CATEGORY THEORY

JC Moore

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Chapter 1. Basic properties of categories and functors

§1. Categories, subcategories, functors, and some special classes of morphisms and objects.

1.1 Definition. A *category* $\mathcal{X}$ consists of

1) a class obj $(\mathcal{X})$ called the collection of objects of $\mathcal{X}$,

2) for each ordered pair (X',X'') of objects of $\mathcal{X}$, a set $\mathcal{X}(X',X'')$ called the set of morphisms in $\mathcal{X}$ from X' to X'', and

3) for each ordered triple (X',X,X'') of objects of $\mathcal{X}$, a function $\mathcal{X}(X,X'') \times \mathcal{X}(X',X) \longrightarrow \mathcal{X}(X',X'')$, the image of (f'',f') under this function is denoted by $f'' \circ f'$ and called the composite of f'' with f',

further the preceding elements of structure are assumed to satisfy the following conditions:

1) if $X \in \text{obj}(\mathcal{X})$, there is a unique $1_X \in \mathcal{X}(X,X)$ such that

2) if $X' \in \text{obj}(\mathcal{X})$ and $f' \in \mathcal{X}(X',X)$, then $1_X \circ f' = f'$, and

22) if $X'' \in \text{obj}(\mathcal{X})$ and $f'' \in \mathcal{X}(X,X'')$, then $f'' \circ 1_X = f''$, and

2) if $X_j \in \text{obj}(\mathcal{X})$ for $j = 1,2,3,4$, and $f_j \in \mathcal{X}(X_j,X_{j+1})$ for $j = 1,2,3$, then $f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1$.

If $X \in \text{obj}(\mathcal{X})$ the element 1_X of $\mathcal{X}(X,X)$ is the *identity morphism of* $\mathcal{X}$. It is sometimes denoted by X.

If X', X'' are objects of $\mathcal{X}$, the symbolism $f: X' \longrightarrow X''$ is often used to indicate that $f \in \mathcal{X}(X', X'')$. In this situation the object X' is the <u>domain of f</u> and the object X'' is the <u>range of f</u>.

Observe that in the definition of category it has been assumed that there is at hand a set theory, and that this set theory is of the Gödel-Bernays type in that there is a distinction between the notion of class and the notion of set, and that a set or class is specified when its members or elements are known.

The notion of category could have been defined somewhat differently as follows: A category $\mathcal{X}$ consists of two classes obj($\mathcal{X}$) (the class of objects in $\mathcal{X}$) and mor($\mathcal{X}$) (the class of morphisms in $\mathcal{X}$). For each morphism f in $\mathcal{X}$, there is specified two objects of $\mathcal{X}$, one called the domain of f and one called the range of f. The symbolism $f: X' \longrightarrow X''$ indicates that f is a morphism with domain X' and range X''. If $f': X' \longrightarrow X$, $f'': X \longrightarrow X''$ are morphisms in $\mathcal{X}$, there is defined a morphism $f'' \circ f': X' \longrightarrow X''$ in $\mathcal{X}$ called the composite of f'' with f'. If X is an object of $\mathcal{X}$, there is specified a morphism $1_X = X \longrightarrow X$. The preceding elements of structure of the category $\mathcal{X}$ are subject to the following conditions:

1) if $f: X' \longrightarrow X''$ is a morphism in $\mathcal{X}$, then $1_{X''} \circ f = f$, and $f \circ 1_{X'} = f$, and

2) if $f_j : X_j \longrightarrow X_{j+1}$ is a morphism in $\mathcal{X}$ for $j = 1,2,3$, then $f_3 \circ (f_2 \circ f_1) = f_3 \circ f_2) \circ f_1$.

The definition of category given above almost coincides with that given in (1.1). The difference is that in the situation above, if when (X',X'') is an ordered pair of objects of $\mathcal{X}$, there is no guarantee that the class $\mathcal{X}(X',X'')$ of morphisms in $\mathcal{X}$ with domain X' and range X'' is a set and not just a class. If one imposes the additional condition on the conditions 1) and 2) above that for every ordered pair (X',X'') of objects of $\mathcal{X}$, the class $\mathcal{X}(X',X'')$ of morphisms in $\mathcal{X}$ with domain X' and range X'' is a set then a definition of category equivalent with (1.1) has been given.

In this work, it will always be assumed that in a category $\mathcal{X}$, the class of morphisms between any two objects is a set. This hypothesis is made in spite of the fact that some authors have recently found the old and more general definition of category suitable for their purposes. For problems of the type considered in this work, the more general definition is cumbersome and not needed.

<u>1.2 Example</u>. The category $\mathcal{S}$ of sets and functions.

An object of $\mathcal{S}$ is a set. If X', X'' are sets, then an element $f \in \mathcal{S}(X', X'')$ is a function from the set X' to the set X''. If $f': X' \longrightarrow X$, $f'': X \longrightarrow X''$ are functions, then $f'' \circ f': X' \longrightarrow X''$ is the usual composite function from X' to X''.

For any function $f: X \longrightarrow Y$ if $x \in X$, the value of f at x is denoted by $f(x)$. Hence if also $g: Y \longrightarrow Z$ is a function, then $(g \circ f)(x) = g(f(x))$

If $f: X \longrightarrow Y$ is a function, then

1) f is a <u>surjection</u> or <u>surjective</u> function if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$,

2) f is an <u>injection</u> or <u>injective</u> function if whenever $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$ then $x_1 = x_2$, and

3) f is a <u>bijection</u> or <u>bijective</u> function if it is both an injection and a surjection.

<u>1.3 Definitions.</u> If $\mathcal{X}$ is a category and $f: X' \longrightarrow X''$ is a morphism in $\mathcal{X}$, then

1) f is an <u>epimorphism</u> if whenever $g, h: X'' \longrightarrow X$ are morphism in $\mathcal{X}$ such that $g \circ f = h \circ f$; then $g = h$,

2) f is a <u>monomorphism</u> if whenever $g, h: X \longrightarrow X'$ are morphism in $\mathcal{X}$ such that $f \circ g = f \circ h$, then $g = h$,

3) f is a <u>bimorphism</u> if it is both a monomorphism and an epimorphism, and

4) f is an <u>isomorphism</u> if there exists a morphism $g: X'' \longrightarrow X'$ such that $f \circ g = 1_{X''}$ and $g \circ f = 1_{X'}$.

Observe that in the category $\mathcal{S}$ of sets and functions, if $f: X' \longrightarrow X''$ is a morphism, then

1) f is an epimorphism if and only if f is a surjective function,

2) f is a monomorphism if and only if f is an injective function, and

3) the notions of bimorphism, bijection, and isomorphism coincide in the category $\mathcal{S}$.

Note that in any category $\mathcal{X}$ an isomorphism is a bimorphism.

<u>1.4 Lemma</u>. If $\mathcal{X}$ is a category, and $f':X' \longrightarrow X$, $f'': X \longrightarrow X''$ are morphisms in $\mathcal{X}$, then

1) if f' and f'' are epimorphisms, then $f'' \circ f'$ is an epimorphism,

2) if $f''\circ f'$ is an epimorphism, then f'' is an epimorphism,

3) if f' and f'' are monomorphisms, then $f''\circ f'$ is a monomorphism,

4) if $f''\circ f'$ is a monomorphism, then f' is a monomorphism,

5) if f' and f'' are isomorphisms, then $f'' \circ f'$ is an isomorphism, and

6) if $f'' \circ f'$ is an isomorphism, then f' is an isomorphism if and only if f'' is an isomorphism.

The preceding lemma follows immediately from the definitions.

<u>1.5 Definitions</u>. If $\mathcal{X}$ and $\mathcal{Y}$ are categories, then a <u>functor</u> $T: \mathcal{X} \longrightarrow \mathcal{Y}$ consists of

1) for each object X of $\mathcal{X}$, an object $T(X)$ of $\mathcal{Y}$,

2) for each morphism $f: X' \longrightarrow X''$ in $\mathcal{X}$, a morphism $T(f): T(X') \longrightarrow T(X'')$ in $\mathcal{Y}$ such that

1) if X is an object of $\mathcal{X}$, then $T(1_X) = 1_{T(X)}$, and

2) if $f': X' \longrightarrow X, f'': X \longrightarrow X''$ are morphisms in $\mathcal{X}$, then $T(f'' \circ f') = T(f'') \circ T(f')$.

If $T: \mathcal{X} \longrightarrow \mathcal{Y}$ is a functor, then

1) T <u>reflects epimorphisms</u> if whenever $f: X' \longrightarrow X''$ is a morphism in $\mathcal{X}$ such that $T(f): T(X') \longrightarrow T(X'')$ is an epimorphism in $\mathcal{Y}$, then f is an epimorphism in $\mathcal{X}$.

2) T <u>reflects monomorphisms</u> if whenever $f: X' \longrightarrow X''$ is a morphism in $\mathcal{X}$ such that $T(f): T(X') \longrightarrow T(X'')$ is a monomorphism in $\mathcal{Y}$, then f is a monomorphism in $\mathcal{X}$.

3) T is <u>faithful</u> if whenever $f_1, f_2 \colon X' \longrightarrow X''$ are morphisms in $\mathcal{X}$ such that $T(f_1) = T(f_2) \colon T(X') \longrightarrow T(X'')$, then $f_1 = f_2$.

<u>1.6 Proposition</u>. If $T \colon \mathcal{X} \longrightarrow \mathcal{Y}$ is a faithful functor, then T reflects epimorphisms and monomorphisms.

<u>Proof</u>. Suppose $f \colon X' \longrightarrow X''$ is a morphism in $\mathcal{X}$ such that $T(f)$ is an epimorphism. If $g,h \colon X'' \longrightarrow X$ are morphisms in $\mathcal{X}$ such that $g \circ f = h \circ f$, then $T(g) \circ T(f) = T(g \circ f) = T(h \circ f) = T(h) \circ T(f)$, and $T(g) = T(h)$ since $T(f)$ is an epimorphism. Now $g = h$ since T is faithful, and it follows that f is an epimorphism. Then T reflects epimorphisms. The fact that T reflects monomorphisms is established similarly.

<u>1.7 Examples</u>. The category $\mathcal{T}$ of spaces and maps.

The objects of $\mathcal{T}$ are topological spaces. They are called merely spaces. If X', X'' are spaces, a morphism $f \colon X' \longrightarrow X''$ in $\mathcal{T}$ is a continuous function from X' to X''. Morphisms in $\mathcal{T}$ are called maps. Composition of maps in $\mathcal{T}$ is induced by composition of functions.

If X is a space, let $S(X)$ denote the underlying set of X. If $f \colon X' \longrightarrow X''$ is a map, let $S(f) \colon S(X') \longrightarrow S(X'')$ denote the underlying function of f. Now $S(\ ) \colon \mathcal{T} \longrightarrow \mathcal{S}$ is a functor. Evidently it is a faithful functor.

<u>1.8 Definitions</u>. Let $\mathfrak{X}$ be a category and $T: \mathfrak{X} \to \mathcal{S}$ a faithful functor.

If $f: X' \to X''$ is a morphism in $\mathfrak{X}$, then

1) f is <u>surjective relative to T</u> if T(f) is surjective,

2) f is <u>injective relative to T</u> if T(f) is injective, and

3) f is <u>bijective relative to T</u> if T(f) is bijective.

In the definitions above the words relative to T are dropped when T is clear from the context. In particular when dealing with the category $\mathcal{T}$, the canonical functor $S: \mathcal{T} \to \mathcal{S}$ of (1.7) is always understood. Hence the notions of surjection, injection, and bijection are defined in $\mathcal{T}$.

<u>1.9 Definitions</u>. If $\mathfrak{X}$ is a category, then a <u>subcategory $\mathfrak{X}'$ of $\mathfrak{X}$</u> is a category $\mathfrak{X}'$ such that

1) the class obj($\mathfrak{X}'$) is a subclass of obj($\mathfrak{X}$),

2) if X_1, X_2 are objects of $\mathfrak{X}'$, then $\mathfrak{X}'(X_1, X_2)$ is a subset of $\mathfrak{X}(X_1, X_2)$, and

3) composition of morphisms in $\mathfrak{X}'$ conncides with composition of morphisms in $\mathfrak{X}$, and if X is an object of $\mathfrak{X}'$ and 1_X is the identity morphism of X in $\mathfrak{X}$, then 1_X is a morphism in $\mathfrak{X}'$.

The subcategory $\mathfrak{X}'$ of $\mathfrak{X}$ is a <u>full subcategory of $\mathfrak{X}$</u> if whenever X_1, X_2 are objects of $\mathfrak{X}'$, then

$\mathcal{X}'(X_1,X_2) = \mathcal{X}(X_1,X_2)$.

If $\mathcal{X}'$ is a subcategory of $\mathcal{X}$, the <u>inclusion functor</u> $I: \mathcal{X}' \longrightarrow \mathcal{X}$ is the functor such that if X is an object of $\mathcal{X}$, then $I(X) = X$, and if f is a morphism in $\mathcal{X}'$, then $I(f) = f$.

Observe that condition 3) above is equivalent to the inclusion from $\mathcal{X}'$ to $\mathcal{X}$ being a functor.

In order to specify a full subcategory $\mathcal{X}'$ of $\mathcal{X}$ it suffices to specify which objects of $\mathcal{X}$ are objects of $\mathcal{X}'$. Given a class of objects $\mathcal{O}$ in $\mathcal{X}$, the full subcategory of $\mathcal{X}$ generated by $\mathcal{O}$ is that subcategory $\mathcal{X}'$ such that $obj(\mathcal{X}') = \mathcal{O}$.

<u>1.10 Examples</u>. Let $\mathcal{T}_{CL}$ be the subcategory of $\mathcal{T}$ having the same objects as $\mathcal{T}$, and such that if $f:X' \longrightarrow X''$ is a map in $\mathcal{T}$, then $f \in \mathcal{T}_{CL}(X',X'')$ if and only if whenever A' is a closed subspace of X' the image of A' under f is a closed subspace of X''. The category $\mathcal{T}_{CL}$ is the category of spaces and closed maps. It is a subcategory of $\mathcal{T}$, but not a full subcategory.

Let $\mathcal{T}_{OP}$ be the subcategory of $\mathcal{T}$ having the same objects as $\mathcal{T}$, and such that if $f:X' \longrightarrow X''$ is a map in $\mathcal{T}$, then $f \in \mathcal{T}_{OP}(X',X'')$ if and only if whenever U' is an open subspace of X' the image of U' under f is an open subspace of X". The category $\mathcal{T}_{OP}$ is the category of spaces and open maps. It is not a full subcategory of $\mathcal{T}$.

A space X is a Hausdorff space, or a separated space if whenever x_0, x_1 are distinct points of X there exist disjoint open subsets U_0, U_1 of X such that $x_0 \in U_0$ and $x_1 \in U_1$. The category of Hausdorff spaces and maps in the full subcategory $\mathcal{T}_H$ of $\mathcal{T}$ generated by the Hausdorff spaces.

<u>1.11 Definition.</u> If $\mathfrak{X}$ is a category, the <u>dual category</u> of $\mathfrak{X}$ is the category $\mathfrak{X}^*$ defined by the following conditions:

1) $\text{obj}(\mathfrak{X}^*) = \text{obj}(\mathfrak{X})$, however when an object X of $\mathfrak{X}$ is considered as an object of $\mathfrak{X}^*$ it is denoted by X^*,

2) if X and Y are objects of $\mathfrak{X}$, then $\mathfrak{X}^*(X^*, Y^*) = \mathfrak{X}(Y,X)$, and if $f: Y \longrightarrow$ is a morphism in $\mathfrak{X}$, the corresponding morphism in $\mathfrak{X}^*$ is denoted by $f^*: X^* \longrightarrow Y^*$, and

3) if $f^*: X^* \longrightarrow Y^*$, $g^*: Y^* \longrightarrow Z^*$ are morphisms in $\mathfrak{X}^*$; then $g^* \circ f^* = (f \circ g)^*: X^* \longrightarrow Z^*$.

Observe that if $\mathfrak{X}'$ is a subcategory of $\mathfrak{X}$, then $(\mathfrak{X}')^*$ is a subcategory of $\mathfrak{X}^*$.

<u>1.12 Definitions.</u> Let $\mathfrak{X}$ be a category and $\mathfrak{X}'$ a subcategory of $\mathfrak{X}$.

If X is an object of $\mathcal{X}$, a <u>reflection of X in $\mathcal{X}'$</u> is an object $R(X)$ of $\mathcal{X}'$ and a morphism $\rho(X): R(X) \longrightarrow X$ in $\mathcal{X}$ such that if X' is an object of $\mathcal{X}'$ and $f: X' \longrightarrow X$ is a morphism in $\mathcal{X}$, then there is a unique morphism $\bar{f}: X' \longrightarrow R(X)$ in $\mathcal{X}'$ such that $\rho(X) \circ \bar{f} = f$. The subcategory $\mathcal{X}'$ of $\mathcal{X}$ is a <u>reflective subcategory</u> if every object of $\mathcal{X}$ has a reflection in $\mathcal{X}'$.

If X is an object of $\mathcal{X}$, a <u>coreflection of X in $\mathcal{X}'$</u> is an object $R(X)$ of $\mathcal{X}'$ and a morphism $\lambda(X): X \longrightarrow R(X)$ in $\mathcal{X}$ such that if X' is an object of $\mathcal{X}'$ and $f: X \longrightarrow X'$ is a morphism in $\mathcal{X}$, then there is a unique morphism $\bar{f}: R(X) \longrightarrow X'$ in $\mathcal{X}'$ such that $\bar{f} \circ \lambda(X) = f$. The subcategory $\mathcal{X}'$ of $\mathcal{X}$ is a <u>coreflective subcategory</u> if every object of $\mathcal{X}$ has a coreflection in $\mathcal{X}'$.

<u>1.13 Proposition</u>. If $\mathcal{X}$ is a category and $\mathcal{X}'$ is a subcategory of $\mathcal{X}$, then

1) if X is an object of $\mathcal{X}$, $R(X)$ is an object of $\mathcal{X}'$, and $\rho(X): R(X) \longrightarrow X$ is a morphism in $\mathcal{X}$, then $\rho(X)$ is a reflection of X in $\mathcal{X}'$ if and only if $\rho(X)^*: X^* \longrightarrow R(X)^*$ is a coreflection of X^* in $(\mathcal{X}')^*$.

2) $\mathcal{X}'$ is a reflective subcategory of $\mathcal{X}$ if and only if $(\mathcal{X}')^*$ is a coreflective subcategory of $\mathcal{X}^*$.

The proposition follows at once from the definitions.

<u>1.14 Proposition</u>. If $\mathfrak{X}$ is a category and $\mathfrak{X}'$ is a subcategory of $\mathfrak{X}$, then

1) if X is an object of $\mathfrak{X}$, and $\rho(X):R(X) \longrightarrow X$, $\rho_1(X): R_1(X) \longrightarrow X$ are reflections of X in $\mathfrak{X}'$ there is a unique isomorphism $U: R_1(X) \longrightarrow R(X)$ in $\mathfrak{X}'$ such that $\rho(X) \circ U = \rho_1(X)$, and

2) if $\mathfrak{X}'$ is a full subcategory of $\mathfrak{X}$, and X is an object of $\mathfrak{X}'$, then $1_X = X \longrightarrow X$ is a reflection of X in $\mathfrak{X}'$.

<u>Proof</u>. The definition of reflection of X in $\mathfrak{X}'$ implies there exist unique morphisms $U:R_1(X) \longrightarrow R(X)$, $U_1:R(X) \longrightarrow R_1(X)$ such that $\rho(X) \circ U = \rho_1(X)$, $\rho_1(X) \circ U_1 = \rho(X)$. Hence $\rho(X) = \rho_1(X) \circ U_1 = \rho(X) \circ U \circ U_1$, but $1_{R(X)}$ is the unique morphism in $\mathfrak{X}'$ such that $\rho(X) \circ 1_{R(X)} = \rho(X)$. Thus $U \circ U_1 = 1_{R(X)}$, $U_1 \circ U = 1_{R_1(X)}$, and part 1) of the proposition is proved. Since part 2) follows at once from the definitions, the proposition is proved.

<u>1.15 Proposition.</u> If $\mathfrak{X}$ is a category, $f: X \longrightarrow Y$ is a morphism in $\mathfrak{X}$ with dual $f^* = Y^* \longrightarrow X^*$ in $\mathfrak{X}^*$, then

1) f is a monomorphism if and only if f^* is an epimorphism, and

2) f is an isomorphism if and only if f^* is an isomorphism.

The proposition follows at once from the definitions.

<u>1.16 Definitions</u>. If $\mathcal{X}$ is a category and $f: X \longrightarrow Y$ is a morphism in $\mathcal{X}$, then

1) f is a <u>retract</u> if there exists $g: Y \longrightarrow X$ such that $g \circ f = 1_X$, and

2) f is a <u>coretract</u> if there exists $g: Y \longrightarrow X$ such that $f \circ g = 1_Y$. eg $A \oplus B \to A$

<u>1.17 Proposition</u>. If $\mathcal{X}$ is a category and $f: X \longrightarrow Y$ is a morphism in $\mathcal{X}$, then

1) f is a retract if and only if $f^*: Y^* \longrightarrow X^*$ is a coretract in $\mathcal{X}^*$, and

2) if f is a retract, then f is a monomorphism, and if further f is an epimorphism, then f is an isomorphism.

<u>Proof</u>. Assuming f to be a retract choose $g: Y \longrightarrow X$ such that $g \circ f = 1_X$. If $h_1, h_2: X_0 \longrightarrow X$ are such that $f \circ h_1 = f \circ h_2$, then $h_1 = 1_X \circ h_1 = g \circ f \circ h_1 = g \circ f \circ h_2 = 1_X \circ h_2 = h_2$. Thus f is a monomorphism. If f is an epimorphism, then $f = f \circ 1_X = (f \circ g) \circ f$. Thus $f \circ g = 1_Y$ and part 2) of the proposition is proved. Since part 1) follows at once from the definitions, the proposition itself is also proved.

Observe that the dual of part 2) of the proposition above is that a coretract is an epimorphism, and that a coretract which is a monomorphism is an isomorphism.

<u>Convention</u>. When convenient, the dual of a lemma, proposition, or theorem will be referred to by the notation of the original followed by the duality symbol (*).

<u>1.18 Definitions</u>. Let $\mathcal{X}$ be a category.

An object P of $\mathcal{X}$ is <u>projective</u> if whenever

$$\begin{array}{ccc} & & X \\ & & \downarrow f \\ P & \longrightarrow & Y \end{array}$$

is a diagram in $\mathcal{X}$ with f an epimorphism, there exists $\bar{g}: P \longrightarrow X$ such that $f \circ \bar{g} = g$.

An object I of $\mathcal{X}$ is <u>injective</u> if whenever

$$\begin{array}{ccc} Y & \xrightarrow{h} & I \\ \downarrow f & & \\ X & & \end{array}$$

is a diagram in $\mathcal{X}$ with f a monomorphism, there exists $\bar{h}: X \longrightarrow I$ such that $\bar{h} \circ f = h$.

<u>1.19 Proposition</u>. If $\mathcal{X}$ is a category, then

1) the object P of $\mathcal{X}$ is projective if and only if the object P^* of $\mathcal{X}^*$ is injective,

2) if $f: P \longrightarrow X$ is a coretract in $\mathcal{X}$ and P is projective, then X is projective, and   Dir. summands of proj's are proj.

3) if $f: X \longrightarrow P$ is an epimorphism in $\mathcal{X}$ and P is projective, then f is a coretract.   ie, $X \to P \to 0$ splits.

Proof. Part 1) follows at once from the definitions. Suppose the conditions of part 2) obtain, and

$$\begin{array}{ccc} & & Y' \\ & & \downarrow h \\ X & \xrightarrow{g} & Y'' \end{array}$$

is a diagram in $\mathcal{X}$ with h an epimorphism. Choose $v: X \longrightarrow P$ such that $f \circ v = 1_X$. Let $\hat{g}: P \longrightarrow Y'$ be a morphism such that $h \circ \hat{g} = g \circ f$. Let $\bar{g}: X \longrightarrow Y'$ be $\hat{g} \circ v$. Now $h \circ \bar{g} = h \circ \hat{g} \circ v = g \circ f \circ v = g$, and part 2) is proved.

Suppose the conditions of part 3) obtain. Now

$$\begin{array}{ccc} & & X \\ & & \downarrow f \\ P & \xrightarrow{1_P} & P \end{array}$$

is a diagram in $\mathcal{X}$ with f an epimorphism. Since P is projective there exists $g: P \longrightarrow X$ such that $f \circ g = 1_P$. This proves part 3), and hence the proposition.

1.20 Proposition. If $T: \mathcal{X} \longrightarrow \mathcal{Y}$ is a functor, and if the following conditions are satisfied:

1) for X^* an object of $\mathcal{X}^*$, $T^*(X^*) = T(X)^*$, and

2) if $f^*: X^* \longrightarrow Y^*$ a morphism in $\mathcal{X}^*$, $T^*(f^*) = T(f)^*$, then $T^*: \mathcal{X}^* \longrightarrow \mathcal{Y}^*$ is a functor and it is faithful if T is faithful.

The proposition follows at once from the definition. The functor T^* is the <u>dual of the functor T</u>.

Note that if $\mathcal{X}$ is a category, then $(\mathcal{X}^*)^* = \mathcal{X}$, and if T is a functor, then $(T^*)^* = T$.

Exercises.

1. In the category $\mathcal{T}$ of spaces and maps prove that every epimorphism is surjective and every monomorphism is injective. Give an example of a bijection which is not an isomorphism.

2. In the category $\mathcal{T}_H$ (1.10), prove that every monomorphism is injective and give an example of an epimorphism which is not surjective. Prove that a map $f:X \longrightarrow Y$ in $\mathcal{T}_H$ is an epimorphism if and only if the closure of the set theoretic image of f in Y is Y, i.e. $\overline{f(X)} = Y$.

3) Prove that in the category $\mathcal{S}$, every object is projective and every non-empty object is injective. Prove that every epimorphism is a coretract and every monomorphism with non-empty domain is a retract.

4) Prove that for a category $\mathcal{X}$, the following

conditions are equivalent:

1) every epimorphism in $\mathcal{X}$ is a coretract,

2) every object in $\mathcal{X}$ is projective.

5. Prove that a space X is a projective in $\mathcal{T}$ if and only if its topology is discrete, i.e. every subspace of X is open in X. Prove that the full subcategory of $\mathcal{T}$ generated by the discrete spaces is a reflective subcategory isomorphic with the category $\mathcal{S}$.

6. Prove that a space X is an injective in $\mathcal{T}$ if and only if it is non-empty and its topology is trivial, i.e. the only open subspaces of X are the empty subspace and the entire space X. Prove that the full subcategory of $\mathcal{T}$ generated by the spaces with trivial topology is a coreflective subcategory of $\mathcal{T}$ isomorphic with the category $\mathcal{S}$.

7. Prove that in the category $\mathcal{T}_H$ a space X is injective if and only if its underlying set has exactly one point. Hint: Prove that if J is an injective in $\mathcal{T}_H$ whose underlying set has more than one point, then the notion of surjective map and epimorphism in $\mathcal{T}_H$ must coincide. Since this is not the case, no such J exists.

8. A space X is totally disconnected if whenever X_1, X_2 are distinct points of X there exist disjoint closed subsets F_1, F_2 of X whose union is X and such that $X_1 \in F_1$, $X_2 \in F_2$. Prove that the full subcategory of $\mathcal{T}$ generated by the totally disconnected spaces is a coreflective subcategory of $\mathcal{T}$.

§2. Some considerations concerning categories and functors.

__2.1 Definition__. If $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ are categories and $S: \mathfrak{Y} \longrightarrow \mathfrak{Z}$, $T: \mathfrak{X} \longrightarrow \mathfrak{Y}$ are functors, then $S \circ T: \mathfrak{X} \longrightarrow \mathfrak{Z}$ is defined by

1) $(S \circ T)(X) = S(T(X))$ for X an object of $\mathfrak{X}$, and
2) $(S \circ T)(f) = S(T(f))$ for f a morphism in $\mathfrak{X}$.

__2.2 Proposition__. If $T_j: \mathfrak{X}_j \longrightarrow \mathfrak{X}_{j+1}$ is a functor for $j = 1,2,3,$ then

1) $T_{j+1} \circ T_j: \mathfrak{X}_j \longrightarrow \mathfrak{X}_{j+2}$ is a functor for $j = 1,2,$ and
2) $T_3 \circ (T_2 \circ T_1) = (T_3 \circ T_2) \circ T_1$.

The proposition follows at once from the definitions.

If $S: \mathfrak{Y} \longrightarrow \mathfrak{Z}$, $T = \mathfrak{X} \longrightarrow \mathfrak{Y}$ are functors, then the functor $(S \circ T): \mathfrak{X} \longrightarrow \mathfrak{Z}$ is the __composite functor of S and T__.

For any category $\mathfrak{X}$, the __identity functor of $\mathfrak{X}$__ is the functor $1_{\mathfrak{X}}: \mathfrak{X} \longrightarrow \mathfrak{X}$ such that $1_{\mathfrak{X}}(X) = X$ for X an object of $\mathfrak{X}$ and $1_{\mathfrak{X}}(f) = f$ for f a morphism in $\mathfrak{X}$.

__2.3 Proposition__. If $S: \mathfrak{Y} \longrightarrow \mathfrak{Z}$, $T: \mathfrak{X} \longrightarrow \mathfrak{Y}$ are functors, then

1) $T \circ 1_{\mathfrak{X}} = T$, $1_{\mathfrak{Y}} \circ T = T$,
2) if S and T are faithful, then $S \circ T$ is faithful, and
3) if $S \circ T$ is faithful, then T is faithful.

The proposition is evident from the definitions.

<u>2.4 Definition.</u> If $S, T: \mathcal{X} \longrightarrow \mathcal{Y}$ are functors, then a <u>morphism of functors</u> $\alpha: S \longrightarrow T$ is for each object X of $\mathcal{X}$ a morphism $\alpha(X): S(X) \longrightarrow T(X)$ in $\mathcal{Y}$ such that if $f: X_1 \longrightarrow X_2$ is a morphism in $\mathcal{X}$, then $\alpha(X_2) \circ S(f) = T(f) \circ \alpha(X_1)$.

The preceding is equivalent to saying that if $f: X_1 \longrightarrow X_2$ is a morphism in $\mathcal{X}$, then the diagram

$$\begin{array}{ccc} S(X_1) & \xrightarrow{S(f)} & S(X_2) \\ \Big\downarrow \alpha(X_1) & & \Big\downarrow \alpha(X_2) \\ T(X_1) & \xrightarrow{T(f)} & T(X_2) \end{array}$$

in $\mathcal{Y}$ is commutative.

<u>2.5 Proposition.</u> If $T_j: \mathcal{X} \longrightarrow \mathcal{Y}$ is a functor for $j=1,2,3,4$, and $\alpha_j: T_j \text{—} T_{j+1}$ is a morphism for $j=1,2,3$, then

1) if $\alpha_{j+1} \circ \alpha_j: T_j \longrightarrow T_{j+2}$ is defined by $(\alpha_{j+1} \circ \alpha_j)(X) = \alpha_{j+1}(X) \circ \alpha_j(X)$ for $j=1,2$ and X an object of $\mathcal{X}$, then $\alpha_{j+1} \circ \alpha_j$ is a morphism for $j=1,2$, and

2) $\alpha_3 \circ (\alpha_2 \circ \alpha_1) = (\alpha_3 \circ \alpha_2) \circ \alpha_1$.

Details of the proof of this routine proposition are left to the reader.

If $T', T, T'': \mathcal{X} \longrightarrow \mathcal{Y}$ are functors, $\alpha' = T' \longrightarrow T$, $\alpha'': T \longrightarrow T''$ are morphisms then the morphism $\alpha'' \circ \alpha': T' \longrightarrow T''$ is the <u>composite of α' and α''</u>.

A morphism of functors was classically called a <u>natural transformation of functors</u>. However, in the sequel the terminology morphism or morphism of functors will be used.

If $T: \mathcal{X} \longrightarrow \mathcal{Y}$ is a functor, the identity morphism of T, $1_T: T \longrightarrow T$ is the morphism such that for $X \in obj(\mathcal{X})$ $1_T(X) = 1_{T(X)}$. If $S,T: \mathcal{X} \longrightarrow \mathcal{Y}$ are functors and $\alpha: S \longrightarrow T$ is a morphism, then $\alpha \circ 1_S = \alpha$ and $1_T \circ \alpha = \alpha$.

<u>2.6 Proposition</u>. If $\mathcal{X}$ is a category, $\mathcal{X}'$ is a subcategory of $\mathcal{X}$, and $I: \mathcal{X}' \longrightarrow \mathcal{X}$ is the natural inclusion functor, then the following are equivalent:

1) $\mathcal{X}'$ is a reflective subcategory of $\mathcal{X}$, and

2) there is a functor $R: \mathcal{X} \longrightarrow \mathcal{X}'$ and a morphism $\rho: I \circ R \longrightarrow 1_{\mathcal{X}}$ such that if X is an object of $\mathcal{X}$, then $\rho(X): R(X) \longrightarrow X$ is a reflection of X in $\mathcal{X}'$.

<u>Proof</u>. Suppose 1). For each object X of $\mathcal{X}$ choose a reflection of X in $\mathcal{X}'$, $\rho(X): R(X) \longrightarrow X$. If $f: X_1 \longrightarrow X_2$ is a morphism in $\mathcal{X}$, let $R(f): R(X_1) \longrightarrow R(X_2)$ be the unique morphism in $\mathcal{X}'$ such that $\rho(X_2) \circ R(f) = f \circ \rho(X_1)$. Now using the uniqueness property of reflections, it follows that $R(\ ): \mathcal{X} \longrightarrow \mathcal{X}'$ is a functor, and $\rho: I \circ R \longrightarrow 1_{\mathcal{X}}$ is a morphism. Thus 1) implies 2). Since the fact that 2) implies 1) is evident, the proposition is proved.

<u>2.7 Definitions</u>. If $S, T: \mathfrak{X} \longrightarrow \mathcal{Y}$ are functors, and $\alpha: S \longrightarrow T$ is a morphism, then

1) α is a <u>local epimorphism</u> if for every object X of $\mathfrak{X}$, $\alpha(X): S(X) \longrightarrow T(X)$ is an epimorphism in $\mathcal{Y}$,

2) α is a <u>local monomorphism</u> if for every object X of $\mathfrak{X}$, $\alpha(X): S(X) \longrightarrow T(X)$ is a monomorphism in $\mathcal{Y}$,

3) α is an <u>epimorphism</u> if whenever $T'': \mathfrak{X} \longrightarrow \mathcal{Y}$ is a functor and $\lambda_1, \lambda_2: T \longrightarrow T''$ are morphisms such that $\lambda_1 \circ \alpha = \lambda_2 \circ \alpha$, then $\lambda_1 = \lambda_2$, and

4) α is a <u>monomorphism</u> if whenever $S': \mathfrak{X} \longrightarrow \mathcal{Y}$ is a functor and $\rho_1, \rho_2: S' \longrightarrow S$ are morphisms such that $\alpha \circ \rho_1 = \alpha \circ \rho_2$, then $\rho_1 = \rho_2$.

Define $\alpha^*: T^* \longrightarrow S^*$ by $\alpha^*(X^*) = \alpha(X)^*$ for X an object of $\mathfrak{X}$.

<u>2.8 Proposition</u>. If $S, T: \mathfrak{X} \longrightarrow \mathcal{Y}$ are functors and $\alpha: S \longrightarrow T$ is a morphism, then

1) $\alpha^*: T^* \longrightarrow S^*$ is a morphism,

2) α is a local epimorphism if and only if α^* is a local monomorphism,

3) α is an epimorphism if and only if α^* is a monomorphism, and

4) if α is a local epimorphism, then α is an epimorphism.

Parts 1), 2), and 3) of the proposition follow at once from the definitions. As for part 4), suppose $T'': \mathfrak{X} \longrightarrow \mathcal{Y}$ is a functor and $\lambda_1, \lambda_2: T \longrightarrow T''$ are morphisms such that $\lambda_1 \circ \alpha = \lambda_2 \circ \alpha$. If α is a local epimorphism, then for every object X of $\mathfrak{X}$, $\lambda_1(X) \circ \alpha(X) = \lambda_2(X) \circ \alpha(X)$, and $\lambda_1(X) = \lambda_2(X)$ since $\alpha(X)$ is an epimorphism. Hence $\lambda_1 = \lambda_2$, and part 4) is proved.

There is no need to distinguish between the notion of local isomorphisms of functors and ieomorphisms of functors. Indeed using the notation above, suppose $\alpha(X): S(X) \longrightarrow T(X)$ is an isomorphism for every object X of $\mathfrak{X}$. Let $\beta(X): T(X) \longrightarrow S(X)$ the inverse isomorphism of $\alpha(X)$. Now $\beta: T \longrightarrow S$ is a morphism, $\alpha \circ \beta = 1_T$, and $\beta \circ \alpha = 1_S$.

<u>2.9 Proposition</u>. If $S, T: \mathfrak{X} \longrightarrow \mathcal{Y}$ are functors and $\alpha: S \longrightarrow T$ a morphism, then

1) if α is a local epimorphism and T is faithful, then S is faithful, and

2) if α is a local monomorphism and S is faithful, then T is faithful.

<u>Proof</u>. Suppose the conditions of 1) obtain, and $f_1, f_2: X' \longrightarrow X''$ are such that $S(f_1) = S(f_2)$. Now $T(f_1) \circ \alpha(X') = \alpha(X'') \circ S(f_1) = \alpha(X'') \circ S(f_2) = T(f_2) \circ \alpha(X')$, and since $\alpha(X')$ is an epimor-

phism $T(f_1) = T(f_2)$. Since T is faithful $f_1 = f_2$, and 1) follows. Part 2) of the proposition follows similarly, or indeed by duality.

<u>2.10 Definitions</u>. Suppose $T', T'': \mathfrak{X} \longrightarrow \mathcal{Y}$ are functors, and $\alpha: T' \longrightarrow T''$ is a morphism.

If $S: \mathfrak{X}' \longrightarrow \mathfrak{X}$ is a functor, then $\alpha S: T' \circ S \longrightarrow T'' \circ S$ is defined by $(\alpha S)(X') = \alpha(S(X'))$ for X' an object of $\mathfrak{X}'$.

If $S: \mathcal{Y} \longrightarrow \mathcal{Y}''$ is a functor, then $S\alpha: S \circ T' \longrightarrow S \circ T''$ is defined by $(S\alpha)(X) = S(\alpha(X))$ for X an object of $\mathfrak{X}$.

<u>2.11 Definitions</u>. An <u>adjoint pair of functors</u> consists of

1) functors $T: \mathfrak{X} \longrightarrow \mathcal{Y}$, $S: \mathcal{Y} \longrightarrow \mathfrak{X}$, and
2) morphisms $\alpha: S \circ T \longrightarrow 1_{\mathfrak{X}}$, $\beta: 1_{\mathcal{Y}} \longrightarrow T \circ S$ such that
   1) $T\alpha \circ \beta T = 1_T$, and
   2) $\alpha S \circ S_\beta = 1_S$.

The adjoint pair of functors is denoted by $(\alpha, \beta): S \dashv T : (\mathfrak{X}, \mathcal{Y})$. This notation is shortened to $(\alpha, \beta): S \dashv T$, or to $S \dashv T$ when the other elements of the notation are clear from the context.

In the situation above the functor T is the <u>adjoint</u> (right) of the adjoint pair, and the functor S is the <u>coadjoint</u> of (left) the adjoint pair.

<u>2.12 Example</u>. Let $D: \mathcal{S} \longrightarrow \mathcal{T}$ be the functor which assigns

to every set X the space $D(X)$ with underlying set X and the discrete topology, and to every function $f:X' \longrightarrow X''$ the map $D(f): D(X') \longrightarrow D(X'')$ with underlying function f. Let $S:\mathcal{T} \longrightarrow \mathcal{S}$ be the canonical faithful functor (1.7). For X a space let $\alpha(X): (D\circ S)(X) \longrightarrow X$ be the unique map which is the identity on underlying sets. Note that $S\circ D = 1_{\mathcal{S}}$ and let β be the identity morphism of $1_{\mathcal{S}}$. Now $(\alpha,\beta):D \dashv S:(\mathcal{T},\mathcal{S})$ is an adjoint pair of functors with adjoint S and coadjoint D.

<u>2.13 Proposition</u>. If $T: \mathfrak{X} \longrightarrow \mathcal{Y}$, $S: \mathcal{Y} \longrightarrow \mathfrak{X}$ are functors and $\alpha: S\circ T \longrightarrow 1_{\mathfrak{X}}$, $\beta:1_{\mathcal{Y}} \longrightarrow T\circ S$ morphisms, then (α,β): $S \dashv T: (\mathfrak{X}, \mathcal{Y})$ is an adjoint pair if and only if (β^*,α^*): $T^* \dashv S^*: (\mathcal{Y}^*, \mathfrak{X}^*)$ is an adjoint pair.

The proposition follows at once from the definitions. Observe that T is the adjoint of the first adjoint pair above while its dual T^* is the coadjoint of the second adjoint pair.

<u>2.14 Definitions</u>. The functor $T: \mathfrak{X} \longrightarrow \mathcal{Y}$ is an <u>adjoint functor</u> if there exists an adjoint pair $(\alpha,\beta):S \dashv T:(\mathfrak{X}, \mathcal{Y})$. The functor S of the adjoint pair is a <u>coadjoint for T</u>.

The functor $S: \mathcal{Y} \longrightarrow \mathfrak{X}$ is a <u>coadjoint functor</u> if there exists an adjoint pair $(\alpha,\beta):S \dashv T:(\mathfrak{X}, \mathcal{Y})$. The functor T of the adjoint pair is an <u>adjoint for S</u>.

Observe that $T: \mathfrak{X} \longrightarrow \mathcal{Y}$ is an adjoint functor if and only if $T^*: \mathfrak{X}^* \longrightarrow \mathcal{Y}^*$ is a coadjoint functor and that $S: \mathcal{Y} \longrightarrow \mathfrak{X}$ is a coadjoint for T if and only if $S^*: \mathcal{Y}^* \longrightarrow \mathfrak{X}^*$ is an adjoint for T^*.

<u>2.15 Proposition</u>. If $\mathfrak{X}$ is a category and $\mathfrak{X}'$ a subcategory of $\mathfrak{X}$, then the following are equivalent:

1) $\mathfrak{X}'$ is a reflective subcategory of $\mathfrak{X}$, and

2) the natural inclusion functor $I: \mathfrak{X}' \longrightarrow \mathfrak{X}$ is a coadjoint functor.

<u>Proof</u>. Suppose 1) and let $R: \mathfrak{X} \longrightarrow \mathfrak{X}'$, $\rho: I \circ R \longrightarrow 1$ be as in 2.6, 2). For X an object of $\mathfrak{X}'$, let $\beta(X): X \longrightarrow RI(X)$ be the unique morphism in $\mathfrak{X}'$ such that $\rho(I(X))\ \beta(X) = 1_X$. Now $\beta: 1_{\mathfrak{X}} \longrightarrow RI$ is a morphism and $(\rho, \beta): I \longrightarrow R: (\mathfrak{X}, \mathfrak{X}')$ is an adjoint pair of functors. Hence 1) implies 2). Using 2.6 it is evident that 2) implies 1), and the proposition is proved.

<u>2.16 Definitions</u>. Let $T: \mathfrak{X} \longrightarrow \mathcal{Y}$ be a functor.

The <u>functor</u> T <u>preserves epimorphisms</u> if whenever f is an epimorphism in $\mathfrak{X}$, then $T(f)$ is an epimorphism in $\mathcal{Y}$.

The <u>functor</u> T <u>preserves monomorphisms</u> if whenever f is a monomorphism in $\mathfrak{X}$, then $T(f)$ is a monomorphism in $\mathcal{Y}$.

Note that T preserves epimorphisms if and only if T^* preserves monomorphisms.

<u>2.17 Proposition</u>. If $S: \mathcal{Y} \longrightarrow \mathfrak{X}$ is a coadjoint functor, then

S preserves epimorphisms.

<u>Proof</u>. Choose an adjoint pair $(\alpha,\beta):S \longrightarrow T:(\mathcal{X},\mathcal{Y})$. Suppose $g:Y' \longrightarrow Y$ is an epimorphism in $\mathcal{Y}$ and $f_1,f_2:S(Y) \longrightarrow X$ are morphisms in $\mathcal{X}$ such that $f_1\circ S(g) = f_2\circ S(g)$. Now $T(f_1)\circ\beta(Y)\circ g = T(f_1\circ S(g))\circ\beta(Y') = T(f_2\circ S(g))\circ\beta(Y') = T(f_2)\circ\beta(Y)\circ g$. Thus $T(f_1)\circ\beta(Y) = T(f_2)\circ\beta(Y)$ since g is an epimorphism. Hence $(S\circ T)(f_1)\circ S(\beta(T)) = (S\circ T)(f_2)\circ S(\beta(Y))$, and $f_1 = f_1\circ 1_{S(Y)} = f_1\circ \alpha(S(Y))\circ S(\beta(Y)) = \alpha(X)\circ(S\circ T)(f_1)\circ S(\beta(Y))$,

$$f_2 = f_2\circ 1_{S(Y)} = f_2\circ \alpha(S(Y))\circ S(\beta(Y)) = \alpha(X)\circ(S\circ T)(f_2)\circ S(\beta(Y)).$$

Then $f_1 = f_2$ and the proposition is proved.

Observe that proposition 2.17* asserts that an adjoint functor preserves monomorphisms.

<u>2.18 Proposition</u>. If $(\alpha,\beta):S \longrightarrow T:(\mathcal{X},\mathcal{Y})$ is an adjoint pair of functors, then the following are equivalent:

1) T is faithful,
2) T reflects epimorphisms, and
3) $\alpha:S\circ T \longrightarrow 1_{\mathcal{X}}$ is a local epimorphism.

<u>Proof</u>. By 1.6, 1) implies 2). Hence suppose 2). If X is an object of $\mathcal{X}$, then $1_{T(X)} = T(\alpha(X))\circ \beta(T(X))$. Thus $T(\alpha(X))$ is a coretraction and a fortiori an epimorphism. Since T reflects epimorphism, $\alpha(X)$ is an epimorphism, α is a local

epimorphism and 2) implies 3).

Suppose α is a local epimorphism. Now $1_{\mathfrak{X}}$ is certainly faithful. Hence $S \circ T$ is faithful by 2.9, and T is faithful by 2.3. Thus 3) implies 1), and the proposition is proved.

<u>2.19 Proposition</u>. If $(\alpha', \beta'): S' \dashv T': (\mathfrak{X}', \mathfrak{X})$ and $(\alpha'', \beta''): S'' \dashv T: (\mathfrak{X}, \mathfrak{X}'')$ are adjoint pairs of functors, $T = T'' \circ T'$, $S = S' \circ S''$, $\alpha = \alpha' \circ S'\alpha'' T': S \circ T \longrightarrow 1_{\mathfrak{X}'}$ and $\beta = \beta'' \circ T''\beta' S'': 1_{\mathfrak{X}''} \longrightarrow T \circ S$, then $(\alpha, \beta): S \dashv T: (\mathfrak{X}', \mathfrak{X}'')$ is an adjoint pair of functors.

The proposition follows from a short routine calculation.

<u>2.20 Corollary</u>. If $T': \mathfrak{X}' \longrightarrow \mathfrak{X}$, $T'': \mathfrak{X} \longrightarrow \mathfrak{X}''$ are adjoint functors, then $T = T'' \circ T': \mathfrak{X}' \longrightarrow \mathfrak{X}''$ is an adjoint functor.

<u>2.21 Definitions</u>. Let $S: \mathcal{Y} \longrightarrow \mathfrak{X}$ be a functor.

If X is an object of $\mathfrak{X}$, an <u>S-reflection of X</u> is an object $T(X)$ of $\mathcal{Y}$ and a morphism $\alpha(X): S(T(X)) \longrightarrow X$ in $\mathfrak{X}$ such that if $Y \in \text{obj}(\mathcal{Y})$ and $g: S(Y) \longrightarrow X$ is a morphism in $\mathfrak{X}$, then there is a unique $\bar{g}: Y \longrightarrow T(X)$ in $\mathcal{Y}$ such that $g = \alpha(X) \circ S(\bar{g})$.

If X is an object of $\mathfrak{X}$, and <u>S-coreflection of X</u> is an object $T(X)$ of $\mathcal{Y}$ and a morphism $\beta(X): X \longrightarrow S(T(X))$ in $\mathfrak{X}$ such that if $Y \in \text{obj}(\mathcal{Y})$ and $h: X \longrightarrow S(Y)$ is a morphism in $\mathcal{X}$, then there is a unique $\bar{h}: T(X) \longrightarrow Y$ in $\mathcal{Y}$ such that $h = S(\bar{h}) \circ \beta(X)$.

Observe that if $\mathcal{Y}$ is a subcategory of $\mathcal{X}$ and S is the natural inclusion functor, then an S-reflection of X is just a reflection of X in $\mathcal{Y}$.

<u>2.22 Proposition</u>. If $S: \mathcal{Y} \longrightarrow \mathcal{X}$ is a functor and $X \in \mathrm{obj}(\mathcal{X})$, then

1) if $T(X)$ is an object of $\mathcal{Y}$ and $\alpha(X): S(T(X)) \longrightarrow X$ is a morphism in $\mathcal{X}$, then $\alpha(X)$ is an S-reflection of X if and only if $\alpha(X)^*: X^* \longrightarrow S^*(T(X)^*)$ is an S^*-coreflection of X^*, and

2) if $\alpha(X): S(T(X)) \longrightarrow X$, $\alpha_1(X): S(X)) \longrightarrow X$ are S-reflections of X there is a unique isomorphism $U: T_1(X) \longrightarrow T(X)$ in $\mathcal{Y}$ such that $\alpha(X) \circ S(U) = \alpha_1(X)$.

<u>Proof</u>. Part 1) follows at once from the definitions. The definition of S-reflection of $\mathcal{X}$ implies there exist morphisms $U: T_1(X) \longrightarrow T(X)$, $U_1: T_1(X) \longrightarrow T(X)$ in $\mathcal{Y}$ such that $\alpha(X) \circ S(U) = \alpha_1(X)$ and $\alpha_1(X) = \alpha_1(X) \circ S(U_1)$. As in 1.14, $U \circ U_1 = 1_{T(X)}$, $U_1 \circ U = 1_{T_1(X)}$, and U is unique. Hence the proposition is proved.

<u>2.23 Proposition</u>. If $S: \mathcal{Y} \longrightarrow \mathcal{X}$ is a functor, then the following are equivalent:

1) every object of $\mathfrak{X}$ has an S-reflection, and

2) S is a coadjoint functor.

Proof. Suppose 1). For every object X of $\mathfrak{X}$, choose an S-reflection $\alpha(X)\colon S(T(X)) \longrightarrow X$ of X. If $f\colon X_1 \longrightarrow X_2$ is a morphism in $\mathfrak{X}$, let $T(f)\colon T(X_1) \longrightarrow T(X_2)$ be the unique morphism in $\mathcal{Y}$ such that $\alpha(X_2)\circ S(T(f)) = f\circ\alpha(X_1)$. For Y an object of $\mathcal{Y}$, let $\beta(Y)\colon Y \longrightarrow T(S(Y))$ be the unique morphism in $\mathcal{Y}$ such that $\alpha(S(Y))\circ S(\beta(Y)) = 1_{S(Y)}$. Now $(\alpha,\beta)\colon S \dashv T$ is an adjoint pair, and 1) implies 2). If $(\alpha,\beta)\colon S \dashv T$ is an adjoint pair, then $\alpha(X)\colon S(T(X)) \longrightarrow X$ is an S-reflection of X for every object X of $\mathfrak{X}$. Hence 2) implies 1) and the proposition is proved.

2.24 Proposition. If $S\colon \mathcal{Y} \longrightarrow \mathfrak{X}$ is a functor, and $(\alpha,\beta)\colon S \dashv T$, $(\alpha_1,\beta_1)\colon S \dashv T_1$ are adjoint pairs, then there is a unique isomorphism $\lambda\colon T_1 \longrightarrow T$ such that $\alpha\circ S\lambda = \lambda_1$.

Proof. Applying 2.22 and 2.23, for every object X of $\mathfrak{X}$, there is a unique morphism $\lambda(X)\colon T_1(X) \longrightarrow T(X)$ such that $\alpha(X)\circ S(\lambda(X)) = \alpha_1(X)$, $\lambda(X)$ is an isomorphism, and $\alpha\circ S\lambda = \alpha_1$. Hence the proposition is proved.

2.25 Conventional abbreviations. If $\mathfrak{X}$ is a category, and $f'\colon X' \longrightarrow X$, $f''\colon X \longrightarrow X''$ are morphism in $\mathfrak{X}$, the composite of f'' with f' will usually be denoted by $f''f'$ rather than

$f''\circ f'$. Similarly if $T'\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$, $T''\colon \mathfrak{X} \longrightarrow \mathfrak{X}''$ are functors, the composite of T'' with T' will usually be denoted by $T''T'$, or if $T', T, T''\colon \mathfrak{X} \longrightarrow \mathcal{Y}$ are functors, and $\alpha'\colon T' \longrightarrow T$, $\alpha''\colon T \longrightarrow T''$ morphisms of functors the composite of α'' with α' will ordinarily be denoted by $\alpha''\alpha'$.

Exercises

1. Prove that the canonical functor $S\colon \mathcal{T} \longrightarrow \mathcal{S}$ is a coadjoint functor, but that $S \mid \mathcal{T}_H\colon \mathcal{T}_H \longrightarrow \mathcal{S}$ is not a coadjoint functor.

2. Let $\mathcal{R}$ be the category whose objects are rings (with unit), whose morphisms are structure-preserving functions, and such that composition is induced by composition of functions. The category $\mathcal{R}$ is the category of rings and morphisms. Let $T\colon \mathcal{R} \longrightarrow \mathcal{S}$ be the functor which assigns to every ring its underlying set, and to every morphism in $\mathcal{R}$ its underlying function. Show that T is a faithful functor and that every monomorphism in $\mathcal{R}$ is an injection.

3. Let $\mathbb{Z}$ denote the ring of rational integers, and φ its field of fractions, the rational numbers. Show that the canonical morphism $\varphi\colon \mathbb{Z} \longrightarrow Q$ in $\mathcal{R}$ is a bimorphism in $\mathcal{R}$ which is not an isomorphism. Conclude that $T\colon \mathcal{R} \longrightarrow \mathcal{S}$ does not preserve epimorphisms and hence is not a coadjoint functor.

4. Show that $\mathbf{Z}$ is a projective in $\mathcal{R}$ and that if $\mathcal{P}$ is a projective in $\mathcal{R}$, then $\mathcal{P}$ is isomorphic with $\mathbf{Z}$.

5. Let $\mathcal{R}^c$ denote the full subcategory of $\mathcal{R}$ generated by the commutative rings. Show that $\mathcal{R}^c$ is a coreflective subcategory of $\mathcal{R}$.

6. Show that the canonical functor $T: \mathcal{R} \longrightarrow \mathcal{S}$ is an adjoint functor.

7. Let $(\alpha, \beta): F \dashv T: (\mathcal{R}, \mathcal{S})$ be an adjoint pair, and $f: R \longrightarrow R''$ a morphism in $\mathcal{R}$. Show that f is surjective if and only if whenever X is a set and

$$\begin{array}{ccc} & & R \\ & & \downarrow{\scriptstyle f} \\ F(X) & \xrightarrow{g} & R'' \end{array}$$

is a diagram in $\mathcal{R}$, there exists a morphism $\bar{g}: F(X) \longrightarrow R$ in R such that $f\,\bar{g} = g$.

8. Recall that the radical of a ring R is the intersection of the maximal left ideals of R, or equivalently the intersection of the maximal right ideals. A ring R is without radical if its radical is zero. Show that the full subcategory $\mathcal{R}'$ of $\mathcal{R}$ generated by the rings without radical is a coreflective subcategory of $\mathcal{R}$.

§3. Functors of several variables, products, and coproducts.

__3.1 Definition.__ If $\mathcal{X}$ and $\mathcal{Y}$ are categories, a __contravariant functor__ $T: \mathcal{X} \longrightarrow \mathcal{Y}$ consists of

1) for each object X of $\mathcal{X}$, an object $T(X)$ of $\mathcal{Y}$,

and 2) for each morphism $f: X' \longrightarrow X''$ in $\mathcal{X}$, a morphism $T(f): T(X'') \longrightarrow T(X')$ in $\mathcal{Y}$ such that

1) for X an object of $\mathcal{X}$, $T(1_X) = 1_{T(X)}$, and

2) if $f': X' \longrightarrow X$, $f'': X \longrightarrow X''$ are morphisms in $\mathcal{X}$, then $T(f''f') = T(f')\,T(f'')$.

__3.2 Example.__ For $\mathcal{X}$ a category, define $D_{\mathcal{X}}: \mathcal{X} \longrightarrow \mathcal{X}^*$ by $D_{\mathcal{X}}(X) = X^*$ for X an object of $\mathcal{X}$, and $D_{\mathcal{X}}(f) = f^*$ for f a morphism in $\mathcal{X}$. We have at once that $D_{\mathcal{X}}$ is a contravariant functor.

__3.3 Definitions.__ If $\mathcal{X}$ and $\mathcal{Y}$ are categories, a __pseudo-functor__ $T: \mathcal{X} \longrightarrow \mathcal{Y}$ consists of

1) for each object X of $\mathcal{X}$, an object $T(X)$ of $\mathcal{Y}$,

and 2) for each morphism f in $\mathcal{X}$, a morphism $T(f)$ in $\mathcal{Y}$.

If $T': \mathcal{X}' \longrightarrow \mathcal{X}$, $T'': \mathcal{X} \longrightarrow \mathcal{X}''$ are __pseudo-functors__, define $T''T': \mathcal{X}' \longrightarrow \mathcal{X}''$ by $T''T'(X') = T''(T'(X'))$ for X' an object of $\mathcal{X}$, and $T''T'(f) = T''(T'(f))$ for f a morphism in $\mathcal{X}'$.

Note that functors and contravariant functors are pseudo-functors. Further, the composition of pseudo-functors defined above agrees with the composition of functors defined earlier if the pseudo-functors being dealt with are in fact functors.

3.4 Proposition. If $T: \mathcal{X} \longrightarrow \mathcal{Y}$ is a pseudo-functor, then the following conditions are equivalent:

1) T is a contravariant functor,

2) $TD_{\mathcal{X}^*}: \mathcal{X}^* \longrightarrow \mathcal{Y}$ is a functor, and

3) $D_{\mathcal{Y}}T : \mathcal{X} \longrightarrow \mathcal{Y}^*$ is a functor.

3.5 Proposition. If $T_j: \mathcal{X}_j \longrightarrow \mathcal{X}_{j+1}$ is a pseudo-functor for $j = 1,2,3$, then

1) $T_3(T_2T_1) = (T_3T_2)T_1$,

2) $T_j 1_{\mathcal{X}_j} = T_j$ and $1_{\mathcal{X}_{j+1}} T_j = T_j$ for $j = 1,2,3$,

3) if T_1, T_2 are contravariant functors, then T_2T_1 is a functor, and

4) if either T_1 is a functor and T_2 is a contravariant functor, or T_1 is a contravariant functor and T_2 is a functor, then T_2T_1 is a contravariant functor.

The two preceding propositions follow immediately from the definitions involved.

<u>3.6 Example</u>. Let $\mathcal{P} : \mathcal{S} \longrightarrow \mathcal{S}$ be defined as follows: if S is a set, then $\mathcal{P}(S)$ is the set of all subsets of S, and if $f: S' \longrightarrow S''$ is a function, then $\mathcal{P}(f): \mathcal{P}(S'') \longrightarrow \mathcal{P}(S')$ is the function such that if A is a subset of S'' then $\mathcal{P}(f)(A)$ is the subset of S' consisting of points $s' \in S'$ such that $f(s') \in A$, i.e. $\mathcal{P}(f)(A) = f^{-1}(A)$. Now $\mathcal{P}$ is a contravariant functor. Moreover, it is faithful i.e. if $f,g: S' \longrightarrow S''$ are functions such that $\mathcal{P}(f) = \mathcal{P}(g)$, then $f = g$.

<u>3.7 Definition</u>. If $\mathcal{X}$ and $\mathcal{Y}$ are categories, the <u>product category</u> $\mathcal{X} \times \mathcal{Y}$ is the category such that

1) an object of $(\mathcal{X} \times \mathcal{Y})$ is an ordered pair (X,Y) where $X \in obj(\mathcal{X})$ and $Y \in obj(\mathcal{Y})$, and

2) a morphism from (X',Y') to (X'',Y'') in $\mathcal{X} \times \mathcal{Y}$ is an ordered pair (f,g) where $f \in \mathcal{X}(X',X'')$, $g \in \mathcal{Y}(Y',Y'')$, and

3) if $(f',g'): (X',Y') \longrightarrow (X,Y), (f'',g''): (X,Y) \longrightarrow (X'',Y'')$ are morphisms in $\mathcal{X} \times \mathcal{Y}$, then $(f'',g'')(f',g') = (f''f', g''g')$.

<u>3.8 Proposition</u>. If $\mathcal{X}$ and $\mathcal{Y}$ are categories, $\mathcal{P}_{\mathcal{X}}: \mathcal{X} \times \mathcal{Y} \longrightarrow \mathcal{X}$ and $\mathcal{P}_{\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \longrightarrow \mathcal{Y}$ are defined by $\mathcal{P}_{\mathcal{X}}(X,Y) = X$, $\mathcal{P}_{\mathcal{Y}}(X,Y) = Y$ for $(X,Y) \in obj(\mathcal{X} \times \mathcal{Y})$, and $\mathcal{P}_{\mathcal{X}}(f,g) = f$, $\mathcal{P}_{\mathcal{Y}}(f,g) = g$ for $(f,g) \in mor(\mathcal{X} \times \mathcal{Y})$, then

1) $\mathcal{P}_{\mathcal{X}}$ and $\mathcal{P}_{\mathcal{Y}}$ are functors, and

2) if $S: \mathcal{A} \longrightarrow \mathcal{X}$, $T: \mathcal{A} \longrightarrow \mathcal{Y}$ are functors, then there

is a unique functor $S \top T: \mathcal{A} \longrightarrow \mathfrak{X} \times \mathcal{Y}$ such that $P_{\mathfrak{X}}(S \top T) = S$ and $P_{\mathcal{Y}}(S \top T) = T$.

The proposition follows at once from the definitions.

<u>3.9 Definitions.</u> If $S: \mathfrak{X}' \longrightarrow \mathfrak{X}''$, $T: \mathcal{Y}' \longrightarrow \mathcal{Y}''$ are pseudo-functors, then $S \times T: \mathfrak{X}' \times \mathcal{Y}' \longrightarrow \mathfrak{X}'' \times \mathcal{Y}''$ is the pseudo-functor such that $(S \times T)(X',Y') = (S(X'),T(Y'))$ for $(X',Y') \in \operatorname{obj}(\mathfrak{X}' \times \mathcal{Y}')$, and $(S \times T)(f,g) = (S(f),T(g))$ for $(f,g) \in \operatorname{mor}(\mathfrak{X}' \times \mathcal{Y}')$.

If $\mathfrak{X}$, $\mathcal{Y}$, and $\mathfrak{Z}$ are categories, then the pseudo-functor $T: \mathfrak{X} \times \mathcal{Y} \longrightarrow \mathfrak{Z}$ is a <u>functor contravariant in the first variable and covariant in the second variable</u> if the composite

$$\mathfrak{X}^* \times \mathcal{Y} \xrightarrow{D_{\mathfrak{X}^*} \times 1_{\mathcal{Y}}} \mathfrak{X} \times \mathcal{Y} \xrightarrow{T} \mathfrak{Z}$$

is a functor.

<u>3.10 Definition.</u> If $\mathfrak{X}$ is a category, define $\mathfrak{X}(\,,\,): \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathcal{S}$ as follows:

1) for $(X',X'') \in \operatorname{obj}(\mathfrak{X} \times \mathfrak{X})$, $\mathfrak{X}(X',X'')$ is the set of morphisms in $\mathfrak{X}$ from X' to X'', and

2) if $(f,g): (X_0',X_0'') \longrightarrow (X_1',X_1'')$ is a morphism in $\mathfrak{X} \times \mathfrak{X}$, then $\mathfrak{X}(f,g): \mathfrak{X}(X_1',X_0'') \longrightarrow \mathfrak{X}(X_0',X_1'')$ is the function such that $\mathfrak{X}(f,g)(h) = ghf$ for $h \in \mathfrak{X}(X_1',X_0'')$.

$\mathfrak{X}(\,,\,): \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathcal{S}$ is the <u>basic structural functor of the category $\mathfrak{X}$</u>.

<u>3.11 Proposition.</u> If $\mathfrak{X}$ is a category, then $\mathfrak{X}(\,,\,): \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathcal{S}$

is a functor contravariant in the first variable and covariant in the second variable.

The proposition follows at once from the definitions.

<u>3.12 Proposition.</u> If $\mathfrak{X}$ is a category and $f\colon X' \longrightarrow X''$ is a morphism in $\mathfrak{X}$, then

1) f is a monomorphism if and only if for every object X of $\mathfrak{X}$, $\mathfrak{X}(X,f)\colon \mathfrak{X}(X,X') \longrightarrow \mathfrak{X}(X,X'')$ is injective in $\mathcal{S}$, and

2) f is an epimorphism if and only if for every object X of $\mathfrak{X}$, $\mathfrak{X}(f,X)\colon \mathfrak{X}(X'',X) \longrightarrow \mathfrak{X}(X',X)$ is injective in $\mathcal{S}$.

<u>3.13 Proposition.</u> If $\mathfrak{X}$ is a category and X is an object of $\mathfrak{X}$, then

1) X is projective if and only if for every epimorphism $f\colon X' \longrightarrow X''$ in $\mathfrak{X}$, $\mathfrak{X}(X,f)\colon \mathfrak{X}(X,X') \longrightarrow \mathfrak{X}(X,X'')$ is surjective in $\mathcal{S}$, and

2) X is injective if and only if for every monomorphism $f\colon X' \longrightarrow X''$ in $\mathfrak{X}$, $\mathfrak{X}(f,X)\colon \mathfrak{X}(X'',X) \longrightarrow \mathfrak{X}(X',X)$ is surjective in $\mathcal{S}$.

The two preceding propositions though tautologies, illustrate a point of view which differs mildly from that used in the original definitions. Note that in their statement we have used the convention that one sometimes uses the same notation for an object and its identity morphism (1.1).

<u>3.14 Proposition.</u> If $\mathfrak{X}$ is a category and $f: X' \longrightarrow X''$ is a morphism in $\mathfrak{X}$, then

1) f is a coretract if and only if for every object X in $\mathfrak{X}$, $\mathfrak{X}(X,f): \mathfrak{X}(X,X') \longrightarrow \mathfrak{X}(X,X'')$ is surjective in $\mathcal{S}$, and

2) f is a retract if and only if for every object X in $\mathfrak{X}$, $\mathfrak{X}(f,X): \mathfrak{X}(X'',X) \longrightarrow \mathfrak{X}(X',X)$ is surjective in $\mathcal{S}$.

<u>Proof.</u> If $\mathfrak{X}(X'',f)$ is surjective there exists $g: X'' \longrightarrow X'$ such that $fg = 1_{X''}$. Now for every object X of $\mathfrak{X}$ the composite $\mathfrak{X}(f,X)\ \mathfrak{X}(g,X)$ is the identity of $\mathfrak{X}(X,X'')$ and part 1) follows. Part 2) is merely the dual of part 1.

<u>3.15 Definitions.</u> If $T',T'': \mathfrak{X} \times \mathcal{Y} \longrightarrow \mathcal{Z}$ are functors contravariant in the first variable and covariant in the second, then a morphism $b: T' \longrightarrow T''$ consists of a morphism $b(X,Y): T'(X,Y) \longrightarrow T''(X,Y)$ for each object (X,Y) of $\mathfrak{X} \times \mathcal{Y}$ such that if $(f,g): (X',Y') \longrightarrow (X'',Y'')$ is a morphism in $\mathfrak{X} \times \mathcal{Y}$, then the diagram

$$\begin{array}{ccc} T'(X'',Y') & \xrightarrow{T'(f,g)} & T'(X',Y'') \\ \downarrow{\scriptstyle b(X'',Y')} & & \downarrow{\scriptstyle b(X',Y'')} \\ T''(X'',Y') & \xrightarrow{T''(f,g)} & T''(X',Y'') \end{array}$$

in $\mathcal{Z}$ is commutative.

Suppose $S\colon \mathcal{Y} \longrightarrow \mathcal{X}$, $T\colon \mathcal{X} \longrightarrow \mathcal{Y}$ are functors. Note that $\mathcal{X}(\ ,\ )(S \times 1_{\mathcal{X}}), \mathcal{Y}(,)(1_{\mathcal{Y}} \times T)\colon \mathcal{Y} \times \mathcal{X} \longrightarrow \mathcal{S}$ are functors contravariant in the first variable and covariant in the second.

In the situation above

1) if $\beta\colon 1_{\mathcal{Y}} \longrightarrow TS$ is a morphism, for (Y,X) an object of $\mathcal{Y} \times \mathcal{X}$, let $b_\beta(Y,X)\colon \mathcal{X}(S(Y),X) \longrightarrow \mathcal{Y}(Y,T(X))$ be defined by $b_\beta(Y,X)(f) = T(f)\beta(Y)$, and

2) if $b\colon \mathcal{X}(\ ,\ )S \times 1_{\mathcal{X}} \longrightarrow \mathcal{Y}(\ ,\ )1_{\mathcal{Y}} \times T$ is a morphism for Y an object of $\mathcal{Y}$, let $\beta_b(Y)\colon Y \longrightarrow TS(Y)$ be $b(Y,S(Y))(1_{S(Y)})$.

<u>3.16 Proposition.</u> If $S\colon \mathcal{Y} \longrightarrow \mathcal{X}$, $T\colon \mathcal{X} \longrightarrow \mathcal{Y}$ are functors, then

1) if $\beta\colon 1_{\mathcal{Y}} \longrightarrow TS$ is a morphism, then $b_\beta\colon \mathcal{X}(\ ,\ )S \times 1_{\mathcal{X}} \longrightarrow \mathcal{Y}(\ ,\ )1_{\mathcal{Y}} \times T$ is a morphism, and if $b = b_\beta$, then $\beta = \beta_b$, and

2) if $b\colon \mathcal{X}(\ ,\ )S \times 1_{\mathcal{X}} \longrightarrow \mathcal{Y}(\ ,\ )1_{\mathcal{Y}} \times T$ is a morphism, then $\beta_b\colon 1_{\mathcal{Y}} \longrightarrow TS$ is a morphism, and if $\beta = \beta_b$, then $b = b_\beta$.

The proposition follows by routine verification.

<u>3.17 Proposition.</u> If $S\colon \mathcal{Y} \longrightarrow \mathcal{X}$, $T\colon \mathcal{X} \longrightarrow \mathcal{Y}$ are functors, and $\beta\colon 1_{\mathcal{Y}} \longrightarrow TS$ is a morphism, then the following are equivalent:

1) there exists $\alpha: ST \longrightarrow 1_{\mathfrak{X}}$ such that $(\alpha,\beta): S \dashv T: (\mathfrak{X}, \mathcal{Y})$ is an adjoint pair of functors, and

2) $b_\beta: \mathfrak{X}(\ ,\ )S \times 1_{\mathfrak{X}} \longrightarrow \mathcal{Y}(\ ,\ )1_{\mathcal{Y}} \times T$ is an isomorphism.

<u>Proof</u>. Suppose 1). For (Y,X) an object of $\mathcal{Y} \times \mathfrak{X}$, define $a(Y,X): \mathcal{Y}(Y,T(X)) \longrightarrow \mathfrak{X}(S(T),X)$ by $a(Y,X)(g) = \alpha(X)ST(g)$. Now $a(Y,X)b_\beta(Y,X)$ and $b_\beta(Y,X)a(Y,X)$ are identity functions, and it follows that 1) implies 2).

Suppose 2). Let $a: \mathcal{Y}(\ ,\ )1_{\mathcal{Y}} \times T \longrightarrow \mathfrak{X}(\ ,\ )S \times 1_{\mathfrak{X}}$ be the inverse of b_β. For X an object of $\mathfrak{X}$, let $\alpha(X): ST(X) \longrightarrow X$ be $a(T(X),X)(1_{T(X)})$. Now by 3.16^*, $\alpha: ST \longrightarrow 1_{\mathfrak{X}}$ is a morphism. A routine verification shows that $(\alpha,\beta): S - T: (\mathfrak{X}, \mathcal{Y})$ is an adjoint pair of functors. Hence 2) implies 1), and the proposition is proved.

Note that the preceding proposition affords an alternative formulation of the notion of adjoint pairs of functors.

<u>3.18 Definitions</u>. Let $\mathfrak{X}$ be a category, and X_1, X_2 objects of $\mathfrak{X}$.

A <u>product of</u> X_1 <u>and</u> X_2 is a pair of morphisms $p_1: X \longrightarrow X_1$, $p_2: X \longrightarrow X_2$ in $\mathfrak{X}$ such that if $f_1: X' \longrightarrow X_1$, $f_2: X' \longrightarrow X_2$ are morphisms in $\mathfrak{X}$, there is a unique morphism $f_1 \top F_2: X' \longrightarrow X$ such that $p_1(f_1 \top f_2) = f_1$ and $p_2(f_1 \top f_2) = f_2$.

A <u>coproduct of</u> X_1 <u>and</u> X_2 is a pair of morphisms $i_1: X_1 \longrightarrow X$, $i_2: X_2 \longrightarrow X$ in $\mathfrak{X}$ such that if $f_1: X_1 \longrightarrow X'$,

$f_2: X_2 \longrightarrow X'$ are morphisms in $\mathcal{X}$, there is a unique morphism $f_1 \perp f_2 : X \longrightarrow X'$ such that $(f_1 \perp f_2)i_1 = f_1$ and $(f_1 \perp f_2)i_2 = f_2$.

<u>3.19 Proposition.</u> If $\mathcal{X}$ is a category, then

1) the morphisms $p_1: X \longrightarrow X_1$, $p_2: X \longrightarrow X_2$ are a product of X_1 and X_2 in $\mathcal{X}$ if and only if the morphisms $(p_1)^*: X_1^* \longrightarrow X^*$, $p_2^*: X_2 \longrightarrow X^*$ are a coproduct of X_1^* and X_2^* in $\mathcal{X}^*$, and

2) if $p_1: X \longrightarrow X_1$, $p_2: X \longrightarrow X_2$, and $\bar{p}_1: \bar{X} \longrightarrow X_1$, $\bar{p}_2: \bar{X} \longrightarrow X_2$ are products of X_1 and X_2 in $\mathcal{X}$, then there is a unique isomorphism $u: X \longrightarrow \bar{X}$ such that $\bar{p}_1 u = p_1$ and $\bar{p}_2 u = p_2$.

<u>Proof.</u> Part 1) follows at once from the definitions. As for part 2), the definition of product guarantees there are unique morphisms $u: X \longrightarrow \bar{X}$ such that $\bar{p}_1 u = p_1$, $\bar{p}_2 u = p_2$ and $v: \bar{X} \longrightarrow X$ such that $p_1 v = \bar{p}_1$, $p_2 v = \bar{p}_2$. Further $\bar{p}_1 u\, v = \bar{p}_1$, $\bar{p}_2 u\, v = \bar{p}_2$, but $1_{\bar{X}}$ is the unique morphism having this property. Thus $u\, v = 1_{\bar{X}}$, and for the same reasons $v\, u = 1_X$. Hence part 2) and also the proposition are proved.

<u>3.20 Definition.</u> If $\mathcal{X}$ is a category, the <u>diagonal functor of $\mathcal{X}$</u> is the functor $\Delta: \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$ such that

1) if $X \in \text{obj}(\mathcal{X})$, then $\Delta(X) = (X,X)$, and

2) if $f \in \text{mor}(\mathcal{X})$, then $\Delta(f) = (f,f)$.

<u>3.21 Proposition.</u> If $\mathfrak{X}$ is a category and (X_1,X_2) is an object of $\mathfrak{X} \times \mathfrak{X}$, then

1) the following are equivalent:

i) $p_1\colon X \longrightarrow X_1$, $p_2\colon X \longrightarrow X_2$ is a product of X_1,X_2 in $\mathfrak{X}$, and

ii) $(p_1,p_2\colon \Delta(X) \longrightarrow (X_1,X_2)$ is a Δ-reflection of (X_1,X_2) in $\mathfrak{X}$, and

2) the following are equivalent:

i) $i_1\colon X_1 \longrightarrow X$, $i_2\colon X_2 \longrightarrow X$ is a coproduct of X_1,X_2 in $\mathfrak{X}$, and

ii) $(i_1,i_2)\colon (X_1,X_2) \longrightarrow \Delta(X)$ is a Δ-coreflection of (X_1,X_2) in $\mathfrak{X}$.

The proposition follows at once from the definitions.

<u>3.22 Definitions.</u> Let $\mathfrak{X}$ be a category, J a set, and X_j an object of $\mathfrak{X}$ for $j \in J$.

A <u>product of</u> $(X_j)_{j\in J}$ is a J-indexed set of morphisms $(p_j)_{j\in J}$, $p_j\colon X \longrightarrow X_j$ in $\mathfrak{X}$ such that if $(f_j)_{j\in J}, f_j\colon X' \longrightarrow X_j$ in a J-indexed set of morphisms, then there is a unique morphism $\top_{j\in J} f_j\colon X' \longrightarrow X$ such that $p_k(\top_{j\in J} f_j) = f_k$ for $k \in J$.

A <u>coproduct of</u> $(X_j)_{j\in J}$ is a J-indexed set of morphisms $(i_j)_{j\in J}$, $i_j\colon X_j \longrightarrow X$ in $\mathfrak{X}$ such that if $(f_j)_{j\in J}$, $f_j\colon X_j \longrightarrow X'$ is a J-indexed set of morphisms, then there is a unique morphism $\perp_{j\in J} f_j\colon X \longrightarrow X'$ such that $(\perp_{j\in J} f_j) i_k = f_k$ for $k \in J$.

If $(X_j)_{j\in J}$ is a J-indexed set of objects of $\mathfrak{X}$ having a product $(p_j)_{j\in J}$, $p_j : X \longrightarrow X_j$, then for $k \in J$ the morphism p_k is the <u>projection to k-th factor of the product</u>. If $(i_j)_{j\in J}, i_j : X_j \longrightarrow X$ is a coproduct of $(X_j)_{j\in J}$, then for $k \in J$ the morphism i_k is the <u>injection from the k-th cofactor of the coproduct</u>.

<u>3.23 Proposition</u>. If $\mathfrak{X}$ is a category, J is a set, and $(X_j)_{j\in J}$ is a J-indexed set of objects of $\mathfrak{X}$, then

1) $(p_j)_{j\in J}$, $p_j : X \longrightarrow X_j$ is a product of $(X_j)_{j\in J})$ in if and only if $(p_j^*)_{j\in J}$, $p_j^* : X_j^* \longrightarrow X^*$ is a coproduct of $(X_j^*)_{j\in J}$ in $\mathfrak{X}^*$, and

2) if $(p_j)_{j\in J}$, $p_j : X \longrightarrow X_j$ and $(\bar{p}_j)_{j\in J}$, $\bar{p}_j : \bar{X} \longrightarrow X_j$ are products of $(X_j)_{j\in J}$, then there is a unique morphism u i $X \longrightarrow \bar{X}$ such that $\bar{p}_j u = p_j$ for $j \in J$, and u is an isomorphism.

Part 1) of the proposition follows at once from the definitions, while part 2) is proved exactly as 3.19,2) using J rather than {1,2} as index set.

<u>3.24 Definitions</u>. The <u>category $\mathfrak{X}$ has products</u> if whenever J is a set and $(X_j)_{j\in J}$ is a J-indexed set of objects of $\mathfrak{X}$ a product of $(X_j)_{j\in J}$ exists in $\mathfrak{X}$. It has <u>countable products</u> if the condition above is satisfied for any countable set J,

and finite products if it is satisfied for any finite set J.

The category $\mathfrak{X}$ has coproducts if whenever J is a set and $(X_j)_{j\in J}$ is a J-indexed set of objects of $\mathfrak{X}$ a coproduct of $(X_j)_{j\in J}$ exists in $\mathfrak{X}$. It has countable coproducts if the condition above is satisfied for any countable set J, and finite coproducts if it is satisfied for any finite set J.

3.25 Example. The category $\mathcal{S}$ has products and coproducts.

If J is a set and $(S_j)_{j\in J}$ is a J-indexed set of objects of $\mathcal{S}$, then in a rough way a product of $(S_j)_{j\in J}$ may be thought of as follows: Let $\times_{j\in J}S_j$ be the set of functions s with domain J and such that $s(j) \in S_j$ for $j \in J$, and let $p_k : \times_{j\in J}S_j \longrightarrow S_k$ be the function such that $p_k(s) = s(k)$ for $k \in J$, $s \in \times_{j\in J}S_j$. Now $(p_j)_{j\in J}$ is a product of $(S_j)_{j\in J}$ in $\mathcal{S}$. Indeed if $f_j^!: S' \longrightarrow S_j$ is a function for $j \in J$, then $\Pi_{j\in J}f_j^!: S' \longrightarrow \times_{j\in J}S_j$ is the function such that $p_k(\Pi_{j\in J}f_j^!)(s') = f_k(s')$ for $s' \in S'$, $k \in J$.

Usually in $\mathcal{S}$ the product of two sets S', S'' is denoted by $S' \times S''$ and thought of explicitly as the set of ordered pairs (s', s'') such that $s' \in S'$, $s'' \in S''$. When using this notation if $f: S_0' \longrightarrow S_1'$, $g: S_0'' \longrightarrow S_1''$ are functions, then $f \times g: S_0' \times S_0'' \longrightarrow S_1' \times S_1''$ is the function such that $(f \times g)(x,y) = (f(x), g(y))$.

If J is a set and $(S_j)_{j\in J}$ is a J-indexed set of objects in $\mathcal{S}$, a coproduct of $(S_j)_{j\in J}$ is a disjoint union of the

J-indexed family of sets. If a coproduct is $(i_j)_{j\in J}, i_j : S_j \longrightarrow S$, then for any set X, $(\mathcal{S}(i_j,X))_{j\in J}, \mathcal{S}(i_j,X): \mathcal{S}(S,X) \longrightarrow \mathcal{S}(S_j,X)$ is a product of $(\mathcal{S}(S_j,X))_{j\in J}$ in $\mathcal{S}$.

<u>3.26 Proposition.</u> If $\mathcal{X}$ is a category, J is a set, and $(X_j)_{j\in J}$ is a J-indexed set of objects of $\mathcal{X}$, then if $(p_j)_{j\in J}, p_j: X \longrightarrow X_j$ is a J-indexed set of morphisms in $\mathcal{X}$, then the following are equivalent:

1) $(p_j)_{j\in J}$ is a product of $(X_j)_{j\in J}$ in $\mathcal{X}$, and

2) for every object X' of $\mathcal{X}$, $(\mathcal{X}(X',p_j))_{j\in p} \mathcal{X}(X',p_j)$: $\mathcal{X}(X',X) \longrightarrow \mathcal{X}(X',X_j)$ is a product of $(\mathcal{X}(X',X_j))_{j\in J}$ in $\mathcal{S}$.

<u>3.27 Proposition.</u> If $\mathcal{X}$ is a category, J is a set, and $(X_j)_{j\in J}$ is a J-indexed set of objects of $\mathcal{X}$, then if $(i_j)_{j\in J}, i_j: X_j \longrightarrow X$ is a J-indexed set of morphisms in $\mathcal{X}$, then the following are equivalent:

1) $(i_j)_{j\in J}$ is a coproduct of $(X_j)_{j\in J}$ in $\mathcal{X}$, and

2) for every object X' of $\mathcal{X}$, $(\mathcal{X}(i_j,X'))_{j\in J}$, $\mathcal{X}(i_j,X'): \mathcal{X}(X,X') \longrightarrow \mathcal{X}(X_j,X')$ is a product of $(\mathcal{X}(X_j,X'))_{j\in J}$ in $\mathcal{S}$.

In each of the two preceding propositions, the two parts are merely reformulations of each other.

<u>3.28 Notation</u>. Usually in a category $\mathfrak{X}$ such that every ordered pair of objects has a product, a product of the objects X_1 and X_2 is denoted by $p_1\colon X_1 \Pi X_2 \to X_1, p_2\colon X_1 \Pi X_2 \to X_2$. If also $p_1'\colon X_1' \Pi X_2' \to X_1', p_2'\colon X_1' \Pi X_2' \to X_2'$ is a product of X_1' and X_2', and $f\colon X_1' \to X_1$, $g\colon X_2' \to X_2$ are morphisms in $\mathfrak{X}$, then $f \Pi g\colon X_1' \Pi X_2' \to X_1 \Pi X_2$ is the unique morphism such that $p_1(f \Pi g) = fp_1'$ and $p_2(f \Pi g) = gp_2'$.

Similarly if every ordered pair of objects of $\mathfrak{X}$ has a coproduct, the usual notation for a coproduct of X_1 and X_2 in $i_1\colon X_1 \to X_1 \amalg X_2$, $i_2\colon X_2 \to X_1 \amalg X_2$. If also $i_1'\colon X_1' \to X_1' \amalg X_2'$, $i_2'\colon X_2' \to X_1' \amalg X_2'$ is a coproduct of X_1' and X_2', and $f\colon X_1' \to X_1$, $g\colon X_2' \to X_2$ are morphisms in $\mathfrak{X}$, then $f \amalg g\colon X_1' \amalg X_2' \to X_1 \amalg X_2$ is the unique morphism such that $(f \amalg g)i_1' = i_1 f$ and $(f \amalg g)i_2' = i_2 g$.

If J is a set, $\mathfrak{X}$ a category with J-indexed products, and $(p_j)_{j\in J}$ is a product of $(X_j)_{j\in J}$ in $\mathfrak{X}$, the domain of the morphism p_j is usually denoted by $\Pi_{j\in J} X_j$. If $(p_j')_{j\in J}$, $p_j'\colon \Pi_{j\in J} X_j' \to X_j'$ is another J-indexed product and $f_j\colon X_j' \to X_j$ is a morphism in $\mathfrak{X}$ for $j \in J$, then $\Pi_{j\in J} f_j\colon \Pi_{j\in J} X_j' \to \Pi_{j\in J} X_j$ is the unique morphism such that $p_k(\Pi_{j\in J} f) = f_k p_k'$ for $k \in J$.

If J is a set, $\mathfrak{X}$ is a category with J-indexed coproducts, and $(i_j)_{j\in J}$ is a coproduct of $(X_j)_{j\in J}$ in $\mathfrak{X}$, the range of the morphism i_j is usually denoted by $\coprod_{j \in J} X_j$. If $(i_j')_{j \in J}$, $i_j' \colon X_j' \longrightarrow \coprod_{j \in J} X_j'$ is another J-indexed coproduct and $f_j \colon X_j' \longrightarrow X_j$ is a morphism in $\mathcal{X}$ for $j \in J$, then $\coprod_{j \in J} f_j \colon \coprod_{j \in J} X_j' \longrightarrow \coprod_{j \in J} X_j$ is the unique morphism such that

$$(\coprod_{j \in J} f_j) i_k' = i_k f_k \quad \text{for } k \in J.$$

The preceding notation is often abbreviated. Thus a product of X_1 and X_2 in $\mathcal{X}$ is denoted by $X_1 \prod X_2$ the projections to the factors being understood from the notation or a coproduct of X_1 and X_2 is denoted by $X_1 \coprod X_2$ the injections from the cofactors being understood by the notation. If J is a set, a product of $(X_j)_{j \in J}$ is denoted by $\prod_{j \in J} X_j$ or a coproduct by $\prod_{j \in J} X_j$ in this abbreviated notation, the appropriate projections or injections being understood implicitly.

In some special categories the standard notation for products or coproducts differs from the standard general notation (e.g. in the category $\mathcal{S}$ the product of S' and S'' is denoted by $S' \times S''$, as in example 3.25.

<u>3.29 Definitions.</u> Let $T \colon \mathcal{X} \longrightarrow \mathcal{Y}$ be a functor.

The <u>functor T preserves products</u> if whenever J is a set and $(p_j)_{j \in J}$, $p_j \colon X \longrightarrow X_j$ is a product of $(X_j)_{j \in J}$ in $\mathcal{X}$, then $(T(p_j))_{j \in J}$, $T(p_j) \colon T(X) \longrightarrow T(X_j)$ is a product of $(T(X_j))_{j \in J}$ in $\mathcal{Y}$. It <u>preserves countable products</u> if the preceding condition holds whenever J is a countable set, or it preserves finite products if it holds whenever J is a finite set.

The functor T preserves coproducts if whenever J is a set and $(i_j)_{j\in J}$, $i_j\colon X_j \longrightarrow X$ is a coproduct of $(X_j)_{j\in J}$ in $\mathcal{X}$, then $(T(i_j))_{j\in J}$, $T(i_j)\colon T(X_j) \longrightarrow T(X)$ is a coproduct of $(T(X_j))_{j\in J}$ in $\mathcal{Y}$. It preserves countable coproducts if the preceding condition holds whenever J is a countable set, or it preserves finite products if it holds whenever J is a finite set.

3.30 Proposition. If $T\colon \mathcal{X} \longrightarrow \mathcal{Y}$ is an adjoint functor, then T preserves products.

Proof. Let $(\alpha,\beta)\colon S \dashv T$ be an adjoint pair. Suppose J is a set and $\prod_{j\in J}X_j$ is a J-indexed product in $\mathcal{X}$. For Y an object of $\mathcal{Y}$, there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{X}(S(Y), \prod_{j\in J}X_j) & \longrightarrow & \mathcal{Y}(Y, T(\prod_{j\in J}X_j)) \\
\downarrow & & \downarrow \\
\times_{j\in J}\,\mathcal{X}(S(Y), X_j) & \longrightarrow & \times_{j\in J}\,\mathcal{Y}(Y, T(X_j))
\end{array}$$

in $\mathcal{S}$, where the horizontal morphisms are isomorphism by 3.17, and the left vertical morphism is an isomorphism by 3.26. This implies that the right vertical morphism is an isomorphism. Since this is the case for every object Y of $\mathcal{Y}$, applying 3.26 again the proposition is proved.

Note that we have already seen that an adjoint functor preserves monomorphisms 2.16. Further, the dual of the preceding proposition asserts that a coadjoint functor preserves coproducts.

<u>3.31 Definitions and comments.</u> Let $\mathfrak{X}$ be a category.

$\mathfrak{X}$ has a terminal point if there is an object $*_1$ of $\mathfrak{X}$ such that if X is any object of $\mathfrak{X}$ there is a unique morphism $q(X): X \longrightarrow *_1$ in $\mathfrak{X}$. If $*_1$ and $\bar{*}_1$ are terminal points of $\mathfrak{X}$ there is a unique isomorphism $u: *_1 \longrightarrow \bar{*}_1$ in $\mathfrak{X}$. Observe that the hypothesis that $\mathfrak{X}$ has a terminal point is equivalent with the hypothesis that $\mathfrak{X}$ has products indexed on the empty set. Thus a category $\mathfrak{X}$ with finite products has a terminal point.

<u>$\mathfrak{X}$ has an initial point</u> if there is an object $*_0$ of $\mathfrak{X}$ such that if X is any object of $\mathfrak{X}$ there is a unique morphism $\eta(X): *_0 \longrightarrow X$ in $\mathfrak{X}$. If $*_0$ and $\bar{*}_0$ are initial points of $\mathfrak{X}$, there is a unique isomorphism $u: *_0 \longrightarrow \bar{*}_0$ in $\mathfrak{X}$. Observe that the hypothesis that $\mathfrak{X}$ has coproducts indexed on the empty set is equivalent with the hypothesis that $\mathfrak{X}$ has an initial point. Thus a category $\mathfrak{X}$ with finite coproducts has an initial point.

Note also that an object $*_0$ of $\mathfrak{X}$ is an initial point if and only if its dual $(*_0)^*$ is a terminal point of $\mathfrak{X}^*$. The empty set φ is the initial point of $\mathcal{S}$.

<u>3.32 Proposition.</u> If $\mathfrak{X}$ is a category, $\mathfrak{X}'$ is a reflective subcategory of $\mathfrak{X}$, and $R: \mathfrak{X} \longrightarrow \mathfrak{X}'$ is a reflection from $\mathfrak{X}$ to $\mathfrak{X}'$, i.e. an adjoint for the natural inclusion $I: \mathfrak{X}' \longrightarrow \mathfrak{X}$, then

1) if J is a set, and $(p_j)_{j \in J}$, $p_j: X \longrightarrow X_j$ is a product of $(X_j)_{j \in J}$ in $\mathfrak{X}$, then $(R(p_j))_{j \in J}$, $R(p_j): R(X) \longrightarrow R(X_j)$ is a product of $(R(X_j))_{j \in J}$ in $\mathfrak{X}'$, and

2) if $\mathfrak{X}'$ is a full subcategory of $\mathfrak{X}$, then $\mathfrak{X}'$ has products, countable products, or finite products according as $\mathfrak{X}$ has products, countable products, or finite products.

<u>Proof.</u> Part 1) is a special case of 3.30. Observing that if $\mathfrak{X}'$ is a full subcategory it may be assumed that $R(X) = X$ for $X \in \text{obj}(\mathfrak{X}')$, by 1.14, part 2) follows at once from part 1) and the universal property of reflections. Hence the proposition is proved.

Note that in the situation of the preceding proposition it is possible that $\mathfrak{X}'$ has products even though $\mathfrak{X}$ does not.

<u>3.33 Definitions.</u> Let $\mathfrak{X}$ be a category.

A <u>weak separation subcategory of $\mathfrak{X}$</u> is a full coreflective subcategory $\mathfrak{X}'$ of $\mathfrak{X}$ such that if $f: X' \longrightarrow X$ is a retract in $\mathfrak{X}$ and X is an object of $\mathfrak{X}'$, then f is a morphism in $\mathfrak{X}'$.

A <u>weak coseparation subcategory of $\mathfrak{X}$</u> is a full reflective subcategory $\mathcal{X}'$ of $\mathcal{X}$ such that if $f: X \longrightarrow X''$ is a coretract in $\mathcal{X}$ and X is an object of $\mathcal{X}'$, then f is a morphism in $\mathcal{X}'$.

Note that $\mathcal{X}'$ is a weak separation subcategory of $\mathcal{X}$ if and only if $(\mathcal{X}')^*$ is a weak coseparation subcategory of $\mathcal{X}^*$. When dealing with a weak coseparation subcategory $\mathcal{X}'$ of $\mathcal{X}$ one assumes that the reflection $R(\ ):\mathcal{X} \longrightarrow \mathcal{X}', \rho(\ ): R(\ ) \longrightarrow 1_{\mathcal{X}}$ has been so chosen that $\rho(X) = 1_X$ for $X \in obj(\mathcal{X}')$, and dually.

<u>3.34 Proposition</u>. If $\mathcal{X}'$ is a weak coseparation subcategory of $\mathcal{X}$, then

1) if J is a set, $X_j \in obj(\mathcal{X}')$ for $j \in J$, and $(i_j)_{j\in J}$ $i_j: X_j \longrightarrow X$ is a coproduct of $(X_j)_{j\in J}$ in $\mathcal{X}$, then it is also a coproduct of $(X_j)_{j\in J}$ in $\mathcal{X}'$, and

2) $\mathcal{X}'$ has coproducts, countable coproducts, or finite coproducts if $\mathcal{X}$ has coproducts, countable coproducts, or finite coproducts.

<u>Proof</u>. Let $\rho(\ ): R(\ ) \longrightarrow 1_{\mathcal{X}}$ be the reflection of $\mathcal{X}$ in $\mathcal{X}'$. Under the conditions of part 1), there is a unique $v: X \longrightarrow R(X)$ such that $\rho(X)\, v\, i_j = \rho(X) R(i_j) = i_j$ for $j \in J$. Now $\rho(X)\, v = 1_X$. Thus $\rho(X)$ is a morphism in $\mathcal{X}'$, $\rho(X) = 1_X$ and part 1) is proved. Part 2) follows at once from part 1) and the proposition is proved.

<u>3.35 Definitions.</u> Let $\mathfrak{X}$ be a category.

The object P of $\mathfrak{X}$ is a <u>generator of $\mathfrak{X}$</u> if the functor $\mathfrak{X}(P, \ ): \mathfrak{X} \longrightarrow \mathcal{S}$ is a faithful functor. The category $\mathfrak{X}$ has a generator if there exists a generator P of $\mathfrak{X}$.

The object I of $\mathfrak{X}$ is a <u>cogenerator of $\mathfrak{X}$</u> if the contravariant functor $\mathfrak{X}(\ ,I): \mathfrak{X} \longrightarrow \mathcal{S}$ is faithful. The category $\mathfrak{X}$ has a cogenerator if there exists a cogenerator I of $\mathfrak{X}$.

<u>3.36 Proposition.</u> If $\mathfrak{X}$ is a category, then

1) the object P of $\mathfrak{X}$ is a generator of $\mathfrak{X}$ if and only if the object P^* of $\mathfrak{X}^*$ is a cogenerator of $\mathfrak{X}^*$, and

2) suppose $\mathfrak{X}$ is a category with products, and I is an object of $\mathfrak{X}$; and that for X an object of $\mathfrak{X}$, $(p_f)_{f \in \mathfrak{X}(X,I)}$, $p_f: T(X) \longrightarrow I$ is a product of copies of I indexed on the set $\mathfrak{X}(X,I)$, $\beta(X): X \longrightarrow T(X)$, and $p_f\beta(X) = f$ for $f \in \mathfrak{X}(X,I)$, then the following are equivalent:

i) I is a cogenerator of $\mathfrak{X}$, and

ii) for every object X of $\mathfrak{X}$, $\beta(X): X \longrightarrow T(X)$ is a monomorphism in $\mathfrak{X}$.

<u>Proof.</u> Part 1) follows at once from the definitions.

Suppose the conditions of ii) obtain and that $f_1, f_2: X' \longrightarrow X$ are morphisms in $\mathfrak{X}$ such that $\beta(X)f_1 = \beta(X)f_2$. Then

$ff_1 = ff_2$ for every $f \in \mathfrak{X}(X,I)$, $\mathfrak{X}(f_1,I) = \mathfrak{X}(f_2,I) : \mathfrak{X}(X,I) \longrightarrow \mathfrak{X}(X',I)$, and $f_1 = f_2$ since $\mathfrak{X}(\ ,I)$ is faithful. Thus 2i) implies 2ii). Suppose 2ii) and that $f_1, f_2 = X' \longrightarrow X$ are morphisms in $\mathfrak{X}$ such that $\mathfrak{X}(f_1,I) = \ (f_2,I)$. Then for every $f \in \ (X,I)$, $ff_1 = ff_2$. Hence $\beta(X) f_1 = \beta(X) f_2$, $f_1 = f_2$, 2ii) implies 2i), and the proposition is proved.

Exercises

1. If $\mathfrak{X}$ is a category with a terminal point, show that the following are equivalent:

i) every ordered pair (X_1, X_2) of objects of $\mathfrak{X}$ has a product in $\mathfrak{X}$, and

ii) $\mathfrak{X}$ has finite products.

2. Show that the category R has products. Show that the category R^C is a weak separation subcategory of R , and conclude that R^C has products. Show that the ring of rational integers $\mathbf{Z}$ is an initial point of R , and that a ring whose underlying set has exactly one element is a terminal point of R . Note that it is assumed that every ring R has a unit, the unit is not necessarily different from zero.

3. Show that the categories $\mathcal{T}$ and $\mathcal{T}_H$ have products and coproducts and that the natural inclusion functor $\mathcal{T}_H \longrightarrow \mathcal{T}$ preserves both products and coproducts.

4. Suppose that $\mathfrak{X}$ is a category, J is a set, and $X = \prod_{j \in J} X_j$ is a product of $(X_j)_{j \in J}$ in $\mathfrak{X}$. Show that if each X_j is injective, then X is injective.

5. Show that the category $\mathcal{R}$ has coproducts, and that the natural inclusion functor $\mathcal{R}^c \longrightarrow \mathcal{R}$ does not preserve coproducts.

6. A space X is a Kolmogoroff or T_0 space if whenever x_0, x_1 are distinct points of X there is an open subset of X containing one but not the other. Let $\mathcal{T}_K$ be the full subcategory of $\mathcal{T}$ generated by the Kolmogoroff spaces. Let T_0 be the space with underlying set $\{0,1\}$, and closed subspaces $\emptyset$, $\{0\}$, $\{0,1\}$. Show that T_0 is a cogenerator of $\mathcal{T}_K$ but not of $\mathcal{T}$. Find a cogenerator of $\mathcal{T}$.

7. Show that $\mathcal{T}_K$ is a weak separation subcategory of $\mathcal{T}$, and that it contains $\mathcal{T}_H$. Show that the monomorphisms in $\mathcal{T}_K$ are the injective maps and that the epimorphisms in $\mathcal{T}_K$ are the surjective maps.

8. Let I denote the unit interval with its usual topology. Recall that a completely regular space X is a separated space such that whenever A is a closed subspace of X and x_0 a point of X - A, there exists a map $f\colon X \longrightarrow I$ such that $f(a) = 0$ for $a \in A$ and $f(X) = 1$. Let $\mathcal{T}_{CR}$ be the full subcategory of $\mathcal{T}$ generated by the completely regular

spaces. Show that I is a cogenerator of $\mathcal{T}_{CR}$ but not of $\mathcal{T}_H$, and that $\mathcal{T}_{CR}$ is a weak separation subcategory of $\mathcal{T}$.

9. Show that if $X \in obj(\mathcal{T}_{CR})$ and A is a subspace of X, then $A \in obj(\mathcal{T}_{CR})$. Show that the monomorphisms in $\mathcal{T}_{CR}$ are the injective maps and the epimorphisms are those maps which viewed in $\mathcal{T}_H$ are epimorphisms (§1, Ex. 2).

10. Recall that a compact space C is a separated space on which every ultra filter is convergent, or alternatively a Hausdorff space such that every open covering has a finite sub-covering. Let $\mathcal{C}$ denote the full subcategory of $\mathcal{T}$ generated by the compact spaces. Show that $\mathcal{C}$ is a weak separation sub-category of $\mathcal{T}_{CR}$ or alternatively of $\mathcal{T}$. Prove that I is an injective cogenerator of $\mathcal{C}$ (Tietze's extension theorem). Prove that every epimorphism in $\mathcal{C}$ is surjective and every monomorphism in $\mathcal{C}$ is injective.

11. A subspace A of a space X is <u>compactly closed in X</u> if whenever $f: C \longrightarrow X$ is a map with C compact, then $f^{-1}(A)$ is closed in C. The space X has a compactly generated topology if every compactly closed subspace of X is a closed subspace. Let $\mathcal{T}^C$ denote the full subcategory of $\mathcal{T}$ generated by the spaces having a compactly generated topology. Show that $\mathcal{T}^C$ is a weak coseparation subcategory of $\mathcal{T}$ which contains $\mathcal{C}$. Show that the epimorphisms in $\mathcal{T}^C$ are the surjective maps and the monomorphisms are the injective maps.

12. Suppose that C is a compact space and Y is a compactly generated space. Show that if $p: X \longrightarrow C$, $q: X \longrightarrow Y$ is a product of C and Y in $\mathcal{T}^C$, then it is also a product in $\mathcal{T}$. Show that the natural inclusion functor $\mathcal{T}^C \longrightarrow \mathcal{T}$ does not preserve finite products.

13. Show that the category $\mathcal{R}$ has a generator which is commutative and without radical (§2, Ex. 8). Show that the full subcategory of $\mathcal{R}$ generated by the rings without radical is a weak separation subcategory of $\mathcal{R}$.

14. An element x of a ring R is a non-trivial nilpotent element if $x \neq 0$ and $x^n = 0$ for some positive integer n. A commutative ring with no non-trivial nilpotent elements is called a reduced ring. Show that if R is a commutative ring without radical, then R is a reduced ring. Show that the full subcategory of $\mathcal{R}^C$ generated by the reduced rings is a weak separation subcategory of $\mathcal{R}^C$.

§4. Limits and Colimits

4.1 Definitions. Let $\mathcal{K}$ be a category.

If $f_1, f_2: X' \longrightarrow X''$ are morphisms in $\mathcal{K}$, then

1) $u: X \longrightarrow X'$ is an equalizer of f_1 and f_2 if $f_1 u = f_2 u$, and whenever $g: Y \longrightarrow X'$ is a morphism in $\mathcal{K}$ such that $f_1 g = f_2 g$, there is a unique $\bar{g}: Y \longrightarrow X$ such that $g = u\,\bar{g}$, and

2) $v: X'' \longrightarrow X$ is a coequalizer of f_1 and f_2 if $vf_1 = vf_2$, and whenever $h: X'' \longrightarrow Y$ is a morphism in $\mathcal{K}$ such that $hf_1 = hf_2$, there is a unique $\bar{h}: X \longrightarrow Y$ such that $h = \bar{h}v$.

4.2 Proposition. If $\mathcal{K}$ is a category, and $f_1, f_2: X' \longrightarrow X''$ are morphisms in $\mathcal{K}$, then

1) $u: X \longrightarrow X'$ is an equalizer of f_1, f_2 if and only if $u^*: (X')^* \longrightarrow X^*$ is a coequalizer of f_1^*, f_2^* in $\mathcal{K}^*$, and

2) if $u: X \longrightarrow X'$, $u_0: X_0 \longrightarrow X'$ are equalizers of f_1, f_2, there is a unique isomorphism $w: X_0 \longrightarrow X$ such that $u_0 = uw$.

Part 1 of the proposition follows at once from the definitions. Part 2 is a routine verification of a type often made earlier in this chapter.

4.3 Definitions.

The category $\mathcal{K}$ has equalizers if whenever $f_1, f_2: X' \longrightarrow X''$ are morphisms in $\mathcal{K}$ an equalizer of f_1, f_2 exists in $\mathcal{K}$.

The category $\mathfrak{X}$ has coequalizers if whenever f_1, f_2: $X' \longrightarrow X''$ are morphisms in $\mathfrak{X}$ a coequalizer of f_1, f_2 exists in $\mathfrak{X}$.

<u>4.4 Proposition.</u> If $\mathfrak{X}$ is a category, $f_1, f_2: X' \longrightarrow X''$ are morphisms in $\mathfrak{X}$, and $u: X \longrightarrow X'$ is an equalizer of f_1, f_2, then u is a monomorphism.

The proposition follows at once from the definition of equalizer.

<u>4.5 Definitions.</u> Let $T: \mathfrak{X} \longrightarrow \mathcal{Y}$ be a functor.

The <u>functor T preserves equalizers</u> if whenever $u: X \longrightarrow X'$ is an equalizer of $f_1, f_2: X' \longrightarrow X''$ in $\mathfrak{X}$, then $T(u): T(X) \longrightarrow T(X')$ is an equalizer of $T(f_1), T(f_2): T(X') \longrightarrow T(X'')$ in $\mathcal{Y}$.

The <u>functor T preserves coequalizers</u> if whenever $v: X'' \longrightarrow X$ is a coequalizer of $f_1, f_2: X' \longrightarrow X''$ in $\mathfrak{X}$, then $T(v): T(X'') \longrightarrow T(X)$ is a coequalizer of $T(f_1), T(f_2): T(X') \longrightarrow T(X'')$ in $\mathcal{Y}$.

<u>4.6 Proposition.</u> If $T: \mathfrak{X} \longrightarrow \mathcal{Y}$ is an adjoint functor, then T preserves equalizers.

<u>Proof.</u> Let $(\alpha, \beta): S \dashv T$ be an adjoint pair.

Suppose $f_1, f_2: X' \longrightarrow X''$ are morphisms in $\mathfrak{X}$, and $u: X \longrightarrow X'$ is an equalizer of f_1, f_2. Suppose $g: Y \longrightarrow T(X')$

is a morphism in $\mathcal{Y}$ such that $T(f_1)g = T(f_2)g$. Now $\alpha(X')S(g)$: $S(Y) \longrightarrow X'$, and further $f_1\alpha(X')S(g) = \alpha(X'')S(T(f_1)g) = \alpha(X'')S(T(f_2)g) = f_2\alpha(X')S(g)$. Hence there is a unique $h:S(Y) \longrightarrow X$ such that $uh = \alpha(X')S(g)$. Let $\bar{g} = T(h)\beta(Y):Y \longrightarrow T(X)$. Note that $T(u)\bar{g} = T(uh)\beta(Y) = T(\alpha(X'))TS(g)\beta(Y) = T(\alpha(X'))\beta(T(X'))g = g$. By 4.4, u is a monomorphism, and by 2.17*, $T(u)$ is a monomorphism. Thus $\bar{g}$ is the unique morphism such that $T(u)\bar{g} = g$, $T(u)$ is an equalizer of $T(f_1)$, $T(f_2)$, and the proposition is proved.

Observe that now it has been proved that an adjoint functor preserves monomorphisms, products, and equalizers, and by duality a coadjoint functor preserves epimorphisms, coproducts, and coequalizers, 2.17, 3.30, and 4.6.

<u>4.7 Proposition</u>. If $\mathcal{X}'$ is a weak coseparation subcategory of $\mathcal{X}$ with reflection $R : \mathcal{X} \longrightarrow \mathcal{X}'$, $\rho: R \longrightarrow 1_{\mathcal{X}}$, then

1) if $f_1,f_2: X' \longrightarrow X''$ are morphisms in $\mathcal{X}'$, and $u: X \longrightarrow X'$ is an equalizer of f_1,f_2 in $\mathcal{X}$, then $R(u)$: $R(X) \longrightarrow X'$ is an equalizer of f_1,f_2 in $\mathcal{X}'$, and

2) if $f_1,f_2: X' \longrightarrow X''$ are morphisms in $\mathcal{X}'$, and $v: X'' \longrightarrow X$ is a coequalizer of f_1,f_2 in $\mathcal{X}$, then v is a coequalizer of f_1,f_2 in $\mathcal{X}'$.

<u>Proof</u>. Part 1 follows at once from the preceding proposition. As for part 2), $R(v)f_1 = R(v)f_2$, and there is a unique morphism

$w: X \longrightarrow \mathcal{R}(X)$ such that $\mathcal{R}(v) = w\, v$. Thus $v = \rho(X)\mathcal{R}(v) = \rho(X)w\, v$, and $\rho(X)w = 1_X$ since v is an epimorphism. Hence since $\mathfrak{X}'$ is a weak coseparation subcategory of $\mathfrak{X}$, $\rho(X) = 1_X$, part 2) is proved, and the proposition is proved.

4.8 Corollary. If $\mathfrak{X}$ is a category and $\mathfrak{X}'$ is a weak coseparation subcategory of $\mathfrak{X}$, then

1) if $\mathfrak{X}$ has equalizers, then $\mathfrak{X}'$ has equalizers, and

2) if $\mathfrak{X}$ has coequalizers, then $\mathfrak{X}'$ has coequalizers.

4.9 Definitions. A category $\mathfrak{X}$ is complete if it has equalizers and products, it is countably complete if it has equalizers and countable products, and it is finitely complete if it has equalizers and finite products.

A category $\mathfrak{X}$ is cocomplete if it has coequalizers and coproducts, it is countably cocomplete if it has coequalizers and countable coproducts, and it is finitely cocomplete if it has coequalizers and finite coproducts.

4.10 Proposition. The category $\mathfrak{X}$ is complete, countably, or finitely complete if and only if the category $\mathfrak{X}^*$ is cocomplete, countably cocomplete, or finitely cocomplete.

4.11 Definitions. The category $\mathfrak{X}$ is bicomplete if it is complete and cococomplete, it is countably bicomplete if it is countably complete and countably cocomplete, and it is finitely bicomplete if it is finitely complete and finitely cocomplete.

Observe that bicompleteness is a self dual notion.

4.12 Examples. The category $\mathcal{S}$ is bicomplete.

Suppose that $f_1, f_2 \colon S' \longrightarrow S''$ are functions. Let $S = \{x \mid x \in S' \text{ and } f_1(x) = f_2(x)\}$, and let $u \colon S \longrightarrow S'$ be the natural inclusion function. One sees at once that u is an equalizer of f_1, f_2. Hence $\mathcal{S}$ has equalizers.

Suppose again that $f_1, f_2 \colon S' \longrightarrow S''$ are functions. Let S be the set obtained from S'' by identifying $f_1(x)$ with $f_2(x)$ for every $x \in S'$, i.e. one divides S'' by the least equivalence relation on S'' having the property that $f_1(x)$ is equivalent with $f_2(x)$ for every $x \in S'$. Let $v \colon S'' \longrightarrow S$ be the function that takes an element of S'' into its equivalence class. One verigies at once that v is a coequalizer of f_1, f_2. Hence $\mathcal{S}$ has coequalizers.

Since it has already been observed that $\mathcal{S}$ has products and coproducts, it now follows that $\mathcal{S}$ is bicomplete.

Recall that a set S is finite if whenever $f \colon S \longrightarrow S$ is an injection then f is a bijection or equivalently if whenever f is a surjection then f is a bijection. Let $\mathcal{S}_f$ be the full subcategory of $\mathcal{S}$ generated by the finite sets. The category $\mathcal{S}_f$ is finitely bicomplete, and the natural inclusion functor $\mathcal{S}_f \longrightarrow \mathcal{S}$ preserves equalizers, coequalizers, finite products and finite coproducts.

<u>4.13 Proposition</u>. If $\mathfrak{X}'$ is a weak coseparation subcategory of $\mathfrak{X}$, then

1) $\mathfrak{X}'$ is complete, countably complete, or finitely complete if $\mathfrak{X}$ is complete, countably complete, or finitely complete, and

2) $\mathfrak{X}'$ is cocomplete, countably cocomplete, or finitely cocomplete if $\mathfrak{X}$ is cocomplete, countably cococomplete, or finitely cocomplete.

The proposition follows at once from 4.8, 3.32, and 3.34.

<u>4.14 Definitions</u>. Let $\mathfrak{X}$ be a category, and

$$\begin{array}{ccc} X_0' & \xrightarrow{f_0} & X_0'' \\ \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u''} \\ X_1' & \xrightarrow{f_1} & X_1'' \end{array}$$

a commutative diagram in $\mathfrak{X}$.

The diagram is a <u>cartesian square</u> if whenever $g:Y \longrightarrow X_1'$, $h:Y \longrightarrow X_0''$ are morphisms in $\mathfrak{X}$ such that $f_1 g = u''h$ there is a unique $v:Y \longrightarrow X_0'$ such that $u'v = g$ and $f_0 v = h$.

The diagram is a <u>cocartesian square</u> if whenever $g:X_1' \longrightarrow Y$, $h:X_0'' \longrightarrow Y$ are morphisms in $\mathfrak{X}$ such that $ju' = hf_0$ there is a unique $v: X_1'' \longrightarrow Y$ such that $vf_1 = g$ and $vu'' = h$.

Observe that a diagram in $\mathfrak{X}$ is a cartesian square if and only if the dual diagram in $\mathfrak{X}^*$ is a cocartesian square. Cartesian squares have sometimes been called pull-back diagrams and cocartesian squares have then been called push-out diagrams.

<u>4.15 Proposition</u>. If $\mathcal{X}$ is a category which has a terminal point, the following conditions are equivalent:

1) $\mathcal{X}$ is a finitely complete,

2) every diagram

$$\begin{array}{ccc} & & X''_0 \\ & & \downarrow{\scriptstyle u''} \\ X'_1 & \xrightarrow{f_1} & X''_1 \end{array}$$

in $\mathcal{X}$ can be completed to a cartesian square in $\mathcal{X}$.

<u>Proof</u>. Suppose 1), and that

$$\begin{array}{ccc} & & X''_0 \\ & & \downarrow{\scriptstyle u''} \\ X'_1 & \xrightarrow{f_1} & X''_1 \end{array}$$

is a diagram in $\mathcal{X}$. Let $p:X'_1 \Pi X''_0 \longrightarrow X'_1$, $q:X'_1 \Pi X''_0 \longrightarrow X''_0$ be a product of X'_1, X''_0 in $\mathcal{X}$, and $v:X'_0 \longrightarrow X'_1 \Pi X''_0$ an equalizer of $f_1 p, u'' q$. Letting $u' = pv$, $f_0 = qv$, one checks at once that

$$\begin{array}{ccc} X'_0 & \xrightarrow{f_0} & X''_0 \\ \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u''} \\ X'_0 & \xrightarrow{f_1} & X''_1 \end{array}$$

is a cartesian square in $\mathcal{X}$. Hence 1) implies 2).

Suppose 2). Let $*_1$ be a terminal point of $\mathcal{X}$. If X_1, X_2 are objects of $\mathcal{X}$, let

$$\begin{array}{ccc} X & \xrightarrow{p_2} & X_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \varepsilon(X_2)} \\ X_1 & \xrightarrow{\varepsilon(X_1)} & *_1 \end{array}$$

be a cartesian square in $\mathcal{X}$. Using the fact that $*_1$ is a terminal point of $\mathcal{X}$ it follows at once that $p_1 : X \longrightarrow X_1$, $p_2 : X \longrightarrow X_2$ is a product of X_1 and X_2 in $\mathcal{X}$. Details of finishing the verification that $\mathcal{X}$ has finite products are left to the reader (§3, Ex. 1). Suppose $f_1, f_2 : X' \longrightarrow X''$ are morphisms in $\mathcal{X}$. Let

$$\begin{array}{ccc} X & \xrightarrow{v} & X'' \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle 1_{X''} \top 1_{X''}} \\ X' & \xrightarrow{f_1 \top f_2} & X'' \Pi X'' \end{array}$$

be a cartesian square in $\mathcal{X}$. One verifies at once that u is an equalizer of f_1, f_2. Thus $\mathcal{X}$ has equalizers, 2) implies 1), and the proposition is proved.

<u>4.16 Proposition</u>. If $\mathcal{X}$ is a category, then the following are equivalent:

1) $f : X' \longrightarrow X''$ is a monomorphism in $\mathcal{X}$, and

2)

$$\begin{array}{ccc} X' & \xrightarrow{1_{X'}} & X' \\ \downarrow{\scriptstyle 1_{X'}} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{f} & X'' \end{array}$$

is a cartesian square in $\mathcal{X}$.

The proposition follows at once from the definitions.

4.17 Proposition. If $\mathfrak{X}$ is a category, $\mathcal{Y}$ is a category with terminal point $*_1$, and $T_1 : \mathfrak{X} \longrightarrow \mathcal{Y}$ is the functor such that if $X \in \mathrm{obj}(\mathfrak{X})$, $T_1(X) = *_1$, and if $f \in \mathrm{mor}(\mathfrak{X})$, then $T_1(f) = 1_{*_1}$, then if $T: \mathfrak{X} \longrightarrow \mathcal{Y}$ is a functor, there is a unique morphism of functors $\varepsilon : T \longrightarrow T_1$.

Proof. For $X \in \mathrm{obj}(\mathfrak{X})$, let $\varepsilon(X): T(X) \longrightarrow *_1$ be the unique morphism in $\mathcal{Y}$ from $T(X)$ to $*_1$.

4.18 Proposition. If $T, T'': \mathfrak{X} \longrightarrow \mathcal{Y}$ are functors, $\mathcal{Y}$ is a finitely complete category, and $\alpha: T \longrightarrow T''$ is a morphism of functors, then there is a functor $T': \mathfrak{X} \longrightarrow \mathcal{Y}$ and morphisms of functors $\alpha_1, \alpha_2: T' \longrightarrow T$ such that for $X \in \mathrm{obj}(\mathfrak{X})$ the diagram

$$\begin{array}{ccc} T'(X) & \xrightarrow{\alpha_2(x)} & T(X) \\ \Big\downarrow {\scriptstyle \alpha_1(X)} & & \Big\downarrow {\scriptstyle \alpha(X)} \\ T(X) & \xrightarrow{\alpha(X)} & T''(X) \end{array}$$

is a cartesian square in $\mathcal{Y}$.

Proof. For $X \in \mathrm{obj}(\mathfrak{X})$ choose a cartesian square satisfying the conditions of the square of the diagram of the proposition. If $f: X' \longrightarrow X''$ is a morphism in $\mathfrak{X}$, then $\alpha(X'')T(f)\alpha_1(X') = T''(f)\alpha(X')\alpha_1(X')$, $\alpha(X'')T(f)\alpha_2(X') = T''(f)\alpha(X')\alpha_2(X')$, and hence $\alpha(X'')T(f)\alpha_1(X') = \alpha(X'')T(f)\alpha_2(X')$. Thus there is a unique $T'(f)\colon T'(X') \longrightarrow T'(X'')$ such that $\alpha_1(X'')T'(f) = T(f)\alpha_1(X')$ and $\alpha_2(X'')T'(f) = T(f)\alpha_2(X')$. Consequently $T'\colon \mathfrak{X} \longrightarrow \mathcal{Y}$ is a functor, $\alpha_1, \alpha_2\colon T' \longrightarrow T$ are morphisms satisfying the required conditions, and the proposition is proved.

4.19 Corollary. If $T, T''\colon \mathfrak{X} \longrightarrow \mathcal{Y}$ are functors, and $\mathcal{Y}$ is a finitely complete category, then the following are equivalent:

1) $\alpha\colon T \longrightarrow T''$ is a monomosphism of functors,

2) $\alpha\colon T \longrightarrow T''$ is a local monomorphism of functors.

The proposition follows at once from 4.18, 4.17, and the definitions 2.7, 4.14.

4.20 Definitions. The category $\mathcal{J}$ is a small category if the the class $\mathrm{mor}(\mathcal{J})$ is a set, it is a countable category if $\mathrm{mor}(\mathcal{J})$ is a countable set, and a finite category if $\mathrm{mor}(\mathcal{J})$ is a finite set.

Observe that $\mathcal{J}$ is small, countable, or finite, if and only if $\mathcal{J}^*$ is small, countable or finite.

4.21 Examples and definitions. An ordered set J is a set $T(J)$ together with a subset $\mathcal{R}(J)$ of $T(J) \times T(J)$ called the order relation of J such that if one denotes by $j_1 \leq j_2$ the fact that the ordered pair $(j_1, j_2) \in \mathcal{R}(J)$, then

1) $j \leq j$ for all $j \in T(J)$,

2) $j_1 \leq j_2$ and $j_2 \leq j_3$ implies $j_1 \leq j_3$ for $j_1, j_2, j_3 \in T(J)$,

and

3) $j_1 \leq j_2$ and $j_2 \leq j_1$ implies $j_1 = j_2$ for $j_1 j_2 \in T(J)$.

Condition 1) says that the order relation is reflexive and condition 2) that it is transitive.

If J', J'' are ordered sets, an order preserving function $f: J' \longrightarrow J''$ is a function $T(f): T(J') \longrightarrow T(J'')$ such that $T(f)(j_1) \leq T(f)(j_2)$ for $(j_1, j_2) \in \mathcal{R}(J')$. The category ord($\mathcal{S}$) of ordered sets and order preserving functions is the category whose objects are ordered sets, whose morphisms are order preserving functions and such that composition is induced by composition of functions.

For $n \in \mathbf{Z}$, $n \geq 0$ let Δ_n be the ordered set whose elements are integers $j \in \mathbf{Z}$ such that $0 \leq j \leq n$, and such that the ordering of Δ_n is induced by the standard ordering of the natural numbers. The simplicial category Δ is the full subcategory of ord($\mathcal{S}$) generated by the objects Δ_n, $n \in \mathbf{Z}$, $n \geq 0$. The category Δ is a countable category such that if Δ_m, Δ_n are objects of Δ, then $\Delta(\Delta_m, \Delta_n)$ is a finite set. The cosimplicial category Δ^* is the dual of the category Δ.

An ordered set J is a small category such that $\mathrm{obj}(J) = T(J)$, and if j_1 and j_2 are objects of J there is a morphism from j_1 to j_2 in J if and only if $j_1 \leq j_2$, and in this case the morphism is unique. Composition of morphisms is determined by the transitivity of the order relation.

If $\mathcal{J}$ is a small category such that there is at most one morphism between any two objects of $\mathcal{J}$ and such that every isomorphism is an identity morphism, then $\mathcal{J}$ is an ordered set with set obj($\mathcal{J}$) and order relation $j_1 \leq j_2$ if there is a morphism in $\mathcal{J}$ from j_1 to j_2.

An ordered set J is <u>discretely ordered</u> if $j_1 \leq j_2$ implies $j_1 = j_2$, i.e. the only morphisms in the small category J are identity morphisms.

<u>4.22 Definitions</u>. If $\mathcal{J}$ is a small category and $\mathfrak{X}$ is a category, the functor category $[\mathcal{J}, \mathfrak{X}]$ is the category such that

1) an object of $[\mathcal{J}, \mathfrak{X}]$ is a functor $T: \mathcal{J} \longrightarrow \mathfrak{X}$,

2) if T', T'' are objects of $[\mathcal{J}, \mathfrak{X}]$ a morphism $\alpha: T' \longrightarrow T''$ in $[\mathcal{J}, \mathfrak{X}]$ is a morphism of functors, and

3) composition of morphisms is composition of morphisms of functors.

If $\mathfrak{X}$ is a category, the <u>simplicial category over $\mathfrak{X}$</u> is the functor category $[\Delta^*, \mathfrak{X}]$ and the <u>cosimplicial category over $\mathfrak{X}$</u> is the functor category $[\Delta, \mathfrak{X}]$.

The condition that $\mathcal{J}$ is a small category insures that for any category $\mathfrak{X}$, if (T', T'') is an ordered pair of objects of $[\mathcal{J}, \mathfrak{X}]$, then $[\mathcal{J}, \mathfrak{X}](T', T'')$ is a set.

<u>4.23 Proposition.</u> If $\mathcal{J}$ is a small category and $\mathcal{X}$ is a category, the category $[\mathcal{J},\mathcal{X}]^*$ is the category $[\mathcal{J}^*, \mathcal{X}^*]$.

The proposition is evident. Note that if $\mathcal{J}$ is a small category, the category of contravariant functors from $\mathcal{J}$ to $\mathcal{X}$ may be viewed either as $[\mathcal{J}^*, \mathcal{X}]$ or $[\mathcal{J}, \mathcal{X}^*[$. The category of simplicial objects over $\mathcal{X}$, $[\Delta^*, \mathcal{X}]$ is the category of contravariant functors from the simplicial category Δ to the category $\mathcal{X}$, and its dual is the category $[\Delta, \mathcal{X}^*]$ of cosimplicial objects over $\mathcal{X}^*$.

<u>4.24 Proposition.</u> If $\mathcal{J}$ is a small category and $\mathcal{X}$ is a category with equalizers, then $[\mathcal{J}, \mathcal{X}]$ is a category with equalizers, and if $\alpha\colon T' \longrightarrow T$, $\alpha_1,\alpha_2\colon T \longrightarrow T''$ are morphisms in $[\mathcal{J}, \mathcal{X}]$, the following are equivalent:

1) α is an equalizer of α_1,α_2 and

2) for $J \in \text{obj}(\mathcal{J})$, $\alpha(J)$ is an equalizer of $\alpha_1(J),\alpha_2(J)$.

<u>Proof.</u> Suppose $\alpha_1,\alpha_2\colon T \longrightarrow T''$ are morphisms in $[\mathcal{J}, \mathcal{X}]$. For every object J of $\mathcal{J}$, let $\alpha(J)\colon T'(J) \longrightarrow T(J)$ be an equalizer of $\alpha_1(J)$, $\alpha_2(J)$ in $\mathcal{X}$. If $j\colon J' \longrightarrow J''$ is a morphism in $\mathcal{J}$, then $\alpha_1(J'')T(j)\alpha(J') = T''(j)\alpha_1(J')\alpha(J') = T''(j)\alpha_2(J')\alpha(J') = \alpha_2(J'')T(j)\alpha(J')$. Hence there is a unique morphism $T'(j)\colon T'(J') \longrightarrow T'(J'')$ such that $\alpha(J'')T'(j) = T(j)\alpha(J')$. Now $T'\colon \mathcal{J} \longrightarrow \mathcal{X}$ is a functor, $\alpha\colon T' \longrightarrow T''$ is an equalizer of α_1,α_2 in $[\mathcal{J}, \mathcal{X}]$, and the proposition is proved.

<u>4.25 Proposition.</u> If $\mathcal{J}$ is a small category, I is a set, and $\mathcal{X}$ is a category which has I-indexed products, then

1) $[\mathcal{J},\mathcal{X}]$ has I-indexed products, and

2) if $(p_i)_{i\in I}$, $p_i\colon T \longrightarrow T_i$ is an I-indexed set of morphisms in $[\mathcal{J},\mathcal{X}]$, then the following are equivalent:

i) $(p_i)_{i\in I}$ is a product of $(T_i)_{i\in I}$ in $[\mathcal{J},\mathcal{X}]$, and

ii) for every $J \in \mathrm{obj}(\mathcal{J})$, $(p_i(J))_{i\in I}$ is a product of $(T_i(J))_{i\in I}$ in $\mathcal{X}$.

<u>Proof.</u> Suppose $(T_i)_{i\in I}$ is an I-indexed set of objects of $[\mathcal{J},\mathcal{X}]$. For $J \in \mathrm{obj}(\mathcal{J})$ let $(p_i(J))_{i\in I}$, $p_i(J)\colon T(J) \longrightarrow T_i(J)$ be a product of $(T_i(J))_{i\in I}$ in $\mathcal{X}$. If $j\colon J' \longrightarrow J''$ is a morphism in $\mathcal{J}$, let $T(j)\colon T(J') \longrightarrow T(J'')$ be the unique morphism such that $p_i(J'')T(j) = T_i(j)\,p_i(J')$ for $i \in I$. Now T is an object of $[\mathcal{J},\mathcal{X}]$, $(p_i)_{i\in I}$, $p_i\colon T \longrightarrow T_i$ is a product of $(T_i)_{i\in I}$ in $[\mathcal{J},\mathcal{X}]$, $[\mathcal{J},\mathcal{X}]$ has I-indexed products, and the construction of products in $[\mathcal{J},\mathcal{X}]$ from products in $\mathcal{X}$ insures the validity of part 2) of the proposition. Consequently the proposition is proved.

<u>4.26 Proposition.</u> If $\mathcal{J}$ is a small category and $\mathcal{X}$ is a category, then the functor category $[\mathcal{J},\mathcal{X}]$ is complete, countably complete, or finitely complete according as $\mathcal{X}$ is complete, countably complete, or finitely complete.

The proposition follows at once from 4.24 and 4.25.

Observe that the duals of the preceding propositions assure that coequalizers, coproducts, etc. pass appropriately to functor categories.

<u>4.27 Definition</u>. If $\mathcal{J}$ is a small category and $\mathcal{X}$ is a category, the <u>constant functor</u> $C_{\mathcal{J}}(\ ): \mathcal{X} \longrightarrow [\mathcal{J}, \mathcal{X}]$ is the functor such that

1) if $X \in obj(\mathcal{X})$, then for $J \in obj(\mathcal{J})$, $C\ (X)(J) = X$, and for $j \in mor(\mathcal{J})$, $C_{\mathcal{J}}(X)(j) = 1_X$, and

2) if $f: X' \longrightarrow X''$ is a morphism in $\mathcal{X}$, then $C_{\mathcal{J}}(f)(J) = f$ for $J \in obj(\mathcal{J})$.

<u>4.28 Definitions</u>. Let $\mathcal{J}$ be a small category and $\mathcal{X}$ a category.

If T is an object of $[\mathcal{J}, \mathcal{X}]$ a <u>limit of T</u> is an object $\lim_{\mathcal{J}}(T)$ of $\mathcal{X}$ and a morphism $\alpha(T): C_{\mathcal{J}}(\lim_{\mathcal{J}}(T)) \longrightarrow T$ which is a $C_{\mathcal{J}}(\ )$ reflection of T. A <u>colimit of T</u> is an object colim (T) of $\mathcal{X}$ and a morphism $\beta(T): T \longrightarrow C_{\mathcal{J}}(\mathrm{colim}_{\mathcal{J}}(T))$ which is a $C_{\mathcal{J}}(\ )$ coreflection of T.

Limits are also called <u>projective limits</u> or <u>inverse limits</u>, while colimits are also called <u>injective limits or direct limits</u>.

<u>4.29 Theorem</u>. If $\mathcal{X}$ is a category, then

1) the following conditions are equivalent:

i) $\mathfrak{X}$ is finitely complete, and

ii) for every finite category $\mathcal{J}$, $C_{\mathcal{J}}(\ ): \mathfrak{X} \longrightarrow [\mathcal{J}, \mathfrak{X}]$ is a coadjoint functor,

2) the following conditions are equivalent:

i) $\mathfrak{X}$ is countably complete, and

ii) for every countable category $\mathcal{J}$, $C_{\mathcal{J}}(\ ): \mathfrak{X} \longrightarrow [\mathcal{J}, \mathfrak{X}]$ is a coadjoint functor, and

3) the following conditions are equivalent:

i) $\mathfrak{X}$ is complete, and

ii) for every small category $\mathcal{J}$, $C_{\mathcal{J}}(\ ): \mathfrak{X} \longrightarrow [\mathcal{J}, \mathfrak{X}]$ is a coadjoint functor.

Proof. Suppose either (i), ii), or 3i), and that $\mathcal{J}$ is a category satisfying the corresponding condition. For T an object of $[\mathcal{J}, \mathfrak{X}]$, let $(P(J))_{J \in obj(\mathcal{J})}$, $P(J): C^0(T) \longrightarrow T(J)$ be a product of $(T(J))_{J \in obj(\mathcal{J})}$ in $\mathfrak{X}$. For $j \in mor(\mathcal{J})$, let $R(j)$ be the range of $\mathcal{J}$, and let $(P(j))_{j \in mor(\mathcal{J})}$, $P(j): C^1(T) \longrightarrow T(R)j))$ be a product of $(T(R(j))_{j \in mor(\mathcal{J})}$ in $\mathfrak{X}$. Let $\delta^0, \delta^1: C^0(T) \longrightarrow C^1(T)$ be the morphisms in $\mathfrak{X}$ such that for $j \in mor(\mathcal{J})$, $P(j)\,\delta^0 = P(R(j))$ and $P(j)\,\delta^1 = T(j)\,P(D(j))$ where $D(j)$ is the domain of j. Let $\delta: \lim_{\mathcal{J}}(T) \longrightarrow C^0(T)$ be an equalizer of δ^0, δ^1, and $\alpha(T): C_{\mathcal{J}}(\lim(T)) \longrightarrow T$ the morphism in $[\mathcal{J}, \mathfrak{X}]$ such that $\alpha(T)(J) = P(J)\delta$ for $J \in obj(\mathcal{J})$. One verifies at once that $\alpha(T)$ is a $C_{\mathcal{J}}(\ )$

reflection of T. Consequently every object in $[\mathcal{J}, \mathcal{X}]$ has a $C_{\mathcal{J}}(\ )$ reflection, i.e. a limit. The fact that i) implies ii) under the conditions of either part 1), 2), or 3) of the theorem now follows from 2.23.

Suppose 1ii). Let $\mathcal{J}$ be the small category with two objects J', J'' and four morphisms $1_{J'}, 1_{J''}$, and i_0, i_1: $J' \longrightarrow J''$. Let (α, β): $C_{\mathcal{J}}(\ ) \dashv \lim(\ )$ be an adjoint pair. If f_0, f_1: $X' \longrightarrow X''$ are morphisms in $\mathcal{X}$, let T be the object of $[\mathcal{J}, \mathcal{X}]$ such that $T(i_0) = f_0$, $T(i_1) = f_1$. Now $\alpha(J)(J')$: $\lim_{\mathcal{J}}(T) \longrightarrow X'$ is an equalizer of f_0, f_1. Hence $\mathcal{X}$ has equalizers.

If I is a set, consider it as a discretely ordered set, and hence as a small category. The objects of $[I, \mathcal{X}]$ are now just I-indexed sets of objects of $\mathcal{X}$, and a $C_I(\ )$ reflection of an object $(X_1)_{i\in I}$ of $[I, \mathcal{X}]$ in $\mathcal{X}$ is exactly a product of $(X_i)_{i\in I}$ in $\mathcal{X}$. The preceding observations complete the proof of the theorem.

<u>4.30 Notation and comments.</u> If $\mathcal{J}$ is a small category and $\mathcal{X}$ is a category, if an adjoint for the constant functor $C_{\mathcal{J}}(\ )$: $\mathcal{X} \longrightarrow [\mathcal{J}, \mathcal{X}]$ exists, it is usually denoted by $\lim_{\mathcal{J}}(\ ): [\mathcal{J}, \mathcal{X}] \longrightarrow \mathcal{X}$, and the adjoint pair by $(\alpha_{\mathcal{J}}, \beta_{\mathcal{J}})$: $C_{\mathcal{J}} \longrightarrow \lim_{\mathcal{J}}: ([\mathcal{J}, \mathcal{X}], \mathcal{X})$. Often abbreviations of the notation are used such as dropping the subscripts $\mathcal{J}$ when they are clear from the context. A

standard exception to the preceding is if I is a set, i.e. a discrete small category, the adjoint for $C_I: \mathcal{X} \longrightarrow [1, \mathcal{X}]$ is denoted by $\prod_I(\ ): [I, \mathcal{X}] \longrightarrow \mathcal{X}$ in conformity with the notations used earlier it was not yet known that products are limits and thus a special case of reflections.

Observe that the dual of $C_{\mathcal{J}}(\ ): \mathcal{X} \longrightarrow [\mathcal{J}, \mathcal{X}]$ is $C_{\mathcal{J}^*}(\ ): \mathcal{X}^* \longrightarrow [\mathcal{J}^*, \mathcal{X}^*]$. If a coadjoint for the functor $C_{\mathcal{J}}(\ ): \mathcal{X} \longrightarrow [\mathcal{J}, \mathcal{X}]$ exists it is usually denoted by $\operatorname{colim}_{\mathcal{J}}(\ ): [\mathcal{J}, \mathcal{X}] \longrightarrow \mathcal{X}$, and the adjoint pair by $(\alpha_{\mathcal{J}}, \beta_{\mathcal{J}})$: $\operatorname{colim}_{\mathcal{J}}(\ ) \dashv C_{\mathcal{J}} : (\mathcal{X}, [\mathcal{J}, \mathcal{X}])$ or some appropriate abbreviation. Note that although neither of the limit functors are unique, they are unique up to functional isomorphism (2.24). A standard exception to the usual colimit notation is that in the case of a set I, a coadjoint for $C_I: \mathcal{X} \longrightarrow [1, \mathcal{X}]$ is denoted by $\coprod_I(\ ): [I, \mathcal{X}] \longrightarrow \mathcal{X}$ in conformity with the earlier notation used for I-indexed coproducts.

There are other special exceptions to the preceding. They usually occur in dealingwith some special classical category such as $\mathcal{S}$ where for a set I, the usual product notation is $\times_I(\ ): [I, \mathcal{S}] \longrightarrow \mathcal{S}$.

It is perhaps worth observing that diagrams

$$\begin{array}{ccc} & & X_0 \\ & & \downarrow f_0 \\ X_1 & \xrightarrow{f_1} & X_2 \end{array}$$

in a category $\mathcal{X}$, are nothing but objects in the functor category $[\mathcal{J},\mathcal{X}]$ where $\mathcal{J}$ is a small category with three objects and five morphisms, the objects being J_0, J_1, J_2 and the morphisms in $\mathcal{J}$ other than identity morphisms being $j_0\colon J_0 \longrightarrow J_2$, $j_1\colon J_0 \longrightarrow J_2$. The limit of an object of $[\mathcal{J},\mathcal{X}]$ may be viewed as a cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\;i_0\;} & X_0 \\ \Big\downarrow{\scriptstyle i_1} & & \Big\downarrow{\scriptstyle f_0} \\ X_1 & \xrightarrow{\;f_1\;} & X_2 \end{array}$$

in $\mathcal{X}$ where the lower right-hand part is the original object of $[\mathcal{J},\mathcal{X}]$. In the proof of the preceding proposition, this fact could have been used in proving the implication (ii) implies (i) rather than properties of equalizers.

<u>4.31 Proposition.</u> If $(\alpha,\beta)\colon S \dashv T\colon (\mathcal{X},\mathcal{Y})$ is an adjoint pair of functors, $S_0\colon \mathcal{X} \longrightarrow \mathcal{Z}$ is a functor, Z an object of $\mathcal{Z}$, $T_0(Z)$ an object of $\mathcal{X}$, and $\alpha_0(Z)\colon S_0(T_0(Z)) \longrightarrow Z$ an S_0 reflection of Z, then the composite $S_0ST(T_0(Z)) \xrightarrow{S_0\alpha(T_0(Z))} S_0(T_0(Z)) \xrightarrow{\alpha_0(Z)} Z$ is an SS_0 reflection of Z.

<u>Proof.</u> Suppose $g : S_0S(Y) \longrightarrow Z$ is a morphism in $\mathcal{Z}$, where $Y \in \mathrm{obj}(\mathcal{Y})$. There is a unique $\bar{g}\colon S(Y) \longrightarrow T_0(Z)$ in $\mathcal{X}$ such that $g = \alpha_0(Z)S_0(\bar{g})$ since $\alpha_0(Z)$ is an S_0 reflection of Z.

There is a unique $\hat{g}: Y \longrightarrow TT_0(Z)$ in $\mathcal{Y}$ such that $\bar{g} = \alpha(T_0(Z))S(\hat{g})$ since $\alpha(T_0(Z))$ is an S reflection of $T_0(Z)$. Now $\alpha_0(Z)S_0\alpha(T_0(Z))S_0S(\hat{g}) = g$, and $\hat{g}$ is the unique morphism having this property, which proves the proposition.

<u>4.32 Definitions</u>. Let $\mathcal{J}', \mathcal{J}''$ be small categories, $\mathcal{X}', \mathcal{X}''$ categories, $S: \mathcal{J}' \longrightarrow \mathcal{J}''$, $T: \mathcal{X}' \longrightarrow \mathcal{X}''$ functors. Let $[S,T]: [\mathcal{J}'', \mathcal{X}'] \longrightarrow [\mathcal{J}', \mathcal{X}'']$ be the functor such that

1) if V is an object of $[\mathcal{J}'', \mathcal{X}']$, then $[S,T](V) = TVS$, and 2) if $\alpha: V' \longrightarrow V''$ is a morphism in $[\mathcal{J}'', \mathcal{X}']$, then $[S,T](\alpha) = (T\alpha)S = T(\alpha S)$.

In the preceding if $\mathcal{J}' = \mathcal{J}''$ and $S = 1_{\mathcal{J}'}$, the functor $[S,T]$ is usually denoted by $[\mathcal{J}', T]$.

The <u>functor</u> $T: \mathcal{X}' \longrightarrow \mathcal{X}''$ <u>preserves limits</u> if whenever $\mathcal{J}$ is a small category, V is an object of $[\mathcal{J}, \mathcal{X}']$ and $\alpha(V): C_{\mathcal{J}}(\lim_{\mathcal{J}}(V)) \longrightarrow V$ is a limit of V, then $[\mathcal{J},T](\alpha(V)): C_{\mathcal{J}}(T(\lim_{\mathcal{J}}(V))) \longrightarrow [\mathcal{J},T](V)$ is a limit of $[\mathcal{J},T](V)$.

The <u>functor</u> $T: \mathcal{X}' \longrightarrow \mathcal{X}''$ <u>preserves colimits</u> if whenever $\mathcal{J}$ is a small category, V is an object of $[\mathcal{J}, \mathcal{X}']$, and $\beta(V): V \longrightarrow C_{\mathcal{J}}(\operatorname{colim}_{\mathcal{J}}(V))$ is a colimit of V, then $[\mathcal{J},T](\beta(V)): [\mathcal{J},T](V) \longrightarrow \mathcal{C}_{\mathcal{J}}(T(\operatorname{colim}_{\mathcal{J}}(V))$ is a colimit of $[\mathcal{J},T](V)$.

Thus a functor T preserves limits if and only if its dual preserves colimits.

<u>4.33 Proposition</u>. If $\mathcal{X}$ is a complete category and $T: \mathcal{X} \longrightarrow \mathcal{Y}$ is a functor, the following are equivalent:

1) T preserves products and equalizers,

2) T preserves limits.

The proposition follows at once from the construction of limits from products and equalizers in a complete category, 4.29.

<u>4.34 Proposition</u>. If $T: \mathcal{X} \longrightarrow \mathcal{Y}$ is an adjoint functor, then T preserves limits.

The proposition follows at once from 4.31. For the case $\mathcal{X}$ complete it also follows from 3.30, 7.6, and 4.33.

Observe that the dual of the preceding proposition is that a coadjoint functor preserves colimits.

<u>4.35 Proposition</u>. If $\mathcal{J}', \mathcal{J}''$ are small categories, then $\mathcal{J}' \times \mathcal{J}''$ is a small category, and if $\mathcal{X}$ is a category, there is an isomorphism of categories $\theta: [\mathcal{J}' \times \mathcal{J}'', \mathcal{X}] \longrightarrow [\mathcal{J}', [\mathcal{J}'', \mathcal{X}]]$ such that

1) if $J' \in obj(\mathcal{J}')$, $T \in obj[\mathcal{J}' \times \mathcal{J}'', \mathcal{X}]$, then
$\theta(T)(J')(J'') = T(J', J'')$ for $J'' \in obj(\mathcal{J}'')$, and
$\theta(T)(J')(j) = T(1_{J'}, j)$ for $j \in mor(\mathcal{J}'')$, and

2) if $j' \in mor(\mathcal{J}')$, $T \in obj[\mathcal{J}' \times \mathcal{J}'', \mathcal{X}]$, then
$\theta(T)(j')(J'') = T(j', 1_{J''})$ for $J'' \in obj(\mathcal{J}'')$.

<u>Proof</u>. The fact that $\mathcal{J}' \times \mathcal{J}''$ is a small category if $\mathcal{J}', \mathcal{J}''$ are follows at once.

Let $\Gamma: [\mathcal{J}', [\mathcal{J}'', \mathcal{X}]] \longrightarrow [\mathcal{J}' \times \mathcal{J}'', \ ]$ be the functor such that for $T \in obj([\mathcal{J}', [\mathcal{J}'', \mathcal{X}]]$, $\Gamma(T)(J', J') = T(J')(J'')$ for $(J', J'') \in obj(\mathcal{J}' \times \mathcal{J}'')$, and if $(j', j''): (J_0', J_0'') \longrightarrow (J_1', J_1'')$ is a morphism in $\mathcal{J}' \times \mathcal{J}''$, then $\Gamma(T)(j', j'') = T(j')(J_1'')\, T(J_0')(j'')$. Note that the diagram

$$\begin{array}{ccc} T(J_0')(J_0'') & \xrightarrow{T(j')(J_0'')} & T(J_1')(J_0'') \\ \downarrow{\scriptstyle T(J_0')(j'')} & & \downarrow{\scriptstyle T(J_1')(j'')} \\ T(J_0')(J_1'') & \xrightarrow{T(j')(J_1'')} & T(J_1')(J_1'') \end{array}$$

in $\mathcal{X}$ is commutative. The conditions of the proposition insure that the functor θ is well defined. Further $\Gamma\theta$ is the identity functor of $[\mathcal{J}' \times \mathcal{J}'', \mathcal{X}]$ and $\theta\Gamma$ the identity of $[\mathcal{J}', [\mathcal{J}'', \mathcal{X}]]$. Hence the proposition is proved.

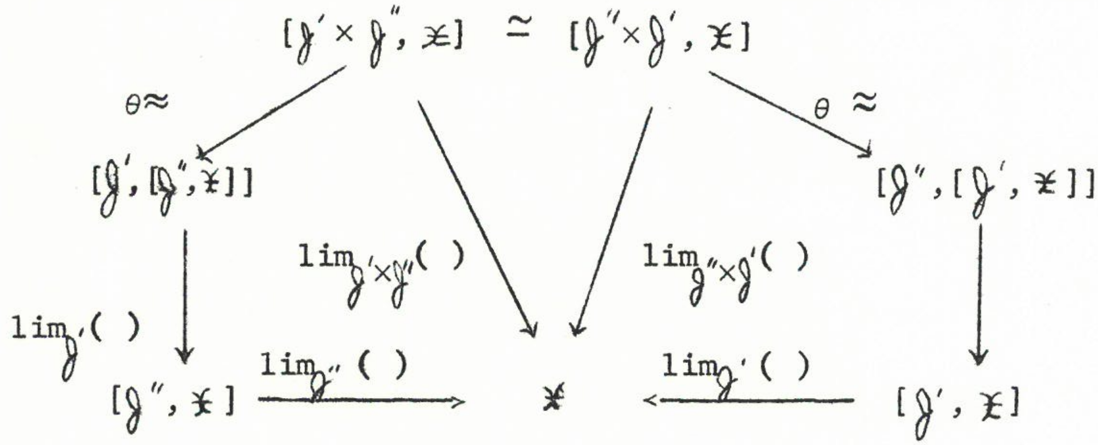
Observe that if $\mathcal{X}', \mathcal{X}''$ are categories, the category $\mathcal{X}' \times \mathcal{X}''$ is canonically isomorphic with the category $\mathcal{X}'' \times \mathcal{X}'$, the isomorphism $\mathcal{X}' \times \mathcal{X}'' = \mathcal{X}'' \times \mathcal{X}'$ being induced by the canonical interchange of entries in the ordered pairs involved in the definition of these categories.

<u>4.36 Proposition</u>. If $\mathcal{J}', \mathcal{J}''$ are categories, $\mathcal{X}$ is a category, and one of the following conditions is satisfied:

1) $\mathcal{J}', \mathcal{J}''$ are small and $\mathcal{X}$ is complete,

2) $\mathcal{J}', \mathcal{J}''$ are countable and $\mathcal{X}$ is countably complete, or

3) $\mathcal{J}', \mathcal{J}''$ are finite and $\mathcal{X}$ is finitely complete, then there exists a commutative diagram of categories and functors:



Proof. The proposition follows at once from the composition proposition for adjoint pairs of functors 2.19, 4.26, 4.29, and 4.35, together with the fact that the composites of the appropriate constant functors are again constant functors.

If $\mathcal{J}', \mathcal{J}''$ are small categories and $\mathcal{X}$ is a category, then using the natural isomorphism between $[\mathcal{J}'', [\mathcal{J}', \mathcal{X}]]$ and $[\mathcal{J}', [\mathcal{J}'', \mathcal{X}]]$ the constant functor $C_{\mathcal{J}''}(\ ): [\mathcal{J}', \mathcal{X}] \longrightarrow [\mathcal{J}'', [\mathcal{J}', \mathcal{X}]]$ may be identified with the functor $[\mathcal{J}', C_{\mathcal{J}''}(\ )]: [\mathcal{J}', [\mathcal{J}'', \mathcal{X}]]$. This identification is made for convenience in the next proposition showing the general local character of limits when they are universally defined.

<u>4.37 Proposition.</u> If $\mathcal{J}', \mathcal{J}''$ are categories, $\mathcal{X}$ is a category, and one of the following conditions is satisfied:

1) $\mathcal{J}', \mathcal{J}''$ are small and $\mathcal{X}$ is complete,

2) $\mathcal{J}', \mathcal{J}''$ are countable and $\mathcal{X}$ is countably complete, or

3) $\mathcal{J}', \mathcal{J}''$ are finite and $\mathcal{X}$ is finitely complete,

then if $T \in obj[\mathcal{J}', [\mathcal{J}'', \mathcal{X}]]$, $\lim_{\mathcal{J}''}(T) \in obj[\mathcal{J}', \mathcal{X}]$ and $\alpha(T): [\mathcal{J}', C_{\mathcal{J}''}(\ )]\ (\lim_{\mathcal{J}''}(T)) \longrightarrow T$ is a morphism in $[\mathcal{J}', [\mathcal{J}'', \mathcal{X}]]$, then the following are equivalent:

1) $\alpha(T)$ is a limit of T, and

2) for every object $J' \in obj(\mathcal{J}')$, $\alpha(T)(J')$ is a limit of $T(J')$.

<u>Proof.</u> Proposition 4.24 is the special case of the preceding where $\mathcal{J}''$ is the small category used to define equalizers (4.29), and proposition 4.25 the special case where $\mathcal{J}''$ is a discrete category. The construction of general limits from products and equalizers (4.29) insures that the two special cases imply the general case. Hence the proposition is proved.

<u>Exercises</u>

1. Show that the category $\mathcal{T}$ is bicomplete. Conclude that the categories $\mathcal{T}_K$, $\mathcal{T}_{CR}$, and $\mathcal{C}$ are bicomplete since they are weak separation subcategories of $\mathcal{T}$ (§3, ex. 7, 8, and 10), and that the category $\mathcal{T}^C$ is bicomplete since it is a weak coseparation subcategory of $\mathcal{T}$ (§3, ex. 11).

2. Show that the natural inclusion functor $\mathcal{T}_K \to \mathcal{T}$, $\mathcal{T}_{CR} \to \mathcal{T}_K$ preserve coproducts, but not coequalizers, and that the natural inclusion functors $\mathcal{C} \to \mathcal{T}_{CR}$, $\mathcal{C} \to \mathcal{T}_H$, $\mathcal{C} \to \mathcal{T}_{WH}$ preserve finite coproducts and coequalizers, but not countable coproducts. Show that the natural inclusion functor $\mathcal{T}^C \to \mathcal{T}$ does not preserve equalizers.

3. Define a functor $T_K(\ ): \mathrm{ord}(\mathcal{S}) \to \mathcal{T}_K$ having the following properties:

1) if J is an ordered set, the space $T_K(J)$ has the same underlying set as does the ordered set J, and

2) the functor $T_K(\ )$ induces an isomorphism of $\mathrm{ord}(\mathcal{S})$ with a weak coseparation subcategory of $\mathcal{T}_K$. Conclude $\mathrm{ord}(\mathcal{S})$ is bicomplete.

4. A space X is a Frechet space if every one point subspace of X is a closed subspace. Let $\mathcal{T}_F$ denote the full subcategory of $\mathcal{T}$ generated by the Frechet spaces. Show that $\mathcal{T}_F$ is a weak separation subcategory of $\mathcal{T}_K$. Prove that the epimorphisms in $\mathcal{T}_F$ are the surjective maps and the monomorphisms are the injective maps.

5. Let N denote the natural numbers, i.e. the set of positive integers. Consider N as topologized with the topology such that $\{n\}$ is open in N for $n \in N$, $n \neq 0$, and the neighborhoods of $\{0\}$ are the subsets containing $\{0\}$ and having a finite complement. With this topology N is a compact

space. A space X is sequentially separated if whenever $f: N \longrightarrow X$ is a map, the subspace of X determined by the set theoretic image of f is a closed subspace. Let $\mathcal{T}_{SF}$ denote the full subcategory of $\mathcal{T}$ generated by the sequentially separated spaces. Show that $\mathcal{T}_{SF}$ is a weak separation subcategory of $\mathcal{T}_F$, and that the monomorphisms in $\mathcal{T}_{SF}$ are the injective maps.

5. A subspace A of a space X is sequentially closed in X if whenever $f: N \longrightarrow X$ is a map, $f^{-1}(A)$ is closed in N. Show that if $g: X \longrightarrow Y$ is a map in $\mathcal{T}_{SF}$, then g is an epimorphism in $\mathcal{T}_{SF}$ if and only if the least sequentially closed subspace of Y containing the set theoretic image of g is Y itself. A space X has a sequential topology if every sequentially closed subspace is a closed subspace. Let $\mathcal{T}^F$ denote the full subcategory of $\mathcal{T}$ generated by the spaces having a sequential topology. Show that $\mathcal{T}^F$ is a weak coseparation subcategory of $\mathcal{T}^C$ (§3, ex. 11). Prove that the epimorphisms in $\mathcal{T}^F$ are the surjective maps and the monomorphisms are the injective maps.

6. A space X satisfies the first axiom of countability if every point of X has a countable basis for its filter of neighborhoods. Let $\mathcal{T}^I$ denote the full subcategory of $\mathcal{T}$ generated by the spaces satisfying the first axiom of countability. Show that $\mathcal{T}^I$ is countably complete and has coproducts.

Prove that the natural inclusion functor $\mathcal{T}^I \longrightarrow \mathcal{T}$ preserves coproducts and countable limits.

8. Show that $\mathcal{T}^I$ is a subcategory of $\mathcal{T}^F$. Prove that if $X \in \mathrm{obj}(\mathcal{T}^I)$ and X is sequentially separated, then X is separated.

9. A space X is a compactly separated space or a weak Hausdorff space if whenever $f: C \longrightarrow X$ is a map with C compact, the subspace of X determined by the set theoretic image of f is a closed subspace. Let $\mathcal{T}_{WH}$ be the full subcategory of $\mathcal{T}$ generated by the compactly separated spaces. Show that $\mathcal{T}_{WH}$ is a weak separation subcategory of $\mathcal{T}_{SF}$, and that $\mathcal{T}_H$ is a weak separation subcategory of $\mathcal{T}_{WH}$. Show that the morphism $g: X \longrightarrow Y$ in $\mathcal{T}_{WH}$ is an epimorphism in $\mathcal{T}_{WH}$ if and only if the least compactly closed subspace of Y containing the set theoretic image of g is Y itself.

10. Show that if X is a space, the following are equivalent:

1) X is a sequentially separated space, and
2) the diagonal subspace of $X \prod X$ is sequentially closed.

Conclude that if $X \in \mathrm{obj}(\mathcal{T}^F)$, the following are equivalent:

1) X is a sequentially separated space, and
2) the diagonal subspace of the product $X \prod X$ in $\mathcal{T}^F$ is a closed subspace.

11. Show that if X is a space, the following are equivalent:

1) X is a compactly separated space, and

2) the diagonal subspace of $X \prod X$ is compactly closed.

Conclude that if $X \in \mathrm{obj}(\mathcal{T}^C)$, the following are equivalent:

1) X is a compactly separated space, and

2) the diagonal subspace of the product $X \prod X$ in $\mathcal{T}^C$ is a closed subspace.

12. Prove that the natural inclusion functors $\mathcal{T}_{CR} \to \mathcal{T}_H$, $\mathcal{T}_{WH} \to \mathcal{T}_F$, $\mathcal{T}_H \to \mathcal{T}_{WH}$, $\mathcal{T}_{SF} \to \mathcal{T}_F$, and $\mathcal{T}_F \to \mathcal{T}_K$ preserve coproducts, but not coequalizers.

13. Prove that the category R of rings and morphisms is bicomplete. Conclude that the full subcategory generated by the rings without radical is also bicomplete (§2, ex. 8, §3, ex. 13). Conclude also that the category R^C, and the full subcategory generated by the reduced commutative rings are bicomplete (§3, ex. 14).

14. Prove that the full subcategory of R generated by those rings whose underlying set is countable is finitely complete and countably cocomplete.

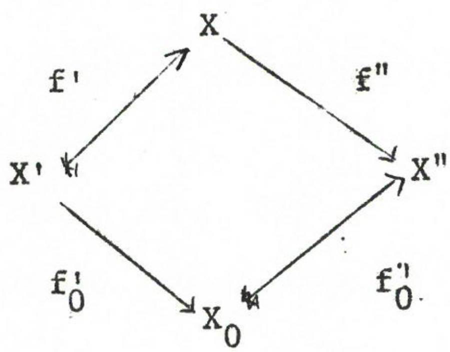
15. Prove that the full subcategory of R generated by the rings whose radical is nilpotent is a finitely complete category.

Chapter 2. Some special classes of categories and some of their basic properties

§1. Factorization of morphisms and some properties of limits.

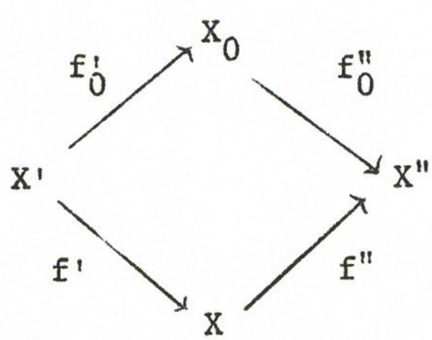
1.1 Definitions. Let $\mathcal{X}$ be a category.

An epimorphism $f'\colon X' \longrightarrow X$ is an *extremal epimorphism* if whenever



is a commutative diagram in $\mathcal{X}$ with f''_0 a monomorphism, there is a morphism $h\colon X \longrightarrow X_0$ such that $hf' = f'_0$.

A monomorphism $f''\colon X \longrightarrow X''$ is an *extremal monomorphism* if whenever



is a commutative diagram in $\mathcal{X}$ with f'_0 an epimorphism there is a morphism $h\colon X_0 \longrightarrow X$ such that $f''h = f''_0$.

Observe that in both cases above, the morphism n is unique and if added to the diagram makes the entire diagram commutative.

1.2 Definitions

If $\mathfrak{X}$ is a finitely complete category, the morphism $f': X' \longrightarrow X$ is an <u>effective epimorphism</u> if when

$$\begin{array}{ccc} X_0' & \xrightarrow{p_1} & X' \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f'} \\ X' & \xrightarrow{f'} & X \end{array}$$

is a cartesian square in $\mathfrak{X}$, then f' is a coequalizer of $p_1, p_2: X_0' \longrightarrow X'$.

If $\mathfrak{X}$ is a finitely cocomplete category, the morphism $f'': X \longrightarrow X''$ is an <u>effective monomorphism</u> if when

$$\begin{array}{ccc} X & \xrightarrow{f''} & X'' \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle i_2} \\ X'' & \xrightarrow{i_2} & X_0'' \end{array}$$

is a cocartesian square in $\mathfrak{X}$, then f'' is an equalizer of $i_1, i_2: X'' \longrightarrow X_0''$.

<u>1.3 Proposition.</u> If $\mathfrak{X}$ is a category, then

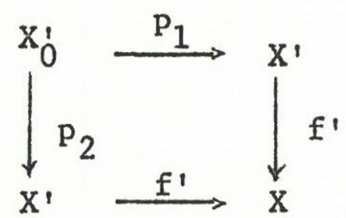
1) the morphism $f': X' \longrightarrow X$ in $\mathfrak{X}$ is an extremal epimorphism if and only if the morphism $(f')^*: X^* \longrightarrow (X')^*$ in $\mathfrak{X}^*$ is an extremal monomorphism, and

2) if $\mathfrak{X}$ is finitely complete, then

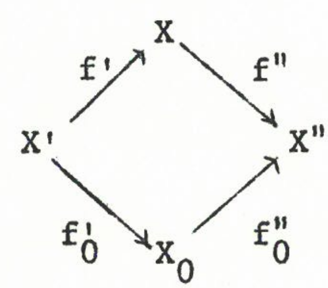
i) the morphism $f'\colon X' \longrightarrow X$ in $\mathfrak{X}$ is an effective epimorphism if and only if the morphism $(f')^*\colon X^* \longrightarrow (X')^*$ in $\mathfrak{X}^*$ is an effective monomorphism, and

ii) if $f'\colon X' \longrightarrow X$ is an effective epimorphism, then f' is an extremal epimorphism.

<u>Proof</u>. Both parts 1) and 2i) of the proposition follow at once from the definitions. Hence suppose $\mathfrak{X}$ is finitely complete and $f'\colon X' \longrightarrow X$ is an effective epimorphism. Let



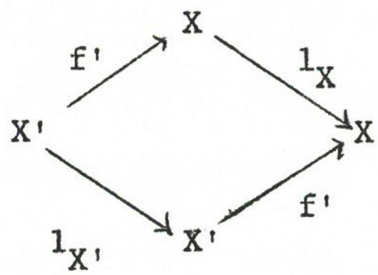
be a cartesian square in , and



a commutative diagram in $\mathfrak{X}$ such that f''_0 is a monomorphism. Now $f''_0 f'_0 p_1 = f'' f' p_1 = f'' f' p_2 = f''_0 f'_0 p_2$, and $f'_0 p_1 = f'_0 p_2$ since f''_0 is a monomorphism. Thus since f' is a coequalizer of p_1, p_2 there is a unique $h\colon X \longrightarrow X_0$ such that $hf' = f'_0$, and the proposition is proved.

<u>1.4 Proposition.</u> If $\mathcal{X}$ is a category and $f'\colon X' \longrightarrow X$ is an extremal epimorphism which is a monomorphism, then f' is an isomorphism.

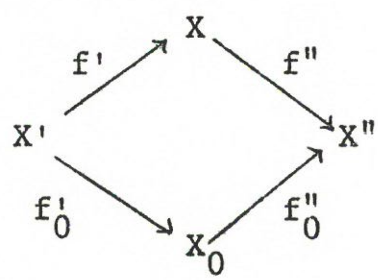
<u>Proof.</u> The diagram



is a commutative diagram such that f' is a monomorphism. Hence there exists $h\colon X \longrightarrow X'$ such that $hf' = 1_{X'}$, $f'h = 1_X$, and the proposition is proved.

<u>1.5 Proposition.</u> If $\mathcal{X}$ is a category and $f' = X' \longrightarrow X$ is a coretraction in $\mathcal{X}$, then f is an extremal epimorphism.

<u>Proof.</u> Suppose



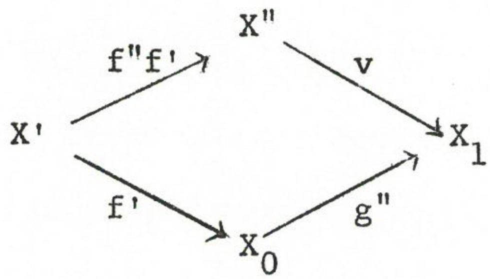
is a commutative diagram in $\mathcal{X}$ such that f''_0 is a monomorphism. Choose $g\colon X \longrightarrow X'$ such that $f'g = 1_X$. Let $h = f'_0 g\colon X \longrightarrow X_0$. Now $f''_0 hf' = f''_0 f'_0 gf' = f''f'gf' = f''f' = f'_0$, and $hf' = f'_0$ since f''_0 is a monomorphism. Consequently the proposition is proved.

<u>1.6 Proposition.</u> If $\mathfrak{X}$ is a category and $f': X' \longrightarrow X$, $f'': X \longrightarrow X''$ are morphisms in $\mathfrak{X}$, then

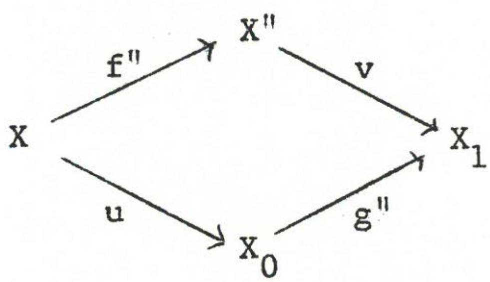
1) if f', f'' are extremal epimorphisms, then $f''f' = X' \longrightarrow X''$ is an extremal epimorphism, and

2) if $f''f': X' \longrightarrow X''$ is an extremal epimorphism, then f'' is an extremal epimorphism.

<u>Proof.</u> Suppose the conditions of part 1) are satisfied and that the diagram

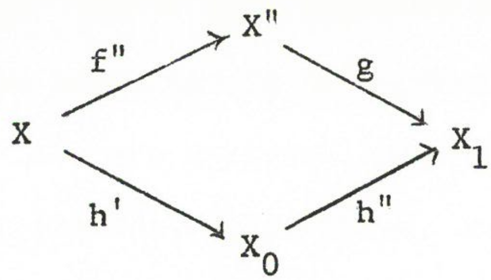


in $\mathfrak{X}$ is commutative with g'' a monomorphism. Since f' is an extremal epimorphism, there exists $u: X \longrightarrow X_0$ such that $uf' = g'$. Thus the diagram



in $\mathfrak{X}$ is commutative and there exists $v: X'' \longrightarrow X_0$ such that $vf'' = u$. Hence $vf''f' = uf' = g'$, and part 1) is proved.

If the conditions of part 2) are satisfied, and

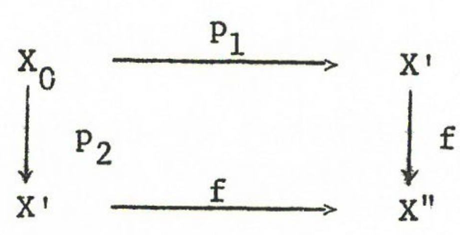


is a commutative diagram in $\mathcal{X}$ with h'' a monomorphism, then since $f''f'$ is an extremal epimorphism, there exists $u: X'' \longrightarrow X_0$ such that $uf''f' = h'f'$. Thus $h''uf'' = gf'' = h''h'$, and $uf'' = n'$ since h'' is a monomorphism. Thus part 2) of the proposition is proved.

<u>1.7 Definition</u>. If $\mathcal{X}$ is a category and $f: X' \longrightarrow X''$ is a morphism in $\mathcal{X}$, then a <u>factorization of f</u> is a diagram $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ in $\mathcal{X}$ such that $f''f' = f$.

<u>1.8 Proposition</u>. If $\mathcal{X}$ is a category, $f: X' \longrightarrow X''$ is a morphism in $\mathcal{X}$, and $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ is a factorization of f such that f'' is a monomorphism, then the following are equivalent:

1)



is a cartesian sequence in $\mathcal{X}$, and

2)

$$\begin{array}{ccc} X_0 & \xrightarrow{p_1} & X' \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f'} \\ X' & \xrightarrow{f'} & X \end{array}$$

is a cartesian square in $\mathfrak{X}$.

<u>Proof</u>. Since $f = f''f'$ and f'' is a monomorphism, if $g_1, g_2 \colon Y \longrightarrow X'$ are morphisms in $\mathfrak{X}$, the assertion $fg_1 = fg_2$ is equivalent with the assertion $f'g_1 = f'g_2$. The proposition now follows at once from the definition of cartesian square.

<u>1.9 Definitions</u>. Let $\mathfrak{X}$ be a category and $f \colon X' \longrightarrow X''$ a morphism in $\mathfrak{X}$.

A <u>coimage of f</u> is an extremal epimorphism $f' \colon X' \longrightarrow X$ such that there exists a monomorphism $f'' \colon X \longrightarrow X''$ such that $f''f' = f$. If $\mathfrak{X}$ is finitely complete, the coimage f' of f is an <u>effective coimage of f</u> if f' is an effective epimorphism.

An <u>image of f</u> is an extremal monomorphism $f'' \colon X \longrightarrow X''$ such that there exists an epimorphism $f' \colon X' \longrightarrow X$ such that $f''f' = f$. If $\mathfrak{X}$ is finitely cocomplete, the image f'' of f is an <u>effective image of f</u> if f'' is an effective monomorphism.

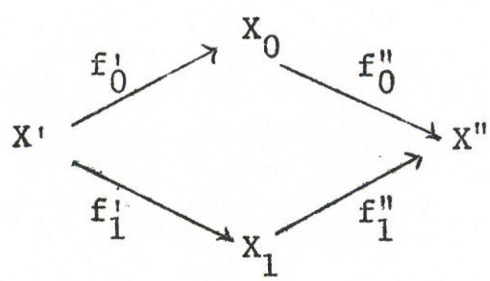
<u>1.10 Proposition</u>. If $\mathfrak{X}$ is a category and $f \colon X' \longrightarrow X''$ is a

morphism in $\mathfrak{X}$, then

1) $f': X' \longrightarrow X$ is a coimage of f in $\mathfrak{X}$ if and only if $(f')^*: X^* \longrightarrow (X')^*$ is an image of $f^*: (X'')^* \longrightarrow (X')^*$ in $\mathfrak{X}^*$, and

2) if $f_0': X' \longrightarrow X_0$, $f_1': X' \longrightarrow X_1$ are coimages of f in $\mathfrak{X}$, there is a unique isomorphism $u: X_o \longrightarrow X_1$ such that $uf_0' = f_1'$.

<u>Proof</u>. Part 1) of the proposition follows at once from the definitions. Under the conditions of part 2), let $f_0'' = X_0 \longrightarrow X''$, $f_1'': X_1 \longrightarrow X''$ be monomorphisms such that $f_0''f' = f = f_1''f_1'$, and note that f_0'', f_1'' are unique. The diagram



is a commutative diagram in $\mathfrak{X}$ such that f_1'' is a monomorphism. Since f_0' is an extremal epimorphism, there is a unique $u: X_0 \longrightarrow X_1$ such that $uf_0' = f_1'$. Similarly there is a unique $v: X_1 \longrightarrow X_0$ such that $vf_1' = f_0'$. Now $vu = 1_{X_0}$, and the proposition is proved.

<u>1.11 Definitions</u>. The category $\mathfrak{X}$ has <u>projective factorization</u> if every morphism in $\mathfrak{X}$ has a coimage; it has <u>injective factor-</u>

ization if every morphism in $\mathfrak{X}$ has an image. If $f: X' \longrightarrow X''$ is a morphism in $\mathfrak{X}$, then a projective factorization of f is a factorization $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ such that f' is a co-image of f while an injective factorization of f is a diagram $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ which is a factorization of f with f'' an image of f.

Note that a category $\mathfrak{X}$ has projective factorization if and only if $\mathfrak{X}^*$ has injective factorization, and that $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ is a projective factorization of $f = f''f'$ in $\mathfrak{X}$ if and only if $(X'')^* \xrightarrow{(f'')^*} X^* \xrightarrow{(f')^*} (X')^*$ is an injective factorization of f^* in $\mathfrak{X}^*$. Observe further that if $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ is a projective factorization of f, and $X' \xrightarrow{f_0'} X_0 \xrightarrow{f_0''} X''$ is any other factorization of f such that f_0'' is a monomorphism, then there is a unique $u: X \longrightarrow X_0$ such that $uf' = f_0'$.

1.12 Definition. The functor $T: \mathfrak{X} \longrightarrow \mathcal{Y}$ is an extremely faithful functor if it is a faithful functor which reflects isomorphisms.

Note that $T: \mathfrak{X} \longrightarrow \mathcal{Y}$ is extremely faithful if and only if $T^*: \mathfrak{X}^* \longrightarrow \mathcal{Y}^*$ is extremely faithful.

1.13 Proposition. If $(\alpha, \beta) : S \dashv T: (\mathfrak{X}, \mathcal{Y})$ is an adjoint pair of functors and $\mathfrak{X}$ has projective factorization, then the following are equivalent:

1) T is extremely faithful,

2) T reflects extremal epimorphisms, and

3) for $X \in obj(\mathcal{X})$, $\alpha(X)\colon ST(X) \longrightarrow X$ is an extremal epimorphism.

<u>Proof</u>. Suppose 1) and that $f\colon X' \longrightarrow X''$ is a morphism in $\mathcal{X}$ such that $T(f)$ is an extremal epimorphism. Let $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ be a projective factorization of f. Now $T(X') \xrightarrow{T(f')} T(X) \xrightarrow{T(f'')} T(X'')$ is a factor of $T(f)$, and hence by 1.6 2), $T(f'')$ is an extremal epimorphism. Further $T(f'')$ is a monomorphism since an adjoint functor preserves monomorphisms (Chapter 1, 2.17). Hence by 1.4, $T(f'')$ is an isomorphism. Since T is extremely faithful, f'' is an isomorphism and f is an extremal epimorphism. Consequently 1) implies 2).

Suppose 2). For $X \in obj(\mathcal{X})$, $1_{T(X)} = T(\alpha(X))\beta(T(X))$, and thus $T(\alpha(X))$ is a coretraction. By 1.5, $T(\alpha(X))$ is an extremal epimorphism. Hence $\alpha(X)$ is an extremal epimorphism and 2) implies 3).

Suppose 3). By 2.18, T is faithful. Suppose $f\colon X' \longrightarrow X''$ is a morphism in $\mathcal{X}$, and $T(f)$ is an isomorphism. Then f is a monomorphism since T reflects monomorphisms because it is a faithful functor. Now

$$\begin{array}{ccc} ST(X') & \xrightarrow{ST(f)} & ST(X'') \\ \downarrow{\scriptstyle \alpha(X')} & & \downarrow{\scriptstyle \alpha(X'')} \\ X' & \xrightarrow{\quad f \quad} & X'' \end{array}$$

is a commutative diagram in $\mathfrak{X}$, ST(f) is an isomorphism, and $\alpha(X'')$ is an extremal epimorphism. Thus by 1.6, f is an extremal epimorphism, and by 1.4, f is an isomorphism. Hence 3) implies 1), and the proposition is proved.

1.14 Definitions. The category $\mathfrak{X}$ has full factorization if it has both projective and injective factorization. It has unique factorization if it has full factorization and every bimorphism in $\mathfrak{X}$ is an isomorphism.

If $\mathfrak{X}$ is any category and $f: X' \longrightarrow X''$ is a morphism in $\mathfrak{X}$, then a full factorization of f is a diagram $X' \xrightarrow{f'} X_0' \xrightarrow{\bar{f}} X_0'' \xrightarrow{f''} X''$ in $\mathfrak{X}$ such that f' is a coimage of f, f'' is an image of f, and $f''\bar{f}f' = f$.

Observe that the condition that a category $\mathfrak{X}$ have full factorization is equivalent with the condition that every morphism in $\mathfrak{X}$ have a full factorization. If $\mathfrak{X}$ has unique factorization, then if $f: X' \longrightarrow X''$ is a morphism in $\mathfrak{X}$ there is a factorization $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ such that f' is a coimage of f and f'' is an image of f. If $X' \xrightarrow{f_0'} X_0 \xrightarrow{f_0''} X''$ is a 2-nd factorization of f where f_0' is an epimorphism and f_0'' is a monomorphism, there is a unique isomorphism $u: X \longrightarrow X_0$ characterized either by the property that $uf' = f_0'$ or the property that $f_0'' u = f''$. In a category with unique factorization every epimorphism is an extremal epimorphism and every monomorphism is an extremal monomorphism.

<u>1.15 Proposition.</u> If $\mathcal{X}$ is a finitely complete category with projective factorization and $\mathcal{J}$ is a small category, then

1) $[\mathcal{J}, \mathcal{X}]$ is a finitely complete category with projective factorization , and

2) if $\alpha' \colon T' \longrightarrow T$ is a morphism in $[\mathcal{J}, \mathcal{X}]$ the following are equivalent:

i) α' is an extremal epimorphism and

ii) for every object J of $\mathcal{J}$, $\alpha'(J) \colon T'(J) \longrightarrow T(J)$ is an extremal epimorphism.

<u>Proof</u>. Suppose $\alpha \colon T' \longrightarrow T''$ is a morphism in $[\mathcal{J}, \mathcal{X}]$. For $J \in \mathrm{obj}(\mathcal{J})$, let $T'(J) \xrightarrow{\alpha'(H)} T(J) \xrightarrow{\alpha''(J)} T''(J)$ be a projective factorization of $\alpha(J)$. If $j \colon J' \longrightarrow J''$ is a morphism in $\mathcal{J}$, then there is a commutative diagram in $\mathcal{X}$,

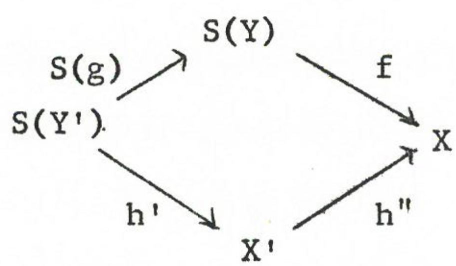
$$\begin{array}{ccccc} & & T(J') & & \\ & \overset{\alpha'(J')}{\nearrow} & & \overset{T''(j)\alpha''(J'')}{\searrow} & \\ T'(J') & & & & T''(J'') \\ & \underset{\alpha'(J'')T'(j)}{\searrow} & & \underset{\alpha''(J'')}{\nearrow} & \\ & & T(J'') & & \end{array}$$

where $\alpha'(J')$ is an extremal epimorphism and $\alpha''(J'')$ is a monomorphism. Hence there is a unique $T(j) \colon T(J') \longrightarrow T(J'')$ such that $T(j)\alpha'(J') = \alpha'(J'')T'(j)$. Note that $\alpha''(J'')T(j) = T''(j)\alpha''(J')$. Thus $T' \xrightarrow{\alpha'} T \xrightarrow{\alpha''} T''$ is a factorization of

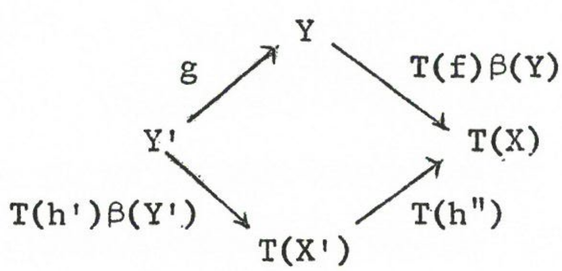
α such that α'' is a local monomorphism and for $J \in obj(\mathcal{J})$, $\alpha'(J)$ is an extremal epimorphism. By 4.19, Chapter 1, every monomorphism in $[\mathcal{J}, \mathcal{X}]$ is a local monomorphism. Thus $\alpha': T' \longrightarrow T$ is an extremal epimorphism, $[\mathcal{J}, \mathcal{X}]$ has projective factorization, and part 2) of the proposition is proved. By 4.26, Chapter 1, $[\mathcal{J}, \mathcal{X}]$ is finitely complete, and the proposition is proved.

<u>1.16 Proposition.</u> If $S: \mathcal{Y} \longrightarrow \mathcal{X}$ is a coadjoint functor, then S preserves extremal epimorphisms.

<u>Proof</u>. Suppose $(\alpha, \beta): S \dashv T$ is an adjoint pair, and that $g: Y' \longrightarrow Y$ is an extremal epimorphism in $\mathcal{Y}$. If



is a commutative diagram in $\mathcal{X}$ with h'' a monomorphism, then



is a commutative diagram in $\mathcal{Y}$ with $T(h'')$ a monomorphism. Therefore there exists $v: Y \longrightarrow T(X')$, $vg = T(h')\beta(T')$. Let

$u = \alpha(X')S(v): S(Y) \longrightarrow X'$. Now $uS(g) = \alpha(X')S(vg) = h'$, and the proposition is proved.

<u>1.17 Definitions</u>. Let $\mathcal{X}$ be a category.

A separation subcategory $\mathcal{X}'$ of $\mathcal{X}$ is a weak separation subcategory $\mathcal{X}'$ of $\mathcal{X}$ such that if $f: X \longrightarrow X''$ is an extremal monomorphism in $\mathcal{X}$ and $X'' \in obj(\mathcal{X}')$ then f is a morphism in $\mathcal{X}'$. A <u>strong separation subcategory</u> $\mathcal{X}'$ of $\mathcal{X}$ is a weak separation subcategory $\mathcal{X}'$ of $\mathcal{X}$ such that if $f: X \longrightarrow X''$ is a monomorphism in $\mathcal{X}$ and $X'' \in obj(\mathcal{X}')$ then f is a morphism in $\mathcal{X}'$.

A <u>coseparation subcategory</u> $\mathcal{X}'$ of $\mathcal{X}$ is a weak coseparation subcategory $\mathcal{X}'$ of $\mathcal{X}$ such that if $f: X' \longrightarrow X$ is an extremal epimorphism in $\mathcal{X}$ and $X' \in obj(\mathcal{X}')$ then f is a morphism in $\mathcal{X}'$. A <u>strong coseparation subcategory</u> $\mathcal{X}'$ of $\mathcal{X}$ is a weak coseparation subcategory $\mathcal{X}'$ of $\mathcal{X}$ such that if $f: X' \longrightarrow X$ is an epimorphism in $\mathcal{X}$ and $X' \in obj(\mathcal{X}')$ then f is a morphism in $\mathcal{X}'$.

<u>1.18 Proposition</u>. If $\mathcal{X}'$ is a weak separation subcategory of $\mathcal{X}$ with coreflection $R: \mathcal{X} \longrightarrow \mathcal{X}'$, $\lambda: 1_{\mathcal{X}} \longrightarrow R$, then

1) if R preserves extremal monomorphisms, $f: X' \longrightarrow X''$ is a morphism in $\mathcal{X}'$, and $f'': X \longrightarrow X''$ is an image of f in $\mathcal{X}$, then $R(f''): R(X) \longrightarrow X''$ is an image of f in $\mathcal{X}'$, and hence if $\mathcal{X}$ has injective factorization, so also does $\mathcal{X}'$,

2) if $\mathfrak{X}'$ is a separation subcategory such that the inclusion functor $\mathfrak{X}' \longrightarrow \mathfrak{X}$ preserves epimorphisms, $f: X' \longrightarrow X''$ is a morphism in $\mathfrak{X}'$, and $f'': X \longrightarrow X''$ is an image of f in $\mathfrak{X}$, then f'' is an image of f in $\mathfrak{X}'$ and if $\mathfrak{X}$ has injective factorization, so also does $\mathfrak{X}'$,

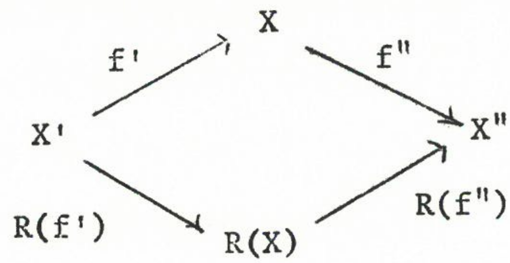
3) if the inclusion functor $\mathfrak{X}' \longrightarrow \mathfrak{X}$ preserves extremal epimorphisms, $f: X' \longrightarrow X''$ is a morphism in $\mathfrak{X}'$, and $f': X' \longrightarrow X$ is a coimage of f in $\mathfrak{X}$, then f' is a coimage of f in $\mathfrak{X}'$, and hence if $\mathfrak{X}$ has projective factorization, so also does $\mathfrak{X}'$, and

4) if $\mathfrak{X}$ has projective factorizations and $\mathfrak{X}'$ is a strong separation subcategory of $\mathfrak{X}$, the inclusion functor $\mathfrak{X}' \longrightarrow \mathfrak{X}$ preserves extremal epimorphisms.

Proof. Under the conditions of part 1), let $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ be an injective factorization of f. Since f' is an epimorphism and R is a coadjoint functor, $R(f')$ is an epimorphism in $\mathfrak{X}'$ (though not necessarily in $\mathfrak{X}$). Since R preserves extremal monomorphisms and $f = R(f) = R(f'')R(f')$, part 1) is proved.

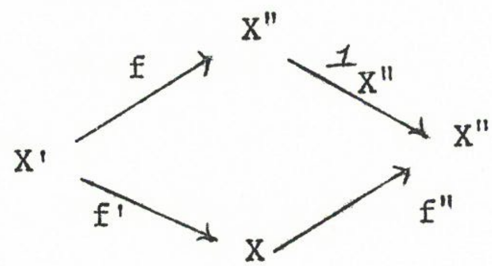
Part 2) follows at once from the definitions and the conditions stated.

Under the conditions of part 3), let $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ be a projective factorization of f in $\mathfrak{X}$. There is a commutative diagram



in $\mathcal{X}$, $R(f')$ is an extremal epimorphism in $\mathcal{X}'$ since R is a coadjoint functor (1.16), and hence since $\mathcal{X}' \longrightarrow \mathcal{X}$ preserves extremal epimorphisms, $R(f')$ is an extremal epimorphism in $\mathcal{X}$. Thus by 1.6, $\lambda(X)$ is an extremal epimorphism in $\mathcal{X}$. However $\lambda(X)$ is a monomorphism since f'' is a monomorphism. Thus by 1.4, $\lambda(X)$ is an isomorphism, $\lambda(X) = 1_X$ and part 3) is proved

If $\mathcal{X}'$ is a strong separation subcategory, $\mathcal{X}$ has projective factorization, and $f: X' \longrightarrow X''$ is an extremal epimorphism in $\mathcal{X}'$, there is a commutative diagram



in $\mathcal{X}$ with f' a coimage of f in $\mathcal{X}$. Hence f'' is a monomorphism in $\mathcal{X}$, and f'' is a morphism in $\mathcal{X}'$ since $X'' \in$ obj( ) Thus f' is also a morphism in $\mathcal{X}'$, and there exists a morphism $u: X'' \longrightarrow X$ in $\mathcal{X}'$ such that $uf = f'$, $f''u = 1_{X''}$. Hence u is an isomorphism, and part 4) is proved, which proves the proposition.

1.19 Proposition. If $\mathcal{X}$ is a finitely complete category, the following are equivalent:

1) every morphism in $\mathcal{X}$ has an effective coimage, and

2) $\mathcal{X}$ has projective factorization and every extremal epimorphism in $\mathcal{X}$ is an effective epimorphism.

Proof. Suppose 1). If $f: X' \longrightarrow X''$ is a morphism in $\mathcal{X}$, there is a factorization $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ of f such that f' is an effective coimage of f and f'' is a monomorphism. This is a projective factorization of f. If f is an extremal epimorphism, so is f'' by 1.6, and by 1.4, f'' is an isomorphism. Thus 1) implies 2). Since certainly 2) implies 1), the proposition is proved.

1.20 Proposition. If $\mathcal{X}$, $\mathcal{Y}$ are finitely complete categories and $S: \mathcal{Y} \longrightarrow \mathcal{X}$ is a coadjoint functor, then S preserves effective epimorphisms.

Proof. Suppose $g: Y' \longrightarrow Y''$ is an effective epimorphism in $\mathcal{Y}$, and

$$\begin{array}{ccc} Y_0 & \xrightarrow{p_2} & Y' \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{g} & Y'' \end{array}$$

in a cartesian square in $\mathcal{Y}$. Let

$$\begin{array}{ccc} X_0 & \xrightarrow{\overline{p_2}} & S(Y') \\ {\scriptstyle \overline{p_1}}\downarrow & & \downarrow{\scriptstyle S(g)} \\ S(Y') & \xrightarrow{S(g)} & S(Y'') \end{array}$$

be a cartesian square in $\mathcal{X}$. There is a unique $w: S(Y_0) \longrightarrow X_0$ such that $\overline{p_1}\, w = S(p_1)$, $\overline{p_2} w = S(p_2)$. Suppose $f: S(Y') \longrightarrow X$ is a morphism in $\mathcal{X}$, and $f\, \overline{p_1} = f\, \overline{p_2}$. Let $(\alpha, \beta): S \dashv T$ be an adjoint pair. Now $T(f)\beta(Y'): Y' \longrightarrow T(X)$, and $T(f)\beta(Y')p_1 = T(f\beta(Y')p_2$. Thus g being the coequalizer of p_1, p_2, there is a unique $\tilde{f} : Y'' \longrightarrow T(X)$ such that $\tilde{f}\, g = T(f)\beta(Y')$. Let $\bar{f} = \alpha(X)S(\tilde{f}) : S(Y'') \longrightarrow X$. Now $f = \bar{f}\, S(g)$, and $S(g)$ is an epimorphism. Hence the proposition is proved.

<u>1.21 Definitions</u>. A finitely complete category $\mathcal{X}$ has <u>effective projective factorization</u> if every morphism in $\mathcal{X}$ has an effective coimage. A finitely cocomplete category $\mathcal{X}$ has <u>effective injective factorization</u> if every morphism in $\mathcal{X}$ has an effective image. A finitely bicomplete category $\mathcal{X}$ has <u>effective full factorization</u> if it has both effective projective factorization and effective injective factorization. It has <u>unique effective full factorization</u> if further it has unique factorization.

<u>1.21 Proposition</u>. If $\mathcal{X}$ is a finitely complete category with effective projective factorization and $\mathcal{X}'$ is a weak separation

subcategory of $\mathfrak{X}$ with coreflection R: $\mathfrak{X} \longrightarrow \mathfrak{X}'$, λ: $1_{\mathfrak{X}} \longrightarrow R$ then if either

1) R preserves monomorphisms, or

2) the inclusion functor $\mathfrak{X}' \longrightarrow \mathfrak{X}$ preserves extremal epimorphisms,

then $\mathfrak{X}'$ is a finitely complete category with effective projective factorization.

<u>Proof</u>. By 4.13*, Chapter 1, $\mathfrak{X}'$ is finitely complete. Suppose f: X' ⟶ X" is a morphism in $\mathfrak{X}'$, and $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ is a projective factorization of f in $\mathfrak{X}$. Now f = R(f), and $X' \xrightarrow{R(f')} R(X) \xrightarrow{R(f'')} X''$ is a factorization of f in $\mathfrak{X}'$. By 1.20, R(f') is an effective epimorphism in . Thus if R preserves monomorphisms R(f') is an effective coimage of f in $\mathfrak{X}'$. If the inclusion preserves extremal epimorphisms, then by 1.18 3), R(X) = X, R(f') = f', and R(f") = f". Hence the proposition is proved.

<u>1.22 Proposition</u>. If $\mathfrak{X}$ is a finitely cocomplete category with effective injective factorization, $\mathfrak{X}'$ is a weak separation subcategory of $\mathfrak{X}$ with coreflection R: $\mathfrak{X} \longrightarrow \mathfrak{X}'$, λ:$1_{\mathfrak{X}} \longrightarrow R$, and R preserves effective monomorphisms, then $\mathfrak{X}'$ is a finitely cocomplete category with effective injective factorization.

<u>Proof</u>. By 4.13, Chapter 1, $\mathfrak{X}'$ is finitely cocomplete. Suppose f: X' ⟶ X" is a morphism in $\mathfrak{X}'$, and $X' \xrightarrow{f'} X \xrightarrow{f''} X''$

is an injective factorization of f in $\mathcal{X}$. Since R preserves effective monomorphism and R(f) = f, $X' \xrightarrow{R(f')} R(X) \xrightarrow{R(f'')} X''$ is an injective factorization of f in $\mathcal{X}'$, and it has the desired property.

<u>1.23 Proposition.</u> If $\mathcal{X}$ is a finitely complete category with effective projective factorization and $\mathcal{J}$ is a small category, then $[\mathcal{J}, \mathcal{X}]$ is a finitely complete category with effective projective factorization.

<u>Proof.</u> By Chapter 1, 4.26, $[\mathcal{J}, \mathcal{X}]$ is finitely complete. By 1.15, $[\mathcal{J}, \mathcal{X}]$ has projective factorization, and by 1.15 and Chapter 1, 4.24 and 4.25, it follows that every extremal epimorphism in $[\mathcal{J}, \mathcal{X}]$ is an effective epimorphism. Hence the proposition is proved.

<u>Exercises</u>

1. Prove that the category $\mathcal{S}$ has unique effective full factorization. Prove that the category $\mathcal{T}$ has effective full factorization but not unique effective full factorization.

2. Let T_0 be the canonical Kolmogoroff space (Chapter 1, §3, ex. 6). Prove that if $f\colon A \longrightarrow X$ is a map, the following are equivalent:

i) f is an effective monomorphism,

ii) f is injective, and $\mathcal{T}(f,T_0)\colon \mathcal{T}(X,T_0) \longrightarrow \mathcal{T}(A,T_0)$ is surjective.

Prove that $\mathcal{T}_K$ is a strong separation subcategory of $\mathcal{T}$ with coreflection $R_K: \mathcal{T} \longrightarrow \mathcal{T}_K$, $\lambda_K: 1_{\mathcal{T}} \longrightarrow \mathcal{T}_K$ such that R_K preserves effective monomorphisms, and further the inclusion $\mathcal{T}_K \longrightarrow \mathcal{T}$ preserves epimorphisms. Conclude that $\mathcal{T}_K$ has effective full factorization and that if f is a map in $\mathcal{T}_K$ its full factorization in $\mathcal{T}$ coincides with its full factorization in $\mathcal{T}_K$.

3. Prove that $\mathcal{T}_F$, $\mathcal{T}_{SF}$, $\mathcal{T}_{WH}$, and $\mathcal{T}_H$ are strong separation subcategories of $\mathcal{T}$ with effective full factorization. Show that the inclusion $\mathcal{T}_F \longrightarrow \mathcal{T}_K$ preserves full factorization, but the inclusions $\mathcal{T}_{SF} \longrightarrow \mathcal{T}_F$, $\mathcal{T}_{WH} \longrightarrow \mathcal{T}_F$, $\mathcal{T}_H \longrightarrow \mathcal{T}_F$ preserve projective factorization but not injective factorization.

4. A space X is nearly completely regular if whenever x_0, x_1 are distinct points of X there exists a map $f: X \longrightarrow I$ such that $f(x_0) = 0$, $f(x_1) = 1$. Let $\mathcal{T}_{NCR}$ be the full subcategory of $\mathcal{T}$ generated by the nearly completely regular spaces. Show that $\mathcal{T}_{NCR}$ is a strong separation subcategory of $\mathcal{T}$ with effective full factorization and that I is a cogenerator of $\mathcal{T}_{NCR}$.

5. Show that if $X \in obj(\mathcal{T}_{NCR})$, C is a compact subspace of X, and $x \in X-C$, there exists $f: X \longrightarrow I$ such that $f(c) = 0$ for $c \in C$, and $f(x) = 1$.

6. Show that $\mathcal{T}_{NCR}$ is a strong separation subcategory of $\mathcal{T}_H$ such that the inclusion $\mathcal{T}_{NCR} \longrightarrow \mathcal{T}_H$ preserves full factorization. Show that $\mathcal{T}_{CR}$ is a separation subcategory of $\mathcal{T}$ which is not a strong separation subcategory. Show that $\mathcal{T}_{CR}$ is a separation subcategory of $\mathcal{T}_{NCR}$ with coreflection $R: \mathcal{T}_{NCR} \longrightarrow \mathcal{T}_{CR}$, $\lambda: 1_{\mathcal{T}_{NCR}} \longrightarrow R$ such that $\lambda(X): X \longrightarrow R(X)$ is bijective for $X \in obj(\mathcal{T}_{NCR})$. Show that $\mathcal{T}_{CR}$ has effective full factorization and that $\mathcal{T}_{CR} \longrightarrow \mathcal{T}_{NCR}$ preserves injective factorization .

7. Show that $\mathcal{C}$ has unique effective full factorization, and that $\mathcal{C} \longrightarrow \mathcal{T}$ preserves full factorization. Show that $\mathcal{C}$ is a separation subcategory of $\mathcal{T}_{WH}$ but not of $\mathcal{T}_F$.

8. Recall that a locally compact space is a separated space such that every point has a compact neighborhood. Prove that if $f: X \longrightarrow X''$ is an effective epimorphism in $\mathcal{T}$ and L is a locally compact space, then $f \Pi 1_L: X \Pi L \longrightarrow X'' \Pi L$ is an effective epimorphism in $\mathcal{T}$.

9. If X is a space, prove the following are equivalent:

i) the topology of X is compactly generated, and

ii) there exists an effective epimorphism $f: L \longrightarrow X$ in $\mathcal{T}$ with L locally compact.

10. Use exercises 8,9 and chapter 1, §3, ex. 12 to show that in $\mathcal{T}^C$ finite products of effective epimorphisms are effective epimorphisms. Show this to be false in $\mathcal{T}$.

11. Show that $\mathcal{T}^C$ is a coseparation subcategory of $\mathcal{T}$ but not a strong coseparation subcategory. Show that if $R^C\colon \mathcal{T} \longrightarrow \mathcal{T}^C$, $\rho^C\colon R^C \longrightarrow 1_{\mathcal{T}}$ is the reflection, then $\rho^C(X)$ is bijective for any space X. Show that $\mathcal{T}^C$ has effective full factorization and that the inclusion $\mathcal{T}^C \longrightarrow \mathcal{T}$ preserves projective factorization but not injective factorization.

12. If X is a space, prove the following are equivalent:

i) the topology of X is a sequential topology,

ii) there exists an effective epimorphism $f\colon L \longrightarrow X$, L locally compact metrizable, and

iii) there exists an effective epimorphism $f\colon Y \longrightarrow X$, Y satisfying the first axiom of countability.

13. Prove that $\mathcal{T}^F$ is a coseparation subcategory of $\mathcal{T}$ but not a strong coseparation subcategory. Show that $\mathcal{T}^F$ has effective full factorization and that $\mathcal{T}^F \longrightarrow \mathcal{T}$ preserves projective factorization but not injective factorization. Show that the reflection, $R^F\colon \mathcal{T} \longrightarrow \mathcal{T}^F$, $\rho^F\colon R^F \longrightarrow 1_{\mathcal{T}}$ is such that $\rho^F(X)$ is bijective for any space X.

14. Recall that a subspace A of a space X is locally closed in X if it is the intersection of a closed subspace and an open subspace. Show that if $X \in \mathrm{obj}(\mathcal{T}^C)$ and A is a locally closed subspace of X, then $A \in \mathrm{obj}(\mathcal{T}^C)$. Show that if $X \in \mathrm{obj}(\mathcal{T}^F)$ and A is a locally closed subspace of X then $A \in \mathrm{obj}(\mathcal{T}^F)$.

15. Prove that the natural inclusion functor $\mathcal{T}^F \longrightarrow \mathcal{T}^C$ preserves finite products. Use this together with Chapter 1, §4, ex. 9, 10 to show that if $X \in \mathrm{obj}(\mathcal{T}^F)$ and X is sequentially separated, then X is compactly separated.

16. Let $\mathcal{T}(C)$ be the full subcategory of $\mathcal{T}$ generated by those spaces whose topology is both compactly generated and compactly separated. Show that $\mathcal{T}(C)$ is a strong separation subcategory of $\mathcal{T}^C$ and a coseparation subcategory of $\mathcal{T}_{WH}$.

17. Let $\mathcal{T}(F)$ be the full subcategory of $\mathcal{T}$ generated by those spaces whose topology is a sequential topology which is sequentially separated. Show that $\mathcal{T}(F)$ is a strong separation subcategory of $\mathcal{T}^F$ and a coseparation subcategory of $\mathcal{T}_{SF}$.

18. Show that the categories $\mathcal{T}(C)$ and $\mathcal{T}(F)$ have effective full factorization. Prove that $\mathcal{T}(F)$ is a coseparation subcategory of $\mathcal{T}(C)$ such that the inclusion $\mathcal{T}(F) \longrightarrow \mathcal{T}(C)$ preserves finite limits. Prove that both the category $\mathcal{T}(F)$ and the category $\mathcal{T}(C)$ have the property that finite products of effective epimorphism are effective epimorphisms.

19. Prove that if $f: R' \longrightarrow R''$ is a morphism in the category $\mathcal{R}$ of rings, the following are equivalent:

i) f is surjective, and

ii) f is an effective epimorphism.

Show that the canonical functor $T: \mathcal{R} \longrightarrow \mathcal{S}$ (Chapter 1, §2, ex.2) is an extremely faithful adjoint functor which preserves effective epimorphisms.

20. Show that $\mathcal{R}$ has effective projective factorization. Prove that $\mathcal{R}^c$ is a strong separation subcategory of $\mathcal{R}$. Prove that the category of reduced commutative rings (Chapter 1, §3, ex. (4) is a strong separation subcategory of $\mathcal{R}^c$.

§2. Introduction to pointed categories.

2.1 Definitions. If $\mathcal{X}$ is a category, the object $*$ of $\mathcal{X}$ is a point of $\mathcal{X}$ if it is simultaneously an initial point and a terminal point of $\mathcal{X}$.

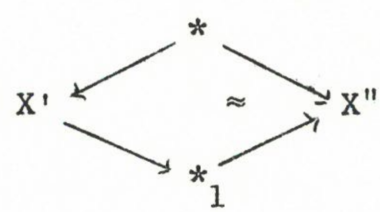
A pointed category $\mathcal{X}$ is a category which has a point.

If $\mathcal{X}$ and $\mathcal{Y}$ are pointed categories, a pointed functor $T\colon \mathcal{X} \longrightarrow \mathcal{Y}$ is a functor such that if $*$ is a point of $\mathcal{X}$, then $T(*)$ is a point of $\mathcal{Y}$.

If $\mathcal{X}$ is a pointed category with point $*$, and X', X'' are objects of $\mathcal{X}$, the trivial morphism from X' to X'' is the composite $X' \longrightarrow * \longrightarrow X''$.

Notice that $\mathcal{X}$ is a pointed category if and only if its dual $\mathcal{X}^*$ is a pointed category. Further, if $\mathcal{X}$ and $\mathcal{Y}$ are pointed categories, then the functor $T\colon \mathcal{X} \longrightarrow \mathcal{Y}$ is a pointed functor if and only if its dual $T^*\colon \mathcal{X}^* \longrightarrow \mathcal{Y}^*$ is a pointed functor.

If $*$ and $*_1$ are points of a category $\mathcal{X}$, there is a unique isomorphism $* \longrightarrow *_1$ in $\mathcal{X}$. Further, if X', X'' are objects of $\mathcal{X}$ there is a commutative diagram



in $\mathcal{X}$. Hence the point of $\mathcal{X}$ is unique up to canonical isomorphism, and the trivial morphism between two objects of is unique.

Standard practice is to denote the trivial morphism from X' to X'' in a pointed category $\mathcal{X}$ by the same symbol as is used for the point of $\mathcal{X}$. The two most usual symbols for the point are $*$ and 0. The symbol $*$ is usually preferred when $\mathcal{X}$ does not have additive structure and the symbol 0 when $\mathcal{X}$ does have additive structure.

In a pointed category $\mathcal{X}$ the dual of the trivial morphism $*: X' \longrightarrow X''$ in $\mathcal{X}$ is the trivial morphism from $(X)^*$ to $(X')^*$ in $\mathcal{X}^*$. Further, if $f: X' \longrightarrow X''$ is a morphism in $\mathcal{X}$, $*: X \longrightarrow X'$ is the trivial morphism then $f * = *: X \longrightarrow X''$, or if $*: X'' \longrightarrow X$ is the trivial morphism then $*f = *: X' \longrightarrow X$.

If $\mathcal{X}$ and $\mathcal{Y}$ are pointed categories, $T: \mathcal{X} \longrightarrow \mathcal{Y}$ is a pointed functor and $*: X' \longrightarrow X''$ is the trivial morphism in $\mathcal{X}$ from X' to X'', then $* = T(*): T(X') \longrightarrow T(X'')$ is the trivial morphism from $T(X')$ to $T(X'')$ in $\mathcal{Y}$.

<u>2.2 Definitions</u>. Let $\mathcal{X}$ be a pointed category with point $*$, and $f: X' \longrightarrow X''$ a morphism in $\mathcal{X}$.

A <u>kernel of f</u> is a morphism $k: N \longrightarrow X'$ in $\mathcal{X}$ such that $fk = *$, and if $g: X \longrightarrow X'$ is a morphism in $\mathcal{X}$ such that $fg = *$ then there is a unique $\bar{g}: X \longrightarrow N$ such that $k\bar{g} = g$.

A cokernel of f is a morphism $j: X'' \longrightarrow C$ in $\mathfrak{X}$ such that $jf = *$, and if $h: X'' \longrightarrow X$ is a morphism in $\mathfrak{X}$ such that $hf = *$ then there is a unique $\bar{h}: C \longrightarrow X$ such that $\bar{h}j = h$.

<u>2.3 Proposition</u>. If $\mathfrak{X}$ is a pointed category with point $*$, and $f: X' \longrightarrow X''$ is a morphism $\mathfrak{X}$, the following are equivalent:

1) $k: N \longrightarrow X'$ is a kernel of f in $\mathfrak{X}$,
2) $k^*: (X')^* \longrightarrow N^*$ is a cokernel of f^* in $\mathfrak{X}^*$, and
3) $k: N \longrightarrow X'$ is an equalizer of $f,*: X' \longrightarrow X''$ in $\mathfrak{X}$.

The proposition follows at once from the definition . Observe that the fact that 1) is equivalent with 3) implies that kernels are limits, and hence when they exist they are unique up to canonical isomorphism.

<u>2.4 Definitions</u>. Let $\mathfrak{X}$ be a pointed category.

The <u>category $\mathfrak{X}$ has kernels</u> if every morphism in $\mathfrak{X}$ has a kernel.

The <u>category $\mathfrak{X}$ has cokernels</u> if every morphism in $\mathfrak{X}$ has a cokernel.

Observe that $\mathfrak{X}$ has kernels if and only if $\mathfrak{X}^*$ has cokernels. Further, if $\mathfrak{X}$ has equalizers, then $\mathfrak{X}$ has kernels (2.3).

2.5 Example. The category $\mathcal{S}_0$ of pointed sets and functions.

The canonical terminal point of $\mathcal{S}$ is $\{\emptyset\}$, the set whose only element is the empty set. This set is usually denoted by 1. An object S of $\mathcal{S}_0$ is a function $\eta(S): 1 \longrightarrow T(S)$ in $\mathcal{S}$. If S', S'' are objects of $\mathcal{S}_0$, a morphism $f: S' \longrightarrow S''$ in $\mathcal{S}_0$ is a commutative diagram

$$\begin{array}{ccccc} & & 1 & & \\ & \swarrow^{\eta(S')} & & \searrow^{\eta(S'')} & \\ T(S') & & \xrightarrow{T(f)} & & T(S'') \end{array}$$

in $\mathcal{S}$. Composition of morphism in $\mathcal{S}_0$ is induced by composition of functions. The morphisms in $\mathcal{S}_0$ are called pointed functions. There is a canonical functor $T(\ ): \mathcal{S}_0 \longrightarrow \mathcal{S}$ which assigns to every pointed set its underlying set and to every pointed function its underlying function. The canonical point of $\mathcal{S}_0$ is $1_1: 1 \longrightarrow 1$. Observe that T is an extremely faithful functor which preserves terminal points. Note also that the underlying set of a pointed set is never empty for certainly it has at least one element.

The category $\mathcal{S}_0$ has kernels and cokernels. If $f: S' \longrightarrow S''$ is a pointed function, the kernel $k: N \longrightarrow S'$ of f is the unique morphism in $\mathcal{S}_0$ such that $T(k)$ is a natural inclusion function and the diagram

$$\begin{array}{ccc} T(N) & \longrightarrow & 1 \\ \Big\downarrow T(k) & & \Big\downarrow \eta(S'') \\ T(S') & \xrightarrow{T(f)} & T(S'') \end{array}$$

is a cartesian sequence in $\mathcal{S}$. The cokernel $j : S'' \longrightarrow C$ of f is the unique pointed function such that $T(j): T(S'') \longrightarrow T(C)$ is a projection induced by dividing $T(S'')$ by an equivalence relation and the diagram

$$\begin{array}{ccc} T(S') & \xrightarrow{T(f)} & T(S'') \\ \Big\downarrow & & \Big\downarrow T(j) \\ 1 & \xrightarrow{(C)} & T(C) \end{array}$$

is a cocartesian sequence in $\mathcal{S}$.

2.6 Conventions

If $\mathcal{X}$ is a pointed category, the basic structural functor of $\mathcal{X}$ is viewed as taking values in $\mathcal{S}_0$ rather than $\mathcal{S}$ in the following way:

1) if X', X'' are objects of $\mathcal{X}$, then (X', X'') is the pointed set $1 \xrightarrow{\eta} T(\mathcal{X}(X',X''))$ such that $T(\mathcal{X}(X',X''))$ is the set of morphisms in $\mathcal{X}$ from X' to X'', and $\eta(\emptyset)$ is the trivial morphism from X' to X'', and

2) the composite $\mathcal{X} * \mathcal{X} \xrightarrow{\mathcal{X}(,)} \mathcal{S}_0 \xrightarrow{T} \mathcal{S}$ is the functor contravariant in the first variable and covariant in the second variable which would previously have been considered

to be the basic structural functor of $\mathfrak{X}$.

Observe that the basic structural functor of a pointed category $\mathfrak{X}$ is a pointed functor.

<u>2.7 Proposition</u>. If $\mathfrak{X}$ is a pointed category, and $k: N \longrightarrow X'$, $f: X' \longrightarrow X''$, $j: X'' \longrightarrow C$ are morphisms in $\mathfrak{X}$, then

1) the following are equivalent:

i) k is a kernel of f, and

ii) for every $X \in \mathrm{obj}(\mathfrak{X})$, $\mathfrak{X}(X,k): \mathfrak{X}(X,N) \longrightarrow \mathfrak{X}(X,X')$ is a kernel of $\mathfrak{X}(X,f): \mathfrak{X}(X,X') \longrightarrow \mathfrak{X}(X,X'')$ in $\mathcal{S}_0$, and

2) the following are equivalent:

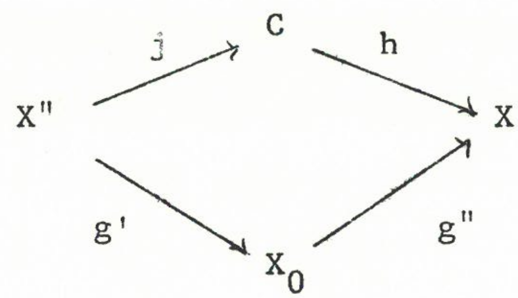
i) j is a cokernel of f, and

ii) for every $X \in \mathrm{obj}(\mathfrak{X})$, $\mathfrak{X}(j,X): \mathfrak{X}(C,X) \longrightarrow \mathfrak{X}(X'',X)$ is a kernel of $\mathfrak{X}(f,X): \mathfrak{X}(X'',X) \longrightarrow \mathfrak{X}(X',X)$ in $\mathcal{S}_0$.

The proposition is a tautology. Notice however, that both kernels and cokernels in $\mathfrak{X}$ are characterized using kernels in $\mathcal{S}_0$.

<u>2.8 Proposition</u>. If $\mathfrak{X}$ is a pointed category, $f: X' \longrightarrow X''$ is a morphism in $\mathfrak{X}$ and $j: X'' \longrightarrow C$ is a cokernel of f, then j is an extremal epimorphism.

<u>Proof</u>. Suppose $g_1, g_2: C \longrightarrow X$ are morphisms in $\mathfrak{X}$ such that $g_1 j = g_2 j$. Then $g_1 jf = * = g_2 jf$, and there is a unique morphism $\bar{g}: C \longrightarrow X$ such that $g_1 j = \bar{g} j = g_2 j$. Hence $g_1 = \bar{g} = g_2$ and j is an epimorphism. Suppose



is a commutative diagram in $\mathcal{X}$ such that g'' is a monomorphism. Then $g''g'f = hjf = *$, and $g'f = *$ since g'' is a monomorphism. Thus since j is a cokernel of f there is a unique $u : C \longrightarrow X_0$ such that $u j = g'$. Hence the proposition is proved.

<u>2.9 Definitions</u>. Let $\mathcal{X}$ be a pointed category and $f:X' \longrightarrow X''$ a morphism in .

The morphism f is a <u>normal epimorphism</u> if it has a kernel $k: N \longrightarrow X'$ and is a cokernel of k.

The morphism f is a <u>normal monomorphism</u> if it has a cokernel $j: X'' \longrightarrow C$ and is a kernel of j.

Observe that the notion of normal epimorphism is dual to that of normal monomorphism.

<u>2.10 Definitions</u>. Let $\mathcal{X}$ be a pointed category.

If $f: X' \longrightarrow X''$ is a morphism in $\mathcal{X}$, the morphism $f': X' \longrightarrow X$ is a <u>normal coimage</u> of f if it is a normal epimorphism which is a coimage of f, and the morphism $f'':X \longrightarrow X''$ is a <u>normal image</u> of f if it is a normal monomorphism which is an image of f.

The category $\mathfrak{X}$ has <u>normal projective factorization</u> if every morphism in $\mathfrak{X}$ has a normal coimage. It has <u>normal injective factorization</u> if every morphism in $\mathfrak{X}$ has a normal image.

<u>2.11 Proposition.</u> If $\mathfrak{X}$ is a pointed category with normal projective factorization, then $\mathfrak{X}$ is a category with kernels, and the following are equivalent:

1) $f : X' \longrightarrow X''$ is a normal epimorphism in $\mathfrak{X}$, and

2) $f : X' \longrightarrow X''$ is an extremal epimorphism in $\mathfrak{X}$.

<u>Proof.</u> Suppose $f : X' \longrightarrow X''$ is a morphism in $\mathfrak{X}$. Let $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ be a projective factorization of f. Since f' is a normal epimorphism, it has a kernel $k: N \longrightarrow X'$ and since f'' is a monomorphism, k is also a kernel of $f = f''f'$. Hence $\mathfrak{X}$ has kernels. If f is an extremal epimorphism, then so also if f'' by 1.6. Thus since f'' is a monomorphism, it is an isomorphism by 1.4, and 2) implies 1). By 2.8, 1) implies 2).

<u>2.12 Proposition.</u> If $\mathfrak{X}$ is a pointed category with normal projective factorization the following are equivalent:

1) $f : X' \longrightarrow X''$ is a monomorphism in $\mathfrak{X}$, and

2) $* \longrightarrow X'$ is a kernel of f.

<u>Proof.</u> Certainly 1) implies 2). Suppose 2) and that $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ is a projective factorization of f.

Since f'' is a monomorphism, $* \longrightarrow X'$ is a kernel of f', but since f' is a normal epimorphism, f' is a cokernel of $* \longrightarrow X'$. However, $1_{X'} : X' \longrightarrow X'$ is a cokernel of $* \longrightarrow X'$. Thus f' is an isomorphism and 2) implies 1).

<u>2.13 Proposition</u>. If $\mathcal{X}$ is a pointed category and $\mathcal{J}$ is a small category, then $[\mathcal{J}, \mathcal{X}]$ is a pointed category, and if $\mathcal{X}$ has kernels, then $[\mathcal{J}, \mathcal{X}]$ has kernels. Further, if $\alpha : T' \longrightarrow T''$ is a morphism in $[\mathcal{J}, \mathcal{X}]$, the following are equivalent:

1) $\beta : T_0 \longrightarrow T'$ is a kernel of α, and

2) $\beta : T_0 \longrightarrow T'$ is a morphism in $[\mathcal{J}, \mathcal{X}]$ such that for $J \in obj(\mathcal{J})$, $\beta(J)$ is a kernel of $\alpha(J)$.

<u>Proof</u>. If $*$ is a point of $\mathcal{X}$ and $C_{\mathcal{J}}(\ ) : \mathcal{X} \longrightarrow [\mathcal{J}, \mathcal{X}]$ is the constant factor, then $C_{\mathcal{J}}(*)$ is a point of $[\mathcal{J}, \mathcal{X}]$ and thus $[\mathcal{J}, \mathcal{X}]$ is pointed.

Suppose $\alpha : T' \longrightarrow T''$ is a morphism in $[\mathcal{J}, \mathcal{X}]$. For $J \in obj(\mathcal{J})$, let $\beta(J) : T_0(J) \longrightarrow T'(J)$ be a kernel of $\alpha(J)$. If $j : J' \longrightarrow J''$ is a morphism in $\mathcal{J}$, $\alpha(J'')T'(j)\beta(J' =$ $T''(j)\alpha(J')\beta(J') = *$ and there is a unique $T_0(j) : T_0(J') \longrightarrow T_0(J'')$ such that $T'(j)\beta(J') = \beta(J'')T_0(j)$. Now $\beta : T_0 \longrightarrow T$ is a kernel of α. Thus $[\mathcal{J}, \mathcal{X}]$ has kernels and the construction of kernels in $[\mathcal{J}, \mathcal{X}]$ from kernels in $\mathcal{X}$ shows the equivalence of 1) and 2). Hence the proposition is proved.

2.14 Proposition. If $\mathcal{X}$ is a pointed category with normal projective factorization and $\mathcal{J}$ is a small category, then $[\mathcal{J}, \mathcal{X}]$ is a pointed category with normal projective factorization.

Proof. By 2.11 and 2.13, $[\mathcal{J}, \mathcal{X}]$ is a pointed category with kernels. Suppose $\alpha : T' \longrightarrow T''$ is a morphism in $[\mathcal{J}, \mathcal{X}]$, and let $\beta : T_0 \longrightarrow T'$ be a kernel of α. For $J \in \text{obj}(\mathcal{J})$, let $T'(J) \xrightarrow{\alpha'(J)} T(J) \xrightarrow{\alpha''(J)} T''(J)$ be a projective factorization of $\alpha(J)$ in $\mathcal{X}$. By 2.14, $\alpha'(J)$ is a cokernel of $\beta(J)$ for $J \in \text{obj}(\mathcal{J})$. If $j : J' \longrightarrow J''$ is a morphism in $\mathcal{J}$, then $\alpha(J'')T'(j)\beta(J') = T''(j)\alpha(J')\beta(J') = *$, and $\alpha'(J'')T'(j)\beta(J') = *$ since $\alpha''(J'')$ is a monomorphism. Thus there is a unique $T(j) : T(J') \longrightarrow T(J'')$ such that $\alpha'(J'')T'(j) = T(j)\alpha'(J')$. Now $\alpha' : T' \longrightarrow T$ is a normal coimage of α in $[\mathcal{J}, \mathcal{X}]$ and the proposition is proved.

2.15 Proposition. If $\mathcal{X}$ is a pointed category and $\mathcal{X}'$ is a weak separation subcategory of $\mathcal{X}$, then

1) if $*$ is a point of $\mathcal{X}$, then $*$ is a point of $\mathcal{X}'$,

2) if $f : X' \longrightarrow X''$ is a morphism in $\mathcal{X}'$, and $k : N \rightarrow X'$ is a kernel of f in $\mathcal{X}$, then k is a kernel of f in $\mathcal{X}'$, and

3) if $R : \mathcal{X} \longrightarrow \mathcal{X}'$, $\lambda : 1_{\mathcal{X}} \longrightarrow R$ is the coreflection of $\mathcal{X}$ in $\mathcal{X}'$, $f : X' \longrightarrow X''$ is a morphism in $\mathcal{X}'$ and $j : X'' \longrightarrow C$

is a cokernel of f in $\mathfrak{X}$, then $R(j) : X'' \longrightarrow R(C)$ is a cokernel of f in $\mathfrak{X}'$.

Proof. If $*$ is a point of $\mathfrak{X}$, $\lambda(*) : * \longrightarrow R(*)$, and $\varepsilon(R(*)\,\lambda(*) = 1_*$. Hence $\lambda(*) = 1_*$, and part 1 is proved. Parts 2 and 3 are special cases of Chapter 1, 4.7^*, and hence the proposition is proved.

2.16 Proposition. If $\mathfrak{X}$ is a pointed category and $\mathfrak{X}'$ is a weak separation subcategory of $\mathfrak{X}$ with coreflection $R: \mathfrak{X} \longrightarrow \mathfrak{X}'$, $\lambda: 1_{\mathfrak{X}} \longrightarrow R$, then

1) if R preserves kernels, $f : X' \longrightarrow X''$ is a morphism in $\mathfrak{X}'$ and $f'': X \longrightarrow X''$ is a normal image of f in $\mathfrak{X}$, then $R(f''): R(X) \longrightarrow X''$ is a normal image of f in $\mathfrak{X}'$, and hence if $\mathfrak{X}$ has normal injective factorization, so also does $\mathfrak{X}'$,

2) if the inclusion functor $\mathfrak{X}' \longrightarrow \mathfrak{X}$ preserves normal epimorphisms, $f : X' \longrightarrow X''$ is a morphism in $\mathfrak{X}'$ and $f':X' \longrightarrow X$ is a normal coimage of f in $\mathfrak{X}$, then f' is a normal coimage of f in $\mathfrak{X}'$, and hence if $\mathfrak{X}$ has normal projective factorization, so also does $\mathfrak{X}'$, and

3) if $\mathfrak{X}'$ is a strong separation subcategory of $\mathfrak{X}$ and $\mathfrak{X}$ has normal projective factorization, then the inclusion functor $\mathfrak{X}' \longrightarrow \mathfrak{X}$ preserves normal epimorphisms.

Proof. If the conditions of part 1) are satisfied, and $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ is an injective factorization of f in $\mathfrak{X}$, then $X' \xrightarrow{R(f')} R(X) \xrightarrow{R(f'')} X''$ is an injective factorization of f in $\mathcal{X}$ since $R(f')$ is an epimorphism in $\mathcal{X}'$ due to the fact that R is a coadjoint functor and $R(f'')$ is a normal monomorphism since R preserves kernels. Hence part 1) is proved. Parts 2 and 3 follow from 1.18 parts 3) and 4), 2.8 and 2.11. Hence the proposition is proved.

<u>2.17 Example</u>. The category $\mathcal{S}_0$ of pointed sets and functions has projective factorization and normal injective factorization, but not normal projective factorization. These facts follow at once upon examination of the construction of kernels and cokernels in $\mathcal{S}_0$ (2.5).

<u>2.18 Definitions and observations</u>.

Let $\mathcal{X}$ be a finitely bicomplete category.

An <u>initial point</u> $*_I$ of $\mathcal{X}$ <u>is effective</u> if for every object X of $\mathcal{X}$ the projection to the second factor $*_I \Pi X \longrightarrow X$ is an effective epimorphism. A <u>terminal point</u> $*_T$ of $\mathcal{X}$ <u>is</u> effective if for every object X of $\mathcal{X}$ the injection from the second cofactor $X \longrightarrow *_T \amalg X$ is an effective monomorphism.

The terminal point 1 of $\mathcal{S}$ is effective, but the initial point $\emptyset$ is not effective.

If both the terminal point and the initial point of $\mathcal{X}$ are effective, then $*_I \longrightarrow *_T \amalg *_I \cong *_T$ is an effective monomorphism, and $*_I \cong *_I \Pi *_T \longrightarrow *_T$ is an effective epimorphism. Hence

$*_I \longrightarrow *_T$ is an isomorphism since it is both an effective epimorphism and an effective monomorphism, and it follows that $\mathfrak{X}$ is a pointed category.

If the terminal point $*_T$ of $\mathfrak{X}$ is effective, the <u>pointed category</u> $\mathfrak{X}_0$ <u>associated with</u> $\mathfrak{X}$ is the category such that an object X of $\mathfrak{X}_0$ is a morphism $\eta(X)\colon *_T \longrightarrow T(X)$ in $\mathfrak{X}$, and a morphism $f\colon X' \longrightarrow X''$ in $\mathfrak{X}_0$ is a commutative diagram

$$\begin{array}{ccc} & *_T & \\ {\scriptstyle \eta(X')} \swarrow & & \searrow {\scriptstyle \eta(X'')} \\ T(X') & \xrightarrow{T(f)} & T(X'') \end{array}$$

in $\mathfrak{X}$. Composition of morphisms in $\mathfrak{X}_0$ is induced by composition of morphisms in $\mathfrak{X}$. There is a canonical extremely faithful functor $T(\ )\colon \mathfrak{X}_0 \longrightarrow \mathfrak{X}$. The point of $\mathfrak{X}_0$ is $1_{*_T}\colon *_T \longrightarrow *_T$. The category $\mathfrak{X}$ is pointed if and only if the canonical functor $T\colon \mathfrak{X}_0 \longrightarrow \mathfrak{X}$ is an isomorphism of categories.

If the initial point $*_I$ of $\mathfrak{X}$ is effective, the <u>pointed category</u> $\mathfrak{X}_0$ <u>associated with</u> $\mathfrak{X}$ is the category such that an object X of $\mathfrak{X}_0$ is a morphism $\varepsilon(X)\colon S(X) \longrightarrow *_I$ in $\mathfrak{X}$, and a morphism $f\colon X' \longrightarrow X''$ in $\mathfrak{X}_0$ is a commutative diagram

$$\begin{array}{ccc} S(X') & \xrightarrow{S(f)} & S(X'') \\ {\scriptstyle \varepsilon(X')} \searrow & & \swarrow {\scriptstyle \varepsilon(X'')} \\ & *_I & \end{array}$$

in $\mathfrak{X}$. Composition of morphisms in $\mathfrak{X}_0$ is induced by composition of morphisms in $\mathfrak{X}$. There is a canonical extremely faithful functor $S(\ ): \mathfrak{X}_0 \longrightarrow \mathfrak{X}$. The point of $\mathfrak{X}_0$ is $1_{*_I} : *_I \longrightarrow *_I$. The category $\mathfrak{X}$ is pointed if and only if the canonical functor $S: \mathfrak{X}_0 \longrightarrow \mathfrak{X}$ is an isomorphism of categories.

Observe that the initial point of $\mathfrak{X}$ is effective if and only if the terminal point of $\mathfrak{X}^*$ is effective. If the initial point of $\mathfrak{X}$ is effective, then $(\mathfrak{X}_0)^* = (\mathfrak{X}^*)_0$ and the dual of the canonical functor $S: \mathfrak{X}_0 \longrightarrow \mathfrak{X}$ is the canonical functor $T: (\mathfrak{X}^*)_0 \longrightarrow \mathfrak{X}^*$. If $\mathfrak{X}$ should happen to be pointed both of these functors become identity functors.

<u>2.19 Proposition</u>. If $\mathfrak{X}$ is a finitely bicomplete category with an effective terminal point, then $\mathfrak{X}_0$ is a finitely bicomplete pointed category and the canonical functor $T: \mathfrak{X}_0 \longrightarrow \mathfrak{X}$ is an extremely faithful adjoint functor.

<u>Proof</u>. Suppose $f_1, f_2: X \longrightarrow X''$ are morphisms in $\mathfrak{X}_0$. Let $T(f): T(X') \longrightarrow T(X)$ be the equalizer of $T(f_1), T(f_2)$ in $\mathfrak{X}$, and let $\eta(X'): *_T \longrightarrow T(X')$ be the unique morphism such that $T(f)\eta(X') = \eta(X)$. There is a unique $f: X' \longrightarrow X$ in $\mathfrak{X}_0$ such that $T(f)$ is as above, f is an equalizer of f_1, f_2 in $\mathfrak{X}_0$ and $\mathfrak{X}_0$ has equalizers.

Let J be a finite set and suppose X_j is an object of $\mathfrak{X}_0$ for $j \in J$. Let $T(\prod_{j\in J} X_j) = \prod_{j\in J} T(X_j)$ with projection

$T(p_k): T(\prod_{j\in J} X_j) \longrightarrow T(X_k)$ to the k-factor for $k \in J$. Let $\eta(\prod_{j\in J} X_j): *_T \longrightarrow T(\prod_{j\in J} X_j)$ be the unique morphisms such that $T(p_k)\ \eta(\prod_{j\in J} X_j) = \eta(X_k)$ for $k \in J$. There are unique morphisms $p_k: \prod_{j\in J} X_j \longrightarrow X_k$ in $\mathcal{X}_0$ for $k \in J$ such that $T(p_k)$ is as above. These morphisms define a product of $(X_j)_{j\in J}$ in $\mathcal{X}_0$. Hence $\mathcal{X}_0$ has finite products. Since it also has equalizers, it is finitely complete and the construction of products and equalizers in $\mathcal{X}_0$ shows that $T: \mathcal{X}_0 \longrightarrow \mathcal{X}$ preserves finite limits.

For X an object of $\mathcal{X}$, let $S(X)$ be the object of $\mathcal{X}_0$ such that $TS(X) = *_T \amalg X$ and $\varepsilon(S(X)): *_T \longrightarrow TS(X)$ is the injection from the first cofactor. If $f: X' \longrightarrow X''$ is a morphism in $\mathcal{X}$, let $S(f): S(X') \longrightarrow S(X'')$ be the unique morphism in $\mathcal{X}_0$ such that $TS(f) = 1_{*_T} \amalg f$. Now $S: \mathcal{X} \longrightarrow \mathcal{X}_0$ is a functor. For $X \in obj(\mathcal{X})$ let $\beta(X): X \longrightarrow TS(X) = *_T \amalg X$ be the injection from the second cofactor. Now $\beta: 1_{\mathcal{X}} \longrightarrow TS$ is a morphism and $\beta(X)$ is an effective monomorphism for $X \in obj(\mathcal{X})$.

For $X \in obj(\mathcal{X}_0)$, let $\alpha(X): ST(X) \longrightarrow X$ be the unique morphism in $\mathcal{X}_0$ such that $T(\alpha(X)) = 1_{*_T}\ \ 1_{T(X)}$. Now $\alpha: ST \longrightarrow 1_{\mathcal{X}_0}$ is a morphism and $(\alpha, \beta): S \dashv T: (\mathcal{X}_0,\ \ )$ is an adjoint pair of functors.

Suppose that

$$\begin{array}{ccc} X & \xrightarrow{f'} & X' \\ {\scriptstyle f''}\downarrow & & \\ X'' & & \end{array}$$

is a diagram in $\mathfrak{X}_0$. There is a diagram

$$\begin{array}{ccc} X & \xrightarrow{f'} & X' \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f'_0} \\ X'' & \xrightarrow[f''_0]{} & X_0 \end{array}$$

in $\mathfrak{X}_0$ such that

$$\begin{array}{ccc} T(X) & \xrightarrow{T(f')} & T(X') \\ {\scriptstyle T(f'')}\downarrow & & \downarrow{\scriptstyle T(f'_0)} \\ T(X'') & \xrightarrow[T(f''_0)]{} & T(X_0) \end{array}$$

is a cocartesian square in $\mathfrak{X}$. This implies that the preceding diagram is a cocartesian square in $\mathfrak{X}_0$. Hence by 4.15*, Chapter 1, $\mathfrak{X}_0$ is finitely cocomplete and the proposition is proved.

<u>2.20 Proposition</u>. If $\mathfrak{X}$ is a finitely bicomplete category with an effective terminal point and $(\alpha, \beta)\colon S \dashv T\colon (\mathfrak{X}_0, \mathfrak{X})$ is the canonical adjoint pair, then

1) T preserves cocartesian squares, and

2) S is an extremely faithful functor.

<u>Proof</u>. Part 1) follows from the construction of cocartesian squares in $\mathfrak{X}_0$ at the end of the proof of 2.19. While part 2) follows from the fact that if $X \in \mathrm{obj}(\mathfrak{X})$, $\beta(X)\colon X \longrightarrow TS(X)$ is an effective monomorphism together with 1.3* and 1.13*.

Note the preceding does not imply that T preserves finite colimits, for it does not preserve finite coproducts except in the case $\mathcal{X}$ is pointed.

<u>2.21 Definition</u>. A <u>terminal category</u> $\mathcal{X}$ is a finitely bicomplete category $\mathcal{X}$ with an effective terminal point and effective injective factorization such that

1) if $f_0 \colon X'_0 \longrightarrow X_0$ and $f_1 \colon X'_1 \longrightarrow X_1$ are effective monomorphisms, then $f_0 \amalg f_1 \colon X'_0 \amalg X'_1 \longrightarrow X_0 \amalg X_1$ is an effective monomorphism, and

2) if

$$\begin{array}{ccc} X & \xrightarrow{\;f'\;} & X' \\ {\scriptstyle f''}\big\downarrow & & \big\downarrow{\scriptstyle f'_0} \\ X'' & \xrightarrow[\;f''_0\;]{} & X_0 \end{array}$$

is a cocartesian square in $\mathcal{X}$ with f'' an effective monomorphism, then f'_0 is an effective monomorphism, and if further f' is a coretract, then the square is cartesian.

An <u>initial category</u> $\mathcal{X}$ is a finitely bicomplete category $\mathcal{X}$ with an effective initial point and effective projective factorization such that

1) if $f_0 \colon X_0 \longrightarrow X''_0$, $f_1 \colon X_1 \longrightarrow X''_1$ are effective epimorphisms, then $f_0 \prod f_1 \colon X_0 \prod X_1 \longrightarrow X''_0 \prod X''_1$ is an effective epimorphism, and

2) if

$$\begin{array}{ccc} X_0 & \xrightarrow{\;f''_0\;} & X'' \\ {\scriptstyle f'_0}\big\downarrow & & \big\downarrow{\scriptstyle f''} \\ X' & \xrightarrow[\;f'\;]{} & X \end{array}$$

is a cartesian square in $\mathfrak{X}$ with f'' an effective epimorphism, then f_0' is an effective epimorphism, and if further f' is a retract, then the square is cocartesian.

Note that $\mathfrak{X}$ is a terminal category if and only if $\mathfrak{X}^*$ is an initial category.

<u>2.22 Proposition</u>. If $\mathfrak{X}$ is a terminal category, then $\mathfrak{X}_0$ is a pointed terminal category with normal injective factorization.

<u>Proof</u>. By 2.19, $\mathfrak{X}_0$ is a finitely bicomplete pointed category. Suppose $f' : X' \longrightarrow X$ is a morphism in $\mathfrak{X}_0$ such that $T(f')$ is an effective monomorphism and that $f'' : X \longrightarrow X''$ is a cokernel of f'. Now

$$\begin{array}{ccc} T(X') & \xrightarrow{T(f')} & T(X) \\ \Big\downarrow{\scriptstyle \varepsilon(X')} & & \Big\downarrow{\scriptstyle T(f'')} \\ *_T & \xrightarrow{\eta(X'')} & T(X'') \end{array}$$

is a cocartesian square in $\mathfrak{X}$ with $T(f')$ an effective monomorphism, and $\varepsilon(X')$ a coretract. Since $\mathfrak{X}$ is a terminal category, the square is a cartesian square and f' is a kernel of f''.

Suppose $f : X' \longrightarrow X''$ is a morphism in $\mathfrak{X}_0$. There is a factorization $X' \xrightarrow{f'} X \xrightarrow{f''} X''$ of f in $\mathfrak{X}_0$ such that $T(X') \xrightarrow{T(f')} T(X) \xrightarrow{T(f'')} T(X'')$ is an injective factorization of $T(f)$ in $\mathfrak{X}$. Since T is faithful, f' is an epimorphism, and by the preceding paragraph, f'' is a normal monomorphism. Thus $\mathfrak{X}_0$ has normal injective factorization. Suppose that

$$\begin{array}{ccc} X & \xrightarrow{f''} & X'' \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle i_2} \\ X'' & \xrightarrow[i_1]{} & X_0 \end{array}$$

is a cocartesian square in $\mathfrak{X}_0$. Its image under T is a cocartesian square in $\mathfrak{X}$ by 2.20. Now the fact that $T(F'')$ is an equalizer of $T(i_1)$, $T(i_2)$ implies that f'' is an equalizer of i_1, i_2. Hence the notion of effective monomorphism in $\mathfrak{X}_0$ coincides with the notion of normal monomorphism in $\mathfrak{X}_0$.

If

$$\begin{array}{ccc} X & \xrightarrow{f'} & X' \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f'_0} \\ X'' & \xrightarrow[f''_0]{} & X_0 \end{array}$$

is a cocartesian square in $\mathfrak{X}_0$ its image under T is a cocartesian square in $\mathfrak{X}$, and if f'' is an effective monomorphism, then $T(f'')$ is an effective monomorphism, $T(f'_0)$ is an effective monomorphism, and hence f'_0 is an effective monomorphism. If further f' is a coretract, then $T(f')$ is a coretract and the fact that the image by T of the square is a cartesian square in $\mathfrak{X}$ implies the original square is a cartesian square in $\mathfrak{X}_0$. Consequently the proposition is proved.

2.23 Proposition. If $\mathcal{X}$ is a terminal category and $\mathcal{J}$ is a small category, then $[\mathcal{J}, \mathcal{X}]$ is a terminal category and $[\mathcal{J}, \mathcal{X}]_0 = [\mathcal{J}, \mathcal{X}_0]$.

Proof. By Chapter 1, 4.26 and 4.26*, $[\mathcal{J}, \mathcal{X}]$ is a finitely bicomplete category, and by 1.23*, $[\mathcal{J}, \mathcal{X}]$ has effective injective factorization. Now the local character of the cartesian and cocartesian squares in $[\mathcal{J}, \mathcal{X}]$ (Chapter 1, 4.37, 4.37*) insures that since $\mathcal{X}$ is a terminal category, so also is $[\mathcal{J}, \mathcal{X}]$. Given the preceding, the fact that $[\mathcal{J}, \mathcal{X}]_0 = [\mathcal{J}, \mathcal{X}_0]$ is immediate. Hence the proposition is proved.

2.24 Proposition. If $\mathcal{X}$ is a terminal category and $\mathcal{X}'$ is a weak separation subcategory of $\mathcal{X}$ with coreflection $R: \mathcal{X} \longrightarrow \mathcal{X}'$, $\lambda: 1_{\mathcal{X}} \longrightarrow R$ and R preserves finite limits, then $\mathcal{X}'$ is a terminal category.

The proof is routine using 1.20*, and Chapter 1, 4.13, 4.13*.

Exercises.

1. Prove that $\mathcal{S}$ is a terminal category, and that $\mathcal{S}_0$ is a pointed bicomplete terminal category with unique factorization. Give an example of an epimorphism in $\mathcal{S}_0$ which is not a normal epimorphism. Prove that every object in $\mathcal{S}_0$ is both projective and injective. Show that $\mathcal{S}_0$ has a generator and a cogenerator.

2. Prove that if $\mathfrak{X}$ is a finitely bicomplete category with an effective terminal point, then

i) if $\mathfrak{X}$ is complete or countably complete, then $\mathfrak{X}_0$ is complete or countably complete,

ii) if $\mathfrak{X}$ is cocomplete or countably cocomplete, then $\mathfrak{X}_0$ is cocomplete or countably cocomplete, and

iii) if $\mathfrak{X}$ has a generator, then $\mathfrak{X}_0$ has a generator.

3. Prove that $\mathcal{T}$ is a terminal category, but that $\mathcal{T}_K$ and $\mathcal{T}_F$ are not terminal categories.

4. Prove that $\mathcal{T}_{SF}$ and $\mathcal{T}_{WH}$ are terminal categories, but that $\mathcal{T}_H$, $\mathcal{T}_{NCR}$, and $\mathcal{T}_{CR}$ are not terminal categories.

5. Prove that $\mathcal{T}^C$ and $\mathcal{T}^F$ are terminal categories with effective projective factorization. Show that $\mathcal{T}(C)$ and $\mathcal{T}(F)$ are terminal categories with effective projective factorization.

6. Prove that $\mathcal{C}$ is a terminal category and that $\mathcal{C}_0$ is a pointed bicomplete terminal category with unique factorization. Show that $\mathcal{C}_0$ has a projective generator and an injective cogenerator.

7. Let $\mathcal{C}^{met}$ denote the full subcategory of $\mathcal{C}$ generated by the metrizable compact spaces. Show that $\mathcal{C}^{met}$ is a countably complete terminal category such that the inclusion $\mathcal{C}^{met} \longrightarrow \mathcal{C}$ preserves countable limits and finite colimits. Prove that $\mathcal{C}^{met}$ has a projective generator and an injective cogenerator.

8. Show that the categories $\mathcal{T}^C \cap \mathcal{T}_K$, $\mathcal{T}^C \cap \mathcal{T}_F$, $\mathcal{T}^C \cap \mathcal{T}_H$, $\mathcal{T}^C \cap \mathcal{T}_{NCR}$, and $\mathcal{T}^C \cap \mathcal{T}_{CR}$ are not terminal categories.

9. Prove that the categories $\mathcal{R}$ and $\mathcal{R}^C$ are initial categories. Show that $(\mathcal{R}^C)_0$ is a strong separation subcategory of $\mathcal{R}_0$.

10. Let R be a commutative ring. If A and B are R-modules, let $A \otimes B$ denote the tensor product of A and B over R. An <u>R-coalgebra</u> C is an R-module C together with morphisms of R-modules $\Delta(C)\colon C \longrightarrow C \otimes C$ and $\varepsilon(C)\colon C \longrightarrow R$ such that

1) the diagram

$$\begin{array}{ccc} C & \xrightarrow{\Delta(C)} & C \otimes C \\ \Big\downarrow {\scriptstyle \Delta(C)} & & \Big\downarrow {\scriptstyle C \otimes \Delta(C)} \\ C \otimes C & \xrightarrow{\Delta(C) \otimes C} & C \otimes C \otimes C \end{array}$$

of R-modules is commutative,

2) the diagram

$$\begin{array}{ccc} & & C \otimes R \\ & \overset{\approx}{\nearrow} & \Big\downarrow {\scriptstyle C \otimes \varepsilon(C)} \\ C & \xrightarrow{\Delta(C)} & C \otimes C \\ & \underset{\approx}{\searrow} & \Big\downarrow {\scriptstyle \varepsilon(C) \otimes C} \\ & & R \otimes C \end{array}$$

is commutative.

The morphism $\Delta(C)$ is the diagonal or comultiplication of C, while $\varepsilon(C)$ is the unit of C.

If C', C'' are R-coalgebras, then a morphism $f: C' \longrightarrow C''$ of R-coalgebras is a morphism of R-modules such that the diagrams

$$\begin{array}{ccc} C' & \xrightarrow{\quad f \quad} & C'' \\ \Big\downarrow{\scriptstyle \Delta(C')} & & \Big\downarrow{\scriptstyle \Delta(C'')} \\ C' \otimes C' & \xrightarrow{\quad f \otimes f \quad} & C'' \otimes C'' \end{array}$$

and

$$\begin{array}{ccc} C' & \xrightarrow{\quad f \quad} & C'' \\ & {\scriptstyle \varepsilon(C')} \searrow \quad \swarrow {\scriptstyle \varepsilon(C'')} & \\ & R & \end{array}$$

are commutative.

The category of R-coalgebras, coalg(R) is the category whose objects are the R-coalgebras, whose morphisms are the morphisms of R-coalgebras, and with composition induced by composition in the category of R-modules mod(R).

Show that coalg(R) is a cocomplete category with a terminal point and that there is an extremely faithful colimit preserving functor $S: \text{coalg}(R) \longrightarrow \text{mod}(R)$.

11. If R is a commutative ring, and C is an R-coalgebra, then C is <u>commutative</u> if the diagram of R-modules

$$\begin{array}{ccc} & & C \otimes C \\ & \overset{\Delta(C)}{\nearrow} & \\ C & & \big\downarrow \tau \\ & \underset{\Delta(C)}{\searrow} & \\ & & C \otimes C \end{array}$$

is commutative. The category of commutative R-coalgebras $coalg^C(R)$ is the full subcategory of coalg(R) generated by the commutative R-coalgebras.

Show that the category $coalg^C(R)$ is a cocomplete category with a terminal point and finite products. Show that the inclusion functor $coalg^C(R) \longrightarrow coalg(R)$ preserves colimits.

12. Show that if R is a commutative ring, then the categories coalg(R) and $coalg^C(R)$ have effective injective factorization .

13. Show that if k is a field (commutative) then the category $coalg^C(k)$ of commutative coalgebras over k is a terminal category.

14. If R is a commutative ring, define the category of R-algebras in such a manner that if alg(R) denotes the category of R-algebras, then

i) if $R = \mathbb{Z}$, then $alg(\mathbb{Z}) = \mathcal{R}$, the category of rings, and

ii) there is an extremely faithful coadjoint functor $S: mod(R) \longrightarrow alg(R)$.

15. Show that if R is a commutative ring, the category alg(R) is a bicomplete initial category.

16. If R- is a commutative ring, define the category of commutative R-algebras so that

i) if $R = \mathbb{Z}$, then the category $alg^C(R)$ of commutative R-algebras is the category $\mathcal{R}^C$ of commutative rings,

ii) $alg^C(R)$ is a strong separation subcategory of $alg(R)$, and

iii) the composite

$$mod(R) \xrightarrow{S} alg(R) \xrightarrow{V} alg^C(R)$$

is an extremely faithful functor when V is the coreflection of $alg(R)$ in $alg^C(R)$ and S is the functor of exercise 14.