{Hom}(A,Y^O) \xrightarrow{\operatorname{Hom}(i_A,d^O)} \operatorname{Hom}(A,Y^{1}) \text{ is exact,}$$
so $\operatorname{H}^O(\operatorname{Hom}(A,Y)) = \ker(\operatorname{Hom}(i_A,d^O))$ is isomorphic to $\operatorname{Hom}(A,B)$.

Proposition 4.7: The following are equivalent:

- 1) A is projective
- ii) $\operatorname{Ext}^{1}(A,B) = 0$ for all B
- iii) $\operatorname{Ext}^{q}(A,B) = 0$ for all $q \ge 1$ and for all B.

Proof: This follows from proposition 2.10 of chapter 3.

Proposition 4.7': The following are equivalent:

- i) B is injective
- ii) $\operatorname{Ext}^{1}(A,B) = 0$ for all A.
- iii) $\operatorname{Ext}^{q}(A,B) = 0$ for all $q \ge 1$ and for all A.

Proof: This follows from proposition 3.5 of chapter 3.

Note that since there is no analog to proposition 5.16 of chapter 3, we do not obtain a complete analog to proposition 3.7.

Remark 4.8: If we had used right modules throughout this section, we would have obtained analogous results. If Λ is commutative, then the values of Ext(A,B) are Λ -modules.

Proposition 4.9: If A is a finitely generated module over a commutative Noetherian local ring Λ with maximal ideal M, then A is free if and only if $\operatorname{Ext}^1(A, \Lambda/M) = 0$.

Proof: If A is free, then $\operatorname{Ext}^1(A, \Lambda/M)$ is clearly zero. Conversely, we proceed exactly as in the proof of proposition 3.9, obtaining an exact sequence

 $0 \longrightarrow B \longrightarrow F \longrightarrow A \longrightarrow 0$ when F is free and F/MF is isomorphic to A/MA. Hom(A, A/M) = Hom(A/M A, A/M) so Hom(A, A/M) is isomorphic

to $\operatorname{Hom}(F, \Lambda/M)$. $O \longrightarrow \operatorname{Hom}(B, \Lambda/M) = \operatorname{Hom}(B/MB, \Lambda/M) \longrightarrow \operatorname{Ext}^1(A, \Lambda/M) = O$ is exact, $\operatorname{Hom}(B/MB, \Lambda/M) = O$, B/MB = O and by proposition 4.1 of chapter 5, B = O. Thus A is isomorphic to F.

5. Categories and functors.

In this section, we introduce terminology which greatly simplifies the statements of homological algebra.

Definitions 5.1: Let \mathcal{F} be a set with elements denoted by f, f_1, f_2 , etc. such that for certain pairs (f_1, f_2) a product $f_1 f_2$ is defined in \mathcal{F} . An element is \mathcal{F} such that if $f_1 = f_1$ and $f_2 = f_2$ whenever if f_1 and $f_2 = f_2$ are defined, is called an identity. \mathcal{F} is called a system of abstract maps provided that

- i) If either $f_1(f_2f_3)$ or $(f_1f_2)f_3$ is defined, then so is the other and the two are equal.
- ii) If f_1f_2 and f_2f_3 are defined, then so are $(f_1f_2)f_3$ and $f_1(f_2f_3)$.
- iii) If $f \in \mathcal{F}$, then there exist (unique) identities i_1 and i_2 in \mathcal{F} such that $f i_1$ and $i_2 f$ are defined.

Definitions 5.2: A category $\mathbb C$ is a system $\mathcal F$ of abstract maps together with objects C, C_1, C_2, \ldots which are in 1-1 correspondence with the identities of $\mathcal F$. If $f \in \mathcal F$, then the unique objects C_1 and C_2 such that fi_{C_1} and $i_{C_2}f$ are defined, are called the domain and range of f, and we write $f: C_1 \longrightarrow C_2$. If for each pair C_1 and C_2 of objects in $\mathbb C$, the set of all maps $f: C_1 \longrightarrow C_2$ has a natural structure as an Abelian group, then $\mathbb C$ is called an additive category. If for each pair C_1 and C_2 the set of maps $f: C_1 \longrightarrow C_2$ has a structure as a left (right) Λ -module, then $\mathbb C$ is called a left (right) Λ -category.

The set of left (or right) Λ -modules and their morphisms is an example of an additive category. If Λ is commutative, the set of Λ -modules and their morphisms is a Λ -category. Diagrams of Λ -modules and their translations give further examples of additive categories (a translation $f: D \longrightarrow D'$ of two similar diagrams D and D' is a family of morphisms $f_j: D_j \longrightarrow D'_j$ such that for each pair (j,k),

$$\begin{array}{ccc}
D_{j} & \xrightarrow{\phi_{jk}} & D_{k} \\
\downarrow^{f_{j}} & & \downarrow^{f_{k}} \\
D'_{j} & \xrightarrow{\phi'_{jk}} & D'_{k}
\end{array}$$

is a commutative diagram where $D_j, D_k, \ldots, D_j, D_k, \ldots$ are the modules and $\phi_{jk}, \ldots, \phi_{jk}, \ldots$ are the component morphisms of the diagrams D and D^i).

<u>Definitions 5.3</u>: Let C and A be categories. Suppose that for each object $C \in C$, an object $T(C) \in A$ is given and for each map $f: C \longrightarrow C'$ in C a map $T(f): T(C) \longrightarrow T(C')$ is given such that

- i) If $f = i_c$ then $T(f) = i_{T(C)}$.
- ii) If f'f is defined, T(f'f) = T(f') T(f).

Then T is said to form a covariant functor from C to \mathcal{N} . If $T(f): T(C') \longrightarrow T(C)$ and T(f'f) = T(f)T(f'), T is said to be a contravariant functor from C to \mathcal{N} .

We extend the definition to N variables as follows: Let C_1,\ldots,C_N , \mathcal{C}_N be categories. Let $C_1,C_1,\ldots,f_1,f_1,\ldots$ be objects and maps in C_1 and let the set $\{1,\ldots,N\}$ be divided into disjoint subsets I and J. Assume that for each set $C_1,\ldots C_N$ of

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Destablished 5.3: Lot C and D be entranced. Supplies that for each septemb object C.s. , so object M(Chap) is siven and for each sept. : 0.--> 0' in C a map Y(I): I/O) --> 7(C') is given such that

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- (1) If \mathbb{Z}^2 is refined, $\mathbb{Z}(\mathbb{Z}^2) = \mathbb{Z}(\mathbb{Z}^2)$ $\mathbb{Z}(\mathbb{Z}^2)$.

Then T is well in rows a constraint functor from G to G . If $T(T): T(T): T(T) \rightarrow T(T) \rightarrow T(T): T(T): T(T) \rightarrow T(T): T($

the extent the declimingon to T variables as follows: let $C_0, \cdots, C_{n-1}, T_1, T_1, \cdots$ be let $C_0, \cdots, C_{n-1}, T_1, T_1, \cdots$ be objects and maps in C_1 and let the cet (x_1, \dots, H) be divided into diagons and release T and T. Assume that the case set T

objects there is given an object $T(C_1,\ldots,C_N) \in \mathcal{O}$, and for each set $f_1,\ldots f_N$ of maps, $f_i\colon C_i \longrightarrow C'_i$ for $i\in I$, $f_j\colon C'_j \longrightarrow C_j$ for $j\in J$, there is given a map in \mathcal{O} $T(f_1,\ldots,f_N)\colon T(C_1,\ldots,C_N) \longrightarrow T(C'_1,\ldots,C'_N)$. Then T is a functor, covariant in the variables in I, contravariant in those in J, provided that

- i) If f_1, \dots, f_N are identities, then so is $T(f_1, \dots, f_N)$.
- ii) If $f_1, \dots, f_N, f_1, \dots f_N$ are such that $f_1^i f_1$, is I, and $f_j f_j^i$, js J, are defined, then $T(\dots, f_1^i f_1, \dots, f_j f_j^i, \dots)$ $= T(f_1^i, \dots, f_N^i) T(f_1, \dots, f_N^i).$

If C_1,\ldots,C_N , \mathcal{L} are all additive categories, and f_1,\ldots,f_N , g_1,\ldots,g_N are maps in C_1,\ldots,C_N such that f_1 and g_1 have the same domain and range for each i, then if $T(f_1,\ldots,f_r+g_r,\ldots,f_N)=T(f_1,\ldots,f_r,\ldots,f_N)+T(f_1,\ldots,g_r,\ldots f_N),$ $i\leq r\leq N$, T is said to be an additive functor. We will only be concerned with additive functors.

Definitions 5.4: Let T and U be functors of N variables from (C_1,\ldots,C_N) to \emptyset . Let $\{1,\ldots,N\}$ be partitioned into I and J with both T and U covariant in the variables C_i , is I, contravariant in C_j , js J. Denote (C_1,\ldots,C_N) by (C). If for each set (C), there is a map $\mu_{(C)}\colon T(C)\longrightarrow U(C)$ such that whenever $f_1,\ldots f_N$ are maps, $f_i\colon C_i\longrightarrow C_i$, is I, $f_i\colon C_i\longrightarrow C_j$, js J, then the diagram

$$T(C) \xrightarrow{T(f_1,...,f_N)} T(C')$$

$$\downarrow^{\mu}(c) \qquad \qquad \downarrow^{\mu}(c')$$

$$U(C) \xrightarrow{U(f_1,...,f_N)} U(C') \qquad \text{is commutative,}$$

we say that μ is a natural transformation of T into U . If $\mu_{(c)}$ is an equivalence for all (C) , we say that $\mu_{(c)}$ is a natural equivalence

objects there is given an object $\mathfrak{A}(C_1,\ldots,C_N)$, and for each set $\{1,\ldots,f_N\}$ of maps, $\{1,\ldots,f_N\}$ for tell, $\{1,\ldots,f_N\}$ of for jet, and for each set in the map in $\{1,\ldots,f_N\}$. $\mathfrak{A}(C_1,\ldots,C_N) \longrightarrow \mathfrak{A}(C_{1,\ldots},C_{N})$. Then I do a function, constrait in the yeldinales, in I, conservations in the grainales, in I, conservations in the provided that

. (i) If $\mathcal{I}_1,\dots,\mathcal{I}_N$ are identities, then so is $\mathcal{P}(\mathcal{I}_1,\dots,\mathcal{I}_N)$.

If C_1,\ldots,C_N are all additive determined and c_1,\ldots,c_N , ..., c_{N} and c_1,\ldots,c_N are cared in ..., c_N and that c_1 and c_1,\ldots,c_N and that c_1 and c_2 have the parameters and range for each c_1 , when if c_1,\ldots,c_N and c_1,\ldots,c_N an

Destinition of [0, 0] for [0, 0] and [0, 0] be particularly from [0, 0] to [0, 0] be particularly and [0, 0] be carticularly both [0, 0] to [0, 0] be carticular [0, 0] and [0, 0] and [0, 0] by [0, 0] by [0, 0] for each set [0, 0] because is a map [0, 0]; [0, 0] and that whenever the representation [0, 0]; [0, 0] and [0, 0] and that whenever then the diagram

we say that μ is a matural transformation of T into U . If $\mu(a)$ is a natural equivalence is en equivalence for all (0), we say that $\varphi(a)$ is a natural equivalence

or a natural isomorphism. (A map f is an equivalence if there exists a map g such that fg and gf are both identities.)

Let Λ_1 , Λ_2 , Λ be rings. We restrict ourselves (for notational convenience) for the remainder of this section to an additive functor T(A,C) define for Λ_1 -modules A and Λ_2 -modules C with values as Λ -modules. We assume that T is covariant in A, contravariant in C.

Proposition 5.5: If $A_{\alpha} \xrightarrow{i_{\alpha}} A \xrightarrow{j_{\alpha}} A_{\alpha}$ and $C_{\beta} \xrightarrow{k_{\beta}} C \xrightarrow{\ell_{\beta}} C_{\beta}$ are finite direct sum representations of A and C, then $T(A_{\alpha}, C_{\beta}) \xrightarrow{T(i_{\alpha}, \ell_{\beta})} T(A, C) \xrightarrow{T(j_{k}, k_{\beta})} T(A_{\alpha}, C_{\beta})$ is a direct sum representation of T(A, C).

Proof: $T(j_{\alpha}, k_{\beta})T(i_{\alpha}, l_{\beta}) = T(j_{\alpha}, i_{\alpha}, l_{\beta}k_{\beta})$, is the identity if $(\alpha, \beta) = (\alpha', \beta')$ and zero otherwise. Also

 $\Sigma_{\alpha,\beta} T(i_{\alpha}, \ell_{\beta}) T(j_{\alpha}, k_{\beta}) = \Sigma_{\alpha,\beta} T(i_{\alpha}j_{\alpha}, k_{\beta}\ell_{\beta}) = T(\Sigma_{\alpha,\beta}i_{\alpha}j_{\alpha}, \Sigma_{\alpha,\beta}k_{\beta}\ell_{\beta})$ = identity.

Corollary 5.6: If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ and $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$ are split exact sequences, then so are $0 \longrightarrow T(A',C) \longrightarrow T(A,C) \longrightarrow T(A'',C) \longrightarrow 0$ and $0 \longrightarrow T(A,C'') \longrightarrow T(A,C) \longrightarrow 0$.

Definitions 5.7: Let $A' \to A \to A''$, $C' \to C \to C''$ be exact sequences. If $T(A',C) \to T(A,C) \to T(A'',C)$ and $T(A,C'') \to T(A,C) \to T(A,C') \text{ are also exact, then } T \text{ is said to be an exact functor.} \text{ Let } 0 \to A' \to A \to A'' \to 0 \text{ and } 0 \to C' \to C \to C'' \to 0 \text{ be exact sequences.}$

or a natural incurrytian. (A map f is an equivalence if there erdsts a men g such that fg and gr ero beth identifies.)

Not Λ_1^2 , Λ_2^2 , Λ to rings. We restrict ourselves (for notational convenience) for the remainder of this section to an additive functor $T(\Lambda, C)$ define for Λ_1 -modules Λ and Λ_2 -modules. We assume that if in covariant in Λ_1 , confinewant in C.

Proposition 5.5: if $A_{cc} \xrightarrow{i_{C}} A \xrightarrow{j_{C}} A_{cc} \xrightarrow{i_{C}} C_{g} \xrightarrow{i_{C}} C$ and $C_{g} \xrightarrow{i_{C}} C_{g}$ are finite fixed sum representations of A and C_{g} then $T(A_{Cc}C_{g}) \xrightarrow{i_{C}} T(A_{g}C_{g}) \xrightarrow{i_{C}} T(A_{g}C_{g})$ is a fixed sum representation of T(A,C).

Proof: $T(j_{\alpha}, j_{\beta}, T(j_{\alpha}, j_{\beta})) = T(j_{\alpha}, j_{\alpha}, j_{\alpha}, j_{\beta}, j_{\beta}, j_{\alpha}, j_{\alpha}$

 $\Sigma_{(i,p)} \mathbb{P}(\mathbf{1}_{G}, \mathcal{Z}_{p}) \mathbb{P}(\mathbf{1}_{G}, \mathcal{Z}_{p}) = \Sigma_{G,p} \mathbb{P}(\mathbf{1}_{G}, \mathcal{Z}_{G}, \mathcal{Z}_{p}) = \mathbb{P}(\Sigma_{G,p}, \mathbf{1}_{G}, \mathcal{Z}_{G}, \mathcal{Z$

Corollary 5.6: If $0 \to A' \to A \to A'' \to 0$ and $0 \to C' \to C \to C'' \to 0$ are applied exact nequences, then so are $0 \to T(A',C) \to T(A,C') \to T(A,C') \to T(A,C') \to T(A,C') \to C$.

If $0 \longrightarrow T(A',C) \longrightarrow T(A,C) \longrightarrow T(A'',C)$ and $0 \longrightarrow T(A,C'') \longrightarrow T(A,C) \longrightarrow T(A,C'')$ are exact, then T is said to be left exact. If $T(A',C) \longrightarrow T(A,C) \longrightarrow T(A'',C) \longrightarrow 0$ and $T(A,C'') \longrightarrow T(A,C) \longrightarrow T(A,C') \longrightarrow 0$ are exact, then T is said to be right exact.

The proofs of the following propositions are straightforward and will be omitted.

Proposition 5.8: T is exact if and only if T is both right and left exact.

Proposition 5.9: The following are equivalent:

- i) T is left exact
- ii) If $0 \to A' \to A \to A''$ and $C' \to C \to C'' \to 0$ are exact, then so are $0 \to T(A',C) \to T(A,C) \to T(A'',C)$ and $0 \to T(A,C'') \to T(A,C) \to T(A,C')$.
- iii) If $0 \to A' \to A \to A''$ and $C' \to C \to C'' \to 0$ are exact, then so is $0 \to T(A',C'') \to T(A,C) \xrightarrow{\psi} T(A'',C) \oplus T(A,C')$ where ψ has coordinates $T(A,C) \to T(A'',C)$; $T(A,C) \to T(A,C')$

Proposition 5.10: The following are equivalent:

- i) T is right exact
- ii) If $A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ and $0 \longrightarrow C' \longrightarrow C \longrightarrow C''$ are exact, then so are $T(A',C) \longrightarrow T(A,C) \longrightarrow T(A'',C) \longrightarrow 0$ and $T(A,C'') \longrightarrow T(A,C) \longrightarrow T(A,C') \longrightarrow 0$.
- iii) If $A' \to A \to A'' \to 0$ and $0 \to C' \to C \to C''$ are exact, then so is $T(A',C) \oplus T(A,C'') \xrightarrow{\varphi} T(A,C) \to T(A'',C') \to 0$, where ϕ has coordinates $T(A',C) \to T(A,C)$ and $T(A,C'') \to T(A,C)$.

We observe that by propositions 1.10, 2.7 and 2.8 of chapter 3, the functor \otimes_{Λ} is right exact and the functor \hom_{Λ} is left exact.

TO 0 -- T(A', E) -- (U, A', E) -- (U, W) -- C TI

 $0 \Rightarrow \mathbb{F}(A, \mathbb{C}^n) \to \mathbb{F}(A, \mathbb{C}) \Rightarrow \mathbb{F}(A, \mathbb{C}^n)$ use exact, then \mathbb{T} is said to be left smeet. If $\mathbb{T}(A, \mathbb{C}) \Rightarrow \mathbb{T}(A, \mathbb{C}) \to \mathbb{T}(A, \mathbb{C}) \Rightarrow \mathbb{T}(A, \mathbb{C}^n) \Rightarrow \mathbb{T}(A, \mathbb{C}^n)$

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Proposition 5.8: If is except if and only if I hour right sad left exact.

Proposition 5.9: The following are equivalent:

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14) If $0 \to A' \to A'' = 0$ (1.4), $0 \to C'' \to C'' \to 0$ (2.5)

14. (4), $0 \to C'(A,C) \to C(A,C) \to C(A,C) \to C(A,C)$ 15. (4), $0 \to C'(A,C) \to C'(A,C) \to C'(A,C)$

111) If $0 \longrightarrow A = A = A^{-1}$ and $0 \longrightarrow C \longrightarrow C^{-1} \longrightarrow A$ are specified as $T(A',C'') \longrightarrow T(A,C)$. Since $T(A',C'') \longrightarrow T(A,C')$ and T(A,C') are specified as $T(A',C'') \longrightarrow T(A,C')$.

Proposition Sald: "The full owing one applyalent:

- Joseph Singler ed T. (1 ..
- 11) If $(X') \rightarrow X \rightarrow A'' \rightarrow 0$ and $(Y') \rightarrow (Y') \rightarrow (Y') \rightarrow (Y') \rightarrow (Y') \rightarrow (X') \rightarrow (X'$
- Lii) If $L^1 = A \rightarrow A^{-} \rightarrow 0$ and $0 \rightarrow 0' \rightarrow 0' \rightarrow 0''$ are exact, when so is $T(A',C) \in T(A,C'') \rightarrow T(A,C) \rightarrow T(A'',C') \rightarrow T(A,C') \rightarrow T(A,C'') \rightarrow T(A,C'')$ where M are coordinated $T(A',C) \rightarrow T(A,C)$ and $T(A,C'') \rightarrow T(A,C'') \rightarrow T(A,C'')$. We observe that by interestinal 1.10, 2.7 and 2.6 of chapter 3, the innerest A is T and the innerest A is T and the exact.

Definitions 5.11: A connected sequence of covariant functors is a family $T = \{T^N\}$ of covariant functors (of one variable) together with connecting morphisms $T^N(A^n) \longrightarrow T^{N+1}(A^n)$ defined for each exact sequence $0 \longrightarrow A^n \longrightarrow A \longrightarrow A^n \longrightarrow 0$ subject to the conditions:

- i) ... $\rightarrow T^{N-1}(A'') \rightarrow T^{N}(A') \rightarrow T^{N}(A) \rightarrow T^{N}(A'') \rightarrow T^{N+1}(A') \rightarrow \cdots$ is a zero sequence.
- ii) If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is a commutative diagram $\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad 0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$

with exact rows then $T^N(A") \longrightarrow T^{N+1}(A')$ is a commutative diagram $\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$ $T^N(B") \longrightarrow T^{N+1}(B').$

If the roles of A' and A" are reversed, $\{T^N\}$ is a connected sequence of contravariant functors.

Generalizing, a multiply connected sequence of functors is a sequence {TN} of functors of the same variables and variance such that there are connecting morphisms with respect to each variable for which

- i) $\{T^{N}\}$ is a connected sequence of functors with respect to each variable separately.

are commutative diagrams with exact rows, then $T^{N}(A'',C) \longrightarrow T^{N+1}(A',C)$ $\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$ $T^{N}(A'',C_{1}) \longrightarrow T^{N+1}(A',C_{1})$

Definitions 5.11: A connected sequence of constlain functions is a family $T = \{T^{R}\}$ of countert functions (of one vertable) together with connecting merghises $T^{R}(A^{0}) \longrightarrow T^{R+1}(A^{0})$ defined for each exact sequence $0 \longrightarrow A^{0} \longrightarrow A \longrightarrow A^{0} \longrightarrow 0$ subject to the conditions:

(1) ... $\longrightarrow I^{R+1}(A^{0}) \longrightarrow I^{R}(A^{1}) \longrightarrow T^{R}(A^{0}) \longrightarrow T^{R}(A^{0}) \longrightarrow T^{R+1}(A^{0}) \longrightarrow T^{R+1$

mangald evitainance a at 0 <- "A <- 'A <- 0 II (it

0 2 18 4 1 1 1 1 1 4 4 - 10

with enget rows then $T'(A'') \longrightarrow T^{N-1}(A')$ is a commutative diagram

2^{II}(B').

Genevalizing, a multiply counceded sequence of tonework is a sequence (T) of functors of the cons variables one variables for thet there are connecting according to the respect to each variable for which

the separately $_{i,j}$ $_{i,$

are commutative diagrams with exact ways, then $e^{-1}(\lambda^n,0) \rightarrow e^{-1}(\lambda^n,0)$. $e^{-1}(\lambda^n,0) \rightarrow e^{-1}(\lambda^n,0)$

and $T^N(A,C') \longrightarrow T^{N+1}(A,C'')$ are commutative diagrams (where we $T^N(A_1,C') \longrightarrow T^{N+1}(A_1,C'')$

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have chosen $\{T^N\}$ all covariant in A, contravariant in C, as a typical example). If $\{T^N\}$ and $\{U^N\}$ are multiply connected sequences of functors, a morphism $\Phi\colon \{T^N\} \longrightarrow \{U^N\}$ is a sequence of natural transformations $\Phi^N\colon T^N \longrightarrow U^N$ which commute with the connecting morphisms.

 $\{\operatorname{Tor}_{N}^{\Lambda}(A,C)\}\$ and $\{\operatorname{Ext}_{N}^{N}(A,C)\}\$ are examples of exact multiply connected sequences of functors (that condition ii) is satisfied follows from the constructions, and propositions 1.8, 1.8', 2.8 and 2.8'.). The method by which Tor^{A} and Ext_{A} were constructed is actually quite general. Let $T(A_1, ..., A_r)$ be any functor of r variables. Assume that A_1, \dots, A_r are all graded. Define A_1, \dots, A_r are all graded. Define A_1, \dots, A_r are all graded. Define A_1, \dots, A_r where A_1, \dots, A_r where A_1, \dots, A_r where A_1, \dots, A_r are all graded. A is a covariant variable, -1 if contravariant, and define $\mathbf{T}^{\mathbf{N}}(\mathbf{A}_{1},\ldots,\mathbf{A}_{r}) = \mathbf{T}^{\mathbf{N}_{1},\ldots,\mathbf{N}_{r}}(\mathbf{A}_{1},\ldots,\mathbf{A}_{r}) \quad \text{[for } \otimes_{\mathbf{A}}, \quad \text{if all}$ complexes are left complexes as in the construction of Tor^{Λ} , this is the same as $\mathbf{x}_{\mathbf{N}_{1} \leftarrow \mathbf{N}} \mathbf{x}^{\mathbf{N}_{1}, \dots, \mathbf{N}_{r}} (\mathbf{A}_{1}, \dots, \mathbf{A}_{r})$]. If $\mathbf{A}_{1}, \dots, \mathbf{A}_{r}$ is another set of graded modules and $f_i: A_i \longrightarrow A_i'$ are given for A_i a covariant variable, $f_i \colon A_i' \longrightarrow A_i$ for A_i contravariant, where f_i is of degree p_i , define $T^{1_1, \ldots, N_r}(f_1, \ldots f_r)$ on $T^{N_1, \ldots, N_r}(A_1, \ldots, A_r)$ as $(-1)^{\ell} T(f_1^{-1}, \dots, f_r^{-r}) : T(A_1^{-1}, \dots, A_r^{-r}) \longrightarrow T(A_1^{-1}, \dots, A_r^{-r}) \xrightarrow{\epsilon_r(N_r + P_r)},$ where $\mathcal{F} = \sum_{i \in I} N_i p_i$, $\ell_i = N_i$ if T is covariant in A_i , $\ell_i = -(N_i + p_i)$ if T is contravariant in A_i . If $g_i: A_i' \longrightarrow A_i''$ (resp. $g_1: A_1^n \longrightarrow A_1^n$) are morphisms of degree q_1^n , we then verify that

 $T(g_1f_1,\ldots,g_rf_r)=(-1)^{\gamma_i}T(g_1,\ldots,g_r)T(f_1,\ldots,f_r), \text{ where}$ $\mathcal{H}=\sum_{i}\sum_{j}p_iq_j. \text{ Now suppose each } A_i \text{ is a complex with differentiation}$ $d_i \text{ and let } \delta_i=T(i_{A_1},\ldots,d_1,\ldots,i_{A_r}). \text{ The } \delta_i \text{ anticommute,}$ hence define $T(A_1,\ldots,A_r)$ as a complex with differentiation $\Sigma \delta_i. \text{ If } f_1,\ldots,f_r \text{ and } f'_1,\ldots,f'_r \text{ are respectively homotopic}$ translations of complexes and $s_i, 1 \leq i \leq r$ are homotopies, then for $\sigma_i=T(i_{A_1},\ldots,s_i,\ldots,i_{A_r}), \Sigma \sigma_i$ defines a homotopy between $T(f_1,\ldots,f_r)$ and $T(f'_1,\ldots,f'_r).$

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Now if we are given a functor T of modules and we replace all covariant variables A, by projective resolutions X, all contravariant variables A, by injective resolutions X, we obtain a left complex. We define $IT(A_1,...,A_r)$ as $HT(X_1,...,X_r)$. By propositions 1.8, 1.8' and the previous paragraph, LT is independent of the choices of the X_i. IT is graded, and the component of degree N gives a functor L_N T called the Nth left derived functor of T. $L_NT = 0$ for n < 0. By corollary 5.6, lemma 2.1, and propositions 1.8, 1.8', 2.8 and 2.8', $\{L_N^T\}$ is an exact multiply connected sequence of functors. The augmentation morphisms $\epsilon_i: X_i \longrightarrow A_i$ (resp. $\epsilon_i: A_i \longrightarrow X_i$) induce a natural transformation $\gamma_0: L_0T(A_1, \ldots, A_r) \longrightarrow T(A_1, \ldots, A_r). \quad \gamma_0$ is a natural equivalence if and only if T is right exact: the condition is necessary since L_{o}^{T} is right exact; if T is right exact, considering our typical case T(A,C), and letting X be a projective resolution of A, Y an injective resolution of C, the sequence $T(X_1,Y^0) \oplus T(X_0,Y^1) \xrightarrow{\text{d}} T(X_0,Y^0) \longrightarrow T(A,C) \longrightarrow 0$ is exact by proposition 5.10; since $H_{O}(T(X,Y)) = \text{coker } (\phi)$, this gives the result. For this reason, left derived functors are of interest for

the study of right exact functors. Similarly, if we replace all covariant variables by injective resolutions and all contravariant variables by projective resolutions in a functor T of modules, we obtain a multiply connected sequence of functors $\{R^NT\}$, called the right derived functors of T. R^OT is naturally equivalent to T if and only if T is left exact.

3

It is clear that $\{\operatorname{Tor}_N^\Lambda\}$ are the left derived functors of \otimes_Λ and that $\{\operatorname{Ext}_N^\Lambda\}$ are the right derived functors of $\operatorname{Hom}_\Lambda$.