Chapter 1: The elementary theory of Noetherian rings and modules.

§1. General properties of Noetherian modules.

Definition 1.1: The Λ-module A is Noetherian if each submodule of A is finitely generated. The ring Λ is (left) Noetherian if when considered as a module it is Noetherian.

There are examples of rings which are left Noetherian; but not right Noetherian. Since, however, all of this chapter except this paragraph will be devoted to commutative algebra we will omit any consideration of such problems.

Proposition 1.2: If $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is an exact sequence of Λ-modules, then B is Noetherian if and only if both B' and B'' are Noetherian.

Proof: Let A be a submodule of B. We then have a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

with exact rows and columns where if we suppose $B' \subset B$, then $A' = A \cap B'$, and $A'' = A/A'$. If B' and B'' are Noetherian, then A' and A'' are finitely generated, and hence A is finitely generated.

However, since A was an arbitrary submodule of B this says that B is Noetherian.

The proof that if B is Noetherian then B' and B'' are Noetherian is immediate.

Proposition 1.3: If Λ is a Noetherian ring, then any finitely generated Λ-module is Noetherian.

Proof: If B is a Λ-module with 1-generator, there is an epimorphism $f: \Lambda \longrightarrow B$ and so B is Noetherian. Suppose that we have proved that every Λ-module with less than or equal to n-generators is Noetherian, and that B has $n+1$ generators. Now there is an exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ such that B' has n generators and B'' has 1 generator. Applying the preceding proposition B is Noetherian, and by induction this proposition follows.

Notation: If X and Y are sets $X \subset Y$, and $X \neq Y$, we write $X < Y$.

Proposition 1.4: Let A be a Λ-module. The following conditions on A are equivalent:

i) A is Noetherian,

ii) if $A_1 \subset A_2 \subset \cdots \subset A_r \subset A_{r+1} \subset \cdots$ is an ascending sequence of submodules of A, then for some integer n, $A_n = A_r$ for $r \geq n$, and

iii) every non-empty family of submodules of A has a maximal element.

Proof: Suppose that A is Noetherian, and that $A_1 \subset A_2 \subset \dots$ is an ascending sequence of submodules of A. Let $A' = \cup_n A_n$. Now A' is a submodule of A and hence finitely generated. Thus if we choose a finite set of generators for A' there must exist an integer n such that these generators belong to A_n. Consequently $A_n = A'$, and if $r \geq n$, $A_n = A_r = A'$, which says that i) implies ii).

Suppose that A satisfies condition ii). Let $\mathcal{A}$ be a non empty set of submodules of A. Choose $A_1 \in \mathcal{A}$, choose $A_2 \in \mathcal{A}$ so that $A_1 \subset A_2$ and so that if possible $A_1 < A_2$. Proceed in this way to obtain an ascending sequence of submodules of A. Let A_n be an element of this sequence such that $A_r = A_n$ for $r \geq n$. Now A_n is a maximal element of $\mathcal{A}$, and so ii) implies iii).

If A satisfies condition iii) and B is a submodule of A, let B be the family of finitely generated submodules of B. We have B is non empty since $0 \in B$, and so B has a maximal element B'. Certainly B' must equal B. Thus since B was an arbitrary submodule of A, iii) implies i) and the proposition is proved.

Definition 1.5: If B is a Λ-module, then A is an irreducible submodule of B if there do not exist submodules A_1, A_2 of B such that $A < A_1$, $A < A_2$, and $A = A_1 \cap A_2$.

<u>Proposition 1.6</u>: If A is a submodule of B, and B/A is Noetherian, then A is a finite intersection of irreducible submodules of B.

<u>Proof</u>: Let $\mathcal{A}$ be the set of submodules of B such that if $X \in \mathcal{A}$, then $A \subset X$, and X is not a finite intersection of irreducible submodules of B. Since B/A is Noetherian, there is a maximal element in $\mathcal{A}$, if $\mathcal{A}$ is not empty. Suppose Y is such a maximal element. Since $Y \in \mathcal{A}$, Y is not irreducible, therefore $Y = Y_1 \cap Y_2$ where $Y < Y_1$, $Y < Y_2$, and Y_1 and Y are submodules of B. Now by the maximality of Y, we have $Y_1, Y_2 \notin \mathcal{A}$, therefore there exist irreducible submodules $X_1, \ldots, X_n$ of B such that $Y_1 = X_1 \cap \ldots \cap X_r$, $Y_2 = X_{r+1} \cap \ldots \cap X_n$. Thus $Y = X_1 \cap \ldots \cap X_n$ which is impossible and $\mathcal{A}$ is empty. This proves the proposition.

§2. Noetherian modules over commutative rings.

In proceeding with the study of Noetherian modules we need to recall a few properties of commutative rings. A commutative ring Λ is an integral domain if whenever $x,y \in \Lambda$ are such that $xy = 0$ and $y \neq 0$, then $x = 0$. An ideal P in Λ is a prime ideal if Λ/P is an integral domain. In other words P is prime if $P \neq \Lambda$, and whenever $xy \in P$ and $y \notin P$ then $x \in P$. Thus Λ is an integral domain if and only if 0 is a prime ideal in Λ. An ideal I in Λ is a primary ideal if $I \neq \Lambda$, and whenever $xy \in I$ and $y \notin I$ there exists a positive integer n such that $x^n \in I$.

Definition 2.1: Let I be an ideal in the commutative ring Λ. The radical of I is the set of all elements $x \in \Lambda$ such that some power of x is in I; it is denoted by $\sqrt{I}$.

Proposition 2.2: Let Λ be a commutative ring. If I is an ideal in Λ, then

i) $\sqrt{I}$ is an ideal in Λ, and $I \subset \sqrt{I}$,

ii) if J is an ideal in Λ such that $I \subset J \subset \sqrt{I}$, then $\sqrt{I} = \sqrt{J}$,

iii) if I is a primary ideal, then $\sqrt{I}$ is a prime ideal, and

iv) if $I = I_1 \cap I_2$, then $\sqrt{I} = \sqrt{I_1} \cap \sqrt{I_2}$.

The proof of the preceding proposition is easy, and we leave it to the reader.

Definition 2.3: If B is a Λ-module, the annihilator of B is the ideal in Λ consisting of those elements $\lambda \in \Lambda$ such that $\lambda B = 0$. Suppose that Λ is a commutative ring, A is a submodule of B and I is the radical of the annihilator of B/A, then A is a primary submodule of B if $A \neq B$, and whenever $\lambda \in \Lambda$, $b \in B$ and $\lambda b \in A$ either $\lambda \in I$ or $b \in A$.

Proposition 2.4: If Λ is a commutative ring and A is a primary sub Λ-module of B, then the annihilator of B/A is a primary ideal in Λ.

Proof: Let J be the annihilator of B/A. Suppose $xy \in J$, $y \notin J$, then for some $b \in B$, $yb \notin A$. However since $xyb \in A$, we have $x \in \sqrt{J}$, and thus J is primary.

Definition 2.5: If Λ is a commutative ring, and A is a primary submodule of B, then the radical of the annihilator of B/A is the associated prime ideal of A in B.

Proposition 2.6: Let Λ be a commutative ring, B a Λ-module, and A_1, A_2 primary submodules of B with associated prime ideal P, then $A_1 \cap A_2$ is a primary submodule of B with associated prime ideal P.

Proof: Suppose $\lambda \in \Lambda$, $b \in B$, and $\lambda b \in A_1 \cap A_2$. If $b \notin A_1 \cap A_2$, then $b \notin A_1$ or $b \notin A_2$, and in either case $\lambda \in P$ since A_1 and A_2 are primary submodules with associated prime P.

Proposition 2.7: If Λ is a commutative ring, A is a proper irreducible sub Λ-module of B, and B/A is Noetherian, then A is a primary submodule of B.

Proof: Suppose $\lambda \in \Lambda$, $b \in B$, and $\lambda b \in A$. Let I be the radical of the annihilator of B/A. Suppose $b \notin A$. Let $\mathcal{B}$ be the family of submodules B' of B such that $A < B'$, and λ belongs to the radical of the annihilator of B'/A. Since $A + \Lambda b \in B$, we have that $\mathcal{B}$ is non empty. Let A' be a maximal element of $\mathcal{B}$, and let n be an integer such that λ^n belongs to the annihilator of A'/A. Suppose $y \in A' \cap (A + \lambda^n B)$. In this case $y = a + \lambda^n z$ for some $z \in B$, and $\lambda^n y = \lambda^n a + \lambda^{2n} z \in A$. Consequently $\lambda^{2n} z \in A$, and by the maximality of A', $z \in A'$, $\lambda^n z \in A$, and $A = A' \cap (A + \lambda^n B)$. Since A is irreducible, and $A < A'$, this implies $A = A + \lambda^n B$, and $\lambda \in I$. In other words A is a primary submodule of B.

Definition 2.8: Let Λ be a commutative ring, and A a sub Λ-module of B. A primary decomposition of A in B is a finite set $A_1, \ldots, A_n$ of primary submodules of B such that $A = \cap_{j=1}^{n} A_j$. Such a decomposition is a reduced primary decomposition if

1) $A_j \not\supset \cap_{i \neq j} A_i$ for $j = 1, \ldots, n$, and

2) if $i \neq j$ then the associated prime ideal of A_i is different from the associated prime ideal of A_j.

It is understood that a primary decomposition of an ideal I in Λ refers to a decomposition of I as a submodule of Λ .

Theorem (Lasker- Noether) 2.9: If Λ is a commutative ring, and A is a proper sub Λ-module of B such that B/A is Noetherian, then there is a reduced primary decomposition of A in B .

Proof: Applying propositions 1.6 and 2.7 some primary decomposition of A in B exists. Applying proposition 2.6 this decomposition may be replaced by one such that the primary submodules of B involved in the decomposition have distinct associated prime ideals. Having obtained such a primary decomposition, a reduced primary decomposition may be obtained by throwing away so many primary submodules involved in the decomposition that no one of the remaining contains the intersection of the others, always being sure to keep enough of them so as to have a primary decomposition of A in B .

Definition 2.10: Let Λ be a commutative ring, and suppose A is a proper sub Λ-module of C . The prime ideal P in Λ is an associated prime ideal of A in C if there exists a submodule B of C such that $A \cap B$ is a primary submodule of B with associated prime ideal P . An associated prime ideal of A in C is an isolated prime ideal of the imbedding of A in C if it contains no other associated prime ideal of A in C ; otherwise it is an imbedded prime ideal.

In the special case where $A = I$ is an ideal in Λ, an associated prime ideal of I in Λ is called merely an associated prime ideal of I. Similarly an isolated prime ideal of I is just an isolated prime ideal of I considered as a submodule of Λ.

<u>Theorem 2.11</u>: Let Λ be a commutative ring, A a proper sub Λ-module of C, and $A_1,\ldots,A_n$ a reduced primary decomposition of A in C where the associated prime ideal of A_i in C is P_i for $i = 1,\ldots,n$. A prime ideal P in Λ is an associated prime ideal of A in C if and only if $P \in \{P_1,\ldots,P_n\}$.

<u>Proof</u>: Suppose P is an associated prime ideal of A in C. Let B be a submodule of C such that $A \cap B$ is a primary submodule of B with associated prime ideal P. Denote by Q_i the radical of the annihilator of $B/A_i \cap B$ for $i = 1,\ldots,n$. Now $P = Q_1 \cap \ldots \cap Q_n$, and $P_i \subset Q_i$ for $i = 1,\ldots,n$. Suppose $b \in B - A_i \cap B$ and $\lambda \in Q_i$, then for some integer r we have $\lambda_r\, b \in A_i \cap B$, whence $\lambda \in P_i$. Thus for i between 1 and n either $P_i = Q_i$, or $B \subset A_i$. Since $A \cap B \neq B$ there is at least one integer i such that $P_i = Q_i$. Consequently we may suppose $P_i = Q_i$ for $i = 1,\ldots,k$, $k \geq 1$, and $B \subset A_i$ for $i = k+1,\ldots,n$. Thus if $i > k$, $Q_i = \Lambda$, and $P = P_1 \cap \ldots \cap P_k$. Since P is prime $P \supset P_i$ for some i between 1 and k, and thus $P = P_i$

and $P \in \{P_1,\dots,P_n\}$.

It remains to show that P_i is an associated prime ideal of A in C for $i = 1,\dots,n$. Let $A^i = \cap_{j \neq i} A_j$. Now $A^i \neq A$ since $A_1,\dots,A_n$ was a reduced primary decomposition of A in C . Let I_i be the annihilator of A^i/A for $i = 1,\dots,n$. Suppose $\lambda \in \Lambda$, $x_i \in A^i - A$ and $\lambda x_i \in A$, then $\lambda \in P_i$, so for some integer r we have $\lambda^r C \subset A_i$, and $\lambda^r \in I_i$. This shows that A is a primary submodule of A^i with associated prime ideal $\sqrt{I_i} \subset P_i$ for $i = 1,\dots,n$. However since if $\lambda \in P_i$, $\lambda^r C \subset A_i$ for some r , and this implies $\lambda^r A^i \subset A$, we have $\sqrt{I_i} = P_i$ for $i = 1,\dots,n$, and the theorem follows.

Observe that the preceding theorem shows that the number of primary submodules which occur in a reduced primary decomposition does not depend on the choice of decomposition for there must be exactly one for each associated prime ideal of A in C .

Before proceeding we insert a criterion for an ideal I to be primary.

<u>Proposition 2.12</u>: If Λ is a commutative ring and I is an ideal in Λ such that $\sqrt{I}$ is a maximal ideal, then I is a primary ideal.

<u>Proof</u>: Recall that a proper ideal M is maximal if and only if it is contained in no proper ideal other than itself, or equivalently if and only if Λ/M is a field.

Now let $M = \sqrt{I}$. Suppose that $x, y \in \Lambda$, $xz \in I$, and $y \notin I$. Since M is maximal and $y \notin I$, we have that $\Lambda = M + \Lambda y$, and $1 = m + \lambda y$ for some $m \in M$, $\lambda \in \Lambda$. Thus $x = mx + \lambda xy$, and $x \in M$. Since $M = \sqrt{I}$ saying that $x \in M$ is equivalent to saying that some power of x is in I , and the proposition is proved.

Corollary 2.13: If Λ is a commutative ring and M is a maximal ideal in Λ , then for any positive integer n , M^n is a primary ideal and $\sqrt{M^n} = M$.

Proposition 2.14: If Λ is a commutative ring, I is an ideal in Λ , $I_1, \ldots, I_n$ is a reduced primary decomposition of I , $\sqrt{I_j}$ is an isolated prime ideal of I for $i \leq k$, and $\sqrt{I_j}$ is an imbedded prime ideal of I for $i > k$, then $\sqrt{I_1}, \ldots, \sqrt{I_k}$ is a reduced primary decomposition of $\sqrt{I}$.

The proof of the proposition is easy, and it is left to the reader.

Lemma 2.15: If Λ is a commutative Noetherian ring and I is an ideal in Λ , there exists an integer n , such that $(\sqrt{I})^n \subset I$.

Proof: Let J be an ideal in Λ maximal among those ideals which have some power contained in I . Now for some integer n , $J^n \subset I$. Suppose $\lambda \in \sqrt{I} - J$, then for some integer m , $\lambda^m \in I$, and $(J + \Lambda\lambda)^{n+m} \subset I$. However, by the maximality of J this is impossible, so $\sqrt{I} \subset J$. Clearly $J \subset \sqrt{I}$ and the lemma follows.

Proposition 2.16: If Λ is a commutative Noetherian ring, I is an ideal in Λ , C is a Noetherian Λ-module, and $A = \cap_n I^n C$, then $IA = A$.

Proof: We may as well suppose $IA \neq C$ for in that case the proposition is trivial. Now let $A_1, \ldots, A_n$ be a reduced primary decomposition of IA in C . Suppose P_i is the associated prime ideal of A_i for $i = 1, \ldots, n$. If $x \in A - A_i$ and $\lambda \in I$, then $\lambda x \in A_i$ so $\lambda \in P_i$. Thus for each i between 1 and n we have either $I \subset P_i$ or $A \subset A_i$. If $I \subset P_i$, we have $P_i^n C \subset A_i$ for some integer n , by the preceding lemma. Consequently $A \subset I^n C \subset P_i^n C \subset A_i$, so $A \subset A_i$ for $i = 1, \ldots, n$, and $A = IA$ as was to be proved.

The preceding proposition is the main proposition in the proof of Krull's theorem. We will not however prove that theorem now, but delay until we have some linear algebra at our disposal. We close this paragraph with one more proposition concerning Noetherian modules over Noetherian rings.

Proposition 2.17: If Λ is a commutative Noetherian ring, I is an ideal in Λ , C is a Noetherian Λ module, and A is a submodule of C , then there exists a submodule A' of C such that

i) $IA = A \cap A'$, and

ii) for some integer n , $I^n C \subset A'$.

<u>Proof</u>: Let $A_1, \ldots, A_n$ be a reduced primary decomposition of IA in C , and let P_i be the associated prime ideal of A_i in C for $i = 1, \ldots, n$. Let A' be the intersection of those A_i's such that $I \subset P_i$, and A'' the intersection of the remaining A_i's . Now $IA = A' \cap A''$. For some integer n , $I^n C \subset A'$, and further $A \subset A'$ as may be seen by looking at the proof of the preceding proposition. Thus

$$IA = A' \cap A'' \supset A' \cap A \supset IA ,$$

and the proposition is proved.

§3. Polynomial algebras and Noetherian rings

At this stage of events we have seen no examples of the preceding theory. This situation will be partially rectified by beginning the study of polynomial algebras.

<u>Definitions 3.1:</u> Let K be a commutative ring. A K-algebra (or algebra over K) is a ring Λ which is a K-module, and such that if $\varphi\colon \Lambda \times \Lambda \longrightarrow \Lambda$ is the multiplication map, then $\varphi(kx,y) = k\,\varphi(x,y) = \varphi(x,ky)$ for $x,y \in \Lambda$, $k \in K$. In other words $(kx)y = k(xy) = x(ky)$. There is a canonical map $\eta\colon K \longrightarrow \Lambda$ defined by $\eta(k) = k\cdot 1$.

If Λ and Γ are K-algebras a <u>morphism</u> $f\colon \Lambda \longrightarrow \Gamma$ is a morphism of rings which is simultaneously a morphism of K-modules.

Observe that η is a morphism of K-algebras. Further, if η is a monomorphism then for $f\colon \Lambda \longrightarrow \Gamma$ to be a morphism of K-algebras it is sufficient that f be a morphism of rings.

<u>Definition 3.2:</u> Let K be a commutative ring, and X a set. A <u>polynomial algebra over K generated by X</u> consists of a commutative K-algebra Λ and a map $i\colon X \longrightarrow \Lambda$ such that if Γ is any commutative K-algebra and $f\colon X \longrightarrow \Gamma$ any map, then there is a unique morphism of K-algebras $\tilde{f}\colon \Lambda \longrightarrow \Gamma$ such that $\tilde{f}i = f$.

<u>Proposition 3.3:</u> If K is a commutative ring, and X is a set, there exists a polynomial algebra Λ, $i\colon X \longrightarrow \Lambda$ over K generated by X. Further if Λ', $i'\colon X \longrightarrow \Lambda'$ is a second such polynomial

algebra there is a unique isomorphism $f: \Lambda \longrightarrow \Lambda'$ such that $fi = i'$.

Proof: We prove the second part of the proposition first. Let $\tilde{i}': \Lambda' \longrightarrow \Lambda$ be the unique morphism of K-algebras such that $\tilde{i}\, i' = i$. Now $\tilde{i}\,\tilde{i}'i = \tilde{i}\, i' = i$ and $\tilde{i}'\tilde{i}\, i' = \tilde{i}\, i = i'$, so $\tilde{i}\,\tilde{i}'$ is the identity morphism of Λ and $\tilde{i}'\tilde{i}$ is the identity morphism of Λ'. Let $f = \tilde{i}'$, and the second part of the proposition is proved.

Let $\bar{X}$ be the set of functions $h: X \longrightarrow \mathbb{Z}$ such that

i) if $x \in X$, then $h(x) \geq 0$, and

ii) $\{x \mid x \in X$ and $h(x) \neq 0\}$ is a finite set. Now let Λ be the free K-module generated by $\bar{X}$. To make Λ into a ring it suffices to define a multiplication between elements of $\bar{X}$ since they form a basis for Λ. Then if $h_1, h_2 \in \bar{X}$, let h_1h_2 be the element of $\bar{X}$ such that $(h_1h_2)(x) = h_1(x) + h_2(x)$. This multiplication makes Λ into a K-algebra, the unit $1 \in \Lambda$ being the element of $\bar{X}$ which assigns 0 to each element of X. Define $i: X \longrightarrow \Lambda$ by letting $i(x)$ be the element of $\bar{X}$ which assigns 1 to x and 0 to all other elements of X.

Suppose now Γ is a commutative K-algebra and $g: X \longrightarrow \Gamma$, is a map. Suppose $h \in \bar{X}$, and let $x_1, \ldots, x_n$ be the elements of X such that $h(x_2) \neq 0$. Define $\tilde{g}(h) = g(x_1)^{h(x_1)} \ldots g(x_n)^{h(x_n)}$, and $\tilde{g}(1) = 1 \in \Gamma$. Now since Λ is the free K-module generated by $\bar{X}$ we have a unique morphism of K-modules $\tilde{g}: \Lambda \longrightarrow \Lambda$ such that $\tilde{g}(h)$

is as above for $h \in \bar{X}$. Observing that $\tilde{g}$ is a morphism of K-algebras and $\tilde{g}\, i = g$ we have that Λ, $i: X \longrightarrow \Lambda$ is a polynomial algebra over K generated by X , and the proposition is proved.

Notations and conventions 3.3: Let K be a commutative ring. Having proved that given any set X there exists a unique polynomial algebra over K generated by the set X , we denote this algebra by $K[X]$, and write $X \subset K[X]$ instead of $i: X \longrightarrow K[X]$. The elements are called indeterminates of the polynomial algebra $K[X]$.

In the notation of the preceding proof if $h \in \bar{X}$ and $x_1, \ldots, x_n$ are the only elements of X on which h does not vanish, then the integer $\Sigma_{j=1}^{n} h(x_j)$ is called the degree of h . The element $1 \in \bar{X}$ is of degree 0 . The K-submodule of $K[X]$ generated by the elements $h \in \bar{X}$ of degree n is denoted by $K[X]_n$, and elements of this submodule are called homogeneous polynomials of degree n . Observe that $K[X] = \oplus_{n \geq 0} K[X]_n$.

An arbitrary element $f \in K[X]$ is of degree n if f is a linear combination of elements of $\bar{X}$ of degree less than or equal to n but not a linear combination of elements of $\bar{X}$ of degree less than or equal to $n-1$, i.e. $f \in \Sigma_{j \leq n} K[X]_j$ and $f \notin \Sigma_{j \leq n-1} K[X]_j$.

Sometimes a slight change in notation is made. If $X = \{x_1, \ldots, x_n\}$ is a finite set one frequently writes $K[x_1, \ldots, x_n]$ instead of $K[X]$. Further if $h \in \bar{X}$ is of degree n , and

$x_1,\ldots,x_m$ are the elements of X on which h does not vanish one often writes $x_1^{h(x_1)}\ldots x_m^{h(x_m)}$ instead of h .

In the proof of proposition 3.2 we tacitly assumed the set X to be non empty. If X is empty by convention $K[X] = K$, this definition is consistent with the definition of polynomial algebra over K since if Λ is a K-algebra the morphism $\eta: K \longrightarrow \Lambda$ is the only morphism of K-algebra from K to Λ .

<u>Proposition 3.4</u>: If K is a commutative ring and X,Y are sets, then

$$K[X \cup Y] = K[X][Y] .$$

<u>Proof</u>: Let Λ be a commutative K-algebra, and $f: X \cup Y \longrightarrow \Lambda$ a map. Now there is a unique morphism of K-algebras $\tilde{f}_1: K[X] \longrightarrow \Lambda$ such that $\tilde{f}_1|X = f|X$. Via $\tilde{f}_1$, Λ may be considered as an algebra over K[X] and there is a unique morphism of K[X] algebras $\tilde{f}: K[X][Y] \longrightarrow \Lambda$ such that $\tilde{f}\,|Y = f|Y$. Now $\tilde{f}$ is a morphism of K-algebras, and it is certainly the only morphism of K-algebras such that $\tilde{f}\,|X \cup Y = f$ which proves the proposition.

<u>Proposition 3.5</u>: Let K be a commutative ring, and X a set. Then if $f,g \in K[X]$,

i) degree $(f+g) \leq \max\{\text{degree } f, \text{degree } g\}$,

ii) degree $(fg) \leq$ degree f + degree g .

<u>Proposition 3.6</u>: If K is an integral domain, and X is a set, then $K[X]$ is an integral domain.

The proofs of the preceding proposition are easy, and they are left to the reader.

Until now we have seen no connection between polynomial algebras and Noetherian rings. We must now introduce a few definitions in order to see such a connection.

<u>Definitions 3.7</u>: Let $K[x]$ be the polynomial algebra over the commutative ring K in one indeterminate x. Let $\varphi_n: K[x] \longrightarrow K$ be the morphism of K-modules such that $\varphi_n(x^n) = 1$ and $\varphi_n(x^q) = 0$ for $q \neq n$ where n is a positive integer.

Let $F_p\, K[x] = \Sigma_{q \leq p}\, K[x]_q$ for p a positive integer. Abbreviate the notation $F_pK[x]$ to F_p when no confusion will arise.

If I is an ideal in $K[x]$ let I_n = image $\varphi_n: I \cap F_n \longrightarrow K$. The ideals $I_o, I_1, \ldots, I_n, \ldots$ are called the associated ideals of I.

<u>Lemma 3.8</u>: If I, J are ideals in $K[x]$, and $I \subset J$, then $I = J$ if and only if $I_n = J_n$ for all positive integers n.

<u>Proof</u>: Suppose $I_n = J_n$ for all n. Notice that $I \cap F_n / I \cap F_{n-1} \xrightarrow{\approx} I_n$, and $J \cap F_n / J \cap F_{n-1} \xrightarrow{\approx} J_n$ for $n > 0$, and $I \cap F_o \xrightarrow{\approx} I_o$, $J \cap F_o \xrightarrow{\approx} J_o$. Therefore we have $I \cap F_o = J \cap F_o$.

Suppose $I \cap F_q = J \cap F_q$ for $q \leq n$. Now we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \cap F_n & \longrightarrow & I \cap F_{n+1} & \longrightarrow & I_{n+1} & \longrightarrow & 0 \\ & & \downarrow\approx & & \downarrow & & \downarrow\approx & & \\ 0 & \longrightarrow & J \cap F_n & \longrightarrow & J \cap F_{n+1} & \longrightarrow & J_{n+1} & \longrightarrow & 0 \end{array}$$

where the rows are exact. Thus $I \cap F_{n+1} = J \cap F_{n+1}$, and we have by induction that $I \cap F_n = J \cap F_n$ for all n . Thus

$$I = \cup_n I \cap F_n = \cup_n J \cap F_n = J \ .$$

If $I = J$ it is clear that $I_n = J_n$ for all n , and the lemma is proved.

<u>Theorem (Hilbert) 3.9</u>: If K is a commutative Noetherian ring and X is a finite set, then $K[X]$ is a Noetherian ring.

<u>Proof</u>: In view of proposition 3.4 it suffices to prove the theorem in case X has one element x . Consequently we deal with the polynomial algebra in one indeterminate $K[x]$. Let J be an ideal in $K[x]$, and let $J_o, \ldots, J_q, \ldots$ be the associated ideals of J .

Let $\lambda\colon K[x] \longrightarrow K[x]$ be the morphism such that $\lambda(f) = xf$. It follows from the commutativity of the diagram

$$\begin{array}{ccc} F_q \cap J & \xrightarrow{\varphi_q} & K \\ \downarrow \lambda & \nearrow \varphi_{q+1} & \\ F_{q+1} \cap J & & \end{array}$$

that $J_q \subset J_{q+1}$ for all q. Since K is Noetherian there exists an integer n such that $J_q = J_n$ for $q \geq n$. Now for $q = 0,\ldots,n$, let Y_q be a finite subset of $F_q \cap J$ such that $\varphi_q(Y_q)$ generates J_q. The set $Y = \cup_{q=o}^{n} Y_q$ is a finite subset of J. Let I be the ideal in $K[x]$ generated by Y. We have $I \subset J$, and further if $I_o, I_1,\ldots,I_q,\ldots$ is the set of associated ideals of I, then $I_q = J_q$ for all q. Thus $I = J$, and J is finitely generated. Since J was an arbitrary ideal in $K[x]$, the theorem is now proved.

Exercises

1. In the algebra $\mathbb{Z}[x]$, let I be the ideal generated by $9, 3x$; J the ideal generated by $9, x$; $M = \sqrt{J}$, and P the ideal generated by 3 .

   i) Show that J is a primary ideal and M is a maximal ideal.

   ii) Show that J, P and M^2, P are both reduced primary decomposition of I .

   iii) Show that $\sqrt{I} = P$.

2. In the polynomial algebra $K[x,y]$ over the field K , let I be the ideal generated by x^2 and xy . Find at least two distinct reduced primary decompositions of the ideal I .

3. Let $K[x]$ be the polynomial algebra in one indeterminate over the commutative ring K . Show that K is an integral domain if and only if for every $f, g \in K[x]$, degree (fg) = degree f + degree g .

4. Let X be a finite non empty set and K a commutative ring. Show that the polynomial algebra $K[X]$ is a principal ideal domain if and only if K is a field and X is a set with one element.

   Recall that a <u>principal ideal domain</u> is an integral domain Λ such that every ideal is generated by one element.

5. Let K be a commutative ring, $I = \sqrt{0}$, and A a K-module such that $IA = A$, i.e. such that if $a \in A$ there exist elements $x_1, \ldots, x_n \in I$, $a_1, \ldots, a_n \in A$ such that $a = \Sigma_{j=1}^{n} x_j a_j$. Show that $A = 0$ if either K is Noetherian or A is finitely generated.

6. Let K be a commutative ring, and Λ a commutative K-algebra. If X is a subset of Λ the image Γ of the morphism of K-algebras $K[X] \longrightarrow \Lambda$ induced by the inclusion map $X \longrightarrow \Lambda$ is the sub K-algebra of Λ generated by X . Show that if K is Noetherian and X is a finite set, then Γ is Noetherian.

7. Let Γ be the subring (or sub Z algebra) of $\mathbb{Z}[x]$ generated by $3x, x^2, x^3$. Let P be the ideal in Γ generated by $3x, x^2, x^3$. Show that P is a prime ideal and that P^2 is not a primary ideal.

8. Find a finitely generated commutative algebra over a field and a prime ideal in this algebra whose square is not primary.

9. Let G be an abelian group such that if the group operation is written multiplicatively, for every $y \in G$, there exists $x \in G$ such that $x^2 = y$, and such that if $y \in G$ there exists an integer n such that $y^{2^n} = 1$. Let $K = \mathbb{Z}/2\mathbb{Z}$, and let $K(G)$ be the vector space over K with basis G . Make $K(G)$ into an algebra by assuming that if $x,y \in G$ their product in $K(G)$ is the element $xy \in G$. Let $I = \sqrt{0}$ in the algebra $K(G)$. Find a basis for the ideal I considered as a vector space over K . Show that $I^2 = I$.

Suggested Reading

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Moderne Algebra, Vol. 1, Berlin 1937, vol. 2, Berlin 1940.

Particularly read chapters 3, 4, 12 and 13.

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Commutative Algebra, vol. 1, Princeton 1958.

Particularly read chapters 3 and 4.