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Lectures on Characteristic Classes
by John Milnor
Notes by James Stasheff
(Spring 1957)

I. n-plane bundles:

In the study of characteristic classes, we will be concerned with n-dimensional vector space bundles or, briefly, n-plane bundles.

Definition: An n-plane bundle consists of a triple \((E, B, \pi)\) with \( \pi \) a map (i.e. continuous function) from a Hausdorff space \( E \) onto a Hausdorff space \( B \), and the structure of an n-dimensional vector space over the reals \( \mathbb{R} \) in the \( \pi \)'th fibre \( \pi^{-1}(b) \) for all \( b \in B \), satisfying the further requirements that

1) there exist a distinguished class of open sets \([U]\)
   covering \( B \) and \( n \) maps \( c_i: U \to E \) for each \( U \), such that

2) each \( c_i \) is a cross-section, that is \( \pi(c_i(b)) = b \) for each \( b \in U \), and

3) the map \( U \times \mathbb{R}^n \to \pi^{-1}(U) \) defined by
   \((b, \lambda_1, \ldots, \lambda_n) \to \sum \lambda_i c_i(b), \lambda_i \in \mathbb{R}, \) is a homeomorphism. (This is the local product structure on \( E \).

We call \( B \) the base space, \( E \) the total space, \( \pi \) the projection and denote the triple and structure by \( \xi \) Greek letter i.e. \( \xi = (E, B, \pi) \). A superscript on a bundle indicates the dimension of the fibre \( \pi^{-1}(b) \), e.g. \( \xi^n \).

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Remark: Although not necessary for what follows, it should be noted that an n-plane bundle is an example of a fibre bundle. (See Steenrod; Topology of Fibre Bundles, 1951.) In fact, an n-plane bundle is exactly a fibre bundle with real n-dimensional vector space as fibre and GL(n,R) as structural group.

Examples of n-plane bundles:

1) The product bundle B x R^n

2) The tangent bundle T^n of a differentiable manifold M^n of class C^1 or more. Here B = M^n and E is the set of all pairs (b, contravariant vector at b).

3) The normal k-plane bundle \nu^k of a differentiable manifold M^n \subset R^{n+k} (For a differentiable manifold "\subset" is always to be read "differentiably imbedded in".) Here the base space B is again M^n and E is the set of all pairs (b, normal vector at b).

4) The 1-plane bundle or line bundle \xi_{1n} over real projective n-space P^n defined as follows. Consider P^n as the set of all unordered pairs \{x,-x\} where x ranges over all unit vectors in R^{n+1}. The total space E is to be the set of all pairs \{[x,-x], \lambda x\} with \lambda a real number.

Remark 1. Every cross-section of this bundle (i.e. a map \phi: B \to E such that \pi \phi = identity on B) is somewhere zero, \phi(b) = (b,0,0,...,0) for some b. We call a "non-zero cross-section" one which is never zero. Proof that the bundle \xi_{1n} of

(4) has no non-zero cross-sections, n ≥ 1: Given any cross-section \phi: P^n \to E we can define a map \lambda: S^n \to R by \phi([x,-x]) = ([x,-x], \lambda(x)x). Since \lambda(-x) = -\lambda(x) and S^n is connected, there is a point x for which \lambda(x) = 0 or \phi([x,-x]) = ([x,-x],0).

Remark 2. The following alternative description of \xi_{1n} will be useful later. As total space E_1 take S^n \times R with the identification (x,\lambda) = (-x,-\lambda). Evidently the element \{(x,\lambda), (-x,-\lambda)\} of E_1 can be identified with \{[x,-x], \lambda x\} \in E. Therefore this new bundle is equivalent (see next paragraph) to the one defined above.

Bundle maps and induced bundles:

Definition: A bundle map \tau: \xi \to \eta, where \xi = (E, E, \pi) and \eta = (E', B', \pi') are n-plane bundles, is a pair of maps (f_E, f_B) such that

1) the following diagram is commutative

2) \tau = f^{-1}_B \tau (i.e. \tau f_E = f_B \tau) and

3) f_E|_{\tau^{-1}(b)} is linear and non-singular for each b in B.
Special Case: \( B = B' \)

Definition: Two n-plane bundles \( \xi, \eta \) over \( B \) are equivalent if there is a bundle map \( f: \xi \rightarrow \eta \) with \( f_B^* = \text{Identity on } B \). This is an equivalence relation and using it we define

Definition: An n-plane bundle is trivial if it is equivalent to the product bundle \( B \times \mathbb{R}^n \).

Remark: A bundle is trivial if and only if there exist \( n \) independent cross-sections. (We use them to define \( f_B^* \))

Using this concept, we have

Definition: A differentiable manifold \( M^n \) is parallelizable if the tangent bundle \( T^n(M^n) \) is trivial.

Induced bundle: Given a bundle \( \xi \) with \( \pi: E \rightarrow B' \), another space \( B' \) and a map \( f_{B'}: B' \rightarrow B \), there is a construction by which we get another bundle \( (E', B', \pi') \) and a bundle map \( f = (f_{B'}, f_B) \). Let \( E' \) be the subset of \( B' \times E \) consisting of all pairs \((b', e')\) with \( b' \in B' \), \( e \in E \) such that \( f_{B'}(b') = \pi(e) \). Define \( \pi': E' \rightarrow B' \) by \( \pi'(b', e) = b' \). Each fibre \( \pi'^{-1}(b') \) will have the structure of a vector space isomorphic to \( \pi^{-1}(f_{B'}(b')) \). Thus we have constructed the induced bundle, the bundle induced by \( f_{B'} \). It is easy to verify that the projection map \( f_{B'}(b', e) = e \) gives a bundle map \( (f_{B'}, f_B) \) of the induced bundle into the original bundle.

Remark: Given two bundles \( \xi \) and \( \eta \) and a map \( f_B \) of their base spaces as indicated in the diagram, it is often possible to define a map \( f_{E'} \) so that the pair is a bundle map. This is possible, if and only if \( \xi \) is equivalent to the bundle induced by \( f_B \) from \( \eta \). For example, \( F_k^1 \subset F^{k+1}_k \), \( E_1^{k+1} \subset E_1^*_{k+1} \) and there is the obvious bundle map \( f = (1_{E}, 1_{B}) \) where \( 1_{E} \), \( 1_{B} \) are the indicated inclusion maps. Thus \( \xi_1^{k} \) is equivalent to the i-plane bundle over \( F^k \) induced from \( \xi_{k+1}^{1} \) by \( 1_{B}^*: F^k \rightarrow F^{k+1} \).

We need one more relation between n-plane bundles:

The Whitney Bundle Sum: Given an m-plane bundle \( \xi = (E, B, \pi) \) and an n-plane bundle \( \eta = (E', B, \pi') \), let \( E'' \) be the subset of \( E \times E' \) consisting of all pairs \((e, e')\) such that \( \pi(e) = \pi'(e') \).

Define \( \rho: E'' \rightarrow E \) by \( \rho(e, e') = e \)

\( \rho': E'' \rightarrow E' \) by \( \rho'(e, e') = e' \)

\( \pi'': E'' \rightarrow B \) by \( \pi'' = \pi \circ \rho' \)

Since \( \pi, \pi' \) are projections of m, n-plane bundles respectively, \( \pi'' \) is the projection of an \((m+n)\)-plane bundle, the Whitney sum, \( \xi \oplus \eta = (E'', B, \pi'') \).
Example 5: For a differentiable manifold $N^n = R^{n+k}$ we have that $S^n \otimes S^k$ is trivial (equivalent to the product bundle $N^n \times R^{n+k}$).

II. Stiefel-Whitney classes:

We begin to look at the cohomology of $n$-plane bundles. Henceforth unless otherwise stated, we will use some cohomology theory with $Z_2$ as coefficients. $H^i(X)$ will mean $H^i(X; Z_2)$ and $H^s(X)$, the direct sum $h^0(X) \oplus h^1(X) \oplus \ldots$. We have the following, similar to the axioms for Chern classes given in Hirzebruch, *Neue topologische Methoden in der Algebraischen Geometrie*, Berlin 1956 p. 60:

Axioms for Stiefel-Whitney Classes

1) To each $n$-plane bundle $\xi$ over a paracompact base space $B$, there corresponds an element $W(\xi) = 1 + W_1(\xi) + \cdots + W_n(\xi)$ of $H^* (B)$ where $W_k \in H_k (B)$, such that

2) For a bundle map $\xi = (f_1, f_2) : \eta \to \eta$, we have

$$f_1^* (W(\eta)) = W(\xi)$$

3) The Whitney Product Theorem holds: $W(\xi \oplus \eta) = W(\xi) W(\eta)$

i.e. $W_k (\xi \oplus \eta) = \sum_{i+j=k} W_i (\xi) W_j (\eta)$


4) For the non-trivial line bundle over $S^1$ (which can be represented as the open Möbius band or, since $S^1 = R^1$, as $\xi^1_1$ of Ex 4)

$$W_1 (\xi^1_1) \neq 0.$$ We will call $W_1 (\xi)$ the Stiefel-Whitney classes and $W (\xi)$ the total Stiefel-Whitney class.

Consequences and examples.

41) For the bundle $\xi_1^n$ of example 4, $W (\xi_1^n) = 1 + \alpha$ where $\alpha$ is the non-zero element of $H^1 (P^n)$.

For we have $S^1 = P^1 \subset P^2 \subset \cdots \subset P^n$ and using the inclusion maps in the bundle spaces as well, we define bundle maps

$$\xi^1_1 \longrightarrow \xi^2 \longrightarrow \cdots \longrightarrow \xi^n$$

Call the composition $f = (f_1, f_2, \ldots, f_n) : \xi^1_1 \longrightarrow \xi^n_1$.

Then for $f_1^* : H^* (P^n) \longrightarrow H^* (S^1)$ we have

$$f_1^* (W_1 (\xi_1^n)) = W_1 (\xi_1^1) \neq 0,$$

hence $W_1 (\xi_1^n)$ is the non-zero element of $H^1 (P^n)$. The Stiefel-Whitney classes $W_1 (\xi_1^n)$, $i > 1$, are zero by Axiom 1.
Axiom 4) may be used instead of 4) in which case it would not be necessary to specify in Axiom 1 that $W(\xi^n)$ has at most $n$-dimensional classes.

B. Axiom 2) gives us that $W$ is a function of equivalence classes. In particular, if $\xi$ is trivial then $W(\xi) = 1$ since $f_B : B \times R^n \rightarrow R^n$; $f_B : B \rightarrow \text{point}$, gives a bundle map and $f_B(\text{point}) = 0$ for $i > 0$.

C. $W(\gamma^n(S^n)) = 1$ where $\gamma^n(S^n)$ is the tangent bundle of the $n$-sphere.

Proof: Let $f_B : S^n \rightarrow p^n$ be the natural map and define the bundle map $f : \gamma^n(S^n) \rightarrow \gamma^n(p^n)$ in the obvious manner. Then

$$W_1(\gamma^n(S^n)) = 0; 0 < i < n \text{ since } H^i(S^n) = 0; 0 < i < n,$$

and $f_B : \gamma^n(p^n) \rightarrow \gamma^n(S^n)$ is zero so that $0 = f_B(W_1(\gamma^n(S^n))) = W_n(\gamma^n(S^n))$. Therefore $W_1(\gamma^n(S^n)) = 1$.

This can also be found through Axiom 3) since

D. Axiom 3) is "solvable", that is, given any two of $W(\xi)$, $W(\eta)$, or $W(\xi \oplus \eta)$ we can solve for the third. For example, given $W(\xi)$ and $W(\xi \oplus \eta)$, let us write $W(\xi) = W = 1 + W_1 + W_2 + \cdots + W_n$, $W(\eta) = W' = 1 + W_1 + \cdots + W_m$, and $W(\xi \oplus \eta) = W'' = 1 + W_1 + \cdots + W_{n+m}$. Expanding Axiom 3) we find

$$W'' = 1 \cdot W_1 + W_{1,1}$$

$$W'' = 1 \cdot W_2 + W_{1,2} + W_{2,1}$$

$$W'' = 1 \cdot W_3 + W_{1,3} + \cdots + W_{r-1} + W_{1,r-1} + W_{r,1} + W_{r,1}$$

We can solve for $W_r$ in terms of $W_1, 1 < r$ and the known $W_j$ and $W_k$:

$$W_r = 1 \cdot W_r = W_n - (W_1 \cdot W_{r-1} + \cdots + W_{r-1} \cdot W_1 + W_{r-1} \cdot W_{r-1})$$

This formula together with $W_1 = W'' - W_1$ to start things off, gives a complete recursive solution for the $W_1$ (Note that this procedure depends on the fact that $W_0$ is always equal to 1.) This discussion can be simplified by noticing that the set of all infinite sequences $1 + \alpha_1 + \alpha_2 + \cdots$ where $\alpha_i \in H^i(X)$ forms a group under $\oplus$, which is abelian since we are working mod 2. For example, $(1 + \alpha_1)^{-1} = 1 + \alpha_1 + \alpha_1^2 + \cdots$. Thus we can write $W(\xi) = W(\xi \oplus \eta)W^{-1}(\eta)$ to indicate the solvability of Axiom 3). In particular, for $N^n \subset R^{n+k}$, we know that $\gamma^n(N^n) \equiv \nu^k(N^n)$ is trivial and that therefore $W(\nu^k) = 1$. Thus we have:

Theorem 1. The Whitney Duality Theorem (see Lectures in Topology, Univ. of Michigan Press, 1941, p. 133, especially [21.9]: If we have a differentiable Manifold $M^n \subset R^{n+k}$ with tangent bundle $\gamma^n$ and normal bundle $\nu^k$, then

$$W(\gamma^n)W(\nu^k) = 1 \text{ or } W(\nu^k) = W^{-1}(\gamma^n).$$

We write $W^{-1} = 1 + W_1 + W_2 + \cdots + W_k$ with as usual $W = 1 + W_1 + W_2 + \cdots + W_n$. (Note that by the above theorem, $W^{-1}(\gamma^n)$ has classes of at most dim $k$.) Solving as above
\[ W_1 = W_1 \]
\[ W_2 = W_1^2 + W_2 \]
\[ W_3 = W_1^2 + W_3 \]
\[ W_4 = W_1^4 + W_2 W_2 + W_2^2 + W_4 \] etc.

In particular, we have another proof for assertion C. Taking the usual imbedding of \( S^n \) in \( R^{n+1} \) we have

\[ W(\xi^1(S^n)) = W^{-1}(\nu^1(S^n)). \] but \( \nu^1(S^n) \) is trivial which implies \( W(\xi^1(S^n)) = 1 \).

\[ III. \] Applications:

We will often be concerned with the situation of example 2, the tangent bundle \( \tau^n \) to a differentiable manifold \( M^n \). Though we have defined \( W \) only for bundles and not for manifolds, we will extend our use of classes by writing \( W(\nu^n) \), defined as \( W(\xi^n) \). For the sake of exposition, these are called the Stiefel class (of a manifold) and the Whitney class (of a tangent bundle) respectively.

Consider, for instance, \( \tau^n(S^n) \). This can be represented in terms of the unit \( S^n \subset R^{n+1} \) as follows.

Let \( E(\xi^n(p^n)) \) be the set of all unordered pairs \( \{(\overline{u}, \overline{v}), (-\overline{u}, -\overline{v})\} \) where \( (\overline{u}, \overline{v}) \) is an ordered pair with \( \overline{v} \) perpendicular to the unit vector \( \overline{u} \). In other words, \( E(\xi^n(p^n)) \) is the set of all pairs \( (\overline{u}, \overline{v}) \) with \( \overline{v} \) perpendicular to the unit vector \( \overline{u} \) modulo the identification \( (\overline{u}, \overline{v}) = (-\overline{u}, -\overline{v}) \).

On the other hand, consider the \((n+1)\)-fold Whitney sum \( \xi^1_n \otimes \cdots \otimes \xi^1_n \) where \( \xi^1_n \) is the line bundle over \( p^n \) of example 4. The bundle space \( E(\xi^1_n \otimes \cdots \otimes \xi^1_n) \) consists of all \((n+2)\)-tuples \( (\overline{u}; t_0, \cdots, t_n) \) where \( (\overline{u}; t_0, \cdots, t_n) \) is identified with \( (-\overline{u}, -t_0, \cdots, -t_n) \), or \( (\overline{u}; v) = (-\overline{u}; -v) \), \( |\overline{u}| = 1 \). (See Remark 2 after Example 4.) Notice \( \overline{u} \) need not be perpendicular to \( \overline{v} \), though at least

\[ E(\nu^n) \subset E(\xi^1_n \otimes \cdots \otimes \xi^1_n). \]

Let \( \eta^1 \) be the 1-plane bundle over \( p^n \) with \( E(\eta^1) = \{(\overline{u}, t\overline{u})\} \) modulo the identification \( (\overline{u}, t\overline{u}) = (-\overline{u}, -t\overline{u}) \). Clearly \( \eta^1 \) is a trivial bundle.

As can readily be seen,

\[ \tau^n \otimes \eta^1 \] is equivalent to \( \xi^1_n \otimes \cdots \otimes \xi^1_n \).
Thus $W((p^n)^1) = W((p^n)^{n+1})$ and since $n^1$ is trivial,

$$W((p^n)^1) = W((p^n)^{n+1}) = (1 + \alpha)^{n+1}, \alpha \in H^1(p^n).$$

Theorem 2: The Stiefel class of projective n-space is given by

$$W(p^n) = (1 + \alpha)^{n+1}$$

where $\alpha$ is the non-zero element of $H^1(p^n)$. In other words, $W_1(p^n) = \binom{n+1}{2} \alpha^2; \alpha \in H^1(p^n)$.

The following is a table of binomial coefficients mod 2.

We do not use the last coefficient since $W$ has no $n+1$-dim component, e.g.

$\begin{array}{c|c}
\text{Coefficient} & \text{Value} \\
\hline
1 & 1 \\
0 & 0 \\
1 & 1 \\
10 & 0 \\
11 & 1 \\
\end{array}$

$W(p^2) = 1 + \alpha + \alpha^2$

$W(p^3) = 1 + \alpha + \alpha^2$

$W(p^4) = 1 + \alpha + \alpha^2$

In making use of these formulas it is important to know that the powers $\alpha, \alpha^2, \ldots, \alpha^n$ are all non-zero. We will assume this to be known. (A proof based on the Gysin sequence will be given later in these notes.)

Parallelizability: Now we are ready to ask: which $p^n$ are parallelizable? We have necessary conditions at hand since parallelizable implies that $W(p^n) = 1$.

Theorem 3: $W(p^n) = 1$ if and only if $n + 1$ is a power of two.

(From the above work, this reduces to an exercise in arithmetic mod 2)

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Proof: Since $(\alpha + \beta)^2 = \alpha^2 + \beta^2 \mod 2$

we have $(1 + \alpha)\alpha^2 = 1 + \alpha^2 \frac{\alpha}{2}$

Therefore, for $n + 1 = 2^m$, $W(p^n) = (1 + \alpha)^{n+1} = 1 + \alpha^{n+1} = 1$.

Conversely $n + 1 = 2^m$, $m \text{ odd} > 1$, implies that

$W(p^n) = (1 + \alpha)^{n+1} = (1 + \alpha^{2^m}) - 1 + \binom{m}{1}\alpha^{2^m} + \cdots = 1 + \alpha^{2^m} + \cdots \neq 1$

since $2^m < n + 1$.

Thus the only $p^n$ which can be parallelizable are $p^1, p^3, p^7, p^{15}, p^{31}, \ldots$. It is known that $p^1, p^3, p^7$ are in fact parallelizable and that $p^{15}$ is not.

---

Immersion:

Definition: An immersion of $M^n \in R^{n+k}$ is a differentiable map $M^n \rightarrow R^{n+k}$ such that the Jacobian is never singular (this means there is a well-defined tangent plane at every point). This differs from an embedding in that "nice" self intersections are permitted.

Theorem 4: If the manifold $M^n$ can be immersed in $R^{n+k}$, then the dual Stiefel-Whitney classes $W_i(M^n)$ must be zero for $i > k$.

Proof: As in the case of an embedding, $i^n \times j^n$ is trivial so that $W_i(p^n) = W_j(p^n)$, but $W_i(p^n) = 0$ for $i > k$.

Applying this to $p^n$ immersed in $R^{n+1}$, we have that

$W(2^1) = 1$ or $1 + \alpha^2$. $W(p^n) = W((p^n)^1) = W^{2^1}(2^1) = 1$

or $1 + \alpha + \alpha^2 + \cdots + \alpha^n$.

We have seen that $W(p^n) = 1$ if and only if $n + 1 = 2^m$. On the other hand if $W(p^n) = (1 + \alpha)^{n+1} = 1 + \alpha + \cdots + \alpha^n$ then $(1 + \alpha)^{n+2} = 1 + \alpha^{n+1} \mod 2$ and again $H^{n+1}(p^n) = 0 = \alpha^{n+1} = 0 = (1 + \alpha)^{n+2} = 1$.
As before this implies that \( n + 2 = 2^r \). Thus the only \( \mathbb{F}^n \) which can be immersible in \( \mathbb{R}^{n+1} \) are \( \mathbb{F}^1, \mathbb{F}^2, \mathbb{F}^3, \mathbb{F}^6, \mathbb{F}^7, \mathbb{F}^{14}, \mathbb{F}^{15}, \ldots \). It is known that \( \mathbb{F}^1, \mathbb{F}^2, \mathbb{F}^3 \) are in fact immersible but that \( \mathbb{F}^{15} \) is not. (See Milnor, "The immersion of n-manifolds in (n+1)-space", Comm. Math. Helv. 30 (1956), pp. 275-284.)

On the other hand, consider the case \( n = 2^r \). Then

\[
W(\mathbb{F}^n) = (1 + \alpha)^{n+1} = (1 + \alpha)^{2^r} (1 + \alpha) = (1 + \alpha^{2^r}) (1 + \alpha) = 1 + \alpha^{2^r},
\]

and

\[
W^{-1}(\mathbb{F}^n) = (1 + \alpha^n)^{-1} (1 + \alpha)^{-1} = (1 + \alpha^n)(1 + \alpha + \alpha^2 + \cdots + \alpha^n) = (1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}) \pmod{2}.
\]

In other words \( W_1(\mathbb{F}^n) = 0 \), \( 1 = n \neq 0 \), \( 1 = 1 \), \( \ldots \), \( n-1 \).

Therefore by Theorem 2, for \( n = 2^r \), \( \mathbb{F}^n \) is not immersible in \( \mathbb{R}^{2n-2} \).

We would like to know how good an answer this is, i.e. for what dimensions \( q > 2n-2 \) can \( \mathbb{F}^n \) be immersed in \( \mathbb{R}^q \)? There is, in fact, the Theorem of Whitney: any \( \mathbb{M}^n, \, n > 1 \), can be immersed in \( \mathbb{R}^{2n-1} \). [See Whitney, "Singularities of a Smooth n-Manifold in (2n-1) space", Ann. Math. 45 (1944) p. 247.] So for \( n = 2^r \) we have the exact result: \( \mathbb{F}^n \) is immersible in \( \mathbb{R}^{2n-1} \), but not immersible in \( \mathbb{R}^{2n-2} \).

Our results can be extended somewhat, as follows:

If \( P^9 \) is immersible in \( \mathbb{R}^{14} \), so is \( P^9 \subset P^9 \), but this we know is impossible, so \( P^9 \) is not immersible in \( \mathbb{R}^{14} \). Similarly, we have in general, if \( n = 2^r + q \) where \( r \) is the largest power of 2 in \( n \), then \( P^n \) is not immersible in \( \mathbb{R}^{2n+1-2} \).

**Imbedding:** Similar results can be obtained for imbedding. We will show later that if \( \mathbb{M}^n \) is imbedded in \( \mathbb{R}^{n+k} \), then the highest Stiefel-Whitney class \( \bar{W}_k(y^k) \) is zero. Hence if \( \mathbb{M}^n \) is imbeddable in \( \mathbb{R}^{n+k} \) then \( \bar{W}_1(M^n) = 0 \) for \( 1 \leq k \). In particular, for \( n = 2^r + q \) as above, \( \mathbb{F}^n \) is not imbeddable in \( \mathbb{R}^{2n+1-2} \).

**Division algebras:** Another application of Stiefel-Whitney classes is in the question: for what \( n \) does there exist a division algebra of dim \( n \) over \( \mathbb{R} \)? (Again we will get necessary but not sufficient conditions.) We are looking for a product operation in \( \mathbb{R}^n \) which

1) is bilinear, and

2) has no zero-divisors.

Suppose such exists and choose a basis \( (e_1, \ldots, e_n) \) for \( \mathbb{R}^n \). Let a vary over \( \mathbb{R}^n \) so that \( a \cdot e_1 \) varies over the unit \( S^{n-1} \) in \( \mathbb{R}^n \), \( (a \cdot e_1, \ldots, a \cdot e_n) \) are linearly independent by 1) and 2), and the projections of \( a \cdot e_2, \ldots, a \cdot e_n \) on the tangent plane to the unit sphere at \( a \cdot e_1 \) are still linearly independent. This in effect gives us \( n-1 \) linearly independent tangent vector fields on \( S^{n-1} \), as \( a \cdot e_1 \) varies over \( S^{n-1} \). If we identify \( a \) and \( -a \) we have that \( \langle a, e_1 \rangle = -\langle a, e_1 \rangle \) is identified with \( a \cdot e_1 \) and thus we have \( n-1 \) linearly independent tangent vector fields on \( \mathbb{R}^{n-1} \).

\( \therefore \mathbb{R}^{n-1} \) is parallelizable or \( n = 2^n \).

In fact, we know that there are the following division algebras:
n = 1 R,
\[ n = 2 \text{ Complex numbers} \]
\[ n = 4 \text{ Quaternions} \]
\[ n = 2 \text{ Complex numbers} \]
\[ n = 8 \text{ Cayley numbers} \]

**IV. Stiefel-Whitney numbers.**

We will now construct a tool which will allow us to compare homology classes of different manifolds. (So far we have only compared W's which could be represented in terms of the cohomology of a fixed manifold, \( F^n \)).

\( W^n \) will be a closed, possibly disconnected differentiable manifold.

Let \( \mu \) be the fundamental class in \( H_n (\mathbb{Z}_2) \). (There is one use we make coefficient group \( \mathbb{Z}_2 \)).

For any \( \gamma \in H^n(X, \mathbb{Z}_2) \), there is defined the Kronecker index \( \gamma \mu \in \mathbb{Z}_2 \) [See *Lefschetz Algebraic Topology*, AMS, 1943, p. 118] usual, write \( W(\mathbb{Z}_2) = 1 + W_1 + W_2 + \ldots + W_n \).

Consider any monomial in \( W_1, \ldots, W_n \) which is an element of \( (\mathbb{Z}_2^n) \), that is, has total dimension \( n \), i.e. \( W_1, \ldots, W_n \) with \( r_1 + 2r_2 + \ldots + nr_n = n \). Each such monomial is of the proper dimension to obtain a Kronecker index; therefore we define

**Definition:** The Stiefel-Whitney number of the manifold \( M^n \) corresponding to the monomial \( \gamma = W_1^{r_1} W_2^{r_2} \ldots W_n^{r_n} \) is the integer mod 2: \( \langle \gamma \mu \rangle \). In using Stiefel-Whitney numbers as a tool, we will usually concerned with the complete set of numbers. When we compare Stiefel-Whitney numbers of different manifolds, we naturally compare the numbers corresponding to the same monomial.

Let us apply this to projective spaces, about all we can work with at this point. For \( n \) even, \( W_n (F^n) = (n+1) \alpha^n \neq 0 \) so that \( \langle W_n \mu \rangle \neq 0 \). Similarly \( W_n (F^n) = (n+1) \alpha^n \neq 0 \) so that \( \langle W_n \mu \rangle \neq 0 \).

(In the special case \( n = 2^m \), we know that \( W(\mathbb{Z}_2) = 1 + \alpha + \alpha^n \), so that these are the only Stiefel-Whitney numbers different from zero.)

For \( n \) odd, on the other hand, we can set \( n + 1 = 2m \).

\( W(\mathbb{Z}_2) = (1 + \alpha)^{2m} = (1 + \alpha^2)^m \). Therefore \( W_n = 0 \) for all odd \( n \).

Any monomial of dim \( n \) contains a factor of odd dimension and therefore is zero. Thus all Stiefel-Whitney numbers are zero.

This gives some indication of how much detail and structure this invariant overlooks. On the other hand, these numbers are very useful as is indicated by the following theorem and its converse.

**Theorem of Pontriagin:** For \( B^{n+1} \), a manifold with boundary \( M^n \), the Stiefel-Whitney numbers of \( M^n \) are all zero. (see Pontriagin, "Characteristic Cycles on Differentiable Manifolds," Math. Sborn. (NS 21 (63), p. 233, AMS Translation 32).

In this case we represent the fundamental class of \( H_n (M^n) \) not by \( \mu \) but by \( \partial \mu \) where \( \mu \) is the fundamental class of \( H_{n+1} (B^{n+1}, M^n) \).

**Proof:** By a standard result for arbitrary cohomology classes.

\[ \langle W_1^{r_1} \ldots W_n^{r_n}, \partial \mu \rangle = \langle \partial (W_1^{r_1} \ldots W_n^{r_n}), \mu \rangle \]

As usual, let \( T^n \) be the tangent bundle to \( M^n \). Let \( B^{n+1} \) be the tangent bundle to \( B^{n+1} \), and let \( B^{n+1} | M^n \) be the restriction of
$\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. (That is the bundle with total space $\pi^{-1}(\mathbb{R}^n)$, base space $\mathbb{R}^n$, and projection $\pi|\mathbb{R}^n$ where $\pi$ denotes the projection map of $\mathbb{R}^{n+1}$.) Choosing a Riemann metric on $\mathbb{R}^{n+1}$ (see next section) there is a unique unit inward normal vector $\mathbf{n}$ to $\mathbb{R}^n$. This generates a trivial bundle $\mathbb{R}^1$. Clearly $\mathbb{R}^{n+1}|\mathbb{R}^n$ is equivalent to the bundle $\mathbb{R}^n\oplus \mathbb{R}^1$. In other words, $\mathbb{R}^n \to \mathbb{R}^{n+1}$ is covered by a bundle map $f: \mathbb{R}^n \oplus \mathbb{R}^1 \to \mathbb{R}^{n+1}$, $f = (f_0, 1)$.

Therefore $i^*(W_1(\mathbb{R}^{n+1})) = W_1(\mathbb{R}^n)$.

But in general we have the exact sequence

$$H^n(B^{n+1}) \xrightarrow{i^*} H^n(\mathbb{R}^n) \xrightarrow{c} H^{n+1}(B^{n+1}, \mathbb{R}^n).$$

Thus, by exactness, $\sum(W_1, \ldots, W_n) = 0$ and so all the Stiefel-Whitney numbers of $\mathbb{R}^n$ are 0.

The converse, due to Thom, is true, although much harder to prove:

**Theorem of Thom:** If all the Stiefel-Whitney numbers of $\mathbb{R}^n$ are zero, then $\mathbb{R}^n$ bounds. [see Thom "Quelques propriétés globales des variétés différentiables", Comm. Math. Helv. 28 (1954) pp. 52-72].

For example, $\mathbb{R}^n \cup \mathbb{R}^n$, where we mean the union of disjoint copies, always bounds (This can be thought of as the two ends bound a cylinder.)

More generally we define cobordism class:

**Definition:** $w_1^n, w_2^n$ belong to the same cobordism class if there exists $\mathbb{R}^{n+1}$ with boundary $w_1^n \cup w_2^n$; and obtain the:

**Theorem:** $w_1^n, w_2^n$ belong to the same cobordism class if and only if corresponding Stiefel-Whitney numbers are equal. [see Thom, op.cit. Cor IV.11].

**V. Paracompactness:**

We next give some basic tools necessary for the study of $n$-plane bundles. First let us define some of our terms:

**Definition:** A partition of unity on $X$ is an indexed collection $(\{p_\alpha\})$

such that

1. each $p_\alpha$ is a map $X \to [0,1]$,
2. each $x \in X$ has a neighborhood $U_x$ such that $p_\alpha(U_x) = 0$ for all but a finite number of $\alpha$'s, and
3. $\sum_{\alpha} p_\alpha(x) = 1$, each $x \in X$.

**Definition:** Given an indexed open covering $(U_\alpha)$ of a space $X$, an associated partition of unity is a partition of unity $(p_\alpha)$ with the same index set such that $p_\alpha = 0$ outside a closed subset $V_\alpha$ of $U_\alpha$.

**Definition:** $X$ is paracompact if $X$ is Hausdorff and given any indexed open covering of $X$ there is an associated partition of unity.
Remark: The usual definition, which is equivalent, is: X is paracompact if X is Hausdorff and if every open covering has an open locally finite refinement. (For this definition and other properties of paracompactness, see Kelley, General Topology, Van Nostrand, 1955, p. 156).

In particular, every metric space is paracompact as is every regular space which is a countable union of compact subsets. [see Norita, Math. Jap. vol 1 (1948) p. 50-66, Thm. 10]. These are all we will need. Note that separable manifolds are paracompact since they fall in both of these categories.

Illustrations of the Use of Paracompactness in Bundle Theory.

First: Definition. A Riemannian metric on an n-plane bundle is an inner product defined in each fibre [e_1 * e_2 = R^R for all e_1 * e_2 \in R] such that \mu(e_1) = \mu(e_2) such that

- e_1 * e_2 is 1) symmetric: e_1 * e_2 = e_2 * e_1;
- 2) bilinear;
- 3) positive definite: e_1 * e_1 > 0 except for 0 * 0 = 0; and
- 4) as a continuous function of two variables (although e_1 * e_2 is defined only for e_1 * e_2 in the same fibre, we require continuity with respect to the topology of \gamma, not just that of the fibre, i.e., if e_1 * e_2 are close

to e_1 * e_2 respectively and if e_1 * e_2 and e_1 * e_2 are defined then e_1 * e_2 is close to e_1 * e_2 in R.)

Remark: The term Riemannian metric is ordinarily used only in the tangent bundle, but this seems like a natural generalization.

Theorem 5: Every n-plane bundle \xi over a paracompact base X admits a Riemannian metric.

Proof: Case I: product bundle. We need only define the product on a basis of each fibre and extend by bilinearity. We can use as a basis for the \xi(x), x \in X the (c_1(x)) given by the cross-section which in the case of a product bundle can be taken to be global. Define c_1(x) * c_2(x) = \delta_{ij} \text{the Kronecker}\delta_{ij}; \text{the Kronecker}\delta_{ij} = 0 \text{if} \ i \neq j.

Case II: In general, let \{U_\alpha\} be the distinguished class of open sets of X giving the local product structure for \xi. Let \{p_\alpha\} be an associated partition of unity and \{(c_1 * c_2)_\alpha\} the associated Riemannian metric defined as in Case I for each \xi \mid U_\alpha.

Define c_1 * c_2 to be \sum \delta_{ij} (\mu(c_1))(c_1 * c_2)_\alpha \text{with the convention 0 * 0 (undefined) = 0 since (c_1 * c_2)_\alpha is defined only for} \mu(c_1) \in U_\alpha. It is easy to verify that this is

1) symmetric - since \mu(c_1) = \mu(c_2);
2) bilinear - since it is a weighted sum of bilinear functions;
3) positive definite;
4) continuous - since locally it is a finite sum of
continuous functions. (For some neighborhood of \( x \), all but a
finite number of \( p_\alpha = 0 \).) QED

**Second Illustration: Grassman Manifolds.**

In classical differential geometry, there is encountered Gauss's
construction of the spherical image of a manifold \( \mathbb{R}^n \subset \mathbb{R}^{n+1} \).

This is a mapping of \( \mathbb{R}^n \) into \( \mathbb{S}^n \) given by mapping a point \( x \) of
\( \mathbb{R}^n \) into the unit vector at the origin of \( \mathbb{R}^{n+1} \) with the same direction
as the normal to \( \mathbb{R}^n \) at \( x \). More generally, for \( \mathbb{R}^n \) immerse in \( \mathbb{R}^{n+k} \)
we associate with \( x \in \mathbb{R}^n \) the \( n \)-plane through the origin parallel to
the tangent plane at \( x \). (Tangent planes correspond 1-1 with the
undirected normal in the case \( k = 1 \).) This gives a map not of \( \mathbb{R}^n \)
into \( \mathbb{S}^n \) but rather into \( G_{n,k} \).

**Definition:** The Grassman manifold \( G_{n,k} \) is the set of all
\( n \)-dimensional subspaces in \( (n+k) \)-space (\( n \)-planes through the origin). This set has a natural structure as a differentiable manifold and is in fact compact. [see Steenrod, op.cit. p. 35] Note that there is no natural structure for the symbol for a Grassman manifold; there is no agreement in the literature.

By the usual duality between \( n \)-dimensional subspaces of \( \mathbb{R}^{n+k} \)
and \( (n+k)-n \) or \( k \)-dimensional subspaces, \( G_{n,k} \subset G_{k,n} \). One example
\( G_{n,k} \) is easy to picture: We obtain \( \mathbb{S}^n \) as the set of unit vectors
in \( \mathbb{R}^{n+1} \) or, what is equivalent, the set of directed lines through the origin of \( \mathbb{R}^{n+1} \). Since the \( n \)-planes in \( G_{n,k} \) are unoriented, we see

that

\[
G_{n,1} \cong \mathbb{S}^n = \mathbb{R}^n.
\]

Now let \( \gamma_{k}^n \) be the \( n \)-plane bundle over \( G_{n,k} \) with
\( E(\gamma_{k}^n) = \text{set of all pairs (plane through origin, vector in that plane}) \).

\( \gamma_{k}^1 (0, k) = k' \)

and we obtain

**Theorem 6:** For \( \mathbb{R}^n \) immersed in \( \mathbb{R}^{n+k} \), there is an associated bundle
map \( f : \mathbb{C}_n(\mathbb{R}^n) \to \gamma_{k}^n \) such that \( f_B \) is the generalized Gauss map:

\[
\mathbb{R}^n \to G_{n,k}
\]

This theorem is expressed by saying \( \gamma_{k}^n \) is "universal" for sufficiently large \( k \); i.e., every tangent bundle maps into it.

The map \( f_B \) is defined in the obvious fashion and the verification that the pair is a bundle map is left to the reader.

In a still more general situation, we define

**Definition:** The infinite Grassman manifold \( G_n \) (i.e., \( k = \infty \)) is
the set of all \( n \)-dimensional subspaces of \( \mathbb{R}^\infty \), countably infinite
dimensional Euclidean space, with the topology given as follows.
Let \( \{b_i\} \) be a basis for \( \mathbb{R}^\infty \), \( i = 1, 2, \ldots \) and let \( \mathbb{R}^n \) be the subspace
spanned by \( b_1, \ldots, b_n \). Then \( \mathbb{R}^1 \subset \mathbb{R}^2 \subset \ldots \subset \mathbb{R}^\infty \), \( G_n \subset G_{n,1} \subset \ldots \subset G_\infty \),
and this sequence of inclusions induces a topology on \( G_n \) by defining
\( H \subseteq G_n \) to be closed if and only if \( H \cap G_n, k \) is closed for all \( k \).

**Note:** \( G_n \) is not metric, but is regular and a countable union of compact subsets \( G_n, k \) and therefore is paracompact. (We will omit the proof that \( G_n \) is regular, since we will see presently that \( G_n \) is actually a CW-complex. Every CW-complex is known to be normal. [See J.H.C. Whitehead, "Combinatorial homotopy I", Bull. Amer. Math. Soc. 55 (1949), pp. 213-245.]

As above we define \( \gamma^n \), an \( n \)-plane bundle over \( G_n \), with total space

\[ E(\gamma^n) = \text{set of all pairs (n-dimensional subspace of } \mathbb{R}^\infty, \text{ vector in that subspace).} \]

The following is a generalization of Theorem 6.

**Theorem 7.** For any \( n \)-plane bundle \( \gamma^n \) over a paracompact base \( X \), there exists a bundle map \( \gamma^n \to \gamma^n \).

(Actually a somewhat stronger result holds. Any two such bundle maps \( \gamma^n \to \gamma^n \) are homotopic. Furthermore any two homotopic maps \( \gamma X \to G_n, k \) induce equivalent bundles. For this reason \( \gamma^n \) is called a universal bundle and \( G_n \) a classifying space for \( n \)-plane bundles.)

**Proof:** Case I: product bundle.

There exists a linear homeomorphism \( h: \gamma \sim X \times \mathbb{R}^n \)

Let \( \varphi \) be the projection: \( X \times \mathbb{R}^n \to \mathbb{R}^n \). Then \( \gamma \) is linear and non-singular in each fibre. Let \( f: \mathbb{R}^n \to \text{origin}, g: \gamma \to \text{origin.} \)

Then \((\varphi, g)\) is a bundle into the bundle \( (\mathbb{R}^n, \mathbb{R}^n) \), which maps into \( \gamma^n \) in the obvious fashion.

**Case II:** There is a countable distinguished covering \( \{U_i\} \).

Let \( \{p_i\} \) be an associated partition of unity. \( \mathbb{R}^\infty \) can be represented as \( \mathbb{R}^n \circ \mathbb{R}^n \circ \cdots \).

Map \( E(\gamma) \to \mathbb{R}^\infty \) by \( F(e) = (p_1(e), f_1(e), p_2(\gamma(e)), f_2(e), \cdots) \),

where \( f_i: \mathbb{R}^N \to \mathbb{R}^n \) as in case I.

\( F \) is continuous and is linear, non-singular on each fibre since each \( f_i \) is.

Let \( \varepsilon_B(x) = \text{subspace } \subseteq \mathbb{R}^\infty \text{spanned by } \{F(e) | e \in \gamma^{-1}(x)\} \).

Therefore \( \varepsilon_B(x) \) is an element of \( G_n \). That \( \varepsilon_B \) is continuous can be checked easily since locally \( \varepsilon_B \) lies in some finite \( G_n, k \subseteq G_n \).

Define \( \varepsilon_B: \gamma \to E(\gamma^n) \) by

\[ \varepsilon_B(e) = (\varepsilon_B(\gamma)), F(e)) \]

Then \((\varepsilon_B, \varepsilon_B)\) is the required bundle map.

Thus we will have proved the theorem as soon as we show

**Lemma:** Given an \( n \)-plane bundle \( \gamma \) over a paracompact base space \( X \), there is a countable covering \( \{U_i\} \) of \( X \) such that the restrictions \( \gamma | U_i \) are trivial.

**Proof:** Let \( \{V_\alpha\} \) be the distinguished covering. Choose an associated partition of unity \( \{p_\alpha\} \). Call the index set \( A \) and for each finite \( S \subseteq A \)

let \( V_S = \{x | \min_{\alpha \in S} p_\alpha(x) > \max_{\beta \in S} p_\beta(x)\} \)

Thus \( V_S \) is open and \( \bigcup_{S \subseteq A} V_S \) is a local trivialization.
(Us) is an open covering of X since
1) Us is open by continuity of all pα, and
2) x ∈ Us for Sx = {α ∈ A | pα(x) > 0} for each x ∈ X.

Let Un be the union of Us over all S with n elements. Again {Un} is an open covering.

Notice the Us in Un are disjoint since the S all have the same length and therefore for S1 ≠ S2, there exist α, β such that α ∈ S1, β ∈ S2, α ≠ β.

Thus for x ∈ S1, pα(x) > pβ(x),
and for x ∈ S2, pα(x) < pβ(x). Therefore Un1 - Un2 = 0.

On the other hand for each S, pα = 0 outside Vα, and therefore Us - Vα. Thus {Un} is a countable open covering giving the local product structure of X.

QED.

VI. The cohomology ring H^n(Gn, Z).

In a little while, we will need to know something of the structure of Gn as a cell complex, and this we investigate by means of matrices over the reals. We need the following notions and theorems of matrix theory. [See, for example, Birkhoff and MacLane, *A Survey of Modern Algebra*, The Macmillan Co., 1946 p. 271.]

**Definition:** Two n x m matrices A, B are row equivalent if A can be obtained from B by a succession of elementary row operations i.e.
1) interchanging any two rows,
2) multiplication of a row by a non-zero scalar,
3) addition of one row to another.

**Definition:** The row space of an nx(n+k) matrix is the subspace of Rn+k spanned by the n row vectors of the matrix (= "range" in Birkhoff and MacLane op. cit.)

**Theorem:** Two matrices are row equivalent if and only if they have the same row space.

**Theorem:** Every matrix is row equivalent to a matrix of canonical form, the reduced echelon matrix, i.e.

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]

where each ri > 0, and the ri x 1 blocks are arbitrary depending on the original matrix.
By these matrix tools, we have reduced the study of \( n \)-dimensional subspaces of \( \mathbb{R}^{n+k} \) to the study of reduced echelon matrices. Since given a pattern as above, we can vary the entries in the blocks independently, each such pattern determines a vector space over \( \mathbb{R} \) or a cell in \( G_{n,k} \). Thus we have a cell subdivision of \( G_{n,k} \). [For full details, see Ehresmann, Ann. Math. 35 (1934) p. 396.]

Example: \[
\begin{pmatrix}
  a & b & l & 0 & 0 & 0 & 0 \\
  c & d & 0 & 1 & 0 & 0 & 0 \\
  e & f & 0 & 0 & g & 1 & 0
\end{pmatrix}
\]

represents a 7-cell in \( G_{3,5} \).

Similarly for \( G_n \), we look at such patterns in \( n \times \infty \) matrices e.g. \[
\begin{pmatrix}
  a & 0 & 1 & 0 & \cdots & \cdots & \cdots \\
  b & 0 & d & 1 & 0 & \cdots & \cdots \\
  c & 0 & e & 0 & f & g & 1 & 0 & \cdots & \cdots
\end{pmatrix}
\]

is a cell in \( G_3 \). That is, for each sequence of \( n \) non-negative integers \( r_1, \ldots, r_n \), we obtain a cell of \( \dim r_1 + 2r_2 + \cdots + nr_n \) in \( G_n \). This gives \( G_n \) the structure of a CW-complex. Thus there is a unique 0-cell in each \[
G_n:\begin{pmatrix}
  1 & 1 & \cdots & \cdots & \cdots \\
  0 & 1 & \cdots & \cdots \\
  \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \cdots & 1
\end{pmatrix}
\]

and a unique 1-cell: \[
\begin{pmatrix}
  1 & 1 & \cdots & \cdots & \cdots \\
  0 & 0 & \cdots & \cdots \\
  \vdots & \vdots & \cdots & \ddots \\
  \vdots & \vdots & \cdots & \cdots & 1
\end{pmatrix}
\]
or the special case \( G_{1,k} = p^k \), we have one cell of each dimensional \( \leq r_1 \leq k \).

Looking at the Stiefel-Whitney classes of \( \gamma^n \), we see:

\textbf{Theorem 6:} The cohomology ring \( H^*(G_{n,\infty}) \) is a polynomial algebra over \( \mathbb{Z}_2 \) generated by \( W_1(\gamma^n), \ldots, W_n(\gamma^n) \).

\textbf{Proof:} First we show

\textbf{Lemma:} There are no relations among the \( W_i(\gamma^n) \).

\textbf{Proof:} If a polynomial \( p(W_1(\gamma^n), \ldots, W_n(\gamma^n)) = 0 \) then \( p(W_1(\gamma^n), \ldots, W_n(\gamma^n)) = 0 \) for any \( \gamma^n \) over a paracompact base \( \mathbb{X} \), since the bundle map given by Theorem 6 induces a homomorphism \( g^* \) such that \( g^*(W_i(\gamma^n)) = W_i(\gamma^n) \) and thus

\[
\begin{align*}
p(W_1(\gamma^n), \ldots, W_n(\gamma^n)) &= p(g^*(W_1(\gamma^n)), \ldots, g^*(W_n(\gamma^n))) \\
&= g^*(p(W_1(\gamma^n), \ldots, W_n(\gamma^n))) \\
&= g^*(0) = 0.
\end{align*}
\]

To prove the lemma, we need only find some \( \gamma^n \) with no relations among the \( W_i(\gamma^n) \).

Consider \( \xi_{k+1} \): \( W(\xi_{k+1}) = 1 + \xi_k \in H^1(p^k, \mathbb{Z}_2) \).

Let \( X = p^k x \ldots x p^k \) with projections \( \pi_i \) into the \( n \) factors, \( p_i^k; i = 1, \ldots, n \).

It is known that \( H^*(p^\infty, \mathbb{Z}_2) \) is the polynomial algebra generated by \( \alpha \in H^2(p^\infty, \mathbb{Z}_2) \) and for \( k = \infty \), \( H^*(X) \) is the polynomial algebra generated by \( \alpha_1, \ldots, \alpha_n \) where \( \alpha_i = \pi_i^* \alpha(i) \).
Let $\zeta^n$ be the bundle $\eta_1^1(\xi) \cdots \otimes \eta^n_1$ over $X$ where $\eta_1^1$ is the bundle over $X$ induced from $\xi_1$ by $\eta_1^1$.

Thus

$$W(\zeta^n) = W(\eta_1^1) \cdots W(\eta^n_1)$$

$$= \eta_1^1 W(\xi_1) \cdots W^n(\xi_1)$$

$$= (1 + \eta_1^1)(1 + \eta_2^1) \cdots (1 + \eta_n^1)$$

In other words

$$W_1(\zeta^n) = \alpha_1 \times \cdots \times \alpha_n$$

$$\vdots$$

$$W_n(\zeta^n) = \alpha_1 \cdots \alpha_n$$

where the polynomials which appear on the right are just the elementary symmetric functions $\alpha_i$ in the $\alpha_i$. From algebra we have [cf. Van der Waerden, *Modular Algebra* Ungar, 1953 p. 79 or 176.]

**Theorem:** For $\Lambda$, a commutative ring with 1 and $x_1, \ldots, x_n$ indeterminate symbols, the symmetric elements of $\Lambda[x_1, \ldots, x_n]$ form a polynomial ring $\Lambda[\alpha_1, \ldots, \alpha_n]$.

This means in particular that if some polynomial $p$ satisfies

$$p(\alpha_1, \ldots, \alpha_n) = 0,$$

then $p = 0$. Thus for $\zeta^n$ as above,

$$p(W_1(\zeta^n), \ldots, W_n(\zeta^n)) = 0$$

implies

$$p(W_1(\zeta^n), \ldots, W_n(\zeta^n)) = 0$$

and thus $p = 0$; or there are no (polynomial) relations among the

$$(W_1(\zeta^n), \ldots, W_n(\zeta^n))$$

which proves the lemma.

Thus we know that $H^i(G_n)$ contains the polynomial algebra generated by $(W_1(\zeta^n))$.

Let $C^1(G_n)$ represent the 1-cochains of $G_n$ and $Z^1(G_n)$, the 1-cocycles. The dimension of $C^1(G_n)$ as a vector space over $\mathbb{Z}_2$ is the number of 1-dimensional cells, which is finite since they correspond to sequences $r_1, \ldots, r_n$ with $r_1 + 2r_2 + \cdots + nr_n = 1$. Moreover it is $\geq \dim Z^1(G_n) = 1^{th}$ Betti number and $\geq$ number of monomials in $(W_1(\zeta^n))$ of total dimension $1$, since $H^1(G_n) \cong 1$-dimensional part of the polynomial algebra generated by $(W_1(\zeta^n))$. On the other hand, such monomials correspond to sequences $r_1, \ldots, r_n$ with

$$r_j \geq 0, r_1 + 2r_2 + \cdots + nr_n = 1.$$ That is, there is a one to one correspondence between cells and monomials of the same dimension. Thus all the above inequalities are in fact equalities or:

$$\dim C^1 = \dim H^1(G_n) = \text{number of monomials of dim 1}.$$ Therefore $H^1(G_n)$ is the 1-dimensional part of the polynomial algebra and $H^1(G_n; \mathbb{Z}_2)$ is the polynomial algebra over $\mathbb{Z}_2$ generated by $W_1(\zeta^n), \ldots, W_n(\zeta^n)$. 

**QED**

Further for $g^*$ as above, $g^*: H^*(G_n) \to H^*(p^\infty \times \cdots \times p^\infty)$ is an isomorphism onto the subalgebra consisting of all symmetric polynomials in $\alpha_1, \ldots, \alpha_n$. 


Uniqueness of Stiefel-Whitney classes:

At this point, we still have not shown that there exists a collection of classes satisfying the given axioms, but before investigating that question we will prove

Theorem 9: There is at most one collection of classes compatible with the axioms.

Proof: Suppose we have two collections \( \{ \bar{w} \} \) and \( \{ \tilde{w} \} \) satisfying the axioms. As we showed in proving alternative Axiom 3), \( \bar{w}(\xi_1 \otimes \cdots \otimes \xi_n)^l = \tilde{w}(\xi_1 \otimes \cdots \otimes \xi_n)^l \), and \( \bar{w}(\xi_1 \otimes \cdots \otimes \xi_n)^l \) must both equal \( l \cdot \alpha \), where \( \alpha \) is the non-zero element of \( H^1(P \otimes X) \). This still holds true for \( P \otimes X \): \( \bar{w}(\xi_1 \cdots \xi_n)^l = \bar{w}(\xi_1 \cdots \xi_n)^l \otimes \bar{w}(\xi_1 \cdots \xi_n)^l \). By naturality of \( \bar{w} \) and \( \tilde{w} \) under mappings, in particular the projections \( P \otimes X \rightarrow P \otimes X \rightarrow \cdots \rightarrow P \otimes X \), of the previous section, \( \bar{w}(\eta_1 \otimes \cdots \otimes \eta_n)^l = \tilde{w}(\eta_1 \otimes \cdots \otimes \eta_n)^l \). By Axiom 3), therefore,\n
\[
\bar{w}(\eta_1 \otimes \cdots \otimes \eta_n)^l = \tilde{w}(\eta_1 \otimes \cdots \otimes \eta_n)^l.
\]

For \( g : P \otimes X \rightarrow P \otimes X \rightarrow \cdots \rightarrow P \otimes X \rightarrow G_n \) before \( \bar{w}(\eta_1 \otimes \cdots \otimes \eta_n)^l = \tilde{w}(\eta_1 \otimes \cdots \otimes \eta_n)^l \) and \( g^* \) is a monomorphism so \( \bar{w}(\eta_1 \otimes \cdots \otimes \eta_n)^l = \tilde{w}(\eta_1 \otimes \cdots \otimes \eta_n)^l \). But \( G_n \) is a classifying space; for any bundle \( \xi_n \) over a paracompact base \( X \), there is a bundle map \( f : \xi_n \rightarrow \eta_n \), and so \( \bar{w}(\eta_1 \otimes \cdots \otimes \eta_n)^l = \bar{w}(\xi_1 \otimes \cdots \otimes \xi_n)^l \), \( g^* \). Thus for every \( \xi_n \) over a paracompact base space, \( \bar{w}(\eta_1 \otimes \cdots \otimes \eta_n)^l = \tilde{w}(\eta_1 \otimes \cdots \otimes \eta_n)^l \).

QED

*Remark:* It is possible to prove this for bundles restricted to manifolds for base, but not just for tangent bundles of manifolds.

VII. Existence of Stiefel-Whitney classes.

We now proceed to prove the existence of Stiefel-Whitney classes by giving a construction in terms of known operations. For any \( n \)-plane bundle \( \xi \) with total space \( E \), base space \( B \) and projection \( \pi \), we denote by \( E_0 \) the set of non-zero elements of \( E \) and by \( E_0^* \), the set of all non-zero elements of \( F = \pi^{-1}(b) \), a fibre. Clearly \( F_0 = F \cap E_0 \).

Using singular theory and one of several techniques (e.g. spectral sequences or that of the appendix) we have that

\[
H^n(P \pi_0; \mathbb{Z}_2) = \begin{cases} 0 & \text{for } 1 < n \\ \mathbb{Z}_2 & \text{for } 1 = n \\ \mathbb{Z}_2 \otimes \mathbb{Z} \otimes \mathbb{Z} & \text{for } 1 = n \end{cases}
\]

(This can be seen intuitively, though not rigorously, without spectral sequences as follows: The unit \( n \)-cell is a deformation retract of \( \mathbb{R}^n \) and the unit \((n-1)\)-sphere is a deformation retract of \( \mathbb{R}^n \)-origin) = \( R_0^n \). For \( B \) paracompact, we know that we can put a Riemannian metric on \( E \). Looking at the cohomology of \((E, E_0)\), we might just as well look at the cohomology of \((E, E_0^*)\) where \( E_1 \) is the set of all elements of \( E \) with norm \( 1 \), \( E_0^* \) the set of all elements of \( E \) with norm \( 1 \), since as indicated above \( E_0 \) and \( E_0^* \) have the same homotopy type as \( E_1 \) and \( E_0^* \) respectively. Now assume that \( B \) is a cell complex. Take a fine enough cell subdivision of \( B \) so that we have a product bundle over each cell \( \alpha \). In \((E_1^*, E_0^*)\) we are looking at \( \alpha^1 \chi(\text{n-cell}) \mod \alpha^1 \chi(\text{the boundary of that n-cell}) \), thus we have a collection of cells covering \( E_1 \) and can extend it in a trivial fashion to give a cell subdivision of \((E, E_0^*)\). The relation between the cell structure of \( B \) and that of \((E, E_0^*)\) indicates why
the dimension of the cohomology gets shifted by \( n \). As can be seen, there are no cells at all of dimension \( < n \) which are not in \( E_0 \).

Rigorously and more explicitly, it is possible to prove (see app.):

**Theorem 10:**

1) \( H^i(E, E_0) = 0 \) for \( i < n \)

2) There exists a unique class \( U \) in \( H^n(E, E_0) \) such that for each fibre \( F = \pi^{-1}(b) \), we have \( j_0^* U = \) the non-zero element \( U_n \) of \( H^n(F, F_0) \) where \( j_0 \) is the inclusion map \( F, F_0 \to E, E_0 \).

3) \( H^i(E) \xrightarrow{\eta} H^{i+n}(E, E_0) \) is an isomorphism for all \( i \).

Now \( \eta^*: H^*(E) \to H^*(E) \) is an isomorphism, since there is the trivial cross-section of \( E \to E \) given by \( b \to (b, 0) \) and the image of \( U \) this cross-section is a deformation retract of \( E \) and homeomorphic to \( E \). Following Thom, we combine these two isomorphisms in a new isomorphism

\[
\beta = (\pi U) \circ \eta^*: H^i(E) \xrightarrow{\eta^*} H^i(E) \xrightarrow{\pi U} H^{i+n}(E, E_0)
\]

and then define the Stiefel-Whitney classes as follows:

\[ W_i(1) = \beta^{-1} Sq^i(1) \cdot \beta \]

To study this definition, we will assume as known the following properties of the Steenrod squares, \( Sq \) (see square upper 1):

1) For spaces \( X, Y \) with \( X \to Y \), \( Sq \) is an additive homomorphism
   \[ Sq^i: H^k(X, Y) \to H^{k+i}(X, Y) \]
   such that

2) it is natural with respect to maps \( f: X \to X', Y' \) i.e.
   \[ Sq^i f^* = f^* Sq^i \]

3) \( Sq^k(\omega) = 0 \) for \( i > k \)

4) \( Sq^k = 0 \) for \( i = k \) where \( k \) indicates the dimension of \( \omega \)

5) (Cartan) \( Sq^k(\omega \cdot \omega) = \sum_{i+j=k} Sq^i(\omega) \cdot Sq^j(\omega) \)

Writing \( Sq(\omega) \) for \( (\omega + Sq^1(\omega) + \ldots + Sq^{k-1}(\omega)) ) \cdot \omega \), property 5)

becomes \( Sq(\omega \cdot \omega) = Sq(\omega) \cdot Sq(\omega) \). (Note that for \( dim \omega = k \), \( Sq(\omega) \) reduces to \( (\omega + Sq^1(\omega) + \ldots + Sq^{k-1}(\omega)) ) \cdot \omega \) .) Thus \( Sq \) is a ring homomorphism

\[ Sq: H^*(X, X) \to H^*(X, X) \]

We can now write our construction of Stiefel-Whitney classes as

\[ W_i(1) = \beta^{-1} Sq^i(1) = \beta^{-1} Sq U \]

**Verification of the Axioms:**

**Axiom 1:** Our construction gives elements of the proper dimension i.e. \( W_i(1) \) is an additive homomorphism \( H^i(X, X) \to H^{i+n}(X, X_0) \) such that

\[ Sq^i: H^k(X, X) \to H^{k+i}(X, X) \]

2) it is natural with respect to maps \( f: X \to X', Y' \) i.e.

\[ Sq^i f^* = f^* Sq^i \]

3) \( Sq^k(\omega) = 0 \) for \( i > k \)

4) \( Sq^k = 0 \) for \( i = k \) where \( k \) indicates the dimension of \( \omega \)

**Axiom 2:** Naturality under bundle maps: For \( f = (f_E, f_B) \), \( f_E \) induce a map \( g: E, E_0 \to E', E_0' \) and by the definition of \( U \), \( g^* U = U \). Thus \( \beta \) is natural and 2) above gives us that \( Sq \) is natural, and so \( W \) is natural.

**Axiom 4:** (We will return to Axiom 3 in a moment.) Let \( \xi \) be as usual the twisted line bundle over \( S = S^1 \), a nowhere presentable as the Möbius band. As can clearly be seen by homotopy type arguments similar to those above, we have \( H^*(E, E_0) \cong H^*(\text{Möbius band}, \text{Boundary of the Möbius band}) \). Since we can obtain a Möbius band by removing a 2-cell from the projective plane, we have \( H^*(E, E_0) \cong H^*(P^2, 2\text{-cell}) \) which we know to be \( H^0 = 0, H^1 = Z_2, H^2 = Z_2 \).

Further it is known that for \( \omega \) the non-zero 1-dimensional class, \( \omega \cdot \omega \) is the non-zero 2-dimensional class. Therefore \( Sq^1(\omega) \neq 0 \), and so \( W_1(\xi) \neq 0 \).
Axiom 3: We prove

**Theorem 11**: \( W(\zeta \otimes \zeta') = W(\zeta)W(\zeta') \).

**Proof**: Let \( \zeta'' = \zeta \otimes \zeta' \) and represent the total space of \( \zeta \) by \( E \), that of \( \zeta' \) by \( E' \), and so on, with similar notation for the respective fibres \( P, P', P'' \), etc. From the structure of the Whitney bundle sum, we know \( P \times P' = P'' \).

Let \( E_a = \bigcup \text{ all fibres of } P \times P' \).

\( E_a = \bigcup \text{ all fibres of } P \times P' \).

Obviously \( E_a \subseteq E_b \subseteq E_c \subseteq E_d \) and it is clear that \( E_a \cup E_b = E_d \).

The following diagrams will be helpful in following the rest of the proof:

\[
\begin{array}{c}
\begin{array}{c}
E_a' \subset E_a \subset E_b' \subset E_b'' \\
E_a' \subset E_b' \subset E_b'' \\
E_a' \subset E_b' \subset E_b''
\end{array}
\end{array}
\]

Diagram 1

\[
\begin{array}{c}
\begin{array}{c}
E_b' \subset E_b \subset E_c' \subset E_c'' \\
E_b' \subset E_b \subset E_c' \subset E_c'' \\
E_b' \subset E_b \subset E_c' \subset E_c''
\end{array}
\end{array}
\]

Diagram 2

We assert that \( q_i''(U) \cup q_i''(U') = U'' \). By the uniqueness of \( U'' \) as given in Theorem 10, we need only show, for each \( j_0 : F_0 \to F_0' \), that \( j_0(q_1''U \cup q_2''U') \) is the non-zero element \( U'' \) of \( H^*(F_0, F_0') \).

Consider the following diagram

\[
\begin{array}{c}
\begin{array}{c}
U \cup U' \in H^*(E, E_0) \otimes H^*(E', E_0') \overset{q_1'' \otimes q_2''}{\longrightarrow} H^*(E''', E_0') \otimes H^*(E''', E_0') \overset{\alpha_1 \otimes \alpha_2}{\longrightarrow} H^*(E_0''', E_0'') \overset{\alpha_1 \otimes \alpha_2}{\longrightarrow} H^*(F_0, F_0') \overset{\alpha_1 \otimes \alpha_2}{\longrightarrow} H^*(F_0', F_0'') \overset{\alpha_1 \otimes \alpha_2}{\longrightarrow} H^*(F_0', F_0'')
\end{array}
\end{array}
\]

Diagram 2

where we have written systematically \( P \) for the arbitrary fibre \( F_0 \), \( F_0' \), etc., and where \( j_1 : F_0'' \to E_0'' \), \( j_2 : F_0' \to E_0' \); \( j_3 : F_0 \to E_0 \); \( j_4 : F_0' \to E_0' \) are the inclusion maps.

The element \( j_0(q_1''U \cup q_2''U') \) is obtained by following the outside edge of the diagram clockwise from \( H^*(E, E_0) \otimes H^*(E, E_0) \) to \( H^*(F_0, F_0') \). By commutativity of the diagram, the same element is reached by the outside counter clockwise path. By the definition of \( U \) and \( U' \) we have \( j_0''U = U_0'j_0''U = U_0'' \). Since the projections \( q_3, q_4 \) yield isomorphisms \( q_3'' \otimes q_4'' \), the element we reach in \( H^*(F_0, F_0') \) is the non-zero element \( U'' \).

QED
Since
\[ q_1^n \cup q_2^n U = U^n \]
we have
\[ SQ^n U = Sq(\alpha_1^n \cup q_2^n U). \]
By property of 5) of Sq,
\[ SQ^n U = Sq(q_1^n U) \cup Sq(q_2^n U). \]
Using the naturality of Sq this becomes
\[ SQ^n U = (q_1^n SQ U) \cup (q_2^n SQ U). \]

Our definition of \( W(\zeta) = \beta^{-1} SQ \beta(1) = \beta^{-1} SQ U \) can be rewritten

3) \[ \pi^n W(\zeta) \cup U = SQ U \]
and similarly for \( \zeta^1 \) and \( \zeta^" \).
Combining 2) and 3) we have

4) \[ SQ^n U = q_1^n (\alpha^n W(\zeta) \cup U) \cup q_2^n (\alpha^n W(\zeta^1) \cup U). \]
We will make use of the relation \( q_2^n (\zeta \cup U) = p_1^n \zeta \cup q_2^n q \) which holds

\( \zeta \in H^n (E, F \cup E) \) (see Diagram 1). And the corresponding relation for \( p_2^n \) and \( q_2^n \). Thus 4) becomes

\[ SQ^n U = (p_1^n \alpha^n W(\zeta) \cup q_2^n U) \cup (p_2^n \alpha^n W(\zeta^1) \cup q_2^n U). \]
By commutativity of \( \cup \) mod 2, we obtain

\[ SQ^n U = p_1^n \alpha^n W(\zeta) \cup p_2^n \alpha^n W(\zeta^1) \cup q_2^n U \cup q_2^n U. \]
Referring to diagram 3,
\[ \zeta \in p_1 = \pi^n p_2 = \pi^n \]
thus, using 1) we have

5) \[ SQ^n U = \pi^n (\alpha(\zeta) \cup W(\zeta^1)) \cup U^n. \]
But the class \( \alpha(\zeta) \) is uniquely defined by the equation

\[ SQ^n U = \pi^n W(\zeta^1) \cup U^n. \]
This completes the proof that \( W(\zeta^1) = \zeta \cup W(\zeta^1) \).

VIII. Oriented Bundles:

Up to this point, we have been working strictly with \( Z_2 \) as coefficients for the cohomology we have used. This of necessity means that we overlook some detail in the structure; now we take a closer look using \( Z \) as coefficient group. Since part of the study we have conducted so far made strong use of the existence of a non-zero element of \( H^n (E, F \cup E) \) which was guaranteed by using \( Z_2 \) as coefficients, we will have to limit ourselves when using \( Z \) as coefficients to oriented bundles, as will be seen in what follows. First, some preliminary definitions:

**Definition:** Two bases of a finite-dimensional vector space are **equivalent** if the determinant of the matrix expressing one in terms of the other is positive.

**Definition:** An **orientation** of a vector space \( V \) of dimension \( n \) is an equivalence class of bases.

This corresponds to choosing a generator (there are two) of \( H_n (V, V_0; Z) \) (and incidently to the intuitive geometric idea of orientation). The correspondence can be given as follows:

Let \( v_1, \ldots, v_n \) be a basis for \( V \) and \( \Delta_n \), the standard \( n \)-simplex with vertices \( A_0, A_1, \ldots, A_n \). The linear map \( \Delta_n \rightarrow V \) given by \( A_0 \rightarrow v_1, A_1 \rightarrow v_2, \ldots, A_n \rightarrow v_n \) for \( i = 1, 2, \ldots, n \) determines a generator
of $H_n(V, u; Z)$ in the singular theory. Two bases will determine the same generator under this correspondence if and only if they are equivalent.

Definition: An oriented n-plane bundle is an n-plane bundle together with an orientation for each fibre such that these orientations are locally compatible, in the following sense. For each point $b_o$ of the base space there should exist a neighborhood $N$ and cross-sections $\alpha_1, \ldots, \alpha_n: N \to E$

such that for each $b \in N$ the vectors $\alpha_1(b), \ldots, \alpha_n(b)$ form a basis for the fibre $F_b$ which is compatible with the given orientation of $F_b$.

In terms of cohomology this means that for each fibre $F_b$ we have a distinguished generator $u_b \in H^n(F_b; F_{b_0}; Z)$. The local compatibility condition can then be put in the following form. For each $b \in B$ there should exist a neighborhood $N$ and a cohomology class $u \in H^n(F_{b_0}; F_{b_0}; Z)$ such that $j_{0b}(u) = u_b$ for each $b \in N$, where $j_{0b}: F_{b_0} \to F_{b_0}$ denotes the inclusion map. The proof that these two definitions of "oriented n-plane bundle" are equivalent is not difficult.

For an oriented bundle $\xi$ with total space $E$, base $B$, and projection $\pi$, Theorem 10 can be generalized as follows:

**Theorem 10:** For $\xi$ an oriented n-plane bundle as indicated,

1. $H^i(E, E_0; Z) = 0$ for $i < n$
2. There exists a unique class $u \in H^n(E, E_0; Z)$ such that $j_{0b}U = u_b$ for all $b \in B$ where $j_{0b}: F_{b_0} \to F_{b_0}$ and $F = H^{n-1}(b)$
3. $H^i(E; Z) \xrightarrow{U} H^{i+n}(E, E_0; Z)$ is an isomorphism for all $i$.

(More generally, any commutative ring with unit may be used as coefficient group.) The proof will be given in the appendix.

Recall that before we had $\pi_*: H^i(B) \to H^i(Z)$ and we defined $\phi_0: \pi_*: H^i(B) \to H^{i+n}(Z, E_0)$ and then working mod $2$ we had $W_0 = \pi_*\pi^nU = \pi^n(U \cup U)$

Now using our new $U$, this last construction can go through with coefficients group $2$ if we omit the reference to $\pi^n$.

**Definition:** The Euler class $x$ of an n-plane bundle $\xi$ is the class of $H^n(B; Z)$ defined by $x = \phi^{-1}(U \cup U)$ where $U$ is as in Theorem 12.

**Remark 1:** $x(\cdot)$ reduced mod 2 is $H_n(\cdot)$. $x(\cdot)$ is a strengthened Steifel-Whitney class.

**Remark 2:** If $n$ is odd, $x$ is of order 2 since for $U$ of odd dimension $U \cup U = -(U \cup U)$.

**Theorem 12:** $x = g^*i^*U$ where $g$ is the homeomorphism of $B$ into $Z$ given by any cross section, and $i: Z \to E_0$ is the injection.

**Proof:** Since $\phi$ is an isomorphism, we need only show that $g^*i^*U$ is equal to $\phi X = U \cup U$. But $\phi = (\cdot \cup \cdot) * n^*$ so $g^*i^*U = (n^* g^*U) \cup U$ since $g(B)$ is a deformation retract of $\varnothing$,

$g^* U \simeq \text{identity } E$ and thus

$g^* i^* U \simeq \text{identity } E$,

Therefore we have $g^* i^* U = (U \cup U)$.

Since $i$ is the injection $E \to E_0$ we have $(U \cup U) = U \cup U$ the cup products being defined in the proper groups and therefore $g^* i^* U = U \cup U$.

It is always possible to define $X$ this way since there is always the zero cross section. If, however, $g$ is a non-zero (never zero) cross section, $g^* i^* = 0$ and thus we have
Corollary: An oriented n-plane bundle \( \xi \) with \( X(\xi) \neq 0 \) cannot have any non-zero cross section.

We won't attempt an axiomatization of Euler classes here, but note that Axiom 2), naturality under bundle maps, holds for Euler classes and Axiom 1) is satisfied except for the modification that we have an Euler class only in the dimension of the fibre. As for Axiom 3), let \( V^n_1 \times V^n_2 \), where \( V^n_1 \) and \( V^n_2 \) have orientations \( v_1, \ldots, v_m \) \( v_1', \ldots, v_n' \), be given the obvious orientation \( v_1, \ldots, v_m, v_1', \ldots, v_n' \).

By the way this means that the orientation of \( V^n_2 \times V^n_1 \) is \((-1)^{mn}\) times the orientation of \( V^n_1 \times V^n_2 \). Corresponding to Axiom 3) for Stiefel-Whitney classes, we have

**Theorem 13:** \( X(\xi \times \eta) = X(\xi)X(\eta) \)

The proof here is completely analogous to the proof of Theorem 11, using the same notation and the uniqueness of \( \nu^n \text{Ch}^m(B^n, Z_2) \) as given by Theorem 10, to prove that \( p^n_1 \omega p^n_2 = u^n \) and thus to show \( X^n = X \omega X' \).

**Note:** Although the product formula looks completely analogous to the formula for Stiefel-Whitney classes, it works out rather differently in practice, since \( H(\xi) \) is a unit in the cohomology ring \( H^1(B; Z_2) \), the complete direct product, while \( H(\xi) \) is never a unit in \( H^1(B; Z_2) \). Given \( X(\xi) \) and \( X(\xi \times \eta) \), this means it is not possible to solve for \( X(\xi) \).

**Corollary:** For \( \xi \) an oriented n-plane bundle, if \( X(\xi) \) is not of order 0, then \( \xi \) is not the sum of two odd-dimensional bundles. In particular, this shows there does not exist a continuous field of oriented odd-dimensional subspaces in the tangent bundle of a manifold with \( X \neq 0 \). (The hypothesis that the subspaces are oriented is not actually necessary.)

**Corollary 2:** \( \xi \), an oriented n-plane bundle over a paracompact base \( B \) with \( X(\xi) \neq 0 \), cannot have any non-zero cross section. (This gives an alternate proof for the corollary to the preceding Theorem 12 under the restricted condition that the base be paracompact.)

For if \( \xi \) has a non-zero cross section, let \( \theta^1 \) be the line bundle spanned by the cross section and let \( \gamma^{n-1} \) be the (n-1)-plane bundle orthogonal to \( \theta^1 \) (in the Riemannian metric which we can assume since the base is paracompact). Since \( \theta^1 \) is trivial, we obtain \( X(\epsilon^1) = 0 \) and hence the contradiction \( 0 \neq X(\xi) = X(\theta^1)X(\gamma^{n-1}) = 0 \).

IX. Computation in a differentiable manifold.

1) The normal bundle

Using Theorem 12, we need knowledge of the maps

\[ B \xrightarrow{\pi} E \xrightarrow{i} E; E_0 \quad (g = \text{zero cross section}, i = \text{inclusion}) \]

in order to study \( X \), but this knowledge is available in a neighborhood of the zero cross section as will be seen in what follows. Let us first consider a simple case to illustrate the situation.

Let \( V^k \) be the normal bundle to a closed differentiable manifold \( M^n \) imbedded in \( R^{n+k} \). Instead of looking at the entire total space \( E \), consider small vectors in each fibre, that is vectors of length \( \leq \varepsilon \) in the Riemannian metric which we know we can define. Denote this subset of \( E \) by \( E_0(\varepsilon) \). Similarly, the non-zero small vectors are to be denoted by \( E_0(\varepsilon) \). The inclusion map \( (\varepsilon) \to E; E_0(\varepsilon) \to E_0 \) is an excision so we have that \( H_0(E; E_0(\varepsilon)) \to H_0(E; E_0) \). Assuming the manifold \( M^n \) is differentiable of class \( C^2 \), since it is compact, we can pick an
6 so that the map which assigns to each vector in $2(e)$ its endpoint in $\mathbb{R}^{N+1}$ is a 1-1 correspondence between $2(e)$ and a neighborhood $N$ of $0$ in $\mathbb{R}^N$. Thus we have $H^k(2(e), x_0(e)) \simeq H^k(N, N - N^1)$. Again by the excision axiom, we know that $H^k(N, N - N^1) \simeq H^k(\mathbb{R}^{N+1}, \mathbb{R}^{N+1} - N^1)$.

Putting these three isomorphisms together we have an isomorphism

$$\psi: H^k(2(e), x_0(e)) \rightarrow H^k(\mathbb{R}^{N+1}, \mathbb{R}^{N+1} - N^1)$$

Now assume that the normal bundle is oriented. (This is equivalent to the assumption that the tangent bundle is oriented.) Then the class $U \in H^k(E, E_0)$ is defined and determines $\psi U \in H^k(\mathbb{R}^{N+1}, \mathbb{R}^{N+1} - N^1)$. The inclusions

$$N^1 \subset \mathbb{R}^{N+1} \subset \mathbb{R}^{N+1} - N^1$$

gives maps of cohomology $H^k(N^1) \rightarrow H^k(\mathbb{R}^{N+1}) \rightarrow H^k(\mathbb{R}^{N+1} - N^1)$ under which, using the above isomorphisms, $\psi U$ goes into $H^k(N^1)$. But $H^k(\mathbb{R}^{N+1}) = 0$ so $X = \psi U U = 0$. Thus we have proved

**Theorem 1:** If $N^1$ is imbedded in $\mathbb{R}^{N+1}$ with an oriented normal bundle $\nu$, then $X(\psi_U) = 0$. (Alternatively, without orientability, the same argument shows $H^1(\nu, \mathbb{R}^{N+1}) = 0$, a fact we used on page 15.)

**Remark:** These results are true for imbedding but definitely do not carry over to immersions. For instance, consider the well known immersion of $S^2$ in $\mathbb{R}^3$ (Boy's surface). According to the Whitney duality theorem, we have $H^1(\nu) \neq 0$. Recently, S. Smale has shown that $S^2$ can be imbedded in $\mathbb{R}^4$ so as to obtain any desired even multiple of the generator of $H^2(S^2; \mathbb{Z})$ for $X(\nu)$. Roughly, this multiple corresponds to the self-intersection number of $S^2$ as imbedded. (See Bull. Amer. Math. Soc. 63, (1957), p. 106).

2) **The tangent bundle of an oriented manifold.**

Now let us turn our attention to the tangent bundle of a manifold $M^N$ which is differentiable of class $C^3$. Such a manifold can be given a Riemannian metric of class $C^2$. Let $F_b(e)$ denote the set of all tangent vectors at $b$ of length $\leq \varepsilon$. Then for $\varepsilon$ sufficiently small a homomorphism

$$F_b(e) \rightarrow N \subset \mathbb{R}^N$$

is defined by mapping each vector $\nu$ into the endpoint of the geodesic which starts at $b$ in the direction of $\nu$ and has length $|| \nu ||$.

The image $N$ is a neighborhood of $b$ in $\mathbb{R}^N$. Thus we have isomorphisms

$$H_n(F_b, F_b, 0) \rightarrow H_n(F_b(e), F_b(0)) \rightarrow H_n(N, N - b) \rightarrow H_n(\mathbb{R}^N, \mathbb{R}^N - b).$$

Call the composite isomorphism, $\psi_b$.

We will say that $N^1$ is oriented if its tangent bundle is oriented. If the orientation of each $F_b$ is specified by a generator $U_b \in H_n(F_b, F_b, 0)$ then the corresponding generator $\psi_b(U_b)$ of $H_n(N, N - b)$ will be denoted by $\bar{\mu}_b$. (Integer coefficients should be understood.)

**Lemma 1.** If $M^N$ is a closed oriented differentiable manifold then there is a unique homology class $\bar{\mu} \in H_n(M^N)$ such that for each point $b$ the inclusion homomorphism $H_n(M^N) \rightarrow H_n(\mathbb{R}^N, \mathbb{R}^N - b)$ carries $\bar{\mu}$ into $\bar{\mu}_b$.

The class $\bar{\mu}$ is called the fundamental class of $M^N$. Proof of

**Lemma 1.** A theorem of Cairns asserts that every differentiable manifold can be triangulated. For a recent proof see Whitney, Geometric Integration Theory, Princeton, 1957. However, under the hypothesis

*The proof is the same as our previous proof of the existence of a Riemannian metric, except that differentiable partitions of unity must be used.*
that $H^n$ is triangulated, a proof of this Lemma has been given by Steenrod, *Fiber Bundles*, p. 200. (Steenrod works with the system of local coefficients $\pi_{n-1}(F_o)$. However the hypothesis that $H^n$ is oriented implies that $\pi_{n-1}(F_o) \cong H_{n-1}(F_o)$, and $H_n(F,F_o)$ is canonically isomorphic to our coefficient group $Z$.) This completes the proof.

**Lemma 2.** If $H^n$ is connected, as well as being closed, oriented, and differentiable, then the homology group $H_n(H^n)$ is infinite cyclic with generator $\mu$. The cohomology group $H^n(H^n)$ is also infinite cyclic with a unique generator such that the Kronecker index $\langle \mu, \mu \rangle$ is $-1$.

This is also proved by Steenrod (See the reference cited above. Compare Eilenberg and Steenrod, *Algebraic Topology*, p. 105.)

$\mu$ will be called the fundamental cohomology class of $H^n$. (It is definitely not defined unless the manifold is connected.)

Now consider the total space $E$ of the tangent bundle. A map $E(\varepsilon) \to H^n \times H^n$ is defined by sending $(x,y)$ into $(x,y)$ where $x$ is the end point of a geodesic, as above. For $\varepsilon$ sufficiently small this gives a homeomorphism of $E(\varepsilon)$ onto a subset $D$ of $H^n \times H^n$. Clearly $D$ is a neighborhood of the diagonal $\Delta$ in $H^n \times H^n$. Thus $H^n(E(\varepsilon), E(\varepsilon))$ is isomorphic to $H^n(D,D- \Delta)$. Let $\nu$ denote the composition of the following isomorphisms:

$$H^n(E(\varepsilon), E(\varepsilon)) \cong H^n(D,D-\Delta) \cong \nu H^n(H^n \times H^n, H^n \times H^n - \Delta) \cong \nu H^n(H^n \times H^n, H^n \times H^n - \Delta).$$

The class $\nu$ in the first group corresponds to a class $\nu \in \nu UG^H(H^n(H^n \times H^n, H^n \times H^n - \Delta)).$ Finally define $U = \nu \nu U$ where $1: H^n \times H^n \to (H^n \times H^n, H^n \times H^n - \Delta)$ is the inclusion map. Thus we have:

$$H^n(E(\varepsilon), E(\varepsilon)) \xrightarrow{\nu} H^n(H^n \times H^n, H^n \times H^n - \Delta) \xrightarrow{\nu} H^n(H^n \times H^n, H^n \times H^n - \Delta).$$

3) Computation of the class $U$.

In the next sections, we will be engaged in investigating properties of Stiefel-Whitney classes and Euler classes through computation of the class $U$. Our most important result will be Wu's formula for the Stiefel class of a manifold $H^n = Sq^l$ where $l$ is characterized by the equation $<l, \mu> = <\alpha, l, V, \mu>$ for all $\alpha \in H^n(H^n)$. This gives a direct computational construction for $W$ which does not require knowledge of the tangent bundle. For Euler classes, we will elucidate a relation the reader has probably been suspecting, that of the Euler class to the Euler characteristic of a manifold. In the course of this development, we will obtain a proof of the Poincaré duality theorem.

Assume that the manifold $H^n$ is connected. For the remainder of the section, we will consider two cases simultaneously.

Case 1: $H^n$ is not necessarily oriented, but the coefficient group is $Z$.

Case 2: $H^n$ is oriented and the coefficient group is a field $\Lambda$, usually the rational numbers, $\mathbb{Q}$. The coefficient homomorphism $Z \to \Lambda$ carries the fundamental class $\nu H^n(H^n; Z)$ into a class in $H^n(H^n; \Lambda)$ which will also be denoted by $\nu$.

In either case the group $H^n(H^n)$ is a one dimensional vector space over the coefficient field with generator $\mu$. Let $\lambda_1, \ldots, \lambda_n$
be a basis for the cohomology of $H^n$. In particular, let $\alpha_1 = 1$.

as the generator of $H^*(\mathbb{R}^2)$ will be some $\alpha_1$. Using a field for
coefficients, we know that $\mathfrak{H}^*(\mathbb{R}) \cong \mathfrak{H}^*(\mathbb{R}^2)$ given by
$k \mathfrak{H}^*(\mathbb{R}^n)$. This is a well-known result on the
cohomology of products of finite complexes. We can represent $V$
consequently in terms of the generators

$$V = \sum_{i,j} c_{ij} \alpha_i \times \omega_j.$$ (Since $V$ is of dimension $n$, $c_{ij} = 0$
unless $\dim x_i + \dim y_j = n$.) Consider the map $f_b : N \times N \times N$ defined by
by $f_b(y) = (b, y)$. As indicated in the diagram below, the compatibility
condition on $V$ reduces to $f_b^* (V) = U$. 

in the other hand, by the very definition of $f_b$ it is clear that

$$f_b^* (\alpha_1 \times \omega_j) = 0 \text{ for dim } \omega_j > 0.$$ 

($\alpha_1$ for dim $\omega_1 = 0$, that is for $\omega_1$, which is
equal to 1.

Thus the coefficient of the $1 \times \mu$ term must be 1 and we have

Formula 1: $U = 1 \times \mu + \sum_{i,j} c_{ij} \alpha_i \times \omega_j$ where the summation $\Sigma$ extends
over all $i,j$ with $\dim \alpha_i > 0$, $\dim \omega_j < n$.

To get more information about $U$, consider the projections $p_1, p_2$
of $\mathbb{R} \times \mathbb{R}^2$ into its first and second factors respectively. Observe
that $p_1|D$ corresponds to $\mathbb{R} \times \mathbb{R}$ under the homomorphism we have set

$$(d, p_2) \xrightarrow{\mu} D \times \mathbb{R}$$

$$(d, p_2) \xrightarrow{\omega} D \times \mathbb{R}.$$ (see illustration). On the other hand, $p_1|\Delta = p_2|\Delta$ and since $\Delta$
is a deformation retract of $D$ it follows that $p_1|D \simeq p_2|D$. $(\sim$ is
to be read "is homotopic to").

Formula 2: $U \cup (1 \times \omega_k) = U \cup (\omega_k \times 1)$ for all $\omega_k$.

Proof: $1 \times \omega_k = p_2^*(\omega_k), \omega_k \times 1 = p_1^*(\omega_k)$.

Consider the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{H}^*(N \times N \times N \times N \times N) & \xrightarrow{f_b^*} & \mathfrak{H}^*(N \times N) \\
\downarrow \psi_b & & \downarrow \psi_b \\
\mathfrak{H}^*(F \times F) & \xrightarrow{f_b^*} & \mathfrak{H}^*(N) \\
\downarrow \psi_b & & \downarrow \psi_b \\
\mathfrak{H}^*(E) & \xrightarrow{f_b^*} & \mathfrak{H}^*(N) \\
\downarrow \psi_b & & \downarrow \psi_b \\
\mathfrak{H}^*(\mathbb{R}^2) & \xrightarrow{f_b^*} & \mathfrak{H}^*(N) \\
\downarrow \psi_b & & \downarrow \psi_b \\
\mathfrak{H}^*(\mathbb{R}) & \xrightarrow{f_b^*} & \mathfrak{H}^*(N) \\
\downarrow \psi_b & & \downarrow \psi_b \\
\mathfrak{H}^*(\mathbb{R}) & \xrightarrow{f_b^*} & \mathfrak{H}^*(N) \\
\downarrow \psi_b & & \downarrow \psi_b \\
\mathfrak{H}^*(\mathbb{R}) & \xrightarrow{f_b^*} & \mathfrak{H}^*(N) \\
\end{array}
$$

We obtained $V$ as the image under $i^*$ of the class $\psi U$ in
$\mathfrak{H}^*(N \times N \times N \times N \times N \times N)$ determined by $U$, so to compare $U \cup (1 \times \omega_k)$ and
$U \cup (\omega_k \times 1)$ we can first cup with $\psi U$ and then apply $i^*$. If the
respective products with $\psi U$ are equal, then the products with $U$ will
also be equal. Further, since the excision homomorphism $i^*$ is an
isomorphism we can check the equality by taking $d^*$ of $p_2^* (\omega_k)$ and
$p_1^* (\omega_k)$ respectively and then cupping with $i^* \psi U$. Since $p_1|D \times p_2|D$
we know that $d^* (p_1^* (\omega_k)) = d^* (p_2^* (\omega_k))$ hence the cup products with $i^* \psi U$.
are equal and lifting back up into \( H^\prime(N \times N, N \times N \setminus \Lambda) \) we have that
\[
\omega U(z^{\mathfrak{a}_{1}, \mathfrak{a}_{2}, \cdots, \mathfrak{a}_{k}^{\prime}}) = \omega U(z^{\mathfrak{a}_{1}, \mathfrak{a}_{2}, \cdots, \mathfrak{a}_{k}^{\prime}})
\]
\[Q.E.D.\]

Calling the coefficient field \( \Lambda \), define a homomorphism \( \gamma: H^\prime(N) \to \Lambda \) by
\[
\gamma(x) = \langle \mathfrak{a}, \overline{\mu} \rangle;
\]
where \( \overline{\mu} \) is the fundamental class of \( H^\prime(N) \). Using this homomorphism, define coefficients \( y_{j,k} \) by \( y_{j,k} = \gamma(x_{j} \cdot \mathfrak{a}_{k}^{\prime}) \). Extend \( \gamma \) to \( H^\prime(N) \times H^\prime(N) \times \Lambda \) by
\[
\gamma \times \gamma: H^\prime(N) \times H^\prime(N) \times \Lambda \to H^\prime(N) \times \Lambda
\]
and denote by \( h \) the corresponding homomorphism \( h: H^{n+1}(M \times N) \to H^{n}(N) \).

Now apply this homomorphism \( h \) to Formula 2.

On the left side we have
\[
h(y \cup (l \times \mathfrak{a}_{k}^{\prime})) = \sum_{i,j} c_{ij} h(l \times x_{j} \cup j \times \mathfrak{a}_{k}^{\prime})
\]
\[
= \sum_{i,j} c_{ij} h(l \times \mathfrak{a}_{k}^{\prime} \cup j \times \mathfrak{a}_{j})
\]
\[
= \sum_{i,j} c_{ij} y_{j,k} \cdot l \times j.
\]

On the right side we have
\[
h(y \cup (c_{k} \times l)) = \sum_{i,j} (-1)^{\text{dim} \cdot k_{i,j}} h(\cup k \times \mathfrak{a}_{j}).
\]

Formula 1 asserts that \( c_{ij} \cdot (\mathfrak{a}_{k}^{\prime}) = 0 \) except for the single term
\[
c_{ij} \cdot \mathfrak{a}_{j} \times \mathfrak{a}_{j} = l \times l.
\]

Therefore
\[
h(y \cup (c_{k} \times l)) = (-1)^{n} \text{dim} \cdot k \cdot c_{k}.
\]

Comparing these two formulas we have
\[\sum_{j} c_{ij} y_{j,k} = \begin{cases} 0 & \text{for } i \neq k \\ (-1)^{\text{dim} \cdot k} c_{k} & \text{for } i = k \end{cases}\]

Let \( C \) be the matrix \((c_{ij})\) and \( Y \) the matrix \((y_{ij})\). Then we have proved:

**Theorem 15.** The class \( Y \in H^\prime(N \times N) \) is equal to \( \sum c_{ij} a_{i} \times a_{j} \)
where the matrix \( C \) of coefficients is, up to sign, the inverse of the matrix \( Y \), where \( Y_{jk} = a_{j} \cup a_{k} \cup \overline{\mu} \).

\[\gamma Y = \pm 1, \pm 1, \pm 1, \pm 1\]

If \( n \) is even, then \( C \) is actually the inverse of \( Y \).

From Theorem 15, there follows as a corollary one of the classic results of combinatorial topology:

**Corollary 1:** (Poincaré Duality Theorem): For \( M \) a closed, connected manifold (oriented unless the coefficient field \( \Lambda \) is \( Z_{2} \)), the groups
\( H^{i}(N, \Lambda) \) and \( H^{n-i}(N, \Lambda) \) have the same rank. Furthermore, these groups are dually paired to \( \Lambda \) by the correspondence \( (a, b) \rightarrow \langle a \cup b, \overline{\mu} \rangle \).

**Proof:** Arrange the basis \( a_{1}, a_{2}, \ldots, a_{n} \)
in increasing order of dimension. \( Y \) will then have the form at the right.

Since \( Y \) is a square matrix and the theorem shows it to be non-singular, each of the blocks in this form must be square, as can readily be seen from an elementary argument with matrices. Thus the ranks of the paired cohomology groups are equal and the pairing of the generators is given by \( \langle a_{k} \cup a_{j}, \overline{\mu} \rangle \).

**Note:** We have given a proof only if \( M \) is differentiable of class \( C^{1} \). The more general result can be obtained by somewhat finer reasoning.
4) **The Euler Characteristic $\chi$**

We will now carry our computations over to the investigations of the Euler class $X$, and we will show

**Theorem 16:** If $M^n$ is a closed connected oriented manifold, differentiable of class $C^3$, then the Euler class $X(\tau^n(M^n))$ is equal to the Euler characteristic $\chi$ (i.e. the alternating sum of the Betti numbers) times the fundamental class $\mu \in H^n(M^n;Z)$. (This result is actually true for a $C^1$ manifold.)

**Proof:** For $n$ odd, we have seen that $X$ is of order 2. Since $H^n(M^n;Z)$ is infinite cyclic, this means that $X = 0$. By the Poincaré Duality Theorem, the Betti numbers in complementary dimensions all cancel out to give $\chi = 0$.

For $n$ even, we will make a computational investigation of $X(\tau^n(M))$ using coefficients in a field, e.g. $Q$, the rationals. The theorem will follow for $Z$ as coefficients since $H^n(M^n;Z) \rightarrow H^n(M^n;Q)$ is an isomorphism into. Recall that according to Theorem 12, $X = g^*1^*U$ where $B \rightarrow E \rightarrow E,E_0$, $g$ is the zero cross section. The following diagram relates these maps to our homeomorphism and $\psi U$:

$$
\begin{array}{cccc}
H^*(E,E_0) & \rightarrow & H^*(E(\epsilon),E_0(\epsilon)) & \rightarrow \rightarrow H^*(D,D-\Delta) & \rightarrow \rightarrow H^*(M \times M,\Delta) \\
\downarrow & & \downarrow & & \downarrow \\
H^*(E) & \rightarrow & H^*(M) & \rightarrow & H^*(M \times M) \\
& & \uparrow & & & \\
\end{array}
$$

where $d: M \rightarrow M \times M$ is the diagonal map which as can be seen corresponds to the zero cross section. Thus $X = d^*U$.

Now representing $U$ again by $\sum c_{1j}\alpha_1 \alpha_j$, we see that

$$X = \sum_{1,j} c_{1j}(\alpha_1 \cup \alpha_j)$$

which for the $\gamma_{1j}$ defined as before shows that $\sum c_{1j}\gamma_{1j}$ or in terms of matrices $X = \text{Trace}(\gamma^T \gamma)$. Since dim $M^n$ is even, $X = \text{Trace}(Y^T Y)$.

Arrange the basis as follows (the ordering of the basis has not been used in our work so far except in corollary 1):

$$\begin{array}{ccc}
\alpha_1 & \cdots & \alpha_r \\
\alpha_{r+1} & \cdots & \alpha_n \\
\text{even dim} & & \text{odd dim} \\
\end{array}$$

With respect to this basis, $Y$ has the form

$$Y = \begin{pmatrix} Y_e & 0 \\ 0 & Y_o \end{pmatrix}$$

where $Y_e$ refers to the even dimensional elements, $Y_o$ the odd.
Thus \( Y^{-1} = \begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix} \) and \( Y^T = \begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix} \) because of the anticommutativity of the cup product.

Therefore

\[
\text{Trace} \left( Y^{-1} \right) = \text{Trace} \left( \begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix} \right) = \sum \text{(even Betti numbers)} - \sum \text{(odd Betti numbers)} = \chi. \quad \text{Q.E.D.}
\]

5) Wu's Formula

Returning to Stiefel-Whitney classes, recall the definition according to Thom, SqU = U \cup \pi^* W. Under our canonical isomorphism

\[ H^\ast(E; \mathbb{Z}_2) \to H^\ast(M \times M; \mathbb{Z}_2) \]

and the inclusion homomorphism

\[ i^\ast : H^\ast(M \times M; \mathbb{Z}_2) \to H^\ast(M; \mathbb{Z}_2), \]

U goes into \( U \) (see page 47) and the above relation becomes

\[ \text{SqU} = U \cup (W \times 1). \]

Again applying \( h \) (see page 50) after substituting \( \text{W} = \Sigma c_{ij} a_i X a_j \), we have first

\[ h \text{SqU} = \Sigma c_{ij} a_i X a_j. \]

Using known properties of \( \text{Sq} \), this gives

\[ \text{W} = \Sigma c_{ij} h(\text{Sq}_j X \text{Sq}_j). \]

Defining \( a_j = \gamma(\text{Sq}_j) = \langle \text{Sq}_j, \mu \rangle \) we can rewrite our formula as

\[ \text{W} = \Sigma c_{ij} a_i \text{Sq}_j, \]

or writing \( V = \Sigma c_{ij} a_i \) we have \( \text{W} = \Sigma V \text{Sq} \). Now, following Wu, observe that \( V \) is characterized by the equation \( \langle \text{Sq}, a \rangle, \mu \rangle = \langle a \cup V, \mu \rangle. \)

In each dimension \( i \) the correspondence \( a \mapsto \langle \text{Sq}^{n-1} a, \mu \rangle \) defines an additive homomorphism of \( H^i(M^n; \mathbb{Z}_2) \) into \( \mathbb{Z}_2 \). According to the Poincaré duality theorem there is a unique element \( V_{n-1} \in H^{n-1}(M^n; \mathbb{Z}_2) \) such that

\[ \langle \text{Sq}^{n-1} a, \mu \rangle = \langle a \cup V_{n-1}, \mu \rangle \]

for each \( a \). (Note that \( V_0 = 1, V_{n-1} = 0 \) for \( n-1 > 1 \).) Defining \( V = V_0 + V_1 + \ldots + V_n = 1 + V_1 + \ldots + V[n/2] \) this formula becomes

\[ \langle \text{Sq}, \mu \rangle = \langle a \cup V, \mu \rangle \]

for all \( a \in H^\ast(M^n; \mathbb{Z}_2) \). The element \( V \) defined in this way is equal to \( \Sigma c_{ij} a_i \). Certainly \( V \) can be expressed in the form \( \Sigma V_k a_k \) for some coefficients \( V_k \). Then the identity

\[ \langle \text{Sq} a_j, \mu \rangle = \langle a_j \cup V, \mu \rangle \]

can be written as

\[ a_j = \Sigma V_k a_k. \]

Now multiplying on the left by \( c_{ij} \) and summing over \( j \) we have

\[ \Sigma c_{ij} a_j = \Sigma b_{ik} V_k = V_i. \quad \text{Q.E.D.} \]

Hence we have

Theorem 17 (Wu): \( W(M) = \text{SqV} \) where \( V \) is characterized by the equation \( \langle a \cup V, \mu \rangle = \langle \text{Sq}, \mu \rangle \) for all \( a \in H^\ast(M) \). Since \( W \) is thus defined entirely in terms of cohomology and homology operations, we have:

Corollary: The Stiefel-Whitney classes of manifolds are invariants of the homotopy type:

Examples:

\[ \mathbb{P}^n(\mathbb{C}) \]: For complex projective \( n \)-space (eight real dimensions)
we have the following system of generators: $1 \in H^0, a \in H^2, a^2 \in H^4, a^3 \in H^6$

$\alpha^4 = \mu \in H^8$, on which $Sq$ operates as follows

$$Sq1 = 1, \quad Sq \alpha = \alpha + \alpha^2, \quad Sq^4 = \alpha^4 (1 - \alpha)^4.$$ 

Thus

$$Sq^4 \alpha^2 = \alpha^4 \quad \text{and} \quad V_4 = \alpha^2, \quad Sq^4 \alpha^3 = \alpha^4 \quad \text{and} \quad V_2 = \alpha.$$ 

or $V = 1 + \alpha + \alpha^2$.

$$W = Sq V = 1 + (a + \alpha^2) + \alpha^2 (1 + 2 \alpha + \alpha^2) = 1 + \alpha + \alpha^4.$$ 

In general, to calculate $W(F^n(C))$ we go through a procedure which is formally identical with the calculation for $W(F^n(R))$. But we already know the results in that case; thus we have:

Theorem 8: $W(F^n(C)) = (1 + \alpha)^{n+1}$ for $\alpha$ the non-zero class in $H^2$.

Similarly $W(F^n(\text{Quaternions})) = (1 + \alpha)^{n+1}$ for $\alpha$ the non-zero class in $H^4$.

$W(\text{Cayley plane}) = 1 + \alpha^2$ for $\alpha$ the non-zero class in $H^8$.

(These are the only known examples of differentiable manifolds $F^n$ such that $H^*(F^n; Z_2)$ is a truncated polynomial ring. In fact, according to a theorem of Adem, if a complex $K$ exists such that $H^*(K; Z_2)$ is generated by $a \in H^2$, $r \geq 1$, with relation $a^k + 1 = 0$, $k \geq 2$, then $r$ must be a power of 2. If $k > 2$, then $r$ must be 1, 2, or 4. Thus for $r < 16$ the above manifolds give the only possible truncated polynomial rings.)

X. Obstructions:

In the section which follows, we will assume familiarity with the definitions of obstruction and primary obstruction. (See, for example, Steenrod, Topology of Fibre Bundles § 32, 35.) With terminology close to that of Steenrod, p. 190, given an $n$-plane bundle $\xi^n$ we have for each $q < n$ the associated $\xi^q$ with base $B$ and fibre $V^q(B, n - q)$, the Stiefel manifold of $(n-q)$-frames in $n$-space. By an $(n-q)$-frame we mean just a set of $n-q$ linearly independent vectors. (Note: Steenrod uses orthogonal unit $(n-q)$-frames in $n$-space; the modification does not affect the argument.) Explicitly, a point in the associated bundle fibre over $b \in B$ can be represented as $(b, \text{frame } (V_1, V_2, \ldots, V_{n-q}))$ in the $n$-plane $\pi^{-1}(b)$. The primary obstruction to a cross section of $\xi^q$ is an element $o_{q+1}$ of $H^q(B, \pi^*V^q(B, n-q))$. This coefficient group is either $Z$ or $Z_2$, depending on the dimensions. In general these are twisted coefficients, but this complication can be avoided by reducing mod 2; thus we write $o_{q+1}$. (In general, we lose nothing by this reduction since $o_{q+1}$ can be recovered from $o_q$ but for the one dimension where we can calculate $X$. See Steenrod p. 195.) Now, it is possible to interpret Stiefel-Whitney classes as follows:

Theorem 19: $o_4(\xi^n) = W_1(\xi^n) - W_2(\xi^n) - W_3(\xi^n) + W_4(\xi^n)$.

**Proof**: Consider the bundle map $f = (f_B, f_B)$ mapping $\xi^n$ into $\gamma^n$ the canonical bundle over $G_n$. Since obstructions are natural with respect to bundle maps, we have $f_B^* o_4(\gamma^n) = o_4(\gamma^n)$. Since $H^*(G_n; Z_2)$ is a polynomial algebra in the $W_j$, for each pair $1,n$ we have that $o_4(\gamma^n) = W_j(\gamma^n)$. The above relation shows that
Theorem 20: For $\xi$ an oriented $n$-plane bundle over a paracompact base $B$, there is an orientation-preserving bundle map $f$ into $\gamma^n$ with $f_B$ the canonical lifting of the map $B \to G_n$. This extends without trouble until we reach a singularity at the poles which can be seen to correspond to a generator of the homotopy group. Thus $\omega_1(\gamma^1)_2 \neq 0$ for this particular $\xi^1$. 

B) Suppose $n > 1$. Repeat with $\xi^n = \eta^1 \otimes G_{n-1}$ where $\eta^1$ is the bundle of A). By the same reasoning, $\omega_1(\xi^n)_2 \neq 0$ and so for all $1, n$ we have shown $\lambda_{1, n} = 1$ or $\omega_1(\xi^1) = W_1(\xi^1)$. To follow the same procedure in order to relate the Euler class to an obstruction, we must work with the integers as coefficients and will introduce the oriented analogue of $\xi^n$. Let $\gamma^n_1$ be the set of all oriented $n$-planes in $\mathbb{R}^n \otimes$ with topology defined to correspond to that of $G_n$. As can easily be seen, the obvious map $\gamma^n_1 \to G_n$ is a two-fold covering. Call $\gamma^n$, the bundle induced by this map from $\gamma^n_1$. Note that $\gamma^n$ is naturally an oriented bundle. For an oriented bundle $\xi^1$, we can lift the map $B \to G_n$ into $G_n^1$ by using the orientation of the fibre $F_b$ to determine which leaf of $G_n$ to map $b$ into (the local compatibility of orientations insures that this will be a continuous map). From this, it is easy to complete the diagram to get an oriented bundle map $\xi^1 \to \gamma^n$. 

Thus we have proved for each $1, n$ there is a number $\lambda_{1, n}$ such that the identity $\omega_1(\xi^1)_2 = \lambda_{1, n} W_1(\xi^1)$ holds for all $n$-plane bundles.

A) Let $i = n$. We know that $\lambda_{1, n} = 1$ or $0$. To prove the theorem in this case we need only show that for each $i$, there exists a bundle $\xi^i$ with $\omega_1(\xi^i)_2 \neq 0$. Let $B = P^i$ and let $n^{-1}(b)$ be the set of all vectors orthogonal to $x$ in $\mathbb{R}^{i+1}$, where $P^i$ is considered as the unit $S^i$ with antipodal points identified i.e. $b = [\pm x]$. We can start with a cross section on the $(i-1)$-skeleton as illustrated in the
Gysin Sequence:

Using $Z$ as coefficients throughout this section and assuming $ξ$ to be oriented, we have determined an element $\psi_1(E; E_0; Z)$ and know that $\cup H^1(E) \to H^{n+1}(E; E_0)$ is an isomorphism, as is $\varphi : H^1(B) \cong H^{n+1}(E; E_0)$. From the exact sequence of the pair $E; E_0$:

$$
\cdots \to H^i(E; E_0) \xrightarrow{i^*} H^i(E) \xrightarrow{i^*} H^i(E_0) \xrightarrow{i^*} H^{i+1}(E; E_0) \to \cdots
$$

we get the lower exact sequence by the indicated isomorphisms. The indicated map is $i^* X$ since

$$
t^* \cong i^*(\pi \cdot \alpha \cup U) = t^* \cong (\pi \cdot \alpha \cup U) = \alpha \cdot t^* = i^* \alpha \cup U = \alpha \cdot X.
$$

That is,

Theorem 21 (Gysin): For an oriented $n$-plane bundle we have an exact sequence

$$
\cdots \to H^i(B) \xrightarrow{\tau} H^{i+n}(B) \xrightarrow{\tau} H^{i+n}(E_0) \xrightarrow{\tau} H^{i+1}(B) \to \cdots
$$

where $\tau$ is the restriction $\tau|_{E_0}$.

Note: For unoriented bundles, we would get a corresponding exact sequence using $Z_2$ as coefficients and $\mathcal{W}_n$ in place of $X$.

The Euler class as an obstruction.

Now consider the top obstruction class $o_n(\gamma_n) \in H^n(B; \alpha_n(V_{n,1}))$. For an oriented bundle the coefficient group

$$
\pi_{n-1}(V_{n,1}) = \pi_{n-1}(F_0) \xrightarrow{\tau} H_{n-1}(F_0) \cong H_n(F, F_0)
$$

is canonically isomorphic to the integers $Z$. Hence the following statement makes sense.

Theorem 22: For an oriented $n$-plane bundle $\gamma_n$, we have $o_n(\gamma_n) = X(\gamma_n)$.

Proof: Consider the Gysin sequence in the special case $B = \tilde{O}_n$:

$$
\cdots \to H^0(\tilde{O}_n) \xrightarrow{-X} H^1(\tilde{O}_n) \xrightarrow{\tau} H^1(E_0) \xrightarrow{\tau} H^1(B) \to \cdots
$$

We want to show the special case of the theorem,

$$
X = o_n(\gamma_n) \in H^n(\tilde{O}_n; \alpha_n(V_{n,1})).
$$

First, we show $\tau o_n(\gamma_n) = 0$. Let $\gamma_n$ be the bundle over $E_0$ induced by $\tau$ from $\gamma_n$. By definition of the induced bundle, a point in $E(\gamma_n)$ is a pair $(e, e')$ where $e$ is a point in $E_0(\gamma_n)$ and $e'$ is any point in $E(\gamma_n)$ which belongs to the same fibre. The projection $E(\gamma_n) \to B(\gamma_n) = E_0(\gamma_n)$ is given by $(e, e') \to e$. Now the map $B(\gamma_n) \to E(\gamma_n)$ given by $e \mapsto (e, e')$ is clearly a non-zero section of $\gamma_n$. Therefore $o_n(\gamma_n) = 0$, but by naturality with respect to bundle maps $o_n(\gamma_n) = \tau o_n(\gamma_n)$. Hence $\tau o_n(\gamma_n) = 0$ as asserted.

By exactness of the above sequence, this implies that $o_n(\gamma_n) = o_n(\gamma_n) = \lambda \cdot X$ for some $\lambda \in H^0(\tilde{O}_n)$. That is $o_n = \lambda_n X$ where $\lambda_n$ is an integer since $H^0(\tilde{O}_n) \cong Z$. We write $\lambda_n$ to emphasize
that the integer $\lambda_n$ depends on the dimension of the bundle and not on the particular bundle, since the above formula relating $\varphi_n$ and $X$ holds for all bundles by naturality (cf. the similar discussion for Stiefel-Whitney classes). Thus we can determine $\lambda_n$ from special cases:

For $n$ even, consider the tangent bundle $T^n(S^n)$. By Theorem 16, we know that $X$ is $X(s^n)$ times the fundamental class, that is twice the fundamental class. On the other hand it is easy to verify that $\varphi_n$ is also twice the fundamental class in this case. Therefore

$$\lambda_n = +1.$$

For $n$ odd, $\lambda_n = 0$ or $1$ since $X$ is already of order 2. To show $\lambda_n = 1$ we need only show that $\varphi_n$ is not zero for all $n$-plane bundles, but we have already done this while relating Stiefel-Whitney classes to obstruction. In fact, we even showed $(\varphi_n)_2$ was not identically 0. Thus we have shown that the relation

$$\varphi_n(x^n) = X(x^n)$$

holds true for all $n$.

**XI. Complex $n$-plane bundles.**

For many investigations in other branches of mathematics, e.g. the study of complex analytic manifolds, the structure of a real $n$-plane bundle is not a sufficient tool; it is therefore natural to give the following generalization of the definition of an $n$-plane bundle:

**Definition:** A complex $n$-plane bundle $\omega^n$ consists of a triple $(E, B, \pi)$ where $\pi$ is a map from a Hausdorff space $E$ onto a Hausdorff space $B$ together with the structure of a complex $n$-dimensional vector space in each fibre $\pi^{-1}(b)$ satisfying the further conditions

1. There exist a distinguished class of open sets $\{U\}$ covering $B$ and $n$ maps $\varphi^U_1: U \rightarrow \mathbb{R}$ for each $U$ such that
2. each $\varphi^U_1$ is a cross section and
3. the map $U \times C^n \rightarrow \pi^{-1}(U)$ defined by $(b, \lambda_1, \ldots, \lambda_n) = \sum \varphi^U_1(b)$ where $\lambda_i \in \mathbb{R}$ is a homeomorphism.

**Note:** Throughout these notes we will represent the complex numbers by $C$.

**Example:** The tangent bundle $T^n$ of a complex analytic manifold $M^n$. A complex analytic $n$-manifold is defined analogously to a differentiable manifold except that we use $n$ complex variables as local coordinates, and require that the functions relating the local coordinate systems must be analytic.

**Remark:** A complex $n$-plane bundle $\omega^n$ can be regarded as a real 2$\times$2-plane bundle $\omega^n$ by ignoring the multiplication by complex numbers.

**Canonical orientation of $\omega^n$**

We can choose a basis $a_1, a_2, \ldots, a_n$ over $C$ for each fibre $\pi^{-1}(b)$. The real fibre, that is, the underlying real vector space of $\pi^{-1}(b)$, has a canonical orientation $a_1, i\lambda_1, a_2, \ldots, a_n, i\lambda_n$. This orientation is independent of the choice of the complex basis $a_1, a_2, \ldots, a_n$, since $GL(n, \mathbb{C})$ is connected and we can pass from this basis to any other continuously, i.e. without change in sign.

**Corollary:** Every complex manifold has a standard orientation. As we have already seen in the real case, an orientation of the tangent bundle corresponds to an orientation of the manifold.
Corollary: For every complex n-plane bundle \( \mathcal{N} \) there is a well-defined Euler class \( \chi(\nu^n)_{\mathbb{R}} \in \text{H}^2(B;\mathbb{Z}) \).

If we take the bundle sum \( \mathcal{N} + \mathcal{E}^k \) of two complex plane bundles \( \mathcal{N} \) and \( \mathcal{E}^k \), with bases \( a_1^1,\ldots,a_n^1 \) and \( b_1^1,\ldots,b_k^1 \) the vectors \( a_1^1,b_1^1,\ldots,a_n^1,b_k^1 \) form a basis for \( \mathcal{N} + \mathcal{E}^k \). This means that the canonical orientation of \( \nu^n + \nu^k \) is

\[
\nu^n, a_1^1, a_2^1, \ldots, a_n^1, b_1^1, b_2^1, \ldots, b_k^1 \to \nu^k, a_1^1, a_2^1, \ldots, a_n^1, b_1^1, b_2^1, \ldots, b_k^1.
\]

Thus we see that in a natural way \( \nu^n + \nu^k \to \nu^n \), \( \nu^k \) as oriented bundles. (This was one reason of defining the canonical orientation this way.) From this it follows that \( \chi(\nu^n + \nu^k)_{\mathbb{R}} = \chi(\nu^n) \chi(\nu^k) \).

**Hirzebruch classes**

We will now give an inductive definition of characteristic classes for a complex n-plane bundle, \( \mathcal{N} \). We define a canonical complex (n-1)-plane bundle \( \mathcal{N}^{n-1} \) over \( E_0(\nu^n) \). (As in the real case, \( E_0(\nu^n) \) denotes the set of all non-zero vectors in \( E(\nu^n) = (\nu^n)^{\mathbb{R}} \).)

A point in \( E_0 \) is specified by a fibre of \( \mathcal{N} \) and a non-zero vector in that fibre. We will obtain \( \mathcal{N}^{n-1} \) by considering the orthogonal (n-1)-space in that fibre. This can be done using the Hermitian metric, which can be defined in any complex n-plane bundle over a paracompact base \( B \) by a procedure analogous to that for real plane bundles (see Theorem 5). Alternatively it can be obtained algebraically by looking at the factor space, \( E(\nu^n)^{n-1} \) will consist of all pairs \( (a_1^1,a_2^1+c_1) \) where \( a_1^1 \) is the non-zero vector, \( a_2^1+c_1 \) is another vector in the same fibre and \( a_2^1+c_1 \) is a coset. The projection \( \pi: E(\nu^n)^{n-1} \to E_0(\nu^n) \) is defined by \( \pi((a_1^1,a_2^1+c_1)) = a_1^1 \).

Recall that for real oriented 2n-plane bundles, we have a Gysin sequence

\[
\rightarrow H^{1+2n}(B) \xrightarrow{i^*} H^1(E_0) \xrightarrow{j^*} H^1(E_0) \xrightarrow{k^*} H^{1+2n}(B) \rightarrow.
\]

For \( i \leq 2n-2, H^{1+2n}(B) = 0 \) and so \( \chi(\nu^{n-1} + \nu^k) = \chi(\nu^{n-1}) = \chi(\nu^k) \).

**Definition:** The Chern classes \( c_1(\nu^n)_{\mathbb{R}} \in \text{H}^1(\mathbb{B};\mathbb{Z}) \) are defined as follows, by induction on \( n \),

\[
c_1(\nu^n)_{\mathbb{R}} = \begin{cases} 
0 & \text{for } i > n \\
\chi(\nu^n) & \text{for } i = n \\
\chi(\nu^{n-1}) c_1(\nu^{n-1}) & \text{for } i < n.
\end{cases}
\]

The last expression is well defined since

\[
\nu^{n-1}: H^1(\mathbb{B}) \to H^1(\mathbb{B})
\]

is an isomorphism for \( i < n \). The expression \( c(\nu^n) = 1 + c_1(\nu^n) + \ldots + c_n(\nu^n) \) is called the total Chern class of \( \nu^n \).

**Lemma:** Chern classes are natural with respect to bundle maps \( f \) for a bundle map \( f = (f_B^*:f_E^*): \mathcal{N} \to \mathcal{N} \) we have \( f_c^*(c(\mathcal{N})) = c(\mathcal{N}) \).

**Proof:**

1) \( f_B^*(c(\mathcal{N})) = c(\mathcal{B}) \) since Euler classes are natural.

2) \( f_B^*: E_0 \to E_0 \) can be covered by a bundle map \( \omega \to \omega' \),

between the canonical (n-1)-plane bundles over \( E_0 \) and \( E_0' \). But \( c_{n-1}(\omega^n) = \chi(\omega^{n-1}) c_{n-1}(\omega^{n-1}) \) and \( c_{n-1}(\omega^{n-1}) = \chi(\omega^{n-1}) \) which is natural with respect to bundle maps. Since \( f_B^*|_{E_0} = (f_B^*|_{E_0})_c \); see that \( c_{n-1}(\omega^n) \) is natural with respect to bundle maps. Descending this way, we show naturality of each \( c_i(\mathcal{N}) \) and so naturality of the total class \( c(\mathcal{N}) \).
Lemma 2: Let \( \Theta^n \otimes \mathbb{C}^k \) be the trivial complex \( k \)-plane bundle, then
\[
\psi^{(n)} = T^{(n)} \otimes \mathbb{C}^k.
\]

Proof: It is sufficient to prove the assertion for \( \Theta^1 \) since the general case then follows by induction. Changing the notation for convenience, write \( \psi^n = \phi^{n-1} \otimes \mathbb{C}^1 \). We want to show that
\[
\chi^{(n)} = \tau^{(n)}.
\]
Since the bundle \( \psi^n \) has a non-zero cross-section, it is certainly true that \( \chi^{(n)} = \psi^{(n)} \otimes \mathbb{C}^1 \). Thus \( \tau^{(n)} \) is covered, in an obvious way, by a bundle map \( \phi^{n-1} \rightarrow \psi^{n-1} \). Thus by Lemma 1, \( \tau^{(n)} \) is equal to \( \tau^{(n)} \) by definition; so that
\[
\chi^{(n)} = \tau^{(n)} = \phi^{(n)} \otimes \mathbb{C}^1 \otimes \mathbb{C}^1 = \phi^{(n)} \otimes \mathbb{C}^1 \otimes \mathbb{C}^1.
\]
This completes the proof.

We continue our complex analogy of real bundle theory with the following. Definition: The complex Grassman manifold \( G_n^{k} \) is the set of all \( n \)-dimensional subspaces in \( C^{n+k} \) (when working with complex structures, dimensional notation will always refer to complex dimension unless otherwise stated.)

Just as in the real case, \( G_n^{k} \) has a natural structure as a differentiable manifold; in fact, \( G_n^{k} \) has a natural structure as a complex analytic manifold. For example, still paralleling the real case, \( G_n^{k} \) is a complex projective space.

Similarly, let \( \nu^{n} \) be the \( n \)-plane bundle over \( G_n^{k} \), where \( B(\nu^n) \) is the set of all pairs \( (n \)-dim subspace, vector in that subspace). We investigate the structure of \( H^n(P_k^n) \). Applying the Gysin sequence to \( \nu^n \) over \( G_{n,k}^{1} \approx P_k^n \) and using the fact \( X(\nu^n) = c_1(\nu^n) \), we have
\[
H^{i+1}(E_0) \rightarrow H^i(P_k^n) \rightarrow \cdots \rightarrow H^{i+2}(P_k^n) \rightarrow H^{i+2}(E_0) \rightarrow H^{i+2}(E_0).
\]
The space \( E_0 = E_0(\nu^n) \) is the set of all pairs \( \langle \nu^n \rangle \) through origin in \( C^{k+1}, \) non-zero vector in that line. Clearly, this is just the set \( C^{k+1} \) of all non-zero vectors, which has the same homotopy type as \( S^{2k+1} \): Hence \( E_0 \) has the same cohomology ring as \( S^{2k+1} \). Thus the sequence becomes
\[
0 \rightarrow H^i(P_k^n) \rightarrow H^i(P_k^n) \rightarrow 0 \quad \text{for} \quad 0 \leq i \leq 2k - 2.
\]
That is, \( H^2(P_k^n) \approx H^2(P_k^n) \approx \cdots \approx H^2(P_k^n) \) and each group \( H^{2i}(P_k^n) \) is infinite cyclic generated by \( c_1(\nu^{2i}) \). For \( i = -1 \) and \( k \) odd, the sequence becomes
\[
H^{-1}(P_k^n) \rightarrow H^1(P_k^n) \rightarrow H^1(E_0) \rightarrow 0.
\]
Combining this with the isomorphism
\[
H^2(P_k^n) \approx H^2(P_k^n) \approx \cdots \approx H^{2k-1}(P_k^n)
\]
we obtain \( H^{2i+1}(P_k^n) = 0 \) for all \( i \). That is:

Theorem 1. \( H^n(G_{n,k}^{1}) = H^n(P_k^n) \) is the truncated polynomial ring terminating in dimension \( 2k \) and generated by \( c_1(\nu^n) \).
A formally identical procedure can be carried through in the
real case to show that $H^n(P^k;Z_2)$ is the truncated polynomial ring
terminating in dimension $k$ and generated by $\mathcal{J}$, the non-zero
element of $H^1(P^k;Z_2)$. In particular this means that $c_1^2, c_2^3, \ldots, c_k^k$
are all different from zero, a fact of which we made extensive use in
sections III and IV.

If we let $k \to \infty$ we have shown explicitly that $H^n(G_n(C))$ is the
polynomial ring generated by $c_1(\gamma^n(C)), \ldots, c_n(\gamma^n(C))$.

**Theorem 24:** $H^n(G_n(C))$ is the polynomial ring generated by
$c_1(\gamma^n(C)), \ldots, c_n(\gamma^n(C))$.

**Proof** (by induction): We have already shown it to be true for
$n = 1$. Using the Hermitian metric defined in $C^{n+k}$, i.e.

$$(\lambda_1, \ldots, \lambda_{n+k}, \mu_1, \ldots, \mu_n) = \sum_{i=1}^{n+k} \lambda_i \mu_i,$$

we know what is meant by orthogonality. For a point of $E_0(\gamma^n(C))$ given by an
$n$-dimensional subspace of $C^{n+k}$ and a non-zero vector therein, we take
the complementary (orthogonal) $(n-1)$-dimensional subspace in the given
subspace and thus obtain a map $\rho: E_0 \to G_{n-1}(C)$. On the other
hand, given an $(n-1)$-plane in $C^{n+k}$, any orthogonal non-zero vector
determines an $n$-plane and hence a point of $E_0$. In other words, $\rho$
is a fibre map and the fibre is $O_{n-1}^{k+1}$. For $1 \leq k$, the Gysin sequence
of this bundle gives $\rho^*: H^i(G_{n-1}(C)) \cong H^i(E_0)$.

Letting $k \to \infty$ as usual, we have

$$\cdots \to H^i(G_n(C)) \xrightarrow{c_0} H^{i+2n}(G_n(C)) \xrightarrow{\rho^*} H^{i+2n}(G_{n-1}(C)) \to H^{i+2n}(G_n(C)) \to \cdots$$

referring to diagram 4, we see that by naturality of Chern classes

under bundle maps, $\rho^*: H^i(G_n(C)) \to H^i(G_{n-1}(C))$ takes the Chern classes of $\gamma^n(C)$ into
those of $\gamma^{n-1}(C)$ which by the induction hypothesis, are the genera-
tors of $H^i(G_{n-1}(C))$. This means that $\rho^*: H^i(G_n(C)) \to H^i(G_{n-1}(C))$
are exact. In other words the exact sequence becomes:

$$\cdots \to H^i(G_n(C)) \xrightarrow{c_0} H^{i+2n}(G_n(C)) \to H^{i+2n}(G_{n-1}(C)) \to \cdots$$

We want to show 1) that every element $a$ of $H^*(G_n(C))$ is a pol-
ynomial in $c_1, \ldots, c_n$ and 2) that no non-trivial polynomial is zero.
We will prove both assertions by induction; 1) will be proved by induction on the dimension of $a$. [At the same time, we have the
induction hypothesis on the structure of $H^*(G_{n-1}(C))$.]

Certainly the assertion is true for $\dim a = n$. Since

$\rho^* \pi_0^*(a) e_0(G_{n-1}(C))$, it is a polynomial in
$c_1(\gamma^{n-1}(C)), \ldots, c_n(\gamma^{n-1}(C))$

i.e. $\rho^* \pi_0^*(a) = p(c_1(\gamma^{n-1}(C)), \ldots, c_n(\gamma^{n-1}(C)))$. To simplify
notation, we will write $c_i$ for $c_i(\gamma^n(C))$ and $c_j^i$ for $c_i(\gamma^{n-1}(C))$, and $\lambda$ for $\gamma^{*,-\sigma}_0$. Thus we have shown $\lambda(a)$ can be written as some polynomial $p(c_1^i, c_2^i, \ldots, c_{n-1}^i)$. Consider $a^i = a^i - p(c_1^i, \ldots, c_{n-1}^i)\in H^*(G_n(C))$. We see that $\lambda(a) = 0$ which by exactness of the above sequence means there is some $a^i = a^i - p(c_1^i, \ldots, c_{n-1}^i)$ such that $a^i = a^i - c_n^i$. Now $a^i$ has a smaller dimension than $a$ and hence by our special induction for $1$ can be written as a polynomial in $c_1^i, \ldots, c_n^i$. Therefore $a^i = a^i - c_n^i$ is a polynomial in $c_1^i, \ldots, c_n^i$. But this implies $a = a^i + p(c_1^i, \ldots, c_{n-1}^i)$ is a polynomial in $c_1^i, \ldots, c_n^i$. QED

As for 2), suppose $p(c_1^i, \ldots, c_n^i) = 0$. Then $\lambda(p(c_1^i, \ldots, c_n^i)) = p(c_1^i, \ldots, c_{n-1}^i, 0) = 0$. This means that $p(x, \ldots, x, 0)$ must be identically zero as a polynomial. In other words, $p(x_1, \ldots, x_n)$ has $x_n$ as a factor; $p = x_n p'$. Again we use a subsidiary induction, this time on the dimension of $p$. Certainly 2) holds for $\dim p = -1$. Having $p = x_n p'$, we know $p(c_1^i, \ldots, c_n^i) = p'(c_1^i, \ldots, c_{n-1}^i) - c_n^i = 0$. Since $c_n^i$ is a monomorphism, this means $p'(c_1^i, \ldots, c_{n-1}^i) = 0$. By the induction hypothesis, $p' \equiv 0$ thus $p = p' - c_n^i \equiv 0$. QED

Just as for real $n$-plane bundles we prove:

**Theorem 22:** Every complex $n$-plane bundle over a paracompact base has a bundle map into $\gamma^n(C)$ covering the generalized Gauss map into $G_n(C)$. (As in the real case, $G_n(C)$ is assumed to have the weak topology.)

**Product theorem for Chern classes**

We will use this universal bundle construction to prove the product theorem for Chern classes. Let $\omega^m$ and $\nu^n$ be complex plane bundles over the same paracompact base $B$. Then there exist bundle maps $\omega^m \rightarrow \gamma_m$ and $\nu^n \rightarrow \gamma_n$. (The $C$'s will be omitted whenever they are clear from the context.) The corresponding maps $B \rightarrow G_m$, $B \rightarrow G_n$ of the base space combine to give a map

$$f_B : B \rightarrow G_m \times G_n.$$

Let $\gamma_1^m$ and $\gamma_2^n$ be the bundles over $G_m \times G_n$ induced by the projection maps $p_1 : G_m \times G_n \rightarrow G_m$, $p_2 : G_m \times G_n \rightarrow G_n$ respectively. Then we have a bundle map $\nu^n \rightarrow \gamma_1^m$, where the dotted arrow in the following diagram is defined so that the diagram is commutative.

Similarly we have a bundle map $\omega^m \rightarrow \gamma_2^n$, and hence a bundle map

$$\omega^m \oplus \nu^n \rightarrow \gamma_1^m \oplus \gamma_2^n.$$

Thus we have proved the following: The bundles $\gamma_1^m$ and $\gamma_2^n$ over $G_m \times G_n$ are universal for pairs of bundles, in the sense that given any two bundles $\omega^m$ and $\nu^n$ of the same dimensions over a paracompact
space $B$, there exist bundle maps $\omega^m \rightarrow \nu^m$, $\nu^n \rightarrow \nu^n$ and $\nu^m \otimes \nu^n \rightarrow \nu^m \otimes \nu^n$, covering the same map $f_B$ of the base space.

The existence of these bundle maps, together with Lemma 1, gives us

$$
(1) \quad c(\gamma^m) = t_B^\nu c(\gamma^m_1), \quad c(\gamma^n) = t_B^\nu c(\gamma^n_2), \quad c(\omega^m \otimes \nu^n) = t_B^\nu c(\gamma^m_1 \otimes \gamma^n_2).
$$

Since $H^8(G_m)$ and $H^8(G_n)$ are both polynomial rings generated by the respective Chern classes, it follows that $H^8(G_m \times G_n)$ is a polynomial ring generated by the Chern classes

$$
c_1(\gamma^m_1), \ldots, c_m(\gamma^m_1), c_1(\gamma^n_2), \ldots, c_n(\gamma^n_2).
$$

(This is a consequence of the Künneth sequence)

$$
0 \rightarrow H^8(G_m) \otimes H^8(G_n) \rightarrow H^8(G_m \times G_n) \rightarrow \text{Tor}(H^8(G_m), H^8(G_n)) \rightarrow 0
$$

which is known to be exact for any pair of finite complexes. This sequence is exact in this case since each subcomplex $G_{n,k}$ is finite.)

Now consider the Chern class $c(\gamma^m_1 \otimes \gamma^n_2)$. Since it belongs to this polynomial ring $H^8(G_m \times G_n)$, there must be a unique polynomial $p_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n)$ with integer coefficients such that

$$
c(\gamma^m_1 \otimes \gamma^n_2) = p_{m,n}(c_1(\gamma^m_1), \ldots, c_m(\gamma^m_1), c_1(\gamma^n_2), \ldots, c_n(\gamma^n_2)).
$$

Applying $t_B^\nu$ to both sides of this equation, and using the naturality conditions (1) we have:

**Lemma:** For any pair of complex plane bundles $\nu^m$, $\nu^n$ over a paracompact base $B$, the formula

$$
c(\omega^m \otimes \nu^n) = p_{m,n}(c_1(\omega^m), \ldots, c_m(\omega^m), c_1(\nu^n), \ldots, c_n(\nu^n))
$$

holds.

Recall the definition of Chern classes. In particular

$$
c_n(\omega^n) = X(\omega^n_R)$$

and so we have the product theorem for the top Chern class from that for Euler classes:

1) $c_m(\omega^m_1) c_n(\nu^n_2) = c_{m+n}(\omega^m_1 \otimes \nu^n_2).

Recall further that we have already proved a special case of the product theorem

2) $c(\nu^n_1 \otimes \theta^1) = c(\omega^n_1)$ (see Lemma 2, of this section).

Now we are ready to prove in general:

**Theorem 26:** $c(\omega^m_1 \otimes \nu^n_2) = c(\omega^m_1) c(\nu^n_2).

In other words, the polynomial $P_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n)$ of Lemma 3 is in fact $(1 + x_1 + \ldots + x_m)(1 + y_1 + \ldots + y_n)$.

**Proof:** By induction on $m+n$. Certainly the assertion is true if $m+n=0$ or 1 or if either $m$ or $n$ is zero. By induction, assume the theorem true for $m+n-1$. Look at $\omega^m_1 \otimes \nu^{n-1} \otimes \theta^1$.

Grouping it one way, $c(\omega^m_1 \otimes \nu^{n-1} \otimes \theta^1) = c(\nu^{n-1} \otimes \theta^1) = 0$.

By 2) we have

$$
c(\omega^m_1 \otimes \nu^{n-1} \otimes \theta^1) = c(\omega^m_1) c(\nu^{n-1}).
$$

By the induction hypothesis,

$$
c(\nu^{n-1}) = c(\nu^{n-1} \otimes \theta^1).
$$

On the other hand, associating the other way:

$$
c(\omega^m_1 \otimes \nu^{n-1} \otimes \theta^1) = c(\omega^m_1 \otimes \nu^{n-1} \otimes \theta^1).
$$

By the lemma and 2) this is

$$
p_{m,n}(c_1(\omega^m_1), \ldots, c_m(\omega^m_1), c_1(\nu^{n-1}), \ldots, c_{n-1}(\nu^{n-1}), 0).
$$

This is true for all complex bundle pairs; in particular, considering the bundle $\gamma^m_1 \otimes \gamma^n_2$ where there are no polynomial relations, this
must be a polynomial identity

\[(1 + x_1 + \ldots + x_m)(1 + y_1 + \ldots + y_{n-1}) = p_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_{n-1}).\]

In other words

\[p_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n) \equiv (1 + x_1 + \ldots + x_m)(1 + y_1 + \ldots + y_n) \mod(y_n),\]

where \(y_n\) is the ideal generated by \(y^2\).

If we repeat the same procedure with \(\mathfrak{C}^{n+1} \equiv \mathfrak{C}^{\mathfrak{C}}\), we find

\[p_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n) \equiv (1 + x_1 + \ldots + x_m)(1 + y_1 + \ldots + y_n) \mod(x_m),\]

It is a simple algebraic consequence that

\[p_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n) \equiv (1 + x_1 + \ldots + x_m)(1 + y_1 + \ldots + y_n) \mod(x_m) \cap (y_n) = (x_m, y_n),\]

That is, \(p_{m,n} = (1 + x_1 + \ldots + x_m)(1 + y_1 + \ldots + y_n) + z(x_m, y_n)\), where \(z\) belongs to the polynomial ring concerned. By \(1\), the only term of dimension \(\geq 2m + 2n\) (the dimension of the top class is twice that of the bundle over \(C\)) is \(x_m, y_n\); that is, \(z = 0\):

QED

Application (again analogous to the real case):

Theorem 27: \(c(T^n(p^n(C))) = (1 + x)^{n+1}\) where \(c\) is the standard generator of \(H^2(p^n(C); Z)\), i.e., the one corresponding to the standard generator of \(H^2(S^2)\) under the inclusion \(S^2 = p^n(C) \subset p^n(C)\).

As a complex manifold, \(S^2\) has a uniquely distinguished generator of \(H^2(S^2; Z)\).

Proof: Complex projective n-space \(p^n(C)\) can be represented as the unit \(S^{2n+1} \subset C^{n+1}\) under the identification \(\bar{u} = \lambda \bar{u}\) for all \(\lambda \in C\), \(\lambda \neq 1\).

Then \(S^n(p^n(C))\) can be represented as the set of all pairs \((\bar{u}, \bar{v})\), with \(\|\bar{u}\| = 1\) and \(\bar{u} \bar{v} = 0\) in the Hermitian metric under the identification \((\bar{u}, \bar{v}) = (A \bar{u}, A \bar{v})\) for all \(A \in C, \lambda \neq 1\).

Consider the complex line bundle \(\xi^n\) over \(p^n(C)\) obtained from \(S^{2n+1} \times C\) by the identification \((\bar{u}, \bar{v}) = (A \bar{u}, A \bar{v})\) where \(A \in C\) and \(\lambda \neq 1\).

\[\xi^n = (1 + x)^{n+1}\]

By taking the orthogonal compliment to this cross section, using the Hermitian metric, \(\xi_1^n \cap \cdots \cap \xi_1^n\) splits into \(\xi_1^n(p^n(C)) \oplus 1\). By the product theorem, we have \(c(T^n(p^n(C))) = c(\xi_1^n(p^n(C))) = (1 + c_1(\xi_1^n)(1 + c_1(\xi_1^n)^n)\)

The inclusions \(S^2 = p^n(C) \subset p^n(C) \subset \cdots\) are covered by bundle maps \(\xi_2^n \leftrightarrow \xi_1^n \leftrightarrow \cdots\) and by naturality \(c_1(\xi_2^n)\) goes into \(c_1(\xi_1^n)\).

In fact, the homomorphism \(H^2(p^n(C)) \to H^2(p^n(C))\) is an isomorphism.

Consider \(S^2 = p^n(C)\) as a complex manifold, there is a distinguished generator \(c^n\) of \(H^2(S^2; Z)\), and \(c_1(\xi_2^n)\) must be some multiple of this standard generator. We have shown that \(c(T^n(p^n(C))) = (1 + c_1(\xi_2^n))^n\) or \(c_1(\xi_2^n(p^n(C))) = 2c_1(\xi_1^n)\). On the other hand, by definition, \(c_1(\xi_1^n(p^n(C))) = X(p^n(C)) = X(S^2)\). As is known, \(X(S^2) = 2\alpha\) which shows that \(c_1(\xi_2^n) = \alpha\) or \(c(T^n(p^n(C))) = (1 + \alpha)^n\).

Corollary: \(\alpha^n\) is the fundamental class of \(H^n(p^n(C))\) since \(\alpha^n = c_1(T^n(p^n(C))) = X(p^n(C)) = X(p^n(C)) = 2\alpha\) and it is known that the Euler characteristic is \(n+1\). (Since \(H^n(p^n(C))\) is the truncated polynomial ring, \(\alpha^n\) is a generator. Here we have settled the ambiguity as to whether it was + or - the fundamental cohomology class.)
classes of complex n-plane bundles, we introduce a new tool.

**Definition:** Two complex n-plane bundles $\omega$ and $\nu$ are conjugate equivalent if there is a map $f : \Sigma(\omega) \to \Sigma(\nu)$ such that 1) $\omega = f(\omega) = \nu$ are equivalent under $f$ and 2) $f_\omega(\delta e) = \bar{f}(\sigma)$ for all $e \in \Sigma(\omega)$. We will denote $\omega$ by $\bar{\omega}$.

**Note:** Conjugate equivalence is not an equivalence relation since in general $\omega$ is not conjugate equivalent to itself. For example, consider $T^1(\mathbb{P}^1(C))$. (Ignoring the complex structure, this is just the tangent bundle of the 2-sphere.) If this bundle were self-conjugate, there would be defined a map of the tangent plane at each point into itself so that the complex structure (rotation by 1) was reversed. The only such maps are obtained by reflection in some line of the plane. We would thus have a continuous field of lines in the tangent bundle of the 2-sphere, but this is impossible according to the corollary to Theorem 13. Hence $T^1(\mathbb{P}^1(C))$ is not self-conjugate.

An alternative proof of this will be given below using Chern classes. Conjugate equivalence is however an involutive relation, like the relation between two oriented bundles which are equivalent except that their orientations are opposite, in that the conjugate equivalent to the conjugate equivalent of a bundle is equivalent to the original bundle. There is a canonical representative of $\bar{\omega}$; namely, the bundle with the identical total space and conjugate structure in each fibre.

**Example:** Over $\mathbb{P}^1(C) = G_{1,1}(C)$ we have made use of two line bundles: $\xi \in G_{1,1}(C)$ and $\gamma \in G_{1,1}(C)$. They are in fact conjugate equivalent.

Looking at the Chern classes of the conjugate bundle we see:

**Theorem 26.** $c(\omega) = 1 - c_1(\omega) + c_2(\omega) - c_3(\omega) + \ldots$

**Proof:** Let $v_1, \ldots, v_n$ be a basis for the complex fibre $F = \pi^{-1}(b)$ of $\omega$ for some arbitrary $b \in B$. This gives $v_1, \bar{v}_1, v_2, \bar{v}_2, \ldots, v_n, \bar{v}_n$ as the orientation of the real fibre. Applying $f$ which gives the conjugate equivalence, $f(v_1), \bar{f}(v_2), \ldots, f(v_n)$ gives $f(v_1), \bar{f}(v_2), \ldots, f(v_n)$ as the orientation of $(\bar{\omega})_R$. On the other hand applying $f$ to the orientation of the real fibre we get $f(v_1), \bar{f}(v_2), \ldots, f(v_n), \bar{f}(v_n)$ which is $(-1)^n$ times the orientation of $(\bar{\omega})_R$. Thus we see $\chi(\bar{\omega}) = (-1)^n \chi(\omega)$ and so $c_n(\omega) = (-1)^n c_n(\omega)$ which checks with the formula.

To check the formula for the lower dimensional classes, recall the definition $c_{n-1}(\omega) = \kappa_{n-1}^0 c_{n-1}(\omega)$; that $c_n(\omega) = \chi(\omega)$ by the above argument therefore, $c_{n-1}(\omega) = (-1)^n c_{n-1}(\omega)$. Descending in this way, we obtain the above formula for the total Chern class. Note: This gives us a new proof of our earlier assertion that $T^1(\mathbb{P}^1(C))$ is not self-conjugate for $c_2(T^1(\mathbb{P}^1(C)) = 2$;

(see proof of Theorem 21) which is not of order 2.

**XII. Pontrjagin Classes**

To complete our study of characteristic classes of n-plane bundles, we need one new tool: the construction of the complex n-plane bundle induced by a real bundle. There are two ways of looking at the new structure although the structure itself is the same.

**Definition:** Given a real n-plane bundle $\xi$ the induced complex n-plane bundle $\xi_c$ with the same base $B$ is obtained by considering as fibre over $b$ the set of all formal sums $x + iy$ where $x, y \in \mathbb{R}$.
fibre of \( \xi \). (Each fibre of \( \xi \) is an \( n \)-dimensional vector space over \( \mathbb{C} \) as desired.)

**Alternative definition:** Given a real \( n \)-plane bundle \( \xi \), the induced complex \( n \)-plane bundle \( \xi \) with the same base space \( \mathbb{B} \) is defined as follows: \( E(\xi) = E(\xi \otimes \mathbb{C}) \) and multiplication over \( \mathbb{C} \) is defined in each fibre by \( 1^*(x,y) = (-y,x) \).

Using this second definition, it is easy to see that

**Lemma:** \( \xi \) is equivalent to its conjugate \( \overline{\xi} \), which is the same as saying \( \xi \) is conjugate equivalent to itself.

**Proof:** Let \( f:E(\xi) \to E(\overline{\xi}) \) be defined by \( f(x,y) = (x,-y) \).

Clearly \( f \) gives the equivalence of the real bundle structures.

Further \( f(1^*(x,y)) = f((-y,x)) = (-y,-x) = 1^*(x,y) \). QED.

If we look at the Chern classes of \( \xi \), we see \( c(\xi) = c(\overline{\xi}) \), which, by our result on the Chern classes of the conjugate bundle, gives us

\[
c(\overline{\xi}) = 1 + c_1(\overline{\xi}) + c_2(\overline{\xi}) + \cdots
\]

\[
c(\xi) = 1 - c_1(\xi) + c_2(\xi) - c_3(\xi) + \cdots
\]

Thus we have

\[
2c_1(\xi) = 2c_1(\overline{\xi}) = \cdots = 0
\]

This means that these odd classes carry a limited amount of information; we therefore confine our attention to the even classes.

**Definition:** For a real \( n \)-plane bundle \( \xi \), the \( i \)-th Pontrjagin class \( p_i(\xi) \) is defined to be \((-1)^i c_{2i}(\xi) \) \( H^{2i}(\mathbb{B}) \). (The reason for \((-1)^i \) is that it is, as is, will appear below.)

The total Pontrjagin class \( p(\xi) \) is defined to be

\[
1 + p_1(\xi) + p_2(\xi) + \cdots + p_{[n/2]}(\xi).
\]

(The highest Chern class is \( c_n \) since \( \xi \) is a complex \( n \)-plane bundle and thus the highest Pontrjagin class corresponds to \( [n/2] \), the integral part of \( n/2 \).

As for the other classes we have studied, we would like the Pontrjagin classes to satisfy the product formula, but we are likely to run into trouble because we have thrown away the odd dimensional Chern classes of \( \xi \). The factors \((-1)^i \) we have introduced will cause no trouble since if \((\xi + c_2 + c_4 + \cdots)(\xi + c_2 + c_4 + \cdots) = (1 + c_2 + c_4 + \cdots)(1 + c_2 + c_4 + \cdots) \) then \((1 - c_2 - c_4 + \cdots)(1 - c_2 - c_4 + \cdots) = (1 - c_2 - c_4 + \cdots) \). In fact, throwing away the odd dimensional classes forces a revision of the product theorem as follows:

**Theorem 20:** \( p(\xi \otimes \eta) = p(\xi)p(\eta) \) is a sum of elements in order 2

**Proof:** \( \xi \otimes \eta \) = \( \xi \otimes \eta \) and by the product theorem for Chern classes

\[
o(\xi \otimes \eta) = o(\xi) o(\eta) = o(\xi) o(\eta) \]

We know the odd dimensional classes are not included in the Pontrjagin classes and are all of order 2. QED.

**Example:** If we look at \( p(T^n(S^r)) \) we see that it is trivially 1 unless \( n = 4k \), in which case \( p(T^n(S^r)) = 1 + p_k \). However, \( \xi ^n \) is trivial as is \( y^1 \) so \( p(\xi^n \otimes y^1) = p(\xi^n)p(y^1) = 1 - (1 + p_k) = -p_k \) must be of order 2. But \( H^n(S^r) = \mathbb{Z} \) has no element of order 2 other than zero. That is, \( p_k(T^n(S^r)) = 0 \) and so \( p(T^n(S^r)) = 1 \) for all \( n \).

We see that the Pontrjagin classes of spheres are uninteresting; it turns out that the things to look at are complex projective spaces.
But at first let us consolidate our gains. At this point, we have a situation which is represented symbolically at the right. Given a real $n$-plane bundle, we now can obtain the induced complex $n$-plane bundle. Given a complex $n$-plane bundle we can look at its underlying real structure to obtain a real oriented $2n$-plane bundle. Given a real oriented $2n$-plane bundle, we can always ignore the orientation to get a real $2n$-plane bundle. In other words, we can start at any point on the circle above and traverse it in the clockwise direction; notice that when we return to the original point we do not have the original bundle but rather one of twice the dimension. We would next like to investigate the behavior of characteristic classes under this sequence of operations. In particular, we have

**Theorem 30:** For $\omega^n$ a complex $n$-plane bundle,

$$(-1)^{\frac{n(n+1)}{2}} (-1) \sum_{k+j=n} (-1)^k c_k(\omega^n) c_j(\omega^n)$$

**Proof:** By definition,

$$p_1(\omega^n_R) = (-1)^{n} c_2(\omega^n_{RC})$$

where $\omega^n_{RC}$ is obtained by neglecting the complex structure to get an oriented real $2n$-plane bundle, ignoring the orientation, and then complexifying to get a complex $2n$-plane bundle. By definition, $E(\omega^n_R) = E(\omega^n_{RC})$ and $E(\omega^n_{RC}) = E(\omega^n_R) \oplus \omega^n_R$. Therefore $E(\omega^n_{RC}) = E(\omega^n_R) \oplus \omega^n_R$. Our problem is to compare the complex structure of $E(\omega^n_{RC})$ with that of $E(\omega^n_R)$. A point in $E(\omega^n_{RC})$ is given by a pair $(x,y)$ where $x$ and $y$ belong to the same fiber of $\omega^n_R$. The multiplication by $1$ defined in $\omega^n_R$ will always appear inside the pair; the multiplication by $1$ for $\omega^n_{RC}$ will appear outside the pair and will be written with $\cdot$, for example, $1^*(x,y)$. Let $E(\omega_1)$ be the subspace of $E(\omega^n_{RC})$ consisting of all pairs $(x,-1x)$. This space $E(\omega_1)$ is invariant under $1^*$ as we have defined it, for $1^*(x,-ix) = (ix,x)$ which is of the required form. Similarly $E(\omega_2)$, defined as the subspace of all pairs of the form $(ix,x)$, is invariant under $1^*$ since $1^*(x,ix) = (-ix,x)$. Now $\omega^n_R = \omega_1 \oplus \omega_2$ since any point $(x,y)$ of $E(\omega^n_{RC})$ can be written as $(\frac{x+iy}{2}, \frac{x-iy}{2}, \frac{x+iy}{2}, \frac{x-iy}{2})$. Moreover $\omega_1$ is equivalent to $\omega^n_R$. Consider the map $f: (x,-ix) \to x$ taking $E(\omega_1)$ into $E(\omega)$. Since $1^*(x,-ix) = (ix,x)$, we have $f(1^*(x,-ix)) = ix = f(x,-ix)$ and $f$ gives the equivalence of the complex bundles. Similarly, $\omega_2$ is equivalent to $\omega^n_R$. Let $g(x,ix) = x$ take $E(\omega_2)$ into $E(\omega^n_R)$, then $g(1^*(x,ix)) = g(-ix,x) = -ix = -ig(x,ix)$ as required. Thus we have shown:

**Lemma:** $\omega^n_{RC}$ is equivalent to $\omega^n_R \oplus \omega^n_R$.

By the product theorem for Chern classes

$$c(\omega^n_{RC}) = c(\omega) c(\omega^n_R) = (1+c_1(\omega^n_R)+c_2(\omega^n_R)+\ldots)(1+c_1(\omega^n_R)+c_2(\omega^n_R)+\ldots)$$

Observe that the minus signs in this formula cooperate to cancel out all the odd dimensional classes in the product. The result can be stated

$$1-P_1(\omega^n_{RC}) + P_2(\omega^n_{RC}) - \ldots = (1+c_1(\omega^n_R)+\ldots)(1-c_1(\omega^n_R)+c_2(\omega^n_R)+\ldots)$$

$$= \sum_{k+j=n} (-1)^k c_k(\omega^n_R) c_j(\omega^n_R)$$

$$= \sum_{k+j=n} (-1)^{k+j} c_k(\omega^n_R) c_j(\omega^n_R)$$

$$= \sum_{k+j=n} (-1)^{k+j} c_k(\omega_R^n) c_j(\omega^n_R)$$
Broken down this is

\[-1 \sum_{k+j=2} (-1)^j c_k(\omega^n) c_j(\omega^n), \text{ QED.} \]

These formulas can be written as follows

\[
\begin{align*}
p_1(\omega^n_R) &= c_1^2(\omega^n) - 2c_2(\omega^n) \\
p_2(\omega^n_R) &= c_2^2(\omega^n) - 2c_1(\omega^n) c_1(\omega^n) + 2c_4(\omega^n) \\
p_3(\omega^n_R) &= c_3^2(\omega^n) - 2c_2(\omega^n) c_2(\omega^n) + 2c_1(\omega^n) c_3(\omega^n) - 2c_6(\omega^n)
\end{align*}
\]

e tc.

Example: We already know \(c(t^n(p^n(C))) = (1 + \alpha)^{n+1}\) where \(\alpha H^2(p^n(C))\). It is clear that \(c(t^n_R) = (1 - \alpha)^{n+1}\) and by the above formula

\[1 - p_1 t^n_R + p_2 t^n_R - p_3 t^n_R + \ldots - c(t^n) e(t^n) = (1 - \alpha^2)^{n+1}\]

Therefore \(p(t^n_R p^n(C)) = (1 + \alpha^2)^{n+1}\).

Since there will be no ambiguity we will write \(p(M^n)\) for \(p(t^n_R(M^n))\) where \(M^n\) is a complex manifold. In particular \(p(t^2(C)) = (1 + \alpha^2)^2 = 1\) where \(\alpha H^2(t^1(C)) = 0\) for \(i > 2\).

(This checks with our previous result since \(p^1(C) = S^2\).

Further

\[
\begin{align*}
p(p^2(C)) &= (1 + \alpha^2)^3 = 1 + 3\alpha^2, \\
p(p^3(C)) &= (1 + \alpha^2)^4 = 1 + 4\alpha^2, \\
p(p^4(C)) &= (1 + \alpha^2)^5 = 1 + 5\alpha^2 + 10\alpha^4, \text{ etc.}
\end{align*}
\]

These last results were obtained from the sequence

complex \(\rightarrow\) oriented real \(\rightarrow\) real \(\rightarrow\) complex

(see diagram above.) If we follow the sequence

oriented real \(\rightarrow\) real \(\rightarrow\) complex \(\rightarrow\) oriented real

instead we find that starting with an oriented \(n\)-plane bundle \(\xi^n\)

we have

\[
\xi^n_{CR} = t(\xi^n \otimes \xi^n)
\]

the only question being the agreement of the orientations.

The orientation in each fibre is given by a basis \(v_1, \ldots, v_n\) for that fibre. The corresponding orientation \(\text{for } t^n v^n_{CR}\) is given by

1) \((v_1,o),(v_2,o),(v_3,o), \ldots, (v_n,o)\).

On the other hand using \(v_1, \ldots, v_n\) as a basis for the complex fibre of \(\xi^n\), the corresponding real basis for \(\xi^n_{CR}\) is given by \((v_1,0),(v_2,0),(v_3,0), \ldots, (v_{n-2},0),(v_n,0)\) or

2) \((v_1,o),(v_2,o),(v_3,o), \ldots, (v_n,o)\).

It is easy to determine the sign of the permutation relating these two bases (and therefore relating the corresponding orientations of \(\xi^n_{CR}\) and \(\xi^n_0(\xi^n)\). The permutation can be effected by moving each \((o,v_1)\) to the left in 1) until it is in the proper place for 2), and the sign can thus be seen to be

\[-1(n-1)+(n-2)\ldots+2+1 = (-1)\frac{1}{2}n(n-1)\]

If we confine our attention to even dimensional bundles (where the Euler class is not necessarily of order 2) we have
Lemma: \( \zeta_{CR}^{2n} = (-1)^{n}(\zeta_{2n}^{2} + \zeta_{2n}^{2} \zeta_{2n}) \) for any oriented 2n-plane bundle \( \xi_{2n} \).

Therefore, looking at the Euler classes we have,

**Theorem 31.** For any oriented 2n-plane bundle \( p_{n}(\xi_{2n}) = (x(\xi_{2n}^{2}))^{2} \).

**Proof:**

\[
p_{n}(\xi_{2n}) = (-1)^{n} c_{2n}(\xi_{2n}) = (-1)^{n} x(\xi_{2n}^{2})
\]

\[
= (-1)^{n} x(-1)^{n}(\zeta_{2n}^{2} + \zeta_{2n}^{2} \zeta_{2n}) = x(\zeta_{2n}^{2} + \zeta_{2n}^{2} \zeta_{2n})
\]

Thus by the product theorem for Euler classes, \( p_{n}(\xi_{2n}) = (x(\xi_{2n}^{2}))^{2} \)

QED.

(This is the one place where we find it convenient to have defined \( p_{1}(\xi_{2n}) \) with the factor \((-1)^{1}\).)

**Structure of \( H^{*}(\tilde{G}_{n}(R); \Lambda) \)**

For \( \Lambda \) a coefficient ring which contains \( 1/2 \) (so that we need not worry about elements of order 2, e.g. the rationals \( \mathbb{Q} \)), we can now give the structure of the cohomology ring \( H^{*}(\tilde{G}_{n}(R); \Lambda) \). (See pg. 59 for definition.)

The result will be only slightly more complicated than the cases \( H^{*}(\tilde{G}_{n}(R); \mathbb{Z}) \) and \( H^{*}(\tilde{G}_{n}(R); \mathbb{Z}) \) which we have already computed.

**Theorem 32.** If \( \Lambda \) is an integral domain containing \( 1/2 \) then the cohomology ring \( H^{*}(\tilde{G}_{2n+1}; \Lambda) \) is a polynomial ring generated by \( p_{1}(\xi_{2n+1}), \ldots, p_{m}(\xi_{2n+1}) \).

The cohomology ring \( H^{*}(\tilde{G}_{2n}; \Lambda) \) is a polynomial ring generated by \( p_{1}(\xi_{2n}), \ldots, p_{m-1}(\xi_{2n}), \) and \( X(\xi_{2n}) \).

This can be summarized by saying that \( H^{*}(\tilde{G}_{n}; \Lambda) \) is the polynomial ring generated by \( p_{1}, \ldots, p_{m} \) and \( X \), modulo the relations

\[
X = 0 \quad \text{for } n \text{ odd}
\]

\[
X^{2} = p_{m}/2 \quad \text{for } n \text{ even}.
\]

**Proof by induction on \( n \).** Since \( \tilde{G}_{0} \) is a point, we can clearly start the induction. Just as in the complex case we have an exact sequence

\[
\rightarrow H^{i}(\tilde{G}_{n}) \rightarrow H^{i+n}(\tilde{G}_{n}) \rightarrow H^{i+n}(\tilde{G}_{n-1}) \rightarrow H^{i+1}(\tilde{G}_{n}) \rightarrow
\]

where \( \cdot = \cdot \circ \chi_{n-1}^{*} \) carries the Pontrjagin classes of \( \tilde{G}_{n} \) into those of \( \tilde{G}_{n-1} \).

**Case 1.** Assume that the theorem is true for \( \tilde{G}_{2n-1} \). That is \( H^{*}(\tilde{G}_{2n-1}) \) is a polynomial ring generated by \( p_{1}, \ldots, p_{m-1} \). Now the argument used in the proof of Theorem 24 shows that \( H^{*}(\tilde{G}_{2m}) \) is a polynomial ring generated by \( p_{1}, \ldots, p_{m-1} \) and \( X \).

**Case 2.** Assume that \( H^{*}(\tilde{G}_{2m}) \) has this form. Since \( X(\chi_{2m+1}) = 0 \) (with coefficient group \( \Lambda \)) the above sequence, for \( n=2m+1 \), \( j=1, \ldots, m+1 \), becomes

\[
0 \rightarrow H^{j}(\tilde{G}_{2m+1}) \rightarrow H^{j}(\tilde{G}_{2m}) \rightarrow H^{j-2m}(\tilde{G}_{2m+1}) \rightarrow 0.
\]

Thus \( H^{*}(\tilde{G}_{2m+1}) \) can be considered as a subring of \( H^{*}(\tilde{G}_{2m}) \). This subring is known to contain the elements \( p_{1}, \ldots, p_{m-1} \) and \( p_{m}X \). Thus, if \( \Lambda^{*} \) denotes the subring generated by \( p_{1}, \ldots, p_{m} \), we have
\[ R^* \cong \text{H}^*(\tilde{\mathfrak{s}}_{2m+1}) \subset \text{H}^*(\tilde{\mathfrak{s}}_{2m}) \]

which implies that

a) \( \text{rank } R^j \leq \text{rank } H^j(\tilde{\mathfrak{s}}_{2m+1}) \).

(For the concept of rank, see for example Eilenberg and Steenrod, p. 52.) From the exact sequence above we see that

\[ \text{rank } H^j(\tilde{\mathfrak{s}}_{2m+1}) + \text{rank } H^{j-2m}(\tilde{\mathfrak{s}}_{2m+1}) = \text{rank } H^j(\tilde{\mathfrak{s}}_{2m}). \]

But the equality

\[ \text{rank } R^j + \text{rank } R^{j-2m} = \text{rank } H^j(\tilde{\mathfrak{s}}_{2m}) \]

is easily verified. (In fact \( H^j(\tilde{\mathfrak{s}}_{2m}) = R^j \otimes \mathbb{R}^{2m} \).)

Therefore

\[ \text{rank } H^j(\tilde{\mathfrak{s}}_{2m+1}) + \text{rank } H^{j-2m}(\tilde{\mathfrak{s}}_{2m+1}) = \text{rank } R^j + \text{rank } R^{j-2m} \]

Using a) for both \( j \) and \( j-2m \), we have \( \text{rank } R^j = \text{rank } H^j(\tilde{\mathfrak{s}}_{2m+1}) \).

From this it follows easily that \( R^j = H^j(\tilde{\mathfrak{s}}_{2m+1}) \) which completes the proof.
XIII. Pontrjagin numbers

1. Partitions

A partition \( \omega \) of an integer \( k \) is an unordered sequence \( i_1 \ldots i_r \) of positive integers with sum \( k \). The set of all such partitions will be denoted by \( \Pi(k) \) and the number of partitions by \( \eta(k) \).

[For \( k = 0, 1, 2, 3, 4 \) the number \( \eta(k) \) is equal to \( 1, 1, 2, 3, 5 \) respectively. As \( k \) tends to infinity, a theorem of Hardy and Ramanujan asserts that

\[
\eta(k) \sim \frac{1}{4k^{1/2}} e^{\sqrt{2k}}.
\]

For further information see Ostmann [14].]

The natural composition operation \( \Pi(k) \times \Pi(k) \rightarrow \Pi(k+1) \) will be denoted by juxtaposition:

If \( \omega = i_1 \ldots i_r \), \( \omega' = j_1 \ldots j_s \), then \( \omega \omega' = i_1 \ldots i_r j_1 \ldots j_s \).

This composition operation is associative, commutative and has an identity element, which is denoted by \( e \). It is also possible to define a partial ordering relation among partitions. A refinement of \( i_1 \ldots i_r \) will mean any partition which can be written in the form \( \omega_1 \ldots \omega_r \) with \( \omega_1 \in \Pi(i_1) \), \( \ldots \), \( \omega_r \in \Pi(i_r) \).]

2. Pontrjagin numbers

Let \( \mathbb{R}^n \) be a compact, oriented, differentiable manifold with tangent bundle \( \mathbb{T}^n \) and fundamental homology class \( \mu_n \). Given any partition \( i_1 \ldots i_r \in \Pi(k) \), define the \( (i_1 \ldots i_r) \)-th Pontrjagin number

\[
P_{1} \ldots P_{1_r} [\mathbb{R}^n] \text{ of } \mathbb{R}^n \text{ to be the integer } \langle P_{1}^{i_1} \ldots P_{1_r}^{i_r} \mu_n, \mu_n \rangle.
\]

Note that this is zero unless \( n = 4k \).

(Compare Stiefel-Whitney numbers page 16.)

As an example consider the complex projective space \( \mathbb{P}^{2n}(C) \). Recall (pg. 82) that

\[
P_{1}^{(2n)}(\mathbb{P}^{2n}(C)) = \binom{2n+1}{1} \alpha_{2n+1}
\]

where \( \alpha \in H^2(\mathbb{P}^{2n}(C); \mathbb{Z}) \) and \( \langle \alpha^{2n}, \mu_{4n} \rangle = 1 \).

Hence

\[
P_{1} \ldots P_{1_r} [\mathbb{P}^{2n}(C)] = \binom{2n+1}{1} \ldots \binom{2n+1}{1_r}
\]

for any \( i_1 \ldots i_r \in \Pi(k) \).

It is frequently useful to consider various linear combinations of the Pontrjagin numbers of a manifold. The rest of this chapter will be concerned with one such set of linear combinations. Others will occur in Chapter \( 15 \).

3. Symmetric functions; the polynomials \( a_n \).

Consider a polynomial ring in \( n \) variables over the integers:

\( \mathbb{Z}[t_1, \ldots, t_n] \). This is made into a graded ring by defining the degree of each \( t_i \) to be \( 1 \). The elementary symmetric functions \( e_1, \ldots, e_n \) are defined by

1) \( \deg e_1 = 1 \), and

2) \( 1 + e_1 t_1 + \ldots + e_n t_1 \ldots (1+t_n). \)
(There is an important connection between symmetric functions and Pontrjagin classes due to Borel. For our purposes this can be formulated as follows. Suppose that a bundle $\mathcal{V}^{2n}$ splits into a sum $\mathcal{V}^0 \oplus \cdots \oplus \mathcal{V}_n^2$ of 2-plane bundles. Then the identity

$$1 + p_1(\mathcal{V}^{2n}) + \cdots + p_n(\mathcal{V}^{2n}) = (1 + p_1(\mathcal{V}_1^2)) \cdots (1 + p_1(\mathcal{V}_n^2))$$

shows that $p_1(\mathcal{V}^{2n})$ is the 1-th elementary symmetric function of $p_1(\mathcal{V}_1^2), \ldots, p_1(\mathcal{V}_n^2)$.

Let $S$ denote the graded subalgebra of $\mathbb{Z}[t_1, \ldots, t_n]$ consisting of the polynomials which are left fixed by all permutations of $t_1, \ldots, t_n$. A standard theorem asserts that $S = \mathbb{Z}[\gamma_1, \ldots, \gamma_n]$, where $\gamma_1, \ldots, \gamma_n$ are algebraically independent.

An alternative description of $S$ is the following: Define two monomials in $t_1, \ldots, t_n$ to be equivalent if some permutation of $t_1, \ldots, t_n$ carries one into the other. Define

$$\sum t_1^{i_1} \cdots t_r^{i_r}$$

to be the summation of all monomials equivalent to $t_1^{i_1} \cdots t_r^{i_r}$.

(For example $\sigma_4 = \sum t_1 t_2 t_3 t_4$)

Lemma. An additive basis for $S^k \subseteq$ subspace of $S$ of dimension $k, k \leq n,$ is given by the set of polynomials

$$\sum t_1^{i_1} \cdots t_r^{i_r}$$

where $i_1, \ldots, i_r$ ranges over all partitions of $k$.

The proof is not difficult.

Now define a polynomial in $k$ variables $a_1^{i_1} \cdots a_r^{i_r}$ belonging to $S^k$, where $i_1, \ldots, i_r \in \Pi(k)$, by the identity

$$a_1^{i_1} \cdots a_r^{i_r} (\gamma_1, \ldots, \gamma_k) = \sum t_1^{i_1} \cdots t_r^{i_r}.$$
4. A product formula; the group $\Gamma \Lambda^k$

It will be convenient to introduce the following concept. Given any graded ring $K$ with unit, define a ring $\Gamma K$ as the cartesian product

$$ A^0 \times A^1 \times A^2 \times \ldots $$

with composition operations

$$(a_0, a_1, \ldots ) \cdot (b_0, b_1, \ldots ) = (a_0 \cdot b_0, a_1 \cdot b_1, \ldots ),$$

$$(a_0, a_1, \ldots )^{-1} \cdot b = (a_0^{-1} \cdot b, \ldots ).$$

Each element $a \in \Gamma K$ will be identified with the sequence $$(a_0, a_1, a_2, \ldots )$$ in $\Gamma K$. Whenever there is no confusion, the sequence $(a_0, a_1, a_2, \ldots )$ will be written as a formal sum $a_0 + a_1 + a_2 + \ldots$.

Let $\Gamma K^*$ denote the subset consisting of sequences $(1, a_1, a_2, \ldots )$ with leading term 1. Then $\Gamma K^*$ is a multiplicative group.

As an example, given any commutative ring with unit $\Lambda$ and given a space $X$, the groups $A^1 = H^1(X; \Lambda)$ give rise to a commutative graded $\Lambda$-algebra, which will be denoted by $H^3(X; \Lambda)$. The total Pontrjagin class

$$ p = p_1 + p_3 + \ldots + p_n = (p_1, p_3, \ldots, p_n, 0, \ldots ) $$

with coefficients in $\Lambda$ of a bundle over $\Lambda$ is an element of $\Gamma_1 H^3(X; \Lambda)$. Recall (p. 79) the identity

$$ p(q) p(q) = p(q \otimes q) $$

holds whenever $H^3(X; \Lambda)$ has no 2-torsion.

Given any partition $\omega \in \Omega(k)$ and given a $x \in \Gamma k$, where $K$ is commutative, define $s_\omega (x) \in \Lambda^k$ to be $s_\omega (a_1, a_2, \ldots, a_k)$.

**Theorem 33.** The polynomials $s_\omega$ satisfy the identity

$$ s_\omega (a \cdot b) = \sum s_\omega (a) \cdot s_\omega (b), $$

to be summed over all pairs $\omega_1, \omega_2$ such that $\omega_1 \cdot \omega_2 = \omega$.

As an example, for $\omega = x$, this formula takes the particularly simple form:

**Corollary 1.** $s_k (a^* b) = s_k (a) + s_k (b)$.

Given any n-plane bundle $\xi$, the elements $s_\omega (p(\xi)) \in H^3(k; \Lambda)$ can be considered as new characteristic classes of $\xi$.

**Corollary 2.** The identity

$$ s_\omega (p(\xi) \cup \eta) = \sum \omega_1 \cdot \omega_2 = \omega \cdot s_\omega (p(\xi)) \cdot s_\omega (p(\eta)) $$

holds modulo 2-torsion.

Now consider a compact, oriented, differentiable manifold $\Sigma^n$. For each $\omega \in \Omega(k)$, define a new characteristic number by the formula

$$ s_\omega (\Sigma^n) = 0 \text{ if } n \neq 4k $$

$$ s_\omega (\Sigma^{4k}) = s_\omega (p(\xi^{4k})) < \sum s_\omega (p(\xi^{4k})) > $$

These numbers are linear combinations of the Pontrjagin numbers, and conversely the Pontrjagin numbers can be expressed as linear combinations of these. However the new numbers satisfy a very simple product formula.

**Corollary 3 (Thom).**

$$ s_\omega (\Sigma_1 \times \Sigma_2) = \sum \omega_1 \cdot \omega_2 = s_\omega (\Sigma_1) \cdot s_\omega (\Sigma_2) $$

Note that most of the terms on the right drop out for dimensional reasons. For example:
Corollary 4. If $M_1$ and $M_2$ both have positive dimension then

$$s_k[M_1 \times M_2] = 0.$$ 

The characteristic numbers $s_k[N^2]$ will turn out to be particularly important.

**Example.** For the manifold $P^{2n}(C)$, since $p = (1 + \alpha^2)^{2n+1}$, the class $p_1$ can be considered as the $i$-th elementary symmetric function in $\alpha^2, \ldots, \alpha^2$. Hence $s_k(p)$ is equal to

$$\sum_1^2 s_k = (2n+1)\alpha^{2k}.$$ 

In particular

$$s_n[P^{2n}(C)] = 2n + 1 \neq 0.$$ 

It follows from Corollary 4 that $P^{2n}(C)$ cannot be expressed as a product of positive dimensional manifolds.

**Proof of Theorem 33.** Consider the special case

$$A^n = Z[t_1, \ldots, t_{2k}],$$

where $t_1 = (1 + t_1) \ldots (1 + t_k)$, $b = (1 + t_{k+1}) \ldots (1 + t_{2k})$, and where the elements $a_i, \ldots, a_k$ and $b_i, \ldots, b_k$ are algebraically independent of degree 1. Then the elements $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ are algebraically independent.

Hence if Theorem 33 is true in this special case, it will be true universally.

Let $\omega = \omega_1 \ldots \omega_r$. By definition $s_{\omega}(a \cdot b)$ is equal to

$$\sum_1^r t_{\alpha_1} \ldots t_{\alpha_r}.$$ 

Each term of this sum has the form

$$t_{\alpha_1} \ldots t_{\alpha_r},$$

where $\alpha_1, \ldots, \alpha_r$ are distinct numbers between 1 and 2k. Let $\omega$ be the partition formed by those exponents $\alpha_i$ such that $1 \leq \alpha_i \leq k$, and let $\omega'$ be the partition formed by the remaining $\alpha_i$. The sum of all the terms corresponding to a given

decomposition $\omega = \omega_1 \omega_2$ is clearly just

$$s_{\omega_1}(a) s_{\omega_2}(b).$$

Since every such decomposition occurs, this completes the proof.

Corollaries 1, 2 and 4 are clear.

**Proof of Corollary 3.** For $i = 1, 2$ the tangent bundle $\tau_i$ of $M_i$, together with the projection $M_1 \times M_2 \to M_i$, induces a bundle $\tau_i$ over $M_i$. The tangent bundle $\tau$ of $M_1 \times M_2$ may be identified with the sum $\tau_1 \oplus \tau_2$. Hence Corollary 2 takes the form

$$s_{\omega}(p(\tau)) = s_{\omega_1}(p(\tau_1)) s_{\omega_2}(p(\tau_2)) \pmod{2 \text{ torsion}}.$$ 

The fact that the Kronecker index with integral coefficients ignores torsion, together with the identities $\mu_1 = \mu_1 \cdot \mu_2$ and $\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle \langle \beta, \gamma \rangle$, completes the proof.

5. **Linear independence of Pontrjagin numbers.**

The object of this section will be to prove the following theorem, which shows that the $\pi(n)$ Pontrjagin numbers of a general $4n$-manifold satisfy no linear relations.

**Theorem 34.** (Thom) The $\pi(n) \times \pi(n)$ matrix

$$\| P_{1,1} \ldots P_{r,1} [P^{2j_1}(C) \times \ldots \times P^{2j_r}(C)] \|,$$

where $j_1, \ldots, j_r$ range over $\pi(n)$, is non-singular.

**Remark.** In place of the manifolds $P^{2j_1}(C)$, $P^{2j_2}(C)$, $\ldots$ one could substitute any sequence $A^4, A^8, \ldots$ of manifolds which satisfy the conditions $s_k[A^{4k}] \neq 0$. 

Example. For \( n = 2 \),
\[
\begin{align*}
& p_1^2 (\mathbb{P}^2(C) \times \mathbb{P}^2(C)) = 18 \quad p_1^2 (\mathbb{P}^4(C)) = 25 \quad \\
& p_2 (\mathbb{P}^2(C) \times \mathbb{P}^2(C)) = 9 \quad p_2 (\mathbb{P}^4(C)) = 10, \\
\end{align*}
\]
so that the determinant is \(-45 \neq 0\). It is evident that the direct approach of simply computing the matrix will not help much in the general case.

Proof of Theorem 34. In place of the Pontrjagin numbers themselves we will use the linear combinations \( s_n(M) \). The following formula is a direct generalization of Theorem 33 Corollary 3.

\[
(1) \quad s_{\omega} \left( M_1 \times \cdots \times M_r \right) = \sum_{\omega_1 \cdots \omega_r = \omega} s_{\omega_1} \left( M_1 \right) \cdots s_{\omega_r} \left( M_r \right).
\]

Suppose that the manifolds \( M_1, \ldots, M_r \) have dimensions \( 41, \ldots, 41 \), respectively. Then the term
\[
s_{\omega_1} \left( M_1 \right) \cdots s_{\omega_r} \left( M_r \right)
\]
is zero unless \( \omega_1 \in \pi(1), \ldots, \omega_r \in \pi(1) \). This proves:

(2) \( s_{\omega} \left( M_1 \times \cdots \times M_r \right) = 0 \) unless \( \omega \) is a refinement of \( 1_1 \cdots 1_r \).

For the special case \( \omega = 1_1 \cdots 1_r \), the formula becomes

\[
(3) \quad s_{1_1 \cdots 1_r} \left( M_1 \times \cdots \times M_r \right) = s_{1_1} \left( M_1 \right) \cdots s_{1_r} \left( M_r \right),
\]

since all the other terms are necessarily zero.

Now choose some sequence \( M_1, M_2, \ldots, M_n \) of manifolds such that \( s_{1_i} \left( M_i \right) \neq 0 \) for \( 1 = 1, 2, \ldots, n \). Let \( M_{1_1 \cdots 1_r} \) denote the product manifold \( M_1 \times \cdots \times M_r \).

Then we will prove:

(4) the matrix \( \left\| s_{\omega} \left( M_{1_1 \cdots 1_r} \right) \right\|_{\omega, \omega'}, \in \mathbb{N}(n) \) is non-singular.

In fact let \( \omega_1, \ldots, \omega_r \) denote the partitions of \( n \), numbered so that, if \( \omega_j \) is a refinement of \( \omega_k \), then \( j < k \). Assertion (2) implies that

\[
s_{\omega_j} \left[ H_{\omega_k} \right] = 0 \text{ for } j < k,
\]

while (3) implies that

\[
s_{1_j} \left[ H_{1_k} \right] \neq 0.
\]

Thus the matrix is triangular and nonsingular. This completes the proof of (4), and therefore of Theorem 34, for the \( P_1 \cdots P_r \left( M_{1_1 \cdots 1_r} \right) \) are linear combinations of the \( s_{\omega} \left[ H_{\omega} \right] \) so that dependence of the former would imply dependence of the latter.

XIV. Cobordism

This chapter will give a presentation of the cobordism theory of Thom [23].

1. The ring \( \pi^n \).

All manifolds considered are to be compact, oriented and differentiable unless otherwise stated. The word "differentiable" will always mean "differentiable of class \( C^{\infty} \)." We construct an operation of addition among manifolds of the same dimension:

\[
\text{Definition. } M_1^n + M_2^n \text{ will represent the disjoint union } M_1^n \cup M_2^n. \text{ It is natural therefore to write } k M^n \text{ for the union of } k \text{ disjoint copies of } M^n, k \geq 0. \text{ Further define } -M^n \text{ to be the same manifold but with the opposite orientation.}
\]

This sum operation has a zero element: namely the vacuous manifold. Note however that \( N^n - H^n \) is not equal to zero.
An equivalence relationship between manifolds of the same dimension is defined as follows (as was indicated briefly on p. 19):

**Definition:** \( N^m \) is a boundary if there exists a compact, oriented, differentiable bounded-manifold \( B^{n+1} \) whose boundary is \( N^m \). The induced differentiable structure on the boundary \( N^m \) should coincide with the differentiable structure originally given. (The differentiable structure on \( B^{n+1} \) may be specified by a coordinate system \((\mathcal{U}_\alpha, \phi_\alpha)\) where

1. The \( \mathcal{U}_\alpha \) are open sets covering \( B^{n+1} \);
2. each \( f_\alpha : U_\alpha \rightarrow \mathbb{R}^n \) is a homeomorphism, either onto \( \mathbb{R}^n \) or onto a closed half-space; and
3. for each \( \alpha, \beta \) the composition

\[
f_\alpha f_\beta^{-1} : f_\alpha \left( \mathcal{U}_\alpha \cap U_\beta \right) \rightarrow \mathbb{R}^n
\]

is differentiable (i.e., can be extended to a differentiable map defined on a neighborhood of \( f_\beta \left( U_\beta \cap U_\alpha \right) \)). For further details see [12] appendix I.

**Definition:** \( N_1^m \times N_2^n \) (read: \( N_1^m \) and \( N_2^n \) belong to the same cobordism class) if \( N_1^m \times N_2^n \) is a boundary.

It is clear that this relation is reflexive and symmetric; that it is transitive can be seen using the obvious construction. If \( B_{12}^{n+1} \) has boundary \( N_1^m \times N_2^n \) and \( B_{23}^{n+1} \) has boundary \( N_2^n \times N_3^n \), then \( B_{12}^{n+1} \) and \( B_{23}^{n+1} \) are identified along the common boundary \( N_2^n \). The resulting structure can be smoothed out to give a \( C^0 \) manifold whose boundary is \( N_1^m \times N_3^n \). (See [12] Appendix I, Lemma 4.) If we denote by + the operation on equivalence classes induced by the operation + on the manifolds, the classes form an Abelian group \( \Omega^m \) under +. \( \Omega^n \) is the cobordism group in dimension \( n \).

**R.**

A bilinear pairing from \( \Omega^m \) and \( \Omega^n \) to \( \Omega^{m+n} \) is defined by the correspondence \( \Omega^m \times \Omega^n \rightarrow \Omega^{m+n} \). Thus the sequence \( \Omega^* = (\Omega^0, \Omega^1, \Omega^2, \ldots) \) of cobordism groups has the structure of a graded ring. It is easily verified that \( \Omega^m_{\mathbb{Z}} \) is isomorphic (as an oriented manifold) to \((-1)^{m+n} \Omega^m \times \Omega^n \). Thus the cobordism ring is anticommutative.

The Pontrjagin numbers provide a basic tool for studying this cobordism ring.

**Theorem 35 (Pontrjagin).** If \( N^m \) is a boundary, then every Pontrjagin number \( p_1, \ldots, p_r \) of \( N^m \) is zero.

**Proof:** The argument is completely analogous to that on page 17. Since the identity

\[
p_1, \ldots, p_r \left[ N_1^m \times N_2^n \right] = p_1, \ldots, p_r \left[ N_1^m \right] + p_1, \ldots, p_r \left[ N_2^n \right]
\]

is clearly satisfied we have:

**Corollary 1.** For each \( i_1, \ldots, i_r \in \Omega (k) \) the correspondence

\[
\Omega^{4k} \rightarrow p_1, \ldots, p_r \left[ \Omega^{4k} \right]
\]

defines a homomorphism of \( \Omega^{4k} \) into \( \mathbb{Z} \).

Comparing Theorems 34 and 35 we have:

**Corollary 2:** The manifolds

\[
p^{2i_1}(C) \times \cdots \times p^{2i_r}(C)
\]

with \( i_1, \ldots, i_r \in \Omega (k) \) represent linearly independent elements of the cobordism group \( \Omega^{4k} \). Hence the group \( \Omega^{4k} \) has rank \( \pi (k) \).

The principal object of Chapter \( \text{IV} \) will be to show that this is a best possible result. (That is \( \Omega^{4k} \) has rank exactly \( \pi (k) \); while \( \Omega^n \) has rank zero for \( n \neq 0 \) (mod 4).)
Remark. The actual structure of the first few groups is the following:

\[ \Omega^0 = \mathbb{Z}; \text{ since the only 0-manifolds are finite sets of points, and the algebraic number of points determines the cobordism class uniquely.} \]

\[ \Omega^2 = \mathbb{N}^2 = \Omega^2 = 0. \text{ It is well known that every 1-manifold or (orientable!)} \text{ 2-manifold bounds. The corresponding assertion for 3-manifolds is non-trivial.} \]

\[ \Omega^4 = \mathbb{Z} \text{ generated by the complex projective plane } \mathbb{P}^2(\mathbb{C}). \]

\[ \Omega^6 = \mathbb{Z}^6 = \Omega^7 = 0, \text{ } \Omega^8 = \mathbb{Z} + \mathbb{Z} \text{ generated by } \mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \text{ and } \mathbb{P}^4(\mathbb{C}). \text{ For further information see Dold [1] and Milnor [13].} \]

2. The Thom space of a bundle.

Let \( \xi \) be an n-plane bundle over a compact base space \( B(\xi) \).

By the Thom space \( T(\xi) \) will be meant the one point compactification of the total space \( E(\xi) \). The base space will be identified with the subset of \( E(\xi) \) corresponding to the zero cross-section. Thus we have

\[ E(\xi) \subseteq E(\xi) \subseteq T(\xi) \]

The point at infinity will be denoted by \( t_0 \).

Remark 1. The following alternative definition is sometimes more convenient. Choosing a Riemannian metric, let \( E^1 \) denote the subspace of \( E \) consisting of vectors of length \( \leq 1 \), and let \( E_0^1 \) denote the subspace of unit vectors. Define \( T^1 \) as the identification space \( E^1 / E_0^1 \). Then the correspondence

\[ \bar{\gamma} \rightarrow \bar{\gamma} / (1 - \| \bar{\gamma} \| ) \]

gives rise to a homeomorphism

\[ h: T^1 \rightarrow T. \]

Remark 2. Thom's notation is as follows. Let \( G \) be a closed subgroup of the orthogonal group \( O_n \), and let \( \xi \) be the n-plane bundle associated with a universal bundle for \( G \). Then Thom denotes \( T(\xi) \) by \( H(G) \).

The following two lemmas describe the structure of the Thom space.

Lemma 1. If \( B(\xi) \) is a finite cell complex then \( T(\xi) \) is an \( (n-1) \)-connected finite cell complex.

Proof. For each open q-cell \( e \) of \( B \), the inverse image \( \pi^{-1}(e) \subseteq E \subseteq T \) is an open \((n+q)\)-cell. If the cells \( \{ e_i \} \) cover \( B \) then the cells \( \{ \pi^{-1}(e_i) \} \), together with the point \( t_0 \), cover \( T \). Note that there are no cells in dimensions \( 1 \) through \( n-1 \).

Let \( D^n \) denote the unit ball in \( \mathbb{R}^n \), and let \( f: D^n \rightarrow B \) be a characteristic map for the cell \( e \). The induced bundle \( \gamma_f \) over \( D^n \) is necessarily a product bundle \( D^n \times \mathbb{R}^n \rightarrow \mathbb{D}^n \times \mathbb{D}^n \). Hence the composition of the natural maps

\[ D^n \times D^n = \mathbb{E}^n, \mathbb{E}^n \rightarrow \mathbb{E}^n = T^1(\xi) \rightarrow T(\xi) \]

gives the required characteristic map for \( \pi^{-1}(e) \). This completes the proof.

Lemma 2. If \( \xi \) is an oriented n-plane bundle, then each cohomology group \( H^{r+nk}(T(\xi), t_0) \) is isomorphic to \( H^k(B(\xi)), k > 0. \)

Proof. There are natural isomorphisms

\[ H^k(B) \rightarrow H^{r+nk}(E, \mathbb{Z}) \text{- excision } H^{r+nk}(T, T-B). \]

(See the appendix for the details of \( \beta \).)

Since the space \( T-B \) is contractible to the point \( t_0 \), this last group can be replaced by \( H^{r+nk}(T, t_0) \), which completes the proof.
3. Regular values of differentiable maps.

Let \( W \) be an open subset of Euclidean space \( \mathbb{R}^n \), and let \( f: W \to \mathbb{R}^k \) be a differentiable map.

**Definition:** A point \( y \in \mathbb{R}^k \) is a regular value of \( f \) if, for each \( x \in f^{-1}(y) \), the Jacobian matrix

\[
\| f'_1(x)/\partial x_j \|
\]

has rank \( k \). (The case \( f^{-1}(y) = \emptyset \) is excluded. For example if \( n < k \) then \( y \) is a regular value only if \( f^{-1}(y) = \emptyset \).)

More generally, for any subset \( C \) of \( W \), we will say that \( y \) is a regular value of \( f|C \) if the Jacobian matrix has rank \( k \) for all \( x \in f^{-1}(y) \cap C \).

Motivation for this definition is provided by

**Lemma 3.** If \( y \) is a regular value of \( f \) then \( f^{-1}(y) \) is a differentiable submanifold of \( W \), with dimension \( n-k \).

**Proof.** This follows immediately from the implicit function theorem. (See for example, Graves [6] p. 136.)

The following extremely delicate theorem shows that regular values exist.

**Theorem of Sard.** If \( f: W \to \mathbb{R}^k \) is differentiable (of class \( C^0 \)) then the set of all \( y \in \mathbb{R}^k \) which are not regular values has measure zero.

For the proof, see Sard [15].

The following lemma is based on this theorem. Let \( C \) be a compact subset of \( W \), and \( \mathcal{V} \) a neighborhood of \( C \), with \( \mathcal{V} \) compact \( \subset W \).

**Lemma 4.** Given any differentiable map \( f: W \to \mathbb{R}^k \), and given \( C > 0 \), there exists a differentiable map \( g: \mathcal{W} \to \mathbb{R}^k \) such that

\[
|g_i(x) - f_i(x)| < \varepsilon, \quad \| g'_i(x)/\partial x_j - f'_i(x)/\partial x_j \| < \varepsilon
\]

for all \( x \in W \), all \( 1 \leq i \leq k \), and all \( 1 \leq j \leq n \).

**Proof.** Let \( \mathcal{W} \subseteq W \) be a differentiable function which takes the value 0 on \( C \) and the value 1 on \( W-W \). (See Steenrod [20] p. 26.) If \( y \) is any regular value of \( f \), then the function \( g \) defined by

\[
g(x) = f(x) - \mu(x)y
\]

will certainly satisfy conditions (1) and (2). But, according to the theorem of Sard, the vector \( y \) can be chosen arbitrarily close to the origin. Hence condition (3) can also be satisfied.

Finally, the following will be needed:

**Lemma 5.** Let \( C \) again denote a compact subset of \( W \), and \( g: W \to \mathbb{R}^k \) a map such that \( g|C \) has 0 as regular value. Then there exists \( \delta > 0 \) such that, if \( h: W \to \mathbb{R}^k \) satisfies

\[
|h_i(x) - g_i(x)| < \varepsilon, \quad \| h'_i(x)/\partial x_j - g'_i(x)/\partial x_j \| < \varepsilon
\]

for all \( x \in C \), then \( h|C \) also has 0 as regular value.

The proof is straightforward.

4. Transverse regularity.

Let \( f: M \to M' \) be a differentiable map, and \( N'' \) a submanifold of \( M' \).

**Definition:** \( f \) is transverse regular on \( N'' \) if, for each \( y \in N'' \) and each \( x \in f^{-1}(y) \), the induced map from the tangent vector space at \( x \) to the normal vector space at \( y \).
\[ f_x \to f'_y \to f'_y \to f''_{y} \]
is onto. (Notice in particular that \( \dim N' \geq \dim M' - \dim M'' \) if \( f(M) \)
intersects \( N'' \), and if \( n = n' + n'' \) then \( f(n) \) must be normal to \( M'' \) at
the intersections.)

Using Lemma 3 it is not hard to see that the inverse image
\( f^{-1}(M'') \) is a differentiable manifold of dimension \( n - n' + n'' \), pro-
viding that \( f \) is transverse regular on \( M'' \).

Consider the following situation: Let \( N^n \) and \( B \) be compact
differentiable manifolds, and let \( \xi \) be a differentiable \( k \)-plane
bundle over \( B \). That is we assume that the total space \( E \) has a
differentiable structure compatible with the bundle structure.
Then \( B \) is a differentiable submanifold of \( E \) with normal bundle
equivalent to \( \xi \).

**Theorem 36.** Every map \( f:N^n \to T(\xi) \) is homotopic to a map \( h \)
which

(I) is differentiable on \( f^{-1}(E) \) (i.e., where ever differen-
tiability makes sense); and

(II) is transverse regular on \( E \).

**Proof:** First choose a map \( f_0:N^n \to T(\xi) \) which coincides with
\( f \) on \( f^{-1}(E) \), and which is differentiable on \( f^{-1}(E) \). (Compare
Steenrod [20] § 6.7.) Let \( (B_j) \) be a covering of \( B \) by coordinate
neighborhoods. Thus the bundle \( \xi \) restricted to \( B_j \) is equivalent
in \( B_j \times R^k \), and the projections of \( B_j \times R^k \) into the two factors
correspond to maps

\[ \iota: f^{-1}(B_j), f^{-1}_j: f^{-1}(B_j) \to R^k. \]

Choose a covering of \( f^{-1}(E) \) by open sets \( U_1, \ldots, U_m \subseteq f^{-1}(E) \).

These sets should be small enough so that

1) each \( U_j \) is diffeomorphic to an open subset of \( R^n \), and

2) each \( f_0(U_j) \) is contained in \( f^{-1}B_j \) for some \( j = j(1) \).

Choose smaller open sets \( U_1, V_1 \) with

\[ U_1 \subseteq V_1, \quad V_1 \subseteq W_1 \]

so that the union \( U = U_1 \cup \cdots \cup U_m \) still contains \( f_0^{-1}B \).

Now Lemma 4 will be used to construct a series of modifications \( f_1, \ldots, f_m \) of \( f_0 \). Each \( f_1 \) will coincide with \( f_1^{-1} \) except on
\( V_1 \). Each projection \( f^{-1}E \to B \) will coincide with
\( \pi^{-1}: f^{-1}E \to B \). Thus to construct these modifications, it is only
necessary to construct maps \( W_1 \to R^k \) which coincide with the com-
position \( g_1 \) if

\[ f_1^{-1}|_{W_1} \to \pi^{-1}(B_j) \to R^k \]

outside of \( V_1 \), where \( j = j(1) \).

Assume by induction that \( f_{i-1}^{-1}: N^n \to T(\xi) \) has been defined, as
above; so that

1) \( f_{i-1}^{-1}U_1 \cup \cdots \cup f_{i-1}^{-1}U_m \) is transverse regular on \( B \), and

2) \( f_{i-1}^{-1}B \subseteq U \).

For the case \( i = 1 \), both conditions are certainly satisfied. Con-
sider the composition \( g_1 \) above, carrying \( W_1 \) into \( R^k \). Choose an
approximation \( g'_1: W_1 \to R^k \), as in Lemma 4, so that

(a) \( g'_1 \) has the origin as a regular value.
(b) \( g'_1 \) coincides with \( g_1 \) outside of \( V_1 \), and
(c) the approximation is sufficiently close so that

\[ (U_1 \cup \cdots \cup U_{i-1}) \cap W_1 \] has the origin as regular value (making use
of Lemma 5); and so that \( g'_1(U_1 \cup \cdots \cup U_{i-1}) \) does not contain the origin.

Now define \( f_1 \) by the conditions.
\[ \forall f_1(x) = \forall f_0(x) \quad \text{for all } x \in f_0^{-1}(\mathbb{B}) \]
\[ f \cdot f_1(x) = f_1'(x) \quad \text{for all } x \in W_1 \]
\[ f_1(x) = f_1^{-1}(x) \quad \text{for all } x \notin W_1. \]

Conditions 1) and 2) above are clearly satisfied, since regularity of the \( f_1 \) corresponds to transverse regularity of the \( f_1' \) along \( \mathbb{B} \).

The required map \( h: \mathbb{N} \to T(\xi) \) is now given by \( h = f_m \). The conditions that:

1) \( f_m | \overline{U} \) is transverse regular on \( \mathbb{B} \), and

2) \( f_m^{-1}(\mathbb{B}) \subseteq \overline{U} \)
guarantee that \( f_m \) is transverse regular on \( \mathbb{B} \).

Remark: Suppose that \( \mathbb{N} \) is an oriented manifold and that \( \xi \) is an oriented bundle. Then the manifold \( h^{-1}(\mathbb{B}) \subseteq \mathbb{N} \) has a standard orientation induced as follows:

1) The map \( h \) induces a bundle map of the normal bundle \( v_k \) of \( h^{-1}(\mathbb{B}) \) in \( \mathbb{N} \) into the normal bundle of \( \mathbb{B} \) in \( \mathbb{E} \), which is equivalent to \( \xi \). Hence \( v_k \) is oriented.

2) For any submanifold there is a bundle map \( n^{-k} \oplus v_k \to \mathbb{N} \).

Hence if the tangent bundle \( t^\mathbb{N} \) and the normal bundle \( v^k \) are oriented, there is an induced orientation for \( t^{n-k} \).

Lemma 6: Let \( f \) and \( g \) be homotopic maps of \( \mathbb{N} \) into \( T(\xi) \) which are both differentiable wherever possible and both transverse regular on \( \mathbb{B} \). Then the oriented manifold \( f^{-1}(\mathbb{B}) \) and \( g^{-1}(\mathbb{B}) \) belong to the same cobordism class.

Proof: The homotopy will give the bounding manifold. That is, choose a homotopy

\[ h_0: M \times [0,5] \to T(\xi) \]

so that \( h_0(x,t) = f(x) \) for \( t \leq 2 \), \( h_0(x,t) = g(x) \) for \( t \geq 3 \), and so that \( h_0 \) is differentiable on \( h_0^{-1}(\mathbb{B}) \).

Then, just as in the proof of Theorem 36, \( h_0 \) can be approximated by a map \( h_m \) which is transverse regular on \( \mathbb{B} \). Furthermore this approximation can be chosen so that \( h_m(x,t) = h_0(x,t) \) for \( t \leq 1 \) or \( t \geq 4 \).

[Choose the open sets \( V_1 \) in \( M^4 \times (1,4) \) so as to cover the compact set \( h_0^{-1}(\mathbb{B}) \times [2,3] \). Then the argument of Theorem 36 shows that \( h_m \) will be transverse regular over \( h_m^m \times [2,3] \). It is only necessary to choose all of the approximations close enough so that transverse regularity is not lost on the remainder of \( h_0^{-1}(\mathbb{B}) \times [0,5] \). The inverse image \( h_m^{-1}(\mathbb{B}) \) will then be the required bounded manifold with boundary diffeomorphic to \( g^{-1}(\mathbb{B}) - f^{-1}(\mathbb{B}) \).

5. The main theorem.

In the place of the manifold \( \mathbb{N} \) of the previous section, substitute the \((n+k)\)-sphere.

Lemma 7. Let \( \frac{1}{k} \) be an oriented differentiable \( k \)-plane bundle. The correspondence which assigns to each transverse regular map \( f: S^{n+k} \to T(\xi^k) \) the manifold \( f^{-1}(B(\xi^k)) \) gives rise to a homomorphism \( \lambda \) of the homotopy group \( \pi_{n+k}(T(\xi^k)) \) into the cobordism group \( \pi_{n+k} \).

Proof: Theorem 36 and Lemma 6 imply that every element of the homotopy group corresponds to a unique element of the cobordism group. It is clear that this correspondence is a homomorphism.
Let \( \gamma^k \to \gamma^\infty_n \) be the generalized Gauss map, defined by the correspondence

normal plane \( \to \) parallel plane through origin

(see p. 22-23) and let

\[ g_T : T(\gamma^k) \to T(\gamma^\infty_n) \]

denote the induced map of the Thom space. Then the composition

\[ g_T \circ \gamma^{n+k} : \gamma^{n+k} \to T(\gamma^\infty_n) \]

is clearly transverse regular on \( B(\gamma^k_n) = \tilde{c}_{k,n} \). Furthermore, the inverse image is

\[ f^{-1} g_T^{-1}(\tilde{c}_{k,n}) = \mathbb{R}^n. \]

Now replacing euclidean space by its one-point compactification \( \mathbb{R}^{n+k} \), this completes the proof that

\[ n : \mathbb{R}^{n+k}(T(\gamma^\infty_h)) \to f^*(\mathbb{R}^n) \]

is onto. The more general case \( h \geq n \) is easily handled by the same method.

6. Homotopy and cohomotopy groups modulo \( \mathcal{C} \).

Let \( \mathcal{C} \) denote the class of all finite abelian groups. A homomorphism

\[ h : A \to B \]

between abelian groups is called a \( \mathcal{C} \)-isomorphism if the kernel and cokernel \( ( = B/hA) \) belong to \( \mathcal{C} \). (This concept is due to Serre [16].)

Lemma 9. Let \( X \) be a finite complex which is \((k-1)\)-connected. Then the Hurewicz homomorphism

\[ \phi : \pi_k(X) \to H_k(X;\mathbb{Z}) \]

Now consider the universal bundle \( \gamma^k \) of oriented \( k \)-planes through the origin in \( k+n \)-dimensional Euclidean space (see p. 59). The main result of cobordism theory is the following:

Theorem of Thom. If \( k \) and \( h \) are sufficiently large, then the homomorphism

\[ \lambda : \mathbb{R}^{n+k}(T(\gamma^\infty_h)) \to \mathbb{R}^n \]

is an isomorphism onto.

(Thom's notation for \( T(\gamma^\infty_h) \) is \( N(SO(k)) \).) Thus the computation the cobordism group is reduced to a problem in homotopy theory.

For our purposes it will be sufficient to prove half of this theorem.

Lemma 8. For \( k, h \geq n \) the homomorphism

\[ \lambda : \mathbb{R}^{n+k}(T(\gamma^\infty_h)) \to f^*(\mathbb{R}^n) \]

is onto.

Proof. Start with any manifold \( M^2 \). According to Whitney [23],

\[ M^2 \]

can be imbedded in \( \mathbb{R}^{n+k} \) providing that \( k \geq n \). Let \( \gamma^k \) denote the normal bundle and \( E(\gamma^k) \) the subset of the total space consisting of normal vectors of length \( < \varepsilon \). Here \( \varepsilon \) should be small enough so that the correspondence

normal vector \( \to \) end point

defines a diffeomorphism \( \phi \) of \( E(\gamma^k) \) onto a neighborhood \( U \) of \( M^2 \).

Define a map

\[ f : \mathbb{R}^{n+k} \to T(\gamma^k) \]

transverse regular along \( M^2 \) by

\[ f(x) = v_x \quad \text{for} \ x \notin U \]

\[ f(e(\gamma^k)) = \frac{v}{(e - \| v \|)} \quad \text{for} \ e(\gamma^k) \in U. \]
is a $C^\infty$-isomorphism for $r \leq 2k - 2$.

Instead of giving a detailed proof, it is sufficient to observe that this Lemma is dual, in the sense of the Spanier-Whitehead duality [19], to Lemma 10 below.

If $X$ is a finite complex of dimension $\leq 2n-2$ then the set of all homotopy classes of maps
$$f:X \to S^n$$
form a group $\pi^n(X)$, called the $n$-th homotopy group. The "co-Hurewicz homomorphism"
$$\beta^*: \pi^n(X) \to H^n(X; Z)$$
is defined by
$$(f) \to \int f^\circ \sigma^n$$
where $\sigma^n$ generates $H^n(S^n; Z)$. (For further details see [18].)

Lemma 10 (Serre): The homomorphism
$$\beta^*: \pi^n(X) \to H^n(X; Z)$$
is a $C^\infty$-isomorphism (for dim $X \leq 2n-2$).

For the proof see Serre [16].

Applying Spanier-Whitehead duality, Lemma 9 follows.

7. The structure of $\Omega^a$ modulo $\mathbb{Q}$

By the rank of an abelian group is meant the maximal number of elements which are linearly independent.

Theorem 37. The cobordism group $\Omega^n$ is finitely generated and has rank
$$\eta(n) \text{ for } n = 4a$$
$$0 \text{ for } n \not\equiv 0 \text{ (mod 4)}.$$

The proof will be based on

Lemma 11. Assume that $p$ and $h$ are sufficiently large. Each of the following groups is finitely generated, and has rank $\eta(n)$ or 0 according as $n = 4a$ or $n \equiv 0 \text{ (mod 4)}$:

1. the cohomology group $H^n(\hat{S}_h, \hat{S}, Z);$
2. the cohomology group $H^{n+k}(\hat{T}(\hat{S}_h)^k, \hat{S}, Z);$
3. the homology group $H_{n+k}(\hat{T}(\hat{S}_h)^k, Z);$ and
4. the homotopy group $\pi^{n+k}(\hat{T}(\hat{S}_h)^k).$

Proof. Assertion (1) follows from Theorem 32. According to Lemma 2 the cohomology groups of the Thom space are isomorphic to those of the base space, with a dimension shift. This proves (2).

Assertion (3) now follows from the universal coefficient theorem, together with the fact that $T(\hat{S}_h)^k$ is a finite complex (Lemma 1).

Assertion (4) now follows since the Hurewicz homomorphism is a $C^\infty$-isomorphism (Lemma 9).

Proof of Theorem 37. According to Lemma 8, $\Omega^n$ is a homomorphic image of $\Omega^{n+k}(\hat{T}(\hat{S}_h)^k)$. Therefore $\Omega^n$ is finitely generated and
$$\text{rank } \Omega^{4a} \leq \eta(a), \text{ rank } \Omega^n = 0 \text{ for } n \not\equiv 0 \text{ (mod 4)}.$$ But according to Theorem 35 Corollary 2:
$$\text{rank } \Omega^{4a} \geq \eta(a).$$

This completes the proof.

Now consider the tensor product of $\Omega^a$ with the rational numbers. The argument shows that the vector space $\Omega^{4a} \otimes \mathbb{Q}$ has rank $\eta(n)$, and also gives an explicit basis: namely the set of products.
This proves:

Corollary 1. The algebra $F^n \otimes Q$ has the structure of a polynomial algebra generated by the complex projective spaces $F^n(C)$, $i = 1, 2, \ldots$.

Corollary 2. If all of the Pontrjagin numbers of $N^n$ are zero, then some multiple $k F^n$, $k > 0$, is a boundary.

For otherwise there would be too many linearly independent elements of $F^n$.

Remark. Thom has made the following conjecture (unpublished).

If all the Pontrjagin numbers and the Stiefel-Whitney numbers of $N^n$ are zero, then $F^n$ is a boundary. This conjecture is supported by the fact that $F^n$ has no odd torsion (Milnor [13]), that is, if $N^n$ is not a boundary, no odd multiple bounds. (Note that Thom has proved a weaker statement if we ignore questions of orientation (p. 10): if all the Stiefel-Whitney numbers are zero, then $N^n$ is an (unoriented) boundary.)

XV The index theorem

The material in this chapter is due to Hirzebruch [7], [9].

1. Multiplicative sequences.

Let $\Lambda$ be a fixed commutative ring with unit. (In the main application $\Lambda$ will be the rational numbers.)

Review of Chapter XIII. The symbol $A^n$ will stand for a graded commutative $\Lambda$-algebra with unit. To each such $A^n$ corresponds a group $\Gamma^1 A^n$ with elements

$$a = (1, a_1, a_2, \ldots) = 1 + a_1 + a_2 + \ldots$$

For each $\omega \in \Pi(n)$ there is a polynomial

$$s_\omega : \Gamma^1 A^n \to A^n,$$

which satisfies the product formula of Theorem 33.

Consider a sequence of polynomials

$$K_1(x_1), K_2(x_1, x_2), K_3(x_1, x_2, x_3), \ldots$$

with coefficients in $\Lambda$ such that, if the variable $x_1$ is assigned degree 1:

(1) each $K_n$ is homogeneous of degree $n$.

Then given $A^n$ as above and given a $\in \Gamma^1 A^n$, define a new element $K(a) \in \Gamma^1 A^n$ by the formula

$$K(a) = (1, K_1(a_1), K_2(a_1, a_2), \ldots)$$

Definition: $\{K_n\}$ is a multiplicative sequence of polynomials if the identity

$$(2) \quad K(a \cdot b) = K(a) \cdot K(b)$$

is satisfied for all $A^n$ and all $a, b \in \Gamma^1 A^n$.

Examples. (I) Given any constant $\lambda \in \Lambda$, the polynomials

$$K_n(x_1, \ldots, x_n) = \lambda^n x_n$$

form a multiplicative sequence. The cases $\lambda = 1$ (identity map) and $\lambda = -1$ (compare Theorem 28 p. 77) are of particular interest.

II The identity $K(a) = a^{-1}$ defines a multiplicative sequence with

$$K_1(x_1) = -x_1, \quad K_2(x_1, x_2) = x_1^2 - x_2, \quad K_3(x_1, x_2, x_3) = x_1^3 + 2x_1 x_2 x_3,$$

and so on. These polynomials describe the relations between the Pontrjagin classes of the tangent bundle and normal bundle of a manifold in
Euclidean space.

(III) The polynomials $K_{2n+1} = 0$,

$$K_{2n}(x_1, \ldots, x_{2n}) = x_n^2 - 2x_{n-1}x_{n+1} - \cdots - 2x_2x_{2n-1} + 2x_{2n}$$

form a multiplicative sequence (compare p. 82).]

The following theorem gives a description of the set of all possible multiplicative sequences. Consider the polynomial ring $\Lambda[t]$, with degree $t = 1$. Then $\Gamma_1 \Lambda[t]$ is the set of all formal power series

$$f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots$$

with coefficients in $\Lambda$. In particular $1 + t$ is an element of $\Gamma_1 \Lambda[t]$.

Theorem 38 (Hirzebruch). Given a formal power series

$$f(t) \in \Gamma_1 \Lambda[t],$$

there is one and only one multiplicative sequence $(K_n)$ satisfying the condition

$$K(1 + t) = f(t).$$

(The condition $K(1 + t) = f(t)$ is equivalent to the condition that the coefficient of $x_1^n$ in each $K_n(x_1, \ldots, x_n)$ be $\lambda_n^n$.)

Definition. $(K_n)$ will be called the multiplicative sequence belonging to the power series $f(t)$.

[Examples. The three multiplicative sequences mentioned above belong to the power series $1 + \lambda_1 t$, $1 - t + t^2 = \cdots$, and $1 + t^2$ respectively.]

Remark. Suppose that $(K_n)$ belongs to $f(t)$. Then the identity

$$X(1 + a_1) = f(a_1)$$

holds for any $\Lambda$ and any $a_\Lambda \in \Lambda$. However this identity is no longer true if something of degree $\neq 1$ is substituted for $a_\Lambda$.

Proof of existence. Given

$$f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots$$

define

$$K_n(x_1, \ldots, x_n) = \sum_{\omega \in \Pi(n)} x_1^{\omega_1} \cdots x_n^{\omega_n} [x_1, \ldots, x_n],$$

where $\omega = (i_1, \ldots, i_n) \in \Pi(n)$. Introducing the abbreviation

$$\lambda_{i_1} \cdots \lambda_{i_n} = \lambda_{i_1} \cdots \lambda_{i_n},$$

this means that

$$K(a) = \sum_{\omega \in \Pi(n)} \lambda_{\omega_1} \cdots \lambda_{\omega_n}(a)$$

or

$$= \sum_{\omega} \lambda_\omega a_\omega(a)$$

where the summation is over all partitions of all the integers. Now

$$K(a'b) = \sum_{\omega} \lambda_\omega a_\omega(a'b) = \sum_{\omega} \lambda_{\omega_1} \lambda_{\omega_2} a_\omega(a) = \lambda_{i_1} \cdots \lambda_{i_n} a_{i_1} a_{i_2}(b)$$

where again the summations are over all partitions of all the integers. Hence

$$K(a'b) = K(a)K(b).$$

For the special case $a = 1 + t$, note that

$$a_\omega(1 + t) = \begin{cases} \omega k & \text{if } \omega = k \\ 0 & \text{if } \omega = i_1 \cdots i_r, r > 1 \end{cases}$$

Hence $K(1 + t) = 1 + \sum_{\omega} \lambda_\omega t^k = f(t)$, as required.

Proof of uniqueness. Consider the special case

$$\lambda = \lambda(t_1, \ldots, t_n)$$

where the $t_i$ are algebraically independent of degree 1, and

$$\sigma = (1 + t_1) \cdots (1 + t_n).$$
Then
\[ K(\sigma) = K(1+t) \ldots K(1 + t_n) = f(t) \ldots f(t_n). \]
Taking the homogeneous part of degree \( n \), it follows that
\[ K_n(\sigma_1, \ldots, \sigma_n) \]
is completely determined by the power series \( f(t) \).
Since \( \sigma_1, \ldots, \sigma_n \) are algebraically independent, this completes the proof.

Remark. Hirzebruch has given the following, more convenient, description of \( \{K_n\} \) in terms of \( f(t) \):
Assertion: The coefficient of \( x_1 \ldots x_r \) in \( K_n(x_1, \ldots, x_n) \) is equal to \( s_1 \ldots s_r (\lambda_1, \ldots, \lambda_n) \).

Comparing this with the uniqueness proof above, the following identity is obtained
\[ K_n(x_1, \ldots, x_n) = \sum \lambda_1 \ldots \lambda_n s_1 \ldots s_r (x_1, \ldots, x_n) \]
\[ = \sum s_1 \ldots s_r (\lambda_1, \ldots, \lambda_n) x_1 \ldots x_n. \]
This evidently expresses a symmetry property of the collection of polynomials \( s_\lambda \).

Definition. Given any multiplicative sequence \( \{s_\lambda(x_1, \ldots, x_r) \} \)
with rational coefficients define the \( n \)-genus \( K(M^n) \) of a (compact, oriented, differentiable) manifold to be zero if \( n \) is not divisible
by 4 and
\[ K_n(M^{4s}) = < K_n(p_1, \ldots, p_n), f_{4s} > \]
for \( n = 4s \), where the \( p_i \) denote the Pontrjagin classes of the tangent bundle.

Lemma 1. The correspondence \( M \to K(M) \) defines a ring homomorphism from the cobordism ring \( \Omega \) to the rational numbers. (Or an algebra homomorphism from \( \Omega \otimes \mathbb{Q} \) to \( \mathbb{Q} \).)

Proof. It is clear that this correspondence is additive, and
that the \( n \)-genus of a boundary is zero. For a product manifold
\( M \times M' \) with Pontrjagin class \( p \times p' \), we have
\[ K(p \times p') = K(p) \times K(p'), \]
hence
\[ < K(p \times p'), f_{4s} >= < K(p), f_{4s} > < K(p'), f_{4s} >, \]
which completes the proof.

Remark. The converse is not hard to prove. Any ring homomorphism \( \Omega \to \mathbb{Q} \) is given by the \( n \)-genus for some uniquely determined \( K \).

2. The index theorem

The index \( I \) of a manifold \( M^n \) is defined to be zero if \( n \) is not a multiple of 4, and as follows for \( n = 4s \). Choose a basis
\( \varphi_1, \ldots, \varphi_r \) for \( H^2s(M^{4s}; \mathbb{Q}) \) so that the symmetric matrix
\[ \| \varphi_\lambda \varphi_\mu \| = < \varphi_\lambda \varphi_\mu, f_{4s} > \]
is diagonal. Then \( I(M^{4s}) \) is the number of positive diagonal entries minus the number of negative ones (i.e., the signature of the quadratic form in the usual terminology). The following three properties will be needed:

1. \( I(M_1 + M_2) = I(M_1) + I(M_2) \),
2. \( I(M_1 \times M_2) = I(M_1) \cdot I(M_2) \)
(for the proof see Hirzebruch [9]), and
3. if \( M \) is a boundary then \( I(M) = 0 \).
(The proof, due to Thom [22], is based on the Poincaré duality theorem. In other words, \( I \) gives rise to a ring homomorphism from \( \Omega \) to the integers.)

Remark. Although these properties will be needed only for differentiable manifolds, they are true for much more general
(compact, oriented) manifolds.

Theorem 39 (Hirzebruch). Let \( L_\mu(p_1, \ldots, p_k) \) be the multiplicative sequence of polynomials belonging to the power series

\[
\frac{1}{\tanh \sqrt{t}} = 1 + \frac{1}{3} t - \frac{1}{45} t^2 + \ldots + (-1)^{k-1} \frac{2^k}{(2k)!} B_k t^k + \ldots
\]

Then the index \( I \) of any (compact, oriented, differentiable) manifold \( M^{4k} \) is equal to the L-genus \( L_\mu(M^{4k}) \).

(Here \( B_k \) denotes the \( k \)-th Bernoulli number: \( B_1 = 1/6, B_2 = 1/60, \ldots \). The first three polynomials are

\[
L_1 = \frac{1}{3} p_1, \quad L_2 = \frac{1}{45}(7p_2 - p_1^2), \quad L_3 = \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3), \ldots
\]

Proof. Since both \( I \) and \( L[ \cdot ] \) define algebra homomorphisms

\[
\mathbb{Z} \otimes \mathbb{Q} \rightarrow \mathbb{Q}
\]

it is sufficient to check this assertion on a set of generators for the algebra \( \mathbb{R}[\mathbb{P}^2] \otimes \mathbb{Q} \). According to Theorem 36 Corollary 1, such a set of generators is provided by the complex projective spaces \( \mathbb{P}^{2k}(\mathbb{C}) \). Since \( H^{2k}(\mathbb{P}^{2k}(\mathbb{C}); \mathbb{Q}) \) is generated by \( \alpha^k \), with \( \langle \alpha^k \cup \alpha^k, \mu_{4k} \rangle = +1 \), it follows that the index of \( \mathbb{P}^{2k}(\mathbb{C}) \) is \( +1 \).

Recall that the Pontrjagin class \( p \) of \( \mathbb{P}^{2k}(\mathbb{C}) \) is \( (1 + \alpha^2)^{2k+1} \).

(See p. 82.) We have

\[
L[p] = (\alpha/ \tanh \alpha)^{2k+1}.
\]

The Kronecker index \( < L(p), \mu_{4k} > \) is equal to the coefficient of \( \alpha^{2k} \) in this power series. Replacing \( \alpha \) by the complex variable \( z \), this coefficient can be evaluated by (1) dividing by \( 2i\pi \) \( (2k+1) \), and (2) integrating around the origin. But the substitution \( u = \tanh z \) shows that

\[
\frac{du}{u(1-u^2)} = \frac{dz}{(1+u^2)(2k+1)}(1+u^2+\ldots)du = 2\pi i.
\]

Therefore \( < L(p), \mu_{4k} > = +1 \), which completes the proof.

Corollary 1. The L-genus of any manifold is an integer.

The index \( I \) is an integer by definition. In other words, the Pontrjagin numbers of any manifold satisfy congruences:

\[
p_1[M^4] \equiv 0 \pmod{3},
\]

\[
7p_2[M^8] - p_1^2[M^8] \equiv 0 \pmod{45}, \text{ etc.}
\]

Corollary 2. The L-genus of a manifold is a homotopy type invariant of the oriented manifold, since \( I \) is a homotopy type invariant by definition. It is likely that the Pontrjagin numbers themselves are not homotopy type invariants. (The Pontrjagin classes are definitely not homotopy type invariants; see Chapter XVI § 5.)

Unsolved Problem: Is the L-genus the only linear combination of the Pontrjagin numbers which is a homotopy type invariant over the integers?

3. An axiomatic description of characteristic classes.

This section will sketch another application of multiplicative sequences, without giving detailed proofs. For further information see Wu[29] as well as [7].

Lemma 2. Let \( \Lambda \) be an integral domain containing \( \frac{1}{2} \). Then the cohomology algebra

\[
H^n(\mathbb{G}_n; \Lambda)
\]

of the real Grassmann space is a polynomial algebra generated by
As an example for any odd prime $q$, consider the reduced $q$-power operation
\[ \mathcal{O}^1: H^r(X; \mathbb{Z}_q) \to H^{r+2(q-1)}(X; \mathbb{Z}_q). \]

In analogy with Thom's definition of Stiefel-Whitney classes \((p,j)\), it is natural to define a characteristic class
\[ q_1(\zeta) \in H^{2(q-1)}(B(\zeta); \mathbb{Z}_q) \]
by the identity \[ q_1(\zeta) = g^{-1} \mathcal{O}^1 g(1). \]

Theorem (Wu) This characteristic class \( q_1(\zeta) \) is equal to \( k_{2q_1}(\zeta) \mod q \) where \( (K_j) \) is the multiplicative sequence
over \( \mathbb{Z}_q \) corresponding to the power series
\[ f(t) = 1 + t + \frac{t^2}{q} + \frac{t^3}{q^2} + \cdots \]
(Thus for \( q = 3 \), \( q_1(\zeta) = p_1(\zeta); \) and
for \( q = 5 \), \( q_1 = p_1^2 - 2p_1p_{1+1} + \cdots \pm 2p_{2+1} \).)

The proof is not difficult.

Remark. Just as in the mod 2 case it can be shown that \( q_1(\zeta) \)
for the tangent bundle \( \mathbb{T}^n \) of a manifold, is a homotopy type
invariant. (Compare Theorem 17 p. 55.) In fact
\[ q_1 = v_1 + \mathcal{O}^1 v_{1-1} + 2^2 v_{1-2} + \cdots \]
where \( v_j \) is characterized by the identity \( \langle \mathcal{O}^1 \mathbb{L}, \mathbb{L}^n \rangle = \langle \mathbb{L}, v_j \rangle \),
for all \( \mathcal{O} \in \text{Hom}-2j(q-1)(\mathbb{Z}_q) \). As indicated earlier, less is
known about the existence of linear combinations of the Pontrjagin
numbers which are homotopy type invariants with integral
coefficients.
For any triangulated manifold $X^n$ Thom [25] has defined classes $L_1 \in H^{41}(X^n, \mathbb{Q})$ which are combinatorial invariants. In the case of a differentiable manifold, suitably triangulated, these coincide with the Hirzebruch classes $L_1(p_1, \ldots, p_4)$ of the tangent bundle $T^n$ since the equations $L_1 = L_1(p_1, \ldots, p_4)$ can be uniquely solved for the Pontrjagin classes:

$$p_1 = 3 \lambda_1, \quad p_2 = \frac{1}{7} (45 \lambda_2 - 9 \lambda_3^2)$$

which follows that the rational Pontrjagin classes $p_1(T^n) \in H^{41}(N^n; \mathbb{Q})$ are combinatorial invariants. (This remark depends on the fact that the square of $p_k$ in $L_k$ is never zero. See Hirzebruch [9].)

Sections 1 to 4 will give a new version of Thom's construction. Section 5 will give two applications.

1. An example is given of two simply connected manifolds which long to the same homotopy type, but are not combinatorially equivalent.

2. An example is given of a combinatorial manifold which does not possess any differentiable structure compatible with the given combinatorial structure.

The differentiable case.

In order to motivate the definition we will first give a new interpretation for the classes $L_1(p_1, \ldots, p_4)$ in a differentiable manifold. The restriction $n \geq 81 + 2$ will be needed at first. Consider a differentiable map $f: N^n \to S^{n-41}$ of class $C^\infty$.

Lemma 1. For almost every point $y \in S^{n-41}$ the inverse image $f^{-1}(y)$ is a differentiable manifold $N^{41} \subset N^n$ having trivial normal bundle.

Here "almost every" means "except on a set of measure zero".

Proof: (Compare Chapter XIV 3.) It follows from the Theorem of Sard that almost every $y \in S^{n-41}$ is a regular value of $f$. But if $y$ is a regular value then $f^{-1}(y)$ is a manifold (Chapter XIV Lemma 31) and the normal bundle of $f^{-1}(y)$ maps into the tangent bundle of $S^{n-41}$ at $y$, hence is a product bundle.

Let $\sigma^k, \mu^k$ denote the standard generators of $H^k(S^k, \mathbb{Z}), H_n(N^n; \mathbb{Z})$ respectively. The class $L_1(p_1(T^n), \ldots, p_4(T^n)) \in H^{41}(N^n; \mathbb{Q})$ will be written as $L_1(T^n)$.

Theorem 32: For every differentiable map $f: N^n \to S^{n-41}$ and almost every $y \in S^{n-41}$ the Kronecker index $<L_1(T^n), \mu^k(\sigma^{n-41}), \mu^n>$ is equal to the index $\tilde{I}$ of the manifold $f^{-1}(y) = N^{41}$. In the case $n \geq 81 + 2$ the class $L_1(T^n)$ is completely characterized by this identity.

Proof: Let $\tau^{41}$ be the tangent bundle of $N^{41}$, and $j: N^{41} \to N^n$ the inclusion map. Then $j$ is covered by a bundle map $\tau^{41} \to j^{-1}(\tau^n) \to \tau^n$. Since the normal bundle $\nu^{n-41}$ is trivial, the $j$ means that $L_1(\tau^{41})$ is equal to $j^*L_1(T^n)$. Hence the index $I(\nu^{41}) = <L_1(T^n), \mu^n>$ is equal to $<L_1(T^n), \mu^k(\sigma^{n-41}), \mu^n>$.

The Poincaré dual of the homology class $j_*\mu^{41} \in H_{41}(N^n; \mathbb{Z})$.
A typical consequence is:

**Corollary 2.** If \( m \equiv 0 \pmod{3} \) then every element of the image \( \tilde{\varphi}^*: H^{m-8}(\mathbb{P}(C); \mathbb{Z}) \to H^{m-8}(\mathbb{P}(C); \mathbb{Z}) \) is divisible by 9.

2. **The combinatorial case**

The following will be a convenient class of objects to work with.

Let \( K \) be a finite simplicial complex.

**Definition:** \( K \) is a (compact, simplicial) (rational) homology \( n \)-manifold if the star boundary of each simplex has the rational homology groups of an \((n-1)\)-sphere. (This condition can also be put in the following topologically invariant form: for each point \( x \in X \) the local homology groups

\[ H_1(K; K - X; \mathbb{Q}) \]

should be zero for \( i \neq n \) and isomorphic to \( \mathbb{Q} \) for \( i = n \).)

Each component of such a complex \( K \) is clearly a simple \( n \)-circuit. (See Eilenberg and Steenrod. [4] p. 106.) Hence it makes sense to require that \( K \) be oriented. The orientation may be specified by an element

\[ \mu \in H_n(K; \mathbb{Z}). \]

Similarly, one can define the concept of a "bounded homology \( n \)-manifold" \( K \). In this case the boundary \( K \) is a homology \((n-1)\)-manifold, and the orientation is specified by \( \mu \in H_n(K; \mathbb{Z}; \mathbb{Z}). \)

Let \( \{ r \} \) denote the boundary of an \((r+1)\)-simplex. The key lemma will be the following.

**Lemma 2.** Let \( f \) be a piecewise linear map from a homology \( n \)-manifold \( K \) to \( \{ r \} \), \( r = n - j \). Then:
(1) For almost every \( y \in \Sigma^F \) the inverse image \( f^{-1}(y) \) is a homology \( n \)-manifold. Given orientations for \( K \) and \( \Sigma^F \), there is an induced orientation for \( f^{-1}(y) \).

(2) The index \( I_f^{-1}(y) \) is independent of \( y \) for almost all \( y \), note this constant value by \( I(f) \). Finally:

(3) The integer \( I(f) \) depends only on the homotopy class of \( f \).

Here "almost every" can be taken to mean "except for \( y \) belonging to some lower dimensional complex".

(Remark: There is some analogy between this definition of \( I(f) \), and the definition of the Hopf invariant.)

The proof will be based on:

Lemma 3: Let \( f: K \to L \) be a simplicial map and let \( y \) belong to the interior \( \Delta \) of a simplex of \( L \). Then \( f^{-1}(\Delta) \) is homeomorphic to \( \Delta \times f^{-1}(y) \).

(This assertion would be false for a closed simplex.)

Proof. Let \( A_0, \ldots, A_r \) denote the vertices of \( \Delta \) and let \( y = t_0 A_0 + \ldots + t_r A_r \) where \( t_i > 0 \), \( \sum t_i = 1 \). Then any \( x \in f^{-1}(\Delta) \) can be expressed uniquely as \( x = t_0' A_0' + \ldots + t_r' A_r' \) with \( A_i', \in f^{-1}(A_i) \) being points of the boundary of the simplex of \( K \) to which \( x \) belongs. The required homeomorphism

\[ f^{-1}(\Delta) \cong \Delta \times f^{-1}(y) \]

is now defined by

\[ x \mapsto (t_0' A_0 + \ldots + t_r' A_r, t_0 A_0' + \ldots + t_r A_r'). \]

Remark: Note that the composition

\[ f^{-1}(\Delta) \to \Delta \times f^{-1}(y) \to \Delta \]

is just the original map \( f \) (restricted to \( f^{-1}(\Delta) \)). Hence \( f^{-1}(y') \) is homeomorphic to \( f^{-1}(y) \) for all \( y' \in \Delta \).

Proof of Lemma 2: (1). Subdivide \( K \) and \( \Sigma^F \) so that \( f \) is simplicial. Assume that \( y \) belongs to the interior \( \Delta \) of a top dimensional simplex. Then by Lemma 3, \( \Delta \times f^{-1}(y) \) has the local homology groups of an \( n \)-manifold. Since \( \Delta \) has the local homology groups of an \((n-1)\)-manifold, it follows easily that \( f^{-1}(y) \) is a homology \( n \)-manifold. Furthermore, given orientations for \( \Delta \) and \( \Delta \times f^{-1}(y) \), there is clearly an induced orientation for \( f^{-1}(y) \).

The remark above stated that \( f^{-1}(y') \) is homeomorphic to \( f^{-1}(y) \) for all \( y' \in \Delta \). Therefore:

(4) If \( y \) is chosen as above, then the index \( I f^{-1}(y') \) is independent of \( y' \) at least for \( y' \) in a neighborhood of \( y \).

Now suppose that \( f \) and \( g \) are homotopic piecewise linear maps \( K \to \Sigma^1 \). Then they are related by a piecewise linear homotopy \( h: K \times [0,1] \to \Sigma^F \).

Subdividing and choosing \( y \in \Delta \) as above, a similar argument shows that \( h^{-1}(y) \) is a bounded homology manifold with boundary \( f^{-1}(y) - g^{-1}(y) \). Since the index of a boundary is zero, this implies that

(5) If \( f \) is homotopic to \( g \), then \( I f^{-1}(y) = I g^{-1}(y) \) for almost all \( y \).

Given \( f: K^N \to \Sigma^F \) let \( y_1 \) and \( y_2 \) be any two points satisfying (4) above. Let

\[ u: \Sigma^F \to \Sigma^F, \]

(4) above.
a piecewise linear homeomorphism of degree +1 carrying \( y_1 \) into \( y_2 \).

Since \( uf \) is homotopic to \( f \), hence

\[
I(f^{-1}u^{-1}z) = I(f^{-1}z)
\]

for almost all \( z \). Choosing \( z \) close to \( y_2 \), so that \( u^{-1}z \) is close to \( y_1 \), it follows from (4) that

\[
I(f^{-1}(y_1)) = I(f^{-1}(y_2))
\]

This proves assertion (2). Since (3) now follows from (5), this completes the proof.

**Lemma 4.** If \( n \geq 61 + 2 \) then the correspondence \( f \to I(f) \) defines a homomorphism from the cohomotopy group \( \pi_n^{\text{cohom}}(K^n) \) to the integers.

**Proof.** It follows from the definition of addition in the cohomotopy group that \( (f + g)^{-1}(y) \) is the disjoint union of \( f^{-1}(y) \) and \( g^{-1}(y) \), providing \( f \) and \( g \) are chosen carefully within their cohomotopy classes, hence \( I(f + g) = I(f) + I(g) \).

The main theorem will now follow easily:

**Theorem 41.** For \( n \geq 61 + 2 \) there exists a unique cohomology class

\[
\lambda \in H^{41}(K^n; \mathbb{Q})
\]

such that the identity

\[
\langle \lambda, \omega \rangle = I(f)
\]

is satisfied for every map \( f: K^n \to \Sigma K^{n-1} \).

(Here \( \sigma \) denotes the standard generator of \( H^{n-41}(\Sigma K^{n-1}; \mathbb{Z}) \).)

**Proof:** Consider the diagram

\[
\begin{array}{c}
\pi^{n-41}(K^n) \xrightarrow{I} \mathbb{Z} \\
\downarrow \\
\pi^{n-41}(K^n; \mathbb{Z}) \xrightarrow{\text{inclusion}} \mathbb{Q}.
\end{array}
\]

Since \( \beta^n \) is a \( C^1 \)-isomorphism (Chapter XIV Lemma 10), the bottom arrow can be filled in uniquely. By the Poincaré duality theorem, the resulting homomorphism \( I^1 \) is given by the formula

\[
\beta \mapsto \langle \lambda, \omega \rangle
\]

for some unique \( \lambda \in H^{41}(K^n; \mathbb{Q}) \). This completes the proof.

3. The compatibility theorem \( \nu_1(N^n) = L_1(T^n) \).

Now it is necessary to compare the combinatorial and differentiable situations.

By a triangulation of a space \( X \) is meant a homomorphism of a simplicial complex onto the space \( X \). J. H. C. Whitehead has shown that a differentiable manifold \( M^n \) has a preferred class of triangulations \( t: K \to M^n \)

which are called \( C^1 \)-triangulations. (See [27], [12].) The complex \( K \) which occurs is uniquely determined, up to combinatorial equivalence ( = piecewise linear homeomorphism). Hence the class \( (t^n)^{-1} \langle \lambda, \omega \rangle \in H^{41}(M^n; \mathbb{Q}) \) does not depend on the particular \( C^1 \)-triangulation \( t \) which is chosen. This class will be denoted by \( \nu_1(M^n) \). (It is of course defined only for \( n \geq 61 + 2 \).)
Theorem 42. The class $\mathcal{C}_1(K^n)$ (defined for a differentiable manifold by a combinatorial procedure) is equal to the Hirzebruch class $L_1$ of the tangent bundle.

Proof: Let $f : M^n \to S^p$ be a differentiable map. We will construct a diagram

$$
\begin{array}{ccc}
K & \xrightarrow{t} & K^n \\
\downarrow f_1 & & \downarrow f \\
K_1 & \xrightarrow{t_1} & S^p
\end{array}
$$

commutative up to homotopy, where $t$ and $t_1$ are $C^1$-triangulations, so that

$$I f_1^{-1}(y_1) = I f^{-1}(y)$$

for almost all $y_1 \in K_1$, $y \in S^p$. Together with Theorems 40 and 41, this will complete the proof.

Let $y \in S^p$ be any regular value of $f$. If $B$ is a sufficiently small ball around $y$, then the inverse image $f^{-1}(B)$ can be considered as the product $B \times f^{-1}(y)$. Choose a $C^1$-triangulation $t_1 : K_1 \to S^p$ so that some subcomplex $K_2$ of $K_1$ triangulates $B$; and choose a $C^1$-triangulation $K_3 \to f^{-1}(y)$. Then the product triangulation $K_2 \times K_3 \to B \times f^{-1}(y)$ can be extended to a triangulation $t : K \to K^n$.

The composition $t_1^{-1}f \circ t : K \to K_1$ will not, in general, be piecewise linear. However its restriction to $K_2 \times K_3$ is just the projection map into $K_2 \setminus K_1$, hence is piecewise linear. Choose a piecewise linear map $f_1 : K \to K_1$ which agrees with $t_1^{-1}f \circ t$ on $K_2 \times K_3$, and approximates it elsewhere. Let $y_1$ denote $f_1^{-1}(y)$. Then $f_1^{-1}(y_1)$ will be homeomorphic to $f(y)$. Hence

$$I f_1^{-1}(y_1') = I f_1^{-1}(y_1) = I f^{-1}(y) = I f_1^{-1}(y)$$

for all $y_1'$, $y'$ close to $y_1$ and $y$ respectively.

This completes the proof.

4. The unrestricted case:

So far the condition $n \geq 81 + 2$ has been imposed. However given $K^n$ one can always form the product space $K^n \times \Sigma^m$ with $m$ large. The class $\mathcal{C}_1(K^n)$ can then be defined as the class induced from $\mathcal{C}_1(K^n \times \Sigma^m)$ be the natural inclusion map. It is not hard to show that this new class is well defined, and has the expected properties (For example, $\langle f_1^{-1}(K^n), f_1^{-1}(K^n) \rangle = I(K^n)$.)

Another extension which can easily be made is to bounded homology manifolds. It is only necessary to substitute the relative cohomotopy groups

$$\eta^{-1}(K^n, K^n)$$

and the Lefschetz duality theorem in the above discussion.

5. Applications

The first example which will be discussed was discovered independently by Thom (24) p. 61, Tamura (21), and Shimada (17).

Lemma 5: Given integers $m$ and $n$ with $n \geq 4$, there exists an $n$-plane bundle $\xi$ over $S^4$ with $p_1(\xi) = 2m\sigma$.

($\sigma$ = standard generator of $H^4(S^4; Z)$.)
Remark: A corresponding assertion for any sphere $S^{2k}$ has recently been proved by Borel, Hirzebruch and Bott. The integer $2k$ must be replaced by

$$(2k-1)! \in \mathbb{C}, \mathbb{D}. \ (k+1, 2).$$

This is a best possible result.

Proof of Lemma 5: First assume that $n \geq 6$. Let $T^8$ denote the tangent bundle of the Quaternion projective plane $p^2(K)$, and let $u$ generate $H^4(p^2(K); Z) \neq Z$. According to Hirzebruch [8], the Pontrjagin class of $T^8$ is $1 + 2u + 7u^2$. Since this space is $3$-connected, there exists a map

$$f: S^4 \to p^2(K)$$

satisfying $f^*(u) = m \sigma$. The induced bundle $\xi^8$ over $S^4$ will now satisfy Lemma 5 for $n = 6$. For $n > 8$, the Whitney sum

$$\xi^8 \oplus \text{trivial (n-3)-plane bundle}$$

will satisfy the lemma.

Next consider the case $n=7$. Using obstruction theory, it is seen that $\xi^8$ has a non-zero cross-section. In other words $\xi^8$ is the Whitney sum of the required 7-plane bundle $\xi^7$ and a trivial line bundle. This construction can be iterated until the case $n=4$ is reached, which completes the proof.

(The obstruction to further iteration is $\chi(\xi^4) \in H^4(S^4; Z)$. This is definitely non-zero since $u_2(\xi^8) \neq 0$. See Theorem 18, p. 56.)

Lemma 6: Let $\xi^n$ be a differentiable $n$-plane bundle over $B$ and let $T^n$ be the tangent bundle of $B$. Then the Pontrjagin class of the total space $E$ of $\xi^n$ is given by

$$p(E) = n^*(p(\xi^n)p(\Gamma^n)).$$

Similarly, if $E'_0$ is the set of unit vectors in $E$, then

$$p(E'_0) = \eta_0^*(p(\xi^n)p(\Gamma^n)).$$

Proof: The tangent bundle of $E$ is the Whitney sum of

(1) the bundle of vectors tangent to the fibre, and

(2) the bundle of vectors normal to the fibre.

Since (1) maps into $\xi^n$ and (2) maps into $\varpi^n$, the first assertion is clear. Since the normal bundle of $E'_0$ in $B$ is trivial, the second assertion follows.

Example 1: Consider an $n$-plane bundle $\xi$ over $S^4$ where (for convenience) $n \geq 6$. Then it follows from the Gysin sequence that the homomorphism

$$H^n_0(H^4(S^4; Z) \to H^4(E'_0; Z)$$

is an isomorphism. If $p(\xi) = 2m \sigma$, it follows that

$$p_1(E'_0) = 2mH_0^*(\sigma)$$

since $p(\varpi^n(S^n)) = 1 \ (p. 79.)$.

Since the Pontrjagin class of $E'_0$ is a combinatorial invariant, it follows that the integer $|m|$ is a combinatorial invariant of $E'_0$.

Thus as $m$ varies we obtain infinitely many manifolds which are combinatorially distinct.

On the other hand, according to James and Whitehead [10], these manifolds $E'_0$, for fixed $n$, fall into a finite number of distinct homotopy types (namely 13). This proves:

Assertion: There exist two differentiable simply connected $n$-manifolds which have the same homotopy type, but are not combinatorially equivalent.
(The dimension 9 can easily be improved to 7.) It is not known whether these manifolds are homeomorphic. In dimension 3 there do exist manifolds which have the same homotopy type but are not homeomorphic, although the proof in that case hinges on the fundamental group.

The next example is due to Thom [25]. (See also Milnor [11, 12] and Shinada [17].)

**Lemma 7:** Given integers \( i, j \) satisfying \( i \equiv 2j \mod 4 \), there exists an oriented 4-plane bundle \( \eta \) over \( S^4 \) having characteristic classes

\[
P_1(\eta) = i \sigma , \quad X(\eta) = j \sigma .
\]

**Remark:** The integers \( i, j \) actually determine the equivalence class of the bundle. This is due to the fact that the homotopy group \( \pi_1(G_2) \cong \pi_5(SO_2) \) is \( \mathbb{Z} \times \mathbb{Z} \).

**Proof of Lemma 7:** First let \( \eta \) range over all oriented 4-plane bundles over \( S^4 \). Observe that the corresponding set of pairs \( p_1(\eta), X(\eta) \) forms a subgroup of the direct sum

\[
\mathbb{H}(S^4; \mathbb{Z}) \oplus \mathbb{H}(S^4; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.
\]

In fact each such bundle \( \eta_1 \) is induced by some map \( f_1 : S^4 \to G_2 \). Given two such maps \( f_1, f_2 \) form the sum or difference \( f_1 \pm f_2 \) as defined in homotopy theory, and let \( \eta \) be the bundle induced by \( f_1 \pm f_2 \). Then it can be seen that the characteristic classes of \( \eta \) are:

\[
P_1(\eta) = p_1(\eta_1) \pm p_1(\eta_2) , \quad X(\eta) = \lambda(\eta_1) \pm \lambda(\eta_2).
\]

Now consider the following two bundles: (1) The tangent bundle \( \xi^4 \) of \( S^4 \), which satisfies

\[
P_1(\xi^4) = 0 , \quad X(\xi^4) = 2 \sigma . \quad \text{(see p. 79.)}
\]

(2) In Lemma 5 take \( n = 4, m = 1 \). The resulting bundle \( \xi^4 \) will satisfy

\[
P_1(\xi^4) = 2 \sigma , \quad \mathbb{w}_4(\xi^4) \neq 0,
\]

since \( \mathbb{w}_4(S^4 ; \mathbb{Z}) \neq 0 \) [See Theorem 13, p. 56] and hence

\[
\lambda(\xi^4) = \text{some odd multiple of } \sigma .
\]

But starting with the pairs \( (0, 2), (2, 2r+1) \) and forming sums and differences, one can obtain any pair \( (i, j) \) which satisfies

\[
i \equiv 2j \mod 4.
\]

This completes the proof.

**Example 2:** In particular consider the bundles for which \( j = 1 \) (that is \( \lambda = \sigma \)). Then \( \lambda \) can be any integer congruent to 2 modulo 4. Using the Gysin sequence, it is seen that the corresponding total spaces \( E^4 \) all have the homotopy type of the 7-sphere. Actually each such \( E^4 \) is homeomorphic to \( S^7 \) (see [11]); and even combinatorially equivalent to \( S^7 \) (see [12]). Now consider the Thom space \( T \) of such a bundle. \( T \) can be formed from the space \( E^4 \) of vectors of length \( \leq 1 \) by attaching a cone over the boundary \( E^3 \). Since \( E^4 \) is a 7-sphere it follows that \( E^4 \) is a compact 8-manifold. Furthermore any \( C^1 \)-triangulation (in the sense of Whitehead) of \( E^4 \) gives rise to a triangulation of \( T \).

It follows from Lemma 2 of Chapter XIV that

\[
H^k(T; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{for } k = 0, 4, 8 \\
0 & \text{otherwise}
\end{cases}
\]

Furthermore it is easily verified that the homomorphisms

\[
H^k(S^4; \mathbb{Z}) \xrightarrow{\mathbb{T}^1} H^k(E^4; \mathbb{Z}) \leftarrow H^k(T; \mathbb{Z})
\]

are isomorphisms. Let \( \sigma^1, \sigma^6 \) be the elements in the second two groups.
corresponding to the standard generator $c$. By the Poincaré duality theorem, $\langle c^i, c^j \rangle_B$ must be $\pm 1$. Hence, choosing the orientation $/B$ properly, the index $I(T)$ is $+1$.

According to Lemma 6, the Pontrjagin class $p_1$ of the bounded manifold $\Sigma'$ is $10^{-1}$. Hence the combinatorial Pontrjagin classes $p_1(E')$ and $p_1(T)$ are the rational classes corresponding to $10^{-1}$ and $1$ respectively. Therefore the Pontrjagin number $p_1^2(T)$ is equal to $1^2$.

Using the index theorem

$$I(T) = \frac{7}{45} p_2[T] - \frac{1}{45} p_1^2[T]$$

it follows that the other Pontrjagin number is given by

$$p_2^2[T] = \frac{45 + 12}{7}.$$ 

But in general this is not an integer. (e.g. for $i = 6$.) Since a Pontrjagin number of a differentiable manifold must be an integer, this implies

Assertion: For $1 \neq 2 \pmod{7}$ the triangulated 8-manifold $T$ possesses no differentiable structure which is compatible with the given triangulation.

As a corollary, it follows that the differentiable 7-manifold $E_0'$ is not diffeomorphic to $S^7$. For otherwise $T$ could be given a differentiable structure which was compatible.

In conclusion, here is an

**Unsolved Problem.** Let $\Lambda_n^R$ or $\Lambda_n^C$ denote the analogues of the cobordism group in which homology n-manifolds (or combinatorial ( = formal) n-manifolds) are used in place of differentiable n-manifolds. What is the structure of these groups, and what can be said about the natural homomorphism $\Lambda_n^R \rightarrow O_n^R \rightarrow I_n^R$? It should be noted that in the combinatorial case, Pontrjagin numbers are not invariants of homotopy type (cf. p. 118) for it can be shown that among the 8-manifolds above for which $p_2[T] = \frac{45 + 12}{7}$ (for any $i \equiv 2(4)$), there are only finitely many homotopy types.

**Appendix: The Thom isomorphism $\Theta$**

Let $\xi$ be an oriented n-plane bundle with projection $\nu: E \rightarrow B$. This Appendix will give a proof that the cohomology group $H^{n+1}(E; E_0; \Lambda)$ is isomorphic to $H^4(B; \Lambda)$. (See Theorem 10, p. 40.) A corresponding theorem for homology groups is included as part of the proof. The corresponding proof for the unoriented case, with coefficient group $Z_2$, is left to the reader.

1. Construction of the cohomology class $u$.

Let $SE$ denote the total singular complex of $E$, and define the relative Eilenberg subcomplex $S_n(E; E_0)$ as the set of all singular simplexes

$$f: \Delta^n \rightarrow E$$

such that $f$ maps the $(n-1)$-skeleton of $\Lambda^n$ into $E_0$. Then the following assertion will be proved.

**Lemma 1:** The inclusion $S_n(E; E_0) \rightarrow SE$ induces isomorphisms of homology groups.

Next a canonical cocycle $d \in Z^n(S_n(E; E_0), SE)$ will be defined. Intuitively speaking, $d(f)$ can be considered as the intersection number of the image $f(\Delta^n)$ with $B = E - E_0$. (Note that every n-simplex $f$ in $S_n(E; E_0)$ maps the boundary of $\Delta^n$ into $E_0$.)
Finally, $u \in H^*(\mathcal{E}, \mathcal{E}_0)$ will be defined as the cohomology class determined by $d$.

**Lemma 2.** The correspondence $\zeta \mapsto u(\zeta)$ is natural with respect to bundle maps. For the special case $\mathcal{B} = \text{point}, \mathcal{E} = \mathbb{R}^n$, $u$ is the standard generator of $H^*(\mathbb{R}^n, \mathbb{R}_0^n)$.

The proofs will be based on the following construction. Given any map

$$f: \Delta^r \to \mathcal{E}$$

let $f_\gamma$ denote the bundle over $\Delta^r$ induced from $\zeta$ by the composition $\gamma f$, and let $f'$ be the unique cross-section of $\gamma$, such that the diagram

$$\begin{array}{ccc}
E(\gamma) & \longrightarrow & \mathcal{E} \\
\downarrow f' & & \downarrow f \\
\Delta^r & \longrightarrow & \Delta^r
\end{array}$$

is commutative. Note that $\gamma$ is necessarily a product bundle. (See Steenrod [20], 11.6.) Hence the pair $E(\gamma), E_0(\gamma)$ is $(n-1)$-connected and $H_n(E(\gamma), E_0(\gamma))$ is infinite cyclic with a preferred generator.

**Proof of Lemma 1:** First observe that the pair $\mathcal{E}, \mathcal{E}_0$ is $(n-1)$-connected, since in fact the above argument shows that any map $f_\gamma: \Delta^r \to \mathcal{E}, \mathcal{E}_0$ can be factored through a pair $E(\gamma), E_0(\gamma)$ which is $(n-1)$-connected.

But now Lemma 1 can be proved by an argument completely analogous to that given by Eilenberg [2] Chapter VI.

**Definition of $d(f)$:** Any $n$-simplex $f$ of $S_n(\mathcal{E}; \mathcal{E}_0)$ gives rise to a map

$$f_1: \Delta^n, \Delta^n \to \mathcal{E}, \mathcal{E}_0.$$

Define $d(f)$ as the degree of the associated map

$$f_1': \Delta^n, \Delta^n \to E(\gamma), E_0(\gamma).$$

This defines a cochain $d \in c^*(S_n(\mathcal{E}; \mathcal{E}_0), \mathcal{E}_0).$

It is easily verified that the coboundary of $d$ is zero.

The proof of Lemma 2 will be left as an exercise.

2. **The homology isomorphism.**

Recall that the cap product of a singular $n$-cochain $c$ with a singular $(n+1)$-simplex is defined to be the product of the "front $i$-face" of the simplex sixth the integer obtained by evaluating $c$ on the "back $n$-face". The following properties will be needed.

1. $\cap(c \cap a) = c \cap \bar{a} + (-1)^i(b \cap a)$,
2. $< c \cup c_0, a > = < c_1, c_0 \cap a >$, and
3. the cap product gives rise to a bilinear pairing $H^n(X, Y) \times H_{n+1}(X, Y) \to H_1(X)$.

**Lemma 3.** The correspondence

$$a \mapsto \pi(u \cup a)$$

defines an isomorphism $\phi$ of $H_{n+1}(\mathcal{E}, \mathcal{E}_0; Z)$ onto $H_1(\mathcal{B}; Z)$.

The proof will be divided into four cases.

**Case 1:** $\xi$ is a product bundle so that $(\xi, \xi_0) = \mathcal{B} \times (\mathbb{R}^n, \mathbb{R}_0^n)$.

Let $\mu$ denote the standard generator of $H_0(\mathbb{R}^n, \mathbb{R}_0^n)$. It follows from the Künneth Theorem that the correspondence

$$a \mapsto a \times \mu,$$
defines an isomorphism of \( H_1(B) \) onto \( H_{n+1}(E, E_0) \). (See Eilenberg and Zilber [5] together with Eilenberg and Cartan [3], p. 113.) But it allows from Lemma 2, together with a short computation, that \( \phi(a \lambda) = \ast(u \lambda (u \lambda)) \) is equal to \( a \).

**Case 2:** \( B \) is the union of open subsets \( B', B'' \) with intersection 
"where the Lemma is known to be true for \( \zeta \) restricted to \( B', B'' \)
is \( B'' \). It will be shown that the following diagram of Mayer Vitosis sequences is commutative:
\[
\begin{array}{c}
H_{n+1}(E', E_0') \to H_{n+1}(E_0', E_0) + H_{n+1}(E_0', E_0') \to H_{n+1}(E_1, E_0') \\
\downarrow \phi' \downarrow \phi' + \phi'' \downarrow \phi \\
H_1(B') \oplus H_1(B') \to H_1(B) \\
\end{array}
\]

Once \( \phi' \), \( \phi'' \), and \( \phi''' \) are known to be isomorphisms, it will follow
from the Five Lemma ([4], p. 16) that \( \phi \) is an isomorphism.

The Mayer-Vietoris sequence can be derived as follows: The natural homomorphism of singular chain groups
\[
C_*(B') \to C_*(B'') \to C_*(B)
\]
is kernel isomorphic to \( C_*(B''') \) and an image \( C_*(B(B', B'')) \) which is again equivalent to \( C_*(B) \). (See [4], p. 197.) Now the short exact sequence
\[
0 \to C_*(B'') \to C_*(B') \to C_*(B'; B', B'') \to 0
\]
lives rise to the required sequence of homology groups. Similarly
is a short exact sequence

\[
0 \to C_*(E''', E_0'') \to C_*(E', E_0') + C_*(E''', E_0'') \to C_*(E', E_0'; E', E'') \to 0
\]
gives rise to the relative Mayer-Vietoris sequence.

Choose a representative cocycle \( z \in \mathcal{Z}_n(E, E_0'; E', E'') \) for the class \( u \), and let \( z', z'', z''' \) be the appropriate restrictions. Then
a chain mapping from the sequence (2) to the sequence (1) is defined by the formulas
\[
a'\to \tau_\#(z''', a''), \ldots, a \to \tau_\#(z' \wedge a).
\]

This chain mapping induces the required homomorphism between the Mayer-Vietoris sequences.

**Case 3:** \( B \) is the union of finitely many distinguished open sets \( V_1, \ldots, V_k \). For \( k = 1 \), the assertion follows from Case 1. For \( k \geq 1 \), it follows by induction, applying Case 2 to the pair
\[
B' = V_1 \cup \cdots \cup V_{k-1}, \quad B'' = V_k.
\]
Note in particular that this argument applies whenever \( B \) is compact.

**General case:** Let \( B^* \) range over all compact subsets of \( B \). Then \( H_1(B) \) is the direct limit of the groups \( H_1(B^*) \); and \( H_{n+1}(E, E_0) \) is the direct limit of the corresponding groups \( H_{n+1}(E^*, E_0^*) \). Since the assertion is true for each \( B^* \), this completes the proof.

**Remark:** The arguments given for cases 2 and 3 would apply equally well to cohomology, however the limiting argument does not apply to cohomology.

3. The cohomology isomorphism.

Consider the homomorphism of Lemma 1 on the chain level, That
is choose a cocycle $z \in Z^n(E, E_0)$ which represents $u$, and define

$$\varphi: C_{n+1}(E, E_0) \to C_1(B)$$

by $\varphi_z(a) = \int (z \wedge a)$. It is easily verified that $\varphi$ is onto. Let $C_1(K)$ denote the kernel of $\varphi$. Then there is a exact sequence

$$\cdots \to H_1(K) \to H_{n+1}(E, E_0) \to H_1(B) \to \cdots$$

It follows from Lemma 3 that the chain complex $K$ has trivial homology.

Now for any coefficient group $\Lambda$ consider the corresponding cohomology sequence

$$\cdots \to H^1(B; \Lambda) \to H^{n+1}(E, E_0; \Lambda) \to H^1(K; \Lambda) \to \cdots$$

It follows from the universal coefficient theorem that $K$ has trivial cohomology, so that $\varphi^*$ is an isomorphism.

The identities

$$\langle \varphi^*(a), b \rangle = \langle a, \varphi^*(b) \rangle = \langle a, \varphi^*(\int (z \wedge a)) \rangle = \langle \varphi^*(\int (z \wedge a)), b \rangle$$

show that $\varphi^*$ is just the correspondence

$$\gamma \mapsto \varphi^*(\gamma) \cup u.$$

Thus we have proved

**Lemma 4.** An isomorphism

$$\varphi: H^1(B; \Lambda) \to H^{n+1}(E, E_0; \Lambda)$$

is given by the correspondence

$$\gamma \mapsto \varphi^*(\gamma) \cup u.$$

Now to complete the proof of Theorem 10, (p. 40), it is only necessary to show that $u$ is characterized by the condition

$$j_b u = u_b \in H^n(\Omega^{-1}b, \mathbb{H}^{-1}b \wedge E_0)$$

for each $b \in B$. But applying the isomorphism $\varphi^{-1}$, this is equivalent to showing that the element

$$1 \in H^0(B)$$

is characterized by the fact that its Kronecker index with each point $b$ is $1$. Since this is clear, this completes the proof.

**References**


12. ________, On the relationship between differentiable manifold and combinatorial manifolds (mimeographed), Princeton University 1956.

13. ________, On the cobordism ring $\Omega^*$, to appear.


