Chapter 3: Tensor products, modules of morphisms, and limits.

In this chapter, except for the exercises at the end, we shall stick with the convention that ring means commutative ring.

§1. Tensor products.

Definition 1.1: If A, B, and C are K-modules, a bimorphism $f\colon A \times B \longrightarrow C$ is a function such that

i) $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$,

ii) $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$, and

iii) $f(ka, b) = kf(a, b) = f(a, kb)$

where $a, a_1, a_2 \in A$; $b, b_1, b_2 \in B$, and $k \in K$.

Now we want to generalize the preceding notion to functions of n-variables. Suppose $X_1, \ldots, X_n$ are sets, and $x = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ is an element of $\times_{i \neq j} X_i$. Define $\alpha_x\colon X_j \longrightarrow \times_{i=1}^{n} X_i$ by $\alpha_x(y) = (x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n)$ for $y \in X_j$. The function α_x is called the injection of X_j in $\times_{i=1}^{n} X_i$ at x .

Definition 1.2: If $A_1, \ldots, A_n$ and C are K-modules an n-morphism $f\colon \times_{i=1}^{n} A_i \longrightarrow C$ is a function such that if j is between 1 and n , and $x \in \times_{i \neq j} A_i$, then $f\alpha_x\colon A_j \longrightarrow C$ is a morphism, where $\alpha_x\colon A_j \longrightarrow \times_{i=1}^{n} A_i$ is the injection at x .

Definition 1.3: If $A_1, \ldots, A_n$ are K-modules, the tensor product of $A_1, \ldots, A_n$ is a K-module denoted by $\otimes_{i=1}^{n} A_i$, or $A_1 \otimes \ldots \otimes A_n$, together with an n-morphism $\theta: \times_{i=1}^{n} A_i \longrightarrow \otimes_{i=1}^{n} A_i$ such that if $f: \times_{i=1}^{n} A_i \longrightarrow C$ is any n-morphism, there is a unique morphism $\bar{f}: \otimes_{i=1}^{n} A_i \longrightarrow C$ such that $\bar{f}\theta = f$.

Notice that since we have defined the tensor product by means of a universal property if it exists it is unique. We now proceed to see that it exists.

Proposition 1.4: If $A_1, \ldots, A_n$ are K-modules, the tensor product $\theta: \times_{i=1}^{n} A_i \longrightarrow \otimes_{i=1}^{n} A_i$ exists.

Proof: Let F be the free K-module generated by $\times_{i=1}^{n} A_i$, and let $\times_{i=1}^{n} A_i \longrightarrow F$ be the natural function. Let R be the submodule of F generated by elements y such that for some j between 1 and n, and some $x \in \times_{i \neq j} A_i$ either $y = i\alpha_x(a+a') - i\alpha_x(a) - i\alpha_x(a')$ where $a, a' \in A$ or $y = i\alpha_x(ka) - ki\alpha_x(a)$ where $a \in A$, $k \in K$.

Let $\otimes_{i=1}^{n} A_i = F/R$, and $\pi: F \longrightarrow F/R$ the natural morphism. Define θ to be $\pi \circ i$, and observe that θ is an n-morphism. If $f: \times_{i=1}^{n} A_i \longrightarrow C$ is an n-morphism there is a unique morphism $\tilde{f}: F \longrightarrow C$ such that $\tilde{f}i = f$. Using the fact that f is an n-morphism, we see that $R \subset \operatorname{Ker} \tilde{f}$, and that there is a unique morphism $\bar{f}: \otimes_{i=1}^{n} A_i \longrightarrow C$ such that $\bar{f}\pi = \tilde{f}$. Now $\bar{f}\theta = \bar{f}\pi i = \tilde{f}i = f$, and $\bar{f}$ is unique since the image of $\times_{i=1}^{n} A_i$ under θ generates $\otimes_{i=1}^{n} A_i$ as a K-module. This proves the proposition.

Usually the notation for θ is omitted, but one denotes $\theta(a_1,\dots,a_n)$ by $a_1\otimes\dots\otimes a_n$, or $\theta(\times_{i=1}^{n} a_i)$ by $\otimes_{i=1}^{n} a_i$.

Proposition 1.5: If A, B, C are K-modules, then

$$(A \otimes B) \otimes C = A \otimes B \otimes C = A \otimes (B \otimes C) .$$

Proof: Let $\theta: A \times B \times C \longrightarrow A \otimes B \otimes C$ be the standard trimorphism. For each $c \in C$, the partial map $\theta_c: A \times B \longrightarrow A \otimes B \otimes C$ is a bimorphism, and there corresponds a morphism $\overline{\theta}_c: A \otimes B \longrightarrow A \otimes B \otimes C$. If $\alpha: (A \otimes B) \times C \longrightarrow A \otimes B \otimes C$ is defined by $\alpha(a\otimes b,c) = \overline{\theta}_c(a\otimes b)$, then α is a bimorphism, and there results a morphism $\overline{\alpha}: (A \otimes B) \otimes C \longrightarrow A \otimes B \otimes C$, where $\overline{\alpha}((a\otimes b)\otimes c) = a\otimes b\otimes c$.

Let $\theta_1: A \times B \longrightarrow A \otimes B$, and $\theta_2(A \otimes B) \times C \longrightarrow (A \otimes B) \otimes C$ be the standard bimorphism. Define $\beta: A \times B \times C \longrightarrow (A \otimes B) \otimes C$ by $\beta(a,b,c) = \theta_2(\theta_1(a,b),c)$. Now β is a trimorphism, and there results a morphism $\overline{\beta}: A \otimes B \otimes C \longrightarrow (A \otimes B) \otimes C$, where $\overline{\beta}(a\otimes b\otimes c) = ((a\otimes b)\otimes c)$. Verifying that $\overline{\alpha} \circ \overline{\beta}$ is the identity of $A \otimes B \otimes C$, and that $\overline{\beta} \circ \overline{\alpha}$ is the identity of $(A \otimes B)\otimes C$, the first part of the proposition follows. The proof that $A \otimes B \otimes C = A \otimes(B \otimes C)$ is similar.

The preceding proposition is called the associative law for tensor products. More general forms of the associative law could be stated, but we will not bother to state or prove these formally, although they will be assumed without further comment. For example, $(A_1\otimes\dots\otimes A_p) \otimes (A_{p+1}\otimes\dots\otimes A_n) = A_1\otimes\dots\otimes A_n$.

<u>Proposition 1.6</u>: If A is a K-module, then

$$K \otimes A = A = A \otimes K .$$

<u>Proof</u>: Define $\theta\colon K \times A \longrightarrow A$ by $\theta(k,a) = ka$. If $f\colon K \times A \longrightarrow C$ is a bimorphism, there is a unique morphism $\bar{f}\colon A \longrightarrow C$ such that $\bar{f}\theta = f$, defined by $\bar{f}(a) = f(1,a)$, and the proposition follows.

<u>Proposition 1.7</u>: If A, B, and C are K-modules, then

i) $(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C)$, and

ii) $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$.

<u>Proof</u>: Define $\beta\colon (A \oplus B) \times C \longrightarrow (A \otimes C) \oplus (B \otimes C)$ by $\theta(a + b,c) = a \otimes c + b \otimes c$, and observe that θ satisfies the universal property necessary to prove i). Verifying ii) in a similar fashion the proposition follows.

The preceding proposition is called the <u>distributive law for tensor products</u>. Notice that K-modules act somewhat as a ring. Addition is defined by the direct sum, multiplication is defined by the tensor product, the zero is the module 0, and the unit is the module K (1.5). With care this analogy can be formally exploited after the fashion of Grothendieck.

<u>Proposition 1.8</u>: If A, B are K-modules, there is a unique isomorphism $T\colon A \otimes B \longrightarrow B \otimes A$ such that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\tilde{T}} & B \times A \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\ A \otimes B & \xrightarrow{T} & B \otimes A \end{array}$$

is commutative, where $\tilde{T}(a,b) = (b,a)$.

The proof of this proposition is immediate from the definition. The isomorphism T is called the twisting isomorphism between $A \otimes B$ and $B \otimes A$.

Proposition 1.9: If $f: A \longrightarrow A'$, $g: B \longrightarrow B'$ are morphisms of K-modules, there is a unique morphism $f \otimes g: A \otimes B \longrightarrow A' \otimes B'$ such that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{f \times g} & A' \times B' \\ \downarrow \theta & & \downarrow \theta \\ A \otimes B & \xrightarrow{f \otimes g} & A' \otimes B' \end{array}$$

is commutative.

Once more the proof of the proposition is immediate from the definitions. Notice that it implies that if $f: A \longrightarrow A'$, $f': A' \longrightarrow A''$, $g: B \longrightarrow B'$, $g': B' \longrightarrow B''$ are morphisms, then $(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g)$.

Proposition 1.10: If $A' \xrightarrow{f'} A \xrightarrow{f} A'' \longrightarrow 0$, and $B' \xrightarrow{g'} B \xrightarrow{g} B'' \longrightarrow 0$ are exact sequences of K-modules then the sequences

$$A' \otimes B \xrightarrow{f' \otimes i} A \otimes B \xrightarrow{f \otimes i} A'' \otimes B \longrightarrow 0, \text{ and}$$

$$A \otimes B' \xrightarrow{i \otimes g'} A \otimes B \xrightarrow{i \otimes g} A \otimes B'' \longrightarrow 0$$

are exact where i denotes the identity morphism of A or B .

Proof: It suffices to prove one of the sequences is exact. The exactness of the other will follow from 1.8. Let $X = \text{Coker } A' \otimes B \xrightarrow{f'\otimes i} A \otimes B$, and let $\pi: A \otimes B \longrightarrow X$ be the natural morphism. The composition $A' \times B \longrightarrow A \times B \xrightarrow{\theta} A \otimes B \xrightarrow{\pi} X$ is zero. There results a bimorphism $\theta': A'' \times B \longrightarrow X$ such that the diagram

$$\begin{array}{ccc} A \times B & \longrightarrow & A'' \times B \\ \downarrow \theta & & \downarrow \theta' \\ A \otimes B & \longrightarrow & X \end{array}$$

is commutative. Suppose $h: A'' \times B \longrightarrow C$ is a bimorphism. Now $h(f \times i): A \times B \longrightarrow C$ is a bimorphism, and there is a unique morphism $\tilde{h}: A \otimes B \longrightarrow C$ such that $\tilde{h}\theta = h(f \times i)$. Since $h(f \times i)(f' \times i) = 0$ it follows that $\tilde{h}(f' \otimes i) = 0$, and there is a unique morphism $\bar{h}: X \longrightarrow C$ such that $\bar{h}\pi = \tilde{h}$. Now $\bar{h}\theta'(f \times i) = \bar{h}\,\pi\,\theta = \tilde{h}\theta = h(f \times i)$, so $\bar{h}\theta' = h$, and $\bar{h}$ is unique since the image of θ' generates X. This proves the proposition because it shows $\theta': A'' \times B \longrightarrow X$ satisfies the appropriate universal property for X to be $A'' \otimes B$.

The preceding propositions of this chapter include almost all of the important general properties of tensor products. The major part of what is known concerning tensor products can be proved using these propositions. The most important missing property is the relation between tensor products and limits which will be taken up in the last paragraph of the chapter.

The exact sequence of modules $0 \longrightarrow A' \xrightarrow{f'} A \xrightarrow{f''} A'' \longrightarrow 0$ is said to be <u>split exact</u> if there exists a morphism $h: A \longrightarrow A'$ such that $hf': A' \longrightarrow A'$ is the identity morphism. Notice that the preceding proposition in conjunction with 1.9 implies that if $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is split exact then $0 \longrightarrow A' \otimes B \longrightarrow A \otimes B \longrightarrow A'' \otimes B \longrightarrow 0$ is split exact. In fact $(h \otimes i): A \otimes B \longrightarrow A' \otimes B$ is a morphism such that $(h \otimes i)(f' \otimes i): A' \otimes B \longrightarrow A' \otimes B$ is the identity morphism.

Suppose $\pi: K \longrightarrow \Lambda$ is a morphism of rings. If A is a K-module, the extended module of A relative to π was defined (Chapter 2, 2.7) to be a Λ-module B together with a morphism of K-modules $\pi_A: A \longrightarrow B$ such that if C is a Λ-module and $f: A \longrightarrow C$ is a morphism of K-modules then there is a unique morphism of Λ-modules $\tilde{f}: B \longrightarrow C$ such that $\tilde{f}\pi_A = f$. Note that any Λ-module X is considered as a K-module by defining kx to be $\pi(k)x$ for $k \in K, x \in X$.

Now in Chapter 2 the general question of the existence of extended modules was left in abeyance, although existence was proved in a special case. We are now in a position to remedy this situation, and to simultaneously derive new information concerning tensor products. If A is a K-module, there is an obvious Λ-module associated with A, namely $\Lambda \otimes A$. We now proceed to show that this module is in fact the extended module of A relative to π.

Proposition 1.11: Let $\pi\colon K \longrightarrow \Lambda$ be a morphism of rings, and A a K-module. If $\pi_A\colon A \longrightarrow \Lambda \otimes A$ is defined by $\pi_A(a) = 1 \otimes a$ for $a \in A$, then π_A is the extended module of A relative to π.

Proof: First note that $\Lambda \otimes A$ is a Λ module, where if $\lambda', \lambda \in \Lambda$, $a \in A$, then $\lambda'(\lambda \otimes a) = \lambda'\lambda \otimes a$. Now if C is a Λ-module, and $f\colon A \longrightarrow C$ is a morphism of K-modules, then $(\lambda, a) \longrightarrow \lambda f(a)$ is a bimorphism of K-modules from $\Lambda \times A \longrightarrow C$. Thus there is a morphism of K-modules $\tilde{f}\colon \Lambda \otimes A \longrightarrow C$ such that $\tilde{f}(\lambda \otimes a) = \lambda f(a)$. Observe that $\tilde{f}$ is in fact a morphism of Λ-modules, and that $\tilde{f}\pi_A = f$. Since $\Lambda \otimes A$ is generated as a Λ-module by the image of π_A, $\tilde{f}$ is unique and the proposition follows.

Definition 1.12: The K-module A is flat if for every exact sequence of K-modules $B' \longrightarrow B \longrightarrow B''$ the sequence of K-modules $A \otimes B' \longrightarrow A \otimes B \longrightarrow A \otimes B''$ is exact.

Proposition 1.13: If $\pi\colon K \longrightarrow \Lambda$ is the ring of fractions of K relative to a submonoid M of K, then Λ is a flat K-module.

Proof: By the preceding proposition if B is any K-module, the extended module of B relative to π is $\Lambda \otimes B$. Now if $B' \longrightarrow B \longrightarrow B''$ is exact so is $\Lambda \otimes B' \longrightarrow \Lambda \otimes B \longrightarrow \Lambda \otimes B''$ by Theorem 2.8 of Chapter 2.

In some situations we would like to take lots of copies of a Module A. Consequently we introduce a formal procedure to do this.

<u>Definition 1.14</u>: If X is a set, and A is a K-module, for $x \in X$, let A_x be the K-module such that an element of A_x is a pair (a,x) where $a \in A$ and such that $a \longrightarrow (a,x)$ is an isomorphism of K-modules. Let $A(X) = \oplus_{x\in X} A_x$, and let $i_x : A \longrightarrow A(X)$ be the composition of $A_x \longrightarrow A(X)$ and the isomorphism $A \longrightarrow A_x$.

<u>Proposition 1.15</u>: If X is a set, $F_K(X)$ the free K-module generated by X , A a K-module and $\theta: F_K(X) \times A \longrightarrow A(X)$ the bimorphism such that $\theta(x,a) = i_x(a)$ for $x \in X$, $a \in A$, then $A(X) = F_K(X) \otimes A$.

<u>Proof</u>: Suppose $f: F_K(X) \times A \longrightarrow C$ is a bimorphism. There is a unique morphism $\bar{f}: A(X) \longrightarrow C$ such that $\bar{f}\cdot i_x(a) = f(x,a)$, as is easily seen from the universal property of direct sums and the fact that $A(X) = \oplus_{x\in X} A_x$. Now $\bar{f}\theta = f$ and $\bar{f}$ is the only morphism from $A(X)$ to C with this property. Thus the proposition is proved.

<u>Proposition 1.16</u>: If F is a free K-module, then F is a flat K-module.

<u>Proof</u>: We may as well assume that $F = F_K(X)$ the free K-module generated by some set X . Now if $B' \longrightarrow B \longrightarrow B''$ is exact, then $B'(X) \longrightarrow B(X) \longrightarrow B''(X)$ is exact for it is just X copies of the preceding exact sequence. However, by the preceding proposition this latter exact sequence is just the sequence

$F \otimes B' \longrightarrow F \otimes B \longrightarrow F \otimes B''$ which proves the proposition.

§2. Modules of morphisms.

Definitions 2.1: If A and B are K-modules, the module of morphisms of A into B is the set of morphisms of A into B, denoted by $\mathrm{Hom}(A,B)$, made into a K-module by defining addition and the operation of K as follows:

i) if $f,g \in \mathrm{Hom}(A,B)$, then $(f+g)(a) = f(a) + g(a)$ for $a \in A$, and

ii) if $k \in K$, $f \in \mathrm{Hom}(A,B)$, then $(kf)(a) = k(f(a))$.

If $f: A' \longrightarrow A$, $g: B \longrightarrow B'$ are morphisms, then $\mathrm{Hom}(f,g): \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A',B')$ is the morphism such that $\mathrm{Hom}(f,g)(h) = g \circ h \circ f$.

If $A_o,\dots,A_n$ are K-modules, define $\xi_n: \times_{k=o}^{n-1} \mathrm{Hom}(A_{n-k}, A_{n-k-1}) \longrightarrow \mathrm{Hom}(A_o,A_n)$ by $\xi_n(f_n,\dots,f_o) = f_n \circ \dots \circ f_o$.

Proposition 2.2: If $A_o,\dots,A_n$ are K-modules, then $\xi_n: \times_{k=o}^{n-1} \mathrm{Hom}(A_{n-k-1}, A_{n-k}) \longrightarrow \mathrm{Hom}(A_o, A_n)$ is an n-morphism.

The proof of this proposition is immediate, and it is left for the reader. Note, however, that this proposition implies that there is a natural morphism $\bar{\xi}_n: \otimes_{k=o}^{n-1} \mathrm{Hom}(A_{n-k-1}, A_{n-k}) \longrightarrow \mathrm{Hom}(A_o,A_n)$.

Proposition 2.3: If A_o, A_1, A_2, A_3 are K-modules, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(A_2,A_3) \otimes \mathrm{Hom}(A_1,A_2) \otimes \mathrm{Hom}(A_o,A_1) & \xrightarrow{\bar{\xi}_2 \otimes i_1} & \mathrm{Hom}(A_1,A_3) \otimes \mathrm{Hom}(A_o,A_1) \\ \downarrow {\scriptstyle i_2 \otimes \bar{\xi}_2} & & \downarrow {\scriptstyle \bar{\xi}_2} \\ \mathrm{Hom}(A_2, A_3) \otimes \mathrm{Hom}(A_o, A_2) & \xrightarrow{\bar{\xi}_2} & \mathrm{Hom}(A_o, A_3) \end{array}$$

is commutative, where i_1 is the identity morphism of $\mathrm{Hom}(A_o, A_1)$ and i_2 is the identity morphism of $\mathrm{Hom}(A_2, A_3)$.

Once more the proof of the proposition is immediate, and it is left for the reader. This proposition is an associative law for the behavior of modules of morphisms.

Proposition 2.4: If A is a K-module, the function $\varphi: A \longrightarrow \mathrm{Hom}(K,A)$ such that $\varphi(a)(k) = ka$ is an isomorphism of K-modules.

In view of this proposition we say that $A = \mathrm{Hom}(K,A)$. It is the analogue of 1.6.

Proposition 2.5: If $f': A'' \longrightarrow A'$, $f: A' \longrightarrow A$, $g: B \longrightarrow B'$, $g': B' \longrightarrow B''$ are morphisms of K-modules, then the diagram

$$\begin{array}{ccc} \mathrm{Hom}(A,B) & \xrightarrow{\mathrm{Hom}(f,g)} & \mathrm{Hom}(A',B') \\ & \searrow & \downarrow {\scriptstyle \mathrm{Hom}(f',g')} \\ \mathrm{Hom}(f \circ f', g' \circ g) & & \mathrm{Hom}(A'',B'') \end{array}$$

is commutative

<u>Proposition 2.6</u>: If A, B, C are K-modules, then

i) $\text{Hom}(A \oplus B, C) = \text{Hom}(A,C) \oplus \text{Hom}(B,C)$,

ii) $\text{Hom}(A, B \oplus C) = \text{Hom}(A,B) \oplus \text{Hom}(A,C)$.

<u>Proposition 2.7</u>: If A is a K-module, and $0 \longrightarrow B' \longrightarrow B \longrightarrow B''$ is an exact sequence of K-modules, then $0 \longrightarrow \text{Hom}(A,B') \longrightarrow \text{Hom}(A,B) \longrightarrow \text{Hom}(A,B'')$ is an exact sequence of K-modules.

<u>Proposition 2.8</u>: If $A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence of K-modules, and B is a K-module, then $0 \longrightarrow \text{Hom}(A'',B) \longrightarrow \text{Hom}(A,B) \longrightarrow \text{Hom}(A',B)$ is an exact sequence of K-modules.

The preceding propositions are all more observations concerning the properties of modules of morphisms than they are anything else. However, the reader is asked to observe closely because these propositions contain the major portion of the general properties of modules of morphism.

Notice that if $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is a split exact sequence, then $0 \longrightarrow \text{Hom}(A,B') \longrightarrow \text{Hom}(A,B) \longrightarrow \text{Hom}(A,B'') \longrightarrow 0$ is split exact, and that if $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is split exact, then so is $0 \longrightarrow \text{Hom}(A'',B) \longrightarrow \text{Hom}(A,B) \longrightarrow \text{Hom}(A',B) \longrightarrow 0$.

<u>Definition 2.9</u>: The K-module X is <u>projective</u> if every exact sequence $0 \longrightarrow Y' \longrightarrow Y \longrightarrow X \longrightarrow 0$ of K-modules is split exact.

Proposition 2.10: If X is a K-module, the following conditions are equivalent:

i) X is projective,

ii) X is a direct summand of a free module,

iii) for every exact sequence $B \longrightarrow B'' \longrightarrow 0$, the sequence $\mathrm{Hom}\,(X,B) \longrightarrow \mathrm{Hom}\,(X,B'') \longrightarrow 0$ is exact, and

iv) for every module B, and every exact sequence $0 \longrightarrow Y' \longrightarrow Y \longrightarrow X \longrightarrow 0$, the sequence $0 \longrightarrow \mathrm{Hom}(X,B) \longrightarrow \mathrm{Hom}(Y,B) \longrightarrow \mathrm{Hom}(Y',B) \longrightarrow 0$ is exact.

Proof: Suppose X is projective. Certainly there is an exact sequence $0 \longrightarrow Y' \longrightarrow Y \longrightarrow X \longrightarrow 0$ where Y is a free module. Since this sequence is split exact ii) follows from i).

Suppose X is a direct summand of F where F is free, i.e. $X \oplus X' = F$ for some X'. If $B \xrightarrow{\pi} B' \longrightarrow 0$ is exact, and $f\colon F \longrightarrow B'$ is a morphism, there is a morphism $\tilde{f}\colon F \longrightarrow B$ such that $\pi\tilde{f} = f$ for if I is a basis for F one can choose a map of sets $\tilde{f}\colon I \longrightarrow B$ such that $\pi\tilde{f} = f$ and this map of sets determines a morphism $\tilde{f}\colon F \longrightarrow B$ having the desired property. This shows $\mathrm{Hom}\,(F,B) \longrightarrow \mathrm{Hom}\,(F,B') \longrightarrow 0$ is exact, and by 2.6, $\mathrm{Hom}\,(X,B) \longrightarrow \mathrm{Hom}\,(X,B') \longrightarrow 0$ is exact.

Suppose that for every exact sequence $B \longrightarrow B' \longrightarrow 0$, $\mathrm{Hom}\,(X,B) \longrightarrow \mathrm{Hom}\,(X,B') \longrightarrow 0$ is exact. If $0 \longrightarrow Y' \longrightarrow Y \xrightarrow{\pi} X \longrightarrow 0$ is exact, then so is

$\mathrm{Hom}\ (X,Y) \longrightarrow \mathrm{Hom}\ (X,X) \longrightarrow 0$. Choose $f: X \longrightarrow Y$ so that $\pi f = i$ the identity morphism of X . The existence of f shows that $0 \longrightarrow Y' \longrightarrow Y \xrightarrow{\pi} X \longrightarrow 0$ is split exact, and thus iii) implies i).

In view of earlier comments i) implies iv), so suppose that iv) holds. Consider the special case $B = Y'$. We have $0 \longrightarrow \mathrm{Hom}\ (X,Y') \longrightarrow \mathrm{Hom}\ (Y,Y') \longrightarrow \mathrm{Hom}\ (Y',Y') \longrightarrow 0$. Choose $f: Y' \longrightarrow Y$ such that $\pi f = i_{Y'}$ where $\pi: Y \longrightarrow Y'$ and $i_{Y'}$ is the identity morphism of Y' . The existence of f shows that $0 \longrightarrow Y' \longrightarrow Y \longrightarrow X \longrightarrow 0$ is split exact, and that iv) implies i) which completes the proof of the proposition.

Proposition 2.11: Any projective module is flat.

This proposition is an immediate corollary of 1.16, 1.7, and the preceding proposition. Notice that we now have that free modules are projective, and that projective modules are flat. The fact that free modules are projective shows that there are plenty of projective modules. In fact if A is any K-module, there is a projective module X , and an epimorphism $\pi: X \longrightarrow A$. Indeed one may take X to be free.

§3. Modules of morphism, tensor products, and injective modules.

Proposition 3.1: If A, B, C are K-modules the morphism $\alpha\colon \mathrm{Hom}(A \otimes B, C) \longrightarrow \mathrm{Hom}(B, \mathrm{Hom}(A, C))$ such that $\alpha(f)(b)(a) = f(a \otimes b)$ is an isomorphism.

Proof: Define $\beta\colon \mathrm{Hom}(B, \mathrm{Hom}(A, C)) \longrightarrow \mathrm{Hom}(A \otimes B, C)$ by $\beta(g)(a \otimes b) = g(b)(a)$. Now $\alpha\beta$ in the identity morphism of $\mathrm{Hom}(B, \mathrm{Hom}(A, C))$, and $\beta\alpha$ is the identity morphism of $\mathrm{Hom}(A \otimes B, C)$ which proves the proposition.

The preceding proposition is the fundamental associativity law relating tensor products and modules of morphisms. It is also true that

$$\alpha\colon \mathrm{Hom}(A \otimes B, C) \longrightarrow \mathrm{Hom}(A, \mathrm{Hom}(B, C))$$

defined by $\alpha(f)(a)(b) = f(a \otimes b)$ is an isomorphism. This can be proved either directly, or by using the twisting isomorphism between $A \otimes B$ and $B \otimes A$ together with the preceding proposition.

Remarks concerning notation 3.2: If A, B are K-modules we have denoted the tensor product of A and B by $A \otimes B$. More properly it should be denoted by $A \otimes_K B$, although when no confusion will arise we will retain the original simpler notation. Similarly $\mathrm{Hom}(A, B)$ should be denoted by $\mathrm{Hom}_K(A, B)$, and once more we will retain the simpler notation when possible.

Suppose $\varphi: K \longrightarrow \Lambda$ is a morphism of commutative rings then every Λ module may be considered as a K-module via φ as we have frequently done. In this case there is a generalization of the preceding proposition.

<u>Definition 3.2</u>: If $\varphi: K \longrightarrow \Lambda$ is a morphism of commutative rings, A is a Λ-module and C is a K-module $\mathrm{Hom}_K(A, C)$ is a Λ-module where $(\lambda f)(a) = f(\lambda a)$.

<u>Proposition 3.3</u>: If $\varphi: K \longrightarrow \Lambda$ is a morphism of commutative rings, A, B are Λ-modules and C is a K-module the morphism $\alpha: \mathrm{Hom}_K(A \otimes_\Lambda B, C) \longrightarrow \mathrm{Hom}_\Lambda(B, \mathrm{Hom}_K(A, C))$ such that $\alpha(f)(b)(a) = f(a \otimes_\Lambda b)$ is an isomorphism of Λ-modules.

The proof of this proposition is identical with the proof of proposition 3.2.

<u>Definition 3.4</u>: The K-module X is <u>injective</u> if every exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ is split exact.

<u>Proposition 3.5</u>: If X is a K-module, the following conditions are equivalent:

i) X is injective,

ii) for every exact sequence $0 \longrightarrow A' \longrightarrow A$ the sequence $\mathrm{Hom}(A, X) \longrightarrow \mathrm{Hom}(A', X) \longrightarrow 0$ is exact

iii) for every module A and every exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$, the sequence $0 \longrightarrow \mathrm{Hom}(A,X) \longrightarrow \mathrm{Hom}(A,Y) \longrightarrow \mathrm{Hom}(A,Y'') \longrightarrow 0$ is exact.

Proof: Suppose X is injective and $0 \longrightarrow A' \xrightarrow{f'} A \xrightarrow{f''} A'' \longrightarrow 0$ is exact. Suppose further $g': A' \longrightarrow X$. Define

$h': A' \longrightarrow X \oplus A'$ by $h'(a') = (g'(a'),a')$;

$h: A' \longrightarrow X \oplus A$ by $h(a') = (g'(a'), f'(a'))$,

$\pi': X \oplus A' \longrightarrow X$ by $\pi(x,a') = -x + g'(a')$, and let B be the cokernel of h . Now we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A' & \xrightarrow{h'} & X \oplus A' & \xrightarrow{\pi'} & X & \longrightarrow & 0 \\
 & & \downarrow i_{A'} & & \downarrow i_X \oplus f' & & & & \\
0 & \longrightarrow & A' & \xrightarrow{h} & X \oplus A & \xrightarrow{\pi} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \alpha & & & & \\
 & & 0 & & A'' & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

where $\alpha(x,a) = f''(a)$. As a result there is a unique morphism $\beta: X \longrightarrow B$ such that $\beta\,\pi' = \pi(i_X \oplus f')$. If $\beta(x) = 0$, then $\pi(x,0) = 0$, so $(x,0) = h(y,a') = (g'(a'), f'(a'))$, i.e., $x = g'(a')$, $0 = f'(a')$, and $x = 0$ since f' is a monomorphism which implies $a' = 0$. This shows β is a monomorphism, and $0 \longrightarrow X \xrightarrow{\beta} B \longrightarrow B/X \longrightarrow 0$ is exact. Hence since X is injective, there exists a morphism $\eta: B \longrightarrow X$ such that $\eta\beta = i_B$ the identity of B . Define $g: A \longrightarrow X$ by $g(a) = \eta\,\pi(0,a)$. Verifying that $gf' = g'$, we have $\mathrm{Hom}(A,X) \longrightarrow \mathrm{Hom}(A',X) \longrightarrow 0$ is exact, and i) implies ii).

Suppose ii). If $0 \longrightarrow X \xrightarrow{i} Y \longrightarrow Y'' \longrightarrow 0$ is exact, then so is $\mathrm{Hom}\,(Y,X) \longrightarrow \mathrm{Hom}\,(X,X)$. Choosing $f\colon Y \longrightarrow X$ such that $fi = i_X$ the identity of X , we have $0 \longrightarrow X \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ is split exact, and ii) implies i).

Certainly i) implies iii). Therefore assume iii). If $0 \longrightarrow X \longrightarrow Y \xrightarrow{\pi} Y'' \longrightarrow 0$ is exact, then letting $A = Y''$ $\mathrm{Hom}(Y'',Y) \longrightarrow \mathrm{Hom}(Y'',Y'') \longrightarrow 0$ is exact, and there exists $f\colon Y'' \longrightarrow Y$ such that $\pi f = i_{Y''}$ which shows $0 \longrightarrow X \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ is split exact, and iii) implies i) which completes the proof of the proposition.

The preceding proposition is dual to proposition 2.10.

<u>Proposition 3.6</u>: If X is a K-module, the following conditions are equivalent:

i) X is injective, and

ii) for every ideal I in K the morphism $\theta\colon X \longrightarrow \mathrm{Hom}(I,X)$ such that $\theta(x)(y) = yx$ is an epimorphism.

<u>Proof</u>: Since if I is an ideal in K , the sequence $0 \longrightarrow I \longrightarrow K \longrightarrow K/I \longrightarrow 0$ is exact, we have that if X is injective the sequence $0 \longrightarrow \mathrm{Hom}(K/I,X) \longrightarrow \mathrm{Hom}(K,X) \longrightarrow \mathrm{Hom}(I,X) \longrightarrow 0$ is exact. However, $\mathrm{Hom}(K,X) = X$, and the natural morphism $\theta\colon X \longrightarrow \mathrm{Hom}(I,X)$ is just the morphism $\mathrm{Hom}(K,X) \longrightarrow \mathrm{Hom}(I,X)$. Thus i) implies ii).

Suppose ii), and that $0 \longrightarrow A' \longrightarrow A$ is exact. Let $f: A' \longrightarrow X$. Let $\mathcal{X}$ be the set of pairs (B,g) such that B is a submodule of A containing A', and $g: B \longrightarrow X$ is a morphism such that $f = g\, y_B$ where $y_{:}: A \longrightarrow B$ is the inclusion morphism. Say that $(B,g) < (B',g')$ if $B \subset B'$ and the diagram

$$\begin{array}{ccc} B & \longrightarrow & B' \\ & {\searrow}^{g} & \big\downarrow g' \\ & & X \end{array}$$

is commutative, i.e. if when g, g' are considered as subsets of $A \times X$, then $g \subset g'$. Applying Zorn's Lemma there is a maximal element (B,g) of $\mathcal{X}$. Suppose $a \in A-B$. Let $I = \{k \mid k \in K$ and $ka \in B\}$. Now I is an ideal in K. Define $g': I \longrightarrow X$ by $g'(k) = g(ka)$ for $k \in I$. Let $x \in X$ be an element such that $\theta(x) = g'$. Define $\bar{g}: B + Ka \longrightarrow X$ by $\bar{g}(b + ka) = g(b) + kx$. We have $(B + Ka, \bar{g})$ is an element of $\mathcal{X}$ greater than (B,g). In view of the maximality of (B,g) this is impossible, so $B = A$. This shows that $\mathrm{Hom}(A,X) \longrightarrow \mathrm{Hom}(A',X) \longrightarrow 0$ is exact, and hence that X is injective, and the proposition is proved.

<u>Proposition 3.7</u>: If K is a principal ideal domain, and $X \longrightarrow X'' \longrightarrow 0$ is an exact sequence of modules with X injective, then X'' is injective.

Proof: Let I be an ideal in K . We have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\theta} & \mathrm{Hom}(I,X) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{Hom}(i_I,\pi)} \\ X'' & \xrightarrow{\theta} & \mathrm{Hom}(I,X'') \\ \downarrow & & \\ 0 & & \end{array}$$

with exact columns. Since I is free, $\mathrm{Hom}(i_I,\pi)$ is an epimorphism. Hence θ: $X'' \longrightarrow \mathrm{Hom}(I,X'')$ is an epimorphism, and X" is injective by the preceding proposition.

Proposition 3.8: If K is an integral domain, and X is a module over $Q(K)$, the field of fractions of K , then X is an injective K-module.

Proof: Let I be an ideal in K , and $f: I \longrightarrow X$ a morphism. If $f = 0$, let $z = 0$. If $f \neq 0$, let $y \in I$ be an element such that $f(y) \neq 0$, and let $z = y^{-1} f(y)$, noting that y has an inverse in $Q(K)$. Now $\theta(z) = f$, and the proposition follows from 3.6:

Proposition 3.9: If K is a principal ideal domain, and A is a K-module, then there is an exact sequence

$$0 \longrightarrow A \longrightarrow X \longrightarrow X/A \longrightarrow 0$$

where X is an injective K-module.

Proof: Choose an epimorphism $\pi: F \longrightarrow A$ with F free. Let $N = \operatorname{Ker} \pi$. We have $0 \longrightarrow K \longrightarrow \mathbb{Q}(K)$ is exact, and F is free. Therefore $0 \longrightarrow K \otimes F \longrightarrow \mathbb{Q}(K) \otimes F$ is exact. However $K \otimes F = F$, and there is a natural monomorphism $j: N \longrightarrow \mathbb{Q}(K) \otimes F$. Let $X = \operatorname{Coker} j$. We have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & N & \longrightarrow & F & \xrightarrow{\pi} & A & \longrightarrow & 0 \\ & & \downarrow \approx & & \downarrow \eta & & & & \\ 0 & \longrightarrow & N & \longrightarrow & \mathbb{Q}(K)\otimes F & \xrightarrow{\pi'} & X & \longrightarrow & 0 \end{array}$$

There results a unique monomorphism $\eta': A \longrightarrow X$ such that $\eta'\pi = \pi'\eta$. By the preceding proposition $\mathbb{Q}(K) \otimes F$ is injective, and by 3.7, X is injective. Thus the proposition is proved.

Proposition 3.10: If A is a K-module, there is an exact sequence

$$0 \longrightarrow A \longrightarrow X \longrightarrow X/A \longrightarrow 0$$

where X is an injective K-module.

Proof: Let $\varphi: \mathbb{Z} \longrightarrow K$ be the natural morphism of rings. Consider A as a $\mathbb{Z}$ module, and choose $0 \longrightarrow A \longrightarrow Y$ an exact sequence of $\mathbb{Z}$ modules with Y injective over $\mathbb{Z}$. Let $X = \operatorname{Hom}_{\mathbb{Z}}(K,Y)$. Suppose $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is an exact sequence of K-modules. The sequence $0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(B'',Y) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(B,Y) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(B',Y) \longrightarrow 0$ is exact, since Y is an injective $\mathbb{Z}$ module. For any K-module C,

$Hom_K(C,X) = Hom_K(C, Hom_{\mathbb{Z}}(K,Y)) = Hom_{\mathbb{Z}}(K \otimes C, Y)$ by 3.1. Thus the preceding exact sequence is the sequence

$$0 \longrightarrow Hom_K(B'',X) \longrightarrow Hom_K(B,X) \longrightarrow Hom_K(B',X) \longrightarrow 0$$

and by 3.5 ii), X is an injective K-module. Now $0 \longrightarrow Hom_{\mathbb{Z}}(K,A) \longrightarrow Hom_{\mathbb{Z}}(K,Y)$ is exact, and $A = Hom_K(K,A) \subset Hom_{\mathbb{Z}}(K,A)$ which implies the proposition.

Another way of stating the preceding proposition is that every K-module A is a submodule of an injective K-module X. This proposition is somewhat less obvious than the essentially dual proposition which says that every K-module A is the quotient of a projective module P.

§4. Some properties of finitely generated projective modules.

Definition 4.1: Let A,B,C be K-modules. Define
α: $Hom(A,B) \otimes C \longrightarrow Hom(A,B \otimes C)$ by $\alpha(f \otimes c)(a) = f(a) \otimes c$.

Notice that in the special case $B = K$, then $K \otimes C = C$, and α: $Hom(A,K) \otimes C \longrightarrow Hom(A,C)$, where $\alpha(f \otimes c)(a) = f(a) \cdot c$.

Proposition 4.2: If A is a K-module, the following conditions are equivalent:

i) A is a finitely generated projective,

ii) for all K-modules B and C ,

α: $Hom(A,B) \otimes C \longrightarrow Hom(A,B \otimes C)$

is an isomorphism,

iii) for all K-modules B and C ,

α: $Hom(B,C) \otimes A \longrightarrow Hom(B,C \otimes A)$

is an isomorphism

iv) α: $Hom(A,K) \otimes A \longrightarrow Hom(A,A)$

is an isomorphism.

Proof: In view of propositions 1.7 and 2.6 in order to prove that i) implies ii) or that i) implies iii) it suffices to consider the case that A is free and finitely generated. Note that a finitely generated projective is always a direct summand of a finitely generated free. Thus applying the same proposition it suffices to consider the case $A = K$.

Now if $A = K$, $\mathrm{Hom}(K,B) \otimes C = B \otimes C = \mathrm{Hom}(K,B \otimes C)$ so i) implies ii). Similarly $\mathrm{Hom}(B,C) \otimes K = \mathrm{Hom}(B,C) = \mathrm{Hom}(B,C \otimes K)$, so i) implies ii). Similarly $\mathrm{Hom}(B,C) \otimes K = \mathrm{Hom}(B,C) = \mathrm{Hom}(B,C \otimes K)$, and i) implies iii).

Clearly iv) is a special case of either ii) or iii), and so is implied by either one of them. Therefore it remains to prove that iv) implies i).

Suppose iv). Choose $f_1,\dots,f_n \in \mathrm{Hom}(A,K)$ and $a_1,\dots,a_n \in A$ so that $\alpha(\Sigma_{j=1}^n f_j \otimes a_j) = i_A$ the identity morphism of A. Now for $x \in A$, $x = \Sigma_{j=1}^n f_j(x)\, a_j$, so that A is finitely generated. In particular it is generated by $a_1,\dots,a_n$. Now let F be the free module with generators $e_1,\dots,e_n$. Define $\pi\colon F \longrightarrow A$ by $\pi(e_i) = a_i$ for $i = 1,\dots,n$, and $\delta\colon A \longrightarrow F$ by $\delta(a) = \Sigma_{j=1}^n f_j(a)\, e_j$. Now $\pi\delta = i_A$, and so by 2.10 ii), A is projective, and iv) implies i).

<u>Proposition 4.3</u>: If A,A' are finitely generated projective modules, and B,B' are K-modules, the morphism

$$\delta\colon \mathrm{Hom}(A,B) \otimes \mathrm{Hom}(A',B') \longrightarrow \mathrm{Hom}(A \otimes A', B \otimes B')$$

defined by $\delta(f\otimes g) = f\otimes g$ is an isomorphism.

<u>Proof</u>: Since A,A' are finitely generated projectives it suffices to prove the proposition assuming $A = A' = K$, and in this case it is obvious.

Definition 4.4: If A is a K-module, let A^* denote the K-module $\mathrm{Hom}(A,K)$. If $T: A \longrightarrow B$ is a morphism of K-modules, let $T^*: B^* \longrightarrow A^*$ be the morphism $\mathrm{Hom}(T,i_B)$, i.e. $T^*(f)(a) = f(T(a))$ for $f \in B^*$, $a \in A$. The module A^* is called the dual module of A, and the morphism T^* is called the dual morphism of T.

Proposition 4.4: If A is a finitely generated projective module, then A^* is a finitely generated projective module, and the morphism $\varphi: A \longrightarrow A^{**}$ defined by $\varphi(a)(f) = f(a)$ for $a \in A$, $f \in A^*$ is an isomorphism.

Proof: For finitely generated free modules the proposition follows from 2.4 and 2.6. Since a finitely generated projective is a direct summand of a finitely generated free module, applying 2.6 once more the proposition follows.

§5. Limits.

Definition 5.1: If I is a set, a partial ordering on I is a subset $\tilde{I}$ of $I \times I$ such that if we write $i \leq j$ for $(i,j) \in \tilde{I}$, then

1) $i \leq j$ and $j \leq k$ implies $i \leq k$, and
2) $i \leq j$ and $j \leq i$ is equivalent with $i = j$.

Note that 2) implies that $i \leq i$ for all $i \in I$.

If I is a partially ordered set, a system of K-modules $\mathbb{A}$ indexed on I consists of

1) a K-module A_i for each $i \in I$,
2) a morphism $\alpha_{i,j}: A_i \longrightarrow A_j$ for $(i,j) \in \tilde{I}$, such that if $i \leq j$, $j \leq k$, then $\alpha_{j,k}\,\alpha_{i,j} = \alpha_{i,k}$ and $\alpha_{i,i}: A_i \longrightarrow A_i$ is the identity morphism.

If $\mathbb{A}$ and $\mathbb{B}$ are systems of modules indexed on I, a morphism $f: \mathbb{A} \longrightarrow \mathbb{B}$ is for each $i \in I$ a morphism $f_i: A_i \longrightarrow B_i$ such that if $i \leq j$, then $f_j\,\alpha_{i,j} = \beta_{i,j}\,f_i$.

The sequence $\mathbb{A}' \xrightarrow{f} \mathbb{A} \xrightarrow{g} \mathbb{A}''$ of systems of modules is exact if for each $i \in I$ the sequence $A'_i \xrightarrow{f_i} A_i \xrightarrow{g_i} A''_i$ is exact.

If B is a module, and $\mathbb{A}$ is a system of modules indexed on I, a morphism $f: \mathbb{A} \longrightarrow B$ is for each $i \in I$ a morphism

$f\colon \mathbb{A} \longrightarrow B$ is for each $i \in I$ a morphism $f_i\colon A_i \longrightarrow B$ such that if $i \leq j$, then $f_j\, \alpha_{i,j} = f_i$.

If $\mathbb{A}$ is a system of modules indexed on I , the <u>direct limit</u> of $\mathbb{A}$ is a module $\varinjlim \mathbb{A}$ together with a morphism $\pi_A\colon \mathbb{A} \longrightarrow \varinjlim \mathbb{A}$ such that if B is a module, and $f\colon \mathbb{A} \longrightarrow B$ is a morphism, then there is a unique morphism $\tilde{f}\colon \varinjlim \mathbb{A} \longrightarrow B$ such that $\tilde{f}\,\pi_A = f$. Suppose $\mathbb{B}$ is a system of modules indexed on I , and $f\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a morphism. Now $\pi_B f\colon \mathbb{A} \longrightarrow \varinjlim \mathbb{B}$ is a morphism, and there is a unique morphism $\varinjlim f\colon \varinjlim \mathbb{A} \longrightarrow \varinjlim \mathbb{B}$ such that $\pi_B f = \varinjlim f\, \pi_A$.

If A is a module, and $\mathbb{B}$ is a system of modules indexed on I , a morphism $g\colon A \longrightarrow \mathbb{B}$ is for each $i \in I$ a morphism $g_i\colon A \longrightarrow B_i$ such that $\beta_{i,j} g_i = g_j$ if $i \leq j$.

If $\mathbb{B}$ is a system of modules indexed on I , the <u>inverse limit</u> of $\mathbb{B}$ is a module $\varprojlim \mathbb{B}$ together with a morphism $\lambda_B\colon \varprojlim \mathbb{B} \longrightarrow \mathbb{B}$ such that if A is a module, and $g\colon A \longrightarrow \mathbb{B}$ is a morphism, then there is a unique morphism $\tilde{g}\colon A \longrightarrow \varprojlim \mathbb{B}$ such that $\lambda_B \tilde{g} = g$. Let $\mathbb{A}$ be a second system, and $g\colon \mathbb{A} \longrightarrow \mathbb{B}$ a morphism, then $g\lambda_A\colon \varprojlim \mathbb{A} \longrightarrow \mathbb{B}$, and there is a unique morphism $\varprojlim g\colon \varprojlim \mathbb{A} \longrightarrow \varprojlim \mathbb{B}$ such that $g\, \lambda_A = \lambda_B\, \varprojlim g$.

<u>Comments and Recollections 5.2</u>: If $\tilde{I}$ is the diagonal of $I \times I$, i.e., $(i,j) \in \tilde{I}$ if and only if $i = j$, then $\varinjlim \mathbb{A}$ is just

$\oplus_{i\in I} A_i$ (Chapter 0, exercise 8), and $\varprojlim \mathbb{A}$ is just $\times_{i\in I} A_i$ (Chapter 0, exercise 9).

Proposition 5.3: If $\mathbb{A}$ is a system of modules indexed on I, then the direct limit of $\mathbb{A}$ exists, and further if $f\colon \mathbb{B} \to \mathbb{C}$ and $g\colon \mathbb{A} \to \mathbb{B}$ are morphisms of systems of modules indexed on I, then $\varinjlim fg = \varinjlim f \varinjlim g\colon \varinjlim \mathbb{A} \to \varinjlim \mathbb{C}$.

Proof: For $i \in I$, let $\varphi_i\colon A_i \to \oplus_{r\in I} A_r$ be the canonical injection. For each $(i,j) \in \tilde{I}$ let $\delta_{i,j}\colon A_{i,j} \to A_i$ be an isomorphism, and define $\bar{\alpha}_{i,j}\colon A_{i,j} \to \oplus_{r\in I} A_r$ to be $(\varphi_i - \varphi_j \alpha_{i,j})\, \delta_{i,j}$. Let $\varphi_{i,j}\colon A_{i,j} \to \oplus_{(r,s)\in\tilde{I}} A_{r,s}$ be the canonical injection, and let $\bar{\alpha}\colon \oplus_{(i,j)\in\tilde{I}} A_{i,j} \to \oplus_{i\in I} A_i$ be the morphism such that $\bar{\alpha}\,\varphi_{i,j} = \bar{\alpha}_{i,j}$. Now let $\varinjlim \mathbb{A} = \operatorname{coker} \bar{\alpha}$. If $f\colon \mathbb{A} \to \mathbb{B}$ is a morphism, there is a unique morphism $\bar{f}\colon \oplus_{i\in I} A_i \to B$ such that $\bar{f}\varphi_i = f_i$. Note that $\bar{f}\bar{\alpha}_{i,j} = 0$, so $\bar{f}\bar{\alpha} = 0$, and $\operatorname{Ker}\bar{f} \supset \operatorname{Im}\bar{\alpha}$. There results a unique morphism $\tilde{f}\colon \varinjlim \mathbb{A} \to B$ such that $\tilde{f}\pi_A = f$ where $(\pi_A)_i\colon A_i \to \varinjlim \mathbb{A}$ is the composition of $\varphi_i\colon A_i \to \oplus_{i\in I} A_i$ and the canonical epimorphism $\tilde{\pi}_A\colon \oplus_{i\in I} A_i \to \varinjlim \mathbb{A}$. This proves the first part of the proposition. The second part of the proposition follows easily.

Proposition 5.4: If $\mathbb{A}' \to \mathbb{A} \to \mathbb{A}'' \to 0$ is an exact sequence of modules indexed on I, then

$$\varinjlim \mathbb{A}' \longrightarrow \varinjlim \mathbb{A} \longrightarrow \varinjlim \mathbb{A}'' \longrightarrow 0$$

is exact.

Proof: We have a commutative diagram

$$\begin{array}{ccccccc}
\oplus_{(i,j)\in\tilde{I}} A'_{i,j} & \longrightarrow & \oplus_{(i,j)\in\tilde{I}} A_{i,j} & \longrightarrow & \oplus_{(i,j)\in\tilde{I}} A''_{i,j} & \longrightarrow & 0 \\
\downarrow \bar{\alpha}' & & \downarrow \bar{\alpha} & & \downarrow \bar{\alpha}'' & & \\
\oplus_{i\in I} A'_i & \longrightarrow & \oplus_{i\in I} A_i & \longrightarrow & \oplus_{i\in I} A''_i & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\varinjlim \mathbb{A}' & \longrightarrow & \varinjlim \mathbb{A} & \longrightarrow & \varinjlim \mathbb{A}'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

such that the columns are exact and the two top rows are exact. Consequently we see by doing a little chasing around the diagram that the bottom row is exact, which implies the proposition.

Proposition 5.5: If $\mathbb{A}$ is a system of modules indexed on I, then the inverse limit of $\mathbb{A}$ exists, and further if $f: \mathbb{B} \longrightarrow \mathbb{C}$ and $g: \mathbb{A} \longrightarrow \mathbb{B}$ are morphisms of systems of modules indexed on I, then $\varprojlim fg = \varprojlim f \varprojlim g: \varprojlim \mathbb{A} \longrightarrow \varprojlim \mathbb{C}$.

Proof: For $i \in I$, let $\pi_i : \times_{r\in I} A_r \longrightarrow A_i$ be the canonical projection. For each $(i,j) \in \tilde{I}$ let $\eta_{i,j}: A_j \longrightarrow A_{i,j}$ be an isomorphism, and define $\underline{\alpha}_{i,j}: \times_{r\in I} A_r \longrightarrow A_{i,j}$ to be $\eta_{i,j}(\pi_j - \alpha_{i,j}\, \pi_i)$. Let $\pi_{i,j}: \times_{(r,s)\in\tilde{I}} A_{r,s} \longrightarrow A_{i,j}$ be the canonical projection, and let $\underline{\alpha}: \times_{i\in I} A_i \longrightarrow \times_{(i,j)\in\tilde{I}} A_{i,j}$ be the morphism such that $\pi_{i,j}\, \underline{\alpha} = \underline{\alpha}_{i,j}$. Now let $\varprojlim \mathbb{A} = \operatorname{Ker} \underline{\alpha}$. The completion of the proof is similar to the last half of the proof of 5.3, and is left to the reader.

Proposition 5.6. If $0 \longrightarrow \mathbb{A}' \longrightarrow \mathbb{A} \longrightarrow \mathbb{A}''$ is an exact sequence of modules indexed on I, then $0 \longrightarrow \varprojlim \mathbb{A}' \longrightarrow \varprojlim \mathbb{A} \longrightarrow \varprojlim \mathbb{A}''$ is exact.

The proof of this proposition is completely dual to the proof of 5.4 as the preceding was to 5.3. It is left to the reader.

Definition 5.7: If I, J are partially ordered sets, a morphism $\theta: I \longrightarrow J$ is a function such that if $i_1 \leq i_2$ for $i_1, i_2 \in I$, then $\theta(i_1) \leq \theta(i_2)$. If $\mathbb{A}$ is a system on I, and $\mathbb{B}$ is a system on J, a morphism $f: \mathbb{A} \longrightarrow \mathbb{B}$ is for each $i \in I$ a morphism $f_i: A_i \longrightarrow B_{\theta(i)}$ such that if $i_1 \leq i_2$, then $f_{i_2}\alpha_{i_1,i_2} = \beta_{\theta(i_1),\theta(i_2)} f_{i_1}$.

Proposition 5.8. If I, J are partially ordered sets $\theta: I \longrightarrow J$ is a morphism, $\mathbb{A}$ is a system on I, $\mathbb{B}$ is a system on J, and $f: \mathbb{A} \longrightarrow \mathbb{B}$ is a morphism, then there is a unique morphism $\varinjlim f: \varinjlim \mathbb{A} \longrightarrow \varinjlim \mathbb{B}$ such that $\varinjlim f\, \pi_A = \pi_B f$. Further there is a unique morphism $\varprojlim f: \varprojlim \mathbb{A} \longrightarrow \varprojlim \mathbb{B}$ such that $\lambda_B \varprojlim f = f \lambda_A$.

The proof of the proposition follows easily from the definition.

Definition 5.8: If I, J are partially ordered sets, then $I \times J$ is a partially ordered set where $(r,s) \leq (r',s')$ if and only if

$r \leq r'$; $s \leq s'$.

If $\mathbb{A}$ is a system on I , $\mathbb{B}$ is a system on J , then $\mathbb{A} \times \mathbb{B}$ is the system on $I \times J$ such that $(A \times B)_{r,s} = A_r \times B_s$, and $(\alpha \times B)_{(r,s),(r',s')} = \alpha_{r,r'} \times \beta_{s,s'}$ for $(r,s) \leq (r',s')$.

Similarly $\mathbb{A} \otimes \mathbb{B}$ is the system on $I \times J$ such that $(A \otimes B)_{r,s} = A_r \otimes B_s$ with the obvious morphisms.

Proposition 5.9: If I, J are partially ordered sets, $\mathbb{A}$ is a system on I , and $\mathbb{B}$ is a system on J , then

1) $\varinjlim \mathbb{A} \times \mathbb{B} = \varinjlim \mathbb{A} \times \varinjlim \mathbb{B}$, and
2) $\varprojlim \mathbb{A} \times \mathbb{B} = \varprojlim \mathbb{A} \times \varprojlim \mathbb{B}$.

Proof: Let $\pi_I: I \times J \longrightarrow I$, $\pi_J: I \times J \longrightarrow J$ be the projection. Thus there is a morphism $\varinjlim \mathbb{A} \times \mathbb{B} \longrightarrow \varinjlim \mathbb{A}$, and a morphism $\varinjlim \mathbb{A} \times \mathbb{B} \longrightarrow \varinjlim \mathbb{B}$. There results a morphism $\pi_{A \times B}: \mathbb{A} \times \mathbb{B} \longrightarrow \varinjlim \mathbb{A} \times \varinjlim \mathbb{B}$. To finish the proof of part 1, it suffices to prove that if $f: \mathbb{A} \times \mathbb{B} \longrightarrow C$ is a morphism, then there is a unique morphism $\tilde{f}: \varinjlim \mathbb{A} \times \varinjlim \mathbb{B} \longrightarrow C$ such that $\tilde{f}\, \pi_{A \times B} = f$, and this follows at once upon looking at the construction used to prove the existence of direct limits (5.3).

Part 2 of the proposition follows similarly.

Definition 5.10: If $\mathbb{A}$ is a system on I, $\mathbb{B}$ is a system on J, a bimorphism $f: \mathbb{A} \times \mathbb{B} \longrightarrow C$ is for each $(i,j) \in I \times J$ a bimorphism $f_{i,j}: A_i \times B_j \longrightarrow C$ such that if $(i,j) \leq (r',j')$ then $f_{i',j'}(\alpha \times \beta)_{(i,j),(i',j')} = f_{i,j}$.

Proposition 5.11. If $\mathbb{A}$ is a system on I, $\mathbb{B}$ is a system on J, and $f: \mathbb{A} \times \mathbb{B} \longrightarrow C$ is a bimorphism, then there is a unique bimorphism $\tilde{f}: \underrightarrow{\lim} (\mathbb{A} \times \mathbb{B}) \longrightarrow C$ such that $\tilde{f}\, \pi_{A \times B} = f$.

Proposition 5.12. If I, J are partially ordered sets, $\mathbb{A}$ is a system on I, and $\mathbb{B}$ is a system on J, then

$$\underrightarrow{\lim}\, \mathbb{A} \otimes \mathbb{B} = \underrightarrow{\lim}\, \mathbb{A} \otimes \underrightarrow{\lim}\, \mathbb{B} .$$

Proof: This proposition follows immediately from the preceding proposition together with 5.9 - 1) using the defining universal property of tensor products.

Note that in order to check 5.11 it suffices once more to look at the construction of 5.3.

Definition 5.13: If I is a partially ordered set, then $T(I)$ is the partially ordered set whose underlying set is I and such that $(i,j) \in \widetilde{T(I)}$ if and only if $(j,i) \in \tilde{I}$.

Notice that if $\mathbb{A}$ is a system on I, and $\mathbb{B}$ is a system on J , then $\mathrm{Hom}(\mathbb{A}, \mathbb{B})$ is a system on $T(I) \times J$ where $\mathrm{Hom}(\mathbb{A}, \mathbb{B})_{i,j} = \mathrm{Hom}(A_i, B_j)$ and if $(i,j) \leq (i',j')$

$\mathrm{Hom}(\alpha_{i',i}, \beta_{j,j'}): \mathrm{Hom}(A_i, B_j) \longrightarrow \mathrm{Hom}(A_{i'}, B_{j'})$.

Proposition 5.14: If I, J are partially ordered sets, $\mathbb{A}$ is a system on I , and $\mathbb{B}$ is a system on J , then

$$\varprojlim \mathrm{Hom}(\mathbb{A}, \mathbb{B}) = \mathrm{Hom}(\varinjlim \mathbb{A}, \varprojlim \mathbb{B}) .$$

Proof: Suppose $f: C \longrightarrow \mathrm{Hom}(\mathbb{A}, \mathbb{B})$ is a morphism. By 3.1, f corresponds to a unique morphism $g: \mathbb{A} \otimes C \longrightarrow \mathbb{B}$, i.e.

$g_{i,j}: A_i \otimes C \longrightarrow B_j$ is a morphism, and if $i \leq i'$, $j \leq j'$, then

$g_{i',j'} \alpha_{i,i'} \otimes i_C = \beta_{j,j'} g_{i,j}$. Thus there results a morphism

$\bar{g}: \varinjlim \mathbb{A} \otimes C \longrightarrow \varprojlim \mathbb{B}$; or a morphism

$\bar{f}: C \longrightarrow \mathrm{Hom}(\varinjlim \mathbb{A} , \varprojlim \mathbb{B})$ again using 3.1.

Now $\pi_A: \mathbb{A} \longrightarrow \varinjlim \mathbb{A}$, and $\lambda_B: \varprojlim \mathbb{B} \longrightarrow \mathbb{B}$, and so $\mathrm{Hom}(\pi_A, \lambda_B): \mathrm{Hom}(\varinjlim \mathbb{A} , \varprojlim \mathbb{B}) \longrightarrow \mathrm{Hom}(\mathbb{A}, \mathbb{B})$, and $\tilde{f}$ is the unique morphism such that $\mathrm{Hom}(\pi_A, \lambda_B)\, \bar{f} = f$ which, using the defining universal property of inverse limits, proves the proposition.

Up until now we have not considered what are usually called directed systems. Consequently we do not have as strong exactness of properties of direct limits as are valid for directed systems.

Definitions 5.15: A partially ordered set I is a direct set if for $i,j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

A system of modules indexed on a direct set is called a <u>direct system of modules</u>.

A partially ordered set J is called an <u>inverse set</u> if for $i,j \in I$ there exists $k \in I$ such that $k \leq i$, and $k \leq j$. A system of modules indexed on an inverse set is called an <u>inverse system of modules</u>.

Note that I is a direct set if and only if $T(I)$ is an inverse set.

<u>Proposition 5.16</u>: If $0 \longrightarrow \mathbb{A}' \longrightarrow \mathbb{A} \longrightarrow \mathbb{A}'' \longrightarrow 0$ is an exact sequence of direct systems of modules, then

$$0 \longrightarrow \varinjlim \mathbb{A}' \longrightarrow \varinjlim \mathbb{A} \longrightarrow \varinjlim \mathbb{A}'' \longrightarrow 0$$

is an exact sequence of modules.

<u>Proof</u>: By 5.4, we have that the sequence

$$\varinjlim \mathbb{A}' \longrightarrow \varinjlim \mathbb{A} \longrightarrow \varinjlim \mathbb{A}'' \longrightarrow 0$$

is exact. Therefore it remains only to show that $0 \longrightarrow \varinjlim \mathbb{A}' \longrightarrow \varinjlim \mathbb{A}$ is exact. Let f denote the morphism $\mathbb{A}' \longrightarrow \mathbb{A}$. Since the index set I is direct, if $x \in \varinjlim \mathbb{A}'$ there exists $i_o \in I$, and $x_i \in A_i$ such that $\pi_{A,i_o}(x_{i_o}) = x$. If $x \in \text{Ker} \varinjlim f$, then $\varphi_{i_o} f_{i_o}(x_i) \in \text{Im}\ \alpha$. Suppose $\varphi_{i_o} f_{i_o}(x_i) = \alpha(y)$. Now for some choice indices $y = \Sigma\, \varphi_{i,j}(y_{i,j})$ where $y_{i,j} \in A_{i,j}$, see 5.3 for notation. Let k be an index greater than all of the indices which enter into the expression for y and also greater than i_o . Now $x = \pi_{A,k} \alpha'_{i_o,k} (x_{i_o})$, and

$$f_k(\alpha'_{i_o,k}(x_{i_o})) = \Sigma\, \alpha_{i,k}\, \delta_{i,j}\, (y_{i,j}) - \alpha_{j,k}\, \alpha_{i,j}\, \delta_{i,j}\, (y_{i,j})$$

$$= \Sigma\, \alpha_{i,k}\, \delta_{i,j}\, (y_{i,j}) - \alpha_{i,k}\, \delta_{i,j}\, (y_{i,j}) = 0 \; .$$

Since f_k is a monomorphism this shows that $\alpha'_{i_o,k} (x_{i_o}) = 0$, and thus that $x = 0$ which proves the proposition.

Notice that the preceding implies that the direct limit of any exact sequence of direct system of modules is an exact sequence of modules. No analogous proposition is true for inverse limits of inverse system of modules.

Proposition 5.17: If $/A$ is a direct system of modules, and for each $i \in I$, A_i is flat, then $\varinjlim /A$ is a flat module.

Proof: Since each A_i is flat, we have for any exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ an exact sequence $0 \longrightarrow /A \otimes B' \longrightarrow /A \otimes B \longrightarrow /A \otimes B'' \longrightarrow 0$. By the preceding proposition there results an exact sequence $0 \longrightarrow \varinjlim /A \otimes B' \longrightarrow \varinjlim /A \otimes B \longrightarrow \varinjlim /A \otimes B'' \longrightarrow 0$. However, using 5.2 we see that this is just the exact sequence

$$0 \longrightarrow (\varinjlim A) \otimes B' \longrightarrow (\varinjlim A) \otimes B \longrightarrow (\varinjlim A) \otimes B'' \longrightarrow 0$$

which proves the proposition.

Notice that here we have identified a module C with the system of modules $\mathfrak{C}$ such that $C_i = C$ for $i \in I$, and if $i \leq j$, then $\eta_{i,j}: C_i \longrightarrow C_j$ is the identity morphism. Several things which have been done earlier in the chapter are conveniently considered from this point of view.

Exercises

1. Show that if A is a K-module, and I is an ideal in K, then ${}^{K}/_{I} \otimes A = {}^{A}/_{IA}$.

2. Show that if I,J are ideals in K, then ${}^{K}/_{I} \otimes {}^{K}/_{J} = {}^{K}/_{I+J}$. Show further that $\mathrm{Ker}({}^{K}/_{I} \otimes J \longrightarrow {}^{K}/_{I} \otimes K) = {}^{I \cap J}/_{IJ}$.

3. Let K be the ring of integers Z. Find ideals I, J in K so that the sequence

$$0 \longrightarrow {}^{K}/_{I} \otimes J \longrightarrow {}^{K}/_{I} \otimes K \longrightarrow {}^{K}/_{I} \otimes {}^{K}/_{J} \longrightarrow 0$$

is not exact.

4. Prove that the ring K is a field if and only if the following conditions are satisfied:

   i) every K-module A is projective, and

   ii) if A, B are K-modules such that $A \otimes B = 0$ then $A = 0$ or $B = 0$.

5. Prove that the ring K is a field if and only if the following conditions are satisfied:

   i) every K-module A is injective, and

   ii) if A, B are K-modules such that $\mathrm{Hom}(A, B) = 0$ then $A = 0$ or $B = 0$.

6. Show that every K-module is injective if and only if every K-module is projective. Give an example of a ring which is not a field such that every K-module is projective.

7. Let K be field. Find an ideal in $K[x,y]$ which is not flat, and hence not projective.

8. Show that if A is a finite abelian group, then

   i) $A \otimes \mathbb{Q} = 0$, and

   ii) $A \otimes \mathbb{Q}/\mathbb{Z} = 0$.

9. Let I be an ideal in $\mathbb{Z}$ which is contained in a unique maximal ideal, and suppose that J is an ideal in $\mathbb{Z}$ such that $I \subset J$ and $\mathbb{Z}/J$ is not a projective $\mathbb{Z}/I$ module. Show that it is impossible to have an exact sequence

$$0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_o \longrightarrow \mathbb{Z}/J \longrightarrow 0$$

where each X_i is a projective $\mathbb{Z}/I$ module, and n is a positive integer.

10. If Λ is a ring (not necessarily commutative), then if A is a right Λ-module, B is a left Λ-module, and C is an abelian group, a function $f: A \times B \longrightarrow C$ is <u>balanced</u> if

i) $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$ for $a_1, a_2 \in A$, $b \in B$,

ii) $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$ for $a \in A$, $b_1, b_2 \in B$,

and

iii) $f(a\lambda, b) = f(a, \lambda b)$ for $a \in A$, $b \in B$, $\lambda \in \Lambda$.

Show that there exists an abelian group denoted by $A \otimes_\Lambda B$, and a balanced function $\theta: A \times B \longrightarrow A \otimes_\Lambda B$ such that if $f: A \times B \longrightarrow C$ is a balanced function, then there is a unique morphism of abelian groups $\tilde{f}: A \otimes_\Lambda B \longrightarrow C$ such $\tilde{f}\theta = f$. Show that if Λ is commutative, then $A \otimes_\Lambda B$ is isomorphic as abelian group with the tensor product of A and B as defined in paragraph 1.

11. Under the same hypotheses as the preceding exercise show that if $A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence of right Λ-modules, and B is a left Λ-module, then

$$A' \otimes_\Lambda B \longrightarrow A \otimes_\Lambda B \longrightarrow A \otimes_\Lambda B'' \longrightarrow 0$$

is an exact sequence of abelian groups.

12. If A,B are left Λ-modules, then $\mathrm{Hom}_\Lambda(A,B)$ is the abelian group of morphisms of Λ-modules $f\colon A \longrightarrow B$. Show that if $A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence of left Λ-modules, and B is a left Λ-module, then

$$0 \longrightarrow \mathrm{Hom}_\Lambda(A'',B) \longrightarrow \mathrm{Hom}_\Lambda(A,B) \longrightarrow \mathrm{Hom}_\Lambda(A',B)$$

is an exact sequence of abelian groups. Show that if A is a left Λ-module, and $0 \longrightarrow B' \longrightarrow B \longrightarrow B''$ is an exact sequence of left Λ-modules, then

$$0 \longrightarrow \mathrm{Hom}_\Lambda(A,B') \longrightarrow \mathrm{Hom}_\Lambda(A,B) \longrightarrow \mathrm{Hom}_\Lambda(A,B'')$$

is an exact sequence of abelian groups.

13. Let $p \in Z$ be a prime, and let $K = \mathbb{Z}/p^2\mathbb{Z}$. Show that K is an injective K-module. Show that for a K-module A the following conditions are equivalent:

i) A is projective,

ii) A is free,

iii) A is flat, and

iv) A is injective.

14. Let K be an integral domain. Show that every flat K-module is projective if and only if K is a field.

15. Let K be a Noetherian ring, and A a K-module. Show that A is a finitely generated projective module if and only if A is a finitely generated flat module.

16. Find an example of a ring K, and a finitely generated flat module A over K such that A is not projective.

17. Show that if I is a set, and for each $i \in I$, P_i is a projective K-module, then $\oplus_{i \in I} P_i$ is a projective K-module, then $\oplus_{i \in I} P_i$ is a projective K-module. Give an example showing that $\times_{i \in I} P_i$ is not always a projective K-module.

18. Show that if I is a set, and for each $i \in I$, X_i is an injective K-module, then $\times_{i \in I} X_i$ is an injective K-module. Show that if K is Noetherian, then $\oplus_{i \in I} X_i$ is injective, and give an example to show that this may be false if K is not Noetherian.

19. Let I be a direct set. Give an example of a ring K, and a direct system of K-modules $\mathcal{A}$ on I such that each A_i is projective, and $\varinjlim \mathcal{A}$ is not projective.

20. A partially ordered set I is semi-direct if for each $i, j_1, j_2 \in I$ such that $i \leq j_1$ and $i \leq j_2$ there exists $k \in I$ such that $j_1 \leq k$ and $j_2 \leq k$. Suppose that I is semi-direct, and that $0 \longrightarrow \mathbb{A}' \longrightarrow \mathbb{A} \longrightarrow \mathbb{A}'' \longrightarrow 0$ is an exact sequence of systems of K-modules on I. Show that $0 \longrightarrow \varinjlim \mathbb{A}' \longrightarrow \varinjlim \mathbb{A} \longrightarrow \varinjlim \mathbb{A}'' \longrightarrow 0$ is an exact sequence of K-modules.

21. Give an example of a partially ordered set I with three elements, and an exact sequence of systems of modules $0 \longrightarrow \mathbb{A}' \longrightarrow \mathbb{A} \longrightarrow \mathbb{A}'' \longrightarrow 0$ on I over some ring K such that the sequence $0 \longrightarrow \varinjlim \mathbb{A}' \longrightarrow \varinjlim \mathbb{A} \longrightarrow \varinjlim \mathbb{A}'' \longrightarrow 0$ is not exact.

22. Consider $\mathbb{Z}$ as an ordered set with the usual ordering. Give an example of an exact sequence of systems of $\mathbb{Z}$-modules indexed on $\mathbb{Z}$, $0 \longrightarrow \mathbb{A}' \longrightarrow \mathbb{A} \longrightarrow \mathbb{A}'' \longrightarrow 0$, such that $0 \longrightarrow \varprojlim \mathbb{A}' \longrightarrow \varprojlim \mathbb{A} \longrightarrow \varprojlim \mathbb{A}'' \longrightarrow 0$ is not exact.

Show that if for each $i, j \in \mathbb{Z}$, $i \leq j$ we have $\alpha'_{i,j}: A'_i \longrightarrow A'_j$ is an epimorphism, then

$0 \longrightarrow \varprojlim \mathbb{A}' \longrightarrow \varprojlim \mathbb{A} \longrightarrow \varprojlim \mathbb{A}'' \longrightarrow 0$ is exact.

23. Let I be a semi-direct set, $0 \longrightarrow \mathbb{A}' \longrightarrow \mathbb{A} \longrightarrow \mathbb{A}'' \longrightarrow 0$ an exact sequence of system of K-modules indexed on I, and B an injective K-module. Consider the exact sequence of systems of K-modules

$0 \longrightarrow \mathrm{Hom}(\mathbb{A}'',B) \longrightarrow \mathrm{Hom}(\mathbb{A},B) \longrightarrow \mathrm{Hom}(\mathbb{A}',B) \longrightarrow 0$ indexed on $T(I)$. Show that

$0 \longrightarrow \varprojlim \mathrm{Hom}(\mathbb{A}',B) \longrightarrow \underleftarrow{\mathrm{Hom}}(\mathbb{A},B) \longrightarrow \underleftarrow{\mathrm{Hom}}(\mathbb{A}'',B) \longrightarrow 0$

is exact.

24. Let I be a semi-direct set, and K a field. Suppose that $0 \longrightarrow \mathbb{A}' \longrightarrow \mathbb{A} \longrightarrow \mathbb{A}'' \longrightarrow 0$ is an exact sequence of systems of vector spaces over K indexed on $T(I)$, and that for $i \in I$, A_i is finite dimensional. Show that

$0 \longrightarrow \varprojlim \mathbb{A}' \longrightarrow \varprojlim \mathbb{A} \longrightarrow \varprojlim \mathbb{A}'' \longrightarrow 0$ is exact.

(Hint: Use the preceding exercise in some way.)

Suggested Reading

Bourbaki, N.
Algèbre, Livre II, Chapitre III, Algèbre multilinéaire, Paris, 1948.

Cartan, H., and Eilenberg S.
Homological Algebra, Princeton, 1956. Particularly read chapter I.

Chevalley, C.
Fundamental Concepts of Algebra, New York, 1956. Particularly read chapter III.