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Abstract

The Real Connective K-theory of Brown-Gritler Spectra

Let $B(k)$ denote the spectrum representing real connective $K$-theory.

It's the objective of this paper to compute $B_{k}(\Sigma^{2}S^{3})$ using the classical Adams spectral sequence:

$$\text{Ext}_A^{*,*}(b_{0}A, \Sigma^{2}S^{3}), \mathbb{Z}/2) = E_2 \Rightarrow \pi^{*}_A(b_{0}A, \Sigma^{2}S^{3}),$$

and $A$ is the Steenrod algebra where $\pi^{*}_A(b_{0}A, \Sigma^{2}S^{3}) = b_{0}A(\Sigma^{2}S^{3})$ by definition. Here be is the spectrum representing real connective $K$-Theory.

Since we are computing stable homotopy, we may regard $\Sigma^{2}S^{3}$ as a stable complex and as such, according to Smale, $\Sigma^{2}S^{3}$ where $[\cdot]$ is the greatest integer function decomposes into a wedge of spectra $B(\lceil \frac{n}{2} \rceil)$ for $n > 0$ called Brown-Gritler spectra. This gives us a firm grip on the cohomology of $\Sigma^{2}S^{3}$, since the $B(k)$'s are known to be certain cyclic $A$-modules.

Better yet the $B(k)$'s can be amplified so as to fit nicely into some computational frameworks set up by Adams and Mahowald.
The information so garnered on the modified $R(k)$'s can be lifted back to the $R(k)$'s using some exact sequences and a bit of clever algebra suggested by Ravenel. This will complete the computation.
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If \( X \) is a non-degenerate, path-connected, compactly generated and Hausdorff space then \( S^3 \times S^3 \rightarrow X \) has the weak homotopy type of \( X \) labelled by points in \( \mathbb{R}^2 \). To be more precise, let \( C(\mathbb{R}^2, X) \) be the equivalence classes of pairs \((S,f)\) where \( S \) is a finite subset of \( \mathbb{R}^2 \) and \( f: S \rightarrow X \) is a function with the equivalence relation being given by \((S,f) \sim (S-x, f) \) if \( f(x) = x_0 \), the basepoint in \( X \). We can filter \( C(\mathbb{R}^2, X) \) by putting \( F_k C(\mathbb{R}^2, X) \) equal to the subspace of points represented by \((S,f)\) where the cardinality of \( S \) is at most \( k \).

According to Smith, \( S^3 \times S^3 \rightarrow X \) is a stable complex where \( F_k = F_k C(\mathbb{R}^2, X) \). The most legible account of this result is outlined by C. F. Cohen [16]. This immediately reduces the problem of computing \( H_k(X, S^3) \) to computing \( H_k(F_k/F_{k-1}) \). I now state a result of Brown and Gitler [4].

**Definition:** There exist spectra \( B(n) \) such that

\[
H^*(B(n), \mathbb{Z}/2) \cong \bigoplus_{n \geq 0} \bigoplus_{i \geq 0} X(S^{n+i})/\mathbb{Z}/2 = M(n)
\]

Such spectra are called the Brown–Gitler spectra. I'll call \( M(n) \) the \( n \text{th} \) Brown–Gitler module.

\( M(n) \) is not defined by construction.
Theorem (Mahowald [8]) \( H^* \left( \mathbb{F}_{n,n-1} \right) \cong M \left( \mathbb{F} \left( \frac{n}{2} \right) \right) \)
as left \( A \)-modules with an appropriate dimension shift.

It has been shown by Peterson et al. [5] that in fact \( \mathbb{F}_{n,n-1} \) is a realization of the spectra \( B(n) \). It's also worth noting the somewhat unexpected fact that the \( B(n) \)'s arise as Thom spectra over the filtration of \( \Omega^{2^k-3} \) according to Mahowald [8].

Note that \( A \left( \frac{n}{2} \right) X(s^k) \mid c > n^2 \cong A \left( \frac{n}{2} \right) X(s^k) \mid c > n^3 \)

Set \( M_1(n) = A(n) X(s^k, X(s^k) \mid c > n^3) \) and let \( T : M_1(n) \to M_1(n) \) be the projection. Much is known about the left \( A \)-modules \( M_1(n) \) and we exploit these results to the fullest. It's interesting to observe that this map \( T \) is realized geometrically since \( M_1(n) \) arises as a Thom spectrum over \( F_k(W) \) where as a space \( \Omega^{2^k-3} \times s^k \times W \)

and \( F_k(W) \) is the filtration induced on \( W \) from the filtration on \( \Omega^{2^k-3} \)

under the projection \( F_k(W) \to \Omega^{2^k-3} \) by the inclusion \( i : W \to \Omega^{2^k-3} \).
The Adams Spectral Sequence

In this section I'd like to give a brief outline of the Adams spectral sequence. I will assume a basic knowledge of the Steenrod algebra for the prime 2 denoted $A$. Also $H^*(X)$ denotes mod 2 homology.

Let $X$ be a complex of finite type and consider $H^*(X)$ as a module over the Steenrod algebra.

For each generator $x_i \in H^*(X)$, let $g_i : X \to K(\pi, n)$ be a map inducing a surjection in that dimension. Taking the product of all such $g_i$'s we obtain a map

$$g : X \to \prod_{j>0} K(H^j(X), j) = K_0$$

which induces a surjection in mod 2 cohomology.

Let $X_1$ be the homotopy theoretic fibre of this map. Apply the same construction to $H^*(X_1)$ as we did to $H^*(X)$ to obtain another product of Eilenberg-MacLane spaces which I will denote $K_1$. Reiterating this process we construct an Adams Resolution for $X$:

$$
\begin{align*}
X & \to K_0 \\
X & \to K_1 \\
X & \to K_2 \\
\vdots
\end{align*}
$$
Note that by the usual Barratt-Puppe construction we obtain maps $Ω K_i \to X_{i+1}$ or $K_i \to \Sigma X_{i+1}$ if we recall that in the stable category fibrations and cofibrations are the same. Therefore we get short exact sequences in the stable range

$$H^*(\Sigma X_{i+1}) \to H^*(K_i) \to H^*(X_i),$$

where $H^*(K_i)$ is a free $A$-module.

We can splice all these short exact sequences together to get an $A$-free resolution of $H^*(X)$.

\[\cdots \to H^*(K) \to H^*(K) \to H^*(K) \to H^*(X)\]

$$\to H^*(\Sigma K_2) \to H^*(\Sigma K_1) \to H^*(K_0) \to H^*(X)$$

The genius of this construction is that now the force of homological algebra can be brought to bear on the following elegant observations:

(i) The fibrations $X_{i+1} \to X_i \to K_i$ yield long exact sequences of homotopy groups.

(ii) $\pi^*_K(K_i) \cong \text{Hom}_A(H^*(K_i), \mathbb{Z}/2)$.

(or more explicitly $[S^{m+t-s}, K_i] \cong \text{Hom}_A(H^*(K_i), \mathbb{Z}/2)$)

where $mK_S$ is the $s$th stage in the Adams resolution for $\Sigma^m X$. 
Thus it can be seen that the differential \(d_1\) in the exact couple of homotopy groups coming from the Adams resolution

\[
d_1 : [S^{n+t-s}, \Omega K_s] \to [S^{n+t-s}, K_{s+1}]
\]

can be construed as a map induced from

\[
\text{Hom}_A^{t-s} (H^*(\Omega K_s), \mathbb{Z}/2) \to \text{Hom}_A^{t-s-1} (H^*(\Omega K_{s+1}), \mathbb{Z}/2).
\]

In short, without proving the convergence properties of the spectral sequence, this outline is enough to indicate that the sequence

\[
K_0 \to \Sigma^{-1} K_1 \to \Sigma^{-2} K_2 \to \ldots
\]

gives a cochain complex of homotopy groups whose cohomology is

\[
\text{Ext}_A (H^*(X), \mathbb{Z}/2).\quad \text{For details see [11], [1].}
\]

**Theorem 2 (Adams, II)** There's a spectral sequence converging to the 2-component of \(\pi_{n+k}(X)\) for \(k < n-1\) with

\[
E^{s,t}_2 = \text{Ext}_A^{s,t} (H^*(X), \mathbb{Z}/2) \quad \text{and} \quad d_r : E^{s,t}_r \to E^{s+r, t+r-1}_r.
\]

The groups \(E^{s,t}_\infty\) form the associated graded group to a filtration of the 2-component of \(\pi_{n+k}(X)\).
The Adams spectral sequence can also be set up to calculate the stable homotopy of a spectrum \( X \) \( \pi^S_* (X) \). The provision that \( H^*(X) \) is finite type guarantees that the spectral sequence will converge.

Let's recall some facts about spectra and the stable homotopy category as covered in Adams [11]. Let \( H\mathbb{Z}/2 \) denote the mod 2 Eilenberg-MacLane spectrum.

1. \( H^*(X) = [X, H\mathbb{Z}/2] \)

2. \( H^*(H\mathbb{Z}/2) = A \)

3. If \( K \) is a wedge of suspensions of \( H\mathbb{Z}/2 \) then \( \pi^S_*(K) = \text{Hom}_A (H^*(K), \mathbb{Z}/2) \).

4. A map \( f : X \to K \) is equivalent to a locally finite collection of elements in \( H^*(X) \) in appropriate dimensions.

5. If a locally finite collection of elements of \( H^*(X) \) is constant, then it is a mod \( \mathbb{Z}/2 \) generator.

6. \( H\mathbb{Z}/2 \wedge X \) is a wedge of suspensions of \( H\mathbb{Z}/2 \) with one wedge summand for each \( \mathbb{Z}/2 \) generator of \( H^*(X) \).
(vi) the composition $X \cong S^0 \wedge X \xrightarrow{\mathbb{L}} \mathbb{H} \wedge X$ induces the $A$-module structure $A \otimes H^*(X) \to H^*(X)$. 
In particular this map is onto.

(vii) If a locally finite collection of elements in $H^*(X)$ generate it as an $A$-module, then the corresponding map $f: X \to K$ induces a surjection in cohomology.

Using the above we can construct an Adams resolution for the spectrum $X = X_0$ by setting $K_0 = X \wedge \mathbb{H} \mathbb{Z}/2$ and taking $X_1$ to be the fiber of the map $f_0: X_0 \to K_0$. We inductively set $K_{i+1} = X_{i+1} \wedge \mathbb{H} \mathbb{Z}/2$ where $X_{i+1}$ is the fiber of $f_i: X_i \to K_i$ and choose $f_{i+1}: X_{i+1} \to K_{i+1}$ to be a map inducing a surjection in cohomology.

Since the $f_i$ is surjective we have short exact sequences $H^*(\sum X_{i+1}) \to H^*(K_i) \to H^*(X_i)$.

These can be spliced together to form an $A$-free.
resolution of $H^*(X)$. Without proving the convergence properties of this spectral sequence I claim that Theorem 2 holds for spectra $X$ provided $H^*(X)$ has finite type. For a detailed account of these convergence properties see Ravenel [11].
\[ \pi_*^S(bo) : \text{The Stable Homotopy of Real Connective K-Theory.} \]

This will be the first application of the Adams Spectral Sequence. According to the previous chapter there is a spectral sequence with \( E_2 \)-term

\[ E_2 = \text{Ext}_A \left( H^*(bo), \mathbb{Z}/2 \right) \Rightarrow \pi_*^A_0. \]

where \( \pi_*^A_0 \) is a graded group associated to \( \pi_*^S(bo) \).

First, we should talk about the spectrum representing real connective K-Theory, \( bo \). It is constructed as follows:

Let \( BO \) be the spectrum representing real connective K-Theory. Define a spectrum \( bo \) by \( bo(8k) = BO[8k, 8k+1, \ldots] \wedge M_2 \) where \( BO[8k, 8k+1, \ldots] \) denotes the \( k-1 \)-connective cover of \( BO \) on its Postnikov cover and \( M_2 \) denotes the Moore space for \( \mathbb{Z}/2 \) localized at 2. The structure maps \( \Sigma^g bo(8k) \rightarrow bo(8(k+1)) \) are adjoints to the identifications

\[ BO[8k, 8k+1, \ldots] \rightarrow \Sigma^g BO[8(k+1), 8(k+1)+1, \ldots] \]

coming from Bott periodicity.

According to Stong \([12]\), \( H^*(bo, \mathbb{Z}/2) = A/\langle a_1 \rangle \),

where \( A_1 = \langle a_1^1, a_1^2 \rangle \).
We want to compute $\pi^S_*(b_0)$ using the Adams spectral sequence, so the first objective in this is to calculate the $E_2$ term $\Ext^s_{A_i}(H^*(b_0), \mathbb{Z}/2)$. We employ a sneaky standard trick called a change of rings theorem.

Let $\mathbb{Z}/2 \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$ be an $A_i$-free resolution of $\mathbb{Z}/2$. We can tensor this resolution with on the left with $A$ over $A_i$ to obtain a free $A$-resolution of $A \otimes_{A_i} \mathbb{Z}/2$

$$A \otimes_{A_i} \mathbb{Z}/2 \leftarrow A \otimes_{A_i} F_0 \leftarrow A \otimes_{A_i} F_1 \leftarrow \cdots$$

In this way we obtain an $A$-free resolution of $H^*(b_0) = A \otimes \mathbb{Z}/2$. In order to calculate $\Ext^s_{A_i}(H^*(b_0), \mathbb{Z}/2)$ we would apply $\text{Hom}_A(\cdot, \mathbb{Z}/2)$ to this resolution and then compute the cohomology of the resulting cochain complex.

But note that $\text{Hom}_A(A \otimes_{A_i} F_i, \mathbb{Z}/2) \cong \text{Hom}_{A_i}(F_i, \mathbb{Z}/2)$ and hence $\Ext^s_{A_i}(H^*(b_0), \mathbb{Z}/2) \cong \Ext^s_{A_i}(\mathbb{Z}/2, \mathbb{Z}/2)$.

This isomorphism greatly reduces our work.
One way of computing $\text{Ext}_{A_1}(\mathbb{Z}/2, \mathbb{Z}/2)$ is to use the spectral sequence associated to the extension of algebras

$$E(\mathbb{Z}_h) \to A_1 \to E(\mathbb{Z}_h^1, \mathbb{Z}_h^2)$$

(see Raneeer [11]) where

$$Q = \mathbb{Z}_h^1 \mathbb{Z}_h^2 + \mathbb{Z}_h^2 \mathbb{Z}_h^1$$

is the Milnor exterior generator. Since $E(\mathbb{Z}_h)$ is a 2-sided ideal in $A_1$, this extension is called normal and hence the Cartan-Eilenberg spectral sequence change of rings spectral sequence is applicable. This is a spectral sequence with

$$\begin{align*}
\text{Ext}^{s_1}_{E(\mathbb{Z}_h^1, \mathbb{Z}_h^2)}(\mathbb{Z}/2, \text{Ext}^{s_2,t}_{E(\mathbb{Z}_h)}(\mathbb{Z}/2, \mathbb{Z}/2)) &= E_2^{s_1, s_2, t} \\
\text{d} : E_1^{s_1, s_2, t} E(\mathbb{Z}_h) &\to E_2^{s_1 + s_2, t} A_1 \\
E_2 &\Rightarrow E_\infty = \text{Ext}^{s_1 + s_2, t}_{A_1}(\mathbb{Z}/2, \mathbb{Z}/2)
\end{align*}$$

It's clear that $\text{Ext}^{s_2, t}_{E(\mathbb{Z}_h)}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{P}[h_2]$ is a polynomial algebra on $h_2$ and that this is the exterior algebra over $E(\mathbb{Z}_h^1, \mathbb{Z}_h^2)$ so we can write

$$\text{Ext}^*_{E(\mathbb{Z}_h^1, \mathbb{Z}_h^2)}(\mathbb{Z}/2, \text{Ext}^*_{E(\mathbb{Z}_h)}(\mathbb{Z}/2, \mathbb{Z}/2)) \cong$$

$$\text{Ext}^*_{E(\mathbb{Z}_h^1, \mathbb{Z}_h^2)}(\mathbb{Z}/2, \mathbb{P}[h_2]) \cong$$

$$\text{Ext}^*_{E(\mathbb{Z}_h^1, \mathbb{Z}_h^2)}(\mathbb{Z}/2, \mathbb{Z}/2) \otimes \mathbb{P}[h_2] \cong \mathbb{P}[h_0, h_1, h_2]$$
a polynomial algebra on 3 generators with
\[ h_0 \in E^{1,0,1}_2, \quad h_1 \in E^{1,0,2}_2, \quad h_2 \in E^{0,1,3}_2. \]

This yields the following picture of the $E_2$ term.

**Lemma 3**

\[
\begin{align*}
\text{d}_2 (h_0) &= 0, \\
\text{d}_2 (h_1) &= 0, \\
\text{d}_2 (h_2) &= h_1 \cdot h_0 \quad \text{d}_2 (h_2^2) &= 0. \\
\end{align*}
\]

**Proof:** $\text{d}_2 (h_0) = 0$ is clear, as is $\text{d}_2 (h_1) = 0$.

It can be shown that $\text{d}_2 (h_2) = h_1 \cdot h_0$, rather by
Inspecting the cobar complex for $\mathbb{Z}/2$ over $A_1$. This is done by setting up the reduced bar resolution as follows: let $\overline{A}_1 = \ker (c : A_1 \to \mathbb{Z}/2)$ be the augmentation ideal. Then the following sequence is exact

$$\mathbb{Z}/2 \leftarrow \overline{A}_1 \leftarrow A_1 \otimes \overline{A}_1 \leftarrow A_1 \otimes A_1 \otimes A_1 \leftarrow \cdots$$

where $i : \overline{A}_1 \to A_1$ is the inclusion. Applying $\text{Hom}_{A_1}(-, \mathbb{Z}/2)$ to this free resolution is a way of obtaining a $\mathbb{Z}/2$-cochain complex from which $\text{Ext}_{A_1}(\mathbb{Z}/2, \mathbb{Z}/2)$ can be calculated. Note that if $\overline{A}_1^m$ denotes the $m$-fold product of $\overline{A}_1$ with itself then $\text{Hom}_{A_1}(A_1 \otimes \overline{A}_1^m, \mathbb{Z}/2) \cong \text{Hom}_{\overline{A}_1}(\overline{A}_1^m, \mathbb{Z}/2) \cong (A_1^*)^m$ where $A_1^*$ is the dual of $A_1$. The resulting complex is called the cobar complex, where the differential is an alternating sum of the diagonal $\psi : A_1^* \to A_1^* \otimes A_1^*$ restricted to $A_1^*$. For details see MacLane [8]. Since we know that $h_0 \cdot h_1$ is represented by $\xi_1/\xi_2^2$ in the cobar complex and $\psi(\xi_1) = \xi_1/\xi_2^2$, the element $h_0 \cdot h_1$ must be hit by something in the Cartan-Eilenberg spectral sequence. After inspecting the gradings, we see that the only possibility is that $d_2(\xi_1) = h_0 \cdot h_1$. This yields the following picture of the $E_2$-term.
Lemma 4: \( d_3(h_2^3) = h_1^3 \).

Proof: Again we employ the cobar complex to do specific calculations. After some simple, but tedious, inspection of the cobar complex, we find that

\[
\begin{align*}
\Theta(\xi_1, \xi_2 | \xi_1^2) &= (\xi_1^2 | \xi_1, \xi_2 | \xi_1^2) + (\xi_1 | \xi_2, \xi_1^2 | \xi_1^2) + (\xi_2 | \xi_1, \xi_1^2 | \xi_1^2) + (\xi_1 | \xi_1^3 | \xi_1^2) \\
\Theta(\xi_1, \xi_2^2 | \xi_2) &= (\xi_1^2 | \xi_1, \xi_2^2 | \xi_2) + (\xi_1 | \xi_2^2, \xi_2 | \xi_2) + (\xi_2 | \xi_1, \xi_2^3 | \xi_2) \\
\Theta(\xi_2, \xi_2) &= (\xi_2^2 | \xi_2) + (\xi_2 | \xi_2, \xi_2 | \xi_2) + (\xi_2 | \xi_1, \xi_2^2 | \xi_1) \\
\end{align*}
\]

and hence

\[
2 \left[ (\xi_1, \xi_2 | \xi_1^2) + (\xi_1, \xi_2^2 | \xi_2) + (\xi_2, \xi_2 | \xi_2) \right] = \xi_1^2 | \xi_1 | \xi_1,
\]

which represents \( h_1^3 \) in the cobar complex.
Since this is the $E_3$-term is the last opportunity for $h^3$ to be hit, it must be the case that
\[ d_3(h^3) = h^1. \]

Hence we have the following picture of the $E_4$-term

\[ \text{Diagram of } E_4 \]

**Lemma 5**: $E_4 = E_\infty$

**Proof**: This is easy to see from the above picture as none of the $d_r$ for $r > 4$ have anything to hit. \[ \square \]
The motivating philosophy behind this calculation is that the Cartan-Eilenberg spectral sequence is a good bookkeeping device, but a bit too vague for explicit calculations. Thus we employ the C.E.S.S to tell us where to inspect the cobar complex for relations. Using the cobar complex itself is unreasonable since it's too dense with information for feasible calculation.

This completes the calculation of \( \text{Ext}_A(\mathbb{Z}/2, \mathbb{Z}/2) \) and hence of \( \pi^*(\mathcal{B}) \). This information will be fed into the Adams Spectral Sequence for \( hO \cdot (B(n)) \).
Applying the Adams spectral sequence

In this section we will compute the real connective $K$-theory of the Brown-Gitler spectra using the classical Adams spectral sequence.

Recall our definitions of the $n^{th}$ Brown-Gitler module $M(n)$, the modified Brown-Gitler module $M_1(n)$ and the map $M(n) \rightarrow M_1(n)$. Initially we will concentrate on some of the properties of $M_1(n)$.

**Theorem 10 (ABP 3.3)**: $H_*(bo) = \mathbb{Z}_2[x_1^2, x_3, x_7, \ldots]$ with left $A_1$-action given by

\[ S_i^1(x_4) = x_i \quad \text{and} \quad S_i^2(x_4) = 0 \]

where \( i = 2^{i-1} \).

**Theorem 7 (Mahowald [15])**: Let $g$ be given by

\[ g|_{M_1(2k)} : \Sigma_1^{4k} M_1(2k) \rightarrow A/\mathbb{A}_1 \]

Then $g$ is an isomorphism of left $A_1$-modules.

In $H_*(bo)$, let $x_i$ have weight $2^{i-1}$. Then this weight is multiplicative and is preserved by the left $A_1$-action. Let $N_{4k} \subset H_*(bo)$ consist of the vector space generated by all monomials of weight $4k$. By the above observation, $N_{4k}$ is an $A_1$-submodule for...
Recall that $G_0$ and $G_1$ act on $H^*(S^2)$ as derivations, hence we can regard $N_{4k}$ as a differential group with respect to these actions. It's been proved by Adams [1] that the homology of $N_{4k}$ with respect to this action of $G_0$ and $G_1$ contains a great deal of information. Before I launch into spelling this out explicitly, let's record some information about $N_{4k}$

**Lemma (ABP [3])**

$H^*_k(N_{4k}, G_i) = \mathbb{Z}/2$ in weight $4k$ and $H^*_k(N_{4k}, G_i) = \mathbb{Z}/2$ in weight $4k$. 

**Corollary**: $H^*_k(M_1(2n), G_i) = \mathbb{Z}/2$ in dimension $C$ and $H^*_k(M_1(2n), G_i) = \mathbb{Z}/2$ in dimension $2[\lambda(n) - \lambda(n)]$.

where $\lambda(n)$ is the number of nonzero coefficients in the dyadic expansion of $n$.

**Proof**: Suppose that $n = \sum_{j=2}^{5} i_j 2^{j-1}$ where $i_j = 0$ or 1.

Then $4n = \sum_{j=2}^{5} i_j 2^{j-1} = \sum_{j=2}^{5} 2i_j (2^{j-1})$. Recall that $x_{2^{j-1}}$ has weight $2^{j-1}$. Since we've made the identification $\sum_{j=2}^{5} M_1(2n) = W(N_{4n})$, we can say $H^*_k(M_1(2n), G_1) = \mathbb{Z}/2$ in dimension $\sum_{j=2}^{5} 2i_j (2^{j-1}) - \sum_{j=2}^{5} 2i_j (2^{j-1}) = \text{WHY?}$
Corollary 9: \( H_\ast(M_1(2\alpha), Q_\ast) \cong \mathbb{Z}/2 \) in degree 0 and 
\( H_\ast(M_1(2\alpha), Q_\ast) \cong \mathbb{Z}/2 \) in degree 2 \((2\alpha - 4\alpha)\).

Proof: The fact that \( H_\ast(M_1(2\alpha), Q_\ast) \cong \mathbb{Z}/2 \) in degree zero follows from the fact that \( \sum \alpha M_1(2\alpha) \cong N_{\ast}^\ast \) as left \( A_1 \)-modules.

\( H_\ast(N_{4\alpha}, Q_\ast) \) is one-dimensional and \( Q_\ast \sim \mathbb{C} \) on \( M_1(2\alpha) \).

The other fact can be observed by considering \( N_{4\alpha} \). Suppose that \( \hat{x} \in M_1 \) is some monomial so that \([x]\) generates \( H_\ast(N_{4\alpha}, Q_\ast) \). This can be arranged by a suitable basis for \( N_{4\alpha} \) such that \( x_{\alpha} \) is a polynomial generator of weight \( 2^{i-1} \) and of degree \( 2^{i-1} - 1 \) and \( x_{\alpha} \neq x_{\alpha} \) for \( j \neq k \) in this representation.

Note that \( x_0 \) doesn't occur in this representation.

It's clear that \( x_j \) must be even for \( 1 \leq j \leq n \) since otherwise \( Q_\ast \sim \mathbb{C} \) by the Cartan formula. Also if \( 2 \leq x \geq 4 \) for some \( k \), then \( m = 2 \sum_{j=1}^{\alpha_k} x_{\alpha_j} \) for some \( k \).

\( Q_\ast \left[ x_{(l+2)} x_{(l+2)} \right] = \sum_{j=1}^{\alpha_k} x_{\alpha_j} \) by our choice of \( m \). Thus \( x_j = 2 \) for \( 1 \leq j \leq n \).

We can now find the weight of \( m \) in terms of the weight of its factors, which is
\[
\sum_{p \equiv 2}^{2 \cdot h_p} (2^{p-1})
\]
\[
    b_p = \begin{cases} 
        1 & \text{if } p = i_j \text{ for some } 1 \leq j \leq n \\
        0 & \text{otherwise}
    \end{cases}
\]

But we know that the weight of \( m \) is \( 4n \). Thus
\[
    4n = \sum_{p=2}^{s} 2 b_p \left( 2^{p-1} \right).
\]
It's useful to observe that
\[
    \text{this representation of } 4n \text{ is twice the unique } \text{dyadic expansion and that}
\]
\[
    \text{the monomial } m \text{ of degree } \sum_{p=2}^{s} 2 b_p \left( 2^{p-1} \right).
\]

We have the identification \( \sum_{4n} M_1(2n) \cong N_{4n}^{*} \)

so the monomial \( m' \in M_1(2n) \) corresponding to \( m \in N_{4n} \)

is of degree
\[
    \sum_{p=2}^{s} 2 b_p \left( 2^{p-1} \right) - \sum_{p=2}^{s} 2 b_p \left( 2^{p-1} \right)
\]
\[
    = \sum_{p=2}^{s} 2 b_p \left( 2^{p-1} \right)
\]
\[
    = 2 \left[ 2n - \alpha(n) \right]. \text{ This completes the proof.} \]

It would be easier to compute \( H_X \left( \alpha(n); \xi_1 \right) \)
directly and proceed from there.
We now set up some machinery of Adams through which to push this information.

**Definition 2.** A module $M$ over $A_1$ is called invertible if
\[
\text{rank } H_*(M, \mathbb{Q}) = \text{rank } H_*(N, \mathbb{Q}) = 1 \quad \text{where the rank \ is \ as \ a } \mathbb{F}_2 \text{ vector space.}
\]

**Definition 3.** Two modules, $M$ and $N$ over $A_1$, are said to be stably equivalent if there exist $A_1$ free modules $F$ and $G$ so that $M \oplus F \cong N \oplus G$. This implies that $\text{Ext}^{s,t}_{A_1}(M, \mathbb{F}_2) \cong \text{Ext}^{s,t}_{A_1}(N, \mathbb{F}_2)$ for $s > 1$.

**Theorem 10 (Adams).** If $M$ is an invertible $A_1$-module where $H_*(M, \mathbb{Q}) \cong \mathbb{Z}/2 \text{ in dimension } b = j$ and $H_*(M, \mathbb{Q}) \cong \mathbb{Z}/2 \text{ in dimension at } 3b = k$ then $M$ is stably equivalent to $\bigoplus_{j} \bigoplus_{i} \mathbb{F}_2 J_{j} \cong A$ where $A$ is the augmentation ideal, $J$ is the "joker", and $A(M)$ is some function of the $\mathbb{F}_2$ rank of $M$. 
M is stably equivalent to \[ \sum_{i=1}^{n} \overline{A_i}^j \cdot \mathcal{J}^4(M) \] where the exponents denote an appropriately truncated tensor product, \( \overline{A_i} \) is the polynomial pictured here:

![Diagram](image)

and \( \mathcal{J}^4(M) \) is some function of the \( \mathcal{J} \)-rank of \( M \). \( \Box \)

We will denote such a stable equivalence by \( M \cong \sum_{i=1}^{n} \overline{A_i}^j \cdot \mathcal{J}^4(M) \).

It is proved by Adams in the BSC paper that \( i^2 \cong \sum F \), a free module. I have left \( \mathcal{J}^4(M) \) indeterminate here since Adams provides a convenient way of omitting it.

**Theorem 11 (Adams)**: The stable type of \( \overline{A}_i/\overline{A}_j \) as a left \( \overline{A}_j \)-module is

\[
(1 \oplus \sum_{i=1}^{3} \overline{A}_i \cdot \mathcal{J}) \otimes (1 \oplus \left( \sum_{i=0}^{2} \sum_{j=1}^{\infty} \overline{A}_j \right) \otimes \left( \sum_{i=1}^{5} \overline{A}_i \right) \otimes \left( \sum_{i=1}^{8} \overline{A}_i \right) \otimes \left( \sum_{i=1}^{9} \overline{A}_i \right))
\]

\[
= 1 \oplus \sum_{i=1}^{3} \overline{A}_i \cdot \mathcal{J} \oplus \sum_{i=1}^{3} \overline{A}_i \cdot \mathcal{J} \oplus \sum_{i=1}^{8} \overline{A}_i \cdot \mathcal{J} \oplus \sum_{i=1}^{9} \overline{A}_i \cdot \mathcal{J} \oplus \ldots \ \Box
\]
Corollary 12: \( M_1(2n) \cong \sum_{c} A_i^c \) where \( c = 2m-x(n) \).

Proof: We know that as left \( A_1 \)-modules,

\[
\bigoplus_{k \geq 1} \sum_{c} M_1(2k) \cong A_i/A_i
\]

and as stable modules,

\[
A_i/A_i \cong_s \left( 1 \oplus \sum_{c} A_i^c \right) \left( 1 \oplus \sum_{c+1} \frac{A_i^{c-1}}{A_i} \right).
\]

From these facts and the Y theorerem, Adams [---] theorem 10 of Adams, the result follows. \( \square \)

Corollary: \( \text{Ext}_{A_1}^{5n} (A_i/(3m), A_i/(3m)) \cong \text{Ext}_{A_1}^{5n} \)

Corollary 13: \( \text{Ext}_{A_1}^{5t} (A_i/(2n), \mathbb{Z}/2) \cong \text{Ext}_{A_1}^{5t+c, t+c} (A_i^m, \mathbb{Z}/2) \).

Proof: This follows from the stable type of \( M_1(2n) \) and the exact sequence \( 0 \rightarrow A_i \rightarrow A_i \rightarrow \mathbb{Z}/2 \) inducing the connecting map

\[
S: \text{Ext}_{A_1}^{5t} (A_i, \mathbb{Z}/2) \cong \text{Ext}_{A_1}^{5t+1} (\mathbb{Z}/2, \mathbb{Z}/2).
\]
Theorem 14: \( M(2m+1) \cong M_1(2m) \otimes M(1) \)

proof. Recall that, as a space, \( S^2 \times S^2 \times S^1 \times W \), this yields that \( H^*_x(W) \cong \mathbb{Z}/2 \langle x_1^2, x_2^2, x_3, x_4 \rangle \) by using the Eilenberg-Zilber theorem. The left action of \( A_1 \) on this is the same as that on \( H^*_x(W) \), so we can write

\[
H^*_x(W) \cong H^*_x(W) \otimes \langle 1, x_1, x_2, x_3, x_4 \rangle.
\]

Filter \( H^*_x(W) \) as follows: let \( P_1 = H^*_x(W) \otimes \langle 1, x_1, x_2, x_3 \rangle \) and \( P_2 = H^*_x(W) \otimes \langle 1, x_1, x_2 \rangle \). Then since the \( A_1 \) action must preserve this weight, there's a split short exact sequence of left \( A_1 \)-modules

\[
P_1 \longrightarrow P_2 \longrightarrow P_2/P_1
\]

where \( P_2/P_1 \) consists of all monomials of weight congruent to 2 mod 4 in \( H^*_x(W) \). It remains to be seen that such monomials correspond to \( X(M(2k+1)^*) \), that is, the dual of the oddly indexed Brown-Carrüber modules. This is accomplished as follows.

Let \( S' \times W \hookrightarrow S^1 \times W \longrightarrow S^1 W \) be a cofibration inducing a long exact sequence

\[
\tilde{H}^*_x(S') \oplus \tilde{H}^*_x(W) \rightarrow \tilde{H}^*_x(S' \times W) \rightarrow \tilde{H}^*_x(S^1 W)
\]

in which \( S' \) is trivial, since it must preserve this weight. Thus there exist short exact sequences

\[
\tilde{H}^*_x(F_m(W)/F_{m-1}(W)) \rightarrow \tilde{H}^*_x(F_m/F_{m-1}) \rightarrow \tilde{H}^*_x(S^1 F_m(W)/S^1 F_{m-1}(W))
\]

for \( m > 1 \).
Recall that \( \tilde{H}_* \left( \frac{F_m}{F_{m-1}} \right) \cong X \left( M \left( \left\lfloor \frac{m}{2} \right\rfloor \right)^* \right) \) and that if \( m \) is even then
\[
\tilde{H}_* \left( \frac{F_m}{F_{m-1}} \right) \cong \tilde{H}_* \left( \frac{F_{m-1}(W)}{F_{m-1}(W)} \right).
\]

Therefore \( \tilde{H}_* (W) \cong \bigoplus_{K \geq 0} X \left( M(K)^* \right) \).

It's easy to see that \( \tilde{H}_* \left( \frac{F_m}{F_{m-1}} \right) \) consists of monomials of weight exactly \( m \). Therefore

(i) \( m \equiv 2 \pmod{4} \) implies that \( X \left( M \left( \left\lfloor \frac{m}{2} \right\rfloor \right)^* \right) \) consists of monomials of weight congruent to 2 mod 4.

(ii) \( m \equiv 0 \pmod{4} \) implies that \( X \left( M \left( \left\lfloor \frac{m}{2} \right\rfloor \right)^* \right) \) consists of monomials of weight congruent to 0 mod 4.

Therefore, remark (i) tells us that

\[
X \left( M \left( 2k+1 \right)^* \right) \cong \tilde{H}_* (W) \otimes \langle x^2, x \rangle
\]

\[
\bigoplus_{m \geq 1} X \left( M \left( \left\lfloor \frac{m}{2} \right\rfloor \right) \right) \otimes X \left( M(1)^* \right),
\]

It follows that

\[
M \left( 2k+1 \right) \cong M \left( 2k \right) \otimes M(1).
\]
Lemma 15. There exist short exact sequences of left $A_1$-modules
\[ \Sigma M_i(2n) \xrightarrow{\mu_i} M(2n+1) \xrightarrow{\Pi_i} M_i(2n) \]
\[ \Sigma M_i(2n-2) \xrightarrow{\mu_i} M(2n) \xrightarrow{\Pi_i} M_i(2n) \]
\[ \Sigma \Sigma^k M(\left[ \frac{k}{2} \right]) \xrightarrow{i} M(k) \xrightarrow{P} M(k-1) \]
(Mahowald [9]

**Proof:** I have already described the maps $\Pi_1$ and $\Pi_2$. Let $K$ denote the kernel of $\Pi_1$. Define $\phi_1: K \to \Sigma M_i(2n)$ by $\phi_1(x(Sq^i_0)) = \Sigma (\Pi_1(x(Sq^j_0)))$ where $J$ is an admissible sequence and

$X(Sq^i_0) = X(Sq^j_0) \cdot Sq^i_0$. This is well defined for if

$X(Sq^i_0) = X(Sq^j_0) \cdot Sq^i_0 = X(Sq^k_0) \cdot Sq^i_0$ for some admissible sequence $K$ then $(X(Sq^j_0) - X(Sq^k_0)) \cdot Sq^i_0 = 0 \in M(2n+1)$

so $\Pi_1(Sq^j_0) = \Pi_1(Sq^k_0) \in M_i(2n) = A/\{x \cup Sq^i_0, Sq^i_0 \}$. I claim that $\phi_1$ is a morphism of left $A_1$-modules.

I also claim that $\phi_1$ is injective, for if $\phi_1(x(Sq^i_0)) = 0$ then $X(Sq^i_0) = \phi_1 X(Sq^j_0) \cdot Sq^i_0$ where $X(Sq^j_0) = X(Sq^k_0) \cdot Sq^i_0$

so $X(Sq^i_0) = X(Sq^k_0) \cdot Sq^i_0 = 0$. This proves the injective claim.
Define $\Psi_1 : \Sigma M_j(2n) \to K_j$ by

$$\Psi_1 \left( \Sigma \pi \left( X(\text{Sq}^j) \right) \right) = X(\text{Sq}^j) \cdot \text{Sq}^i.$$  It's clear that

$\Psi_1$ is well defined. Suppose that $X(\text{Sq}^j) \cdot \text{Sq}^i = 0$. Then $X(\text{Sq}^j) = X(\text{Sq}^k) \cdot \text{Sq}^i$, so $\pi_1(\text{Sq}^j) = 0$. This proves the injectivity of $\Psi_1$. Since both $\Sigma M_j(2n)$ and $K_j$ are finite, the existence of these injective maps guarantees that they are isomorphisms. This proves the exactness of $(a)$.

The proof of the exactness of sequence $(b)$ is analogous to that of $(a)$. Sequence $(c)$ is due to Mahowald.
We now have all the necessary information to start the calculation. First, consider the long exact sequence of Ext groups arising from the short exact sequence

$$
\Sigma^1 \mathbb{Z}/2 \to M(1) \to \mathbb{Z}/2.
$$

$$
\text{Ext}_{A_1}^{3,t} \left( \mathbb{Z}/2, \mathbb{Z}/2 \right) \to \text{Ext}_{A_1}^{2,t} \left( \mathbb{Z}/2, \mathbb{Z}/2 \right) \to \text{Ext}_{A_1}^{3,t} \left( \Sigma \mathbb{Z}/2, \mathbb{Z}/2 \right)
$$

It can be shown that the differentials so indicated are non-trivial by inspecting the exact complex for $\text{Ext}_{A_1}^{3,t} \left( M(1), \mathbb{Z}/2 \right)$. Alternatively, one might argue more simply that $s : \text{Ext}_{A_1}^{0,1} \left( \Sigma \mathbb{Z}/2, \mathbb{Z}/2 \right) \to \text{Ext}_{A_1}^{1,1} \left( \mathbb{Z}/2, \mathbb{Z}/2 \right)$ is non-zero since the previous map

$$
i^* : \text{Hom}_{A_1} \left( M(1), \mathbb{Z}/2 \right) \longrightarrow \text{Hom}_{A_1} \left( \Sigma \mathbb{Z}/2, \mathbb{Z}/2 \right)
$$

must be trivial and $\text{Ext}_{A_1}^{1,1} \left( \mathbb{Z}/2, \mathbb{Z}/2 \right)$ is non-trivial.
The other connecting map

\[ \delta : \text{Ext}^{3,3}_{A_1}(\Sigma \mathbb{Z}/2, \mathbb{Z}/2) \rightarrow \text{Ext}^{4,3}_{A_1}(\mathbb{Z}/2, \mathbb{Z}/2) \]

\[ \text{whose definition is complex. By this result, the above complex is zero by virtue of the periodicity of } \pi_x^3(60) \]
Thus we're left with the picture of the Adams $E_2$-term without the relations indicated by dotted lines.

We shall consider the element $j$.

We now see that these dotted lines do represent relations on the induced by the $\tilde{\pi}_*^2(\mathcal{E}_2)$ multiplication. The method which we employ is due to W.S. Massey and the operations are called appropriately Massey Products. A detailed account is given in Ravenel [7].

I want to prove that $v_1 h_0 h_1^2 = 0$, which is proving the existence of a vertical line in the picture above, where $v = \langle 1, h_0, h_1 \rangle$, is a Massey product.

By a method known as juggling (see Ravenel [7]), we get that $\langle 0, h_0^2 \rangle$. $M(1)$ is not a coalgebra so this Massey product is not defined.
In this picture of the Adams $E_2$-term, we can show that the dotted vertical and diagonal lines do indeed represent relations induced by the $\pi_2(X,\nu)$ multiplication. We can regard this module structure as the Yoneda product of exact sequences, but we prove that this product is nonzero by inspecting the cobar complex for $M(1)$ over $A_1$. Let us first label the element $v \in E_1^{1,3}$. We know that if $1 \in \text{Ext}_{A_1}^1(M(1),\mathbb{Z}/2)$ is represented by $\bar{1}$ in the cobar complex for $M(1)$ then $1 \cdot h_0 = 0$ by inspecting just the $E_2$-term additive structure of the $E_2$-term. By the same means we see that $h_1 \cdot h_0 = 0$ in $\text{Ext}_{A_1}^1(\mathbb{Z}/2,\mathbb{Z}/2)$. Therefore, in the cobar complex for $M(1)$ over $A_1$ there exist $\alpha$ and $b$ so that $\Theta(\alpha) = \bar{1}/\xi_1$, and $\Theta(b) = \xi_1/\xi_2$. It follows that $[\alpha/\xi_2 + \bar{1}/b]$ is a cocycle. We can represent this as a Yoneda product $<1,h_0,h_1>$ and by inspecting dimensions we see that we have constructed an explicit representative for $v$.

The next objectives are to show that $v \cdot h_0 = h_1^2$ and that $v \cdot h_1^2 = 0$. The first fact follows from the associativity

\[
<1,h_0,h_1> \cdot h_0 = 1 <h_0,h_1,h_0>
\]

of Yoneda products and a little bit of computing in the cobar complex. \textit{Claim} for $M(1)$: In this complex, note that $1 \cdot h_0 \cdot h_1 = 0$ since $\Theta(\bar{1}/\xi_2) = \bar{1}/\xi_1$ and $\Theta(\bar{1}/\xi_2) = \xi_1/\xi_2$ and $1 \cdot h_1 \cdot h_0 = 0$ since $\Theta(\bar{1}/X(\xi_2)) = \bar{1}/\xi_1$.

\textit{Remark}: (see back)
Therefore we can write \( \langle h_0, h_1, h_0 \rangle = [\xi_2 | \xi_1 + \xi_1 | \xi_2] \times (\xi_2) \)

\[
\begin{bmatrix}
\xi_2 | \xi_1 + \xi_1 | \xi_2 + \xi_1 | \xi_3
\end{bmatrix}.
\]

But

\[
\partial(\xi_3) \cdot \partial(\xi_2) = \xi_2 | \xi_1 + \xi_2 | \xi_1 + \xi_1 | \xi_3 + \xi_1 | \xi_2.
\]

so

\[
[\xi_2 | \xi_1 + \xi_1 | \xi_2] \times (\xi_2) = [\xi_2 | \xi_2] = \xi_1^2.
\]

Thus \( \langle h_0, h_1, h_0 \rangle = \xi_1^2 \) so

\[
\frac{1}{\langle h_0, h_1, h_0 \rangle} = \frac{1}{\xi_1^2} = 0.
\]

Next we would like to see that \( v - \xi_1^2 = 0 \) thereby proving the existence of a diagonal line in the picture of the Adams Spectral Sequence from \( E_2^{5,2} \) to \( E_2^{7,3} \). Again by associativity we get that \( \langle \xi_1^2, h_0 \rangle = \)

\[
\langle 1, h_0, h_1 \rangle \cdot \xi_1^2 = 1 \cdot \langle h_0, h_1, h_1^2 \rangle.
\]

Thus it remains to be seen that \( \langle h_0, h_1, h_1^2 \rangle \neq 0 \). From previous definitions we see that we could define

\[
\langle \xi_1^3, \xi_1^2 \rangle = [a | \xi_1^2 + \xi_1 | b]
\]

for some \( a \) and \( b \) with \( \partial(\xi_1^3) = \xi_3 \).

\[
\langle h_0, h_1, h_1^2 \rangle = [a | \xi_1^2 | \xi_1^2 + \xi_1 | b] \] for some cochains \( a \) and \( b \) in the cobar complex for \( \mathbb{Z}/2 \) over \( A_1 \), where

\[
\partial(a) = \xi_1 | \xi_1^2 \quad \text{and} \quad \partial(b) = \xi_2 | \xi_1^2 | \xi_1^2.
\]

Note that up to some cocycle we could choose \( a = \xi_2 \) and \( b = \frac{1}{2} \xi_2 + \xi_1 | \xi_1^2 + \xi_1 | \xi_1^2. \)
\[
\begin{bmatrix}
\frac{\varepsilon_2}{3} & \frac{\varepsilon_2}{3} & \frac{\varepsilon_2}{3} \\
\frac{\varepsilon_3}{3} & \frac{\varepsilon_3}{3} & \frac{\varepsilon_3}{3} \\
\frac{\varepsilon_2}{3} & \frac{\varepsilon_2}{3} & \frac{\varepsilon_2}{3}
\end{bmatrix}
= \begin{bmatrix}
\frac{\varepsilon_2^2}{3} & \frac{\varepsilon_2^2}{3} & \frac{\varepsilon_2^2}{3} \\
\frac{\varepsilon_3^2}{3} & \frac{\varepsilon_3^2}{3} & \frac{\varepsilon_3^2}{3} \\
\frac{\varepsilon_2^2}{3} & \frac{\varepsilon_2^2}{3} & \frac{\varepsilon_2^2}{3}
\end{bmatrix}
\]

\text{prf: } \vartheta \left( \frac{\varepsilon_2}{3}, \frac{\varepsilon_3}{3}, \frac{\varepsilon_2}{3} \right) = \frac{\varepsilon_2^2}{3}, \frac{\varepsilon_3^2}{3} + \frac{\varepsilon_3^2}{3}, \frac{\varepsilon_2^2}{3} + \frac{\varepsilon_2^2}{3}, \frac{\varepsilon_3^2}{3}, \frac{\varepsilon_2^2}{3}, \frac{\varepsilon_2^2}{3}.

\text{and } \vartheta \left( \frac{\varepsilon_2}{3}, \frac{\varepsilon_3^2}{3}, \frac{\varepsilon_2}{3} + \frac{\varepsilon_2}{3}, \frac{\varepsilon_3}{3}, \frac{\varepsilon_2^2}{3} - \frac{\varepsilon_2}{3} \right) =
\frac{\varepsilon_3^3}{3}, \frac{\varepsilon_3^3}{3}, \frac{\varepsilon_3^2}{3} + \frac{\varepsilon_3^2}{3}, \frac{\varepsilon_3^3}{3}, \frac{\varepsilon_3^3}{3}, \frac{\varepsilon_3^2}{3}, \frac{\varepsilon_3^2}{3} = 0.

\text{Thus } \begin{bmatrix}
\frac{\varepsilon_2^2}{3} & \frac{\varepsilon_3^2}{3} & \frac{\varepsilon_2^2}{3} + \frac{\varepsilon_3^2}{3} & \frac{\varepsilon_2^2}{3} + \frac{\varepsilon_2^2}{3} & \frac{\varepsilon_2^2}{3} + \frac{\varepsilon_2^2}{3} & \frac{\varepsilon_2^2}{3}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{\varepsilon_2^2}{3} & \frac{\varepsilon_3^2}{3} & \frac{\varepsilon_2^2}{3} + \frac{\varepsilon_3^2}{3} & \frac{\varepsilon_3^3}{3} & \frac{\varepsilon_3^3}{3} & \frac{\varepsilon_3^2}{3}
\end{bmatrix}
\]

\[
= 0. \text{ This proves the claim. } \square
\]

Therefore \( \langle h_0, h_1, h_2 \rangle \neq 0 \) and hence the dotted diagonal line can be filled in.
But if this is the case, we note that since \( \delta_1 = \delta_2 \), we have that
\[
\alpha = \delta_1 - \delta_2 \quad \text{and} \quad \beta = \delta_1 + \delta_2
\]
which are in the range of \( \chi \). This completes the examination of case (v) and proves

\[
\nu - h^2 > 0.
\]

Thus all the dotted lines in the picture for \( \text{Ext}_{A_1} (M(1), \mathbb{Z}/2) \)
can be filled in to indicate multiplicative relations in

\[
\text{Hom}_{A_1} (M(1)) \quad \text{as a} \quad \prod \chi \quad \text{-module}.
\]

To assemble all these facts together for the final solution
we recall that

\[
M_{1, (2n)} \cong \bigoplus \mathbb{Z}/2 \cong \bigoplus^c \mathbb{Z}/2 \cong \bigoplus \mathbb{Z}/2
\]

where

\[
c = 2n - \chi(1)
\]

and

\[
M(2n+1) \cong M_{1, (2n)} \otimes M_{1, (1)}
\]

so

\[
\text{Ext}_{A_1}^s (M(2n+1), \mathbb{Z}/2) \cong \text{Ext}_{A_1}^{s+t} (M_{1, (2n)} \otimes M_{1, (1)}, \mathbb{Z}/2).
\]

It's easy to see that the spectral sequence admits no nontrivial
differentials from here on, hence collapses thus we have
computed \( \text{Hom}_{A_1} (B(2n+1)) \).

We compute \( \text{Hom}_{A_1} (B(2n)) \) by comparing the \( E_2 \) terms
\( \delta \) arising from the short exact sequences of \( A_1 \)-modules

\[
\begin{align*}
(\text{A}) & \quad \Sigma M_{1, (2n-2)} \rightarrow M(2n) \rightarrow M_{1, (2n)} \\
(\text{B}) & \quad \Sigma^m M(n) \rightarrow M(2n) \rightarrow M(2n-1)
\end{align*}
\]
It can be seen from the pictures associated to A and B that the $E_2$ terms are completely complementary in that any differential or extension problem in one is solved in the other. To prove that we get the same picture from either short exact sequence we note that in $E_2$, be $(B(1))$, the "periodic elements" on the lower left head point of each lightening flash lie on the line $s = \frac{1}{2} (t-s)$. We compare that these elements are translated to the same spot twice in the $E_3$ term.

In the case $n = 7(4)$ we inspect the case associated to the short exact sequence $A$.

$c = (n-1) - \alpha (n-1) \frac{2n}{\alpha} \geq \frac{2n}{\alpha} A$ where $s \geq C$.

\[
\text{Ext}^s_{A_1} \left( \bigoplus \mathbb{Z}/2 \right) = \text{Ext}^s_{A_1} \left( \bigoplus \mathbb{Z}/2 \right)
\]

which looks like $\text{Ext}^s_{A_1} \left( \mathbb{Z}/2, \mathbb{Z}/2 \right)$ above the lower $s = C$ and to the right of the line $t-s = 2n$ shoved to the right $2n$. Above the line $s = C$. 

\[
\text{Ext}^s_{A_1} \left( \bigoplus \mathbb{Z}/2 \right) = \text{Ext}^s_{A_1} \left( \bigoplus \mathbb{Z}/2 \right)
\]
Thus all of multiplications in $\text{Exh}((M(m), \mathcal{Z}, \mathcal{C})$ are solved.

$\mathcal{M}$ odd

$\mathcal{M}$ even
In the spectral sequences associated to $A$ it's clear that we get the associated diagrams. To prove this is the case for the spectral sequences of spectral sequences associated to $B$ we compute to see how the line $s = \frac{1}{2}(t-s)$ is translated in the 2 cases $n \equiv 1 \ (\mod \ 4)$ and $n \equiv 3 \ (\mod \ 4)$.

If $n \equiv 1 \ (\mod \ 4)$ then $M(n) \cong M_{1}(n-1) \otimes M(1)$ as $A_{1}$-modules and hence $\text{Ext}^{s,t}_{A_{1}} (\sum_{m} M(n), \mathbb{Z}/2) \cong \text{Ext}^{s+c+t+c}_{A_{1}} (\sum_{m} M(1), \mathbb{Z}/2)$ where $c = (n-1) - \alpha(n-1)$. Note that $J$ is not a factor in the stable decomposition of $M(n)$ since $(n-1) \equiv c \ (\mod \ 4)$. The picture of $\text{Ext}^{s+t+c}_{A_{1}} (\sum_{m} M(1), \mathbb{Z}/2)$ is just that of $\text{Ext}^{s+t}_{A_{1}} (M(1), \mathbb{Z}/2)$ chopped off at $s = c$ and shoved to the right $2n$. Also $M(2n-1) \cong M_{1}(2n-2) \otimes M(1)$ as $A(1)$-modules so

$$\text{Ext}^{s,t}_{A_{1}} (M(2n-1), \mathbb{Z}/2) \cong \text{Ext}^{s+b+t+c}_{A_{1}} (M(1), \mathbb{Z}/2)$$

where $b = 2(n-1) - \alpha(n-1)$. Thus the picture of $\text{Ext}^{s,t}_{A_{1}} (M(2n-1), \mathbb{Z}/2)$ is just that of $\text{Ext}^{s+t}_{A_{1}} (M(1), \mathbb{Z}/2)$ chopped off at the line $s = b$. We can now check that the exact sequence $B$ gives the picture we've indicated. In the case for $\text{Ext}^{s,t}_{A_{1}} (\sum_{m} M(m), \mathbb{Z}/2)$ the line $s = \frac{1}{2}(t-s)$ is translated to

$s + c = \frac{1}{2}((t-s) - 2m)$ so

$s + (m-1) - \alpha(m-1) = \frac{1}{2}(t-s) - n$. In the other case

$s + b = \frac{1}{2}(t-s)$ so

$s + 2m - 2 - \alpha(m-1) = \frac{1}{2}(t-s)$. The fact that if $n \equiv 1 \ (\mod \ 4)$ then $\alpha(n-1) = \alpha(m-1)$ so we get...
\[ s = \frac{1}{2} (t-s+2) - 2n + \alpha \] in the first case and 
\[ s = \frac{1}{2} (t-s+4) - 2n + \alpha \] in the second case where 
\[ \alpha = \alpha(m-1) = \alpha \left( \frac{m-1}{2} \right) \]. This proves the desired result.

In the case of \( n = 3 \) (4) we see that 
\[ M(n) \cong \sum_{c} J \] where \( c = (n-1) - \alpha \left( \frac{m-1}{2} \right) \). Therefore

\[ \text{Ext}_{A_1}^{s,t}(\Sigma^2 M(n), \mathbb{Z}/2) \cong \text{Ext}_{A_1}^{s+t+c}(\Sigma^2 M(1) \otimes J, \mathbb{Z}/2) \]

This is \( \text{Ext}_{A_1}^{s,t}(M(1), \mathbb{Z}/2) \) moved to the right \( 2n+4 \) units and chopped off at \( s = c-2 \). Thus the line \( s = \frac{1}{2}(t-s) \) has been translated to \( s + c - 2 = \frac{1}{2}(t-s-2n-4) \)

or \( s = \frac{1}{2}(t-s) - 2n + 1 - \alpha \) where \( \alpha = \alpha \left( \frac{m-1}{2} \right) = \alpha(n-1) \). The picture for \( \text{Ext}_{A_1}^{s,t}(M(3n-1), \mathbb{Z}/2) \) is the same as before so the line \( s = \frac{5}{2}(t-s) \) on it is translated to \( s = \frac{1}{2}(t-s+4) - 2n + \alpha \). Thus we get the claimed result.

If \( n \) is even then the spectral sequence associated to the short exact sequence \( A \) is clear. To show that we get the picture indicated write \( m = a^k \cdot m \) where \( m \) is relatively prime to 2. Then the vertical tower in \( M(2m) \) is the same as grading as \( \sum_{r} (M(2m)) \) which is on the same same as grading as \( \sum_{r} (M(m)) \) where 
\[ s + \sum_{j=0}^{2m} a^j \cdot m + a^j \cdot 2 \cdot m + \cdot \frac{a^j \cdot m}{2n} = m \left( \frac{a^j \cdot m}{2n} \right) \]
Now observe that if we order the vertical towers appearing in the spectral sequence from left to right, then the first tower in \( \text{Ext}^{s,t}_{A_1} (\Sigma^m M(n), \mathbb{Z}/2) \) appears in the same \( t \)-s degree as the first tower appearing in \( \text{Ext}^{s,t}_{A_1} (\Sigma^{2m} M(n), \mathbb{Z}/2) \). This indicates that we should try to prove the claim by induction on the number of powers of 2 appearing in the prime factorization of \( 2n \).

Assume the result for \( r = 2^l m \) for \( 1 \leq l \leq k \). Then for \( m = 2^k m \) we have the following 2 exact sequences:

(i) \( \Sigma^{2m} M(n) \rightarrow M(2n) \rightarrow M(2n-1) \)

and (ii) \( \Sigma^{2m} M(\frac{m}{2}) \rightarrow M(n) \rightarrow M(n-1) \).

The first vertical tower in \( \text{Ext} \) for \( M(2n) \) appears in the same \( t \)-s grading as \( \Sigma^{2m} M(n) \) in \( \text{Ext} \) for \( \Sigma^{2m} M(n) \). But note that \( M(2n-1) \) is \( \Sigma^{-d} \text{I}^d \text{J} \otimes M(1) \) where \( d = 2(n-1) - \alpha(n-1) \) and \( M(n-1) \) is \( \Sigma^{-e} \text{I}^e \text{J} \otimes M(1) \) where \( e = n-2 - \alpha(\frac{n-2}{2}) \). Note that \( \alpha(n-1) = \alpha(\frac{n-1}{2}) \therefore d \) and \( e \) differ by \( n-1 \). This observation completes a verification of the claim since we've just proved that the picture associated to (i) is one associated to (ii) but shifted to the right \( 2m \).