VIII HOMOLOGICAL ALGEBRA

Modern Classical Algebra
VIII. HOMOLOGICAL ALGEBRA

In this chapter, some basic concepts of homological algebra will be defined and some of their elementary properties developed. The duality of the concepts "projective module" and "injective module" will be systematically employed and, where applicable, dual propositions will be stated.

We will need to modify some of the definitions concerning graded modules. We will mean by a graded module a sequence indexed on the integers rather than on the positive integers. If \( X \) and \( Y \) are graded modules, a morphism of degree \( n \), \( f: X \longrightarrow Y \), is a sequence of morphisms \( f^n: X^n \longrightarrow Y^{n+1} \).

\( \Lambda \) will be a ring, not necessarily commutative. All modules will be assumed left modules unless otherwise specified.

1. Differential operators and resolutions.

**Definitions 1.1:** A differential operator, or complex, is a pair \((X, d)\) where \( X \) is a graded module and \( d: X \longrightarrow X \) is a morphism of degree \(+1\) such that \( d^{q+1}d^q = 0 \) for all \( q \). We define further the \( q \)th co-cycle, \( Z^q(X) \), as \( \ker(d^q) \), the \( q \)th coboundary, \( B^q(X) \), as \( \text{im}(d^{q-1}) \), and the \( q \)th cohomology as \( Z^q(X)/B^q(X) \).

We introduce the convention \( X_N = X^{-N}, d_N = d^{-N}, Z_N = Z^{-N}, \)
\( B_N = B^{-N}, H_N = H^{-N} \). Thus, when \((X, d)\) is written as a complex with subscripts, \( d_q: X_q \longrightarrow X_{q-1} \) and \( d_q d_{q+1} = 0 \); in this case \( Z_q(X) \) is called the \( q \)th cycle, \( B_q(X) \), the \( q \)th boundary, \( H_q(X) \) the \( q \)th homology.

\((X, d)\) is called a right complex if \( X^N = 0 \) for all \( N < 0 \); it is called a left complex if \( X_N = 0 \) for all \( N < 0 \). Thus a right
\[ (\delta X) \text{ where } \delta X \text{ is a change in } X \text{ and } X_0 \text{ is the initial value of } X \]

\[ \text{as } X \to X^+ \text{, we have } \delta X \to 0 \text{ and } X \to X^+ \text{ such that } X_0 + \delta X = X^+ \]

\[ (X)^{n-1} \text{ is the } (n-1)\text{th power of } X \]

\[ X = X_0 + \delta X \text{ and } X^+ = X_0 + \delta X \]

\[ (X)^n \text{ is the } n\text{th power of } X \]

\[ (\delta X)^n = (X_0 + \delta X)^n \]

\[ (X)^n = X^n \]

\[ (X_0 + \delta X)^n = X^n + \text{higher order terms} \]

\[ (\delta X)^n = X^n + \text{higher order terms} \]

\[ \text{where } n \geq 2 \]

\[ (X)^n = X^n \text{ for } n = 1, 2, 3, \ldots \]

\[ (\delta X)^n = \text{higher order terms} = \text{higher degree terms} \]

\[ \text{for } n \geq 3 \]

\[ (X_0 + \delta X)^n = X^n + \text{higher order terms} \]

\[ (\delta X)^n = \text{higher order terms} \]

\[ \text{for } n \geq 3 \]

\[ Q + N \text{ is the } Q^\text{th} \text{ power of } (X)^N \]

\[ N \geq 0 \text{ and } Q \text{ is a positive integer} \]
complex has the form \[ \cdots \to 0 \to 0 \xrightarrow{d^{-1}} x^0 \xrightarrow{d^0} x^1 \xrightarrow{d^1} x^2 \xrightarrow{d^2} \cdots \],
a left complex \[ \cdots \xrightarrow{d_2} x^2 \xrightarrow{d_1} x^1 \xrightarrow{d_0} x^0 \xrightarrow{d_{-1}} 0 \to 0 \to \cdots \].

For notational convenience, a complex \((X,d)\) will hereafter be denoted simply by \(X\).

**Definitions 1.2:** A left complex over a module \(A\) is a left complex \(X\) together with an epimorphism \(\varepsilon: X_0 \to A\) such that \[ \cdots \xrightarrow{d_2} x_2 \xrightarrow{d_1} x_1 \xrightarrow{d_0} x_0 \xrightarrow{d_{-1}} 0 \to 0 \] is a 0-sequence (the composition of any two consecutive morphisms is zero). A left complex is called a left resolution if \[ \cdots \xrightarrow{d_2} x_2 \xrightarrow{d_1} x_1 \xrightarrow{d_0} x_0 \xrightarrow{d_{-1}} A \to 0 \] is exact. A projective (free) resolution of \(A\) is a left resolution such that for each \(q\), \(X_q\) is projective (free).

The concepts of right complex over \(A\), right resolution, and injective resolution are analogously defined. In this case, \[ 0 \to A \xrightarrow{\varepsilon} x^1 \to x^2 \to \cdots \] is a zero (resp., exact) sequence.

**Definitions 1.3:** Let \(X\) and \(Y\) be complexes. A translation \(f: X \to Y\) is a morphism of degree 0 which commutes with the boundary maps, that is, for which

\[
\begin{array}{ccc}
X_q & \xrightarrow{f_q} & Y_q \\
\downarrow{d_q} & & \downarrow{d_q} \\
X_{q-1} & \xrightarrow{f_{q-1}} & Y_{q-1}
\end{array}
\]

is a commutative diagram for all \(q\). If \(f\) is a translation, \(f\) induces morphisms \(H_n(f): H_n(X) \to H_n(Y)\), since \(f(Z_n(X)) \subseteq Z_n(Y)\) and \(f(B_n(X)) \subseteq B_n(Y)\).

If \(f: X \to Y\) and \(g: X \to Y\) are translations, a homotopy between \(f\) and \(g\) is a morphism \(D: X \to Y\) of degree \(-1\) such that
$f_q - g_q = d_{q+1}^{q} + d_{q-1}^{q}$ . The relation of homotopy is an equivalence equation.

**Proposition 1.4:** If two translations of $X$ into $Y$ are homotopic, then the corresponding morphisms $H_n(X) \longrightarrow H_n(Y)$ coincide.

**Proof:** If $f$ and $g$ are the homotopic translations, then $f - g$ is null homotopic (homotopic to zero). Let $x_n \in Z_N(X)$.

$$(f-g)(x_n) = d_{n+1}^{n} + d_{N-1}^{N} x_n$$

$$= d_{N+1}^{N} x_n \in B_N(Y) .$$

Therefore $f - g$ induces a zero map $H_n(X) \longrightarrow H_n(Y)$.

We now obtain a series of propositions concerning projective resolutions of modules. We will then give the dual results concerning injective resolutions.

**Proposition 1.5:** Every module has a projective resolution.

**Proof:** Given a module $A$, construct exact sequences

$$0 \longrightarrow Z_0 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow Z_1 \longrightarrow X_1 \longrightarrow Z_0 \longrightarrow 0$$

$$0 \longrightarrow Z_N \longrightarrow X_N \longrightarrow Z_{N-1} \longrightarrow 0 ,$$

where the $X_i$ are projective. Define $d_N$ as the composition

$$X_N \longrightarrow Z_{N-1} \longrightarrow X_{N-1} .$$

Then $\cdots \longrightarrow X_N \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$ is a projective resolution of $A$.

**Corollary 1.6:** If $A$ is left Noetherian and $A$ is a finitely generated left module, then $A$ has a free resolution $X$ where each $X_N$ is finitely generated.

**Proof:** $X_0$ may be chosen free and finitely generated. Then $Z_0$ is finitely generated, and $X_1$ may be chosen free with a finite base, etc.
Proposition 1.7: Let \( A \) and \( B \) be modules, let \( X \) be a projective resolution of \( A \) and \( Y \) a left resolution of \( B \). If \( f: A \to B \) is a morphism, then there exists a translation \( \tilde{f}: X \to Y \) such that

\[
\begin{array}{ccc}
X_0 & \xrightarrow{\tilde{f}_0} & Y_0 \\
\varepsilon & \downarrow & \varepsilon \\
A & \xrightarrow{f} & B
\end{array}
\]

commutes. \( \tilde{f} \) is said to be over \( f \).

Proof: Since \( X_0 \) is projective, there exists \( \tilde{f}_0: X_0 \to Y_0 \) such that \( \varepsilon \tilde{f}_0 = f \varepsilon \). Since \( \varepsilon \tilde{f}_0 d_1 = f \varepsilon d_1 = 0 \), \( \tilde{f}_0 d_1(\varepsilon) \subseteq \ker(\varepsilon) = \im(d_1) \), and there exists \( \tilde{f}_1: X_1 \to Y_1 \) such that \( \tilde{f}_0 d_1 = d_1 \tilde{f}_1 \). Proceeding inductively, we obtain the proposition.

Proposition 1.8: Under the hypothesis of proposition 1.7, if \( \tilde{f} \) and \( \tilde{g} \) both lie over \( f \), then \( \tilde{f} \) and \( \tilde{g} \) are homotopic.

Proof:

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\tilde{f}_1} & Y_1 \\
\tilde{d}_1 & \xleftarrow{\tilde{f}_0 \tilde{g}_0} & \tilde{d}_1 \\
\varepsilon & \downarrow & \varepsilon \\
A & \xrightarrow{f} & B
\end{array}
\]

\( f \varepsilon = \varepsilon \tilde{f}_0 = \varepsilon \tilde{g}_0 \), so \( \varepsilon (\tilde{f}_0 - \tilde{g}_0) = 0 \). Hence \( \im(\tilde{f}_0 - \tilde{g}_0) \subseteq \ker(\varepsilon) = \im(d_1) \), and, since \( X_0 \) is projective, there exists \( D_0: X_0 \to Y_1 \) such that

\[
\tilde{d}_1 D_0 = \tilde{f}_0 - \tilde{g}_0 \].

Now consider \( \tilde{f}_1 - \tilde{g}_1 - D_0 \tilde{d}_1: X_1 \to Y_1 \),

\[
\begin{array}{ccc}
X_2 & \xrightarrow{\tilde{f}_1 - \tilde{g}_1 - D_0 \tilde{d}_1} & Y_2 \\
\tilde{d}_2 & \xleftarrow{\tilde{f}_1 - \tilde{g}_1 - D_0 \tilde{d}_1} & \tilde{d}_2 \\
X_1 & \xrightarrow{\tilde{d}_1} & Y_1 \\
\varepsilon & \downarrow & \varepsilon \\
A & \xrightarrow{f} & B
\end{array}
\]

\( \tilde{d}_1 (\tilde{f}_1 - \tilde{g}_1 - D_0 \tilde{d}_1) = \tilde{f}_0 \tilde{d}_1 - \tilde{g}_0 \tilde{d}_1 - \tilde{d}_1 D_0 \tilde{d}_1 = 0 \), so \( \im(\tilde{f}_1 - \tilde{g}_1 - D_0 \tilde{d}_1) \subseteq \ker(\tilde{d}_1) = \im(\tilde{d}_2) \), and there exists \( D_1: X_1 \to Y_2 \) such that \( \tilde{d}_2 D_1 = \tilde{f}_1 - \tilde{g}_1 - D_0 \tilde{d}_1 \).
Proceeding inductively, the result is obtained.

**Proposition 1.5':** Every module has an injective resolution.

**Proof:** Given a module \( A \), construct exact sequences

\[
0 \rightarrow A \rightarrow Y^0 \rightarrow Z^0 \rightarrow 0 \\
0 \rightarrow Z^0 \rightarrow Y^1 \rightarrow Z^1 \rightarrow 0 \\
0 \rightarrow Z^{N-1} \rightarrow Y^N \rightarrow Z^N \rightarrow 0 ,
\]

where the \( Y^i \) are injective (proposition 3.10 of ch.3). Define \( d^N \) as the composition \( Y^N \rightarrow Z^N \rightarrow Y^{N+1} \). Then

\[
0 \rightarrow A \rightarrow Y^0 \rightarrow Y^1 \rightarrow \ldots \text{ is an injective resolution of } A .
\]

Note that there is no statement dual to corollary 1.6.

**Proposition 1.7':** Let \( A \) and \( B \) be modules, let \( X \) be a right resolution of \( A \) and \( Y \) an injective resolution of \( B \). If \( f: A \rightarrow B \) is a morphism, then there exists a translation \( \tilde{f}: A \rightarrow B \) over \( f \).

**Proof:** Since \( Y^0 \) is injective, there exists \( \tilde{f}^0: X^0 \rightarrow Y^0 \) such that \( \tilde{f}^0 \epsilon = \epsilon f \). Since \( d^{0 \circ \epsilon} \epsilon = d^0 \epsilon f = 0 \), \( d^{0 \circ \epsilon} \) induces a morphism \( X^0 / \text{im}(\epsilon) \rightarrow Y^1 \). Since \( 0 \rightarrow X^0 / \text{im}(\epsilon) \xrightarrow{d^0} X^1 \) is exact, there exists \( \tilde{f}^1: X^1 \rightarrow Y^1 \) such that \( \tilde{f}^1 d^0 = d^{0 \circ \epsilon} \). Proceeding inductively, we obtain the proposition.

**Proposition 1.8':** Under the hypotheses of proposition 1.7', if \( \tilde{f} \) and \( \tilde{g} \) both lie over \( f \), then \( \tilde{f} \) and \( \tilde{g} \) are homotopic.

**Proof:**

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\epsilon} & & \downarrow{\epsilon} \\
X^0 & \xrightarrow{d^0} & Y^0 \\
\downarrow{f^0, g^0} & & \downarrow{d^0} \\
X^1 & & Y^1 \\
\end{array}
\]

\( \epsilon f = \tilde{f}^0 \epsilon = \tilde{g}^0 \epsilon \), so \( (\tilde{f}^0 - \tilde{g}^0) \epsilon = 0 \). Hence \( \tilde{f}^0 - \tilde{g}^0 \) induces a morphism \( X^0 / \text{im}(\epsilon) \rightarrow Y^0 \), and, since \( 0 \rightarrow X^0 / \text{im}(\epsilon) \rightarrow X^1 \) is exact, there exists a morphism \( D^1: X^1 \rightarrow Y^0 \) such that \( D^1 d^0 = \tilde{f}^0 - \tilde{g}^0 \).
Now consider \( \mathfrak{r}^1 - \mathfrak{g}^1 - d^n D^1 : X^1 \rightarrow Y^1 \).

\[
\begin{array}{c}
\mathfrak{r}^1 - \mathfrak{g}^1 - d^n D^1 \downarrow\mathrlap{d^n} \\
X^1 \xrightarrow{\mathfrak{r}^1} Y^1 \downarrow\mathrlap{d^n} \\
X^2 \xrightarrow{\mathfrak{g}^1} Y^2
\end{array}
\]

\( (\mathfrak{r}^1 - \mathfrak{g}^1 - d^n D^1) d^n = d^n \mathfrak{r}^1 - d^n \mathfrak{g}^1 - d^n D^1 d^n = 0 \). Hence \( \mathfrak{r}^1 - \mathfrak{g}^1 - d^n D^1 \) induces a morphism \( X^1 / \text{im}(d^n) \rightarrow Y^1 \), and, since \( 0 \rightarrow X^1 / \text{im}(d^n) \xrightarrow{d^1} X^2 \) is exact, there exists \( D^2 : X^2 \rightarrow Y^2 \) such that \( D^2 d^1 = \mathfrak{r}^1 - \mathfrak{g}^1 - d^n D^1 \).

Proceeding inductively, the result is obtained.

2. Resolutions of sequences.

Here we obtain some results concerning translations of resolutions over the modules of short exact sequences. We will prove our statements only for left resolutions, since the method of proof for right resolutions is step-by-step dualization just as in the proofs given above.

We first obtain a general lemma that will be of great importance in later applications.

**Lemma 2.1:** Suppose \( 0 \rightarrow X' \xrightarrow{i} X \xrightarrow{j} X'' \rightarrow 0 \) is an exact sequence of complexes and translations. Then there is a canonical exact sequence

\[
\ldots \rightarrow H_q(X') \xrightarrow{H_q(i)} H_q(X) \xrightarrow{H_q(j)} H_q(X'') \xrightarrow{\delta} H_{q-1}(X') \rightarrow \ldots
\]

\( \delta \) is called the connecting morphism.

**Proof:** We are given a commutative diagram with exact rows and whose columns are 0-sequences:

\[
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
0 \rightarrow & X'_{N+1} \xrightarrow{i} & X_{N+1} \xrightarrow{j} \rightarrow X''_{N+1} \rightarrow 0 \\
\downarrow & \downarrow d & \downarrow d \\
0 \rightarrow & X'_{N} \xrightarrow{i} & X_{N} \xrightarrow{j} \rightarrow X''_{N} \rightarrow 0 \\
\downarrow & \downarrow d & \downarrow d \\
0 \rightarrow & X'_{N-1} \xrightarrow{i} & X_{N-1} \xrightarrow{j} \rightarrow X''_{N-1} \rightarrow 0 \\
\downarrow & \downarrow d & \downarrow d \\
\vdots & \vdots & \vdots
\end{array}
\]
i) Definition of $\delta$: Let $\nu \in Z_N(x^n)$. Let $x \in X_N$ be such that $j(x) = x^n$. $jd(x) = 0$, so $d(x) \in \text{im}(i)$. Let $x \in X_{N-1}$ be such that $i(x^i) = d(x)$. $di(x^i) = 0$, so $d(x^i) = 0$. Define $\delta(x^n) = x^i$.
Now assume that $y \in Z_N(x^n)$ is such that $y^n = x^n \mod B_N(x^n)$. Choose any $y$ and $y'$ such that $j(y) = y^n$, $i(y') = d(y)$. We must show that $x^i = y' \mod B_{N-1}(x^i)$. $x^n - y^n \in B_N(x^n)$, say $x^n - y^n = d(z^n)$. Let $z \in X_{N-1}$ be such that $j(z) = z^n$. $j(x - y - d(z)) = x^n - y^n - d(z^n) = 0$, so $x - y - d(z) \in \text{im}(i)$, say $i(z^i) = x - y - d(z)$, $z \in X_{N-1}$.

$$i(x^i - y^i - d(z^i)) = d(x) - d(y) - id(z^i)$$
$$= d(x - y - i(z^i)) = d(z) = 0.$$ Since $i$ is a monomorphism, $x^i - y^i - d(z^i) = 0$, $x^i \equiv y' \mod B_{N-1}(x^i)$. $\delta$ is thus well-defined, and is clearly a morphism.

ii) $H_N(x^i) \to H_N(x) \to H_N(x^n)$ is exact:
Clearly $\ker(H_N(j)) \subseteq \text{im}(H_N(1))$ since $j1 = 0$. Let $x \in Z_N(x)$ be such that $H_N(j)(x) = 0$. $j(x) \in B_N(x^n)$. Let $y \in X_{N-1}$ be such that $dj(y) = j(x)$. $j(x - d(y)) = j(x) - dj(y) = 0$, so $x - d(y) \in \text{im}(i)$, say $i(z') = x - d(y)$. $di(z') = dx - ddy = 0$, so $d(z') = 0$. Thus $H_N(1)(z') = x$, and $\ker(H_N(j)) \subseteq \text{im}(H_N(1))$.

iii) $H_N(x) \to H_N(x^n) \to H_{N-1}(x^i)$ is exact:
$x \in Z_N(x)$ implies $\delta H_N(j)(x) = 0$ by construction of $\delta$, so $\ker(\delta) \subseteq \text{im}(H_N(j))$. Let $x \in Z_N(x^n)$ and $\delta(x^n) = 0$. Let $j(x) = x^n$, $i(x^i) = d(x)$. $x \in B_{N-1}(x^i)$, say $x^i = d(y^i)$. Now $d(x - i(y^i)) = d(x) - i(x^i) = 0$, so $x - i(y^i) \in Z_N(x)$. $H_N(j)(x - i(y^i)) = H_N(j)(x) = x^n$, and $\text{im}(H_N(j)) \supseteq \ker(\delta)$. 
iv) \( H_N(x') \rightarrow H_{N-1}(x') \rightarrow H_{N-1}(x) \) is exact:
If \( x' \in Z_{N-1}(x') \) and \( \overline{x'} = \delta(x') \), then \( i(x') = d(x) \) for some \( x \in X_N \), and \( H_N(i)(\overline{x'}) = 0 \), so \( \ker(H_N(i)) \supset \text{im}(\delta) \). Let \( x' \in Z_{N-1}(x') \) and \( H_N(i)(\overline{x'}) = 0 \). \( i(x') \in B_{N-1}(x) \), say \( d(x) = i(x') \). Let \( x'' = j(x) \).
\( d(x'') = dj(x) - jd(x) = ji(x') = 0 \). \( \delta(x'') = \overline{x'} \) by construction of \( \delta \), and \( \text{im}(\delta) \supset \ker(H_N(i)) \). This completes the proof.

**Corollary 2.2:** If \( 0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0 \) is an exact sequence of complexes and any two of \( X', X, X'' \) are exact, then so is the third.

**Definitions 2.3:** Let \( 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0 \) be an exact sequence. An exact sequence \( 0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0 \) of complexes, where \( X', X, X'' \) are left complexes (or left resolutions, etc.), over \( A', A, A'' \) and \( \tilde{i}, \tilde{j} \) are morphisms over \( i, j \) is called a left complex (or left resolution, etc.), over \( 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0 \).

Right complexes over exact sequences are analogously defined.

**Proposition 2.4:** If \( 0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0 \) is a left complex over \( 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0 \) and if \( X' \) and \( X'' \) are projective complexes, then so is \( X \).

**Proof:** For all \( N \) \( 0 \rightarrow X'_N \rightarrow X_N \rightarrow X''_N \rightarrow 0 \) is split exact, \( X_N \) is isomorphic \( X'_N \otimes X''_N \), hence is a direct summand of a free module.

**Corollary 2.5:** If \( X' \) and \( X'' \) are projective resolutions of \( A' \) and \( A'' \), then \( X \) is a projective resolution of \( A \).

**Proposition 2.6:** Let \( 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0 \) be an exact sequence.

Let \( X' \) be a left resolution of \( A' \), \( X'' \) a projective complex over \( A'' \). Then there exists a left complex \( X \) over \( A \) and maps \( \tilde{i}, \tilde{j} \) over \( i, j \) such that \( 0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0 \) is a left complex over \( 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0 \).
Proof: Set $X_N = X'_N \oplus X''_N$; let $\tilde{\iota}_N: X'_N \to X_N$ and $\bar{\jmath}_N: X''_N \to X_N$ be the canonical injection and projection. We must define a differential operator on $X$ such that the desired commutativity relations are satisfied.

1) Consider degree 0.

$$
0 \to X'_0 \xrightarrow{\tilde{\iota}_0} X''_0 \oplus X''_0 \xrightarrow{\iota} X''_0 \to 0
$$

$$
0 \to A'_0 \xrightarrow{\tilde{\iota}_0} A''_0 \oplus A''_0 \xrightarrow{\iota} A''_0 \to 0
$$

Since $X''_0$ is projective, there exists $f'_0: X''_0 \to A$ such that $jf'_0 = \epsilon''$. Define $\epsilon: X'_0 \oplus X''_0 \to A$ by $\epsilon(x'_0, x''_0) = i\epsilon'(x'_0) + f'_0(x''_0)$. Then $\epsilon(x'_0, x''_0) = i\epsilon'(x'_0)$, $j\epsilon(x'_0, x''_0) = jf'_0(x''_0) = \epsilon''j(x'_0, x''_0)$, and the diagram commutes. We must show that $\epsilon$ is an epimorphism. Let $x \in A$. Let $x''_0 \in X''_0$ be such that $j(x) = \epsilon''(x''_0)$. Let $y = x - f'_0(x''_0)$.

Then $J(y) = j(x) - jf'_0(x''_0) = j(x) - \epsilon''(x''_0) = 0$, so $y \in \text{im}(\iota)$. Let $x'_0 \in X'_0$ be such that $\epsilon'(x'_0) = y$. Then

$$
\epsilon(x'_0, x''_0) = i\epsilon'(x'_0) + f'_0(x''_0) = y + f'_0(x''_0) = x .
$$

ii) For $N \geq 1$, let $f'_N: X'_N \to X'_{N-1}$ be, for the moment, arbitrary morphisms. Define $d'_N(x'_0, x''_0) = (d'_N x'_0 + f'_N x''_0, d''_N x''_0)$. Clearly

$$
0 \to X'_N \to X'_N \xrightarrow{d'_N} X''_N \to 0
$$

$$
0 \to X'_{N-1} \to X'_{N-1} \xrightarrow{d'_N} X''_{N-1} \to 0
$$

is commutative for all $N \geq 1$. We will define the $f'_N$ so that $\epsilon d'_N = 0$, $d'_{N-1} d'_N = 0$ for $N > 1$. 
Proposition 5.23: In the proposition above, if $X$ and $Y$ are projective, then $X$ and $Y$ are necessarily a

\[ T_{x}X = T_{x}Y \]

\[ T_{x}Y = T_{x}X \]

Corollary 5.24: The completion of the product $X \times Y$ exists such that $X \times Y \subseteq T_{x}X$. By induction, and the now is exact by hypothesis, so there exists

\[ T_{x}X \subseteq T_{x}Y : T_{x}Y \]

\[ T_{x}Y \subseteq T_{x}X : T_{x}X \]

Hence, since $T_{x}X$ is projective, there exists a unique $T_{x}Y$ such that $T_{x}X \subseteq T_{x}Y$. Hence, the now is exact. Hence,

\[ T_{x}X \subseteq T_{x}Y : T_{x}Y \]

\[ T_{x}Y \subseteq T_{x}X : T_{x}X \]

Now, $T_{x}X = T_{x}Y$, which we can write:

\[ (0 \circ T_{x}N_{x} + N_{x}T_{x}N_{x}) = T_{x}X \]

\[ T_{x}Y = T_{x}X \]

Hence, the now is exact. Hence,

\[ T_{x}X \subseteq T_{x}Y : T_{x}Y \]

\[ T_{x}Y \subseteq T_{x}X : T_{x}X \]

\[ T_{x}X = T_{x}Y \]

\[ T_{x}Y = T_{x}X \]
Proposition 2.8: Let
\[
\begin{array}{c}
0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0 \\
\downarrow g' \downarrow g \downarrow g'' \\
0 \to B' \xrightarrow{k} B \xrightarrow{\ell} B'' \to 0
\end{array}
\]
be a commutative diagram with exact rows. Let \(0 \to X' \xrightarrow{i'} X \xrightarrow{j'} X'' \to 0\) be a split exact left complex over \(0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0\) and
\[
0 \to Y' \xrightarrow{k'} Y \xrightarrow{\ell'} Y'' \to 0
\]
be a split exact left complex over
\[
0 \to B' \xrightarrow{k} B \xrightarrow{\ell} B'' \to 0.
\]
Further, let \(X''\) be a projective complex and let \(Y'\) be exact. Then if \(\tilde{g}' \colon X' \to Y'\) and \(\tilde{g}'' \colon X'' \to Y''\) are translations over \(g'\) and \(g''\), there exists a translation
\(\tilde{g} \colon X \to Y\) over \(g\) such that
\[
\begin{array}{c}
0 \to X' \xrightarrow{i} X \xrightarrow{j} X'' \to 0 \\
\downarrow \tilde{g}' \downarrow \tilde{g} \downarrow \tilde{g}'' \\
0 \to Y' \xrightarrow{k} Y \xrightarrow{\ell} Y'' \to 0
\end{array}
\]
is a commutative diagram of complexes and translations.

Proof: Writing \(X_N\) as \(X'_N \oplus X''_N\) and \(Y_N\) as \(Y'_N \oplus Y''_N\), we see that \(\tilde{g}_N(x',x'') = (\tilde{g}'_N(x'_N) + q_N(x''_N), \tilde{g}''_N(x''_N))\) is necessary for
\[
\begin{array}{c}
0 \to X'_N \to X_N \to X''_N \to 0 \\
\downarrow \tilde{g}'_N \downarrow \tilde{g}_N \downarrow \tilde{g}''_N \\
0 \to Y'_N \to Y_N \to Y''_N \to 0
\end{array}
\]
to commute, where \(q_N \colon X''_N \to Y'_N\) is to be determined. The problem is to choose the \(q_N\) such that
\[
\begin{array}{c}
\tilde{X} \xrightarrow{\tilde{g}} \tilde{Y} \\
\downarrow \epsilon \downarrow \epsilon \\
A \xrightarrow{g} B
\end{array}
\]
commutes. We write \(d',d,d'',e',e,e''\) and \(\delta',\delta,\delta'',\epsilon',\epsilon,\epsilon''\) for the differentiation and augmentation morphisms of \(X',X,X''\) and \(Y',Y,Y''\).
\[
\begin{align*}
0 & \leftarrow X \quad Y & \leftarrow 0 \\
Y & \leftarrow X \quad Y & \leftarrow 0 \\
X & \leftarrow Y \quad Y & \leftarrow 0 \\
Y & \leftarrow X \quad Y & \leftarrow 0 \\
X & \leftarrow 0 \quad Y & \leftarrow 0 \\
\end{align*}
\]
Let $f_0: X^o \rightarrow A$, $f_N: X^o_N \rightarrow X^o_{N-1}$ and $\varphi_0: Y^o \rightarrow B$, $\varphi_N: Y^o_N \rightarrow Y^o_{N-1}$ be morphisms as in the proof of the previous proposition. (These necessarily exist, since the sequences of complexes are split exact.)

By hypothesis, then, we have relations:

$$e = ie' + f_0, \quad ie'f_1 + f_0d''_1 = 0, \quad d''_{N-1}p_N + f_{N-1}d''_N = 0,$$
$$\varepsilon = ke' + \varphi_0, \quad ke'q_1 + \varphi_0d''_1 = 0, \quad \delta''_{N-1}p_N + \varphi_{N-1}d''_N = 0 \quad \text{and}$$
$$g'e' = e'g''_0, \quad g'_{N-1}d''_N = \delta''_{N-1}d''_N, \quad g''e'' = e''g''_0, \quad g''_{N-1}d''_N = \delta''_{N-1}d''_N.$$

We wish to obtain $ge = e'g''_0, \quad g'_{N-1}d''_N = \delta''_{N-1}d''_N$. These relations then take the forms

i) $ke'q_0 = -\varphi_0g''_0 + gf_0$

$$e'g''_0 = e'(g''_0 + q_0g''_0) = ke'g''_0 + ke'q_0 + \varphi_0g''_0;$$

$$ge = gie' + gf_0 = kg'e' + gf_0 = ke'g''_0 + gf_0.$$

ii) $\delta''_{N-1}d''_N = g'_{N-1}d''_N + q_{N-1}d''_N = \varphi_{N-1}d''_N$

$$g'_{N-1}d''_N = \tilde{g}'_{N-1}(d''_N + f''_Nd''_N) = (\tilde{g}'_{N-1}d''_N + \tilde{g}'_{N-1}f''_N + q_{N-1}d''_N, \varphi_{N-1}d''_N);$$

$$\delta''_{N-1}d''_N = \delta''_{N-1}(g''_{N-1} + q_{N-1}g''_{N-1}) = (\delta''_{N-1}g''_{N-1} + \delta''_{N-1}q_{N-1} + \varphi_{N-1}g''_{N-1}, \delta''_{N-1}g''_{N-1});$$

$$\delta''_{N-1}d''_N = g''_{N-1}d''_N, \quad \delta''_{N-1}d''_N = \tilde{g}'_{N-1}d''_N.$$

Finally, then, we observe that in the following diagrams, the rows are exact and the diagrams commute, so that, by the projectivity of $X^o$, the desired morphisms are obtained:

\[
\begin{array}{c}
\xymatrix{ X^o \ar[rrr]^0 & & & Y^o \ar[rr]^\varepsilon \ar[urr]_{f''} & & B'' \\
\varphi_0 = -e'g''_0 + gf_0 & & & & & \lambda_0g''_0 + 2gf_0 = e''g''_0 + g''f''_0 = -e''g''_0 + g''e'' = 0 \\
Y^o \ar@{->}[rr]^\varepsilon & & B & & \varepsilon \ar[rr]^\lambda & & B''}
\end{array}
\]
\[ \tilde{\mathfrak{e}}^i f_{1} + q_{0} d_{1} \tilde{\mathfrak{g}}_{1} = 0 \]
\[ Y'_1 \xrightarrow{\delta'} \text{ke'} \rightarrow Y'_0 \xrightarrow{\text{ke}'} B \]
\[ = \text{ke'} \tilde{\mathfrak{e}}^i f_{1} + \text{ke'} q_{0} d_{1} \tilde{\mathfrak{g}}_{1} - \text{ke'} \mathfrak{g}_{1} \tilde{\mathfrak{g}}_{1} \]
\[ = k_{0} \epsilon_{1} f_{1} - \varphi_{0} \tilde{\mathfrak{g}}_{1} + \varphi_{0} d_{1} + \varphi_{0} \delta_{1} \tilde{\mathfrak{g}}_{1} = 0 \]

\[ \tilde{\mathfrak{g}}_{N-1} f_{N-1} + q_{N-1} d_{N-1} \tilde{\mathfrak{g}}_{N} \]
\[ Y'_N \xrightarrow{\delta'} \text{ke'} \rightarrow Y'_{N-1} \xrightarrow{\delta'} Y'_{N-2} \]
\[ = \tilde{\mathfrak{g}}_{N-1} + q_{N-1} d_{N-1} \tilde{\mathfrak{g}}_{N} - \delta'_{N-1} \mathfrak{N}_{2} \tilde{\mathfrak{g}}_{N} \]
\[ = \tilde{\mathfrak{g}}_{N-2} (d_{N-1} f_{N-1} + \tilde{\mathfrak{g}}_{N-2} f_{N-1} + q_{N-2} d_{N-1} \tilde{\mathfrak{g}}_{N-1}, \tilde{\mathfrak{g}}_{N-1} d_{N-1} \tilde{\mathfrak{g}}_{N}) - \delta'_{N-1} \mathfrak{N}_{2} \tilde{\mathfrak{g}}_{N} = 0 \]

**Proposition 2.4':** If \( 0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0 \) is a right complex over the exact sequence \( 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0 \) and if \( Y' \) and \( Y'' \) are injective complexes, then so is \( Y \).

**Corollary 2.5':** If \( Y' \) and \( Y'' \) are injective resolutions of \( A' \) and \( A'' \), then \( Y \) is an injective resolution of \( A \).

**Proposition 2.6':** Let \( 0 \rightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \rightarrow 0 \) be an exact sequence. Let \( Y' \) be an injective complex over \( A' \), \( Y'' \) a right resolution of \( A'' \). Then there exists a right complex \( Y \) over \( A \) and maps \( i, j \) over \( i, j \) such that \( 0 \rightarrow Y' \xrightarrow{i} Y \xrightarrow{j} Y'' \rightarrow 0 \) is a right complex over \( 0 \rightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \rightarrow 0 \).

**Corollary 2.7':** In the proposition above, if \( Y' \) and \( Y'' \) are injective resolutions of \( A' \) and \( A'' \), then \( Y \) will necessarily be an injective resolution of \( A \).
Proposition 2.8': Let
\[ 0 \to A' \overset{1}{\to} A \overset{1}{\to} A'' \to 0 \]
\[ \text{and} \quad 0 \to B' \overset{k}{\to} B \overset{l}{\to} B'' \to 0 \]
be a commutative diagram with exact rows. Let
\[ 0 \to X' \overset{1}{\to} X \overset{j}{\to} X'' \to 0 \]
be a split exact right complex over
\[ 0 \to A' \overset{1}{\to} A \overset{j}{\to} A'' \to 0 \quad \text{and} \quad 0 \to Y' \overset{k''}{\to} Y \overset{l''}{\to} Y'' \to 0 \]
be a split exact right complex over \( 0 \to B' \overset{k}{\to} B \overset{l}{\to} B'' \to 0 \). Further, let \( X'' \) be exact and let \( Y' \) be an injective complex. Then if \( \tilde{g}': X' \to Y' \) and \( \tilde{g}'': X'' \to Y'' \) are translations over \( g' \) and \( g'' \), there exists a translation \( \tilde{g}: X \to Y \) over \( g \) such that
\[ 0 \to X' \overset{1}{\to} X \overset{j}{\to} X'' \to 0 \]
\[ \text{and} \quad 0 \to Y' \overset{k}{\to} Y \overset{l}{\to} Y'' \to 0 \]
is a commutative diagram of complexes and translations.

3. Construction of \( \text{Tor}(A,B) \)

Let \( A \) be a right \( \Lambda \)-module, \( B \) a left \( \Lambda \)-module. Then \( A \otimes B \) is an Abelian group (see ex. 10, 11 ch. 3). Recall that if
\[ 0 \to A' \to A \to A'' \to 0 \]
is exact, then
\[ A' \otimes B \to A \otimes B \to A'' \otimes B \to 0 \]
is exact, and if
\[ 0 \to B' \to B \to B'' \to 0 \]
is exact, then so is
\[ A \otimes B' \to A \otimes B \to A \otimes B'' \to 0 \]. In this section we will construct objects by means of which the behavior of tensored exact sequences on the left may be studied.

Let \( X \) be a complex of right \( \Lambda \)-modules, \( Y \) a complex of left \( \Lambda \)-modules. \( X \otimes Y \) is a graded module with \( (X \otimes Y)_N = \oplus_{i+j=N} X_i \otimes Y_j \). For notational convenience, we write \( X_i \otimes Y_j = X^{i+j} \).
Now let $X',X''$ be further complexes of right $A$-modules, $Y'$
and $Y''$ of left $A$-modules. Let $f: X \rightarrow X'$ be a morphism of degree $p$, 
g: $Y \rightarrow Y'$ be a morphism of degree $q$. Define

$$f \otimes g = (-1)^q f \otimes g_j$$

and $(f \otimes g)_N = \otimes_{i+j=N} f \otimes g$. Suppose $f': X' \rightarrow X''$
and $g': Y' \rightarrow Y''$ are morphisms of degrees $p'$ and $q'$. Then $f'f$
and $g'g$ are morphisms of degree $p+p'$ and $q+q'$. Further, we have

$$f'f \otimes g'g = (-1)^{q+q'} f' \otimes f \otimes (f' \otimes g_j)$$

$$= (-1)^{q+q'} (f' \otimes g_j) - (1-p) q' (f' \otimes g_j) - (1-p) q' (f \otimes g_j)$$

$$= (-1)^{q+q'} (f' \otimes g_j) (f \otimes g_j)$$.

We now return to the consideration of $X \otimes Y$. We will define a dif-
ferential operator on $X \otimes Y$ by $d = d_x \otimes 1_y + 1_x \otimes d_y$ where $d_x$ and
$d_y$ are the differential operators on $X$ and $Y$ and $1_x$ and $1_y$ are
the identities of $X$ and $Y$. $d$ is of degree 1, and $dd = 0$ since

$$dd = d_x \otimes d_y + d_x \otimes d_y$$

$$= (1) (1_x \otimes d_y) (d_x \otimes 1_y) + (d_x \otimes 1_y) (1_x \otimes d_y) = 0$$.

Thus $X \otimes Y$ is given the structure of a complex.

Before defining $\text{Tor}(A,B)$, we prove the

**Lemma 3.1:** Suppose $X$ and $Y$ are left complexes, $X_q$ is flat for
all $q$ and $Y$ is exact. Then $X \otimes Y$ is exact.

**Proof:** 1) Suppose $Y_q \neq 0$ for $q=s$ and $q=s+1$ only. Then

$$0 \rightarrow Y_{s+1} \rightarrow Y_s \rightarrow 0$$

is exact. Let $f_s: Y_s \rightarrow Y_{s+1}$ be the inverse

isomorphism to $d_{s+1}$, $f_q = 0$, $q \neq s$. Then $df + fd = 1_y$. Define

$D: X \otimes Y \rightarrow X \otimes Y$ by $D = 1_x \otimes f$. 

Then $dD = d_x \otimes i_x^f + i_x \otimes d_y^f$

$Dd = i_x \otimes d_y^f + i_x \otimes fd_y$

$dD + Dd = i_x \otimes d_x^f + i_x \otimes fd_y$

$= i_x \otimes (d_x^f + fd_y) = i_x \otimes i_y = i_x \otimes i_y$.

Hence the identity of $X \otimes Y$ is homotopic to the zero map and

$H_n(X \otimes Y) = 0$ for all $N$.

II) Proceeding inductively, assume $Y_q = 0$ for $q < r$ and for $q > N+1 > r$. Define $Y'$ by $Y_q' = 0$ for $q \neq N, N+1$,

$Y_N' = B_N(Y), Y_{N+1}' = Y_{N+1}$; $H_q(X \otimes Y') = 0$ for all $q$ by step i).

Define $Y''$ such that $0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0$ is exact. Since $H_q(Y') = H_q(Y) = 0$ for all $q$, $H_q(Y'') = 0$ for all $q$. $Y''_q = 0$ if $q < r$ and if $q > N$, so, by induction, $H_q(X \otimes Y'') = 0$ for all $q$.

Since $X$ is flat, $0 \rightarrow X \otimes Y' \rightarrow X \otimes Y \rightarrow X \otimes Y'' \rightarrow 0$ is exact (and is a sequence of translations), and $H_q(X \otimes Y) = 0$ for all $q$.

iii) Now assume the original hypotheses. For $s \geq 0$,

$(X \otimes Y)_s = \bigoplus_{i=0}^{s} X_i \otimes Y_{s-i}$.

Define $Y'$ by $Y_q' = Y_q$ if $q > s + 1$, $Y_{s+1}' = B_{s+1}(Y)$, $Y_q' = 0$ if $q < s + 1$.

Define $Y''$ such that $0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0$ is exact. $H_q(Y') = 0$ for all $q$ by construction since $H_q(Y) = 0$ for all $q$. Hence $H_q(Y'') = 0$ for all $q$.

$Y''_q = 0$ for $q < 0$, $Y''_q = 0$ for $q > s + 1$, so by step ii)

$H_q(X \otimes Y'') = 0$ for all $q$. $H_s(X \otimes Y') = 0$ by construction, hence $0 \rightarrow H_s(X \otimes Y) \rightarrow 0$ is exact, $H_s(X \otimes Y) = 0$. Since $s$ was arbitrary, $H_q(X \otimes Y) = 0$ for all $q$.

Note that the hypotheses and proof of the lemma are symmetric in $X$ and $Y$: If $X$ is exact and $Y$ is flat, $H_q(X \otimes Y) = 0$ for all $q$. 
\( \mu \in \mathcal{A}_1, \nu \in \mathcal{A}_2, \lambda \in \mathcal{A}_3 \rightarrow \lambda = \lambda' \) \\
\( \rho \in \mathcal{B}_1, \sigma \in \mathcal{B}_2, \tau \in \mathcal{B}_3 \rightarrow \tau = \tau' \)

\[ Y \cong X \] 

where \( Y \cong X \) is the isomorphism. 

\[ \rho \cong \sigma \] 

for \( \rho, \sigma \in \mathcal{B}_1 \) and \( \tau = \tau' \) for \( \tau, \tau' \in \mathcal{B}_3 \).

\[ \mu \cong \lambda \] 

for \( \mu, \lambda \in \mathcal{A}_1 \) and \( \nu = \nu' \) for \( \nu, \nu' \in \mathcal{A}_2 \).

\[ \rho \cong \tau \] 

for \( \rho, \tau \in \mathcal{B}_1 \) and \( \sigma = \sigma' \) for \( \sigma, \sigma' \in \mathcal{B}_2 \).

\[ \mu \cong \nu \] 

for \( \mu, \nu \in \mathcal{A}_1 \) and \( \lambda = \lambda' \) for \( \lambda, \lambda' \in \mathcal{A}_2 \).

\[ \rho \cong \sigma \] 

for \( \rho, \sigma \in \mathcal{B}_1 \) and \( \tau = \tau' \) for \( \tau, \tau' \in \mathcal{B}_3 \).

\[ \mu \cong \lambda \] 

for \( \mu, \lambda \in \mathcal{A}_1 \) and \( \nu = \nu' \) for \( \nu, \nu' \in \mathcal{A}_2 \).

\[ \rho \cong \tau \] 

for \( \rho, \tau \in \mathcal{B}_1 \) and \( \sigma = \sigma' \) for \( \sigma, \sigma' \in \mathcal{B}_2 \).

\[ \mu \cong \nu \] 

for \( \mu, \nu \in \mathcal{A}_1 \) and \( \lambda = \lambda' \) for \( \lambda, \lambda' \in \mathcal{A}_2 \).

\[ \rho \cong \sigma \] 

for \( \rho, \sigma \in \mathcal{B}_1 \) and \( \tau = \tau' \) for \( \tau, \tau' \in \mathcal{B}_3 \).

\[ \mu \cong \lambda \] 

for \( \mu, \lambda \in \mathcal{A}_1 \) and \( \nu = \nu' \) for \( \nu, \nu' \in \mathcal{A}_2 \).

\[ \rho \cong \tau \] 

for \( \rho, \tau \in \mathcal{B}_1 \) and \( \sigma = \sigma' \) for \( \sigma, \sigma' \in \mathcal{B}_2 \).

\[ \mu \cong \nu \] 

for \( \mu, \nu \in \mathcal{A}_1 \) and \( \lambda = \lambda' \) for \( \lambda, \lambda' \in \mathcal{A}_2 \).
Definition 3.2: Let $A$ be a right $\Lambda$-module, $B$ a left $\Lambda$-module, $X$ a projective resolution of $A$, and $Y$ a projective resolution of $B$. Define $\text{Tor}_N(A \otimes B) = H_N(X \otimes Y)$. We must prove that $\text{Tor}_N(A \otimes B)$ is independent of the choice of the projective resolutions $X$ and $Y$.

Instead of proving this directly, we first prove

Proposition 3.3: $H_q(A \otimes Y)$, $H_q(X \otimes Y)$, and $H_q(X \otimes B)$ are isomorphic for all $q$, where $A$ and $B$ are regarded as complexes concentrated in degree 0 with $d = 0$.

Proof: $Y \overset{0}{\longrightarrow} B \longrightarrow 0$ may be regarded as a translation $(\epsilon_q = 0$ for $q \neq 0)$. Define $Y'$ by $Y'_q = Y_q$ for $q \neq 0$, $Y'_0 = \ker(\epsilon_0)$. Then $0 \longrightarrow Y' \longrightarrow Y \longrightarrow B \longrightarrow 0$ is an exact sequence of translations. $H_q(Y') = 0$ for all $q$ since $\ker(\epsilon_0) = B_0(Y) = Y'_0$.

By lemma 3.1, $h_q(X \otimes Y') = 0$ for all $q$. By lemma 2.1, there is an exact sequence

$$\ldots \longrightarrow 0 = H_q(X \otimes Y') \longrightarrow H_q(X \otimes Y) \longrightarrow H_q(X \otimes B) \longrightarrow H_{q-1}(X \otimes Y') = 0 \longrightarrow \ldots$$

so $H_q(X \otimes Y)$ is isomorphic to $H_q(X \otimes B)$ for all $q$. Similarly, $H_q(X \otimes Y)$ is isomorphic to $H_q(A \otimes Y)$ for all $q$.

Proposition 3.4: $\text{Tor}_q(A,B)$ is independent of the choice of $X$ and $Y$.

Proof: Let $X$ and $X'$ be projective resolutions of $A$. By proposition 1.5, there exists $f: X \longrightarrow X'$ and $g: X' \longrightarrow X$ lying over the identity $i_A$ of $A \longrightarrow A$. By proposition 1.8, $gf: X \longrightarrow X$ lying over $i_A$ is homotopic to $i_x: X \longrightarrow X$, say $dD + Di = i_x - gf$.

Then $i_x \otimes i_B - gf \otimes i_B = (i_x - gf) \otimes i_B$

$$= (dD + Di) \otimes i_B$$

$$= dD \otimes i_B + Di \otimes i_B$$

$$= (d \otimes i_B)(D \otimes i_B) + (D \otimes i_B)(d \otimes i_B),$$ and,
since $d_{\delta B}$ is the differential operator of $\mathcal{K}_{\mathfrak{L}B}$, $D_{\delta B}$ is a homotopy between $i_{\delta B}$ and $g_{\delta B}$. \[ g_{\delta B} = (s_{\delta B})(f_{\delta B}) \]

so $H_q(g_{\delta B}) = H_q(s_{\delta B})H_q(f_{\delta B})$, and, arguing similarly for $f_\delta$, since $H_q(i_{\delta B})$
is the identity on $H_q(\mathcal{K}_{\mathfrak{L}B})$ and $H_q(i_{\delta B})$ is the identity on $H_q(\mathcal{K}_{\mathfrak{L}B})$, we have that $H_q(g_{\delta B})$ and $H_q(f_{\delta B})$ are inverse isomorphisms. Thus $H_q(\mathcal{K}_{\mathfrak{L}B})$ and $H_q(\mathcal{K}_{\mathfrak{L}B})$ are canonically isomorphic for all $q$. Arguing similarly, $\text{Tor}_q(A,B)$ is independent of the choice of $Y$.

**Proposition 3.5:** Let $0 \longrightarrow A' \xrightarrow{k} A \xrightarrow{\ell} A'' \longrightarrow 0$ be an exact sequence of right $\Lambda$-modules and $B$ a left $\Lambda$-module. Then there exists a canonical exact sequence

$$
\longrightarrow \text{Tor}_N(A'B') \longrightarrow \text{Tor}_N(A,B) \longrightarrow \text{Tor}_N(A'',B) \longrightarrow \text{Tor}_{N-1}(A',B) \longrightarrow \ldots
$$

$$
\longrightarrow \text{Tor}_o(A',B) \longrightarrow \text{Tor}_o(A,B) \longrightarrow \text{Tor}_o(A'B) \longrightarrow 0.
$$

**Proof:** Let $X'$ and $X''$ be projective resolutions of $A'$ and $A''$. By proposition 2.6 and corollary 2.7, there exists a projective resolution $X$ of $A$ and morphisms $\tilde{k}$ and $\tilde{\ell}$ over $k$ and $\ell$ such that $0 \longrightarrow X' \xrightarrow{\tilde{k}} X \xrightarrow{\tilde{\ell}} X'' \longrightarrow 0$ is a projective resolution of $0 \longrightarrow A' \xrightarrow{k} A \xrightarrow{\ell} A'' \longrightarrow 0$, $0 \longrightarrow X'\mathcal{K}_{\mathfrak{L}B} \longrightarrow X\mathcal{K}_{\mathfrak{L}B} \longrightarrow X''\mathcal{K}_{\mathfrak{L}B} \longrightarrow 0$ is exact, so by lemma 2.1 there is an exact sequence

$$
\longrightarrow H_N(X'\mathcal{K}_{\mathfrak{L}B}) \longrightarrow H_N(X\mathcal{K}_{\mathfrak{L}B}) \longrightarrow H_N(X''\mathcal{K}_{\mathfrak{L}B}) \xrightarrow{\delta(x)} H_{N-1}(X'\mathcal{K}_{\mathfrak{L}B}) \longrightarrow \ldots
$$

$$
\longrightarrow H_o(X'\mathcal{K}_{\mathfrak{L}B}) \longrightarrow H_o(X\mathcal{K}_{\mathfrak{L}B}) \longrightarrow H_o(X''\mathcal{K}_{\mathfrak{L}B}) \longrightarrow 0.
$$

To complete the proof, it suffices to show that the connecting morphism $\delta$ is independent of the choice of $X$. Suppose $0 \longrightarrow X' \longrightarrow Y \longrightarrow X'' \longrightarrow 0$ also lies over $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$. Then by proposition 2.8, lying over the commutative diagram
is a commutative diagram

The latter diagram induces a translation of the sequence 1).

Thus

is a commutative diagram whose columns are identity morphisms, and \( \delta \) is independent of the choice of \( X \).

Note that the proof holds equally well for an exact sequence

yielding a canonical exact sequence

... \( \rightarrow \text{Tor}_N(A,B') \rightarrow \text{Tor}_N(A,B) \rightarrow \text{Tor}_N(A,B'') \rightarrow \text{Tor}_{N-1}(A,B') \rightarrow \ldots \)

... \( \rightarrow \text{Tor}_o(A,B') \rightarrow \text{Tor}_o(A,B) \rightarrow \text{Tor}_o(A,B'') \rightarrow 0 \).

Proposition 3.6: \( \text{Tor}_o(A,B) \) is isomorphic to \( A \otimes B \).

Proof: Let \( X \) be a projective resolution of \( A \).

\[ X_1 \otimes B \rightarrow X_0 \otimes B \rightarrow \text{Tor}_o(A,B) \rightarrow 0 \]

is exact, so \( H_0(X \otimes B) = X_0 \otimes B / \text{im}(d_1 \otimes B) \) is isomorphic to \( A \otimes B \).
Proposition 3.7: The following are equivalent:

1) $A$ is flat
2) $\text{Tor}_1(A, B) = 0$ for all $B$.
3) $\text{Tor}_q(A, B) = 0$ for all $q \geq 1$ and for all $B$.
4) $\text{Tor}_q(A, B) = 0$ for all $q \geq 1$ and for all finitely generated $B$.
5) $\text{Tor}_q(A, B) = 0$ for all $q \geq 1$ and for all cyclic $B$.
6) $\text{Tor}_q(A, A/I) = 0$ for all $q \geq 1$ and for all ideals $I$.

Proof: 1) $\Rightarrow$ 3) Let $B$ be a module and $X$ a projective resolution of $B$. $\ldots \rightarrow \bigoplus_{N} X \rightarrow \ldots \rightarrow \bigoplus_{O} A \rightarrow A \rightarrow 0$ is exact.

3) $\Rightarrow$ 1) is immediate.

2) $\Rightarrow$ 1) Let $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ be exact.

$0 = \text{Tor}_1(A, B'') \rightarrow \bigoplus A \rightarrow \bigoplus A \rightarrow \bigoplus A'' \rightarrow 0$ is exact.

3) $\Rightarrow$ 6) is obvious.

4) $\Rightarrow$ 3): Let $B$ be a module. Let the finitely generated submodules of $B$ be indexed by $I$ where we define $i \leq j$ if $B_i \subseteq B_j$.

Let $\mathcal{B}$ denote the direct system of the finitely generated submodules of $B$ so obtained. Then $\varinjlim \mathcal{B} = B$. $\varinjlim(A \otimes \mathcal{B}) = A \otimes \varinjlim \mathcal{B} = A \otimes B$ by Proposition 5.12 of Chapter 3. If $X$ is a projective resolution of $A$, $\ldots \rightarrow \bigoplus_{N} X \rightarrow \ldots \rightarrow \bigoplus_{O} A \rightarrow 0$ is an exact sequence of direct systems, so that $\ldots \rightarrow \bigoplus_{N} X \rightarrow \ldots \rightarrow \bigoplus_{O} A \rightarrow 0$ is exact, by Proposition 5.16 of Chapter 3.

5) $\Rightarrow$ 4): Let $B$ have $N$ generators, and assume the result for modules with $N-1$ generators. Let $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ be exact, where $B''$ has one generator, $B'$ has $N-1$ generators. Then $\ldots \rightarrow 0 = \text{Tor}_q(A, B') \rightarrow \text{Tor}_q(A, B) \rightarrow \text{Tor}_q(A, B'') = 0 \rightarrow \ldots$ is exact, $\text{Tor}_q(A, B) = 0$ for $q \geq 1$. 
vi) \implies v): Let B have one generator. Then there exists I such that \(0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0\) is exact, B is isomorphic to \(A/I\). Of course, the proposition holds also if the roles of the first and second variables in \(\text{Tor}(A, B)\) are interchanged.

Remark 3.8: If \(A\) is a commutative ring, then \(A \otimes B\) and \(\otimes Y\) are \(A\)-modules, so that \(\text{Tor}_q(A \otimes B)\) will be a \(A\)-module.

For further use, we note the following:

Proposition 3.9: If \(A\) is a finitely generated module over a commutative Noetherian local ring \(A\) with maximal ideal \(M\), the following are equivalent:

i) \(A\) is free

ii) \(A\) is projective

iii) \(A\) is flat

iv) \(\text{Tor}_1(A, A/M) = 0\).

Proof: i) \implies ii) \implies iii) \implies iv) are clear.

iv) \implies i) \(A/MA = A/M\otimes A\) is a finitely generated vector space over \(A/M\). Choose \(x_1, \ldots, x_N \in A\) such that \(x_1, \ldots, x_N\) generate \(A/MA\).

Let \(F\) be free with \(N\) generators \(e_1, \ldots, e_N\). Define \(f: F \rightarrow A\) by \(f(e_i) = x_i\). Since \(A/M\otimes A \rightarrow A/M\otimes A\) is an epimorphism, so is \(F \rightarrow A\) by proposition 4.2 of chapter 5. Let \(B = \ker(f)\).

Since \(A\) is Noetherian, \(B\) is finitely generated.

\(0 \rightarrow B \rightarrow F \rightarrow A \rightarrow 0\) is exact, hence

\(0 = \text{Tor}_1(A, A/M) \rightarrow B \otimes A/M \rightarrow F \otimes A/M \rightarrow A \otimes A/M \rightarrow 0\) is exact. Thus \(B \otimes A/M = 0\); \(B = 0\) by proposition 4.1, chapter 5. Thus \(A\) is isomorphic to \(F\).
4. Construction of $\text{Ext}(A,B)$.

In this section all modules will be assumed to be left $A$-modules. If $A$ and $B$ are modules, $\text{Hom}(A,B)$ is an Abelian group. Recall that if $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ and $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ are exact sequences, then $0 \rightarrow \text{Hom}(A'',B) \rightarrow \text{Hom}(A,B) \rightarrow \text{Hom}(A',B)$ and $0 \rightarrow \text{Hom}(A,B') \rightarrow \text{Hom}(A,B) \rightarrow \text{Hom}(A,B'')$ are exact. Here we will construct objects by means of which we may study the behavior of the latter sequences on the right. The constructions of this section will closely parallel those of the previous section, and most proofs will be outlined only.

Let $X$ and $Y$ be complexes, $X$ written with subscripts, $Y$ with superscripts. $\text{Hom}(X,Y)$ is a graded module with $\text{Hom}(X,Y)^N = \prod \text{Hom}(X_i,Y^j)$. Observe that if $f \in \text{Hom}(X,Y)^N$, then $f_i : X_i \rightarrow Y^{N-i}$; that is, $f$ is a morphism of degree $N$. For notational convenience we write $\text{Hom}(X,Y) = \text{Hom}(X_i,Y^j)$.

Now let $X',X'',Y',Y''$ also be complexes. Let $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ be morphisms of degrees $p$ and $q$. Define $\text{Hom}(f,g) = (-1)^{iq} \text{Hom}(f_{i+p},g^j)$ $(\text{Hom}(f,g) : \text{Hom}(X_i,Y^j) \rightarrow \text{Hom}(X_{i+p},Y^{j+q}))$, and define $\text{Hom}(f,g)^N = \prod_{i+j=N} \text{Hom}(f,g)$. Thus $\text{Hom}(f,g)$ is a morphism of degree $p+q$. Suppose $f' : X' \rightarrow X''$ and $g' : Y' \rightarrow Y''$ are morphisms of degrees $p'$ and $q'$. $f'f$ and $g'g$ are morphisms of degrees $p+p'$ and $q+q'$. We have
\[
\begin{align*}
\text{Hom}(f',g') = (-1)^{i+q'} \text{Hom}(f'_{i+p'} g'_{j+q}) \\
= (-1)^i (q+q') \text{Hom}(f'_{i+p'} g'_{j+q}) \text{Hom}(f'_{i+p'}, g'_{j}) \\
= (-1)^i (q+q') (-1)^{(i+p')q'} \text{Hom} (f'_{i+p'}, g') (-1)^{i+q'} \text{Hom}(f', g) \\
= (-1)^{i+p'} (q', j+q') \text{Hom} (f', g') (-1)^{i+q'} \text{Hom}(f', g).
\end{align*}
\]

We will now define a differential operator on \( \text{Hom}(X,Y) \) by
\[
d = \text{Hom}(d_{x,i}, d_{y}) + \text{Hom}(i_{x'}, d_{y}).
\]
\[
dd = \text{Hom}(d_{x,i}, d_{y}) \text{Hom}(d_{x,i}, d_{y}) + \text{Hom}(d_{x,i}, d_{y}) \text{Hom}(i_{x'}, d_{y}) \\
+ \text{Hom}(i_{x'}, d_{y}) \text{Hom}(d_{x,i}, d_{y}) + \text{Hom}(i_{x'}, d_{y}) \text{Hom}(i_{x'}, d_{y}) \\
= \text{Hom}(d_{x,i}, d_{y}) \text{Hom}(i_{x'}, d_{y}) + \text{Hom}(i_{x'}, d_{y}) \text{Hom}(d_{x,i}, d_{y}) \\
= \text{Hom}(i_{x'}, d_{y}) - \text{Hom}(d_{x,i}, d_{y}) = 0.
\]

As in the construction of \( \text{Tor} \), we will employ

**Lemma 4.1:** Suppose \( X \) is a left complex and \( Y \) is a right complex.

*Then:* i) If \( X \) is projective and \( Y \) is exact, \( \text{Hom}(X,Y) \) is exact.

ii) If \( X \) is exact and \( Y \) is injective, \( \text{Hom}(X,Y) \) is exact.

*Proof:* Note that \( \text{Hom}(X,Y) \) is a right complex. The proof is similar to that of lemma 3.1, using propositions 2.10 and 3.5 of chapter 3 and lemma 2.1 in the second and third steps.

**Definition 4.2:** Let \( A \) and \( B \) be left \( A \)-modules, \( X \) a projective resolution of \( A \) and \( Y \) an injective resolution of \( B \). Define
\[
\text{Ext}^N(A,B) = \text{Ext}^N(\text{Hom}(X,Y)).
\]

**Proposition 4.3:** \( H^q(\text{Hom}(A,Y)) \), \( H^q(\text{Hom}(X,Y)) \) and \( H^q(\text{Hom}(X,B)) \) are isomorphic for all \( q \).

*Proof:* Define \( Y' \) by \( Y'^q = Y^q \) for \( q \neq 0 \), \( Y'^0 = Y^0 / \text{im}(\epsilon) \).

Then \( 0 \rightarrow B \rightarrow Y \rightarrow Y' \rightarrow 0 \) is an exact sequence of translations, \( H^q(Y') = 0 \) for all \( q \), hence by lemma 4.1 \( H^q(\text{Hom}(X,Y')) = 0 \) for all \( q \).
Since by proposition 2.10 of chapter 3, 
\[ 0 \to \text{Hom}(X, B) \to \text{Hom}(X, Y) \to \text{Hom}(X, Y') \to 0 \] is exact, by lemma 2.1 
\[ 0 \to H^q(\text{Hom}(X, B)) \to H^q(\text{Hom}(X, Y)) \to 0 \] is exact for all \( q \).
Arguing similarly using proposition 3.5 of chapter 3, \( H^q(\text{Hom}(X, Y)) \) and \( H^q(\text{Hom}(A, Y)) \) are isomorphic for all \( q \).

**Proposition 4.4:** \( \text{Ext}^q(A, B) \) is independent of the choice of \( X \) and \( Y \).

**Proof:** The proof is similar to that of proposition 3.4.

Propositions 1.5 and 1.8 are used to prove independence of \( X \), 1.5' and 1.8' for \( Y \).

**Proposition 4.5:**

i) Let \( 0 \to A' \to A \to A'' \to 0 \) be an exact sequence and \( B \) a module. Then there exists a canonical exact sequence
\[
0 \to \text{Ext}^0(A'', B) \to \text{Ext}^0(A, B) \to \text{Ext}^0(A', B) \to \cdots
\]
\[
\to \text{Ext}^{N-1}(A', B) \to \text{Ext}^N(A'', B) \to \text{Ext}^N(A', B) \to \text{Ext}^N(A, B) \to \text{Ext}^N(A, B) \to \cdots
\]

ii) Let \( 0 \to B' \to B \to B'' \to 0 \) be an exact sequence and \( A \) a module. Then there exists a canonical exact sequence
\[
0 \to \text{Ext}^0(A, B') \to \text{Ext}^0(A, B) \to \text{Ext}^0(A, B'') \to \cdots
\]
\[
\to \text{Ext}^{N-1}(A, B'') \to \text{Ext}^N(A, B') \to \text{Ext}^N(A, B) \to \text{Ext}^N(A, B'') \to \cdots
\]

**Proof:** The proof is similar to that of proposition 3.5:

i) follows using proposition 2.6, corollary 2.7, proposition 2.10 of chapter 3, lemma 2.1, and proposition 2.8.

ii) follows using proposition 2.6', corollary 2.7', proposition 3.5 of chapter 3, lemma 2.1, and proposition 2.8'.

**Proposition 4.6:** \( \text{Ext}^0(A, B) \) is isomorphic to \( \text{Hom}(A, B) \).
Proof: Let $Y$ be an injective resolution of $B$.

\[
0 \rightarrow \text{Hom}(i_A^e) \xrightarrow{\text{Hom}(i_A^o)} \text{Hom}(A,Y^0) \xrightarrow{\text{Hom}(i_A^o)} \text{Hom}(A,Y^1) \text{ is exact,}
\]

so $H^0(\text{Hom}(A,Y)) = \ker(\text{Hom}(i_A^o))$ is isomorphic to $\text{Hom}(A,B)$.

**Proposition 4.7:** The following are equivalent:

1) $A$ is projective

ii) $\text{Ext}^1(A,B) = 0$ for all $B$

iii) $\text{Ext}^q(A,B) = 0$ for all $q \geq 1$ and for all $B$.

Proof: This follows from proposition 2.10 of chapter 3.

**Proposition 4.7':** The following are equivalent:

1) $B$ is injective

ii) $\text{Ext}^1(A,B) = 0$ for all $A$.

iii) $\text{Ext}^q(A,B) = 0$ for all $q \geq 1$ and for all $A$.

Proof: This follows from proposition 3.5 of chapter 3.

Note that since there is no analog to proposition 5.16 of chapter 3, we do not obtain a complete analog to proposition 3.7.

**Remark 4.8:** If we had used right modules throughout this section, we would have obtained analogous results. If $\Lambda$ is commutative, then the values of $\text{Ext}(A,B)$ are $\Lambda$-modules.

**Proposition 4.9:** If $A$ is a finitely generated module over a commutative Noetherian local ring $\Lambda$ with maximal ideal $M$, then $A$ is free if and only if $\text{Ext}^1(A,\Lambda/M) = 0$.

Proof: If $A$ is free, then $\text{Ext}^1(A,\Lambda/M)$ is clearly zero. Conversely, we proceed exactly as in the proof of proposition 3.9, obtaining an exact sequence

\[
0 \rightarrow B \rightarrow F \rightarrow A \rightarrow 0 \text{ when } F \text{ is free and } F/\text{MF} \text{ is isomorphic to } A/\text{MA}. \text{ Hom}(A,\Lambda/M) = \text{Hom}(A/M, A/\Lambda/M) \text{ so } \text{Hom}(A,\Lambda/M) \text{ is isomorphic}
\]
to $\text{Hom}(F, A/M)$. $0 \rightarrow \text{Hom}(B, A/M) \rightarrow \text{Hom}(B/MB, A/M) \rightarrow \text{Ext}^1(A, A/M) = 0$

is exact, $\text{Hom}(B/MB, A/M) = 0$, $B/MB = 0$ and by proposition 4.1 of chapter 5, $B = 0$. Thus $A$ is isomorphic to $F$.

5. Categories and functors.

In this section, we introduce terminology which greatly simplifies the statements of homological algebra.

**Definitions 5.1:** Let $\mathcal{F}$ be a set with elements denoted by $f, f_1, f_2, f_3, \ldots$ such that for certain pairs $(f_1, f_2)$ a product $f_1 f_2$ is defined in $\mathcal{F}$. An element $x \in \mathcal{F}$ such that $f_1 x = f_1$ and $f_2 x = f_2$ whenever $f_1$ and $f_2$ are defined, is called an identity. $\mathcal{F}$ is called a system of abstract maps provided that

i) If either $f_1 (f_2 f_3)$ or $(f_1 f_2) f_3$ is defined, then so is the other and the two are equal.

ii) If $f_1 f_2$ and $f_2 f_3$ are defined, then so are $(f_1 f_2) f_3$ and $f_1 (f_2 f_3)$.

iii) If $f \in \mathcal{F}$, then there exist (unique) identities $i_1$ and $i_2$ in $\mathcal{F}$ such that $fi_1$ and $i_2 f$ are defined.

**Definitions 5.2:** A category $\mathcal{C}$ is a system $\mathcal{F}$ of abstract maps together with objects $C_1, C_2, \ldots$ which are in 1-1 correspondence with the identities of $\mathcal{F}$. If $f \in \mathcal{F}$, then the unique objects $C_1$ and $C_2$ such that $f_{C_1}$ and $i_{C_2}$ are defined, are called the domain and range of $f$, and we write $f: C_1 \rightarrow C_2$. If for each pair $C_1$ and $C_2$ of objects in $\mathcal{C}$, the set of all maps $f: C_1 \rightarrow C_2$ has a natural structure as an Abelian group, then $\mathcal{C}$ is called an additive category.

If for each pair $C_1$ and $C_2$ the set of maps $f: C_1 \rightarrow C_2$ has a structure as a left (right) $\Lambda$-module, then $\mathcal{C}$ is called a left (right) $\Lambda$-category.
The set of left (or right) \( \Lambda \)-modules and their morphisms is an example of an additive category. If \( \Lambda \) is commutative, the set of \( \Lambda \)-modules and their morphisms is a \( \Lambda \)-category. Diagrams of \( \Lambda \)-modules and their translations give further examples of additive categories (a translation \( f : D \rightarrow D' \) of two similar diagrams \( D \) and \( D' \) is a family of morphisms \( f_j : D_j \rightarrow D'_j \) such that for each pair \( (j, k) \),

\[
\begin{array}{ccc}
D_j & \xrightarrow{\varphi_{jk}} & D_k \\
\downarrow f_j & & \downarrow f_k \\
D'_j & \xrightarrow{\varphi'_{jk}} & D'_k 
\end{array}
\]

is a commutative diagram where \( D_j, D_k', \ldots, D'_j, D'_k, \ldots \) are the modules and \( \varphi_{jk}', \ldots, \varphi'_{jk} \) are the component morphisms of the diagrams \( D \) and \( D' \).

**Definitions 5.3:** Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. Suppose that for each object \( C \in \mathcal{C} \), an object \( T(C) \in \mathcal{D} \) is given and for each map \( f : C \rightarrow C' \) in \( \mathcal{C} \) a map \( T(f) : T(C) \rightarrow T(C') \) is given such that

1) If \( f = i_C \) then \( T(f) = i_{T(C)} \).

2) If \( f'f \) is defined, \( T(f'f) = T(f')T(f) \).

Then \( T \) is said to form a covariant functor from \( \mathcal{C} \) to \( \mathcal{D} \). If \( T(f) : T(C') \rightarrow T(C) \) and \( T(f'f) = T(f)T(f') \), \( T \) is said to be a contravariant functor from \( \mathcal{C} \) to \( \mathcal{D} \).

We extend the definition to \( N \) variables as follows:

Let \( \mathcal{C}_1, \ldots, \mathcal{C}_N, \mathcal{D} \) be categories. Let \( C_1, C'_1, \ldots, f_1, f'_1, \ldots \) be objects and maps in \( \mathcal{C}_1 \) and let the set \( \{1, \ldots, N\} \) be divided into disjoint subsets \( I \) and \( J \). Assume that for each set \( C_1, \ldots, C_N \) of
objects there is given an object $T(C_1, \ldots, C_N) \in \mathcal{D}$, and for each set $f_1, \ldots, f_N$ of maps, $f_i : C_i \to C'_i$ for $i \in I$, $f_j : C'_j \to C_j$ for $j \in J$, there is given a map in \( \mathcal{D} \) $T(f_1, \ldots, f_N) : T(C_1, \ldots, C_N) \to T(C'_1, \ldots, C'_N)$. Then $T$ is a functor, covariant in the variables in $I$, contravariant in those in $J$, provided that

1) If $f_1, \ldots, f_N$ are identities, then so is $T(f_1, \ldots, f_N)$.

ii) If $f_1, \ldots, f_N, f'_1, \ldots, f'_N$ are such that $f'_1 f_i, i \in I$, and $f'_j f'_j, j \in J$, are defined, then $T(\ldots, f'_1 f_i, \ldots, f'_j f'_j, \ldots) = T(f'_1, \ldots, f'_N)T(f_1, \ldots, f_N)$.

If $C_1, \ldots, C_N$ are all additive categories, and $f_1, \ldots, f_N, g_1, \ldots, g_N$ are maps in $C_1, \ldots, C_N$ such that $f_1$ and $g_1$ have the same domain and range for each $i$, then if

$T(f_1, \ldots, f_r + g_r, \ldots, f_N) = T(f_1, \ldots, f_r, \ldots, f_N) + T(f_1, \ldots, g_r, \ldots, f_N)$,

$i \leq r \leq N$, $T$ is said to be an additive functor. We will only be concerned with additive functors.

**Definitions 5.4:** Let $T$ and $U$ be functors of $N$ variables from $(C_1, \ldots, C_N)$ to $\mathcal{D}$. Let $\{1, \ldots, N\}$ be partitioned into $I$ and $J$ with both $T$ and $U$ covariant in the variables $C_i, i \in I$, contravariant in $C_j, j \in J$. Denote $(C_1, \ldots, C_N)$ by $(C)$. If for each set $(C)$, there is a map $\mu(C) : T(C) \to U(C)$ such that whenever $f_1, \ldots, f_N$ are maps, $f_1 : C_i \to C'_i$, $i \in I$, $f_j : C'_j \to C_j$, $j \in J$, then the diagram

$$
\begin{array}{ccc}
T(C) & \xrightarrow{T(f_1, \ldots, f_N)} & T(C') \\
\downarrow^{\mu(C)} & & \downarrow^{\mu(C')}
\end{array}
$$

we say that $\mu$ is a natural transformation of $T$ into $U$. If $\mu(C)$ is an equivalence for all $(C)$, we say that $\mu(C)$ is a natural equivalence
or a natural isomorphism. (A map \( f \) is an equivalence if there
exists a map \( g \) such that \( fg \) and \( gf \) are both identities.)

Let \( \Lambda_1, \Lambda_2, \Lambda \) be rings. We restrict ourselves (for
notational convenience) for the remainder of this section to an
additive functor \( T(A, C) \) define for \( \Lambda_1 \)-modules \( A \) and \( \Lambda_2 \)-modules
\( C \) with values as \( \Lambda \)-modules. We assume that \( T \) is covariant in
\( A \), contravariant in \( C \).

**Proposition 5.5:** If \( \xymatrix{ A_\alpha 
\ar[r]^{i_{\alpha}} & A 
\ar[r]^{j_{\alpha}} & A_\alpha } \) and \( \xymatrix{ C_\beta 
\ar[r]^{k_\beta} & C 
\ar[r]^{l_\beta} & C_\beta } \) are
finite direct sum representations of \( A \) and \( C \), then
\( T(A_\alpha, C_\beta) \xymatrix{ \ar[r]^{T(1_{\alpha}, l_\beta)} & T(A, C) } \xymatrix{ \ar[r]^{T(j_{\alpha}, k_\beta)} & T(A_\alpha, C_\beta) } \) is a direct sum
representation of \( T(A, C) \).

**Proof:** \( T(j_{\alpha'}, k_\beta)T(i_{\alpha}, l_\beta) = T(j_{\alpha'}i_{\alpha}, l_\beta k_\beta) \), is the identity if
\( (\alpha, \beta) = (\alpha', \beta') \) and zero otherwise. Also
\[ \sum_{\alpha, \beta} T(i_{\alpha'}, l_\beta)T(i_{\alpha}, k_\beta) = \sum_{\alpha, \beta} T(i_{\alpha}, l_\beta k_\beta) = T(\sum_{\alpha, \beta} \alpha' \alpha, \sum_{\alpha, \beta} \beta' \beta) \]

= identity.

**Corollary 5.6:** If \( 0 \to A' \to A \to A'' \to 0 \) and
\( 0 \to C' \to C \to C'' \to 0 \) are split exact sequences, then so
are \( 0 \to T(A', C) \to T(A, C) \to T(A'', C) \to 0 \) and
\( 0 \to T(A, C'') \to T(A, C) \to T(A, C') \to 0 \).

**Definitions 5.7:** Let \( A' \to A \to A'' \), \( C' \to C \to C'' \) be exact
sequences. If \( T(A', C) \to T(A, C) \to T(A'', C) \) and
\( T(A, C'') \to T(A, C) \to T(A, C') \) are also exact, then \( T \) is said to
be an exact functor. Let \( 0 \to A' \to A \to A'' \to 0 \) and
\( 0 \to C' \to C \to C'' \to 0 \) be exact sequences.
If $0 \to T(A',C) \to T(A,C) \to T(A'',C)$ and $0 \to T(A,C'') \to T(A,C) \to T(A,C')$ are exact, then $T$ is said to be left exact. If $T(A',C) \to T(A,C) \to T(A'',C) \to 0$ and $T(A,C'') \to T(A,C) \to T(A,C') \to 0$ are exact, then $T$ is said to be right exact.

The proofs of the following propositions are straightforward and will be omitted.

Proposition 5.8: $T$ is exact if and only if $T$ is both right and left exact.

Proposition 5.9: The following are equivalent:

i) $T$ is left exact

ii) If $0 \to A' \to A \to A''$ and $C' \to C \to C'' \to 0$ are exact, then so are $0 \to T(A',C) \to T(A,C) \to T(A'',C)$ and $0 \to T(A,C'') \to T(A,C) \to T(A,C')$.

iii) If $0 \to A' \to A \to A''$ and $C' \to C \to C'' \to 0$ are exact, then so is $0 \to T(A',C'') \to T(A,C) \to T(A'',C) \to T(A,C')$ where $\forall$ has coordinates $T(A,C) \to T(A'',C); T(A,C) \to T(A,C')$.

Proposition 5.10: The following are equivalent:

i) $T$ is right exact

ii) If $A' \to A \to A'' \to 0$ and $0 \to C' \to C \to C''$ are exact, then so are $T(A',C) \to T(A,C) \to T(A'',C) \to 0$ and $T(A,C'') \to T(A,C) \to T(A,C') \to 0$.

iii) If $A' \to A \to A'' \to 0$ and $0 \to C' \to C \to C''$ are exact, then so is $T(A',C) \oplus T(A,C'') \to T(A,C) \to T(A'',C') \to 0$, where $\phi$ has coordinates $T(A',C) \to T(A,C)$ and $T(A,C'') \to T(A,C)$.

We observe that by propositions 1.10, 2.7 and 2.8 of chapter 3, the functor $\otimes_{\lambda}$ is right exact and the functor $\text{hom}_{\lambda}$ is left exact.
Definitions 5.11: A connected sequence of covariant functors is a family \( T = (T^N) \) of covariant functors (of one variable) together with connecting morphisms \( T^N(A') \to T^{N+1}(A') \) defined for each exact sequence \( 0 \to A' \to A \to A'' \to O \) subject to the conditions:

i) \( \ldots \to T^{N-1}(A'') \to T^N(A') \to T^N(A) \to T^N(A'') \to T^{N+1}(A') \to \ldots \)

is a zero sequence.

ii) If \( 0 \to A' \to A \to A'' \to 0 \) is a commutative diagram

\[
\begin{array}{ccc}
 & & \\
 & \downarrow & \\
O & \to B' & \to B & \to B'' & \to O
\end{array}
\]

with exact rows then \( T^N(A'') \to T^{N+1}(A') \) is a commutative diagram

\[
\begin{array}{ccc}
 & & \\
 & \downarrow & \\
T^N(B'') & \to T^{N+1}(B').
\end{array}
\]

If the roles of \( A' \) and \( A'' \) are reversed, \( (T^N) \) is a connected sequence of contravariant functors.

Generalizing, a multiply connected sequence of functors is a sequence \( (T^N) \) of functors of the same variables and variance such that there are connecting morphisms with respect to each variable for which

i) \( (T^N) \) is a connected sequence of functors with respect to each variable separately.

ii) If \( 0 \to A' \to A \to A'' \to 0 \), \( 0 \to C' \to C \to C'' \to 0 \)

\[
\begin{array}{ccc}
 & & \\
 & \downarrow & \\
0 & \to A' \to A_1 \to A'' \to 0 & \to C' \to C \to C'' \to 0
\end{array}
\]

are commutative diagrams with exact rows, then \( T^N(A'',C) \to T^{N+1}(A',C) \)

\[
\begin{array}{ccc}
 & & \\
 & \downarrow & \\
T^N(A'',C_1) & \to T^{N+1}(A',C_1)
\end{array}
\]
and $T^N(A, C') \rightarrow T^{N+1}(A, C'')$ are commutative diagrams (where we have chosen $[T^N]$ all covariant in $A$, contravariant in $C$, as a typical example). If $[T^N]$ and $[U^N]$ are multiply connected sequences of functors, a morphism $\Phi: (T^N) \rightarrow (U^N)$ is a sequence of natural transformations $\Phi^N: T^N \rightarrow U^N$ which commute with the connecting morphisms.

$(\text{Tor}^N_{\Lambda}(A, C))$ and $(\text{Ext}^N_{\Lambda}(A, C))$ are examples of exact multiply connected sequences of functors (that condition ii) is satisfied follows from the constructions, and propositions 1.8, 1.8', 2.8 and 2.8'). The method by which $\text{Tor}^N_{\Lambda}$ and $\text{Ext}^N_{\Lambda}$ were constructed is actually quite general. Let $T(A_1, \ldots, A_r)$ be any functor of $r$ variables. Assume that $A_1, \ldots, A_r$ are all graded. Define

$T^{N,N_1,\ldots,N_r}(A_1, \ldots, A_r) = T(A_1^{N_1}, \ldots, A_r^{N_r})$ where $\epsilon_r = +1$ if $A_r$ is a covariant variable, $-1$ if contravariant, and define

$T^N(A_1, \ldots, A_r) = \sum_{N_1, \ldots, N_r}^{N_1, \ldots, N_r}(A_1, \ldots, A_r)$ [for $C_{\Lambda}$, if all complexes are left complexes as in the construction of $\text{Tor}^N_{\Lambda}$, this is the same as $\sum_{N_1, \ldots, N_r}^{N_1, \ldots, N_r}(A_1, \ldots, A_r)$. If $A_1', \ldots, A_r'$ is another set of graded modules and $f_i: A_i \rightarrow A_i'$ are given for $A_i$ a covariant variable, $f_i: A_i' \rightarrow A_i$ for $A_i$ contravariant, where $f_i$ is of degree $p_i$, define $T^{N_1,\ldots,N_r}(f_1, \ldots, f_r)$ on $T^{N_1,\ldots,N_r}(A_1, \ldots, A_r)$ as

$(-1)^{\sum_{i=1}^r N_i p_j} \epsilon_1(N_1^{p_1}) \epsilon_r(N_r^{p_r}) T(f_1, \ldots, f_r)$,

where $\sum N_i p_j$, $\epsilon_1 = N_1$ if $T$ is covariant in $A_1$, $\epsilon_1 = -(N_1 + p_1)$ if $T$ is contravariant in $A_1$. If $g_i: A_i' \rightarrow A_i''$ (resp. $g_i: A_i'' \rightarrow A_i'$) are morphisms of degree $q_i$, we then verify that
\[ T(g_1 f_1, \ldots, g_r f_r) = (-1)^{\gamma_1} T(g_1, \ldots, g_r) T(f_1, \ldots, f_r), \] where
\[ \gamma_2 = \sum_j p_j q_j. \] Now suppose each \( A_i \) is a complex with differentiation \( d_i \) and let \( \delta_i = T(i_{A_1}, \ldots, d_i, \ldots, i_{A_r}). \) The \( \delta_i \) anticommuting,

hence define \( T(A_1, \ldots, A_r) \) as a complex with differentiation \( \Sigma \delta_i. \) If \( f_1, \ldots, f_r \) and \( f'_1, \ldots, f'_r \) are respectively homotopic translations of complexes and \( s_i, 1 \leq i \leq r \) are homotopies, then

for \( \sigma_i = T(i_{A_1}, \ldots, s_i, \ldots, i_{A_r}), \Sigma \sigma_i \) defines a homotopy between \( T(f_1, \ldots, f_r) \) and \( T(f'_1, \ldots, f'_r). \)

Now if we are given a functor \( T \) of modules and we replace all covariant variables \( A_i \) by projective resolutions \( X_i, \) all contravariant variables \( A_i \) by injective resolutions \( X_i \) we obtain a left complex. We define \( LT(A_1, \ldots, A_r) \) as \( HT(X_1, \ldots, X_r). \) By propositions 1.8, 1.8' and the previous paragraph, \( LT \) is independent of the choices of the \( X_i. \) \( LT \) is graded, and the component of degree \( N \) gives a functor \( L_N^T \) called the \( N \)th left derived functor of \( T. \) \( L_N^T = 0 \) for \( n < 0. \) By corollary 5.6, lemma 2.1, and propositions 1.8, 1.8', 2.8 and 2.8', \( (L_N^T) \) is an exact multiply connected sequence of functors. The augmentation morphisms

\[ \epsilon_i : X_i \rightarrow A_i \] (resp. \( \epsilon_i : A_i \rightarrow X_i \)) induce a natural transformation

\[ \gamma_0 : L_0^T(A_1, \ldots, A_r) \rightarrow T(A_1, \ldots, A_r). \] \( \gamma_0 \) is a natural equivalence if and only if \( T \) is right exact: the condition is necessary since

\( L_0^T \) is right exact; if \( T \) is right exact, considering our typical case \( T(A, C), \) and letting \( X \) be a projective resolution of \( A, Y \) an injective resolution of \( C, \) the sequence

\[ T(X_1, Y^0) \oplus T(X_0, Y^1) \rightarrow T(X_0, Y^0) \rightarrow T(A, C) \rightarrow 0 \] is exact by proposition 5.10; since \( H_0(T(X, Y)) = \text{coker } (\phi), \) this gives the result. For this reason, left derived functors are of interest for
the study of right exact functors. Similarly, if we replace all covariant variables by injective resolutions and all contravariant variables by projective resolutions in a functor $T$ of modules, we obtain a multiply-connected sequence of functors $(R^N T)$, called the right derived functors of $T$. $R^N T$ is naturally equivalent to $T$ if and only if $T$ is left exact.

It is clear that $\{\text{Tor}_N^A\}$ are the left derived functors of $\otimes_A$ and that $\{\text{Ext}_N^A\}$ are the right derived functors of $\text{Hom}_A$. 