$V_{n,r}$ Bundles and Secondary Characteristic Classes

by

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§1. Introduction.

Let ξ be an n-plane bundle over B. ~~Let p: E → B the associated bundle with fibre $V_{n,r}$, the Assume that the mod two Stiefel-Whitney classes $W_{n-r+1}(\xi), \ldots, W_n(\xi)$ are all zero.~~ Let $p: E \to B$ be the associated bundle with fibre $V_{n,r}$, the Stiefel manifold of r-frames in R^n. Assume that the mod two Stiefel-Whitney classes $W_{n-r+1}(\xi), \ldots, W_n(\xi)$ are all zero. Then the fibre $V_{n,r}$ of p is totally non-homologous to zero with Z_2 coefficients. We study the [illegible] structure of $H^*(E)$ as a ring and as a module over the Steenrod algebra. ~~Analogously~~ [illegible] This leads to the introduction of invariants of the bundle ξ, in a way similar to that in [2] and [3].

In section 6, we show that these invariants determine all "secondary characteristic classes" with mod 2 coeff. defined when $W_{n-r+1}(\xi)=0, \cdots, W_n(\xi)=0$. This generalizes the results in [4], where the case $r=1$ was treated. In section 7, we show how these invariants behave when a trivial s-plane bundle is added to ξ.

§2. Some Numbers[3]

Let $s \geq r$ be positive integers. Define $\lambda(s,r)$ as that number $r \leq \lambda(s,r) \leq s$ such that there exists a number t with $2^t \mid \lambda(s,r)$ and ~~$2^t \nmid j$ for all $r \leq j \leq s$ and $j \neq \lambda(s,r)$~~ $2^{t+1} \nmid j$ for all j with $r \leq j \leq s$.

THEOREM 2.1. $\binom{\lambda(s,r)-1}{i-\lambda(s,r)} \not\equiv 0\ (2)$ for all i with $\lambda(s,r) \leq i \leq s$. Also, ~~then~~ for all h with $r \leq h < \lambda(s,r)$, $\binom{h-1}{\lambda(s,r)-h} \equiv 0\ (2)$.

Proof: For simplicity, we write $\lambda(s,r) = \lambda$. Let t be such that $2^t \mid \lambda$, $2^{t+1} \nmid \lambda$. Then $\lambda + 2^t > s$. The diadic expansion of $\lambda - 1$ is $\lambda - 1 = \left(\sum_{q=t+1}^{\infty} a_q 2^q\right) + 2^{t-1} + 2^{t-2} + \cdots + 1$.

Furthermore $i - \lambda \leq s - \lambda < 2^t$. ~~Hence $\binom{\lambda-1}{i-\lambda} \not\equiv 0(2)$~~ Hence its diadic expansion is $i - \lambda = \sum_{q=0}^{t-1} b_q 2^q$.

Thus $\binom{\lambda-1}{i-\lambda} \equiv \prod_{q=0}^{t-1} \binom{1}{b_q} \equiv 1\ (2)$ (see the appendix of [1]).

Let $k < \lambda$ with $r \le k$. Since $2^t \nmid k$, we have

$$k-1 = \sum_{q=0}^{\infty} c_q 2^q \text{ with some } c_q = 0 \text{ with } q < t.$$

Let $\bar{q}$ be the smallest q with $c_q = 0$. Then $k = \sum_{q=0}^{\infty} d_q 2^q$ with $d_{\bar{q}} = 1$, $d_q = 0$ for $q < \bar{q}$. Hence $\lambda - k = \sum_{q=0}^{\infty} e_q 2^q$ with $e_{\bar{q}} = 1$. Thus $\binom{k-1}{\lambda-k} \equiv 0\,(2)$.

REMARK. Note that λ is the smallest integer, $\lambda \ge r$, with $\binom{\lambda-1}{i-\lambda} \not\equiv 0\,(2)$ for all i with $\lambda \le i \le s$.

Let $1 \le r \le n$. Define $\lambda_0(n, n-r+1) = n+1$. By induction, define $\lambda_i(n, n-r+1) = \lambda(\lambda_{i-1}(n, n-r+1) - 1, n-r+1)$. This process stops when $\lambda_j(n, n-r+1) = n-r+1$. Define $\mu(n,r)$ as that j. Also, define $\gamma_i = \lambda_{i-1} - \lambda_i - 1$.

We close this section by translating the results of theorem 2.1 into statements about $H^*(V_{n,r})$ and $H^*(BO(n))$. Let $v_i \in H^i(V_{n,r})$ be the simple system of generators, $n-r \le i \le n-1$. ~~Then the inclusion $P^{n-1}/P^{n-r-1} \to V_{n,r}$ is a homotopy equivalence up to dimension~~ We have

$$Sq^t(v_i) = \binom{i}{t} v_{i+t} \text{ if } i+t \le n-1.$$

In a fibre bundle with fibre $V_{n,r}$ and group $O(n)$, v_i transgresses to W_{i+1}.

PROPOSITION 2.2. *Let* $\lambda_i = \lambda_i(n, n-r+1)$.

$$Sq^l(v_{\lambda_i-1}) = v_{\lambda_i-1+l} \quad \text{for } l = 0, \dots, \lambda_{i-1} - \lambda_i - 1 = \nu_i,$$

where $i = 1, \dots, \mu(n,r)$.

Proof: $Sq^l(v_{\lambda_i-1}) = \binom{\lambda_i-1}{l} v_{\lambda_i-1+l} = \binom{\lambda_i-1}{j-\lambda_i} v_{j-1}$

where $j = l + \lambda_i$. To apply theorem 2.1, we note that $\lambda_i \leq j \leq \lambda_{i-1} - \lambda_i - 1 + \lambda_i = \lambda_{i-1} - 1$.

Let $W_i \in H^i(BO(n))$ denote the universal Stiefel-Whitney classes, $i = 1, \dots, n$. $H^*(BO(n))$ is a polynomial ring on $W_1, \dots, W_n$. Similar to proposition 2.2 we have the following.

PROPOSITION 2.3. $Sq^l(W_{\lambda_i}) \equiv W_{\lambda_i+l}$ (mod ideal generated by $W_{\lambda_i}, \dots, W_{\lambda_i+l-1}$) for $l = 0, \dots, \nu_i$, where $i = 1, \dots, \mu(n,r)$.

Proof: Recall the Wu formula: $Sq^t(W_i) = \sum_{s=0}^{t} \binom{i-t+s-1}{s} W_{t-s} \smile W_{i+s}$. Thus proposition 2.3 is true if $\binom{\lambda_i-1}{l} \not\equiv 0$ in the stated range. This was proved in the proof of proposition 2.2.

§3. $V_{n,r}$ Bundles.

Let ξ be an n-plane bundle over B with group $O(n)$. Let $p(\xi) = p : E \to B$ be the associated bundle with fibre $V_{n,r}$. Let $p_T : E_T \to B$ be the associated bundle with fibre $T(V_{n,r})$, the cone on $V_{n,r}$. E_T may be considered as the mapping cylinder of p. p_T is a homotopy equivalence and we will study E by first considering the pair (E_T, E).

The universal bundle for this situation is when ξ is the canonical bundle over $BO(n)$, the Grassmann manifold of n-planes in R^∞. Here E can be taken to be $BO(n-r)$ and p the map induced by the inclusion $O(n-r) \subset O(n)$. Recall that $H^*(BO(n))$ is a polynomial ring on generators $W_1, \ldots, W_n$ and

$$p^*(W_i) = \begin{cases} W_i & \text{if } 1 \leq i \leq n-r \\ 0 & \text{if } n-r < i . \end{cases}$$

Consider the exact sequence of the pair $(BO(n-r)_T, BO(n-r))$:

$$\begin{tikzcd}
\cdots \arrow[r, "\delta^*"] & H^*(BO(n-r)_T, BO(n-r)) \arrow[r, "j^*"] & H^*(BO(n-r)_T) \arrow[r, "i^*"] & H^*(BO(n-r)) \arrow[r, "\delta^*"] & \cdots \\
& & H^*(BO(n)) \arrow[u, "p_T^*"] \arrow[ur, "p^*"'] & &
\end{tikzcd}$$

p_T^* is an isomorphism, p^* is an epimorphism, hence i^* is an epimorphism, $\delta^* = 0$, and j^* is a monomorphism. Thus, for $i = 1, \dots, \mu$, we may choose a unique element $U_i \in H^{\lambda_i}(BO(n-r)_T, BO(n-r))$ such that $j^*(U_i) = p_T^*(W_{\lambda_i})$.

For ~~the bundle~~ an arbitrary bundle ξ, we may define $U_i(\xi) \in H^{\lambda_i}(E_T, E)$, for $i = 1, \dots, \mu$, by

$U_i(\xi) = f_\xi^*(U_i)$, where $f_\xi : (E_T, E) \to (BO(n-r)_T, BO(n-r))$ is the classifying map.

Let $j_1 : V_{n,r} \to E$ and $j_2 : (T(V_{n,r}), V_{n,r}) \to (E_T, E)$ be the inclusion maps. Let δ' be the coboundary in the exact sequence of the pair $(T(V_{n,r}), V_{n,r})$.

THEOREM 3.1. $\delta'(v_{\lambda_i - 1}) = j_2^*(U_i(\xi))$.

This theorem is an immediate consequence of the following more general lemma which is of interest in itself.

Let $p : E \to B$ be a fibre space, $b_0 \in B$, $F = p^{-1}(b_0)$. The suspension homomorphism, $s : \operatorname{Ker} p^* \to \operatorname{Coker} j_1^*$, $j_1 : F \to E$, is of degree -1 and is, roughly speaking, the inverse of the transgression. Let E_T be the mapping cylinder of p, $p_T : E_T \to B$, and $T(F) = p_T^{-1}(b_0)$. Consider the following diagram:

$$\begin{tikzcd}
 & & H^q(B) \arrow[d, "p_T^*"] \arrow[dr, "p^*"] & \\
H^{q-1}(E) \arrow[r, "\delta^*"] \arrow[d, "j_1^*"] & H^q(E_T, E) \arrow[r, "j^*"] \arrow[d, "j_2^*"] & H^q(E_T) \arrow[r, "i^*"] & H^q(E) \\
H^{q-1}(F) \arrow[r, "\delta'"] & H^q(T(F), F). & &
\end{tikzcd}$$

Then $\delta'^{-1} j_2^* j^{*-1} p_T^*$ defines $\sigma : \operatorname{Ker} p^* \to \operatorname{Coker} j_1^*$.

LEMMA 3.2. $\sigma = -S$.

This lemma is proved by applying a rather general argument, valid for any triad, to the triad $(E_T; E, T(F))$.

§4. Iterated Steenrod ~~Operations~~ Squares in $BO(n)$.

Let $I = (i_1, \ldots, i_r)$ be an admissible sequence, i.e. $i_j \geq 2 i_{j+1}$. We define a set of polynomials $Q_i(I, k, l)$ in $w_1, \ldots, w_n$, whenever $I \neq (l)$, $l = 0, \ldots, \lambda_{i-1} - \lambda_i - 1 = \nu_i$ (give range of k.) so as to satisfy

$$(4.1) \quad Sq^I(w_{\lambda_i}) = \sum_{k=1}^{i} \sum_{l=0}^{\nu_k} Q_i(I, k, l) \cup Sq^l(w_{\lambda_k}).$$

In order to do this, one first writes $w_{\lambda_k + l}$, for $l = 0, \ldots, \nu_k$, $k = 1, \ldots, i$, as a linear combination of $\{Sq^m(W_{\lambda_k})\}$, $m = 0, \ldots, \ell$, with coefficients which are polynomials in $W_1, \ldots, W_n$. This can be done by proposition 2.3 using induction on ℓ. Then, using induction on I and the Wu formula, define polynomials $R_i(I, j)$ by

$$4.2) \quad Sq^I(W_{\lambda_i}) = \sum_{j=\lambda_i}^{n} R_i(I, j) \smile W_j .$$

Finally, substitute the above expressions for $W_{\lambda_k + \ell} = W_j$ into 4.2) to obtain 4.1). Also, denote by $Q(i, I)$ the polynomial $Q_i(I, k, \ell)$ when $\lambda_k + \ell = \lambda_i + n(I)$. If such k and ℓ exist, denote them by $k(i, I)$ and $\ell(i, I)$ respectively. If such k and ℓ do not exist, let $Q(i, I) = 0$. Note that $Q(i, I) = 0$ or 1.

Since $j^*(U_i) = p'^*(W_{\lambda_i})$, 4.1) implies that the Steenrod operations in $H^*(BO(n-r)_T, BO(n-r))$ are given by:

$$4.3) \quad Sq^I(U_i) = \sum_{k=1}^{i} \sum_{\ell=0}^{\gamma_k} p_T^*(Q_i(I, k, \ell)) \smile Sq^\ell(U_k) .$$

§5. $V_{n,r}$ Bundles with ~~the Fibre Totally Non~~ $W_{n-r+1}=0, \ldots, W_n=0$.

Let $p: E \to B$ be the associated bundle to ξ with fibre $V_{n,r}$ and group $O(n)$. Assume that $W_{n-r+1}(\xi)=0$, $W_{n-r+2}(\xi)=0, \ldots, W_n(\xi)=0$. (This is equivalent to the statement that the fibre is totally non-homologous to zero with Z_2 coefficients.) In the cohomology sequence of the pair (E_T, E), p^* is a monomorphism, i^* is a monomorphism, $j^*=0$, and δ^* is an epimorphism.

Let $a=(a_1, \ldots, a_\mu)$, $a_i \in H^{\lambda_i - 1}(E)$ be such that

$\delta^*(a_i) = U_i(\xi)$. Since $\delta' j_1^*(a_i) = j_2^* \delta^*(a_i) = j_2^*(U_i(\xi))$

$= \delta'(v_{\lambda_i - 1})$ by theorem 3.1, and δ' is an isomorphism, we see that

$j_1^*(a_i) = v_{\lambda_i - 1}$.

Now $\{ v_{j_1} \ldots v_{j_s} \}$, $n-r+1 \leq j_1 < j_2 < \cdots < j_s \leq n-1$ form a base for $\tilde{H}^*(V_{n,r})$ as a Z_2-module. Using proposition 2.2 and the fact that ~~the~~ $V_{n,r}$ is totally non-homologous to zero in E, we see that ~~[illegible]~~ $u \in H^*(E)$ ~~$H^*(E)$ is a free module over $H^*(B)$~~ can be written uniquely in the form

$$5.1)\quad u = p^*(u_0) + \sum_{1\le k_1<\cdots<k_s\le\mu} \sum_{l_j=0}^{\nu_{k_j}} p^*(u(k,l,a)) \smile Sq^{l_1}(a_{k_1}) \smile \cdots$$

$\smile Sq^{l_s}(a_{k_s})$, where $k=(k_1,\ldots,k_s)$ and $l=(l_1,\ldots,l_s)$.

If we assume that $H^*(B)$ is known as an algebra over the Steenrod algebra, then in order to know $H^*(E)$ as an algebra over the Steenrod algebra, it sufficies to know the expansion of $Sq^I(a_i)$. $\delta^*(Sq^I(a_i)) =$

$$Sq^I(\delta^*(a_i)) = Sq^I(U_{\lambda_i}(\xi)) = \sum_{k=1}^{i} \sum_{l=0}^{\nu_k} p_T^*(Q_i(I,k,l)(\xi)) \smile Sq^l(U_k(\xi)).$$

Dropping the symbol ξ we have, $\delta^*(Sq^I(a_i)) =$

$$\sum_{k=1}^{i}\sum_{l=0}^{\nu_k} p_T^*(Q_i(I,k,l)) \smile \delta^*(Sq^l(a_k)) = \delta^*\Big(\sum_{k=1}^{i}\sum_{l=0}^{\nu_k} p^*(Q_i(I,k,l)) \smile$$

$Sq^l(a_k)\Big)$. By exactness of the cohomology sequence of (E_T, E), we define $\alpha_i(I,a) \in H^*(B)$ by

$$5.2)\quad Sq^I(a_i) = p^*(\alpha_i(I,a)) +$$

$$\sum_{k=1}^{i}\sum_{l=0}^{\nu_k} p^*(Q_i(I,k,l)) \smile Sq^l(a_k).$$

Note that $s=1$ in the expansion of $Sq^I(a_i)$.

However, a is not unique. If $\delta^*(a_i) = \delta^*(a'_i) = U_i(\xi)$, then there exists $b = (b_1, \dots, b_\mu)$, $b_i \in H^{\lambda_i - 1}(B)$, such that $a' = a + p^*(b)$. Substituting in 5.2), we easily see that

5.3) $\alpha_i(I, a') = \alpha_i(I, a) + Sq^I(b_i) + \sum_{k=1}^{i} \sum_{l=0}^{\nu_k} Q_i(I, k, l) \cup Sq^l(b_k).$

Hence, we may define an invariant of the bundle ξ by

$$\alpha_i(I, \xi) = \{\alpha_i(I, a)\} \in H^{\lambda_i - 1 + n(I)}(B) \big/ A_i(I, \xi),$$

where

$$A_i(I, \xi) = \left(Sq^I + \sum_{l=0}^{\nu_i} Q_i(I, i, l) \cup Sq^l\right)\left(H^{\lambda_i - 1}(B)\right) + \sum_{k=1}^{i-1} \left(\sum_{l=0}^{\nu_k} Q_i(I, k, l) \cup Sq^l\right)\left(H^{\lambda_k - 1}(B)\right).$$

(Here, as above, $Q_i(I, k, l)$ denotes $Q_i(I, k, l)(\xi)$.)

REMARK. It is clear that these invariants are natural with respect to bundle maps.

We conclude this section with a property of $\alpha_i(I, \xi)$.

THEOREM 5.4. If $p: E \to B$ has a cross-section, then $\alpha_i(I, \xi) = 0$.

Proof: Let $a_i' = a_i + p^* s^*(a_i)$, where $s : B \to E$ is a cross-section. Then $s^*(a_i') = 0$ and

$$s^*(Sq^I(a_i')) = Sq^I(s^*(a_i')) = 0 = \alpha_i(I, a') + 0 \text{ by } 5.2).$$

§6. The Universal Example.

The classes $\alpha_i(I, \xi)$ are secondary characteristic classes defined whenever $W_{n-r+1}(\xi) = 0, \ldots, W_n(\xi) = 0$. We now study the universal example of such bundles ξ and show that any secondary characteristic class is determined by the ones already defined.

Let $W : BO(n) \longrightarrow \prod_{i=1}^{\mu} K(Z_2, \lambda_i)$ be defined by $W^*(\iota(i)) = W_{\lambda_i}$, where $\iota(i) \in H^{\lambda_i}(Z_2, \lambda_i)$ is the canonical generator. Define $\pi : K_n^{(n)} \longrightarrow BO(n)$ to be the fibre space induced by W from the path space over $\prod_{i=1}^{\mu} K(Z_2, \lambda_i)$, and let $h : F = \prod_{i=1}^{\mu} K(Z_2, \lambda_i - 1) \longrightarrow K_n^{(n)}$ be the inclusion of the fibre in the total space.

Let $f = f(\xi) : B \to BO(n)$ be the classifying map for bundle ξ. If $W_n(\xi) = f^*(W_n) = 0, \dots, W_{n-r+1}(\xi) = 0$, then there exists an $\bar{f} : B \to K_n^{(r)}$ such that $\pi \bar{f} \simeq f$. π is a principal fibre space; let $\mu : F \times K_n^{(r)} \to K_n^{(r)}$ be the operation of the fibre on the total space. Then if $\bar{f}_i : B \to K_n^{(r)}$ are such that $\pi \bar{f}_i \simeq f$, $i = 1, 2$, there there exists $u : B \to F$ such that $\mu(u, \bar{f}_1) \simeq \bar{f}_2$ (c.f. [5]).

If $\phi \in H^*(K_n^{(r)})$, we define a secondary characteristic class $\Phi(\xi) = \{\bar{f}^*(\phi)\} \subset H^*(B)$, where $\bar{f}$ runs through all maps $\bar{f}$ such that $\pi \bar{f} \simeq f$.

We now study $H^*(K_n^{(r)})$. Consider the cohomology spectral sequence of the fibre-map π. Let τ denote the transgression and $\iota(i) \in H^{\lambda_i - 1}(Z_2, \lambda_i - 1)$ denote the canonical generator. Then $\tau(\iota(i)) = W_{\lambda_i}$. Hence $d_{n(I)+\lambda_i}(Sq^I(\iota(i)))$

$$\tau(Sq^I(\iota(i))) = Sq^I(W_{\lambda_i}) = \sum_{k=1}^{i} \sum_{l=0}^{\nu_k} Q_i(I,k,l) \smile Sq^l(W_{\lambda_k})$$

by 4.1). If $I \neq (l)$, $l = 0, \dots, \nu_i$, and $n(I) + \lambda_i \neq l + \lambda_k$, then the corresponding term on the right hand side is $d_{\ell+\lambda_k}\left(Q_i(I,k,\ell)\otimes Sq^{\ell}(\iota(k))\right)$

I don't understand this ?!

with $\ell+\lambda_k < n(I)+\lambda_i$. Thus, in $E_{n(I)+\lambda_i}$,

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$$d_{n(I)+\lambda_i}\left(Sq^{I}(\iota(i))\right) = d_{n(I)+\lambda_i}\left(Q(i,I)\smile Sq^{\ell(i,I)}(\iota(k(i,I)))\right).$$

Hence we have proved the following proposition.

PROPOSITION 6.1. For every admissible sequence I, there exists $\phi_i(I)\in H^{\lambda_i-1+n(I)}(K_n^{(n)})$ such that $h^*(\phi_i(I)) = Sq^{I}(\iota(i)) + Q(i,I)\smile Sq^{\ell(i,I)}(\iota(k(i,I)))$.

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We now compute the structure of $H^*(K_n^{(n)})$.

THEOREM 6.2. Let $\phi_i(I)$ satisfy 6.1. Then $H^*(K_n^{(n)})$ is a polynomial algebra on generators $\pi^*(W_1),\dots,\pi^*(W_{n-r})$, and $\{\phi_i(I)\}$, where (i,I) runs through all pairs satisfying the following conditions:

1) $i=1,\dots,\mu$,
2) for each i, I is admissible,
3) for each i, $e(I)<\lambda_i-1$ except as in 5),
4) for each i, do not include $I=\cancel{\phi\ \text{or}}\ (\ell)$, $\ell=1,\dots,\gamma_i$,
5) for each i, include $I=(\lambda_i-1)$ and $(\lambda_i-1+\ell,\ell)$ for $\ell=1,\dots,\gamma_i$.

Proof: Consider the spectral sequence discussed above. Using the known structure of $H^*(Z_2, \lambda_i - 1)$, we see that E_2 is a polynomial algebra on generators $W_1, \ldots, W_n$, $\{Sq^I(\iota(i))\}$, where (i, I) runs through all pairs with $i = 1, \ldots, \mu$, I admissible, and $e(I) < \lambda_i - 1$. The proof is based on successive applications of the following easy lemma.

LEMMA 6.3. Let A be an algebra over Z_2 with differentiation d. If A is a polynomial algebra on generators $x_1, x_2, \ldots$, and $d(x_1) = x_2$, $d(x_q) = 0$ if $q > 1$, then $H(A)$ is a polynomial algebra on $x_1^2, x_3, \ldots$.

The first non-trivial d in the spectral sequence is d_{n-r+1} and $d_{n-r+1}(\iota(\mu)) = W_{n-r+1} = W_{\lambda_\mu}$. d_{n-r+1} is zero on all other generators and thus E_{n-r+2} is a polynomial algebra on $W_1, \ldots, W_{n-r}, W_{n-r+2}, \ldots, W_n$, $\{Sq^I(\iota(i))\}$, where (i, I) run through all pairs with $i = 1, \ldots, \mu$, I admissible, $e(I) < \lambda_i - 1$ except (μ, ϕ) which corresponds to $\iota(\mu)$ and we must add the pair $(\mu, (\lambda_\mu - 1))$ which corresponds to $\iota(\mu)^2$. $d_{n-r+2}(Sq^1(\iota(\mu))) = Sq^1(d_{n-r+1}(\iota(\mu))) = Sq^1(W_{n-r+1}) = W_1 \cdot W_{n-r+1} + W_{n-r+2} = \{W_{n-r+2}\}$, and d_{n-r+2} is zero on all other generators.

Thus E_{n-r+3} is a polynomial algebra on $W_1, \dots, W_{n-r}, W_{n-r+3}, \dots, W_n$, $\{Sq^I(\iota(i))\}$, where (i, I) run through all pairs with $i = 1, \dots, \mu$, I admissible, $e(I) < \lambda_i - 1$ except (μ, ϕ) and $(\mu, (1))$ and we must add $(\mu, (\lambda_\mu - 1))$ and $(\mu, (\lambda_\mu, 1))$. Continue in this manner. ~~until we reach~~ $\cancel{E_{\lambda_{\mu-1}}, d_{\lambda_{\mu-1}}(\iota(\mu-1)) = W_{\lambda_{\mu-1}}}$ ~~and we proceed as~~ Starting with E_{n-r+4}, however, it is possible that more than one generator goes into the same element under d. In order to avoid this, we take a new set of generators: ${}'Sq^I(\iota(i)) = Sq^I(\iota(i)) + Q(i, I) \cup Sq^{l(i,I)}(\iota(k(i,I)))$ instead of $Sq^I(\iota(i))$. Then $d(Sq^{l(i,I)}(\iota(k(i,I)))) \neq 0$ but $d({}'Sq^I(\iota(i))) = 0$ and we may continue the process as before. ~~[illegible]~~

Thus, E_∞ is a polynomial algebra on generators $W_1, \dots, W_{n-r}$, $\{{}^{\prime}Sq^I(\iota(i))\}$, where (i, I) run through all pairs satisfying conditions 1) to 5) of theorem 6.2. If $\phi_i(I)$ satisfies 6.1, then it projects to ${}^{\prime}Sq^I(\iota(i))$ in E_∞. Hence there are no polynomial relations between $\pi^*(W_1), \dots, \pi^*(W_{n-r})$, $\{\phi_i(I)\}$ in $H^*(K_n^{(r)})$. By counting elements in each dimension, we see that there can be no other elements and the theorem is proved.

Now let ξ be the n-plane bundle over $K_n^{(r)}$ induced by π from the canonical bundle over $BO(n)$. Let $p(\xi) ~~be the~~ : E_n^{(r)} \to K_n^{(r)}$ be the associated bundle with fibre $V_{n,r}$.

THEOREM 6.4. There exists an $a = (a_1, \dots, a_\mu)$, $a_i \in H^{\lambda_i - 1}(E_n^{(r)})$ such that $\phi_i(I) = \alpha_i(I, a) \in H^{\lambda_i - 1 + n(I)}(K_n^{(r)})$ satisfies 6.1.

As an immediate corollary of theorem 6.4, we have the following.

CO R O L L A R Y 6.5. Any secondary characteristic class is a polynomial in $W_1, \ldots, W_{n-r}$, and those described in theorem 6.2.

Proof of theorem 6.4: Let $\pi_1 : (K_n^{(r)}, F) \to (BO(n), *)$, $\pi_2 : (E_{n\,T}^{(r)}, E_n^{(r)}) \to (BO(n-r)_T, BO(n-r))$, $\pi_3 : E_n^{(r)} \to BO(n-r)$, $P_T : BO(n-r)_T \to BO(n)$, and $P_T^{(r)} : E_{n\,T}^{(r)} \to K_n^{(r)}$. π_3 is a fibre map with fibre F, $h^{(r)} : F \to E_n^{(r)}$ is the inclusion of the fibre in the total space and $p(\xi)h^{(r)} \simeq h$. It is easy to check that the following two compositions are homotopic.

$$(E_{n\,T}^{(r)}, F) \xrightarrow{h_1^{(r)}} (E_{n\,T}^{(r)}, E_n^{(r)}) \xrightarrow{\pi_2} (BO(n-r)_T, BO(n-r))$$

and
$$(E_{n\,T}^{(r)}, F) \xrightarrow{P_T^{(r)}} (K_n^{(r)}, F) \xrightarrow{\pi_1} (BO(n), *) \xrightarrow{P_T^{-1}} (BO(n-r)_T, *) \xrightarrow{j} (BO(n-r)_T, BO(n-r))$$
where P_T^{-1} is the "zero" cross-section of P_T. $P_T^{(r)}$ and P_T^{-1} induce isomorphisms on cohomology. Let $\delta : H^{\lambda_i - 1}(F) \to H^{\lambda_i}(E_{n\,T}^{(r)}, F)$.

By definition of transgression in the fibre space $\pi: K_n^{(n)} \to BO(n)$ and the fact that $j^*(U_i) = W_{\lambda_i}$, we have $p_T^{(n)*}\, \pi_1^*\, p_T^{-1\,*}\, j^*(U_i) = \delta(\iota(i))$. Thus, $h_1^{(n)*}\, \pi_2^*(U_i) = \delta(\iota(i))$. Let $a' = (a'_1, \dots, a'_\mu)$ be such that $\delta^*(a'_i) = \pi_2^*(U_i)$. Then $h^{(n)*}(a'_i)$ is such that $\delta(h^{(n)*}(a'_i)) = h_1^{(n)*}\delta^*(a'_i) = h_1^{(n)*}\pi_2^*(U_i) = \delta(\iota(i))$. Let $j_0^{(n)}: E_n^{(n)} \to E_{nT}^{(n)}$ and $j_1: F \to E_{nT}^{(n)}$. ~~Let~~ then there exists an element $b_i \in H^{\lambda_i - 1}(E_{nT}^{(n)})$ such that $\iota(i) = h^{(n)*}(a'_i) + j_1^*(b_i)$. ~~Then~~ Set $a_i = a'_i + j^{(n)*}(b_i)$ and we have $h^{(n)*}(a_i) = \iota(i)$ and $\delta^*(a_i) = \pi_2^*(U_i)$. By 5.2),

$$Sq^I(a_i) = p(\xi)^*(\alpha_i(I,a)) + \sum_{k=1}^{i}\sum_{l=0}^{\nu_k} p(\xi)^*\pi^*(Q_i(I,k,l)) \smile Sq^l(a_k).$$

Apply $h^{(n)*}$ to this equation. $h^{(n)*}\, p(\xi)^*\, \pi^*(Q_i(I,k,l)) = h^{(n)*}\pi_3^*\, p^*(Q_i(I,k,l)) = 0$ unless the dimension of $Q_i(I,k,l)$ is zero. Thus $h^{(n)*}$ of the above equation becomes $Sq^I(\iota(i)) = h^*(\alpha_i(I,a)) + Q(i,I) \smile Sq^{l(i,I)}(\iota(k(i,I)))$ as $h^{(n)*}\, p(\xi)^* = h^*$.

§7. Stability Properties.

In this section we show that, modulo a possibly larger subgroup, the secondary characteristic classes $\alpha_i(I, \xi)$ are invariants of the stable class of the bundle ξ.

Let ξ be an n-plane bundle over B with $W_{n-r+1}(\xi) = 0, \dots, W_n(\xi) = 0$. Let Θ^s denote the trivial s-plane bundle over B. Let $\eta = \xi \oplus \Theta^s$. Clearly, $W_{n-r+1}(\eta) = 0, \dots, W_{n+s}(\eta) = 0$. Let $p(\xi) : E(\xi) \to B$ and $p(\eta) : E(\eta) \to B$ be the associated $V_{n,r}$ and $V_{n+s,r+s}$ bundles respectively. Clearly, there is a fibre-preserving map $g : E(\xi) \to E(\eta)$ which is the usual inclusion on the fibre. Let $\lambda_i = \lambda_i(n, n-r+1)$, $\lambda'_i = \lambda_i(n+s, n-r+1)$, $\mu = \mu(n,r)$, $\mu' = \mu(n+s, r+s)$. ~~Clearly, there exists a t such that~~ Let $t = \mu' - \mu$. Then

$$\lambda_i = \lambda'_{i+t} \quad \text{if } i = 1, \dots, \mu.$$

LEMMA 7.1. We can choose $a = (a_1, \dots, a_{\mu'})$,

$a_i \in H^{\lambda_i'-1}(E(\eta))$ so that $g^*(a_i) = 0$ if $i = 1, \dots, t$ and so that $g^*(a) = (g^*(a_{t+1}), \dots, g^*(a_{\mu'}))$ is an a for $H^*(E(\xi))$.

<u>Proof:</u> We will denote $BO(n-r)_T$ for the bundle $p: BO(n-r) \to BO(n)$ by $BO(n)$. Let $k: (BO(n), BO(n-r)) \to (BO(n+s), BO(n-r))$. Then $k^*(U_i') = 0$ if $i = 1, \dots, t$ and $k^*(U_i') = U_{i-t}$ if $i = t+1, \dots, \mu'$. Consider the diagram:

$$
\begin{tikzcd}
H^*(E(\eta)_T) \arrow[r, "i(\eta)^*"] \arrow[d, "\approx"] & H^*(E(\eta)) \arrow[r, "\delta"] \arrow[d, "g^*"] & H^*(E(\eta)_T, E(\eta)) \arrow[d, "g_1^*"] & H^*(BO(n+s), BO(n-r)) \arrow[l, "f_\eta^*"'] \arrow[d, "k^*"] \\
H^*(E(\xi)_T) \arrow[r, "i(\xi)^*"] & H^*(E(\xi)) \arrow[r, "\delta"] & H^*(E(\xi)_T, E(\xi)) & H^*(BO(n), BO(n-r)). \arrow[l, "f_\xi^*"']
\end{tikzcd}
$$

$\delta g^*(a_i) = g_1^* \delta(a_i) = g_1^* f_\eta^*(U_i') = f_\xi^* k^*(U_i') = f_\xi^*(U_{i-t})$ if $i = t+1, \dots, \mu'$. If $i = 1, \dots, t$, then $\delta g^*(a_i) = 0$ and $g^*(a_i) = i(\xi)^*(b_i)$. Set $a_i' = a_i + i(\eta)^*(b_i)$ and the lemma is proven.

Recall that $\alpha_i(I, \xi) \in H^{\lambda_i - 1 + n(I)}(B) / A_i(I, \xi)$, where $A_i(I, \xi) = \left(Sq^I + \sum_{l=0}^{\nu_i} Q_i(I, i, l)(\xi) \cup Sq^l\right)(H^{\lambda_i - 1}(B)) +$

$$\sum_{k=1}^{i-1}\left(\sum_{l=0}^{\nu_k} Q_i(I,k,l)(\xi)\smile Sq^l\right)(H^{\lambda_k-1}(B)).$$ Also,

$$\alpha_i(I,\eta)\in H^{\lambda_i'-1+n(I)}(B)\big/A_i(I,\eta),$$ where

$$A_i(I,\eta)=\left(Sq^I+\sum_{l=0}^{\nu_i'} Q_i'(I,i,l)(\eta)\smile Sq^l\right)(H^{\lambda_i'-1}(B))+$$

$$\sum_{k=1}^{i-1}\left(\sum_{l=0}^{\nu_k'} Q_i'(I,k,l)(\eta)\smile Sq^l\right)(H^{\lambda_k'-1}(B)).$$ Since

$\lambda_i=\lambda_{i+1}'$ and $\cancel{Q_i(I,k,l)}$ $Q_i(I,k,l)(\xi)=$

$Q_{i+1}'(I,k+1,l)(\eta)$, we have $A_i(I,\xi)\subset A_{i+1}(I,\eta)$ for

$i=1,\ldots,\mu$.

THEOREM 7.2. $\{\alpha_i(I,\xi)\}=\alpha_{i+1}(I,\eta)\in H^{\lambda_i-1}(B)\big/A_{i+1}(I,\eta)$

Proof: Choose a as in lemma 7.1. Then

$$Sq^I(a_{i+1})=p(\eta)^*(\alpha_{i+1}(I,a))+\sum_{k=1}^{i+1}\sum_{l=0}^{\nu_k'} p(\eta)^* Q_{i+1}'(I,k,l)(\eta)\smile Sq^l(a_k).$$

Appy g^* to this equation to obtain

$$Sq^I(g^*(a_{i+1}))=p(\xi)^*(\alpha_{i+1}(I,a))+\sum_{k=1+1}^{i+1}\sum_{l=0}^{\nu_k'} p(\xi)^* Q_i(I,k-1,l)(\xi)\smile Sq^l(g^*(a_k)).$$

Thus $\alpha_i(I,g^*(a))=\alpha_{i+1}(I,a)$. This proves the theorem.

COROLLARY 7.3. If ξ and ξ' are two n-plane bundles over B ~~which are~~ with $W_{n-r+1}(\xi) = W_{n-r+1}(\xi') = 0, \ldots, W_n(\xi) = W_n(\xi') = 0$ which are stably equivalent, then $\{\alpha_i(I, \xi)\} = \{\alpha_i(I, \xi')\} \in H^{\lambda_i - 1}(B) / A_{i+1}(I, \eta)$, where $\eta = \xi \oplus \theta^s = \xi' \oplus \theta^s$.

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Footnotes

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[2] Throughout this paper, all cohomology groups and rings have Z_2 coefficients.

[3] Some of the formalism is made more complicated by the numbers introduced in this section; however, the analysis of the universal example is simplified.