Chapter 5: Linear algebra and the structure of modules over commutative rings.

In this chapter we will begin to exploit the relationship between modules and their exterior algebras. Often, a three step procedure will be used. First we will obtain results in the case where the ground ring (the ring over which all modules are taken) is a field. These results will be used to obtain results when the ground ring is a local ring. Finally the results obtained about modules over local rings will be used to obtain results about modules over more general commutative rings by the process of localization.

Convention: Throughout this chapter ring will mean commutative ring. Usually the ground ring will be denoted by K .

§1. Free modules and the notion of rank.

Definitions 1.1: If A is a module, the rank of A is the least integer n such that $E(A)_q = 0$ for $q > n$. If no such integer exists the rank of A is infinite. We denote the rank of A by $r(A)$. In case the rank of A is infinite we write $r(A) = \infty$. When we write $r(A) = n$, we mean that the rank of A is finite, and that it is n . The symbol ∞ will be combined with itself and with ∞ according to the rules $n + \infty = \infty = \infty + n$, and $\infty + \infty = \infty$.

If $f: A \longrightarrow B$ is a morphism of modules, then $E(f)_q: E(A)_q \longrightarrow E(B)_q$ for every integer q . If an integer n exists such that $E(f)_q = 0$ for $q > n$, the least such integer is the <u>rank of f</u> . If no such integer exists the rank of f is infinite. In any case the rank of f is denoted by $r(f)$.

<u>Notation 1.2</u>: If i,j are non-negative integers, then the binomial coefficient $\frac{(i + j)!}{i!\ j!}$ is denoted by (i,j) . Recall that $0! = 1$.

<u>Proposition 1.3</u>: If A is a free module with basis $a_1,\ldots,a_n$, then

i) $E(A)_q$ is a free module for any q ,

ii) if $0 \leq q \leq n$, a basis for $E(A)_q$ consists of the $(q, n-q)$ elements $a_{i_1}\ldots a_{i_q}$ such that $0 < i_1 < \ldots < i_q \leq n$, and

iii) $E(A)_q = 0$ for $q > n$.

<u>Proof</u>: The proposition is an immediate corollary of 3.18 of the preceding chapter. However, in order to perform a few calculations with exterior algebras we will give a direct proof.

Let A^q be the free submodule of A generated by $a_1,\ldots,a_q$ for $q = 1,\ldots,n$, and let $[a_q]$ be the free submodule generated by a_q for $q = 1,\ldots,n$. Now $A' = [a_1]$, and certainly the proposition is true for A' . Since $A^{q+1} = A^q \oplus [a_{q+1}]$ for $q = 1,\ldots,n-1$, we have by 3.13 of the preceding chapter that $E(A^{q+1}) = E(A^q) \otimes E([a_{q+1}])$. Thus

$E(A^{q+1})_r = \oplus_{s+t=r} E(A^q)_s \otimes E([a_{q+1}])_t = E(A^q)_r \oplus (E(A^q)_{r-1} \otimes [a_{q+1}])$.
Now using the fact that $(r, q-r) + (r-1, q-r+1) = (r, q+1-r)$ for $r = 1,\ldots,q$ it follows that if the proposition is valid for A^q it is also valid for A^{q+1} . Hence by induction the proposition is proved.

Corollary 1.4: If A is a finitely generated free module, then every basis for A has $r(A)$ elements.

Corollary 1.5: If A is a free module of rank n , then $E(A)_q$ is a free module of rank $(q, n-q)$ for $q = 0,\ldots,n$.

Notice that for any module A, $E(A)_o = K$, and that $A = 0$ if and only if $r(A) = 0$ since $E(A)_1 = A$.

Proposition 1.6: If K is a field, and A is a K-module, then

i) the elements $a_1,\ldots,a_n$ of A are linearly independent if and only if the element $a_1\ldots a_n \in E(A)_n$ is not zero, and

ii) $\{a_1,\ldots,a_n\}$ is a basis for A if and only if $r(A) = n$ and the elements $a_1,\ldots,a_n$ are linearly independent.

Proof: Let A' be the submodule of A generated by $a_1,\ldots,a_n$, and let A'' be a submodule of A such that $A = A' \oplus A''$. If $\{a_1,\ldots,a_n\}$ is a basis for A', then $E(A')_n \neq 0$ by 1.3, and $a_1\ldots a_n$ is a basis element for $E(A')_n$. Since $E(A) = E(A') \otimes E(A'')$ we have that $a_1\ldots a_n \in E(A)_n$ is different from zero. If the elements $a_1,\ldots,a_n$ are not linearly independent a proper subset of $\{a_1,\ldots,a_n\}$ is a basis for A', and $E(A')_n = 0$. Thus part i) of the proposition is proved. Part ii) follows easily using 1.3 once again.

Comments 1.7: Notice that if K is a field then a module over K and a vector space over K are the same thing. Further if A is a vector space over K the rank of A is almost what is usually called the dimension of A. In fact if $r(A)$ is finite then the notion of rank coincides with the notion of dimension. In case $r(A) = \infty$ there is a difference between the notion of rank and that of dimension. The dimension of A is the cardinal number which is the cardinal number of any set of basis elements of A. In order for this to make sense it must be shown that any two bases of A have the same cardinal number. The assertion $r(A) = \infty$ tells you that every basis of A has infinitely many elements, but it does not tell you what the cardinality of a basis of A is.

Proposition 1.8: If K is a field, and A and B are vector spaces over K, then $r(A \oplus B) = r(A) + r(B)$.

Proposition 1.9: If K is a field, and $f: A \longrightarrow B$ is a morphism of vector spaces, then

i) if $r(B) = n$, then f is an epimorphism if and only if $r(f) = n$, and

ii) if $r(A) = m$, then f is a monomorphism if and only if $r(f) = m$.

Both 1.8 and 1.9 follow immediately from 1.6. The details are left to the reader.

§2. Exterior algebras and change of rings.

Notation 2.1: If K is a ring, and A is a K-module, we let $E_K(A)$ denote the exterior algebra of A. When no confusion will result, such as when there is only one ring in the discussion, this notation is abbreviated to the earlier notation $E(A)$.

Comments 2.2: If $f: K \to L$ is a morphism of commutative rings, then L may be considered as a graded algebra over K which is concentrated in degree zero. If Γ is any other graded algebra over K, then $L \otimes_K \Gamma$ is a graded algebra over K such that $(L \otimes_K \Gamma)_q = L \otimes_K \Gamma_q$. Further $L \otimes_K \Gamma$ is a graded algebra over L, where if $x, y \in L$, $\gamma \in \Gamma_q$, then $x(y \otimes_K \gamma) = (xy) \otimes_K \gamma$. Observe that if Γ is a strictly commutative graded algebra over K, then $L \otimes_K \Gamma$ is a strictly commutative graded algebra over L.

Proposition 2.3: If $f: K \to L$ is a morphism of rings, and A is a K-module, then

$$E_L(L \otimes_K A) = L \otimes_K E_K(A) .$$

Proof: Let Γ be a strictly commutative graded algebra over L, and suppose $g: L \otimes_K A(1) \to \Gamma$ is a morphism of graded L-modules, where $A(1)_q = 0$ for $q \neq 1$, and $A(1)_1 = A$. We know that g is uniquely determined by the morphism of graded K-modules, $h: A(1) \to \Gamma$ such that $h(a) = g(1 \otimes_K a)$ (chapter 3, 1.11).

Since Γ is a strictly commutative graded algebra over K, there is a unique morphism of graded algebras $\tilde{h}: E_K(A) \longrightarrow \Gamma$ such that $\tilde{h}_1 = h: A(1) \longrightarrow \Gamma$. Considering $\tilde{h}$ as a morphism of K-modules, it determines a unique morphism of L-modules $\tilde{g}: L \otimes_K E_K(A) \longrightarrow \Gamma$. Observe that $\tilde{g}$ is a morphism of graded algebras over L, $(L \otimes_K E_K(A))_1 = L \otimes_K A$, and $\tilde{g}$ is the only morphism of graded algebras over L such that $\tilde{g}_1 = h: L \otimes_K A(1) \longrightarrow \Gamma$. This proves that $L \otimes_K E_K(A)$ satisfies the required universal property for it to be the exterior algebra of $L \otimes_K A$ over L, and hence proves the proposition.

<u>Corollary 2.4</u>: If I is an ideal in K, then

$$E_{K/I}(K/_I \otimes_K A) = K/_I \otimes_K E_K(A) .$$

<u>Corollary 2.5</u>: If P is a prime ideal in K, then

$$E_{K_P}(A_P) = (E_K(A))_P .$$

These corollaries follow immediately from the proposition. Recall that if P is a prime ideal in K, then K_P is the ring K localized at P, and A_P is the module A localized at P (Chapter 2, §2). Further $A_P = K_P \otimes_K A$ (Chapter 3, 1.11).

§3. Direct decompositions of modules.

Definition 3.1: If Λ is an algebra over K , an element $x \in \Lambda$ is idempotent if $x^2 = x$. The set Φ of idempotents of Λ is a set of orthogonal idempotents if

i) $xy = 0$ for $x,y \in \Phi$ and $x \neq y$, and

ii) $0 \notin \Phi$ unless $\Lambda = 0$.

Comments and recollections 3.2: If A is a K-module, then Hom(A,A) is an algebra over K , where if $f,g \in \mathrm{Hom}(A,A)$, then $(fg)(a) = f(g(a))$ for $a \in A$. Further if f is an idempotent element of Hom(A,A) , then $A = \mathrm{Im}\, f \oplus \mathrm{Ker}\, f$. If Φ is a set of orthogonal idempotents of Hom(A,A) such that for $a \in A$, the set $\Phi_a = \{\varphi | \varphi \in \Phi$ and $\varphi(a) \neq 0\}$ is a finite set, then there is a unique element $\psi \in \mathrm{Hom}(A,A)$ such that $\psi\varphi = \varphi = \varphi\psi$ for $\varphi \in \Phi$, ψ is an idempotent, and $\mathrm{Im}\,\psi = \oplus_{\varphi \in \Phi} \mathrm{Im}\,\varphi$. This idempotent is denoted by $\oplus\Phi$. Note that for $a \in A$, $\oplus\Phi(a) = \Sigma_{\varphi \in \Phi_a} \varphi(a)$. If $\oplus\Phi = 1 \in \mathrm{Hom}(A,A)$ we say that Φ is a decomposition of 1 , and in this case $A = \oplus_{\varphi \in \Phi} \mathrm{Im}\,\varphi$.

Proposition 3.3 (Kaplansky): Let A be a K-module, and Φ a decomposition of 1 such that if $\varphi \in \Phi$ then $\mathrm{Im}\,\varphi$ is a countably generated module. If α is an idempotent element of Hom(A,A) , there exists Ψ a decomposition of 1 such that

i) if $\psi \in \Psi$, $\alpha\psi = \psi\alpha$, and

ii) if $\psi \in \bar{\Psi}$ and $\Phi_\psi = \{\varphi | \varphi \in \Phi$ and $\varphi \psi \neq 0\}$, then Φ_ψ is countable, and $\psi = \oplus \Phi_\psi$.

<u>Proof</u>: Let $\beta = 1 - \alpha$. Choose $\varphi_o \in \Phi$. Let $I_o = \{\varphi | \varphi \in \Phi$ and either $\varphi \alpha \varphi_o \neq 0$ or $\varphi \beta \varphi_o \neq 0\}$. Suppose that subsets of Φ called $I_o, \ldots, I_n$ are defined. Let $I_{n+1} = \{\varphi | \varphi \in \Phi$ and for some $\varphi' \in I_n$ either $\varphi \alpha \varphi' \neq 0$ or $\varphi \beta \varphi' \neq 0\}$. Observe that each I_n is countable. Let $I_\omega = \cup_n I_n$. We have that I_ω is a countable set of orthogonal idempotents. Let $\psi_o = \oplus I_\omega$. We have that $\psi_o \varphi_o = \varphi_o = \varphi_o \psi_o$, and $\alpha \psi_o = \psi_o \alpha$. Further ψ_o is an idempotent such that $\operatorname{Im} \psi_o$ is countably generated. In fact ψ_o satisfies conditions i) and ii) of the proposition for $I_\omega = \Phi_{\psi_o}$.

Now let $\tilde{\bar{\Psi}}$ be the set of all idempotent elements of Hom(A,A) satisfying conditons i) and ii) of the proposition, and let $\bar{\Psi}$ be a subset of $\tilde{\bar{\Psi}}$ maximal among those subsets which consist of orthogonal idempotents. Let $\lambda = \oplus \bar{\Psi}$. Note that for $\varphi \in \Phi$ either $\lambda \varphi = 0$ or $\lambda \varphi = \varphi$, and $\lambda \alpha = \alpha \lambda$. Suppose $\varphi_o \in \Phi$, and $\lambda \varphi_o = 0$. Let ψ_o be defined as in the first paragraph of the proof, and let $\psi_1 = \psi_o - \lambda \psi_o$. Now $\psi_1 \varphi_o = \varphi_o$, $\psi_1 \lambda = \lambda \psi_1 = 0$, and $\psi_1 \alpha = \alpha \psi_1$. Thus $\bar{\Psi} \cup \{\psi_1\}$ is a set of orthogonal idempotents contained in $\tilde{\bar{\Psi}}$ and properly containing $\bar{\Psi}$. Since this is impossible, we have $\lambda \varphi = \varphi$ for every $\varphi \in \Phi$, and $\lambda = 1$. Consequently the proposition is proved.

Corollary 3.4: Let X be a K-module, and suppose $X = \oplus_{i \in I} X_i$ and each X_i is countably generated. If $X = A \oplus B$, then for some index set J, $A = \oplus_{j \in J} A_j$ and each A_j is countably generated.

Proof: Let Φ be the decomposition of $1 \in \mathrm{Hom}(X,x)$ corresponding to the decomposition of X and $\oplus_{i \in I} X_i$. Let α be the idempotent such that $\alpha(a,b) = (a,0)$, and let Ψ be a decomposition of 1 satisfying the conditions of the preceding proposition. For $\psi \in \Psi$, let $A_\psi = \mathrm{Im}\, \alpha \psi$. We have $A = \oplus_{\psi \in \Psi} A_\psi$, and the corollary is proved.

Corollary 3.5: If A is a projective module, then A is a direct sum of countably generated projective modules.

Proof: For some free module X, we have $X = A \oplus B$. Applying the preceding corollary, the result is proved.

In closing this paragraph it should be pointed out that the commutativity of the ring K was never really used in the paragraph, and that all results are valid for an arbitrary ring.

§4. Exterior algebras and modules over local rings.

Proposition 4.1 (Nakayama): If K is a local ring with maximal ideal M, and A is a finitely generated K-module, then $A = 0$ if and only if $K/M \otimes A = 0$.

Proof: Suppose first that A has 1-generator. In this case we may assume that $A = K/I$ where I is an ideal in K. Now $K/M \otimes K/I = K/M + I$. Since M is the unique maximal ideal in K, we have $K/M + I = 0$ if and only if $I = K$, i.e. if and only if $K/I = 0$. Suppose that the proposition has been proved for modules having n-generators or less, and $n \geq 1$. Let A be a module with $n + 1$ generators or less. Certainly there is an exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ where both A' and A'' have at most n generators. There results an exact sequence

$K/M \otimes A' \longrightarrow K/M \otimes A \longrightarrow K/M \otimes A'' \longrightarrow 0$. If $K/M \otimes A = 0$ so does $K/M \otimes A''$, and then $A'' = 0$. This implies that $A' = A$ and that A has at most n-generators. Consequently $A = 0$ by inductive hypothesis. If $A = 0$ certainly $K/M \otimes A = 0$, and the proposition is proved.

Proposition 4.2: If K is a local ring with maximal ideal M, $f\colon A \longrightarrow B$ is a morphism of K-modules, and B is finitely generated, then f is an epimorphism if and only if $i_{K/M} \otimes f\colon K/M \otimes A \longrightarrow K/M \otimes B$ is an epimorphism.

<u>Proof</u>: Certainly if f is an epimorphism, so is $i_{K/M} \otimes f$. Suppose that $i_{K/M} \otimes f$ is an epimorphism. Let $C = \text{Coker } f$. We have an exact sequence $A \longrightarrow B \longrightarrow C \longrightarrow 0$, and a resulting exact sequence $K/M \otimes A \longrightarrow K/M \otimes B \longrightarrow K/M \otimes C \longrightarrow 0$. Since $i_{K/M} \otimes f$ is an epimorphism, it follows that $K/M \otimes C = 0$. Further C is finitely generated because B is finitely generated. Consequently by the preceding proposition $C = 0$, and the desired result follows.

<u>Definition 4.3</u>: Let F be a free K-module with basis $\{e_i\}_{i \in I}$. An element $x \in F$ is of <u>length n</u> with respect to this basis if there is a subset I' of I with n elements such that $x = \Sigma_{j \in I'} k_j e_j$, and further n is the least such integer.

<u>Proposition 4.4 (Kaplansky)</u>: If K is a local ring, and A is a projective module over K, then if $a \in A$ there is a finitely generated free module F, and morphism $\alpha: F \longrightarrow A$, $\beta: A \longrightarrow F$ such that

i) $a \in \text{Im } \alpha$

ii) $\beta\alpha = i_F$ the identity morphism of F.

<u>Proof</u>: Let B be a projective module and X a free module such that $A \oplus B = X$. Choose a basis $\{e_i\}_{i \in I}$ such that the length of a is as short with respect to this basis as it is with respect to any other basis of X.

Suppose that the length of a is n, $I = \{1, \ldots, n\} \cup I'$, and $a = \Sigma_{j=1}^{n} k_j e_j$. Let α be the projection of X on A.

If $k_1 = \Sigma_{j=i}^{n} k_j k'_j$, then $a = \Sigma_{j=i}^{n} k_j (k'_j e_1 + e_j)$, and we could replace e_j by $k'_j e_1 + e_j$ in the basis for X for $j = 2,\ldots,n$, thus obtaining a new basis for X such that the length of a with respect to this basis is $(n-1)$ or less. Since this is impossible, we have that $k_1 \notin I_1$ the ideal generated by $k_2,\ldots,k_n$. Similarly if I_j is the ideal generated by $k_1,\ldots,k_{j-1}, k_{j+1},\ldots,k_n$, we have $k_j \notin I_j$.

Now let $a_j = \alpha(e_j)$ for $y = 1,\ldots,n$. We have $a_j = \Sigma_{i=1}^{n} k_{ji} e_i + y_j$ where y_j belongs to the submodule of X complementary to the one generated by $e_1,\ldots,e_n$, i.e. assuming that $\{1,\ldots,n\}$ and I' are disjoint y_j belongs to the submodule generated by $\{e_i\}_{i\epsilon I'}$. Now $\Sigma_{j=1}^{n} y_j = 0$,
$a = \Sigma_{i=1}^{n} k_i e_i = \Sigma_{i=1}^{n} k_i a_i = \Sigma_{j=1}^{n} \Sigma_{i=1}^{n} k_j k_{ji} e_i$, and
$k_i = \Sigma_{j=1}^{n} k_j k_{ji}$ for $i = 1,\ldots,n$.

Let M be the unique maximal ideal of K , and recall that $k \epsilon M$ if and only if k is not a unit in K . Now $I_j \subset M$ for $y = 1,\ldots,n$, and the calculation of the preceding paragraph shows us that $k_{i,j} \epsilon M$ for $i \neq j$, and $(1 - k_{i,i}) \epsilon M$ for $i = 1,\ldots,n$ since otherwise we would have $k_j \epsilon I_j$ for some j . The fact that $(1 - k_{ii}) \epsilon M$ says that k_{ii} is a unit in K . Thus $\{a_1, e_i,\ldots,e_n\} \cup \{e_i\}_{i\epsilon I'}$ is a basis for X , or proceeding $\{a_1,\ldots,a_n\} \cup \{e_i\}_{i\epsilon I'}$ is a basis for X .

Let F be the free module generated by $\bar{e}_1,\ldots,\bar{e}_n$. Define $\alpha\colon F \longrightarrow A$ by $\alpha(\bar{e}_i) = a_i$ for $i = 1,\ldots,n$. Define $\beta\colon X \longrightarrow F$ by $\beta(a_i) = \bar{e}_i$ for $i = 1,\ldots,n$, and $\beta(e_i) = 0$ for $i \in I'$. Since $A \subset X$, we have defined α and β satisfying the conditions desired in the proposition, and the proposition is proved.

We now state a reformulation of the preceding proposition.

Proposition 4.5: If K is a local ring, and A is a projective module over K, then if $a_1 \in A$ there is a finite set of orthogonal idempotents $\Phi_1 \subset \mathrm{Hom}(A,A)$ such that if $\varphi \in \Phi$, then $\mathrm{Im}\,\varphi$ is free with 1-generator, and $(\oplus\, \Phi_1)\, a_1 = a_1$. Further if a_2 also is an element of A, there is a finite set of orthogonal idempotents $\Phi_2 \subset \mathrm{Hom}\,(A,A)$ such that $\Phi_1 \subset \Phi_2$, if $\varphi \in \Phi_2$, then $\mathrm{Im}\,\varphi$ is free with 1-generator, and $(\oplus\, \Phi_2)\, a_2 = a_2$.

Proof: The first part of the proposition is immediate. To prove the second part it suffices to observe that $A = \mathrm{Im}(\oplus\, \Phi_1) \oplus \mathrm{Ker}(\oplus\, \Phi_1)$, and to apply the first part of the proposition to the element $\bar{a}_2 = (1 - \oplus\, \Phi_1)\, a_2$ of the projective module $\mathrm{Ker}(\oplus\, \Phi_1)$.

Theorem 4.6 (Kaplansky): If K is a local ring, and A is a projective module over K, then A is a free module.

Proof: In view of 3.5 it suffices to prove the theorem assuming that A is countably generated. Thus suppose I is the set of non-negative integers, and that $\{a_i\}_{i \in I}$ is a set of generators of A. Let $\tilde{\Phi}$ be the set of idempotents of A such that if $\varphi \in \tilde{\Phi}$ then $\operatorname{Im} \varphi$ is free with 1-generator. Note that we may assume $A \neq 0$ for otherwise the theorem is trivial.

Let Φ_o be a finite subset of $\tilde{\Phi}$ consisting of orthogonal idempotents and such that $\oplus \Phi_o a_o = a_o$. Suppose that Φ_n is a finite set of orthogonal idempotents contained in $\tilde{\Phi}$ such that $\oplus \Phi_n a_i = a_i$ for $i \leq n$. Choose Φ_{n+1} of the same type and containing Φ_n such that $\oplus \Phi_{n+1} a_i = a_i$ for $i \leq n+1$. This is possible by the preceding proposition. Let $\Phi = \bigcup_n \Phi_n$, and observe that $\oplus \Phi$ is defined and $\oplus \Phi = 1$, i.e. Φ is a decomposition of 1. The existence of Φ proves the theorem.

Up until this point in this paragraph, the fact that the ring K is commutative has not been used. Everything that has been proved is true for left or right modules over an arbitrary local ring. Now we pass on to a few propositions concerning exterior algebras and modules over local rings. Here since we use exterior algebras we do need commutativity of the ring.

<u>Proposition 4.7</u>: If K is a local ring, and A is a finitely generated K-module, then the rank of A over K is the same as the rank of $K/M \otimes A$ over K/M where M is the maximal ideal in K.

<u>Proof</u>: This proposition follows immediately from 4.1, and the fact that $E_{K/M}(K/M \otimes_K A) = K/M \otimes_K E_K(A)$ (2.4).

Proposition 4.8: If K is a local ring, and A and B are finitely generated K-modules, then

$$r(A \oplus B) = r(A) + r(B) .$$

Proof: This proposition follows immediately from 1.8 and 4.7.

Proposition 4.9: If K is a local ring, and A is a finitely generated K-module of rank n, there exist elements $a_1, \ldots, a_n \in A$ which generate A, and every set of generators of A has at least n elements.

Proof: We observe that this proposition is implied by 1.6, 4.2, and 4.7.

Proposition 4.10: If K is a local ring, and A is a finitely generated K-module of rank n, then $E(A)_q$ is a finitely generated K-module of rank $(q, n-q)$ for $q = 0, \ldots, n$.

Proposition 4.11: If K is a local ring with maximal ideal M, $f\colon A \longrightarrow B$ a morphism of K-modules, and B is finitely generated of rank n, then the following statements are equivalent:

i) f is an epimorphism,

ii) $E_K(f)_n\colon E_K(A)_n \longrightarrow E_K(B)_n$ is an epimorphism, and

iii) $E_{K/M}(i_{K/M} \otimes f)_n\colon E_{K/M}(K/M \otimes A)_n \longrightarrow E_{K/M}(K/M \otimes B)_n$ is an epimorphism.

Proof: Certainly i) implies ii), and ii) implies iii) because $E_{K/M}(K/M \otimes A) = K/M \otimes E_K(A)$ by 2.4. By 1.9 we have that iii) implies $K/M \otimes A \longrightarrow K/M \otimes B$ is an epimorphism. Condition i) now follows from 4.2, and the proposition is proved.

§5. Coherent modules.

Definition 5.1: If A is a K-module, and P is a prime ideal in K, the rank of A at P is the rank of A_P over K_P. It is denoted by $r_P(A)$.

Proposition 5.2: If A is a K-module, then

i) $r(A) \geq r_M(A)$ for every maximal ideal M in K,

ii) if $r(A) = n$, there is a maximal ideal M in K such that $r_M(A) = n$, and

iii) if $r(A) = \infty$, then for every integer n there is a maximal ideal M in K such that $r_M(A) > n$.

Proof: By 2.5, we have $K_M \otimes E_K(A) = E_{K_M}(A_M)$ for every maximal ideal M in K. The proposition now follows from Theorem 2.14 of Chapter 2.

Definition 5.3: If A is a K-module of finite rank, then A is coherent if $r(A) = r_M(A)$ for every maximal ideal M in K. If A is a K-module, then A is coherent if A is a direct sum of coherent modules of finite rank.

The main object of study of this paragraph will be coherent projective modules of finite rank. Notice that every free module is coherent, and that 4.6 shows that every projective module over a local ring is coherent.

<u>Proposition 5.4</u>: If A is a projective module of rank n, then

i) $E(A)_q$ is a projective module for every integer q,

ii) $r(E(A)_q) = (q, n-q)$ for $q = 0,\ldots,n$, and

iii) if A is coherent then $E(A)_q$ is coherent.

<u>Proof</u>: Part i) of the proposition is a special case of Chapter 4, 3.20. Now if M is a maximal ideal in K, A_M is a free K_M-module of rank less than or equal to n. Thus $(E(A)_q)_M$ is a free K_M module of rank less than or equal to $(q, n-q)$. Further there is at least one maximal ideal such that $r_M(A) = n$. For any such maximal ideal $r_M(E(A)_q) = (q, n-q)$ for $q = 0,\ldots,n$, and part ii) follows. Part iii) is now immediate.

<u>Proposition 5.5</u>: If A is a coherent projective module and $A \otimes B = 0$ then either $A = 0$ or $B = 0$.

<u>Proof</u>: If $A = 0$ the proposition is immediately true. Therefore, suppose $A \neq 0$. For every maximal ideal M in K, $0 = (A \otimes B)_M = A_M \otimes_{K_M} B_M$. However, A_M is free and different from zero. Therefore $B_M = 0$ for every maximal ideal M in K, and $B = 0$.

<u>Definitions 5.6</u>: If A is a K-module, let $\theta\colon \mathrm{Hom}(A,K) \otimes A \longrightarrow K$ be the morphism such that $\theta(f \otimes a) = f(a)$. The ideal $\mathrm{Im}\,\theta$ is called the <u>trace of A</u>, and is denoted by $tr(A)$.

If A is of rank n, the ideal $tr(E(A)_n)$ is called the <u>determinant of A</u>, and is denoted by $\det(A)$.

Definition 5.7: If X is a module, a coordinate system for X is a set I, a set of elements of X, $\{x_i\}_{i\in I}$, and a set of elements of $\mathrm{Hom}(X,K)$, $\{\varphi_i\}_{i\in I}$ such that

i) for $x \in X$, $\{i \mid i \in I \text{ and } \varphi_i(x) \neq 0\}$ is a finite set, and

ii) for $x \in X$, $x = \Sigma_{i\in I}\, \varphi_i(x)\, x_i$.

Proposition 5.8: The module X is projective if and only if there exists a coordinate system for X.

Proof: Suppose $\{x_i\}_{i\in I}$, $\{\varphi_i\}_{i\in I}$ is a coordinate system for X. Let F be the free module with basis $\{e_i\}_{i\in I}$. Define $\alpha\colon X \longrightarrow F$ by $\alpha(x) = \Sigma_{i\in I}\, \varphi_i(x)\, e_i$, and $\beta\colon F \longrightarrow X$ by $\beta(e_i) = x_i$. Now $\beta\alpha$ is the identity morphism of X, and so X is a projective module.

Suppose X is a projective module, let F be a free module, and $\alpha\colon X \longrightarrow F$, $\beta\colon F \longrightarrow X$ morphisms such that $\beta\alpha$ is the identity morphism of X. Let $\{e_i\}_{i\in I}$ be a basis for F. Let $x_i = \beta(e_i)$ for $i \in I$, and let $\varphi_i\colon X \longrightarrow K$ be the morphism such that $\alpha(x) = \Sigma_{i\in I}\, \varphi_i(x)\, e_i$. Now $\{x_i\}_{i\in I}$, $\{\varphi_i\}_{i\in I}$ is a coordinate system for X, and the proposition is proved.

Lemma 5.9: If A is a module of rank 1, and $f\colon A \longrightarrow K$ is a morphism, then $f(x)y = f(y)x$ for $x,y \in A$.

Proof: Looking at the construction of $E(A)$, we see that $E(A)_2$ is the quotient of $A \otimes A$ by the submodule generated by the elements of the form $x \otimes x$ for $x \in A$ (Chapter 4). The assertion that the rank of A is 1, says that $E(A)_2 = 0$, or that $A \otimes A$ is generated by elements of the form $x \otimes x$ for $x \in A$. Let $T: A \otimes A \longrightarrow A \otimes A$ be the twisting isomorphism, i.e. $T(x \otimes y) = y \otimes x$. Notice that $T(x \otimes x) = x \otimes x$. Since $A \otimes A$ is generated by such element T is the identity morphism of $A \otimes A$, and $x \in y = y \otimes x$ for $x, y \in A$. Now let $\tilde{f}: A \otimes A \longrightarrow A$ be the morphism such that $\tilde{f}(x \otimes y) = f(x)y$. We have $\tilde{f}T = \tilde{f}$, i.e. $f(x)y = f(y)x$, and the lemma is proved.

Lemma 5.10: Let A be a projective module of rank 1, and $\{a_i\}_{i \in I}$, $\{\varphi_i\}_{i \in I}$ a coordinate system for A. If $e_i = \varphi_i(a_i)$ for $i \in I$, then

i) $e_i \in tr(A)$,

ii) if $x \in tr(A)$, then $\{i \mid i \in I \text{ and } e_i x \neq 0\}$ is a finite set, and

iii) if $x \in tr(A)$, then $x = \Sigma_{i \in I}(e_i x)$.

Proof: The ideal $tr(A)$ is generated by elements $f(a)$ where $f \in Hom(A,k)$ and $a \in A$. We have $e_i f(a) = f(e_i a) = f(\varphi_i(a_i)a) = f(a_i \varphi_i(a))$ and since $\varphi_i(a)$ is different from zero for at most a finite number of i's this proves ii). Part i) is immediate.

Now $f(a) = f(\Sigma\ \varphi_i(a)a_i) = f(\Sigma\ \varphi_i(a_i)a) = f(\Sigma\ e_i a) = \Sigma(e_i f(a))$, and the lemma follows.

Note that 5.9 was used twice in the preceding proof.

<u>Proposition 5.11</u>: If A is a projective module of finite rank, then $\det A$ is a projective ideal in K , and $(\det A)^2 = \det A$.

<u>Proof</u>: By 5.4 we may as well assume $r(A) = 1$. Note that if $r(A) =)$, $E(A)_o = K$, and $\det A = K$. Now let $\{e_i\}_{i\in I}$ be elements of $tr(A) = \det A$ as in the preceding lemma. Define $\psi_i : tr(A) \longrightarrow K$ by $\psi_i(x) = e_i x$ for $i \in I$, $x \in tr(A)$. Now $\psi_i(x) \neq 0$ for at most a finite number of i's . For $i \in I$, let $I_i = \{j | j \in I$ and $e_i e_j \neq 0\}$, and let $\bar{e}_i = \Sigma_{j\in I_i} e_j$. Note that I_i is a finite set, and $\bar{e}_i e_i = e_i$ for $i \in I$. If $x \in tr(A)$, then $\Sigma_{i\in I}\ \psi_i(x)\bar{e}_i = \Sigma_{i\in I}\ e_i\ x\bar{e}_i = \Sigma_{i\in I}\ e_i x = x$, so $\{\bar{e}_i\}_{i\in I}$, $\{\psi_i\}_{i\in I}$ is a coordinate system for $tr(A)$, and $tr(A)$ is projective by 5.8. Since $e_i\bar{e}_i = e_i$ for $i \in I$ and both e_i and $\bar{e}_i$ are elements of $tr(A)$, we have $tr(A)^2 = tr(A)$, and the proposition is proved.

<u>Proposition 5.12</u>: If A is a projective module of finite rank, then A is coherent if and only if $\det A = K$.

<u>Proof</u>: If $\det A = K$, we have an exact sequence $\mathrm{Hom}(E(A)_n, K) \otimes E(A)_n \longrightarrow K \longrightarrow 0$ where $n = r(A)$. Thus for every

maximal ideal M in K, we have

$$(\mathrm{Hom}(E(A)_n,K) \otimes E(A)_n)_M \longrightarrow K_M \longrightarrow 0$$

is exact, and $r_M(A) \geq n$. Since $r_M(A) \leq n$ this shows that A is coherent.

Suppose now that A is coherent. In order to prove that $\det A = K$ it suffices to consider the case $r(A) = 1$. Now we have an exact sequence

$$0 \longrightarrow \mathrm{tr}(A) \longrightarrow K \longrightarrow K/\mathrm{tr}\ A \longrightarrow 0 .$$

Note that since $r(A) = 1$, $\det A = \mathrm{tr}(A)$. Since A is projective, this gives rise to an exact sequence

$$0 \longrightarrow \mathrm{Hom}(A,\mathrm{tr}(A)) \longrightarrow \mathrm{Hom}(A,K) \longrightarrow \mathrm{Hom}(A,K/\mathrm{tr}(A)) \longrightarrow 0 .$$

However, looking at the definition of $\mathrm{tr}(A)$, we see that if $f\colon A \longrightarrow K$ is any morphism, then $\mathrm{Im}\ f \subset \mathrm{tr}(A)$. Thus $\mathrm{Hom}(A,\mathrm{tr}(A)) \xrightarrow{\approx} \mathrm{Hom}(A,K)$, and $\mathrm{Hom}(A, K/\mathrm{tr}(A)) = 0$. Observe that if $f\colon A \longrightarrow K/\mathrm{tr}\ A$, then $\mathrm{tr}(A)\ A \subset \mathrm{Ker}\ f$, and so $\mathrm{Hom}(A, K/\mathrm{tr}(A)) = \mathrm{Hom}(A/\mathrm{tr}(A)A\ , K/\mathrm{tr}(A))$. Now $A/\mathrm{tr}(A)A = K/\mathrm{tr}(A) \otimes A$ is a projective module over $K/\mathrm{tr}\ A$. Thus if $A/\mathrm{tr}(A)\cdot A$ is not zero there is a non-zero morphism $f\colon A/\mathrm{tr}\ A\cdot A \longrightarrow K/\mathrm{tr}\ A$. Since this is impossible $K/\mathrm{tr}(A) \otimes A = 0$ and by 5.5, $K/\mathrm{tr}(A) = 0$, i.e. $\mathrm{tr}(A) = K$, and the proposition is proved.

<u>Proposition 5.13</u>: If A is a K-module of rank 1, and $\mathrm{tr}(A) = K$, then

i) A is a finitely generated projective module, and

ii) θ: $\mathrm{Hom}(A,K) \otimes A \longrightarrow K$ is an isomorphism.

Proof: Since $tr(A) = K$, there exist elements $f_1,\ldots,f_n \in \mathrm{Hom}(A,K)$, and $x_1,\ldots,x_n \in A$ such that $1 = \Sigma_{j=1}^{n} f_j(x_j)$. Now if $x \in A$,
$x = x\cdot 1 = \Sigma_{j=1}^{n} x\, f_j(x_j) = \Sigma_{j=1}^{n} f_j(x)x_j$ by 5.9. Thus $\{f_1,\ldots,f_n\}$, $\{x_1,\ldots,x_n\}$ is a coordinate system for A and A is a finitely generated projective module.

Now let λ: $K \longrightarrow \mathrm{Hom}(A,A)$ be the morphism such that $\lambda(k)(a) = ka$, and let δ: $\mathrm{Hom}(A,K) \otimes A \longrightarrow \mathrm{Hom}(A,A)$ be the morphism such that $\delta(f \otimes a)(x) = f(x)a$. By Proposition 4.2 of Chapter 3, δ is an isomorphism. Further $\lambda(\theta(f \otimes a))(x) = \lambda(f(a))x = f(a)\,x = f(x)a$ using 5.9 once again. Thus $\lambda\, \theta = \delta$ which implies that θ is a monomorphism and hence an isomorphism, and thus proves the proposition. Observe that λ is also an isomorphism.

Proposition 5.14: If A is a coherent projective module of finite rank, then A is finitely generated.

Proof: Suppose $r(A) = n$. Using 5.12 and 5.13, we have that $E(A)_n$ is finitely generated. Since $E(A)_n$ is generated by elements of the form $x_1\ldots x_n$, where $x_i \in A$ for $i = 1,\ldots,n$, we have that there exist elements $x_1,\ldots,x_m$ finite in number such that the products $x_{i_1}\ldots x_{i_n}$ of these elements with $1 \leq i_1 < \ldots < i_n \leq m$ generate $E(A)_n$. Let A' be the submodule of A generated by $x_1,\ldots,x_m$.

For every maximal ideal M in K, A_M is free of rank n, so it is finitely generated. Further if $f: A' \longrightarrow A$ is the inclusion morphism, then $E_K(f)_n: E_K(A')_n \longrightarrow E_K(A)_n$ is an epimorphism. Hence $E_{K_M}(A'_M)_n \longrightarrow E_{K_M}(A_M)_n$ is an epimorphism for every maximal ideal M in K. Thus using 4.11, $A'_M = A_M$ for every maximal ideal M, which implies $A' = A$ and proves the proposition.

Proposition 5.15: If $f: A \longrightarrow B$ is a morphism, and B is a finitely generated coherent module of rank n, then the following statements are equivalent:

i) f is an epimorphism,

ii) $E(f)_n: E(A)_n \longrightarrow E(B)_n$ is an epimorphism,

iii) $E_{K_M}(f_M)_n: E_{K_M}(A_M)_n \longrightarrow E_{K_M}(B_M)_n$ is an epimorphism for every maximal ideal M in K, and

iv) $E_{K/M}(i_{K/M} \otimes f)_n: E_{K/M}(K/M \otimes A)_n \longrightarrow E_{K/M}(K/M \otimes B)_n$ is an epimorphism for every maximal ideal M in K.

Proof: The proposition follows using the now familiar process of localizing at each maximal ideal M of K and using 4.11.

Observe that there are other equivalent assertions to those stated in the preceding proposition such as, $K/M \otimes A \longrightarrow K/M \otimes B$ is an epimorphism for each maximal ideal M in K.

Next we turn to the question of which rings have the property that all projective modules of finite rank are coherent. In order to do this we define the notion of idempotent ideal which corresponds in many ways to the notion of idempotent element.

Definition 5.16: An ideal I in K is idempotent if $I^2 = I$, and I is a projective ideal. The ring K is coherent if the only idempotent ideals in K are 0 and K.

Proposition 5.17: If K is a ring, the following conditions are equivalent:

i) the ring K is coherent,

ii) every projective K-module of finite rank is coherent,

iii) if A is any non-zero projective module of finite rank over K, and B is any K-module, then $A \otimes B = 0$ if and only if $B = 0$.

Proof: Using 5.11 and 5.12, we have that conditions i) and ii) are equivalent, and by 5.5 we have that ii) implies iii). Now suppose iii) and assume that I is an idempotent ideal in K. There is an exact sequence

$$0 \longrightarrow I \longrightarrow K \longrightarrow K/I \longrightarrow 0 .$$

There results an exact sequence

$$0 \longrightarrow I \otimes I \longrightarrow I \otimes K \longrightarrow I \otimes K/I \longrightarrow 0$$